Abstract: The paper surveys some recent contributions to the analysis of pseudomonotone maps and their application to economics. It is shown that the concept of pseudomonotonicity is strongly related to a notion of rationality of consumer behaviour which is well known in economics as a weak version of the Weak Axiom of Revealed Preference. In economic models of general equilibrium, a pseudomonotone aggregate excess demand function is seen to guarantee convexity of the set of price equilibria and, in the case of so-called regular economies, even the uniqueness of equilibrium. Some general characterizations of differentiable pseudomonotone maps, which reduce to necessary and sufficient first order conditions under the assumption of regularity, are provided. These conditions are employed in order to prove pseudomonotonicity of aggregate excess demand in economies displaying a particular kind of heterogeneous consumer behaviour.

1 Introduction

In general competitive economic analysis most of the relevant properties of equilibrium are closely related to the structure of the demand side of the economy. As it is well-known, at a general level of analysis the aggregate function of excess demand has not any useful properties other than Walras’ Law and homogeneity of zero degree in prices. This means that not much can be said about uniqueness and stability of equilibrium, therefore questions of practical relevance such as comparative statics effects cannot be answered.

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Placing strong restrictions on the nature of individual behaviour does not seem to be an acceptable solution to the problem of indeterminacy of equilibrium. In fact, as shown by Mantel [21], the excess demand may lack any useful properties even when all consumers have (not identical) homothetic preferences, i.e. when individual demand functions are linear in income.

It is clear that assumptions of a different nature are required. An alternative approach is to take into account the distribution of the individual characteristics as in [14] and [11]. In these early examples it is shown that heterogeneity in either the patterns of consumption or incomes may lead to some structure of aggregate demand. However, which structure is needed?

In this paper we focus on an important property called the Weak Weak Axiom of Revealed Preference (WWA). The WWA is a rather mild requirement of rationality for individual behaviour but, as will be seen, it is a sufficiently good property for aggregate demand.

As it turns out, what lies behind the definition of the WWA is actually a notion of generalized monotonicity known as pseudomonotonicity. We provide some general characterizations of pseudomonotonicity for continuously differentiable functions and show that these results allow for a more convenient specification of the WWA. In addition, differential characterizations are shown to be quite effective, since they help formulate a plausible and economically interpretable hypothesis leading to the WWA in the aggregate and guaranteeing uniqueness of equilibrium in many cases.

The plan of the paper is as follows. Section 2 provides some preliminary definitions and establishes the relationship between pseudomonotonicity and the WWA for both individual and aggregate demand. Section 3 shows that the WWA for aggregate demand guarantees convexity of the equilibrium set in general, and uniqueness of the equilibrium prices in a large class of production economies. In Section 4, some characterizations of pseudomonotonicity for continuously differentiable maps are obtained and are used in Section 5 to derive similar characterizations of the WWA for individual and aggregate demand. Finally, Section 6 reviews a recent contribution by Jerison [15] to the theory of general equilibrium in which the differential characterizations of the WWA are successfully employed in conjunction with a specific formulation of the idea of heterogeneity in consumer behaviour.
2 The Weak Weak Axiom and Pseudomonotonicity

The consumption behaviour of an agent in a competitive economy is described by a continuous function \( f: \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+ \), where \( n \) is the number of goods. At every vector of prices \( p \) and every level of nominal income \( w \), the individual demand function \( f \) associates the corresponding bundle of goods demanded by the agent. This function is required to be homogeneous of degree zero in \( p \) and \( w \), i.e. \( f(\lambda p, \lambda w) = \lambda f(p, w) \) for all \( \lambda > 0 \), and to satisfy the budget identity, i.e. \( p^T f(p, w) = w \). An additional condition which is commonly satisfied by demand is the Weak Weak Axiom of Revealed Preference.

**Definition 1.** The individual demand function \( f \) satisfies the Weak Weak Axiom of Revealed Preference (WWA) if, for all pairs \((p, w)\) and \((q, w')\),

\[
q^T f(p, w) \leq w' \quad \text{implies} \quad p^T f(q, w') \geq w. \tag{1}
\]

The WWA is a milder version of the Weak Axiom of Revealed Preference introduced by Samuelson [25] and requires a certain degree of consistency of consumer’s choice. By definition, at prices \( q \) and income \( w' \) the agent chooses the bundle \( f(q, w') \). If at the same prices and income the consumer can also afford the bundle \( f(p, w) \), i.e. if \( q^T f(p, w) \leq w' \), then we can infer that the consumer considers \( f(q, w') \) to be at least as good as \( f(p, w) \). On the other hand, if at prices \( p \) and income \( w \) the bundle \( f(q, w') \) were cheaper than the chosen bundle \( f(p, w) \), he can also afford a bundle \( x \) that contains more of every commodity than \( f(q, w') \). Assuming the goods to be desirable, the consumer prefers \( x \) to \( f(q, w') \) and, consequently, \( x \) to \( f(p, w) \). This, however, contradicts the fact that he has chosen \( f(p, w) \) although \( x \) was affordable. Therefore, we are led to the conclusion that \( f(q, w') \) cannot be cheaper than \( f(p, w) \) and thus \( p^T f(q, w') \geq w \) as required by the WWA.

Although the WWA is a relatively mild rationality requirement, as compared to the hypothesis of utility maximization, the restrictions it places on demand are substantially similar to those implied by standard consumer theory. Broadly speaking, the WWA amounts to the requirement that the ‘compensated’ demand of a consumer obeys the ‘Law of Demand’, namely, that prices and quantities are inversely related when consumer’s income is adequately compensated (see also Section 5).

If \( f \) satisfies the WWA then, trivially, for any fixed income level \( w \)

\[
q^T f(p, w) \leq w \quad \text{implies} \quad p^T f(q, w) \geq w \tag{2}
\]
for all prices \( p \) and \( q \). On the other hand, if (2) holds for \textit{some} level of income \( w \) and all prices \( p \) and \( q \), then, by using homogeneity, one can easily show that \( f \) satisfies the WWA.\(^1\) Moreover, by the budget identity, (2) can be equivalently written as
\[
(q - p)^T f(p, w) \leq 0 \quad \text{implies} \quad (q - p)^T f(q, w) \leq 0. \tag{3}
\]
Condition (3) is simply the definition of a mathematical property put forward by Karamardian [19] and known as \textit{pseudomonotonicity}.

**Definition 2.** A function \( F: D \to \mathbb{R}^n \), where \( D \subseteq \mathbb{R}^n \), is pseudomonotone (PM) if, for all \( x \) and \( y \) in \( D \),
\[
(y - x)^T F(x) \leq 0 \quad \text{implies} \quad (y - x)^T F(y) \leq 0. \tag{4}
\]

Pseudomonotonicity is an important concept for at least two reasons. First, it characterizes the gradient map of a pseudoconcave real valued function. These functions, in contrast to quasiconcave functions, share with concave functions the nice property that the necessary first order condition for a maximum is also sufficient. Second, and more relevant to our subject, the set of zeros of a continuous pseudomonotone function defined on an open convex domain is convex.

Since the main concern of this paper is with the consumption behaviour of the whole economy, rather than with individual demand, we turn to the description of the \textit{consumption sector} in a general equilibrium setting. The consumption sector consists of a finite set of agents \( A \) and, for each agent \( a \), a pair of individual characteristics, i.e. a positive vector of initial endowments \( \omega_a \in \mathbb{R}^n_+ \) and an individual demand function \( f_a(p, w) \). The consumption behaviour of agent \( a \) is given by the demand function
\[
\varphi_a(p) = f_a(p, p^T \omega_a).
\]
In a general equilibrium setting, individual income is given by the market value of initial endowments, i.e. \( p^T \omega_a \), therefore \( \varphi_a(p) \), unlike \( f_a(p, w) \), is a demand function with price-dependent income. Of course, the properties of \( f_a \) imply that \( \varphi_a \) is homogenous of degree zero in \( p \) and that \( p^T \varphi_a(p) = p^T \omega_a \) for every \( p \).

The \textit{(aggregate) demand} of the consumption sector is the continuous function \( \Phi: \mathbb{R}^n_+ \to \mathbb{R}^n \) given by the sum of individual demand functions, i.e.
\[
\Phi(p) = \sum_a \varphi_a(p).
\]
\(^1\)On the relationship between the WWA, homogeneity, and condition (2) see [16].
Like individual demand, $\Phi$ is homogeneous of degree zero in $p$. Moreover, the individual budget identities imply Walras’ Law, i.e. for all $p$ we obtain $p^T \Phi(p) = p^T \omega$, where $\omega = \sum_\alpha \omega_\alpha$ is the vector of total endowments of the economy. For the demand function $\Phi$ there is a similar definition of the WWA.

**Definition 3.** The demand function $\Phi$ of a consumption sector with total endowments $\omega \in \mathbb{R}^n_+$ satisfies the WWA if

$$q^T \Phi(p) \leq q^T \omega \quad \text{implies} \quad p^T \Phi(q) \geq p^T \omega.$$  

(5)

As we shall see in Section 3, the WWA for $\Phi$ ensures the uniqueness of equilibrium in a large class of economic models. One of the most challenging problems in general equilibrium theory arises from the fact that for aggregate demand the WWA cannot be derived from the corresponding property of individual demand functions (see, e.g., [26]).

The WWA for $\Phi$ can be formulated more conveniently in terms of the (aggregate) excess demand function $Z : \mathbb{R}^n_+ \to \mathbb{R}^n$, which is given by

$$Z(p) = \Phi(p) - \omega.$$  

The function $Z$ is continuous and inherits from demand the properties of homogeneity and Walras’ Law, i.e. $Z(\lambda p) = Z(p)$ and $p^T Z(p) = 0$, for all $p$ and all $\lambda > 0$. By (5), the WWA for $\Phi$ can be equivalently defined in terms of the excess demand function by

$$q^T Z(p) \leq 0 \quad \text{implies} \quad p^T Z(q) \geq 0.$$  

(6)

Condition (6) is also known in the economic literature as Wald’s Weak Axiom (see [13]). Finally, by Walras’ Law, (6) is equivalent to the pseudomonotonicity condition for $Z$, i.e.

$$(q - p)^T Z(p) \leq 0 \quad \text{implies} \quad (q - p)^T Z(q) \leq 0.$$  

(7)

The above discussion makes clear that pseudomonotonicity is the actual mathematical property underlying the definitions of the WWA for both individual and aggregate demand functions. The basic results shown in this section are listed below for later reference.

- The individual demand function $f$ satisfies the WWA if and only if, for some (all) $w$, the function $f(\cdot, w)$ is pseudomonotone.

- The aggregate demand function $\Phi$ of a consumption sector with total endowments $\omega$ satisfies the WWA if and only if the corresponding excess demand function $Z$ is pseudomonotone.
3 Equilibrium and the WWA

In this section the basic features of production economies and the definition of equilibrium are introduced. It will be shown that the WWA is an extremely valuable property for the consumption sector of the economy since it ensures the uniqueness of equilibrium in a large class of models.

In the analysis of competitive equilibrium the relevant information about the economy is summarized by the production set $Y$ and the excess demand function $Z$. The production set $Y \subseteq \mathbb{R}^n$ describes the technology available in the economy. Any vector $y \in Y$ is a production plan which specifies quantities of input and output of a feasible production process. Positive components of $y$ denote quantities of output whereas negative components are associated with positive amounts of inputs.

A few standard assumptions are commonly made on technology. The production set is assumed to be closed and convex and to satisfy the following properties:

(i) free disposal, i.e. $-\mathbb{R}^n_+ \subseteq Y$;

(ii) no free lunch, i.e. $Y \cap \mathbb{R}^n_+ = \{0\}$.

According to assumption (i), any amount of goods can be ‘wasted’ without using any other inputs. Assumption (ii) simply says that production requires some inputs. Convexity of $Y$ means that production plans can be combined, i.e. if $y, y' \in Y$ then $(1 - \lambda)y + \lambda y' \in Y$ for all $0 \leq \lambda \leq 1$. Finally, the assumption that $Y$ is closed is essentially made for mathematical convenience.

A technology exhibits constant returns to scale if all production plans can be scaled up and down, i.e. if $y \in Y$ implies $\lambda y \in Y$ for any $\lambda > 0$ or, equivalently, if $Y$ is a cone. The following analysis is confined to the case of production economies with constant returns to scale, therefore $Y$ will be a closed, convex cone.

The demand side of the economy is described by an excess demand function $Z: P \rightarrow \mathbb{R}^n$, which is continuous and satisfies Walras’ Law and homogeneity. The domain of $Z$ is a convex set of prices $P$ such that $\mathbb{R}^n_{++} \subseteq P \subseteq \mathbb{R}^n \setminus \{0\}$. It is worth noting that here, unlike Section 2 and the rest of the paper where $P = \mathbb{R}^n_+$, the excess demand may be defined even when the price of some goods is zero. Therefore, the analysis of the present section also applies to the case where consumers are satiated.

**Definition 4.** An equilibrium of the production economy $(Y, Z)$ is a price vector $p \in P$ satisfying the following two conditions:

\begin{align*}
  a) \ Z(p) & \in Y \quad \text{and} \quad b) \ p^T y \leq 0 \quad \text{for all} \quad y \in Y.
\end{align*}
Condition (a) requires that, at an equilibrium price, the demand of all the consumers is met. Condition (b) implies that, at equilibrium, firms make zero profits. In fact, by (b), firms maximize profits by choosing the production plan \( y = Z(p) \) since, by Walras’ Law, \( p^T Z(p) = 0 \). Notice also that, by homogeneity, what matters are relative prices. If \( p \) is an equilibrium, then any other vector obtained by multiplying the price of each good by the same positive constant still satisfies Definition 4. Therefore, by ‘uniqueness of equilibrium’ it will be understood that the equilibrium price vector is unique up to positive scalar multiples.

The main point of the present section will follow naturally from the characterization of the equilibrium as the solution to a variational inequality problem.\(^2\) If \( F \) is a continuous function and \( X \) is a subset of its domain, the variational inequality problem \( \text{VI}(X,F) \) consists in finding a vector \( x \in X \) such that
\[
(y - x)^T F(x) \leq 0 \quad \text{for all } y \in X.
\]

To establish the relationship between equilibrium and variational inequality problems let us introduce the polar cone of the production set \( Y \), i.e.
\[
Y^* = \{ x \in \mathbb{R}^n \mid x^T y \leq 0 \quad \text{for all } y \in Y \}. \tag{8}
\]
The set \( Y^* \) is a closed convex cone for which \( (Y^*)^* = Y \) holds (see, e.g., [27]). Moreover, by (8), condition (b) is easily seen to be equivalent to \( p \in Y^* \), and this permits to restrict the analysis of equilibrium to the price set \( Q = P \cap Y^* \). Finally, we notice that \( Q \) is a nonempty convex set whose closure is equal to the polar cone of \( Y \), i.e. \( \overline{clQ} = Y^* \) (see Appendix 1).

**Theorem 1.** Let \( (Y, Z) \) be a constant returns to scale economy and \( Q = P \cap Y^* \). The vector \( p \in P \) is an equilibrium of \( (Y, Z) \) if and only if \( p \) is a solution to the variational inequality problem \( \text{VI}(Q,Z) \), i.e.
\[
p \in Q \quad \text{and} \quad (q - p)^T Z(p) \leq 0 \quad \text{for all } q \in Q. \tag{9}
\]

**Proof.** If \( p \in P \) is an equilibrium then, by (b), \( p \in Y^* \) thus \( p \in Q \). By (a) and Walras’ Law, we have \( (q - p)^T Z(p) = q^T Z(p) \leq 0 \) for all \( q \in Y^* \), since \( Z(p) \in Y \). Therefore, since \( Q \subseteq Y^* \), \( p \) satisfies (9).

Conversely, let \( p \) satisfy (9). Then \( p \in Q \) and (b) holds. Next, we have
\[
0 \geq (q - p)^T Z(p) = q^T Z(p) \quad \text{for all } q \in Q \quad \text{and, clearly, also for all } q \in \overline{clQ}.
\]

\(^2\)On economic applications of variational inequality concepts and techniques see [9] or [3].
As observed above, \( clQ = Y^* \), therefore \( q^T Z(p) \leq 0 \) for all \( q \in Y^* \), which means that \( Z(p) \) is in the polar cone of \( Y^* \), i.e. \( Z(p) \in (Y^*)^* \). Finally, since \( (Y^*)^* = Y \), condition (a) holds and \( p \) is an equilibrium price.

There is an important general property concerning variational inequality problems with pseudomonotone functions. If \( F \) is continuous and pseudomonotone on the convex set \( X \), the set of solutions to \( VI(X,F) \) is convex and any solution \( x \in X \) is characterized by the condition \( (x - y)^T F(y) \geq 0 \) for all \( y \in X \) (see [18, Corollary 2.1]). When the consumption sector satisfies the WWA, the excess demand is pseudomonotone, therefore the above result and Theorem 1 yield directly the convexity of the equilibrium set. The following result is due to Jerison [15, Proposition 1] and, for completeness, will be proved here as a simple corollary of Theorem 1.

Corollary 1. If \( Z \) is pseudomonotone, i.e. the consumption sector satisfies the WWA, any equilibrium price \( p \) of the economy \((Y,Z)\) is characterized by

\[
p \in Q \quad \text{and} \quad p^T Z(q) \geq 0 \quad \text{for all} \quad q \in Q,
\]

therefore, the set of equilibrium prices is convex.

Proof. If \( p \) is an equilibrium price then, by Theorem 1, it satisfies (9); therefore, by pseudomonotonicity of \( Z \) and Walras’ Law, (10) holds.

Conversely, let (10) hold and set \( q_\lambda = (1 - \lambda)p + \lambda q \) for any \( q \in Q \) and \( 0 < \lambda \leq 1 \). Then, \( p^T Z(q_\lambda) \geq 0 \) and, by Walras’ Law, \( q^T Z(q_\lambda) \leq 0 \). Letting \( \lambda \) go to zero, by continuity, \( q^T Z(p) \leq 0 \). Hence \( (q - p)^T Z(p) \leq 0 \) for all \( q \in Q \) and, by Theorem 1, \( p \) is an equilibrium. Finally, convexity of the equilibrium set follows trivially from (10).

Corollary 1 is an important result since, as usually happens in applications, (normalized) equilibria are finite in number. Since a finite convex set cannot have more than one element, the WWA guarantees uniqueness of the equilibrium prices in a large class of models.

4 Characterizations of differentiable pseudomonotone functions

It has been shown in Section 2 that the WWA can be defined by the mathematical concept of pseudomonotonicity. In this section we provide a general
characterization of differentiable pseudomonotone functions. In some important special cases these functions will be seen to be completely described by a first order property.

In the sequel, let $F : D \to \mathbb{R}^n$ be a continuously differentiable function defined on the open convex domain $D \subseteq \mathbb{R}^n$.

Mitjushin and Polterovich [23] already observed that for $F$ to be pseudomonotone it is necessary that $F$ satisfies the following condition:

(A) For all $x \in D$, the Jacobian matrix $\partial F(x)$ at $x$ is negative semidefinite on $F(x)^\perp$, i.e. for all $x \in D$ and all $v \in \mathbb{R}^n$

$$v^T F(x) = 0 \text{ implies } v^T \partial F(x) v \leq 0.$$ 

In general, (A) is not sufficient for pseudomonotonicity. A simple example is given by $F(x) = x^3$ defined on $\mathbb{R}$. However, if $F$ does not vanish on its domain, then (A) implies that $F$ is pseudomonotone. This result has been proved in [12] and, independently, by Crouzeix and Ferland [2]. Both proofs consist of the same two steps which are stated in the following

**Lemma 1.** Let $(y-x)^T F(x) \leq 0$ and $(y-x)^T F(y) > 0$ for some $x, y \in D$ and set $v = y - x$. Then there exist a vector $x'$ on the segment $[x, y]$ (i.e. the set of vectors $x + \tau v$ for $0 \leq \tau \leq 1$) and a scalar $t > 0$ such that

$$v^T F(x') = 0 \quad \text{and} \quad v^T F(x' + tv) > 0 \quad \text{for all} \quad t \in ]0, t[.$$

Moreover, if $F(x') \neq 0$, then condition (A) is violated.

**Proof.** Set $f(t) = v^T F(x + tv)$, with $t \in [0, 1]$. The first part of the lemma follows trivially from continuity of $f$. The last part is proved as in [12, Appendix 2].

If $F$ does not vanish, Lemma 1 yields immediately that (A) is not satisfied if $F$ is not pseudomonotone, i.e. (A) is sufficient for pseudomonotonicity of $F$. Obviously, the same conclusion can be drawn from the following weaker assumption which is due to Crouzeix and Ferland [2]:

(CF) There do not exist $x \in D$, $v \in \mathbb{R}^n$, $\overline{t} > 0$ such that $F(x) = 0$, $v^T \partial F(x)v = 0$ and $v^T F(x + tv) > 0$ for $t \in ]0, \overline{t}[$.

Moreover, (CF) is a necessary condition for pseudomonotonicity. Indeed, if $F$ is pseudomonotone, then $F(x) = 0$ implies $v^T F(x + tv) \leq 0$ for all $t$ with $x + tv \in D$. 

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By combining these arguments, we obtain that \( F \) is pseudomonotone if and only if \( F \) satisfies (A) and (CF). This characterization by Crouzeix and Ferland [2] has been improved by Brighi [1] who has shown that (CF) can be replaced by the weaker condition

(B) There do not exist \( x \in D, v \in \mathbb{R}^n, \bar{t} > 0 \) such that \( F(x) = 0, v^T \partial F(x) = 0, \) and \( v^T F(x + tv) > 0 \) for \( t \in [0, \bar{t}] \).

We give another proof of his result that is the following

**Theorem 2.** \( F \) is pseudomonotone if and only if \( F \) satisfies conditions (A) and (B).

**Proof.** It is clear that conditions (A) and (B) are necessary for pseudomonotonicity of \( F \). In order to prove sufficiency, assume that (A) and (B) hold but \( F \) is not pseudomonotone. By Lemma 1, there exist \( x \in D, v \in \mathbb{R}^n \) and \( t > 0 \) such that

\[
v^T F(x) = 0 \quad \text{and} \quad v^T F(x + tv) > 0 \quad \text{for all} \quad t \in [0, \bar{t}] \tag{11}
\]

and, by condition (A), \( F(x) = 0 \). Finally, condition (B) yields \( v^T \partial F(x) =: u^T \neq 0 \). The result is proved by contradiction if we show that \( u^T u \leq 0 \).

Let \( D' = \{(s, t) \in \mathbb{R}^2 | x + su + tv \in D \} \subseteq \mathbb{R}^2 \) and consider the function \( G : D' \to \mathbb{R}^2 \) defined by

\[
G(s, t) = (u^T F(x + su + tv), v^T F(x + su + tv)).
\]

The Jacobian matrix of \( G \) at 0 is given by

\[
\partial G(0, 0) = \begin{pmatrix}
u^T \partial F(x) u & u^T \partial F(x) v \\
v^T \partial F(x) u & v^T \partial F(x) v
\end{pmatrix}.
\]

By (11) and condition (A) we obtain \( v^T \partial F(x) v = 0 \), therefore the determinant of \( \partial G(0, 0) \) is

\[
|\partial G(0, 0)| = -[u^T \partial F(x) v] \times [v^T \partial F(x) u]
\]

To show that \( \partial G(0, 0) \) is not singular, notice that \( \partial F(x) + \partial F(x)^T \) is a symmetric negative semidefinite matrix and \( v^T [\partial F(x) + \partial F(x)^T] v = 0 \). It is known that this implies \([\partial F(x) + \partial F(x)^T] v = 0\); multiplying by \( u^T \) and rearranging yields

\[
u^T \partial F(x) v = -u^T \partial F(x)^T v = -v^T \partial F(x) u = -u^T u < 0
\]
where the last inequality follows from $u^T \neq 0$. Therefore, $|\partial G(0,0)|$ is different from zero and the matrix $\partial G(0,0)$ is not singular. Applying the Inverse Function Theorem to $G$, it is easily seen that there exists $\varepsilon > 0$ such that $F(x + su + tv) \neq 0$ for all $s, t \in ]-\varepsilon, \varepsilon[$ and $(s, t) \neq (0, 0)$.

Consider some $t > 0$ such that $t < \min\{\varepsilon, \bar{t}\}$. From (11) and $y = x + tv$ we obtain $(y - x)^T F(y) = tv^T F(x + tv) > 0$. This implies $(y - z)^T F(y) > 0$, where $z = x - su$ for all sufficiently small $s \in ]0, \varepsilon[$. Since $F$ does not vanish on the segment $[y, z]$, Lemma 1 and condition (A) imply $(y - z)^T F(z) > 0$. Hence, $0 < (y - x + su)^T F(x - su) = (tv + su)^T F(x - su)$ and

$$\lim_{s \to 0} (tv + su)^T \frac{1}{s} F(x - su) = -tv^T \partial F(x) u \geq 0,$$

i.e. $v^T \partial F(x) u = u^T u \leq 0$, in contradiction to $u^T u > 0$.  

As an immediate consequence, the first-order characterization of pseudomonotonicity for non vanishing functions can be extended to all regular functions. These functions may vanish but their Jacobian matrix at any zero must not be singular. The following result is due to John [17].

**Corollary 2.** Let $F$ be regular, i.e. $F(x) = 0$ implies $\det \partial F(x) \neq 0$. Then $F$ is pseudomonotone if and only if $F$ satisfies (A).

**Proof.** If $F$ is regular then $F(x) = 0$ and $v^T \partial F(x) = 0$ implies $v = 0$. Thus, condition (B) trivially holds. 

### 5 Differential characterizations of the WWA

The results obtained in the previous sections will be used here to derive differential characterizations of the WWA for both individual and aggregate demand functions.

From Section 2 we know that the WWA for an individual demand function $f$ is equivalent to pseudomonotonicity of the map $f(\cdot, w)$ for some (or all) $w > 0$. Noting that a continuously differentiable individual demand function is regular since it never vanishes in its domain, Corollary 2 can be directly applied to show that the WWA for $f$ is characterized only by condition (A) for $f(\cdot, w)$.

Another important characterization of the WWA, which is often used in applications, is concerned with the *Slutsky matrix*. As it is well-known, a price
change affects demand directly and indirectly by modifying the ‘purchasing power’ of consumer’s nominal income. This indirect effect can be neutralized by an income transfer (positive or negative) which allows the consumer to buy exactly the same bundle of goods chosen at the old prices. The (Slutsky) compensated demand at \( p \), which incorporates this compensating income transfer, is given by the map \( q \mapsto f(q, q^T f(p, w)) \). The Jacobian matrix of the compensated demand evaluated at \( q = p \) is the Slutsky matrix,

\[
S(p, w) = \partial_p f(p, w) + \partial_w f(p, w) f(p, w)^T
\]  

whose elements represent the direct effect of a price change on demand, also called the ‘substitution effect’. Rearranging (12) we obtain a famous identity known as the Slutsky equation

\[
\partial_p f(p, w) = S(p, w) - \partial_w f(p, w) f(p, w)^T.
\]  

The Slutsky equation decomposes the Jacobian of demand into the Slutsky matrix and the ‘income effect’ matrix \( \partial_w f(p, w) f(p, w)^T \).

As it was originally shown by Kihlstrom, Mas-Colell and Sonnenschein [20], the WWA is equivalent to the negative semi-definiteness of the Slutsky matrix. For later reference, the main characterization results for individual demand functions are brought together in the following theorem which admits a very simple proof.

**Theorem 3.** Let \( f \) be a \( C^1 \) individual demand function. The following statements are equivalent:

(i) \( f \) satisfies the WWA.

(ii) For some (or all) \( w > 0 \), the function \( f(\cdot, w) \) satisfies condition (A), i.e. \( v^T f(p, w) = 0 \) implies \( v^T \partial_p f(p, w)v \leq 0 \).

(iii) For all \( p \) and \( w \) the Slutsky matrix \( S(p, w) \) is negative semidefinite, i.e. \( v^T S(p, w)v \leq 0 \) for all \( v \in \mathbb{R}^n \).

**Proof.** By Corollary 2, (i) and (ii) are equivalent. Since (iii) trivially implies (ii), to complete the proof we only need to show the implication (ii) to (iii). Any \( v \in \mathbb{R}^n \) can be written as \( v = u + \lambda p \), where \( \lambda = v^T f(p, w)/w \) and the vector \( u \) is orthogonal to \( f(p, w) \). Moreover, by the budget identity and homogeneity, it follows that \( p^T S(p, w) = 0 \) and \( S(p, w)p = 0 \), therefore (12) and (ii) yield \( v^T S(p, w)v = u^T S(p, w)u = u^T \partial f(p, w)u \leq 0 \).
The differential characterization of the WWA for individual demand functions by (iii) has been obtained by Kihlstrom, Mas-Colell and Sonnenschein [20], the one by (ii) is due to Hildenbrand and Jerison [13].

Let us turn now to the WWA for aggregate demand. It is clear that the results of Section 4 directly apply to excess demand functions, so that we are able to derive a differential characterization of the WWA for the whole consumption sector. Moreover, the criteria for pseudomonotonicity of $Z$ can be further strengthened, owing to Walras’ Law and homogeneity. Indeed, as shown in [1], conditions (A) and (B) of Theorem 2 need only be satisfied by the vectors in the orthogonal space to $p$.

**Theorem 4.** Let $Z$ be a $C^1$ excess demand function. $Z$ is pseudomonotone, i.e. the consumption sector satisfies the WWA, if and only if the following conditions hold for all $p \in \mathbb{R}_+^n$ and $v \in \mathbb{R}^n$:

(A') $v^T p = v^T Z(p) = 0$ implies $v^T \partial Z(p)v \leq 0$.

(B') There do not exist $p$, $v$ and $\bar{t} > 0$ such that $v^T p = 0$, $Z(p) = 0$, $v^T \partial Z(p) = 0$ and $v^T Z(p + tv) > 0$ for all $t \in [0, \bar{t}]$.

**Proof.** Given Theorem 2 we only need to show that (A’) and (B’) imply (A) and (B). Let $v^T Z(p) = 0$ and notice that $v$ can be written as $v = u + \alpha p$ with $\alpha = v^T p/p^T p$. Clearly, the vector $u$ is orthogonal to both $p$ and $Z(p)$. By Walras’ Law, $p^T \partial Z(p) = -Z(p)^T$, and by homogeneity, $\partial Z(p)p = 0$. Hence

$$v^T \partial Z(p)v = (u + \alpha p)^T \partial Z(p)(u + \alpha p) = u^T \partial Z(p)u,$$

and (A’) implies (A).

Next we show that if (B) is violated then (B’) cannot hold. Let $Z(p) = 0$, $v^T \partial Z(p) = 0$, $\bar{t} > 0$ and

$$v^T Z(p + tv) > 0 \quad \text{for all} \quad t \in [0, \bar{t}]. \quad (14)$$

Write $v$ as $v = u + \alpha p$ with $\alpha = v^T p/p^T p$ so that the vector $u$ is orthogonal to $p$. Without loss of generality, let $(1 + \alpha t) > 0$ for all $t \in [0, \bar{t}]$. Then, by homogeneity,

$$Z(p + tv) = Z((1 + t\alpha)p + tu) = Z(p + su) \quad (15)$$

where $s = t/(1 + t\alpha)$. Using Walras’ law twice yields

$$p^T Z(p + tv) + tv^T Z(p + tv) = 0 = p^T Z(p + su) + su^T Z(p + su). \quad (16)$$
By (15), \( p^T Z(p+tv) = p^T Z(p+su) \), so that by (14) and (16), \( u^T Z(p+su) > 0 \) for all \( 0 < s < \tilde{s} = \tilde{t}/(1+\alpha \tilde{t}) \). Finally, since \( 0 = v^T \partial Z(p) = (u+\alpha p)^T \partial Z(p) = u^T \partial Z(p) \), condition (B') is violated and this completes the proof.  

The following example, which is essentially taken from [13], provides a nice illustration of Theorem 4 and introduces the next result.

**Example.** Let us consider the simple case of a pure exchange economy with only two goods. The excess demand function is given by

\[
Z(p) = (h(p_1/p_2), -p_1 h(p_1/p_2)/p_2)
\]

where \( h: (0, \infty) \to \mathbb{R} \) is defined by \( h(x) = (x-1)^3/(x+1)^3 \). Since the excess demand vanishes only when the prices of the two goods are equal, there is a unique normalized equilibrium price vector, say \( p^* = (2, 2) \). It is easy to see that \( Z \) is not pseudomonotone so that the consumption sector does not satisfy the WWA. For instance, at \( q = (3, 1) \) we have \((q-p^*)^T Z(p^*) = 0 \) and \((q-p^*)^T Z(q) = 1/2 > 0 \). On the other hand, it can be seen that the Jacobian matrix of \( Z \) at any equilibrium price vector is the null matrix and thus, in two dimensions, condition (A') is trivially satisfied. But then, according to Theorem 4, condition (B') must be violated and indeed this is case. Take, for instance, \( v = q - p^* = (1, -1) \). Thus \( v^T p^* = 0, Z(p^*) = 0, v^T \partial Z(p^*) = 0 \) and \( v^T Z(p^* + tv) = [4/(2 - t)] h[(2 + t)/(2 - t)] > 0 \) for all \( 0 < t < 2 \).

The above example shows that, in general, condition (A') alone is not sufficient to characterize the WWA. However, there is an interesting case where one can get rid of condition (B'). Recall that an excess demand function is said to be *regular* if its Jacobian matrix has maximal\(^3\) rank at all zeros, i.e. if \( Z(p) = 0 \) then \( \text{rank} \partial Z(p) = n - 1 \). For a regular excess demand function, condition (B') trivially holds. Indeed, if \( Z(p) = 0 \) and \( v^T \partial Z(p) = 0 \), regularity and Walras’ Law imply \( v = \lambda p, \lambda \in \mathbb{R} \), thus \( v^T p = 0 \) yields \( v = 0 \). The following result is a straightforward corollary of Theorem 4 and was originally proved by John [17].

**Corollary 3.** A regular \( C^1 \) excess demand function \( Z \) is pseudomonotone if and only if condition (A') holds.

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\(^3\)Because of homogeneity, the Jacobian matrix of \( Z \) cannot have full rank.
6 An Application to Equilibrium Analysis

Since the early work by Hicks [10], it has been clear that properties of individual demand such as those concerned with the Axioms of Revealed Preference are not preserved by aggregation. Therefore, there are no theoretical justifications for the consumption sector of an economy to satisfy the WWA.

Some assumptions on individual behaviour leading to good properties in the aggregate are well-known, but are rather strong. For example, aggregate demand behaves as the demand of a ‘representative agent’ (thus, it satisfies the WWA) when consumers have identical homothetic preferences, i.e. the ‘Engel curves’ $f_a(p, \cdot)$ are linear and identical (see [6]), or when preferences are homothetic (not necessarily identical) and the relative income distribution is fixed, as occurs when initial endowments are collinear (see [4]). Somewhat weaker requirements for the aggregate demand to satisfy the Weak Axiom have been found by Freixas and Mas-Colell [5]. However, these conditions are still extremely stringent and, more importantly, are not supported by empirical evidence.

Another way to obtain desirable structural properties for a consumption sector is to bring into the picture the way individual characteristics are distributed among the population. This line of research has been pursued in different ways by some authors including Hildenbrand [11, 12], Jerison [14, 15], Grandmont [7], Marhuenda [22], and Quah [24]. The idea underlying these contributions is that a sufficient degree of heterogeneity in consumption behaviour and income distribution produces good structural properties for the aggregate demand. By reviewing the recent work by Jerison [15], we will see the role played by the differential characterization of pseudomonotonicity in linking the WWA of the consumption sector to a specific formulation of heterogeneity. Throughout this section the individual demand functions $f_a$ are assumed to be continuously differentiable.

The main argument in Jerison’s analysis rests on the Slutsky equation and the differential characterizations of the WWA. By (13), the Jacobian matrix of the individual excess demand, $z_a(p) = \varphi_a(p) - \omega_a$, can be written as

$$\partial z_a(p) = S_a(p) - m_a(p) z_a(p)^T$$

(17)

where $S_a(p) = S_a(p, p^T \omega_a)$ and

$$m_a(p) = \partial_w f_a(p, w)|_{w=p^T \omega_a}$$

is the vector of marginal propensities to consume. To simplify notation we shall consider average rather than aggregate excess demand. Therefore, the
excess demand will be defined by $Z(p) = E[z_a(p)]$, where $E$ denotes the mathematical expectation over the set of agents. By taking the expectation of (17), we obtain a similar decomposition of the Jacobian matrix of the excess demand, i.e.

$$ \partial Z(p) = S(p) - A(p) $$

(18)

where $S(p)$ is the average of individual Slutsky matrices and $A(p)$ is the average of individual income effect matrices, i.e.

$$ A(p) = E[m_a(p)z_a(p)^T]. $$

(19)

If consumer’s choice is assumed to satisfy the WWA (which is a rather mild requirement of rationality) then, by Theorem 3, the individual Slutsky matrices are negative semidefinite and so is the average Slutsky matrix $S(p)$. Moreover, if the excess demand is regular then, by (18) and Corollary 3, the WWA holds when the income effect matrix $A(p)$ is positive semidefinite on the orthogonal space to $Z(p)$ and $p$, i.e. when, for all $p \in \mathbb{R}^n_+$ and $v \in \mathbb{R}^n$,

$$ v^T p = v^T Z(p) = 0 \quad \text{implies} \quad v^T A(p)v \geq 0. $$

(20)

The next step is to show the relationship between property (20) of the average income effect matrix and heterogeneity of consumption behaviour. To this aim let us consider the hypothetical situation where the nominal income of each consumer is increased by the same small amount $\lambda > 0$. The excess demand of the wealthier agent $a$ is given by

$$ z_a(p, \lambda) = f_a(p, \lambda + p^T \omega_a) - \omega_a $$

and, similarly, the excess demand function of the wealthier population of consumers is $Z(p, \lambda) = E[z_a(p, \lambda)]$. Jerison [15] introduces the notion of ‘dispersion of excess demand’ in the direction $v \in \mathbb{R}^n$ as the variance of the real variable $v^T z_a(p, \lambda)$, i.e.

$$ \sigma^2_v(p, \lambda) = E[v^T z_a(p, \lambda) - v^T Z(p, \lambda)]^2. $$

For $\lambda = 0$, we have the dispersion of excess demand for the actual population of consumers, i.e. $\sigma^2_v(p, 0) = E[v^T z_a(p) - v^T Z(p)]^2$.

The idea of heterogeneity in consumption behaviour proposed by Jerison [15], and similarly by Härdle, Hildenbrand and Jerison [8] and Hildenbrand [12], is that any small generalized increase in income cannot reduce the dispersion of the consumption patterns within the population. To formalize this idea, let us introduce the following expression

$$ \delta(p, v) = \partial_\lambda \sigma^2_v(p, 0). $$

(21)
The way consumers behave responds to a small generalized increase in income is reflected by the sign of $\delta$. Clearly, if the dispersion of excess demand is non-decreasing in income then $\delta$ must be non-negative and vice versa. The following hypothesis, called *Nondecreasing Dispersion of Excess Demand* (NDED), was introduced by Jerison [15].

NDED: The dispersion of the individual excess demand functions $z_a(p)$ in all directions $v \in \mathbb{R}^n$ orthogonal to $Z(p)$ and $p$ is not decreasing in income, thus, for all $p \in \mathbb{R}^{n}_{++}$ and $v \in \mathbb{R}^n$,

$$v^TP = v^TZ(p) = 0 \quad \text{implies} \quad \delta(p,v) \geq 0.$$  

The hypothesis of NDED holds if, as consumers become wealthier, the cloud of excess demand vectors expands, so that the projections on any straight line orthogonal to $p$ and to the excess demand expand as well. NDED does not place any special restrictions either on the form of Engel curves (i.e. the functions $f_a(p, \cdot)$) or the patterns of initial endowments. It is just an hypothesis on the joint distribution of individual characteristics $(f_a, \omega_a)$ and, as pointed out by Jerison [15], it can be empirically tested by using cross-section data on family expenditure surveys, provided that some additional assumptions are met.

To show that NDED actually guarantees the WWA for the whole consumption sector we need the following preliminary result.

**Lemma 2.** Let the consumption sector be described by the $C^1$ individual excess demand functions $z_a$. Then for all $p \in \mathbb{R}^n_+$ and all $v \in \mathbb{R}^n$ such that $v^T Z(p) = 0$,

$$\frac{1}{2} \delta(p,v) = v^T A(p) v,$$

where $A(p)$ and $\delta(p,v)$ are respectively given by (19) and (21). Consequently, (20) holds if and only if the consumption sector satisfies NDED.

**Proof.** See Appendix 2.

Bringing together the findings of this section and Section 3, we obtain the following result by Jerison [15].

**Theorem 5.** Let $(Y,Z)$ be a production economy with a constant returns to scale technology and $P = \mathbb{R}^n_+$ (see Sec. 3). The set of equilibrium prices is convex if the following conditions are met:

i) The individual demand functions $f_a$ are $C^1$ and satisfy the WWA.
ii) The excess demand $Z$ is regular, i.e. $Z(p) = 0$ implies $\text{rank } \partial Z(p) = n - 1$.

iii) The consumption sector satisfies NDED.

If, in addition, $(Y,Z)$ is a pure exchange economy, i.e. $Y = -\mathbb{R}^n_+$, then there is a most one equilibrium (up to positive scalar multiples).

**Proof.** By Lemma 2, NDED implies (20). Since, by Theorem 3, $S(p)$ is negative semidefinite, it follows from (18) and Corollary 3 that $Z$ is pseudomonotone. By Corollary 1, the set of equilibrium prices is convex.

Now, consider the case $Y = -\mathbb{R}^n_+$. If $p$ is an equilibrium price vector, then $Z(p)$ is in $-\mathbb{R}^n_+$ and, by Walras’ Law, $p^T Z(p) = 0$. Since $p \in \mathbb{R}^n_+$, we have $Z(p) = 0$. Assume that $p$ and $q$ are two different equilibria. Setting $v = q - p$, we obtain $Z(p + \lambda v) = 0$ for all $\lambda \in [0,1]$, since the equilibrium set is convex. Hence, the derivative of $Z$ with respect to $\lambda$ at $\lambda = 0$, given by $\partial Z(p)v$, is equal to zero. By homogeneity of $Z$, $\partial Z(p)p = 0$. Since $Z$ is regular it follows that $v = \alpha p$ or $q = (1 + \alpha)p$. \hfill $\square$

It can be pointed out that, in the absence of any further restrictions on the average substitution effect $S(p)$, the hypothesis of NDED cannot be weakened if the WWA is to hold for the whole consumption sector. Indeed, condition (i) of Theorem 5 does not rule out the case $v^T S(p)v = 0$ for $v^T p = v^T Z(p) = 0$. If that case occurs and NDED is violated, then $v^T \partial Z(p)v = v^T A(p)v$ and, by Lemma 2 and Corollary 3, the excess demand $Z$ cannot be pseudomonotone.

Finally, we remark that the uniqueness result of Theorem 5 concerning exchange economies can be extended to the case of production if the weak inequality $\delta(p,v) \geq 0$ in the definition of NDED is replaced by a strict one, i.e. if the dispersion of excess demand is *increasing* (see [15]). Alternatively, uniqueness of equilibrium obtains in production models under the conditions stated in Theorem 5 provided that the number of equilibria is shown to be finite, as usually happens in applications.

**Appendix 1**

A few properties of the sets $Y^*$ and $Q$ used in Section 3 are proved here. By $e_i$, with $i = 1, \ldots, n$, will be denoted the $i$-th unit vector of the standard basis of $\mathbb{R}^n$.  

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$Y^* \subseteq \mathbb{R}^n_+$: Let $q \in \mathbb{R}^n$ and the $i$-th component of $q$ be negative. Then $-e_i^T q > 0$ and, since $-e_i \in Y$, $q \not\in Y^*$. Hence $Y^* \subseteq \mathbb{R}^n_+$.

$Y^* \cap \mathbb{R}^n_+ \neq \emptyset$: If for any $i$ there exists a vector $q^i \in Y^*$ such that $e_i^T q^i > 0$ then the result follows from $Y^* \subseteq \mathbb{R}^n_+$ and the convexity of $Y^*$. Let us argue by contradiction and suppose that, for some $i$, $q^i e_i \leq 0$ for all $q \in Y^*$. Then, by definition, $e_i \in (Y^*)^* = Y$ and property (ii) of $Y$ is violated. This contradiction completes the proof.

$clQ = Y^*$: Notice first that $clQ \subseteq Y^*$, since $Y^*$ is closed and $Q \subseteq Y^*$. To show the converse, let $x \in Y^*$ and take a strictly positive vector $\bar{x} \in Y^*$. The sequence $\{x_k\}$ defined by $x_k = (1/k)\bar{x} + (1 - 1/k)x$, with $k = 1, 2, \ldots$, is in $Q$ since $x_k \in \mathbb{R}^n_+$ and $Y^*$ is convex. Since $\{x_k\}$ converges to $x$ we have $x \in clQ$.

Appendix 2

Proof of Lemma 2. Observe first that

$$\sigma^2_\lambda(p, \lambda) = E\{v^T [z_a(p, \lambda) - Z(p, \lambda)]\}^2$$
$$= E\{v^T [z_a(p, \lambda) - Z(p, \lambda)][z_a(p, \lambda) - Z(p, \lambda)]^T v\}$$
$$= v^T E\{[z_a(p, \lambda) - Z(p, \lambda)][z_a(p, \lambda) - Z(p, \lambda)]^T\} v$$
$$= v^T \text{cov} z_a(p, \lambda) v,$$

where $\text{cov} z_a(p, \lambda)$ denotes the covariance matrix of the individual excess demand functions. This implies

$$\delta(p, v) = \partial_\lambda \sigma^2_\lambda(p, 0) = v^T \partial_\lambda \text{cov} z_a(p, 0) v$$
$$= v^T \partial_\lambda E\{[z_a(p, 0) - Z(p, 0)][z_a(p, 0) - Z(p, 0)]^T\} v$$
$$= v^T \{\partial_\lambda E[z_a(p, 0)] - \partial_\lambda [Z(p, 0)Z(p, 0)^T]\} v$$
$$= v^T \{E[m_a(p)z_a(p)^T] + E[z_a(p)m_a(p)^T] - M(p)Z(p)^T - Z(p)M(p)^T\} v$$
$$= 2v^T E[m_a(p)z_a(p)^T] v - 2v^T Z(p)M(p)^T v$$
$$= 2v^T A(p)v,$$

where $M(p) = E[m_a(p)]$ and the last equality follows from the assumption $v^T Z(p) = 0$. \qed
References


