Modeling Monetary Economies: an Equivalence Result†

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Abstract

This paper offers a methodological contribution to monetary theory. First, it presents a typical cash-in-advance model based on Lucas (1984) and then specializes it to preferences and shocks assumed in the model of Lagos and Wright (2005). Second, it derives the main equations describing allocations under competitive pricing and demonstrates that the two models—which on the surface appear different—are mathematically equivalent. Third, such equivalence result is extended to stationary equilibrium under non-competitive pricing: in each model, allocations depend on a free parameter controlling price markups.

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1 Introduction

This paper is a tale of two models. It studies a typical cash-in-advance economy in the tradition of [7, 8, 9, 11, 10]. The main equations that characterize such general equilibrium model are then compared to those emerging from the monetary model in [6] (hereafter, LW). This is a meaningful exercise because the cash-in-advance model is a workhorse in macroeconomics and LW is offered as a distinct alternative to reduced-form models—such as cash-in-advance. Indeed, LW is the benchmark model of an aptly named “New Monetarist Economics” literature; see [15].

On the surface, the LW framework seems different from a cash-in-advance model. For instance, it has been noted that LW provides explicit micro foundations for money; that it is capable to generate significant quantitative differences from other models; that it allows to consider a rich array of assets, credit arrangements, and intermediary structures; see [6, p.463-4] and [15, p.286]. Here we dig a bit more deeply, motivated by two observations. From a substantive standpoint, differentiation of the theory developed in LW from other theories requires a more systematic investigation. From a modeling standpoint, the LW framework shares some fundamental similarities with a typical cash-in-advance model. First, in both frameworks agents synchronously alternate between a centralized market (CM) and a decentralized market (DM).\(^1\) Both models can be seen as frameworks in which agents

\(^1\) For a lengthy description see [9, p.10], where it is stated that “agents alternate between two different kinds of market situations. Each period, they all attend a securities market in which money and other securities are exchanged. Subsequent to securities trading, agents trade in
purchase money and other assets in the CM, while goods and labor are traded in the DM. Second, the informational structure is similar: asset trading decisions are made in the CM before a random shock is observed in the DM. Third, consumption utility depends on whether the purchase is settled in the DM or in the CM.

How can we objectively differentiate the two models? Paraphrasing VonNeumann [14, p.158], we view an economic model as a set of mathematical relationships which, together with a verbal interpretation, describe economic allocations. Hence, a reasonable metric to compare models is evaluating the set of main equations that describe allocations in each model. We thus proceed as follows. In Section 2 we lay out the typical cash-in-advance framework. To do so, we closely follow the article in [9], which has a clear description of the physical environment and allows for a rich array of financial assets in addition to money. In Section 3 and 4 we gradually specialize the cash in advance model to the structure of shocks (iid) and preferences (separable in DM and CM goods) assumed in LW; we then derive the set of main mathematical relationships describing stationary equilibrium allocations. We report that such main equations are identical in the two models. This is immediately evident when prices equal marginal costs, and it is then demonstrated also for the case when DM prices include markups.2 The two models also exhibit

2A version of LW includes bilateral bargaining, which leads to a price markup above the marginal production cost. In a sense, LW can be seen as a framework in which price markups are parameterized by the relative bargaining power of buyers. We use this intuition to parametrically incorporate price distortions in the standard cash-in-advance model.
identical quantitative implications for the welfare cost of inflation. In particular, the magnitude of such cost is shown to depend on the one parameter that, in each model, regulates the degree of price markups. The lesson here is that the quantitative implications depend on the degree of price distortions assumed in the model, not the model itself. In this sense, the model in [6] is mathematically equivalent to a cash-in-advance economy because both models generate identical stationary equilibrium allocations and have identical quantitative implications for the welfare cost of inflation. Section 5 offers some final considerations on additional elements, not considered here, that may be helpful in establishing additional differentiations between the two models.

2 The typical cash-in-advance model

This section develops a standard general-equilibrium macroeconomic model without capital and with incomplete markets. To motivate money (and the assumed market incompleteness), we take off-the-shelf the model presented in [9, p.20], where money is introduced by means of cash-in-advance constraints.

Time is discrete and infinite, denoted \( t = 0, 1, \ldots \) There is a constant population composed of many infinitely-lived agents, who are ex-ante homogeneous and expected utility maximizers. Let \( s_t \) denote a shock at time \( t \), where the realization (also denoted \( s_t \)) is public knowledge at the beginning of \( t \). The shock is drawn from a time-invariant set. Let \( \{s_t\}_{t=0}^{\infty} \) denote a path of shocks and for \( t \geq 1 \),
$S^t = (s_1, ..., s_t) \in S^t$ a history of shocks, where $S^t$ is the set of all possible histories. Let $f^t(S^t)$ be the joint density of $S^t$ for $t \geq 1$. Each agent is endowed with $L > 0$ units of leisure at the start of each period. Preferences are defined over a nonstorable produced good (= service) and leisure. Each agent owns equal shares in a representative production technology (= firm) that produces the perishable good using the technology $F$, which has labor as the only factor of production.

To introduce money, the model assumes that some trades must be settled immediately with the exchange of money (= cash trades) and others can be settled by delivering goods in a subsequent market interaction (= credit trades). It is assumed that money is injected through lump-sum transfer by a central bank, and that goods purchased with cash are distinct from goods purchased on credit.\textsuperscript{3} The model adopts the convention that in a period traders alternate synchronously between centralized and decentralized markets (see [9, p-21-22]). Each period is divided into two subperiods, say, morning and afternoon. A centralized market (CM) is open only in the morning, while a decentralized market (DM) is open only in the afternoon; trades in this second market are subject to a cash-in-advance constraint.

Letting $s_t = (s_{1t}, s_{2t})$, we have $s_{1t}$ as a shock in the CM and $s_{2t}$ as a shock in the DM of $t$. For example, $s_{2t}$ may be a shock to the money supply and $s_{1t}$ may be a shock to TFP or to preferences. Shocks may be aggregate or not. It is assumed that

\textsuperscript{3}A way to motivate it is that goods 1 and 2 are consumed at separate points during each period; or see the discussion in [11].
CM trades are carried out before $s_{2t}$ is known. Therefore, $S^t = (s_1, ..., s_t)$ is known in the DM of $t$, but only $S^{t-1} = (s_1, ..., s_{t-1})$ and $s_{1t}$ are known in the CM of $t$. Hence, we use $f^t_1(S^{t-1}, s_{1t})$ to denote the density of $(S^{t-1}, s_{1t})$ and $f^t_2(s_{2t}; S^{t-1}, s_{1t}) = f^t(S^t)/f^t_1(S^{t-1}, s_{1t})$ for the density of $s_{2t}$ conditional on $(S^{t-1}, s_{1t})$. Events on a generic date $t$ occur as follows.

**Afternoon of $t$:** The DM is open. Household and firms trade goods 1 and 2, and labor. Households buy $c_{1t}(S^t)$ goods in exchange for money, buy $c_{2t}(S^t)$ goods on credit and supply $h_t(S^t)$ labor to the firm on credit. The firm demands $h^F_t(S^t)$ labor. Credit trades are settled in the CM of $t + 1$.

**Morning of $t$:** The CM is open. Before $s_{2t}$ is known, credit trades executed on $t - 1$ are settled, and trade on a financial market takes place. Firms pay wages for work supplied on $t - 1$ and pay dividends out of profits made on $t - 1$. Households pay for credit goods bought on $t - 1$, and receive a lump-sum money transfer $\theta_t(S^{t-1}, s_{1t})$ from the central bank. The central bank retires the old money supply $\bar{M}_{t-1}(S^{t-2}, s_{1,t-1})$ and issues a new money supply $\bar{M}_t(S^{t-1}, s_{1t})$ through lump-sum transfers. In the financial market households buy $M_t(S^{t-1}, s_{1t})$ money.

We call good 1 the “cash good” and good 2 the “credit good.”

### 2.1 Firm’s optimal choices

On date $t$, given history $S^t$, the constraint of the firm is

$$F(h^F_t(S^t); S^t) = c^F_{1t}(S^t) + c^F_{2t}(S^t)$$ (1)
where \(c_{1t}^{F}(S^t), c_{2t}^{F}(S^t)\) and \(h_t^{F}(S^t)\) are chosen in the DM of \(t\), hence depend on \(S^t\).

Because cash and credit goods are distinct, for full generality let \(p_{1t}(S^t)\) and \(p_{2t}(S^t)\) denote the nominal spot price of goods 1 and 2 in the DM of \(t\), and let \(w_t(S^t)\) be the nominal spot wage in the DM of \(t\). These nominal prices are contingent on the history of shocks \(S^t\). Nominal profits (net dollar inflows) on \(t\), given \(S^t\), are

\[
p_{1t}(S^t)c_{1t}^{F}(S^t) + p_{2t}(S^t)c_{2t}^{F}(S^t) - w_t(S^t)h_t^{F}(S^t),
\]

which are distributed as dividends in the CM of \(t + 1\).

Since the firm sells for cash and for credit in the DM of \(t\), payments accrue as follows: in the CM of \(t\) it receives payments for credit sales in the DM of \(t - 1\); in the DM of \(t\), it receives cash payments for contemporaneous cash-goods sales, which are carried into \(t + 1\) as “overnight” balances. Since in the CM of \(t\) only the history \(S^{t-1}\) and the shock \(s_{1t}\) are known (but not \(s_{2t}\), let \(q_{t}(S^{t-1}, s_{1t})\) denote the date–0 price of a claim to one dollar delivered in the CM of \(t\), contingent on \((S^{t-1}, s_{1t})\).

The date–0 value of a dollar earned by the firm on \(t \geq 0\) is \(\int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}\) because this dollar is paid out only on \(t + 1\).\footnote{Equivalently, \(\int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}\) is the date-0 price of a state-contingent bond that delivers one dollar in the CM of date \(t + 1\), conditional on \(S^t\).} In the CM of date 0 (= at the start
of the economy) the firm chooses sequences of output and labor to solve:

\[
\text{Maximize: } \sum_{t=0}^{\infty} \int \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} \{p_{1t}(S^t)c_{1t}^F(S^t) + p_{2t}(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t)\} dS^t ds_{20}
\]

subject to:

\[
c_{1t}^F(S^t) + c_{2t}^F(S^t) = F(h_t^F(S^t); S^t) \text{ on all } t, S^t,
\]

Substituting for \(c_{1t}^F(S^t)\) from the constraint, the FOCs are

\[
h_t^F(S^t): \quad p_{1t}(S^t)F'(h_t^F(S^t); S^t) - w_t(S^t) = 0 \quad \text{for all } t, S^t, s_{20}
\]

\[
c_{2t}^F(S^t): \quad p_{1t}(S^t) - p_{2t}(S^t) = 0 \quad \text{for all } t, S^t, s_{20}.
\]

Consequently, \(p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t)\) and

\[
p_t(S^t)F'(h_t^F(S^t); S^t) = w_t(S^t) \quad \text{for all } t, S^t, s_{20}. \tag{4}
\]

### 2.2 Households’ optimal choices

Households choose consumption of cash and credit goods, and labor effort in the DM of \(t\) after observing the shock \(s_t = (s_{1t}, s_{2t})\). Consumption on \(t\) is thus conditional on the history \(S^t = (s_1, s_2, \ldots, s_t)\). On date 0 households maximize expected utility

\[
\sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f_t(S^t) dS^t f_2(s_{20}) ds_{20}
\]

subject to a cash in advance constraint for good 1

\[
p_t(S^t)c_{1t}(S^t) \leq M_t(s_{t-1}, s_{1t}) \quad \text{for all } t \text{ and } S^t,
\]

where we have used \(p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t)\). Here \(M_t(s_{t-1}, s_{1t})\) denotes money holdings in the DM of \(t\), which are bought in the CM of \(t - 1\), where \(s_{2t}\) is not yet
known. Given this uncertainty, money may be held for transactions purposes and for precautionary reasons.

The households date−0 budget constraint in nominal prices is

\[
\sum_{t=0}^{\infty} \int \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} [p_t(S^t)(c_{1t}(S^t) + c_{2t}(S^t)) - M_t(S^{t-1}, s_{1t}) - w_t(S^t)h_t(S^t)] + q_t(S^{t-1}, s_{1t})[M_t(S^{t-1}, s_{1t}) - \theta_t(S^{t-1}, s_{1t})] \] \[\leq \Pi + \bar{M} \]

The left hand side is the date−0 present value of net expenditure. The right hand side lists the date−0 sources of funds: \(\bar{M}\) initial money holdings (=initial liabilities of the central bank) and the nominal value of the firm, \(\Pi\). The date−0 present value of net expenditure is calculated using the price of money delivered at the start of \(t\), i.e., in the CM. Net expenditure consists of two elements:

1. Net expenditure in the CM of \(t\): the agent chooses to hold \(M_t(S^{t-1}, s_{1t})\) money balances that have date−0 value \(q_t(S^{t-1}, s_{1t})\). Since the agent receives the transfer \(\theta_t(S^{t-1}, s_{1t})\) in the CM of \(t\), the agent spends \(M_t(S^{t-1}, s_{1t}) - \theta_t(S^{t-1}, s_{1t})\) in the CM of \(t\).

2. Net expenditures in the DM of \(t\): the agent earns \(w_t(S^t)h_t(S^t)\) wages during \(t\), which are paid on \(t + 1\); \(M_t(S^{t-1}, s_{1t}) - p_t(S^t)c_{1t}(S^t)\) money balances are not spent on \(t\) and are carried over to \(t + 1\); \(p_t(S^t)c_{2t}(S^t)\) is the expenditure on credit goods on \(t\), paid on \(t + 1\). These funds are available in the CM of \(t + 1\); hence, their date-0 value is \(\int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1}\).
Clearly, the date-0 value of net expenditure depends on the initial shock $s_{20}$ and the history of shocks $S^t$; hence the double integral.

Consumers choose $c_{1t}(S^t)$, $c_{2t}(S^t)$, $h_t(S^t)$, and $M_t(S^{t-1}, s_{1t})$ for all $t$ and $S^t$ to maximize the Lagrangian:

$$L := \sum_{t=0}^{\infty} \beta^t \int \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f^t(S^t) dS^t f_2^0(s_{20}) ds_{20} + \lambda(\Pi + \bar{M})$$

$$-\lambda \sum_{t=0}^{\infty} \int \left\{ \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} [p_t(S^t) (c_{1t}(S^t) + c_{2t}(S^t)) - M_t(S^{t-1}, s_{1t})]ight. \\
- w_t(S^t) h_t(S^t) + q_t(S^{t-1}, s_{1t}) [M_t(S^{t-1}, s_{1t}) - \theta_t(S^{t-1}, s_{1t})]\} dS^t ds_{20}$$

$$+ \sum_{t=0}^{\infty} \int \int \mu_t(S^t) [M_t(S^{t-1}, s_{1t}) - p_t(S^t)c_{1t}(S^t)] dS^t ds_{20}$$

(5)

where $\mu_t(S^t)$ is the Kühn-Tucker multiplier on the cash in advance constraint on $t$, given history $S^t$.

Omitting the arguments from $U$ and $f$ where understood, taking the partial of $L$ relative to $c_{1t}(S^t)$, $c_{2t}(S^t)$, $h_t(S^t)$ and $M_t(S^{t-1}, s_{1t})$, in an interior optimum we have respectively:

$$\beta^t U_1 f^t f_2^0 - \lambda p_t(S^t) \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} - \mu_t(S^t)p_t(S^t) = 0$$

with $p_t(S^t)c_{1t}(S^t) \leq M_t(S^{t-1}, s_{1t})$

$$\beta^t U_2 f^t f_2^0 - \lambda p_t(S^t) \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} = 0$$

(6)

$$\beta^t U_3 f^t f_2^0 - \lambda w_t(S^t) \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} = 0$$

$$\int \left\{ \lambda \left[ \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} - q_t(S^{t-1}, s_{1t}) \right] + \mu_t(S^t) \right\} ds_{2t} = 0.$$
All but the last expression are valid for all $t, s_{2t}, S^t$, while the last expression is valid for all $t, s_{2t}, s_{1t}$, and $S^{t-1}$. The last expression holds because $M_t(S^{t-1}, s_{1t}) > 0$.

Noticing that $\int q_t(S^{t-1}, s_{1t})ds_{2t} = q_t(S^{t-1}, s_{1t}) \int ds_{2t} = q_t(S^{t-1}, s_{1t})$ we rewrite the last equation in (6) as

$$\lambda \int \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}ds_{2t} - \lambda q_t(S^{t-1}, s_{1t}) + \int \mu_t(S^t)ds_{2t} = 0$$

In a representative agent setting, given market clearing $h_t = h^F_t$, from (4) and (6) we get

$$\frac{U_3}{U_2} = F'(h_t(S^t); S^t) \quad \text{for all } t, s_{20}, S^t.$$

The marginal rate of substitution between goods 1 and 2 is

$$\frac{U_1}{U_2} = \frac{\lambda \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1} + \mu_t(S^t)}{\lambda \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}} \quad \text{for all } t, s_{20}, S^t. \quad (7)$$

As a means of comparison, the main equations for the complete-markets model without money are presented in the Appendix.

### 2.3 Monetary policy

Let $\bar{M} \geq 0$ denote initial money balances, on date 0. In the CM of each period $t$, the central bank issues a new money supply $\bar{M}_t(S^{t-1}, s_{1t})$ using lump-sum transfers $\theta_t(S^{t-1}, s_{1t})$. Hence, the money supply process can be history-dependent, but is not dependent on $s_{2t}$, because that shock is not observed until the DM opens in $t$. In the CM of $t + 1$, the central bank retires the money supply issued on $t$, and issues a new money supply. The date–0 value of assets held by the central bank in the
CM of $t$ is $\bar{M}_t(S^{t-1}, s_{1t})q_t(S^{t-1}, s_{1t})$. The liabilities include the lump-sum transfers $\theta_t(S^{t-1}, s_{1t})q_t(S^{t-1}, s_{1t})$ and the cost of retiring the money supply on $t + 1$, which is $\bar{M}_t(S^{t-1}, s_{1t}) \int \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}ds_2$, given the uncertainty on $s_{2t}$ and $s_{1,t+1}$.

The intertemporal budget constraint of the central bank is thus

$$\bar{M} = \sum_{t=0}^{\infty} \{ \bar{M}_t(S^{t-1}, s_{1t}) [q_t(S^{t-1}, s_{1t}) - \int \int q_{t+1}(S^t, s_{1,t+1})ds_{1,t+1}ds_2] - \theta_t(S^{t-1}, s_{1t})q_t(S^{t-1}, s_{1t}) \} d(S^{t-1}, s_{1,t}),$$

which can be rewritten as a set of flow constraints (see the appendix)

$$\bar{M}_t(S^{t-1}, s_{1t}) - \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) = \theta_t(S^{t-1}, s_{1t}). \quad (8)$$

That is, each money injection equals the lump-sum-transfer in that period.

### 3 A special case of the cash-in-advance model

In this section we specialize the cash-in-advance model described above to the case in which $s_t = s_{2t}$, i.e., there is only one shock per period. To maintain the key assumption that the random shock is not observed when money balances are chosen, we can follow one of two routes. One is to assume that $s_t$ is not known at the start of $t$ (in the CM) but only in the middle of the period (in the DM). An equivalent informational structure is obtained by reversing the order of the CM and DM: the DM opens only at the beginning of each period, while the CM opens after. This second route allows us to let $S^t = (s_1, ..., s_t)$ denote the history of shocks that is known prior to all period $t$ trading. Hence, we choose this second route and let $f^t(S^t)$ denotes the density of the history $S^t$. 12
The change in timing of markets implies that the firm’s profits from DM trades on \( t \) depend on \( S^t \) and are distributed as dividends in the CM of \( t \). Also, credit trades in the DM of \( t \) are settled in the CM of \( t \). Hence, payments to the firm accrue as follows on any \( t \): in the DM it receives cash payments for cash-goods sales in \( t \), in the CM it receives payments for credit sales in the DM of \( t \). Hence, expression (2) still holds (it holds even if cash transactions are settled in the CM of \( t \)) and so does the resource constraint (1).

Let \( q_t(S^t) \) denote the date–0 price of a claim to one dollar delivered in the CM on \( t \), contingent on \( S^t \) (state-contingent nominal bond). The firm’s date–0 profit-maximization problem is a simpler version of (3); it chooses sequences of output and labor \((c_{1t}^F(S^t), c_{2t}^F(S^t), h_t^F(S^t))\) given state-contingent prices \( q_t(S^t) \), to solve

Maximize: \[
\sum_{t=0}^{\infty} \beta^t \int q_t(S^t) \left\{ p_{1t}(S^t)c_{1t}^F(S^t) + p_{2t}(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t) \right\} dS^t
\]

subject to: \[
c_{1t}^F(S^t) + c_{2t}^F(S^t) = F(h_t^F(S^t); S^t)
\]
The optimality conditions are still given by (4).

On date 0 consumers maximize the expected utility

\[
\sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f^t(S^t) dS^t
\]
subject to two constraints. One is the cash in advance constraint for good 1

\[
p_{1t}(S^t)c_{1t}(S^t) \leq M_t(S^{t-1}) \quad \text{for all } t \text{ and } S^t,
\]
where \( M_t(S^{t-1}) \) are money balances held in the DM of \( t \), but chosen in CM of \( t - 1 \), when the shock \( s_t \) was not yet known. The other constraint is the date–0 nominal
intertemporal budget constraint in terms of CM prices:
\[
\sum_{t=0}^{\infty} \int \{ q_t(S^t) [p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_t(S^t)h_t(S^t) - M_t(S^{t-1})] \\
+ q_t(S^t)[M_{t+1}(S^t) - \theta_t(S^t)] \} \, dS^t \leq \Pi + \bar{M}
\]
Sources of funds include $\bar{M}$ initial money holdings and the nominal value of the firm, $\Pi$. The present value of net expenditure is calculated by considering the value of nominal funds available at the end of each period $t$ (i.e., in the CM); it includes

1. Net expenditure in the DM: $w_t(S^t)h_t(S^t)$ wages earned in the DM, paid in the CM; $M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)$ balances unspent in the DM, carried to the CM; $p_{2t}(S^t)c_{2t}(S^t)$ spent on credit goods in the DM, paid in the CM. The date-0 value of one dollar worth of expenditure in the DM of $t$, in terms of CM prices of $t$, is $q_t(S^t)$.

2. Net expenditures in the CM: $M_{t+1}(S^t) - \theta_t(S^t)$ net purchases of money. Their date-0 value is $q_t(S^t)$.

Consumers choose sequences of state-contingent consumption, labor and money holdings $c_{1t}(S^t)$, $c_{2t}(S^t)$, $h_t(S^t)$, and $M_{t+1}(S^t)$ to maximize the Lagrangian:
\[
L := \sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f^t(S^t)dS^t + \lambda(\Pi + \bar{M}) \\
- \lambda \sum_{t=0}^{\infty} \int \{ q_t(S^t)[p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_t(S^t)h_t(S^t) \\
- M_t(S^{t-1})] + q_t(S^t)[M_{t+1}(S^t) - \theta_t(S^t)] \} \, dS^t \\
+ \sum_{t=0}^{\infty} \int \mu_t(S^t)[M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)]dS^t.
\]
The FOCs for all $t$, $S^t$ are

$$c_{1t}(S^t) : \quad \beta^t U_1 f^t(S^t) - \lambda p_{1t}(S^t) q_t(S^t) - \mu_t(S^t) p_{1t}(S^t) = 0$$

$$p_{1t}(S^t) c_{1t}(S^t) \leq M_t(S^{t-1})$$

$$c_{2t}(S^t) : \quad \beta^t U_2 f^t(S^t) - \lambda p_{2t}(S^t) q_t(S^t) = 0$$

$$h_t(S^t) : \quad \beta^t U_3 f^t(S^t) - \lambda w_t(S^t) q_t(S^t) = 0$$

$$M_{t+1}(S^t) : \quad -\lambda q_t(S^t) + \lambda \int q_{t+1}(S^{t+1}) ds_{t+1} + \mu_{t+1}(S^{t+1}) ds_{t+1} = 0.$$  \(10\)

Given $p_{2t}(S^t) = p_{1t}(S^t) = p(S^t)$ and (4) we get

$$\frac{U_3}{U_2} = F'(h_t(S^t); S^t) \quad \text{for all } t, S^t$$

$$\frac{U_1}{U_2} = \frac{\lambda q_t(S^t) + \mu_t(S^t)}{\lambda q_t(S^t)} \quad \text{for all } t, S^t.$$  \(11\)

The last expression is helpful to determine the nominal riskfree interest rate on a one-period bond sold on $t$. Fix $t$ and a history $S^t$. The (reciprocal of the) nominal risk-free interest rate on a bond sold in the CM of $t$ is $\frac{1}{1 + r_t(S^t)}$. This is a relative price: the price of a claim to money (bought on date 0) delivered in the CM of $t + 1$ conditional on $S^t$ but not contingent on $s_{t+1}$, relative to the price of a claim to money delivered in the CM of $t$ conditional on $S^t$. That is,

$$\frac{1}{1 + r_t(S^t)} := \frac{\int q_{t+1}(S^{t+1}) ds_{t+1}}{q_t(S^t)} = \frac{\lambda \int q_{t+1}(S^{t+1}) ds_{t+1} + \int \mu_{t+1}(S^{t+1}) ds_{t+1}}{\int q_{t+1}(S^{t+1}) ds_{t+1} + \int \mu_{t+1}(S^{t+1}) ds_{t+1}}.$$  \(12\)

where the last expression is obtained by using the last line in (10).\(^5\)

\(^5\)No-arbitrage requires that expenditures in period 0 are equivalent. The household can spend $q_t(S^t) \frac{1}{1 + r_t(S^t)}$ to buy $1$ good on date $t$ conditional on $S^t$, and then reinvest on $t$ the receipts in a risk-free bond to get $1$ good on date $t+1$. Alternatively, the agent can spend $\int q_{t+1}(S^{t+1}) ds_{t+1}$ on date 0 to have one unit on date $t + 1$, given $S^t$. 

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From (11) we see that the interest rate makes households indifferent between buying money or risk-free bonds in the CM of $t$. With cash the consumer can buy either good 1 or good 2 in the DM of $t+1$; by holding bonds the consumer can only buy credit goods, because bonds mature in the CM of $t+1$. Therefore the interest rate compensates the consumer for the bond’s illiquidity, which is why $\mu_{t+1}$ appears in the denominator in (12). Using the first two lines in (10) and integrating with respect to $s_{t+1}$, we have

$$
\frac{1}{1 + r_t(S^t)} = \frac{\int_{\underline{p}_{t+1}(S_{t+1})}^{\bar{p}_{t+1}(S_{t+1})} f_{t+1}(S_{t+1}) ds_{t+1}}{\int_{\underline{p}_{t+1}(S_{t+1})}^{\bar{p}_{t+1}(S_{t+1})} f_{t+1}(S_{t+1}) ds_{t+1}}
$$

The central bank constraint is now simpler. In the CM of $t$, it issues $M_{t+1}(S^t)$ money valued at $q_t(S^t)$ in date–0 prices; this money supply is retired in the CM of $t + 1$, when the expected value of money is $\int q_{t+1}(S_{t+1}) ds_{t+1}$. Money is injected in the CM of $t$ via lump-sum transfer $\theta_t(S^t)$ valued at $q_t(S^t)$. Therefore, the date–0 budget constraint is

$$
\bar{M} = \sum_{t=0}^{\infty} \int \left\{ \bar{M}_{t+1}(S^t) \left[ q_t(S^t) - \int q_{t+1}(S_{t+1}) ds_{t+1} \right] - \theta_t(S^t) q_t(S^t) \right\} dS^t.
$$

Equivalently, monetary policy is identified by the set of flow constraints

$$
q_t(S^t) \left[ \bar{M}_{t+1}(S^t) - \bar{M}_t(S^{t-1}) \right] = q_t(S^t) \theta_t(S^t) \text{ for all } t, S^t
$$

4 An equivalence result

This section is devoted to prove the following main result.
Proposition 1 Consider preferences, technologies, shocks and pricing as in [6]. Then, the cash-in-advance model in [9] and the monetary framework in [6] exhibit identical stationary equilibrium allocations.

In what follows we will demonstrate in incremental steps why the model in [6] can be seen as being mathematically equivalent to a version of the cash-in-advance model. This holds when DM prices equal marginal costs and also when they include markups. To show this we start by specializing the cash-in-advance model to the preferences, technologies and shock process assumed in [6]. First, assume away aggregate shocks by setting $F(h_t(S^t); S^t) = F(h_t(S^t))$ for all $t$ and letting $\theta_t(S^t) = \theta$ for all $t$. That is, neither the production technology nor monetary policy depend on the history of shocks $S^t$. It is assumed that $s_t$ is an i.i.d. shock to preferences, such that in each period a randomly drawn portion $1 - \delta \in (0, 1)$ of households has no utility from consuming good 1. Hence, 

$$f^t(S^t) = f^t(s_t; S^{t-1}) = f(s_t)f^{t-1}(S^{t-1}) \quad \text{for all } t \geq 0,$$

where $f$ denotes the distribution of the date-$t$ shock. Here $s_t = (s^i_t)_{all \ i}$ where

$$s^i_t = \begin{cases} 
1 & \text{with probability } \delta \\
0 & \text{with probability } 1 - \delta 
\end{cases} \quad \text{for all } t \geq 0 \text{ and all agents } i,$$

where $s^i_t = 0$ means that household $i$ derives no utility from consuming good 1. The marginal probabilities are thus $\int f(s_t)1_{\{s^i_t=0\}}ds_t = 1 - \delta$ and $\int f(s_t)1_{\{s^i_t=1\}}ds_t = \delta$.

Second, preferences are assumed additively separable with linear labor disutility: 

$$U(c_{1t}, c_{2t}, h_t) = u_1(c_{1t})1_{\{s^i_t=1\}} + u_2(c_{2t}) - h_t \quad \text{for } t \geq 0. \quad (13)$$
Finally, let $F(h_t(S^t)) = h_t(S^t)$ so labor demand is infinitely elastic. Using (4), in competitive equilibrium the price of goods equal marginal cost, i.e.,

$$p_t - w_t = 0 \quad \text{for all } t \text{ and } S^t,$$

where $p_t$ and $w_t$ are history-independent because the marginal product of labor is independent of $S^t$. The relative price of cash to credit goods is thus $\frac{p_t}{w_t} = 1$.

To solve the model consider a generic consumer $i$. On date 0, he can spend $q_t(S^t)$ to buy a claim to one unit of money delivered in the CM of $t$, contingent on the history of shocks $S^t$. Let $q_t$ be the price of one unit of money delivered on $t$ unconditional on $S^t$ (a riskfree discount bond). No-arbitrage requires equal expenditures, i.e., $q_t = \int q_t(S^t) dS^t$. It also implies\(^6\)

$$q_t(S^t) = q_t f^t(S^t).$$

The date—0 intertemporal budget constraint is now

$$\sum_{t=0}^{\infty} \int q_t f^t(S^t) \{[p_{1t} c_{1t}(S^t) + p_{2t} c_{2t}(S^t) - w_t h_t(S^t) - M_t(S^{t-1})]$$

$$+ q_t f^t(S^t)[M_{t+1}(S^t) - \theta_t(S^t)]\} dS^t \leq \bar{M}$$

\(^6\)If $q_t(S^t) < q_t f^t(S^t)$, then $q_t(\tilde{S}^t) > q_t f^t(\tilde{S}^t)$ for some other state $\tilde{S}^t$ since $\int f^t(S^t) dS^t = 1$. In this case, the agent could make large profits with zero net investment by (i) purchasing claims that pay in state $S^t$ at a cheap price $q_t(S^t)$, while selling riskfree claims at price $q_t$; and (ii) selling claims that pay in state $\tilde{S}^t$ at a steep price $q_t(\tilde{S}^t)$, while buying riskfree claims at price $q_t$. Thus non-contingent claims would not be traded at price $q_t$, which is a contradiction.
The problem of agent $i$ is still given by (9), where we simply substitute $q_t(S^t) = q_t f^t(S^t)$ and $U$ from (13). The FOCs, for all $t$ and $S^t$, become

\[
c_{1t}(S^t) : \quad \beta^t u'_1(c_{1t}(S^t)) f^t(S^t) - \lambda p_{1t} q_t f^t(S^t) - \mu_t(S^t) p_{1t} = 0 \quad \text{for } s^i_t = 1
\]

\[p_{1t} c_{1t}(S^t) \leq M_t(S^{t-1})\]

\[
c_{2t}(S^t) : \quad \beta^t u'_2(c_{2t}(S^t)) - \lambda p_{2t} q_t = 0
\]

\[h_t(S^t) : \quad -\beta^t + \lambda w_t q_t = 0
\]

\[M_{t+1}(S^t) : \quad \lambda q_t f^t(S^t) = \lambda q_{t+1} f^t(S^{t+1}) + \int \mu_{t+1}(S^{t+1}) ds_{t+1}.
\]

The last line is derived using $q_{t+1} f^{t+1}(S^{t+1}) = q_{t+1} f(s_{t+1}) f^t(S^t)$ and noticing that

\[\int q_{t+1} f(s_{t+1}) f^t(S^t) ds_{t+1} = q_{t+1} f^t(S^t)\]

because $\int f(s_{t+1}) ds_{t+1} = 1$ by definition.\(^7\)

From the firm’s problem $p_{1t} = p_{2t} = w_t$ for all $t$; hence, the second and third expression in (15) give

\[u'_2(c_{2t}(S^t)) = 1 \quad \text{for all } t, S^t.
\]

That is, $c_{2t}(S^t) = c_2$ for all $t, S^t$ and all agents $i$.

Consider cash goods. Their consumption is heterogeneous because for all $S^t$, if $s^i_t = 0$ for agent $i$, then $c^i_{1t}(S^t) = 0$; this also implies $\mu_t(S^t) = 0$ for agent $i$ because this agent’s cash in advance constraint does not bind. Now consider $s^i_t = 1$. We prove that if an agent desires to consume cash goods, then the quantity consumed is independent of the history of shocks $S^t$ and of the identity of the agent, $i$.

\(^7\)The last equation in (15) implies that all agents who have the same shock $s^i_t$ have the same consumption. This is shown in what follows and, through an alternative proof, in the Appendix.
Lemma 2 Consider a generic agent $i$ when $s_i^t = 1$.

1. If $\mu_t(S^t) = 0$, then $c_{1t}(S^t) = c_1$ for all $t, S^t$, with $u_1'(c_1) = 1$.

2. If $\mu_t(S^t) > 0$, then $c_{1t}(S^t) = \frac{M_t}{p_t} = c_1$ for all $t, S^t$, where $c_{1t}$ satisfies

$$\beta^{t+1} \frac{u_1'(c_{1,t+1})\delta + (1 - \delta)u_2'(c_{2,t+1})}{p_{t+1}} - \frac{\beta^t u_2'(c_{2t})}{p_t} = 0 \text{ for all } t. \quad (16)$$

Proof of Lemma 2. See Appendix

On date $t$, not everyone consumes cash goods ($c_{1t} = 0$ when $s_i^t = 0$) but those who do consume an identical quantity $c_{1t}$ that is independent of the history of shocks. This also implies that everyone saves the same amount of money $M_t(S^{t-1}) = M_t$ on $t - 1$. Therefore, there is a non-degenerate distribution of money.

Using the risk-free interest rate defined in (12), we have

$$\frac{1}{1 + r_t} := \frac{\int q_{t+1}(S_i^{t+1})ds_{t+1}}{q_t(S^t)} = \frac{q_{t+1}f^t(S^t)}{q_tf(S^t)} = \frac{\beta}{\pi_t}.$$ 

The second equality holds by substituting $q_t(S^t) = q_tf(S^t)$ and noting that $q_{t+1}f^{t+1}(S^{t+1}) = q_{t+1}f(s_{t+1})f_t(S^t)$ so that $\int q_{t+1}f(s_{t+1})f_t(S^t)ds_{t+1} = q_{t+1}f_t(S^t)$ because $\int f(s_{t+1})ds_{t+1} = 1$. To perform the final step substitute $\frac{\beta u_2'(c_{2t}(S^t))}{\lambda_{p_{2t}}} = q_t$ from (15), use the result $u_2'(c_{2t}(S^t)) = 1$ for all $t, S^t$, and define the gross inflation rate $\pi_t := \frac{p_{t+1}}{p_t}$.

Due to linear labor disutility, households are indifferent to how much labor they supply at the given wage (equations two and three in (15)). Hence, we consider symmetric choices, i.e., every household supplies $h_t$. By market clearing

$$h_t^F = h_t = c_{2t}^F + c_{1t}^F = c_{2t} + \delta c_{1t} \text{ for all } t \text{ and } S^t.$$
On date $t$, the marginal rate of substitution between goods 1 and 2 for an agent who desires cash goods is

$$\frac{u'_1(c_{1t})}{u'_2(c_{2t})} = \frac{\lambda q_tf^t(S^t) + \mu_t(S^t)}{\lambda q_tf^t(S^t)} \text{ for all } t \text{ and } s^i_t = 1.$$ 

Using $u'_2(c_{2t}) = 1$, we have

$$u'_1(c_{1t}) = \left(1 + \frac{\mu_t(S^t)}{\lambda q_tf^t(S^t)}\right) \text{ for all } t, S^t.$$ 

If the constraint is not binding, $\mu_t(S^t) = 0$, then $c_{1t} = c^*_1$, which is the efficient quantity. Otherwise $c_{1t} = \frac{M_t}{p_t} < c^*_1$.

Since $u'_2(c_{2t}) = 1$ for all $t$, the main equation (16), can be written either as

$$\frac{1}{p_{2t}} = \frac{1}{p_{2,t+1}} \beta \left[u'_1(c_{1,t+1}) \delta + 1 - \delta\right] \text{ for all } t,$$

or—since $\frac{1}{p_t} = \frac{c_{1t}}{M_t}$ when $s^i_t = 1$—it can be written as

$$\frac{c_{1t}}{M_t} = \frac{\beta c_{1,t+1}}{M_{t+1}} \left[\delta u'_1(c_{1,t+1}) + 1 - \delta\right]. \quad (17)$$

The two last equations are identical to the main equation in the LW framework under linear labor disutility and zero bargaining power for sellers; more generally, the main equations in the two models are identical for the case of competitive equilibrium (see [6, equation (20)] and [13]). Notice also that, under the conjecture of price-taking behavior and linear labor disutility, the LW framework and the cash-in-advance-model are mathematically equivalent also outside stationary equilibrium.

---

8To see this, use the notation adopted in [6]; $\frac{1}{p_{2t}} = \phi_t = \frac{q_t}{M_t}$, $c_{1t} = q_t$, $1 - \kappa = \alpha\sigma$. Zero sellers’ bargaining power and linearity of labor disutility imply $z'(q_{t+1}) = 1$ in [6] (because $\theta = 1$ and $c' = 1$ from expression (11)).
Though linear disutility and price taking are often seen in studies based on the LW framework (e.g., [2, 3, 4, 5, 13]), quasilinear labor disutility and bargained prices have also been considered. Hence, we turn to study these cases.

4.1 Variation 1: Quasilinear labor disutility

Amend the model so that the firm can produce goods 1 and 2 in two separate production runs. The firm chooses \( h_{it}^F \) (= labor demanded to produce good \( i = 1, 2 \)) and \( c_{it}^F \) (= supply of good \( i = 1, 2 \)) to solve

Maximize: \[
\sum_{t=0}^{\infty} q_t [p_{1t} c_{1t}^F + p_{2t} c_{2t}^F - w_{1t} h_{1t}^F - w_{2t} h_{2t}^F]
\]

subject to: \( c_{2t}^F = h_{1t}^F \) and \( c_{1t}^F = h_{1t}^F \)

Substituting the constraints, the FOCs for the firm are

\[ p_{it} - w_{it} = 0 \quad \text{for all } t \text{ and } i = 1, 2. \]

Here, wages may differ across production processes; consequently, good prices may differ, too. Clearly, profits are still zero.

For a generic household, let \( h_{it}(S^t) \) denote the labor supplied to produce good \( i = 1, 2 \) in state \( S^t \). Following the formulation in [6, p. 465, equation (1)], consider

\[ U(c_{1t}, c_{2t}, h_{1t}, h_{2t}) := u_1(c_{1t})1_{\{s_i=1\}} + u_2(c_{2t}) - \eta(h_{1t}) - h_{2t}, \]

where \( \eta \) is a strictly convex function. Household \( i \) chooses sequences \( c_{1t}(S^t), c_{2t}(S^t), \)
\( h_{1t}(S^t), h_{2t}(S^t) \) and \( M_{t+1}(S^t) \) to maximize the Lagrangian:

\[
L^i := \sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_{1t}(S^t), h_{2t}(S^t)) f^t(S^t) dS^t + \lambda \bar{M} \\
-\lambda \sum_{t=0}^{\infty} \int q_t f^t(S^t)\{[p_{1t} c_{1t}(S^t) + p_{2t} c_{2t}(S^t) - w_{1t} h_{1t}(S^t) - w_{2t} h_{2t}(S^t)] \\
-M_t(S^{t-1}) + q_t f^t(S^t)[M_{t+1}(S^t) - \theta_t(S^t)]\} dS^t \\
+ \sum_{t=0}^{\infty} \int \mu_t(S^t)[M_t(S^{t-1}) - p_{1t} c_{1t}(S^t)] dS^t
\]

(18)

The FOCs differ from (15) only in the choice of labor:

\[
h_{1t}(S^t) : -\beta^t \eta'(h_{1t}(S^t)) + \lambda w_{1t} q_t = 0 \quad \text{for all } t, S^t
\]

\[
h_{2t}(S^t) : -\beta^t + \lambda w_{2t} q_t = 0 \quad \text{for all } t, S^t
\]

(19)

Output of good 1 is pinned down by the labor supply through

\[
\eta'(h_{1t}(S^t)) = \frac{w_{1t}}{w_{2t}} = \frac{p_{1t}}{p_{2t}}
\]

(20)

because the price of each good is equal to its marginal cost, \( p_{it} = w_{it} \) for \( i = 1, 2 \).

Hence, \( h_{1t}(S^t) = h_{1t} \) for all \( t, S^t \), so the aggregate supply of cash goods is independent of the history \( S^t \). By market clearing

\[
h_{1t}^F = h_{1t} = c_{1t}^F = \delta c_{1t} \quad \text{and} \quad h_{2t}^F = h_{2t} = c_{2t}^F = c_{2t} \quad \text{for all } t \text{ and } S^t.
\]

The relative price of cash to credit goods \( \frac{p_{1t}}{p_{2t}} \) (and the relative wage \( \frac{w_{1t}}{w_{2t}} \)) need not be one because labor disutility differs across production runs. Hence, workers’ wages generally differ depending on the good produced.
Since $-\beta^t + \lambda w_{2t} q_t = 0$ and $w_{2t} = p_{2t}$ (from the firm’s problem), the optimal choice of credit goods in (15) again satisfies $\beta^t u_2'(c_{2t}(S^t)) = \lambda p_{2t} q_t$; this implies $u_2'(c_{2t}(S^t)) = 1$, and hence $c_{2t}(S^t) = c_2$ for all $t, S^t$.

The added restriction (20) changes equilibrium consumption of cash goods relative to the linear labor disutility case. A version of Lemma 2 can be proved.

**Lemma 3** Consider any agent $i$ and let $s^i_t = 1$.

1. If $\mu_t(S^t) = 0$, then $c_{1t}(S^t) = c_1$ for all $t, S^t$, with $\frac{u_1'(c_1)}{\eta'(c_1)} = 1$.

2. If $\mu_t(S^t) > 0$, then $c_{1t}(S^t) = \frac{M_t}{p_{1t}} = c_1$ for all $t, S^t$, where $c_{1,t}$ satisfies

$$\beta^{t+1} \left[ u_1'(c_{1,t+1}) \frac{\delta p_{2,t+1}}{p_{1,t+1}} + (1 - \delta) u_2'(c_{2,t+1}) \right] - \frac{\beta^t u_2'(c_{2t})}{p_{2t}} = 0 \text{ for all } t. \quad (21)$$

**Proof of Lemma 3.** See Appendix

Once again, $c_{1t}(S^t) = \frac{M_t(S^t-1)}{p_{1t}} = c_1$ for all $S^t$, given that $s^i_t = 1$. That is, consumption of cash goods is independent of the history of shocks and on all dates all agents choose to save the same amount of money $M_t(S^t-1) = M_t$. Therefore, there is a non-degenerate distribution of money. The interest rate is still given by $r_t = \frac{\pi_t}{\beta} - 1$. The path of consumption is pinned down by a slightly different equation than (17) because the relative price $\frac{p_{1t}}{p_{2t}}$ need not be one. From consumer’s optimality $\frac{\mu_t}{p_{2t}} = \eta'(h_{1t})$ and $u_2'(c_{2t}) = 1$ for all $t$; hence, we rearrange (21) as

$$\frac{1}{p_{2t}} = \frac{\beta}{p_{2,t+1}} \left[ \frac{u_1'(c_{1,t+1})}{\eta'(\delta c_{1,t+1})} \delta + 1 - \delta \right],$$

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where \( h_{1,t+1} = \delta c_{1,t+1} \) from market clearing. Use \( \frac{1}{p_{2,t}} = \frac{\eta'(h_{1,t})}{p_{2,t}} \) from optimality in labor choices. Given \( \frac{1}{p_{1,t+1}} = \frac{c_{1,t+1}}{M_{t+1}} \) from the cash in advance constraint, we get

\[
\eta'(\delta c_{1,t}) \frac{c_{1,t}}{M_{t}} = \beta \eta'(\delta c_{1,t+1}) \frac{c_{1,t+1}}{M_{t+1}} \left[ \frac{u'_1(c_{1,t+1})}{\eta'(\delta c_{1,t+1})} \delta + 1 - \delta \right].
\] (22)

Now consider steady state equilibrium, which is the outcome typically studied; e.g., see [13]. Let \( M_{t+1} = \gamma M_t \) where \( \gamma \geq \beta \) denotes the rate of growth of the money supply. Consumption and real money balances held in the CM must be constant, i.e., \( \frac{M_{t+1}}{p_{2,t+1}} = \frac{M_t}{p_{2,t}} \); hence, \( \frac{p_{2,t+1}}{p_{2,t}} = \pi_t = \gamma \) and \( r_t = r = \frac{\gamma}{\beta} - 1 \) for all \( t \). Using this in (22) generates the main equation \( \frac{\beta - 1}{\beta} = \delta \left[ \frac{u'_1(c_1)}{\eta'(\delta c_1)} - 1 \right] \), rewritten as

\[
\frac{u'_1(c_1)}{\eta'(\delta c_1)} = \frac{r}{\delta} + 1,
\] (23)

which is identical to the main equation in [6, p.474] when sellers price at marginal cost (=have no bargaining power).9 To examine the case where sellers price above marginal cost we take one more step.

### 4.2 Variation 2: pricing above marginal cost in the DM

In [6] the case in which cash goods are sold at a price above marginal cost is studied by assuming that sellers have some bargaining power. A simpler route is to assume that a share \( 1 - \tau > 0 \) of revenue from cash-sales must be rebated back to the firm’s owners, lump-sum. This is equivalent to imposing a proportional tax on cash-sales,  

\[9\text{To see this notice that } \eta' \equiv z' \text{ (in [6] notation) and that in [6] only } \delta \text{ households can produce good 1 hence } h_{1t} = c_{1t} \text{ because there is one worker per buyer; in our setting there are } \frac{T}{T} \text{ workers per buyers, which is without loss in generality. Clearly, expression (22) is identical to the main equation in the LW framework under competitive equilibrium; see [13, equation (29) p.190].} \]
which distorts cash prices relative to the marginal cost $w_1$. The firm solves

$$\text{Maximize: } \sum_{t=0}^{\infty} q_t[p_{1,t} \tau c_{1t}^F + p_{2t} c_{2t}^F - w_{1t} h_{1t}^F - w_{2t} h_{2t}^F]$$

subject to: $c_{2t}^F = h_{1t}^F$ and $c_{1t}^F = h_{1t}^F$

where $\tau > 0$. Substituting for the constraints, the FOCs are for all $t$

$$(h_{1t}^F) : p_{1t} \tau - w_{1t} = 0$$

$$(h_{2t}^F) : p_{2t} - w_{2t} = 0.$$ 

This implies

$$\frac{w_{1t}}{w_{2t}} = \frac{\tau p_{1t}}{p_{2t}}.$$ 

Because $\frac{1}{\tau} = \frac{w_{1t}}{w_{2t}}$, and $w_t$ is the marginal cost, the ratio $\frac{1}{\tau}$ defines the markup over marginal cost for cash goods. CM goods are still sold at marginal cost.

The households’ problem remains the one described in (18) with the only difference that now the budget constraint is

$$\sum_{t=0}^{\infty} \int q_t f_t(S^t)\{[p_{1t} c_{1t}(S^t) + p_{2t} c_{2t}(S^t) - w_{1t} h_{1t}(S^t) - w_{2t} h_{2t}(S^t) - M_t(S^{t-1})]$$

$$+ q_t f_t(S^t)[M_{t+1}(S^t) - \theta_t(S^t)] - q_t T_t\}dS^t \leq \bar{M},$$

where $q_t T_t$ is a per-capita lump-sum rebate from firms to consumers. In equilibrium $T_t = p_{1,t}(1 - \tau)c_{1t}\delta$ in each $t$. Households still pay $p_{1t}$ for cash goods. The FOCs are still described by (15) and (19) but now we have

$$\eta'(h_{1t}(S^t)) = \frac{w_{1t}}{w_{2t}} = \frac{\tau p_{1t}}{p_{2t}}.$$  \hspace{1cm} (24)
The relative price of cash to credit goods \( \frac{p_1}{p_2} \) is generally not one not only due to quasilinear labor disutility but also due to the price markup, \( 1/\tau \).

A Lemma identical to Lemma 3 can be proved, and (21) still holds. Using (24), \( u_2'(c_{2t}) = 1 \) and \( \delta c_{1,t+1} = h_{1,t+1} \) for all \( t \), we now have

\[
\frac{1}{p_{2t}} = \frac{1}{p_{2,t+1}} \beta \left[ \frac{u_1'(c_{1,t+1})}{\eta'c_{1,t+1}/\tau} \delta + 1 - \delta \right] \quad \text{for all } t.
\]

In steady state equilibrium \( M_{t+1} = \gamma M_t \) and \( \frac{p_{2,t+1}}{p_{2t}} = \gamma \), and the main equation is

\[
\frac{u_1'(c_1)}{\eta'c_{1}/\tau} = 1 + \frac{r}{\delta}.
\]

The amount \( c_1 \) differs from that which satisfies (23) because of the price distortion \( \tau \). Note that \( \tau \) is a free parameter here much as the buyer’s bargaining weight \( \theta \) in the LW framework; in particular, such parameters can be selected to match a specific price markup in the data. For an appropriate \( \tau \), the main equation (25) matches the corresponding main equation in [6, equation (22)].

Pur differently, in both models a free parameter alters the allocation by altering the degree of DM price distortions; for any given \( \theta \) in the LW framework, there is a corresponding value \( \tau \) that generates the same allocation in the cash in advance model. To show this more clearly, we propose a quantitative exercise.

---

\(^{10}\) In [6] if the bargaining power is \( \theta = 1 \), then \( z' = c' \) (as in our model with \( \tau = 1 \)). Instead, if \( \theta < 1 \) then \( z' > c' \). To see this notice that in [6] \( u' > c' \) when \( \theta < 1 \); hence, \( \theta u' + (1 - \theta)c' < u' \). Now, from [6, equation (11)] we see that \( z' = \frac{u'}{\theta u' + (1 - \theta)c'} c' + A \) where \( A > 0 \).
4.3 Comparing the welfare cost of inflation across models

To evaluate possible quantitative differences between the cash-in-advance model and the LW framework, consider the specification in [6, p.475]. The setup is as follows. Labor disutility is linear, i.e., $\eta' = 1$, and preferences are defined by

$$u_1(c_1) = \frac{(c_1 + b)^{1-a} - b^{1-a}}{1-a} \quad \text{and} \quad u_2(c_2) = B \log c_2$$

for some $a > 0$, $b \in (0,1)$ and $B > 0$. Given a money growth rate $\gamma > \beta$, the stationary allocation in the cash-in-advance model is fully described by (23):

$$\frac{\gamma}{\beta} - 1 = \delta \left[ \tau u_1'(c_1) - 1 \right].$$

(26)

Define ex-ante welfare

$$W_{\gamma} := u_2(c_2) - c_2 + \delta [u_1(c_1(\gamma)) - c_1(\gamma)]$$

Considering the compensating variation $\Delta$, welfare at zero inflation is denoted

$$W_1 := u_2(\Delta c_2) - c_2 + \delta [u_1(\Delta c_1(1)) - c_1]$$

The welfare cost of $\gamma$ inflation is the value $1 - \Delta$ where $\Delta$ satisfies $W_1 - W_{\gamma} = 0$.

As already noted, the key parameter in [6, equation (22)] is the buyer’s bargaining weight $\theta$. To examine quantitative differences between the two models we proceed as follows. For any given value of $\theta$ considered in the calibration in [6, p.475] we find the corresponding value $\tau$ that supports an identical allocation in the cash-in-advance model. As shown earlier, when $\theta = 1$ (=sellers have not bargaining
power), we have $\tau = 1$ and (26) is identical to the main equation in [6, equation (22)]. When $\theta < 1$, there is generally a unique $\tau < 1$ that delivers identical allocations. In [6, p.475], $\theta$ is calibrated to match the average price markup (= price over marginal cost) in the data; we can calibrate $\tau$ to match the average price markup.

As an example, we replicated the calibrated annual model in [6, Table 1, p.475]. The results are shown in Table 1.

Table 1 here

There are five different cases with varied parameters. The first panel shows that the cash-in-advance model can produce identical allocations and welfare cost as in [6, Table 1, p.475]. The second panel reports the calibrated value of the markup parameter $\tau$. If $\theta = 1$, then there is no distortion in price, so $\tau = 1$. As $\theta$ drops, the distortion in price increases, hence $\tau$ drops.\footnote{The notation in [6] is as follows. The DM price is $\frac{d}{q}$ dollars spent per quantity purchased ($= \frac{1}{p_1}$ in our model). The real price is $\frac{\phi d}{q}$, where $\phi$ ($= \frac{1}{p_2}$ in our model). With binding cash constraints in the DM, the real price is thus $\frac{\phi M}{q}$. This also corresponds to the markup since their calibration assumes linear labor disutility so that the marginal cost of production for DM goods is one. With linear labor disutility $\phi M = z(q) = c(q) = q$ hence the markup is zero. The share of DM output is easily constructed. The calibration in [6] assumes $\alpha = 1$ (everyone is bilaterally matched in each period). The output in the DM is $\sigma q$ and output in the CM is $B$. Hence total output is $Y = \sigma q + B$ and the DM share is $\frac{\sigma q}{Y}$. In our case the DM share of output is $\frac{c_1}{c_1+c_2}$.

11} In a nutshell, the cash-in-advance model can replicate the same large welfare cost of inflation, once price distortions are accounted for. The large welfare cost does not specifically depend on the use of a specific pricing mechanism (such as bargaining), or a specific element of the model (such as search or bilateral trade).
5 Final comments

The framework in [6] is offered as a distinct alternative for modeling monetary economies. It is the benchmark model of an aptly named “New Monetarist Economics” literature. This study has presented a methodological exercise devoted to uncover substantive differences with another workhorse model in monetary economics: the cash in advance model. On the surface, these two models appear to be different. Yet, we report that they are mathematically equivalent in the sense that they generate identical stationary equilibrium allocations and have identical quantitative implications for the welfare cost of inflation.

To uncover this equivalence, the basic cash-in-advance model in [9] has been specialized to the shocks and preferences’ structure assumed in [6]. In this case, the two models exhibit main equations that are identical. In particular, in each model, allocations and quantitative implications are pinned down by a free parameter that regulates the degree of price markups in decentralized markets. There are perhaps other dimensions that could be explored where substantive differences may be found.
References


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Panel 1: Allocations and Welfare

- $c_1$ for $\gamma = 10\%$ | 0.243 | 0.206 | 0.143 | 0.094 | 0.523 |
- $c_1$ for $\gamma = 0\%$ | 0.638 | 0.618 | 0.442 | 0.296 | 0.821 |
- $c_1$ for $\gamma = \frac{1}{\beta} - 1$ | 1.000 | 1.000 | 0.779 | 0.568 | 1.000 |
- $WC_0$                       | 0.014 | 0.014 | 0.032 | 0.046 | 0.012 |
- $WC_\beta$                   | 0.016 | 0.016 | 0.042 | 0.068 | 0.013 |

Panel 2: markup parameter $1/\tau$

- $\tau$ for $\gamma = 10\%$ | 1.000 | 1.000 | 0.719 | 0.511 | 1.000 |
- $\tau$ for $\gamma = 0\%$ | 1.000 | 1.000 | 0.846 | 0.672 | 1.000 |
- $\tau$ for $\gamma = \frac{1}{\beta} - 1$ | 1.000 | 1.000 | 0.928 | 0.802 | 1.000 |

Notes: The Parameters column reports the notation from [6] (our corresponding notation is in parentheses). $WF_0$ is the welfare cost of 10% inflation as opposed to no inflation; $WF_\beta$ is the cost of 10% inflation as opposed to $(\frac{1}{\beta} - 1)$% inflation (Friedman rule).
Appendix

The basic model with complete markets

In a representative agent setting, given a history of shocks $S^t$ and $h_t(S^t)$ labor, feasibility implies

$$F(h_t(S^t); S^t) = c_t(S^t) \quad \text{for all } t, S^t,$$

(27)

where $c_t(S^t) \geq 0$ denotes the amount of the good consumed on $t$, given $S^t$.

Suppose a complete market of state- and date-contingent claims opens on date $t = 0$. Let $\pi_t(S^t)$ denote the period-0 price of a claim to one good delivered on $t \geq 1$ contingent on history $S^t$. The spot price of the good on date 0 is $\pi_0(s_0) = 1$ for all $s_0$, i.e., it is the numeraire good. Let $\pi_{ht}(S^t)$ denote state-contingent wages, i.e., the price of a claim to a unit of labor on date $t \geq 1$ conditional on history $S^t$. Let $h_t(S^t) \in [0, L]$ denote the amount of labor supplied on $t$. The date−0 value of the consumption plan $\{c_t(S^t)\}_{t,S^t}$ is thus $\sum_{t=0}^{\infty} \int_{S^t} \pi_t(S^t) c_t(S^t) dS^t$ and the value of the labor plan $\{h_t(S^t)\}_{t,S^t}$ is $\sum_{t=0}^{\infty} \int_{S^t} \pi_{ht}(S^t) h_t(S^t) dS^t$.

Let $c^F_t(S^t)$ and $h^F_t(S^t)$ denote the firm’s choices of output and labor on date $t$, contingent on history $S^t$. On date 0 the firm chooses a plan $\{c^F_t(S^t), h^F_t(S^t)\}_{t,S^t}$ given the state-contingent prices $\{\pi_t(S^t), \pi_{ht}(S^t)\}_{t,S^t}$ to solve

maximize: $\sum_{t=0}^{\infty} \int_{S^t} [\pi_t(S^t)c^F_t(S^t) - \pi_{ht}(S^t)h^F_t(S^t)]dS^t$

subject to: $F(h^F_t(S^t); S^t) = c^F_t(S^t)$ for all $t, S^t$.

(28)
Since there is no capital, this is equivalent to solving a series of static maximization problems. Substituting for the constraint, the optimal choice $h_t^F(S^t)$ must satisfy

$$\pi_t(S^t)F'(h_t^F(S^t); S^t) - \pi_h(S^t) = 0 \quad \text{for all } t, S^t. \tag{29}$$

Let $U(c_t(S^t), h_t(S^t))$ denote the period utility of an agent who, on date $t$, consumes $c_t(S^t)$ goods and supplies $h_t(S^t)$ labor. $U$ satisfies the expected utility property and is monotone in both arguments, strictly increasing in the first and decreasing in the second. On date 0, agents choose consumption and labor plans given state-contingent prices to solve

\[
\text{maximize: } \sum_{t=0}^{\infty} \beta^t \int U(c_t(S^t), h_t(S^t))f_t(S^t)\,dS^t \\
\text{subject to: } \sum_{t=0}^{\infty} \int \{\pi_t(S^t)c_t(S^t) - \pi_h(S^t)h_t(S^t)\} \,dS^t = A,
\]

where $A \geq 0$ denotes the maximized value of the firm’s objective function (the value of the firm on date 0). The constraint implies that the net present value of planned expenditure (consumption minus labor income) must equal the value of the firm on date 0. Let $\lambda$ be the Lagrange multiplier on the constraint and let $U_i = \frac{\partial U(x)}{\partial x_i}$ for $x = (x_i)_{i \in I}$. The first order conditions in an interior optimum are

\[
c_t(S^t) : \quad \beta^t U_c(c_t(S^t), h_t(S^t))f_t(S^t) - \lambda \pi_t(S^t) = 0, \quad \text{for all } t, S^t \tag{30} \\
h_t(S^t) : \quad \beta^t U_h(c_t(S^t), h_t(S^t))f_t(S^t) - \lambda \pi_h(S^t) = 0, \quad \text{for all } t, S^t.
\]

The equations in (29) and (30), and the relevant constraints implicitly define the set of stochastic processes for quantities and prices that support a competitive
equilibrium. Equilibrium relative prices satisfy

\[
\frac{\pi_{ht}(S^t)}{\pi_t(S^t)} = \frac{U_h(c_t(S^t), h_t(S^t))}{U_c(c_t(S^t), h_t(S^t))} \quad \text{for all } t, S^t.
\]

If there is no heterogeneity in labor plans, then labor market clearing requires

\[
h^F_t(S^t) = h_t(S^t).
\]

From (29), it follows that equilibrium quantities must satisfy

\[
\frac{U_h(c_t(S^t), h_t(S^t))}{U_c(c_t(S^t), h_t(S^t))} = F'(h_t(S^t); S^t) \quad \text{for all } t, S^t.
\]

Using (30), the relative price for claims to goods delivered on date \( \tau > t \) is

\[
\frac{\pi_{\tau}(S^\tau)}{\pi_t(S^t)} = \beta^{\tau-t} \frac{U_c(c_\tau(S^\tau), h_\tau(S^\tau)) f_\tau(S^\tau)}{U_c(c_t(S^t), h_t(S^t)) f_t(S^t)} \quad \text{for all } t, \tau, S^t, S^\tau
\]

Given a sequence of shocks \( \{s_t\}_{t=0}^\infty \), equation (31) and feasibility (27) define the equilibrium allocation \( \{c_t(S^t), h_t(S^t)\}_{t=0}^\infty \).

**Derivation of the flow constraint in (8)**

On any date \( \tau > 0 \) there are \( \bar{M}_{t-1} \) outstanding liabilities, hence we have

\[
q_t(S^{t-1}, s_{1t}) \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) = \sum_{\tau=t}^{\infty} \int \{ M_\tau(S^{\tau-1}, s_{1\tau}) [q_\tau(S^{\tau-1}, s_{1\tau})
\]

\[
- \int \int q_{\tau+1}(S^\tau, s_{1,\tau+1}) ds_{1,\tau+1} ds_{2\tau}
\]

\[
- \theta_\tau(S^{\tau-1}, s_{1\tau}) q_\tau(S^{\tau-1}, s_{1\tau}) \} d(S^{\tau-1}, s_{1\tau}),
\]

36
and updating this expression we have

\[
q_{t+1}(S^t, s_{1,t+1}) \bar{M}_t(S^{t-1}, s_{1t}) = \sum_{\tau=t+1}^{\infty} \int \{ \bar{M}_\tau(S^{\tau-1}, s_{1\tau}) q_\tau(S^{\tau-1}, s_{1\tau}) \\
- \int \int q_{\tau+1}(S^\tau, s_{1,\tau+1}) ds_{1,\tau+1} ds_{2\tau} \\
- \theta_\tau(S^{\tau-1}, s_{1\tau}) q_\tau(S^{\tau-1}, s_{1\tau}) \} d(S^{\tau-1}, s_{1\tau}).
\]

(33)

To derive a flow constraint for the central bank we now fix \( \bar{M}_t(S^{t-1}, s_{1t}) \), which means that the shocks \((S^{t-1}, s_{1t})\) are known. Take the difference between (32) and (33). The right hand side of (32) minus the right hand side of (33) is

\[
\bar{M}_t(S^{t-1}, s_{1t}) [q_t(S^{t-1}, s_{1t}) - \int \int q_{t+1}(S^t, s_{1,t+1}) ds_{1,t+1} ds_{2t}] - \theta_t(S^{t-1}, s_{1t}) q_t(S^{t-1}, s_{1t}).
\]

The difference of the left hand sides is

\[
q_t(S^{t-1}, s_{1t}) \bar{M}_{t-1}(S^{t-2}, s_{1,t-1}) - q_{t+1}(S^t, s_{1,t+1}) \bar{M}_t(S^{t-1}, s_{1t}).
\]

Equating the differences in LHS and RHS, and integrating both sides with respect to \( s_{1,t+1} \) and \( s_{2t} \), we obtain the flow constraint of the Central Bank

\[
q_t(S^{t-1}, s_{1t}) [\bar{M}_t(S^{t-1}, s_{1t}) - \bar{M}_{t-1}(S^{t-2}, s_{1,t-1})] = q_t(S^{t-1}, s_{1t}) \theta_t(S^{t-1}, s_{1t}).
\]

That is, each money injection equals the lump-sum-transfer in that period.

**Showing that** \( c_t(S^t) \) **in (15) depends only on** \( s_t \)

The last equation in (15) shows that \( \mu_{t+1}(S^{t+1}) \) depends on \( S^t \) only through the probability \( f^t(S^t) \). Hence, all agents who have the same shock \( s_i^t \) have the same
consumption—see the first equation in (15). To see this, define the multiplier $\mu_t(S^t) = \mu_{1t}(S^t)f^t(S^t)$. Recall that $f^t(S^t) = f(s_t)f^{t-1}(S^{t-1})$. Then, the last equation in (15) shows that $\mu_{1t}(S^t)$ depends on $S^t$ only through $s_t$, not the whole history $S^t$, because

$$\lambda q_t = \lambda q_{t+1} + \int \mu_{1,t+1}(S^{t+1})f(s_{t+1})ds_{t+1}.$$  

Thus, by the first equation consumption of depends only on $\mu_{1t}(S^t)$. The history $S^t$ does not matter for current consumption. The only thing that matters is the current shock.

**Proof of Lemma 2**

To prove the first part let $s^i_t = 1$ and $\mu_t(S^t) = 0$. From the first and third expressions in (15) we have

$$\beta^t u'_1(c_{1t}(S^t)) = \lambda p_t q_t = \lambda w_t q_t = \beta^t, \quad \text{for all } t, S^t,$$

which implies $u'_1(c_{1t}(S^t)) = 1$ for all $t, S^t$. That is $c_{1t}(S^t) = c_1$ for all $t, S^t$.

To prove the second part let $s^i_t = 1$ and $\mu_t(S^t) > 0$. Update by one period the first expression in (15) to get

$$\frac{\beta^{t+1}}{p_{1,t+1}}u'_1(c_{1,t+1}(S^{t+1}))f(s_{t+1})f^t(S^t) = \lambda q_{t+1}f(s_{t+1})f^t(S^t) + \mu_{t+1}(S^{t+1}), \quad \text{if } s^i_{t+1} = 1$$

where we substituted $f^{t+1}(S^{t+1}) = f(s_{t+1})f^t(S^t)$. Now substitute $c_{1,t+1}(S^{t+1}) = \frac{M_{t+1}(S^t)}{p_{1,t+1}}$ since $\mu_t(S^t) > 0$. The expression above has the status of an equality only if
\(s^i_{t+1} = 1\). In that case, we can integrate both sides with respect to \(s_{t+1}\), conditional on \(s^i_{t+1} = 1\). For the left-hand-side we get

\[
\frac{\beta^{t+1}}{p_{1,t+1}} \int 1_{\{s^i_{t+1} = 1\}} u'_1(c_{1,t+1}(S^{t+1})) f(s_{t+1}) f^t(S^t) ds_{t+1}
\]

\[
= \frac{\beta^{t+1}}{p_{1,t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \int 1_{\{s^i_{t+1} = 1\}} f(s_{t+1}) f^t(S^t) ds_{t+1}
\]

\[
= \frac{\beta^{t+1}}{p_{1,t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) f^t(S^t) \delta
\]

(34)

For the right-hand-side we get

\[
\int 1_{\{s^i_{t+1} = 1\}} [\lambda q_{t+1} f(s_{t+1}) f^t(S^t) + \mu_{t+1}(S^{t+1})] ds_{t+1}
\]

\[
= \lambda q_{t+1} f^t(S^t) + \int \mu_{t+1}(S^{t+1}) ds_{t+1} - \Phi
\]

\[
= \lambda q_{t} f^t(S^t) - \Phi
\]

(35)

where the last step follows from the last line in (15) and

\[
\Phi := \int 1_{\{s^i_{t+1} = 0\}} [\lambda q_{t+1} f(s_{t+1}) f^t(S^t) + \mu_{t+1}(S^{t+1})] ds_{t+1}
\]

\[
= \int 1_{\{s^i_{t+1} = 0\}} [\lambda q_{t+1} f(s_{t+1}) f^t(S^t)] ds_{t+1} \quad (\mu_{t+1}(S^{t+1}) = 0 \text{ when } s^i_{t+1} = 0)
\]

\[
= \lambda q_{t+1} f^t(S^t) \int 1_{\{s^i_{t+1} = 0\}} f(s_{t+1}) ds_{t+1}
\]

\[
= \lambda q_{t+1} f^t(S^t)(1 - \delta) \quad (\int 1_{\{s^i_{t+1} = 0\}} f(s_{t+1}) ds_{t+1} = 1 - \delta)
\]

\[
= \beta^{t+1} u'_2(c_{2,t+1}(S^{t+1})) f^t(S^t)(1 - \delta) \quad \text{(from 15)}
\]

Equating the expectations of both sides from (34) and (35) we have

\[
\frac{\beta^{t+1}}{p_{1,t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \delta = \lambda q_{t} - \frac{\Phi}{f^t(S^t)}
\]

Substituting \(\Phi\) in the equation above we get

\[
\frac{\beta^{t+1}}{p_{1,t+1}} u'_1 \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \delta = \lambda q_{t} - \frac{\beta^{t+1} u'_2(c_{2,t+1}(S^{t+1}))}{p_{2,t+1}}(1 - \delta),
\]

39
or equivalently
\[ \beta^{t+1} \left[ u_1' \left( \frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \delta \frac{p_{2,t+1}}{p_{1,t+1}} + \frac{1 - \delta}{p_{2,t+1}} u'_2(c_{2,t+1}(S^{t+1})) \right] = \lambda q_t. \]

This expression implies that if \( s_{t+1}^i = 1 \), then \( c_{1,t+1}(S^{t+1}) = \frac{M_{t+1}(S^t)}{p_{1,t+1}} = \frac{M_{t+1}}{p_{1,t+1}} = c_{1,t+1} \) for all \( t \) and \( S^t \) and for all agents \( i \), because \( q_t \) is independent of \( S^t \) and \( u'_2(c_{2,t+1}(S^{t+1})) = u'_2(c_{2,t+1}) = 1 \) for all \( t+1 \) and \( S^{t+1} \). Now substitute \( \lambda q_t = \frac{\beta' u'_2(c_{2t})}{p_{2t}} \) from (15) and we write the equation above as
\[ \frac{\beta^{t+1}}{p_{2,t+1}} \left[ u_1' \left( c_{1,t+1} \right) \delta \frac{p_{2,t+1}}{p_{1,t+1}} + (1 - \delta) u'_2(c_{2,t+1}) \right] - \frac{\beta^t}{p_{2t}} u'_2(c_{2t}) = 0 \quad (36) \]

Finally, recall that \( p_{1t} = p_{2t} = p_t \) on each \( t \). 

**Proof of Lemma 3.** Let \( s_t^i = 1 \). Two scenarios apply.

If \( \mu_t(S^t) = 0 \) then from the first expression in (15) and
\[ \beta^t \eta'(h_{1t}(S^t)) = \lambda w_{1t} q_t \]
we have
\[ \beta^t u_1'(c_{1t}(S^t)) = \lambda p_{1t} q_t = \lambda w_{1t} q_t = \beta^t \eta'(h_{1t}(S^t)). \]

Since \( h_{1t}(S^t) = \delta c_{1t}(S^t) \) from market clearing we have \( \frac{u_1'(c_{1t})}{\eta'(b_{c1t})} = 1 \) for all \( t, S^t \). That is \( c_{1t}(S^t) = c_1 \) independent of \( S^t \) for all \( t \) (given \( s_t^i = 1 \)).

If \( \mu_t(S^t) > 0 \), then the procedure adopted in the proof of Lemma 2 once again gives us expression (36). This proves the claim. ■