6. Continuity of Correspondences and Berge’s Maximum Theorem

(cf. FMEA Ch. 14.1-2; DLF Ch. 2.11)

**Definition 6.1.** Let \((X, d), (Y, d')\) be metric spaces. A correspondence \(F\) from \(X\) to \(Y\), denoted by \(F : X \rightarrow Y\), is called

(1) *upper hemicontinuous (uhc)* at \(x \in X\) if for every open set \(V\) in \(Y\) containing \(F(x)\) there is a neighborhood \(U\) of \(x\) such that \(F(x') \subseteq V\) for all \(x' \in U\).

(2) *lower hemicontinuous (lhc)* at \(x \in X\) if for every open set \(V\) in \(Y\) with \(F(x) \cap V \neq \emptyset\) there is a neighborhood \(U\) of \(x\) such that \(F(x') \cap V \neq \emptyset\) for all \(x' \in U\).

(3) *continuous* at \(x \in X\) if it is uhc and lhc at \(x\).

A correspondence is continuous (resp. uhc, lhc) if it is continuous (resp. uhc, lhc) at every point in its domain.
**Theorem 6.1.** Let \((X, d), (Y, d')\) be metric spaces, \(f : X \rightarrow Y\) be a function, and \(F : X \rightarrow Y\) the correspondence defined by \(F(x) = \{f(x)\}\). Then the following statements are equivalent:

(i) \(f\) is continuous at \(x \in X\).
(ii) \(F\) is uhc at \(x \in X\).
(iii) \(F\) is lhc at \(x \in X\).

**Definition 6.2.**
A correspondence \(F : X \rightarrow Y\) is called \textit{closed} if, considered as a relation, it is a closed subset of \(X \times Y\), i.e.:

For every sequence \((x_n)\) in \(X\) converging to \(x \in X\) and every sequence \((y_n)\) in \(Y\) converging to \(y \in Y\) it follows from \(y_n \in F(x_n)\) for all \(n \in \mathbb{N}\) that \(y \in F(x)\).
**Theorem 6.2.** Let \((X, d), (Y, d')\) be metric spaces and let \(F : X \rightarrow Y\) be a correspondence. Then

(1) If \(F\) is uhc and closed-valued, then \(F\) is closed.

(2) If \(F\) is closed and \(Y\) is compact, then \(F\) is uhc (and compact-valued).

**Theorem 6.3.** Let \((X, d), (Y, d')\) be metric spaces and let \(F : X \rightarrow Y\) be a correspondence. Then

(1) \(F\) is lhc if and only if for every sequence \((x_n)\) in \(X\) converging to \(x\) and every \(y \in F(x)\) there is a sequence \((y_n)\) with \(y_n \in F(x_n)\) converging to \(y\).

(2) \(F\) is compact-valued and uhc if and only if for every sequence \((x_n)\) in \(X\) converging to \(x\) and every sequence \((y_n)\) with \(y_n \in F(x_n)\) there is a subsequence of \((y_n)\) that converges to some \(y \in F(x)\).
Theorem 6.4. Let $(X, d), (Y, d')$ be metric spaces and let $F : X \rightarrow Y$ be compact-valued and uhc. If $K \subseteq X$ is compact, then $F(K)$ is compact.

Theorem 6.5. Let $(X, d), (Y, d'), (Z, d'')$ be metric spaces and let $F : X \rightarrow Y, G : Y \rightarrow Z$ be correspondences. Then the composition $G \circ F : X \rightarrow Z$, defined by

$$G \circ F(x) = G(F(x)) = \bigcup_{y \in F(x)} G(y)$$

is lhc (resp. uhc, compact-valued and uhc) if both $F$ and $G$ are lhc (resp. uhc, compact-valued and uhc).
Definition 6.3. Given a set $P$ of parameters, a set $X$ of basic alternatives, a correspondence $B : P \rightarrow X$ restricting the possible choices in $X$, and an objective function $f : P \times X \rightarrow \mathbb{R}$, the **parametric optimization problem** $(P, X, B, f)$ is to find for every $p \in P$ an $x^* \in B(p)$ such that

$$\forall x \in B(p) : f(p, x^*) \geq f(p, x).$$

Theorem 6.6. (Berge 1959) If $(P, X, B, f)$ is a parametric optimization problem with metric spaces $P$ and $X$ such that $B$ is compact-valued and continuous and $f$ is continuous then

$$M(p) := \{x \in B(p) | \forall y \in B(p) : f(p, x) \geq f(p, y)\}$$

defines a compact-valued, uhc choice correspondence and the maximum value function defined by

$$m(p) := \max_{x \in B(p)} f(p, x)$$

is continuous.
Example 6.1.
Let \( P = \{ p \in \mathbb{R}^2_+ | p_1 + p_2 = 1 \} \) and \( X = \mathbb{R}^2_+ \).
Then
\[
B(p) := \{ x \in X | p \cdot x \leq 1 \}
\]
and
\[
f(p, x) := x_1 + x_2
\]
satisfy the assumptions of Theorem 6.6.
The correspondence \( M \) is not continuous because it is not lhc at \( p = (1/2, 1/2) \).

Example 6.2. Let
\[
P = \{ p \in \mathbb{R}^2_+ | p_1 + p_2 = 1 \},
\]
\[
X = \{ x \in \mathbb{R}^2_+ | x \leq (10, 10) \},
\]
and
\[
B(p) := \{ x \in X | p \cdot x \leq p \cdot e \}
\]
with \( e = (1, 0) \).
The correspondence \( B \) is not continuous because it is not lhc at \( p = (0, 1) \).