On the Existence and Prevention of Speculative Bubbles*

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April 28, 2011

Abstract

We develop a model of rational bubbles based on the assumptions of unknown liquidity and limited liability of traders. In a bubble, the price of an asset rises dynamically above its steady-state value, justified by rational expectations about future price developments. The larger the expected future price increase, the more likely it is that the bubble will burst. We give a general condition for the possibility of bubbles, depending on the risk-free rate, uncertainty about liquidity, and traders' degree of leverage. Several policy measures for the prevention of bubbles are discussed.

Keywords: Bubbles, Rational Expectations, Leveraged Investment, Bonuses, Compensation Schemes, Financial Crises, Financial Policy.

JEL-Codes: G01, G12, E44,

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*We thank Felix Bierbrauer, John Boyd, Christoph Engel, Martin Hellwig, Andreas Irmen, and participants of the econ workshop at the Max Planck Institute Bonn, the research seminar of the Deutsche Bundesbank, the Austrian National Bank, the German Economic Association, the German Finance Association, and the Universities of Bonn, Göttingen, Hannover, Heidelberg, Innsbruck, Karlsruhe, Mainz, Mannheim, and Tübingen for discussion. All errors are our own. A former version of this paper was called "The Birth and Burst of Asset Price Bubbles."

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1 Introduction

Under what conditions are asset price bubbles possible? Which policies can prevent them? In the light of recent economic experience, these questions seem important and topical. The last 15 years have seen at least two prominent market developments that are now considered as bubbles. Both the so-called dot-com bubble in the late 1990s and the recent housing bubble in the United States and elsewhere produced large reallocations of wealth during their buildup and after their crashes, thereby affecting real macroeconomic variables. Although the phenomenon of bubbles has long been recognized, economic policy has not been able to prevent their repeated occurrence. Neither does a commonly accepted model of bubbles exist that can be generally used for policy advice. Our paper contributes to the development of such an understanding, which might eventually help guide policymakers. To this end, we build a theoretical model, derive conditions under which multiple pricing equilibria occur, and discuss policy measures that help prevent such bubbles.

Our simple workhorse model of a bubble is based on the assumption that the potential amount of liquidity in the market is not precisely known. Assuming imprecise information about market potential seems natural in the light of increasingly complex and opaque financial markets. In our model, within a bubble, a trader is only willing to invest if she believes there can be yet another market participant in the future to whom he can sell at an even higher price. As observed by Tirole (1982), if the maximum market liquidity were known, the highest possible price of an asset could be computed by the traders. By backward induction, no bubble could emerge in the first place. On the other hand, Blanchard (1979) shows that bubbles are always possible if liquidity is infinite. Our assumption of finite but unknown liquidity is an intermediate case.[1]

[1]Also, Santos and Woodford (1997) show that the conditions for the existence of bubbles are very restrictive if one assumes a fixed number of households that participate in the asset market and own finite aggregate endowments. Tirole (1985) analyzes "rational" bubbles that are possible

The second important feature of our model is limited liability. In addition to investing their own funds, traders borrow. Being protected by limited liability, traders behave as if they were risk-loving. They profit strongly from advancing stock prices, but their losses are limited. The same results obtain if one assumes that some investor delegates investment to a fund manager, whose compensation includes bonus payments that cannot become negative.[2] Hence the model directly applies to any kind of intermediated finance with limited liability, such as through banks, investment banks, insurance companies, and private equity firms, as well as to non-intermediated, debt-financed investments. Looking at the effects of limited liability alone, we find that the induced risk appetite of traders pushes asset prices above their fundamental values (as already shown by Allen and Gale, 2000). Because of limited liability in case of a low or zero return, the trader can increase her expected payoff by engaging in riskier assets. Equilibrium asset prices are therefore driven above fundamentals, but in a static way. These price deviations are not induced by expectations and are not subject to sudden corrections (bursts).

Adding the assumption of uncertain market liquidity extends the space of possible price paths drastically: expectations-induced bubbles with a dynamic price paths may emerge. In such a bubble, traders are aware that they are in a bubble.[3] A high price reduces the probability that current asset holders will find future buyers

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in overlapping-generations models if the real interest rate is smaller than the growth rate of the economy. The present paper does not require this assumption. Note that a stochastic growth rate would induce an uncertain future market size as is assumed here.

[2]According to the OECD database on institutional investors' assets, in 2007 institutional investors in the U.S. managed assets worth 211.2% of GDP, showing their prominent role in investment decisions. Furthermore, their size has grown steadily over the last decade, with a yearly average growth rate of 6.6% from 1995-2005 within the OECD(17) area (see Gonnard, Kim, and Ynesta, 2008).

[3]Referring to the dot-com bubble, Brunnermeier and Nagel (2004) provide evidence that hedge funds were riding the bubble, a result similar to a previous finding by Wermers (1999). They relate this to, among others, a short-term horizon of the managers. This is in line with our model.

at a yet higher price. Given this increased risk, they demand a higher expected gain from the asset. This accelerator mechanism drives prices up over time until the bubble collapses because either the previously unknown maximum market liquidity is reached or the underlying fundamental breaks down (e. g., due to bankruptcy of the issuing firm).[4]

Importantly, the model allows for bubbles in some parameter ranges, but not in others. The above-explained feedback between higher prices and an increased fragility of the bubble may not have a fixed point, at some future date or right from the start. In either case, there is no possible price path besides the one induced by fundamentals; bubbles are unfeasible. Depending on the interaction of limited liability,

[4]Note that in this model the existence of a bubble requires neither heterogeneous traders nor asymmetric information among them. This stands in contrast to a series of existing papers on bubbles. In Allen, Morris, and Postlewaite (1993), private information can drive a price above its fundamental value. Scheinkman and Xiong (2003) and Bolton, Scheinkman, and Xiong (2006) assume that buyers of an asset hope to sell it to overoptimistic agents in the next period. This is only possible in the case of heterogeneous beliefs. In Hong, Scheinkman, and Xiong (2006), the presence of overoptimistic agents and short-sale constraints create bubbles, where prices drop after an increase in asset float. Allen and Gorton (1993) and Barlevi (2008) show that asymmetry of information between investors and heterogeneous managers can lead to deviations of prices from fundamentals if liquidity needs are stochastic. The model of Brunnermeier and Abreu (2003) relies on dispersed opinions that, in combination with coordination failure, can trigger bubbles. Also, in Pástor and Veronesi (2006, 2009), there is initial noisy information and learning over time, leading to stock price behavior that can be confused with a bubble. Froot, Scharfstein, and Stein (1992) analyze which information can influence trading, potentially leading to herding equilibria. In DeLong, Shleifer, Summers, and Waldmann (1990), rational traders' behavior is influenced by noise traders, who follow positive-feedback strategies. Allen, Morris, and Shin (2006) highlight the role of higher-order expectations if traders have asymmetric information. Liquidity needs of firms that are not fully met by uninformed investors can lead to bubbles in Farhi and Tirole (2011). In Plantin and Shin (2010), not all traders have market access at a given point in time. A different strand of literature relies on near-rational investors. In Conlon (2004), agents are rational to the nth degree, giving bounded rationality a crucial role. Agents in Adam and Marcet (2010) use Bayesian updating about asset price movements, leading to bubbles.

uncertainty about the market size, riskiness of the asset, and the risk-free interest rate, the prerequisites for bubbles can be fulfilled or not.[5] We can thus analyze policies to prevent bubbles. One of the widely discussed possible policy measures is a financial transaction (Tobin) tax. We find that such a measure can actually backfire and enlarge the parameter range where bubbles are possible, if levied on *all* financial assets. If imposed on potential bubble-assets only, it prevents the emergence of bubbles. Similarly, a system that reduces bonus payments can actually backfire and increase the parameter range where bubbles are feasible. Only a low enough cap on bonuses will effectively prevent the emergence of bubbles. Also a monetary policy rule that takes asset price inflation into account can render bubbles impossible. Finally, mandatory long-term compensation and/or capital requirements fulfill the same purpose.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 2.2 develops a steady-state (rational expectations) equilibrium price process. Section 3 constructs a simple example of a non-steady-state (rational expectations) equilibrium price process, which we call a bubble. We give a necessary and sufficient condition for the existence of such example bubbles. In section 4, we show that this very condition is necessary and sufficient for the existence of bubbles in general. The condition lends itself to basic policy analysis, which is done in section 5 by discussing several policy measures. Interpreting the trader's payoff scheme as a compensation package that pays bonuses but does not let traders participate in losses, allows us to evaluate the policy measures of a cap on bonuses and long-term compensation in section 5.4. Section 6 concludes. All proofs are in the appendix.

[5] Taking the housing bubble as an example, we find that all conditions that are favorable for the emergence of bubbles were fulfilled. Low interest rates prevailed for a long period, while increasingly international financial flows and more complex financial instruments obscured the potential market liquidity. Furthermore, the 2004 decision of the Securities and Exchange Commission to allow the large investment banks to take on more debt increased their limited liability and increased uncertainty about the maximum market liquidity even further.

2 The Model

2.1 Setup

Consider an infinite horizon economy with overlapping generations of traders.[6] In period t, a continuum of measure N traders is born, each with an initial endowment of E dollars. N is stochastic but fixed over time, i. e., the distribution $F(N)$ and density $f(N)$ are public information, but not the actual realization of N. Traders invest in period t. They earn their returns, withdraw from the market, and consume in the next period, $t+1$. To undertake investment, they borrow additional funds. Each trader, owning E dollars, can borrow D dollars at an interest rate r_D, where D, E and r_D are exogenous. Thus, the trader can invest $E + D$. If y is the gross return rate, then the income from investment is $y\,(E + D)$, and the profit is $y\,(E + D) - (1 + r_D)\,D$ if that is positive, otherwise the profit is zero due to limited liability. The trader's target function is thus

$$\max\{\alpha\,(y - \beta);\, 0\} \tag{1}$$

with $\alpha \equiv (E + D)$ and $\beta \equiv (1 + r_D)\,D/(E + D)$. Traders maximize the expected target function. In section 5.4, we will interpret α and β as parts of a compensation contract in a delegation problem.

There are two assets, a safe asset (short: storage) of unlimited supply and a single risky asset (short: asset) of volume 1. Storage bears a risk-free net interest of r.[7] The risky asset can be interpreted as shares of a firm. This firm pays dividends of d in each period.[8] However, in each period, there is a probability $1 - q$ that the firm

[6]Like Allen and Gale (1997), we use the OLG structure as a metaphor for other sources of non-modeled market imperfections, such as heterogeneous liquidity preferences.

[7]We assume $r > r_D > 0$, such that the trader has an incentive to take up outside financing.

[8]One may also interpret the asset as real estate. If the house is used as rental property, d is the rent per period.

will go bankrupt and cease to pay dividends forever; its shares are then no longer traded. Hence, the time of bankruptcy is determined by a Poisson process. The risky asset is traded on a competitive market in each period. Its price follows an endogenous time-discrete stochastic process $\{\tilde{p}_t\}_{t \geq 0}$.

2.2 The Steady-State Price

Consider the following simple stochastic process $\{\tilde{p}_t\}_{t \geq 0}$. The price of the asset is a constant, $\tilde{p}_t = \bar{p}$. Only if the underlying firm goes bankrupt (with probability $1-q$) and cash ceases to flow, the price drops to $\tilde{p}_t = 0$. Hence, the price follows a simple binomial process with $\Pr_t\{\tilde{p}_{t+1} = \bar{p} | p_t = \bar{p}\} = q$. Zero is an absorbing state. Let us derive the price $\bar{p}$ for which this process is a rational expectations equilibrium.

In a market equilibrium, prices must be such that the traders' expected return is the same for storage and for the risky asset. If a trader opts for storage, her compensation is $\max\{\alpha\,(1+r-\beta);\,0\} = \alpha\,(1+r-\beta)$. If the trader buys shares of the firm at a price $p_t = \bar{p}$, she benefits from the dividend with probability q. She thus earns d/p_t with probability q. In absence of a bankruptcy, the price remains at $\tilde{p}_{t+1} = \bar{p}$, and the trader additionally gets $p_{t+1}/p_t = \bar{p}/\bar{p} = 1$ from selling the asset. This stochastic process is depicted in figure 1.

In the steady state, a trader's expected compensation from holding the risky asset on date t is

$$E_t \max\left\{0;\, \alpha\left(\frac{\tilde{p}_{t+1}}{p_t} + \frac{d}{p_t} - \beta\right)\right\} = q\,\alpha\left(\frac{\bar{p}+d}{\bar{p}} - \beta\right) \tag{2}$$

For the market to clear, traders must be indifferent between storage and the risky investment,

$$\alpha\,(1+r-\beta) = q\,\alpha\left(\frac{\bar{p}+d}{\bar{p}} - \beta\right), \tag{3}$$

$$\Rightarrow \quad \bar{p} = \frac{d\,q}{(1-\beta)\,(1-q)+r}. \tag{4}$$

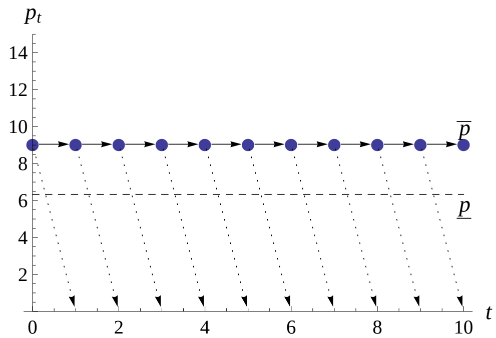

Figure 1: The Price Process in Steady State.

Parameters are $\beta = 0.9$, $q = 95\%$, $d = 1$, and $r = 10\%$.

The *steady-state price* $\bar{p}$ price depends therefore on the compensation scheme (β). The *fundamental value* obtains in the absence of limited liability,

$$\underline{p} := \frac{d\,q}{1 - q + r}. \tag{5}$$

Hence, only if $\beta = 0$ (unlimited liability) or if $q = 1$ (no risk), fundamental value and steady-state price are equal, $\underline{p} = \bar{p}$.[9] The effect that traders with limited liability push prices of risky assets above their fundamental levels has been analyzed before by Allen and Gale (2000). This systematic deviation from fundamentals is not driven by expectations, hence it is no "speculative" bubble in the sense of Harrison and Kreps (1978). We get the following comparative statics.

Remark 1 *The ratio between steady-state price and fundamental value $\bar{p}/\underline{p}$ is higher for a low risk-free rate r, high fundamental risk (large q), and substantial limited liability (high β).*

Let us make one important clarification. In the numerical example for figure 1, the fundamental value is $\underline{p} = 6.33$, but the steady-state price is $\bar{p} = 9.05$. This price deviation is due to the limited liability of traders. However, it is a *static*

[9] Remember that $\beta = 0$ if traders use only their own funds E for investments.

deviation, which is driven by fundamentals (q, d, and r) and the traders' financial contract (β and α, where α is irrelevant). The price deviation is hence driven by future risk (q) and dividends (d), but *not* by traders' expectations about future price developments. The deviation is constant over time and cannot burst, so its existence is less interesting from a financial stability perspective. Nevertheless, this deviation can magnify price movements resulting from, e. g., changing dividend payments. By contrast, the bubble described in the following section is dynamic by nature. It can be sustained only if the price is expected to continue increasing in the future. Increasingly large price deviations will be fueled by the *expectation* that future traders will buy at an even higher price. A bubble grows dynamically, and it can burst at any time.

3 An Example of a Bubble

Consider now a situation where the price p_t is above the steady-state price $\bar{p}$ at some date t. The only conceivable reason to buy is that traders expect the price to rise even further, at least with some probability. Otherwise, it would be a dominant strategy for traders to store rather than to invest in the asset. We will call this expectations-driven price deviation a *bubble*. In this section, we discuss the existence of bubbles under two specific assumptions, one concerning possible price paths $\{\tilde{p}_t\}_t$ and one concerning the distribution of liquidity $F(N)$. Both assumptions will be generalized in section 4.

First, let us concentrate on a *trinomial* process with

$$\tilde{p}_{t+1} = \begin{cases} 0, & \text{with probability } 1-q \\ \bar{p}, & \text{with probability } q-Q_t \\ p_{t+1}, & \text{with probability } Q_t \end{cases} \tag{6}$$

with $Q_t \leq q$.[10] All variables $\{p_t, Q_t\}_t$ will be determined endogenously. For our purposes, trinomial processes are the simplest ones, allowing for a fundamental default of the firm (case one), a bursting of the bubble (case two), and the continuation of the bubble (case three).

Assumption 1 *For now, consider only trinomial price processes, as in (6).*

Second, let us concentrate on a parameterized version of the number of traders $F(N)$. To be concrete, we assume that $\log N$ is exponentially distributed; thus, $F(N) = 1 - e^{-\gamma(\log N - \log N_0)} = 1 - (N/N_0)^{-\gamma}$ for some $N_0 > 0$ and $\gamma > 1$. Here, N_0 is a lower bound on the number of investors, and γ measures the thinness of the tail.[11] Because traders borrow additional money, the aggregate amount of liquidity L is $(E + D)\, N$, which has the distribution function

$$\begin{aligned} \Pr\{(E+D)\, N \leq L\} &= \Pr\{N \leq L/(E+D)\} = F\big(L/(E+D)\big) \\ &= 1 - \Big(\frac{L}{(E+D)\, N_0}\Big)^{-\gamma} = 1 - \Big(\frac{L}{L_0}\Big)^{-\gamma}. \end{aligned} \tag{7}$$

with $L_0 \equiv (E + D)\, N_0$. Slightly abusing notation, we call $F(L)$ the distribution of liquidity. γ measures the precision of the information on liquidity; for $\gamma \to \infty$, liquidity is $L = L_0$ with certainty.

Assumption 2 *For now, assume that* $\log N$ *is exponentially distributed.*

Because each trader can invest $D + E$ dollars, an asset's price p can never exceed $L = N\,(E + D)$. Hence, L can be interpreted as an unknown upper bound on liquidity in the market. The following figure 2 shows the distributions and density functions for $N_0 = 20$ and shape parameters $\gamma = 2$ (solid) and $\gamma = 4$ (dashed).

[10]Note the notational difference between $\tilde{p}_{t+1}$ and p_{t+1}. $\tilde{p}_{t+1}$ is the stochastic price at date $t+1$ that can assume three different values. $p_{t+1} > p_t$ is the largest of these realizations.

[11]The unlimited support of the function implies that in a bubble agents can never be 100% sure that liquidity will be exhausted in the next period, which seems realistic.

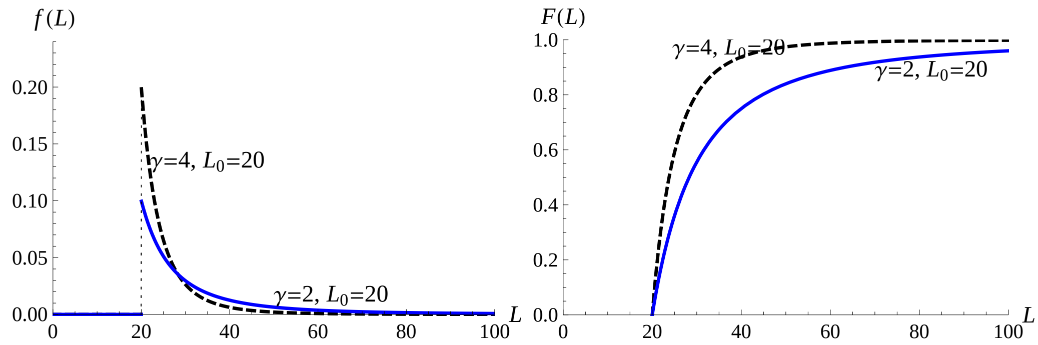

Figure 2: Examples of Density and Distribution Functions

Construction of a Bubble. Now we turn to the discussion of bubble paths. A possible price process is depicted in figure 3. The process starts at some price $p_0 > \bar{p}$; the resulting bubble can then grow further and further, $p_0 < p_1 < p_2 < \ldots$ For a given price increase from p_t to p_{t+1}, more liquidity will be absorbed by the market in $t+1$ than in t. As a consequence, p_{t+1} may exceed the liquidity potential L at some date. In this case, the price hits a ceiling, no further price increases are possible, i. e., traders cannot expect to sell the asset at a higher price in the future. Hence, the bubble must collapse back to the steady-state price $\tilde{p}_{t+1} = \bar{p}$. This ceiling L is not pictured in the figure as it is unknown. The date at which the bubble bursts is (and must be) unknown, but with certainty the ceiling will be hit at some date. Initially, the amount of liquidity absorbed by the risky asset is relatively small. But as the bubble grows, more and more traders are attracted by the risky asset, and it sucks in more and more liquidity. This liquidity is diverted from investment into the safe asset (storage).

Alternatively, if the underlying firm goes bankrupt, the price will drop to $\tilde{p}_{t+1} = 0$. Consequently, the conditional probability of a continuation (*non*-collapse) of the bubble is

$$\begin{aligned} Q_t &= q \ \Pr\{p_{t+1} \leq L | p_t \leq L\} = q \, \frac{1 - F(p_{t+1})}{1 - F(p_t)} \\ &= q \, \frac{L_0^\gamma / p_{t+1}^\gamma}{L_0^\gamma / p_t^\gamma} = q \, p_t^\gamma / p_{t+1}^\gamma, \end{aligned} \tag{8}$$

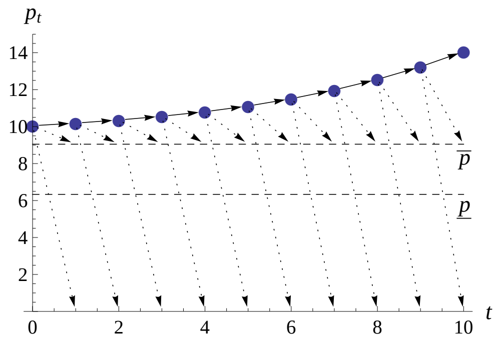

Figure 3: A Trinomial Price Process with a Bubble

As in figure 1, parameters are $\gamma = 2$, $\beta = 0.9$, $q = 95\%$, $d = 1$, and $r = 10\%$.

where q is the probability that a firm continues to operate and Q_t is the probability that the firm's asset price continues to rise. The probability that the bubble bursts although the firm is still solvent is thus $1 - Q_t - (1 - q) = q - Q_t = q\,(1 - p_t^\gamma / p_{t+1}^\gamma)$.

Since no dividends are paid and the shares are no longer traded if the firm is insolvent, the bonus to the trader is $\alpha \max\{0/p_t + 0/p_t - \beta; 0\} = 0$ in this case. If, alternatively, the share price falls because a bubble bursts, the price will drop to $\bar{p}$. For now, let us simply assume that there is no bonus if a bubble bursts. We give a condition and analyze the alternative in the proof of proposition 1 in the appendix. The asset market can only be in equilibrium if a modified version of (3) holds, taking into account the probability of a burst and the increased bonus if the bubble does not burst,

$$\begin{aligned} \alpha\,(1 + r - \beta) &= Q_t\,\alpha \left(\frac{p_{t+1} + d}{p_t} - \beta\right), \\ &= q \left(\frac{p_t}{p_{t+1}}\right)^\gamma \alpha \left(\frac{p_{t+1} + d}{p_t} - \beta\right), \\ \Rightarrow \quad \frac{1 + r - \beta}{q} &= \left(\frac{p_t}{p_{t+1}}\right)^\gamma \left(\frac{p_{t+1}}{p_t} + \frac{d}{p_t} - \beta\right). \end{aligned} \tag{9}$$

Equation (9) implicitly determines a price process in a rational expectations equilibrium. For any given $p_0 > \bar{p}$, (9) implicitly defines p_1, and (8) defines the according Q_0, so all variables for $\tilde{p}_1$ in (6) are defined. Then, starting from p_1 in a next step,

(9) and (8) define p_2 and Q_1, so $\tilde{p}_2$ is defined. Following this procedure gives the complete process recursively. One such process is shown in figure 3.

Existence of Bubble Processes. Equation (9) does not necessarily yield a solution for any set of parameters. As discussed above, the higher the potential future price p_{t+1}, the more likely it is that the maximum liquidity L will be hit and the bubble will burst. But the more likely a bursting of the bubble is, the larger the expected price increase must be to compensate traders for the risk they face. A feedback multiplier effect evolves, which does not necessarily reach an equilibrium price p_{t+1} for all p_t. In this case, there is a $\hat{p}$ above which potential future price increases cannot compensate for the accompanying higher risk of a burst. Because all market participants can calculate the date t at which this $\hat{p}$ is reached, if it exists, a bubble will burst with certainty at some date $t+1$, i. e., $Q_t = 0$. If the bubble cannot be sustained at a date $t+1$, traders will anticipate this. By backward induction, the bubble will not be sustainable right from the start. An example is given in figure 4. At date 7, the price has risen too high, i. e., above the dashed line representing $\hat{p}$, and the hypothetical bubble can no longer be sustained. Consequently, the according initial price p_0 cannot be part of a rational expectations equilibrium process in the first place.

We are interested in conditions under which a bubble can or cannot be sustained. In order to be sustainable, the implicit equation (9) must have a solution at any date t, or equivalently, for any initial price p_t. Rewriting (9) by defining the auxiliary variable $\phi_t = p_{t+1}/p_t$ as the relative price increase yields

$$\phi_t^{\gamma}\,(1+r-\beta) = q\left(\phi_t + \frac{d}{p_t} - \beta\right) \tag{10}$$

The left-hand side of the equation is independent of p_t (it only depends on its ratio to p_{t+1}), but the right-hand side depends on the starting point p_t. Figure 5 shows the left-hand side (thick blue curve) and the right-hand side for two starting prices, $p_t = \bar{p}$ (dashed black line) and $p_t = \infty$ (thin black line). First, consider $p_t = \bar{p}$.

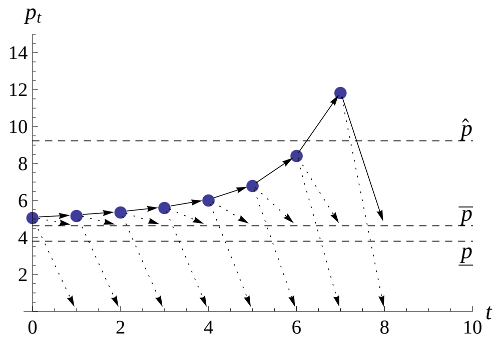

Figure 4: A Trinomial Price Process with a Non-sustainable Bubble

Parameters are $\gamma = 2$, $\beta = 0.9$, $q = 95\%$, and $d = 1$, but now $r = 20\%$. Note that this hypothetical bubble bursts with certainty after date 7, so it cannot emerge in the first place.

From the figure, one can see that the intersection with the thick curve is at $\phi_t = 1$, which implies that $p_{t+1} = \phi_t \, p_t = p_t$, so there is no price increase. Starting with $p_t = \bar{p}$, we are of course in steady state, and the price does not change over time. This steady state is a limiting case.

If the initial price is slightly above $\bar{p}$ due to higher expectations, the straight line shifts downward, implying that it intersects with the curve at some $\phi_t > 1$. In the next period, the price will be higher still, so the intersection ϕ_{t+1} will be even higher. A bubble emerges, and the growth rate $\phi_t = p_{t+1}/p_t$ increases with time. For $\lim p_t \to \infty$, the limiting line $q\,(\phi - \beta)$ is reached (solid line). Because the intersection point moves right as p_t increases, the bubble becomes less and less stable; the probability of a burst, $1 - Q_t = 1 - q/\phi_t$ increases.

Remark 2 *In a bubble process, the relative price increase $\phi_t = p_{t+1}/p_t$ grows over time and Q_t falls over time, so the bubble becomes less stable.*

In figure 4, we saw a hypothetical example of a bubble that was not feasible. Plotting the left and right sides of equation (10) for this set of parameters yields figure 6. Here, as the asset price increases and the straight line moves downward, at some

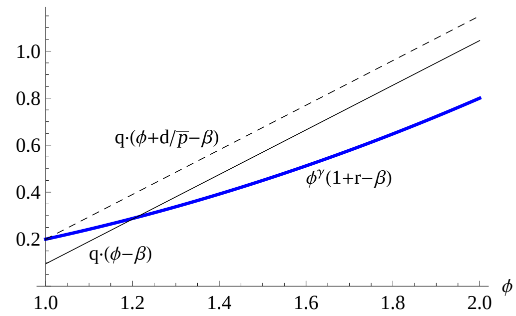

Figure 5: Possibility of a Bubble

As in figure 3, parameters are $\gamma = 2$, $\beta = 0.9$, $q = 95\%$, $d = 1$, and $r = 10\%$.

point an intersection between the line and the bold curve ceases to exist. As a consequence, the asset price cannot rise without bounds. However, because of the upper bound, a backward induction argument applies, and no rational price deviation can exist in the first place.

A Condition for Existence. In order to show that a bubble can be sustained in a market, it therefore suffices to consider large prices p_t. In the limit of $p_t \to \infty$, equation (10) simplifies to

$$\phi^{\gamma}\,(1 + r - \beta) = q\,(\phi - \beta). \tag{11}$$

The equation does not depend on time, so we have dropped the index t. If (11) has a solution for $\phi > 1$, the corresponding market can sustain a bubble.[12] This implies that for arbitrarily high prices p_t, there is always a price p_{t+1} that is high enough to make traders buy at date t. If (11) does not have a solution for ϕ, then there exists a price $\hat{p}$ beyond which no further increase is impossible. Nobody will buy, and the

[12]There are at most two solutions to (11) with $\phi \geq 1$ (values of $\phi < 1$ would stand for bubbles with falling prices and, formally, negative probabilities of a burst). We do not consider the high solution in the following because the corresponding equilibrium is unstable.

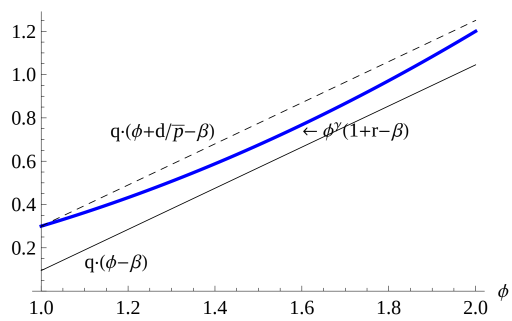

Figure 6: Non-Existence of Bubbles

Like in figure 4, parameters are $\gamma = 2$, $\beta = 0.9$, $q = 95\%$, $d = 1$, and $r = 20\%$.

bubble will burst. Hence, using backward induction, the bubble cannot begin to form at date $t = 0$. The only possible initial price is then $p_0 = \bar{p}$.

In figure 5, one can see that the solution for ϕ ceases to exist if the thin black line no longer intersects with the thick blue curve, like in figure 6. Although one cannot give a closed-form solution of 11, one can give a necessary and sufficient condition for existence. This leads to the following proposition, illustrated in Figure 7.

Proposition 1 *Under assumptions 1 and 2, in a rational expectations equilibrium, a price process can exhibit a trinomial bubble if and only if* $\gamma < q/(1 + r - \beta)$ *and*

$$\gamma^{\gamma} \left(\frac{\beta}{\gamma - 1} \right)^{\gamma - 1} \leq \frac{q}{1 + r - \beta}. \tag{12}$$

Remark 3 *Bubbles tend to be possible for a low risk-free rate* r*, low fundamental risk (large* q*), large uncertainty about liquidity* γ*, and substantial limited liability (high* β*).*

Remember that a higher risk of a bubble bursting implies a larger potential price increase to compensate traders and that a larger potential price increase makes bursting more likely. If and only if this problem has a fixed point at all times, a

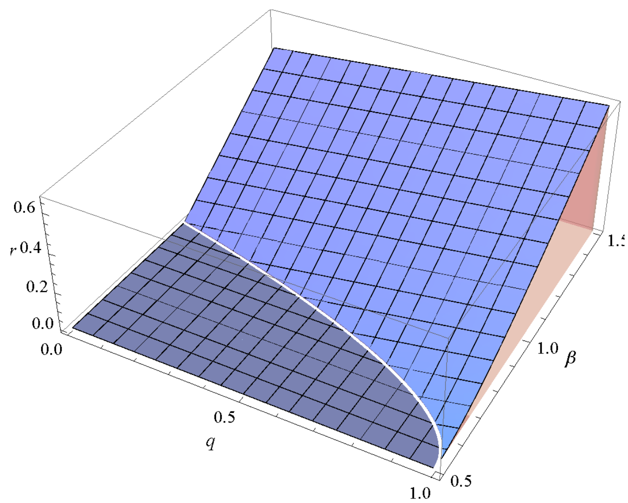

Figure 7: Parameter Range where Bubbles are Possible

Here, $\gamma = 2$. For parameters *below* the surface, bubbles are possible.

bubble can emerge. If risk-free rates are higher, storage becomes more attractive to traders; they need to be compensated by a larger potential price increase of the risky asset in order to hold it. But this makes a burst yet more likely, which impedes the convergence to a new fixed point. Hence, for a larger risk-free rate r, bubbles may cease to be possible. This is in line with the intuitions that central banks can puncture bubbles by increasing interest rates and that bubbles are especially likely to exist if interest rates are low. Furthermore, bubbles can exist especially if q is high; that is, if the underlying asset is rather safe, which decreases the likelihood of a burst. The parameter γ captures the uncertainty in the market. The smaller the value of γ, the larger are the mean and variance of the distribution, and the more uncertain is the potential market size. The parameter N_0 does not appear in the analysis, which shows that, for the existence of a bubble, only the shape of the upper tail matters; bubbles tend to exist for smaller values of γ. In the extreme case of $\gamma \to 1$, the expected market size becomes infinite, and $\gamma^\gamma \left(\beta/(\gamma-1)\right)^{\gamma-1} \to 1$. Hence, a bubble can emerge if $q > 1 + r - \beta$. On the other hand, if $\gamma \to \infty$, the market liquidity is almost certainly L_0, and a bubble can never be sustained independent of

	Static deviation (Remark 1)	Bubble condition (Remark 3)
γ	—	$\searrow$
r	$\searrow$	$\searrow$
β	$\nearrow$	$\nearrow$
q	$\searrow$	$\nearrow$
d	— (but multiplier)	—

Table 1: Effects of Variations of Parameters

the values of other parameters. This is the traditional backward induction argument of Tirole (1982). Finally, the parameter β describes the degree of limited liability or leverage. The larger the value of β, the more traders rely on external financing, and the more prominent the effect of the limited liability of the trader becomes. Hence, we have the result that the emergence of bubbles may become possible in the context of high degree of leverage.

Note again the difference between the *dynamic* price deviation from the steady-state price in this section and the *static* deviation of the steady-state price from the fundamental value of section 2.2. Table 1 summarizes comparative statics for both. An upward arrow means that the *static* deviation (left column) becomes larger or that the necessary condition for a bubble to emerge is met for a larger range of all other parameters (right column).

There is one major difference between the two columns. A static deviation is larger for inherently risky assets, but bubbles tend to emerge for inherently safe assets. Note one subtle but important difference between the inherent and the financial risk of an asset. To give an example, building a house may be inherently safer than buying stock in a firm. But taking into account financial risk, a house may be a riskier investment, especially if it is built during a bubble. Our model distinguishes between these notions of risk. Inherent risk is captured by $1 - q$, the risk of failure of the underlying asset. Additional financial risk can occur if movements in fundamentals are strongly magnified because the asset price deviates from the fundamental value in a static way or because condition (12) holds and a bubble can form. Interestingly, the two sources of financial risk react similarly to most parameter changes, but their reactions with respect to the underlying risk $1 - q$ are directly opposed.

This fine differentiation suggests different explanations for the two most prominent bubbles in recent decades. Real estate and mortgages are inherently safe, so according to proposition 1, a speculative bubble on these assets should be feasible. In this sense, the theory matches the recent real estate bubble.[13] On the other hand, dot-com firms are inherently risky. According to proposition 1, a speculative bubble may be impossible, but the static deviation above potentially changing fundamentals will be especially large. As a consequence, following our theory, the bursting of the dot-com "bubble" may have been the correction of expectations, bloated by a large multiplier.[14]

The Half-Life of a Bubble. Building on the above discussion, we can calculate the half-life of a bubble. Note first that the firm itself can default. In each period, it survives with probability q; hence, after t periods, it survives with probability q^t. The half-life T is the period after which firms have collapsed with probability 50%, so $q^T = 1/2$ and thus $T = \log \frac{1}{2} / \log q$. Given that a bubble can burst for two reasons, one fundamental (the firm defaults) and one financial (market liquidity

[13]As argued in footnote 5, the development of the housing bubble was also promoted by the constellation of the remaining parameters, i. e., low interest rates, opaque financial markets, and the 2004 decision of the Securities and Exchange Commission to allow the large investment banks to take on more debt, which corresponds to a higher β (see section 5.2). Allen and Gale (2000), among others, also point to an empirical connection between financial liberalization, credit expansion, and bubble emergence.

[14]Pástor and Veronesi (2009) model the learning process about the productivity of new technologies and apply it to the introduction of railroad and internet technologies. The mentioned multiplier is then an amplification of movements generated by learning.

is exhausted), the half-life of a bubble will always fall short of T. The conditional bursting probability increases over time, and it has no closed-form solution. However, we can derive comparative statics for a bubble that is fully developed, thus for large p_t. For this limit, $\phi = p_{t+1}/p_t$ is implicitly defined by (11), and $Q = q/\phi^\gamma$, so the half time T' of the bubble is given by

$$T' = \frac{\log \frac{1}{2}}{\log Q} = \frac{\log \frac{1}{2}}{\log q - \gamma \, \log \phi} < T. \tag{13}$$

Ceteris paribus, as ϕ increases, the half time decreases. In calculating the effects of parameter changes on the half-life of a bubble, we find that the same changes that can impede a bubble's formation also shorten the half-life of a bubble.

Remark 4 *The half-life of a bubble is shorter for a low risk-free rate r, low fundamental risk (large q), large uncertainty about liquidity γ, and substantial limited liability (high β).*

4 Bubbles in General

This section delivers the main results of the paper. We have already shown that if liquidity $\log L$ is exponentially distributed (assumption 2), a trinomial process (assumption 1) can develop a bubble if and only if condition (12) holds. More formally, if (12) holds, there are multiple rational expectations equilibria within the class of trinomial processes; otherwise, the equilibrium is unique.

This section shows that the condition is more general. First, and most importantly, we can generalize assumption 1. Hence, if (12) is not satisfied, the steady-state equilibrium is unique, without further qualifications. No bubbly equilibria are possible, whether trinomial or of any other shape. Second, we can also generalize assumption 2 about the probability distribution of liquidity.

4.1 Generalizing the Bubble Price Path

We have argued that a very special kind of bubble process, the trinomial bubble according to assumption 1, exists if and only if (12) holds. We now generalize this result by showing that if (12) fails to hold, the *only* rational expectations equilibrium process is the non-bubble process with the steady-state price $\bar{p}$. There are then no conceivable bubble paths. The argument is intuitive and proven in the appendix. A trinomial bubble does not exist if the thick curve and the thin straight line in figure 6 do not intersect, i. e., if there is no solution for ϕ, and hence prices in the bubble eventually increase too quickly to be sustainable. The point in the graph where the line and the curve are closest corresponds to the price increase with the lowest ratio of risk (of a bubble burst, related to ϕ^γ) to potential gains (of a price increase, related to ϕ). Concentrating all probability mass on this point maximizes the attractiveness of an investment in the risky asset, thereby creating favorable conditions for the emergence of bubbles. Distributing probability mass to other price increases, i. e., deviating from the assumption of a trinomial price process, reduces the willingness of traders to invest in the risky asset and can therefore eliminate the possibility of bubbles. Thus if, for a given set of parameters, not even a trinomial bubble path exists, no bubble can exist at all.

Proposition 2 *Under assumption 2, in a rational expectations equilibrium, a price process can exhibit a general bubble if and only if $\gamma < q/(1+r-\beta)$ and*

$$\gamma^\gamma \left(\frac{\beta}{\gamma-1}\right)^{\gamma-1} \leq \frac{q}{1+r-\beta}, \tag{14}$$

hence if (12) *holds.*

4.2 Generalizing the Distribution of Liquidity

Up to now, we have assumed in assumption 2 that $\log N$, and therefore also $\log L$, is exponentially distributed. In this subsection, we will derive a bubble condition for general distributions of the liquidity constraint L. However, the bubble condition (12) used the distribution-specific parameter γ. Most importantly, γ described the fatness of the tail of the distribution, thereby measuring the uncertainty about the maximum market liquidity. The following proposition states that only the tail behavior of $F(L)$ is relevant.

Proposition 3 *For a general distribution $F(L)$ with density $f(L)$, define*

$$\gamma := \min\left\{n : \lim_{x\to\infty} x^n\, f(x) \neq 0\right\} - 1. \tag{15}$$

Then, in a rational expectations equilibrium, a price process can exhibit a general bubble if and only if $\gamma < q/(1+r-\beta)$ and

$$\gamma^\gamma \left(\frac{\beta}{\gamma-1}\right)^{\gamma-1} \leq \frac{q}{1+r-\beta}, \tag{16}$$

hence, if (12) *holds. If $\gamma = \infty$, bubbles are impossible.*

We now discuss some intuitions for equation (15). We already know that the possibility of a bubble depends on the shape of the distribution $f(L)$ for large L, thus for large prices. When $\log L$ is exponentially distributed, the parameter γ actually gives the order of convergence of $F(L) \to 1$ for $L \to \infty$. It also gives the order of convergence if $f(L) \to 0$ for $L \to \infty$, less one. Intuitively, for a density function $f(N)$ of any other shape, the relevant statistic for the potential existence of a bubble is the order of convergence towards zero, as defined by (15). If $f(L)$ decays very quickly, the information about an upper liquidity ceiling is rather precise, so a bubble cannot exist. If $f(L)$ decays slowly, there is always room for further price increases without hitting the liquidity ceiling with high probability, which makes the emergence of bubbles possible. The proof of the proposition in the appendix is a more accurate version of this intuition.

5 Policy Measures

In this section, we examine whether certain policy measures that have been suggested in the public debate can prevent the creation of bubbles in our model. Specifically, we look at an asset-price augmented Taylor rule, capital requirements, and a financial transaction (Tobin) tax. In section 5.4, we interpret α and β as parts of a compensation package. We can then also discuss caps on bonuses and mandatory long-term compensation as possible policy measures.

5.1 Monetary Policy

We have already seen that a central bank can puncture a bubble by increasing interest rates. Let us now analyze the impact of an automatic, pre-announced interest rate increase in the case of a bubble, following a Taylor rule that takes asset price inflation into account. Specifically, assume a version of the rule used in Bernanke and Gertler (1999, 2001),

$$r_t = \bar{r} + \psi_\pi (\pi_t - \bar{\pi}) + \psi (p_t/p_{t-1} - \bar{\pi}), \tag{17}$$

where π_t is gross consumer price index (CPI) inflation and p_t/p_{t-1} asset price inflation of the only risky asset in the economy, as defined above. For simplicity, we neglect the influence of asset price inflation on CPI inflation by setting CPI inflation equal to its target rate $\bar{\pi}$, which is itself set to unity.[15] This does not influence our conclusions below. As in the above analysis, in a bubble, p_{t+1}/p_t converges towards a constant ϕ. Inserting (17) into (11) yields

$$\phi^\gamma \big(1 + \bar{r} + \psi (\phi - 1) - \beta\big) = q (\phi - \beta). \tag{18}$$

[15]Hence, r_t is the real interest rate. We implicitly assume that the central bank has at least some impact on short-term real rates, as it is standard in monetary economic theory.

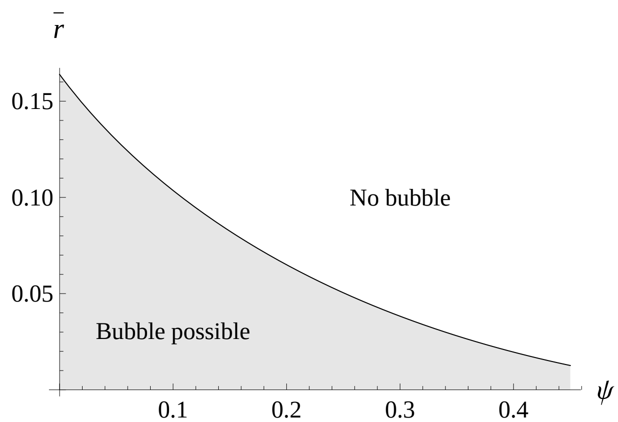

Figure 8: Effects of the Taylor Rule

As for (12), we can derive a condition for parameters $\bar{r}$, ψ, β, γ and q to determine whether (18) has a solution for $\phi > 1$. Unfortunately, the condition is algebraically complex. A bubble equilibrium exists if and only if

$$q\,(\phi - \beta) \geq \phi^{\gamma}\left(1 + \bar{r} + \psi\,(\phi - 1) - \beta\right) \text{ with}$$
$$\phi = \frac{1}{2\,\psi\,\gamma}\Big((1-\beta-\gamma)(1-\psi) + \bar{r} + \beta\gamma - \bar{r}\gamma + \beta\gamma\psi$$
$$+\sqrt{\left(\bar{r}+(1-\beta)(1-\psi)\right)\left((1-\gamma)^2(1+\bar{r}-\psi)-\beta((1+\gamma^2)(1-\psi)-2\gamma(1+\psi))\right)}\Big) > 1.$$

Figure 8 shows parameters $\bar{r}$ and ψ for which bubbles can exist for $\gamma = 2$, $\beta = 0.9$ and $r = 10\%$. The figure shows that, to prevent the emergence of bubbles, the central bank can either raise the steady-state interest rate $\bar{r}$ or threaten to raise interest rates in the future if a bubble should occur by committing to a Taylor rule with a positive ψ. If the central bank opts for the latter option, it never actually needs to raise interest rates: interest rate increases occur only as a consequence of asset price movements, but because of the credible announcement of this policy (with a sufficiently large ψ), asset prices do not rise and bubbles are prevented.[16]

Remark 5 *Monetary policy that systematically reacts to asset price increases re-*

[16]In this respect, the model differs from Bernanke and Gertler (1999, 2001), who show that monetary policy should not react to asset prices based on the assumption of *exogenous* bubbles.

duces the range of parameters under which bubbles are possible.

This argument shows that an augmented Taylor rule could cause fewer distortions than discretionary interest-rate policies do. However, if the central bank cannot differentiate between price movements due to bubbles and changes in the underlying fundamentals (such as the probability of bankruptcy, $1-q$) or if it is uncertain which assets to monitor, it faces a tradeoff between preventing bubbles and the risk of unnecessarily changing the interest rate in times without bubbles. A thorough examination of this trade-off would require a fully specified dynamic stochastic general equilibrium model, which is beyond the scope of this paper.

5.2 Capital Requirements

Capital requirements are straightforward to discuss. For E dollars of equity, a trader can borrow D dollars. The balance sheet total is thus $E+D$, and the equity ratio is thus $E/(E+D)$. If the regulator stipulates stricter capital requirements, the equity ratio must increase, hence D must drop. Because $\beta \equiv (1+r_D)\, D/(E+D)$, a smaller D leads to a smaller β. But from proposition 1, we know that a smaller β tends to allow only for a unique rational expectations equilibrium, without a bubble.

Remark 6 *Bubbles can be prevented by stricter capital requirements.*

Remember that, if the price path is too steep, the reason why bubbles do not exist is that the potential bubble is very likely to burst. Therefore traders cannot be compensated for an investment into an overpriced asset by further phantasies about price increases. But if traders are highly leveraged, they do not shun investment into overpriced assets and can easily be compensated. Because capital requirements push down leverage, they can eliminate potential bubbles.

5.3 Financial Transaction Tax

There are several possible ways to implement a so-called Tobin tax. In the following, assume that the tax must be paid by the buyer of an asset. We denote the tax rate on transactions of the safe asset τ and the potentially different tax rate on the risky asset τ'. Under such a tax regime, the no-arbitrage condition (11) changes to

$$\phi^\gamma\,(1 + r - \beta - \tau) = q\,(\phi - \beta - \tau'). \tag{19}$$

The modified conditions for the existence of bubbles are then $\gamma < [q\,(1-\tau')]/[1 + r - \beta - \tau]$ and

$$\gamma^\gamma \left(\frac{\beta + \tau'}{\gamma - 1}\right)^{\gamma-1} \leq \frac{q}{1 + r - \beta - \tau}. \tag{20}$$

The derivative of the left-hand side of the above expression with respect to τ' is positive, i. e., increasing the tax on transactions of the risky asset can make bubbles impossible. However, the way in which the tax is implemented is important. If it is levied on all financial assets, including the safe one, τ equals τ' and the derivative of the right-hand side w. r. t. the common tax rate is larger than the derivative of the left-hand side. Hence, in such a case, the possibility of bubbles can actually be created by the Tobin tax.

Remark 7 *If the financial transaction tax is levied on the risky asset only, bubbles can be prevented. However, placing the tax on the safe and the risky assets alike can make bubbles possible.*

5.4 Policy Measures for Intermediated Investment

In the following, we want to evaluate two more frequently proposed policy measures: caps on bonuses and long-term compensation. For this purpose, we interpret the above payoff scheme as a contract in an intermediated-investment setup. Assume that households do not have direct access to investment, but must employ institutional traders. These traders are compensated according to (1).[17] The parameter β is then the hurdle rate the trader has to surpass in order to get a bonus, which is proportional to her success according to the parameter α.

Caps on Bonuses. After the financial crisis, the amount of bonus payments has been heavily debated. In this context, we want to analyze whether bubbles can be prevented by cutting bonus payments. The bonus payment to a trader is $B = \alpha\left(\phi_t + d/p_t - \beta\right)$ if the underlying asset continues to pay off (probability q) and if the bubble does not burst (probability $1-Q$). Absent a bubble, this bonus payment is a constant. Let us first ask whether a potential cap on this bonus would bind early or late in the life of a bubble. In both cases, the bubble would have to burst with probability 1 at some date $\bar{t}$, so a backward induction argument would show that the bubble cannot exist in the first place. In terms of the bonus payment, ϕ_t increases over time, but d/p_t decreases. Summing up, due to (10), we have

$$B_t = \alpha\left(\phi_t + d/p_t - \beta\right) = \alpha\,\phi_t^{\gamma}\,(1 + r - \beta)/q.$$

Hence, bonuses increase over time in a bubble, and caps on bonus payments become binding in later stages of a bubble. As a consequence, we can concentrate on large prices p_t so that ϕ_t approaches a constant and the maximum bonus is

$$B = \alpha\left(\phi - \beta\right) = \alpha\,\phi^{\gamma}\,(1 + r - \beta)/q.$$

Now assume that the regulator puts restrictions on the bonus. There are several ways in which the regulation can be implemented. First, the compensation scheme could be adjusted such that bonuses are uniformly lower, for example by reducing

[17] There are several ways to endogenize such a contract. For example, a model with costly state verification, moral hazard or risk aversion on the side of households would lead to this kind of compensation scheme. In the latter case, the parameters α and β, and the shape of the contract, are then the solution of an optimal risk-sharing problem.

α or increasing β. However, α does not have an effect on the existence of bubbles, and an increase in β would favor the emergence of bubbles. Hence, this policy could backfire and render bubbles possible.

Second, one could put a cap $\bar{B}$ on bonuses. The compensation scheme accordingly adjusts to $\min\{\max\{\alpha\left((p_{t+1}+d)/p_t-\beta\right); 0\}; \bar{B}\}$. The bubble will then burst with certainty at some specific point if $\alpha\left(\phi-\beta\right) > \bar{B}$, thus if $\phi > \bar{B}/\alpha+\beta$. Economically speaking, from a certain point onwards, traders' bonuses cannot rise further to compensate them for the still-increasing risk of a bursting bubble. Consequently, for a given compensation scheme with parameters α and β, a cap on bonus payments $\bar{B}$ will render a bubble unfeasible if $\bar{B}/\alpha+\beta < \phi$, with ϕ implicitly defined by (11).

Remark 8 *To prohibit the emergence of a bubble by cutting bonuses, raising β is counterproductive, reducing α is irrelevant, and putting a cap $\bar{B}$ on bonuses is effective if the cap is low enough.*

The model can indicate which types of assets might need a cap on bonus payments and for which assets the cap must be lower. First, relatively safe assets (high q) tend to develop bubbles, and the price increase ϕ is especially low for safe assets. This implies that traders trading in markets with relatively safe assets (e. g., mortgages, bond markets) should have a ceiling in their bonus contracts that should be relatively low. Second, bubbles can emerge especially for high hurdle rates β, and the limit price increase ϕ is lower for large β. Because $\bar{B} < \alpha\,(\phi-\beta)$, the minimal effective cap $\bar{B}$ depends negatively on β. With a more ambitious benchmark, the cap must be stricter.

Long-term Compensation. In the recent political discussion, it has often been argued that traders' incentives should be made more sustainable such that they concentrate more on long-term goals and avoid short-termism. The same argument might apply to the traders in our model. To analyze this question, let us assume that the trader receives $\max\{0;\, \alpha\,(y-\beta)\}$ as before, but that she is liable with her compensation for potential future losses. Hence, she will get nothing if the accumulated yield is negative in the next period. In steady state, the market price will be

$$\alpha\,(1+r-\beta) = q^2\,\alpha\,\big((p_t+d)/p_t-\beta\big),$$
$$p_t = \bar{p} := \frac{d\,q^2}{(1-\beta)\,(1-q^2)+r},$$

i. e., smaller than without long-term liability. If a bubble exists, the probability that the bubble does not burst after two periods is

$$Q = q^2\,p_t/p_{t+2} = q^2/\phi^{2\,\gamma}.$$

As a consequence, the one-period price increase ϕ is determined by

$$\alpha\,(1+r-\beta) = Q\,\alpha\,\big(\phi-\beta\big) = q^2/\phi^{2\,\gamma}\,\alpha\,\big(\phi-\beta\big),$$
$$\phi^{2\,\gamma}\,(1+r-\beta) = q^2\,\big(\phi-\beta\big).$$

The equation is similar to (11), but γ is replaced by $2\,\gamma$, and q is substituted by q^2. Because bubbles exist especially for small γ and large q, according to proposition 1, we find that long-term liability prevents the formation of bubbles. For an even longer liability period, the effect would be larger.

Remark 9 *If traders are liable for future developments with their bonuses, bubbles can be prevented.*

6 Conclusion

Our model endogenizes two reasons why the price of an asset may deviate from its fundamental value. *First*, as also analyzed by Allen and Gale (2000), fund managers may drive up the price of risky assets because of their limited liability. This effect is larger for riskier assets. The price deviation is *not* driven by expectations and is constant over time; it involves no dynamics. *Second*, a fund manager may be willing to spend more than the fundamental value on an asset because she expects to earn even more when she sells the asset. This price deviation is completely driven by *expectations* and is dynamic, typically involving large, unpredictable abnormal returns until the bubble bursts.

These two stories are in line with anecdotal evidence. During the dot-com bubble (1998–2001), fantasies about the potential of internet firms were exuberant. The asset prices of these firms may have been even more exaggerated due to the limited liability of traders. Hence, traders' limited liability let the exuberance appear magnified. When expectations became more realistic, asset prices collapsed because the correction of expectations was again magnified. This complete argument follows from the *first* story. According to our model, it is applicable especially for risky assets, just like the stock of dot-com firms.

Following the "as long as the music is playing, you've got to get up and dance" explanation for the recent U. S. housing bubble, managers bought securities because they assumed they could sell them at a higher price later, driving up prices. This argument follows the *second* story. According to our model, it is applicable especially for fundamentally safe assets, just like real estate.

Our model suggests some possible ways to avoid such bubbles. One can increase interest rates, implement a Taylor rule that reacts to asset-price developments, or put a ceiling on bonus payments to fund managers. By virtue of its relative simplicity, the model lends itself to further discussions. For example, one could consider multiple assets and discuss whether the collapse of a bubble in one market can be contagious for other markets. One could also plug bubbles into macro models and investigate business cycle and growth effects. Especially after the recent burst of the housing bubble, possible applications seem numerous and relevant.

Appendix

Proof of remark 1. We have

$$\frac{\bar{p}}{\underline{p}} = \frac{(1-q)+r}{(1-\beta)(1-q)+r}.$$

Simple analysis proves the remark. ■

Proof of proposition 1. In the exposition in the main text, we have treated only the case in which no bonus is paid if a bubble bursts. Hence, we start the proof of the proposition by giving a condition for this case and analyzing the alternative.

If a bubble bursts, the firm still pays the dividend, so the bonus payment to the manager is

$$\alpha \max\left\{\frac{d}{p_t} + \frac{\bar{p}}{p_t} - \beta; 0\right\} = \alpha \max\left\{\frac{d + \frac{d\,q}{(1-\beta)\,(1-q)+r}}{p_t} - \beta; 0\right\}. \tag{21}$$

This implies that, if the price is only slightly above the steady-state price $\bar{p}$ (i. e., the bubble is small), the manager will earn a bonus even when the bubble bursts. The corresponding condition is

$$p_t < \hat{p} := \left(d + \frac{d\,q}{(1-\beta)\,(1-q)+r}\right)/\beta. \tag{22}$$

Now, if p_t is less than $\hat{p}$ such that (22) is satisfied, a modified version of (9) applies. In market equilibrium,

$$\alpha\,(1+r-\beta) + S = Q_t\,\alpha\,\big((p_{t+1}+d)/p_t - \beta\big) + (q-Q_t)\,\alpha\,\big((\bar{p}+d)/p_t - \beta\big) + S,$$

$$\frac{1+r-\beta}{q} = \left(\frac{p_t}{p_{t+1}}\right)^{\gamma}\frac{p_{t+1}}{p_t} + \left(1 - \left(\frac{p_t}{p_{t+1}}\right)^{\gamma}\right)\frac{\bar{p}}{p_t} + \frac{d}{p_t} - \beta. \tag{23}$$

Again, starting from p_t, we have an implicit equation for p_{t+1} in a rational expectations equilibrium. Substituting $p_{t+1} = \phi_t\,p_t$, we obtain

$$\phi_t^{\gamma}\,\frac{1+r-\beta}{q} = \phi_t + (\phi_t^{\gamma}-1)\,\frac{\bar{p}}{p_t} + \phi_t^{\gamma}\left(\frac{d}{p_t} - \beta\right). \tag{24}$$

However, in a bubble, the price p_t increases over time and eventually exceeds the threshold $\hat{p}$. Therefore, to find out whether bubbles are feasible, it suffices to consider the case $p_t > \hat{p}$, as we have done in the main text.

We have already argued that the probability that a bubble bursts increases with p_t. However, because p_t is an increasing function of t, a bubble is sustainable if and only if it is sustainable for $p_t \to \infty$. Hence, if (11) has a solution for ϕ, the bubble is sustainable. Now consider the limiting case in which the line $q\,(\phi - \beta)$ and the curve $\phi^\gamma\,(1 + r - \beta)$ only just touch. At the point of contact, the slopes must be equal, so

$$(1 + r - \beta)\,\gamma\,\phi^{\gamma-1} = q,$$

which implies that the point of contact is $\phi = \beta\,\gamma/(\gamma - 1)$. Substituting this solution into (11), we find that the limiting case is reached at

$$\left(\frac{\beta\,\gamma}{\gamma - 1}\right)^\gamma (1 + r - \beta) = q\left(\frac{\beta\,\gamma}{\gamma - 1} - \beta\right).$$

Some algebra yields (12). We have checked the conditions under which (11) has a solution. However, in a bubble, prices must increase, so $\phi > 1$. Considering the geometry of the problem, this is the case if the *right*-hand side of (11) is steeper in ϕ than the left-hand side is at the point $\phi = 1$. Otherwise, the curves would intersect for $\phi < 1$. Evaluating the derivatives of the left and right-hand sides of (11) at the point $\phi = 1$, we obtain the condition $\gamma < q/(1 + r - \beta)$. ■

Proof of remark 3. There are two conditions that may become increasingly strict or lax. First, condition (12) is satisfied iff

$$(1 + r - \beta)\,\gamma^\gamma \left(\frac{\beta}{\gamma - 1}\right)^{\gamma-1} - q \leq 0.$$

The derivative of this term with respect to q is negative, so the condition is more likely to be satisfied for large q. The derivative with respect to r is positive; hence,

(12) holds for small r. The derivative with respect to γ is

$$(1+r-\beta)\,\gamma^{\gamma}\left(\frac{\beta}{\gamma-1}\right)^{\gamma-1}\log\frac{\beta\,\gamma}{\gamma-1}.$$

Now remember that the point at which the curves touch is $\phi = \beta\,\gamma/(\gamma-1)$. The above logarithm is therefore positive, and the complete derivative with respect to γ is positive. A larger γ makes bubbles less likely. Finally, the derivative with respect to β is

$$\gamma^{\gamma}\left(\frac{\beta}{\gamma-1}\right)^{\gamma-1}\frac{(1+r)\,(\gamma-1)-\beta\,\gamma}{\beta}.$$

Again, at the touching point, ϕ must exceed 1, so $\beta \geq (\gamma-1)/\gamma$. For the limiting $\beta = (\gamma-1)/\gamma$, the numerator of the above fraction becomes $(1+r)\,(\gamma-1) - \beta\,\gamma = -r\,(\gamma-1) < 0$. Hence, for any β larger than the limiting $(\gamma-1)/\gamma$, the numerator must be negative. Thus, the whole derivative is negative, and a larger β makes bubbles more likely. The second condition, $\gamma < q/(1+r-\beta)$, has the same comparative statics. ■

Proof of remark 4. We start by discussing dT'/dr. Because ϕ is implicitly given by (11), let us define $\Psi = \phi^{\gamma}\,(1+r-\beta) - q\,(\phi-\beta)$, such that $\Psi = 0$ at the equilibrium ϕ. Then, the implicit function theorem yields

$$\frac{d\phi}{dr} = -\frac{\partial\Psi/\partial r}{\partial\Psi/\partial\phi} = -\frac{\phi^{\gamma}}{\gamma\,\phi^{\gamma-1}\,(1+r-\beta)-q}.$$

Because $\partial\Psi/\partial\phi < 0$ and $\phi^{\gamma} > 0$, we know that $d\phi/dr > 0$. Now, because in (13), $dT'/d\phi < 0$, the half life of a bubble drops as interest rates increase, $dT'/dr < 0$.

Now consider $dT'/d\beta$. Following the same procedure as above, note that $\partial\Psi/\partial\beta = q - \phi^{\gamma}$. Because $q \leq 1$ but $\phi > 1$, the term is negative. Consequently, $d\phi/d\beta < 0$, and thus $dT'/d\beta > 0$.

Now let us turn to dT'/dq. There are two effects. As a direct effect, considering (13), T' increases as q increases. A real default of the firm would also make the bubble burst. Hence, as the real half life of the firm increases, the half life of the bubble is also lengthened. The second effect is indirect. $\partial\Psi/\partial q = \beta - \phi < 0$, which implies $d\phi/dq < 0$, implying a longer half life. Thus, both effects go in the same direction, which implies that $dT'/dq > 0$.

Finally, consider $dT'/d\gamma$. The total derivative of Q is

$$\begin{aligned}\frac{dQ}{d\gamma} &= \frac{\partial Q}{\partial \gamma} + \frac{\partial Q}{\partial \phi} \cdot \frac{d\phi}{d\gamma} = \frac{\partial Q}{\partial \gamma} - \frac{\partial Q}{\partial \phi} \cdot \frac{\partial\Psi/\partial\gamma}{\partial\Psi/\partial\phi} \\ &= \frac{q^2\,\phi^{-\gamma}\,\log\phi}{(1+r-\beta)\,\gamma\,\phi^{\gamma-1} - q}.\end{aligned}$$

The numerator is negative, so the whole fraction $dQ/d\gamma$ is negative. As a result, the half life decreases as γ increases, $dT'/d\gamma < 0$. Remembering that the variance of the distribution is inversely related to γ, we can state that the half life of a bubble increases with rising uncertainty about aggregate liquidity. ■

Proof of proposition 2. Assume that a price process exhibits a bubble, that $p_t > \bar{p}$ at a date t, and that $\tilde{p}_{t+1}$ is distributed with distribution $F(\tilde{p}_{t+1})$. Then, in a rational expectations equilibrium,

$$\begin{aligned}\alpha\,(1+r-\beta) + S &= \int_0^\infty Q_t\,\alpha\,\max\left\{\frac{\tilde{p}_{t+1}+d}{p_t} - \beta; 0\right\} df(\tilde{p}_{t+1}) + S, \\ \frac{1+r-\beta}{q} &= \int_0^\infty h(\tilde{p}_{t+1})\,df(\tilde{p}_{t+1}),\ \text{where} \qquad (25)\\ h(\tilde{p}_{t+1}) &= \max\left\{\frac{p_t^\gamma}{\tilde{p}_{t+1}^\gamma}\left(\frac{\tilde{p}_{t+1}+d}{p_t} - \beta\right); 0\right\}\end{aligned}$$

is an auxiliary function. The p_{t+1} implicitly defined by (9) solves this equation for a distribution that has probability mass only at one point p_{t+1} (and zero and $\bar{p}$). The question is, from this three-point distribution, can we shift probability mass to other prices such that the above (25) still holds? The answer depends on the shape of $h(\tilde{p}_{t+1})$. Some straightforward analysis shows that $h(\tilde{p}_{t+1})$ is zero up to $\tilde{p}_{t+1} = \beta\,p_t - d$, then increases and decreases again. For $\tilde{p}_{t+1} \to \infty$, it again approaches zero asymptotically. The maximum of the integral is reached if all probability mass is located at

$$\tilde{p}^*_{t+1} = \gamma \, \frac{\beta \, p_t - d}{\gamma - 1} > \beta \, p_t - d.$$

Hence, a trinomial process with the possible events p^*_{t+1}, $\bar{p}$, and 0 maximizes the right-hand side of (25). Shifting probability mass to other parts of $h(\tilde{p}_{t+1})$ reduces the value of the integral. Note that no bubble can emerge if the left-hand side of (25) is larger than the right side for any price path. We can therefore conclude that if no trinomial bubble process exists, no other bubble process can exist either. On the other hand, if a trinomial bubble process exists, it is an example of a general bubble process. As a consequence, (12) is the general condition for the existence of bubble processes in rational expectations equilibrium. ■

Proof of proposition 3. For general $F(L)$ with infinite support, (11) reads

$$\lim_{p\to\infty} \frac{1 - F(p)}{1 - F(\phi \, p)} \, (1 + r - \beta) = q \, (\phi - \beta).$$

Define the first term as

$$G(\phi) := \lim_{p\to\infty} \frac{1 - F(p)}{1 - F(\phi \, p)} = q \, \frac{\phi - \beta}{1 + r - \beta}.$$

If this equation has a solution for $\phi > 1$, bubbles are feasible. We know that for $\phi = 1$, the left-hand side exceeds the right-hand side,

$$G(1) = 1 > q \, \frac{1 - \beta}{1 + r - \beta} \tag{26}$$

because $1+r-\beta > q\,(1-\beta)$. Therefore, bubbles are feasible if and only if $G(\phi)$ crosses $q \, \frac{\phi-\beta}{1+r-\beta}$ from above for some ϕ. A necessary condition would be $G'(\phi) < \frac{q}{1+r-\beta}$ for some $\phi > 1$. Let us delve deeper into the possible structure of $G(\phi)$. L'Hôpital's rule yields

$$G(\phi) = \lim_{p\to\infty} \frac{1 - F(p)}{1 - F(\phi \, p)} = \lim_{p\to\infty} \frac{f(p)}{\phi \, f(\phi \, p)} = \frac{1}{\phi} \lim_{p\to\infty} \frac{f(p)}{f(\phi \, p)}. \tag{27}$$

In the original example, under assumption 2, $f(p) = \gamma \, L_0^\gamma \, p^{-(1+\gamma)}$, so $G(\phi) = \phi^\gamma$. Now what shapes can $G(\phi)$ potentially take if assumption 2 is dropped? Because of

(15), $\gamma{+}1$ is the smallest number such that $x^{\gamma}\, f(x)$ does not converge to zero. In Landau notation, $f(L) = c_1\, x^{-\gamma-1}{+}O(|x|^{-c_2})$, where c_1 is a constant, and $c_2 > \gamma{+}1$ such that the error term decays faster than the main component $x^{-\gamma-1}$. Consequently, with this approximation,

$$G(\phi) = \lim_{p\to\infty} = \frac{1}{\phi} \lim_{p\to\infty} \frac{c_1\, p^{-\gamma-1} + O(|p|^{-c_2})}{c_1\, (\phi\, p)^{-\gamma-1} + O(|\phi\, p|^{-c_2})} = \phi^{\gamma}. \tag{28}$$

Thus, the behavior of $G(\phi)$ only depends on the order of convergence of $f(p)$ towards zero for large p. ■

Proof of remark 7. Let a tax on transactions of the safe asset be denoted by τ, a tax on selling the risky asset by τ'', and a tax on buying the risky asset by τ'. Hence, $\tau'' = 0$ means that the buyer of an asset has to pay the tax (the case analyzed in the main text), while τ' equals zero if the seller pays the tax. The equilibrium condition (11) is then

$$\phi^{\gamma}\, (1 + r - \beta - \tau) = q\, [\phi\, (1 - \tau'') - \beta - \tau'].$$

Following the steps taken in the proof of proposition 1 yields $\phi = \gamma(\beta + \tau')/[(\gamma - 1)(1 - \tau'')]$ as the touching point of both sides of the above equation. The condition for $\phi > 1$ turns into

$$\gamma < \frac{q\,(1 - \tau'')}{(1 + r) - \beta - \tau}. \tag{29}$$

The modified condition for the existence of bubbles is then

$$\left(\frac{\gamma}{1 - \tau''}\right)^{\gamma} \left(\frac{\beta + \tau'}{\gamma - 1}\right)^{\gamma-1} \le \frac{q}{1 + r - \beta - \tau}.$$

The right-hand side of this equation increases in τ, while the left-hand side increases in τ'' and τ'. Considering equal tax rates on the risky and the safe assets, one can use equation (29) to show that the derivative of the right-hand side w. r. t. the common tax rate is larger than the derivative of the left-hand side for both cases, $\tau = \tau''$ and $\tau' = 0$, or alternatively that $\tau = \tau'$ and $\tau'' = 0$. ■

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