A Tale of Two Correlations:

Evidence and Theory Regarding

the Phase Shift between the Price Level and Output

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Abstract: We examine the relationship between the price level and output at business-cycle frequencies. In the postwar period, there is evidence a phase shift between the price level and output. Such a phase shift is manifested in the price level being countercyclical and the inflation rate being procyclical or acyclical, depending on the detrending method used. Our examination takes three approaches. First, we apply bootstrapping methods to characterize the two correlations, though the methodology could easily be extended to any set of facts. Second, we specify a model economy with a form of rational inattention, showing numerically that this model economy can match the observed pair of correlations. Third, we apply robust control theory, deriving conditions in which the price level is countercyclical and the inflation rate is procyclical.

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1. Introduction

In the post-World War II period, there is evidence of phase shift between the price level and output at business cycle frequencies. We see evidence of the phase shift in the unconditional contemporaneous correlation between the price level and output and the inflation rate and output: the price level is procyclical and the inflation rate is either procyclical or acyclical, depending on the detrending method used.

In this paper, we offer three sets of findings:

1. We develop a data-disciplined methodology that answers questions aimed at characterizing the likelihood of joint facts;
2. With a form of rational inattention, we show numerically that we can account for the countercyclical prices and acyclical inflation;
3. We apply robust control theory, deriving conditions in which the price level will be countercyclical and the inflation rate procyclical.

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This is a tale of two correlations and the phase shift that unifies them. This paper offers a first step toward treating the two correlations jointly.

1.1 Some background

Kydland and Prescott (1990) initiated this research line when they reported that after filtering for the business-cycle component, the price level is negatively correlated with output.[2] Other researchers began testing the robustness of this finding. Cooley and Ohanian (1991) and Smith (1992) extend the sample. The two papers broadly agree that before World War II, the evidence suggests the price level was procyclical.[3] After World War II, however, the evidence is consistent with Kydland and Prescott's finding that the price level has been countercyclical.

Wolf (1991) asks whether the price level is countercyclical over the post-war sample period. He distinguishes between business cycles before and after the 1973 recession. Wolf's view builds on the question, Are all business cycles alike?[4] In particular, Wolf presents evidence from price indexes constructed from consumer expenditure categories. Based on the temporal break and from the correlation between movements in output and the price indexes, he concludes that the price level has been countercyclical since the 1973 recession, but not before.

Cooley and Ohanian also consider the set of business cycle facts to examine the relationship between the cyclical component of the inflation rate and output. They report that inflation rate is procyclical during the post-war sample. Webb (2003) and Kanstantakapoulou , Efthymois and Kollintzas (2009) are more recent contributions. Webb is especially clear on the stakes involved in the issue of procyclicality of the price level. He states, "The issue is of particular importance to macroeconomists who must choose which model to work with."[5] Kanstantakapoulou et al. investigate the robustness of countercyclicality of the price level and procyclicality of the inflation rate for 9 OECD countries using quarterly data, 1960-2004. For example, they state, "We examine the stylized facts …prices are countercyclical; inflation is procylical."[6] Given the deterministic relationship between the price level and the inflation rate, the qualitative difference deserves attention.

Researchers have debated the implications associated with a countercyclical price level. One debate has centered on the competing role of demand shocks and supply shocks as a source or business cycle

[2] Kydland and Prescott reported this result because it contradicted the maintained hypothesis that the price level was procyclical. In their view, theory was needed to account for the negative relationship.
[3] In Cooley and Ohanian, the evidence is ambiguous before World War I. During the interwar period, they report that the price level is procyclical. In Smith, the evidence is that the price level is procyclical before the Great Depression.
[4] See Blanchard and Watson (1986) for a detailed discussion on this question.
[5] See Webb (2003, page 69).
[6] See Kanstantakapoulou et al. (2009, page 1).

fluctuations. Kydland and Prescott asserted the following: "*We caution that any theory in which procyclical prices figure crucially in accounting for postwar business cycle fluctuations is doomed to failure.*" (p.17, 1990).

Researchers responded along two fronts. One approach focused on the relationship between prices and output at business cycle frequencies. In particular, den Haan (2000) asked why the unconditional correlation coefficient the appropriate measure of comovement between the price level and output? In doing so, den Haan proposes using correlations of k-step-ahead forecast errors from VARs which allows the researcher a richer dynamic set of correlations. Based on this evidence, den Haan counters with the argument that "*a theory in which prices do not have some procyclical feature is, at best, missing a part of the explanation of U.S. business cycle fluctuations.*" (p. 5, 2000).

The other approach focused on showing that sticky-price models can account for relationship between the price level and output. Chadha and Prasad (1994), Ball and Mankiw (1995) and Judd and Trehan (1995) specify versions of sticky price models in which only demand shocks are considered. Each shows that model economies are capable of accounting for the negative unconditional correlation coefficient observed between some detrended versions of output and the price level.[7] Rotemberg (1996) examines forecastable movements in output and the price level, showing that they are negatively correlated. In a sticky-price model with only demand shocks, Rotemberg can account for the relationship between the forecastable parts of output and the price level.

Webb (2003) stresses the difference in behavior between commodity prices, final goods prices, and wages over the business cycle. Webb argues that there are key differences in the price-setting institutions: commodity prices are set in spot markets, final goods are set by slower-moving methods, and wages in which the prices of different types of labor tend to move even slower than commodity prices over the business cycle. Webb points out that the monetary regime plays a significant role; during the gold standard, a procyclical price level is likely to become countercyclical in a fiat money regime. Webb cites Wesley Clair Mitchell as providing the original logic for procyclical price level during the gold standard period. Webb further argues that the procyclical property holds when "price level" is replaced by "rate of change of the price level".

While it is true that the relationship between the price level and output has been extensively studied, but there is a hole in the literature regarding the two unconditional correlations. Namely, to our knowledge no one has tried to develop a model economy that can account for both the countercyclical price level and the procyclical inflation rate. Haslag and Hsu (2012) examine the degree to which a phase

[7] The intuition rests on output being mean reverting. Given a positive aggregate demand shock, output increases. Because prices were sticky, output would begin falling (mean reverting) and prices would increase eventually, causing real balances to decline and output to decline further. Of course, introduction of sticky prices into the model economy raises another question; specifically, why are prices sticky?

shift can account two facts. Specifically, Haslag and Hsu document the size of the phase shift that could account for the pair of reported correlation coefficients. The approach does not explain why there is a phase shift.

Thus, there are two broad questions that come to mind in light of the literature. First, we focus on two business cycle facts. One goal, therefore, is to develop a methodology that characterizes the joint distribution function; our aim is to present a methodology that allows one to accurately characterize and measure the level of uncertainty when there are joint patterns. Such a methodology permits one to represent the probability of any particular data pattern in an empirically disciplined way that also respects "model uncertainty." We apply this methodology to the particular question of the correlation between the price level and output and the correlation between the inflation rate and output. In our view, the methodology could easily be extended to characterizing the likelihood of multiple business-cycle correlations. Second, we are interested in providing some theoretical model to account for these two facts. To our knowledge, there is not a paper that has been able to account for the phase shift that is embodied in the two unconditional correlations. There are various "sticky price" models, e.g. Rotemberg , in which the model economy can account for a positive correlation between output *growth* and the inflation rate, but ignore the relationship between output growth and the *change* in the inflation rate. We investigate a form of "rational inattention" as a type of friction that could account for this pair of facts. We then provide quantitative results from a model economy to support this view. We view this exercise as an attempt to "measure" the strength of potential heterogeneity in expectations where some of the expectational types are "backwards-looking" as a potential force in accounting for these two facts. More generally, we investigate how far plausible dynamics of expectations themselves can go in explaining the two facts. On this front, we show that one can account for the phase shift by matching the two correlations. As such, our results constitute a contribution to the business cycle literature in the sense that the introduction of rational inattention and robust control methods can account for the pair of correlations.

In addition, we examine the pair of correlation coefficients from two different filtering approaches. Researchers have applied both trend-stationary and difference-stationary methods to identify the cyclical component of economic time series.[8] We consider both approaches. In both methods, the cyclical component of the price level is negatively correlated with the cyclical component of output. In the trend-stationary approach, which used the H-P filter, the cyclical component of the inflation rate is positively correlated with the cyclical component of output. However, in the difference-stationary case, the cyclical component of the inflation rate is not systematically related to the cyclical component of output. This is

[8] Nelson and Plosser (1982) identified this source of model uncertainty in their study of macroeconomic time series.

consistent with Webb (2003, page 75) where he points out that the absolute value of the correlation is much smaller using log-differenced data than linear detrended data.

1.2 Model Uncertainty

The methodological contribution is related to the stance that was presented in Brock, Durlauf and West (hereafter (BDW)) (2003) and (2007). In BDW, the stance was built on the notion of Bayesian Model Uncertainty where prior probabilities were assigned to each model specification. After estimating each model, the posterior probabilities were assigned based on relative likelihoods. Here, we simply filter out the low-frequency components by applying two different methods of making the time series (covariance) stationary; that is, trend stationary and difference stationary in the sense of Nelson and Plosser (1982). We then estimate models with the resulting filtered data and report how the probability of the pattern of interest depends upon the method of detrending. We can compute the probability that the pattern of interest in a data disciplined way by using the estimated standard errors (under appropriate distributional assumptions, e.g. Gaussian) of the parameters of each model fitted to the filtered data. Of course, one could do this same exercise taking into account the uncertainties in the estimated parameters of the trend model also. We ignore this extra source of uncertainty in this paper in the interest of simplicity. We report histograms that show the part of the space where the pattern of interest holds with a set of "skyscrapers" whose height gives the probability of that grid of the space. For example, we are basically just using bootstrapping (Efron (1982), Efron and Tibshirani (1986)) to estimate the probability of the pattern of interest, i.e. the probability that the "stylized fact" under scrutiny in this paper holds under each method of detrending. All this will be explained in greater detail below.

One of the main points we want to get across in this paper is that we think this methodology is a useful way to present data patterns of interest in macroeconomics together with a measure of the uncertainties surrounding each such pattern. To put it another way, we are arguing that such a methodology can be helpful when presenting joint "stylized facts" in macroeconomics.

In this particular application, we apply a single-equation, univariate autoregressive filter to create time series of the price level and output. We then compute the contemporaneous cross-correlations for the price level and output and the inflation rate and output. With the correlation coefficients, we can compute the likelihood that the filter yields countercyclical price level and procyclical inflation. By adopting the autoregressive filter, the approach stresses the role of persistence and goodness-of-fit to infer the likelihood. In other words, there are three key factors that play roles in determining the likelihood of the observed cross-correlations; namely, (i) the estimated values of the relative persistence in the cyclical components, the standard errors of the parameter estimates and (ii) the unexplained variation in the cyclical components. The single-equation methodology allows us to assess the importance of each factor.

In addition, we apply a VAR approach to simulate the time series for each variable. In this setting, we observe a better fit in the sense that the distribution is massed over the joint event that the price level is countercyclical and the inflation is procyclical. In contrast, the maximum probability of this joint event range is 62 percent in the single-equation method. When we use the first-difference approach to filtering out the cyclical component, the inflation rate is acyclical. We take the opportunity to demonstrate that the methodology is flexible; indeed, we find that the joint likelihood is 82 percent that the price level correlation lies within the range of 0.0 to -0.2 together the inflation rate correlation lies in the range of -0.1 to 0.1. In general, one can compute the frequency for any joint values of the two correlations. In our view, it can be very useful for researchers to compute the likelihood of joint empirical regularities as a way to gauge the joint strength of a wide range of empirical regularities.

1.3 On rational expectations, rational inattention and robust control

"Rational inattention" is one of those labels that has been used to describe more than one approach used in the literature, e.g. Sims (2003) and Reis (2006) are two prominent examples. Hence it is important for us to be precise and describe exactly how we are using the label, "rational inattention." We are using "rational inattention" in the sense that there are "backwards-looking" expectations as in Branch and Evans (2011), De Grauwe (2011), and Massaro (2013), as well as "forward-looking", i.e. rational expectations, also see Brock and Hommes (1997). Brock and Hommes present a framework which is much simpler than the more realistic frameworks of the above authors, where it is costly to acquire the information necessary to form rational expectations, whereas backwards-looking expectations are free. Here, we use an off-the-shelf money-in-the-utility function model, adding a form of rational inattention to determine whether this type of information friction can account for the phase shift in filtered price level. We study a special case of these costs in which agents purchase inexpensive "backward-looking" price expectations. Our quantitative analysis is encouraging in the sense that there is a reasonable set of parameters that result in countercyclical prices and acyclical inflation. The numerical results are consistent with the presence of a phase shift and qualitatively match the results obtained when we assume the cyclical components are derived from a difference-stationary process.

To further illustrate we present the solution to the forward-looking rational expectations equilibrium. In this way, we can contrast the notion of price stickiness that is imparted by our particular form of rational inattention. In other words, the comparison helps one see why rational inattention works the way it does. Our results tend to confirm the role that forward-looking rational expectations plays; namely, that prices adjust so quickly to new shocks that the direction of the change in the price level dominates the direction of change in the inflation rate. We explore the possibility that interaction with persistence of other forces, e.g. taste shocks and the desire of our representative agent for robustness against possible mis-specification of its economic environment may blunt the usual effect of rapid adjustment of the price level under rational expectations. We believe the modest extension of the framework we use—i.e., that of Woodford (2003)—may be of independent interest. Returning to the standard effect of rational expectations, to put it in other words, we wish to stress that, forward-looking rational expectations imparts phase-synchronicity between movements in output and movements in the price level. Such synchronous movements explain why models economies incorporating rational expectations cannot account for the price level being countercyclical and the inflation rate being either procyclical or acyclical. After having made this well-known point to set the stage, we show how minimalist backing off from rational expectations in the sense of maintaining rational expectations on long term trends but allowing departures from rational expectations on shorter term fluctuations about trend can explain the two facts as well as generate plausible dynamics of shorter term fluctuations about trend. Furthermore our treatment of shorter term dynamics of expectations via perturbation techniques around dynamic intertemporal equilibria of the model and our introduction of robustness concerns on the part of the representative agent in the model may be of independent interest.

In addition, we apply robust control theory as a form of model uncertainty. The basic idea is that agents are uncertain with respect to the dynamics of the inflation expectations. Robust control provides a means of dealing with this type of model uncertainty. We derive a closed-form solution of the equilibrium price level, using it to obtain a set of sufficient conditions that yield countercyclical price level and procyclical inflation.

The paper outline is as follows. In Section 2, we consider an example that analytically illustrates the methodological approach. We construct the numerical analyses in Section 3 for both the H-P filter and first-difference measures of the cyclical components. Section 4 develops the model economy and reports the numerical results. In particular, we use the model economy to provide some analytical support and understanding of forces that produce negative correlation of the cyclic component of the price level (positive correlation of inflation) with the cyclic component of output by using the model of Woodford (2003, Chapter 2). Here is where we show that minimalist departures from rational expectations where rational expectations are maintained on trends but relaxed for shorter term fluctuations in expectations (which we justify by reference to recent work on the dynamics of inflation expectations at the Cleveland Fed and elsewhere) easily generates plausible shorter term dynamics that are consistent with the two correlations. We also show how to extend this standard model to robustness in the spirit of Hansen and Sargent (2008) and the recent work of Anderson, Brock, Hansen and Sanstad (2014) in Section 5. Finally, Section 6 is a brief summary, conclusions, and suggestions for future research.

2. Model Uncertainty by Linear Approximations: An Analytic Example

In this section, we consider a specific autoregressive process to illustrate how the various components affect the likelihood that the correlations will exhibit the pattern in the data.

Suppose both the price level *(p)* and output *(y)* (in log levels) are capable of being decomposed into trend components and cyclical components. Formally, $x_t = x_t^T + x_t^C$, where $x = p, y$. The superscript T denotes the trend component while the superscript *C* stands for the cyclical component. We start by estimating low order AR (q) models to the cyclical components. We find that an AR(1) and an AR(2) fit well as specified in (1) and (2) below. The low orders of these processes allow us to prove the two Lemmas and Proposition 1 below. These simple results are useful to uncover sufficient conditions on the Data Generating Processes (DGP) for the cyclical components for the cyclical component of the price level to be countercyclical and the cyclical component of inflation to be procyclical.

Henceforth, for this initial part of the paper, we assume that the cyclical component of the price level follows an AR(1) process while the cyclical component of output follows an AR(2) process. Thus,

$$p_t^C = b_1 p_{t-1}^C + u_t \tag{1}$$

$$y_t^C = a_1 y_{t-1}^C + a_2 y_{t-2}^C + e_t \tag{2}$$

The cyclical components are computed as deviations from trend. Assume that the cyclical components for both the price level and output are mean zero, stationary processes. We drop the superscripts to write the implication of the stationary process as$E[p_t y_t] = E[p_{t-1} y_{t-1}] = E[py]$, where $E[py]$ denotes the covariance of the cyclical components of the price level and output. Under these assumptions, the sign of the covariance determines the sign of the contemporaneous cross-correlation.

We derive the expected value of the product of the cyclical component of the price level and output by substituting equations (1) and (2) into the covariance expression, yielding

$$a_1 b_1 E[py] + +a_2 (b_1)^2 E[py] + E[ue] = E[py]$$

After rearranging and simplifying, we obtain

$$E[py] = \frac{E[ue]}{1 - b_1[a_1 + a_2 b_1]} \tag{3}$$

Based on equation (3), we derive the following lemma.

Lemma 1: With $E[ue] < (>)$ 0 and with $1 - b_1[a_1 + a_2 b_1] > (<) 0$, then the sign of $E[py]$ is negative.

Equation (3) tells us that the sign of the cross-correlation coefficient depends on the sign of the covariance of the unexplained errors from the two AR processes that characterize the cyclical components of the price level and output, respectively. If the residuals are independent, then the correlation coefficient is zero. However, if the unexplained errors are negatively correlated and the denominator is positive, for example, then the correlation coefficient is negative. Note further that the denominator depends on the persistence of the two cyclical components. With $E[ue] < 0$, for example, greater persistence in price level or in output makes it less likely that the correlation between the price level and output will be negative.

Next, we turn to the covariance between inflation and output. Let $E[\pi y] = E[(p_t - p_{t-1})y_t]$ denote the covariance between the inflation rate and output. Substitute for the date-t price level from equation (1) and for output from equation (2), yielding

$$E[\pi y] = E[ue]\left\{\frac{1-\theta}{1-b_1\theta}\right\} \tag{4}$$

where $\theta \equiv a_1 + a_2 b_1$. From which, we derive the following Lemma.

Lemma 2: With $E[ue] < (>) 0$, the sign of $E[\pi y] > 0$ if and only if $\frac{1-a_1-a_2b_1}{1-b_1[a_1+a_2b_1]} < (>) 0$.

Equation (4) indicates that the sign of the correlation between inflation and output again depends on the correlation between residuals. If the residuals are negatively correlated, for example, the bracketed term must be negative for the cyclical components of the inflation rate and output to be positive.

Note that the condition in Lemma 2 involves the denominator in Lemma 1. Therefore, the combination of conditions in Lemmas 1 and 2 create a partition for the space of estimated coefficients that must be satisfied for the price level to be countercyclical and the inflation rate to be procyclical. We derive those conditions in the following proposition.

Proposition 1: Based on the conditions derived in Lemmas 1 and 2, the price level is countercyclical and the inflation rate is procyclical for $\frac{1}{b_1} > a_1 + a_2 b_1 > 1$ if $E[ue] < 0$ and $a_1 + a_2 b_1 > max\left[1, \frac{1}{b_1}\right]$ if $E[ue] > 0$.

By writing down the conditions in Lemma 1 and Lemma 2 and solving for the inverse of the persistence coefficient in price level equation, we obtain the conditions in Proposition 1. Note that the denominator that yields $E[py] < 0$ depends on the sign of the covariance between the unexplained terms. For example, if $E[ue] < 0$, then $E[py] < 0$ if the denominator is positive. It follows from Lemma 2 that $E[\pi y] > 0$ if the numerator is negative.

Combining the conditions in Lemmas 1 and 2, the two correlation coefficients are opposite signs if $\frac{1}{b_1} > a_1 + a_2 b_1 > 1$ if $E[ue] < 0$. In contrast, if $E[ue] > 0$, the condition is $a_1 + a_2 b_1 > max\left[1, \frac{1}{b_1}\right]$.

The upshot of Proposition 1 is that there exists a range of parameter values that are consistent with the joint observation that the price level is countercyclical and the inflation rate is procyclical. In other words, Proposition 1 derives the values for one specific illustration of a linear autoregressive process that will yield the joint business cycle fact. If the residuals from the two autoregressive processes are negatively correlated, for example, Proposition 1 indicates that the price level cannot be "too persistent" relative to the persistence in the output equation for the joint business-cycle observation to hold. Conversely, if the residuals are positively correlated, the price level must be persistent enough relative to the persistence observed in the output equation for the price level to be procyclical and the inflation rate to be countercyclical.

To be more concrete, consider a case in which the cyclical component of the price level is a random walk. If the residuals are negatively correlated, for example, Proposition 1 tells us that the condition which jointly satisfies countercyclical price level and procyclical inflation rate cannot be satisfied. The first inequality in the sequence fails to be greater than one. For the case in which the residuals are positively correlated, then with positive coefficients in the AR process for output, the condition can be satisfied.

Consider a case in which both the output and the price level follow AR(1) processes. With $y_t^C = a_1 y_{t-1}^C + e_t$, we present the conditions for the pair of correlations in the following proposition:

Proposition 2: If $E[ue] < 0$, then the price level is countercyclical and the inflation rate is procyclical, in expected value, if $a_1 b_1 < 1 < a_1$. (See Appendix)

Proposition 2 reduces the expected sign of the correlation coefficients to persistence in the cyclical components. The key is that the cyclical component of output lies outside the unit root and the product of the two persistence coefficients is less than one. It follows that the persistence in the cyclical component of the price level is "low enough" to satisfy the condition.

While we will see below that the AR(q) processes for detrended price level and detrended real output that we identified by standard time series methods are higher order than 1 and 2, we believe our Lemmas and Propositions help understand what kinds of persistence properties are needed into order to be consistent with procyclical inflation rate and countercyclical price level.

3. Model Uncertainty and Histograms

The data are quarterly observations from the United States for the period 1947:1 through 2007:4. Output is measured by real GDP and the chain-weight index for personal consumption expenditures. We use the H-P filter with $\lambda = 1600$ to obtain the filtered, or cyclical, values of output and the price level. In addition, we consider cases in which the cyclical component is constructed by first differencing the data. Formally, $p_t^C = p_t - p_{t-1}$ and $y_t^C = y_t - y_{t-1}$.[9] For completeness, note that the DS-cyclic component of the inflation rate is defined as follows: $\pi_t^C = p_t^C - p_{t-1}^C$.

[9] In the first-differencing case, the cyclical components have other natural interpretations. By first-differencing the log of output and the log of the price level, the correlation, $\hat{\rho}(py)$, is between output growth and inflation. Further, $\hat{\rho}(\pi y)$ is between output growth and the change in the inflation rate.

We begin by estimating the correlation coefficients for the different measures of the cyclical components. Table 1 reports the summary statistics for the HP filtered and the DS-cyclic measures. We are particularly interested in the contemporaneous correlations. In the HP filtered series, the cross correlations are different. The Bartlett standard error of this estimate is $\frac{1}{\sqrt{T-1}} = 0.062$ for our sample. Consequently, the evidence indicates that the price level is countercyclical and the inflation rate is procyclical. With the DS components, however, the contemporaneous correlation coefficient for the price level and output is -0.145 and the contemporaneous correlation coefficient for the inflation rate and output is -0.013. Thus, an important difference emerges based on the approach used to construct the cyclical components; we can reject the null hypothesis that the price level is zero and hence it is countercyclical, but with the DS measures, the inflation rate is acylical.

Overall, the summary statistics paint a different picture based on how the cyclical components are measured. Even with the quantitative difference present in the two detrending methods, there is a common thread that emerges regardless of how the cyclical components of the price level and output are measured. Both the HP filter and the first-difference approach are consistent with the notion that there is evidence of phase shift between the cyclical component of the price level and the cyclical component of output. The phase is not as pronounced when we use the DS approach, but the phase shift can account for the different signs in the unconditional correlation coefficients.

3.1 Single Equation Approach

We begin by applying single, linear regressions to fit the time series. We use AIC to select the lag length in the estimation part. For the price level, the AIC selects four lagged values of the price level for both the HP and the first-difference measurements of the cyclical components. For output, AIC chooses three lagged values when x_t^C is measured by the HP filter and five lagged values when measured by first differences. The coefficients and standard errors are reported in Table 2.

Armed with these equations and the residuals, our aim is to employ standard bootstrapping methods to generate simulated time series. The simulated series can then be used to estimate correlation coefficients and we can assess the likelihood that a particular observed pattern of the correlation coefficients is present. In more detail we wish to estimate the probability that $\mathrm{cov}(p^C, y^C) < 0, \mathrm{cov}(\Delta p^C, y^C) > 0$ where the two covariances are estimators under the DGP's (1) and (2). Since the two covariances must be estimated on our sample of length T, therefore we use the time series bootstrap, e.g., as described in Berkowitz and Kilian's survey (2000). Specifically, we use a time series bootstrapping procedure to compute the likelihood that the joint correlation coefficient is represented by

$$\hat{\mathrm{P}}\{\hat{\rho}\left(p^C y^C\right) < 0, \hat{\rho}\left[\left(\Delta p^C\right) y^C\right] > 0\} \qquad (5)$$

where

$$\hat{\rho}\left(p^C y^C\right) \equiv (1/T)\sum_{t=1}^{T} p_t^C y_t^C, \hat{\rho}\left[(\Delta p^C) y^C\right] \equiv (1/T)\sum_{t=2}^{T}(p_t^C - p_{t-1}^C) y_t^C \qquad (6)$$

$$\begin{aligned}&\hat{\mathrm{P}}\{\hat{\rho}\left(p^C y^C\right) < 0, \hat{\rho}\left[(\Delta p^C) y^C\right] > 0\} \\ &\equiv (1/B)\sum_{b=1}^{B}\{1[(1/T)\sum_{t=1}^{T} p_t^b y_t^b < 0, (1/T)\sum_{t=2}^{T}(p_t^b - p_{t-1}^b) y_t^b > 0]\}\end{aligned} \qquad (7)$$

$$\hat{\mathrm{P}}\{(\hat{\rho}\left(p^C y^C\right), \hat{\rho}\left[(\Delta p^C) y^C\right])\varepsilon A\} \equiv (1/B)\sum_{b=1}^{B}\{1[((1/T)\sum_{t=1}^{T} p_t^b y_t^b, (1/T)\sum_{t=2}^{T}(p_t^b - p_{t-1}^b) y_t^b)\varepsilon A]\} \quad (8)$$

Where the superscript "b" indicates the bootstrapped sample generated by the estimated equations. The implementation of Equations (7) and (8) is described in detail below. We estimate the single equation using our sample of length T for a given method of detrending. We compute the sample variances and covariances of the estimated residuals to get an estimate of the variance matrix. We then used our estimate of the variance matrix of the residuals to compute standardized residuals. Call these standardized residuals, $\{\hat{\varepsilon}_{yt}, \hat{\varepsilon}_{pt}, t = 1, 2, ..., T\}$ which have variance one and covariances zero by construction.

Overall, we followed the procedure outlined in Berkowitz and Kilian (2000) for each of the pair of AR(q)'s estimated for the detrended logs of the price level and real GDP per capita with a small modification to estimate the covariance between the estimated residuals between the price level and real GDP per capita for our standardization procedure. We start with the HP filtered measure of the detrended log price level and log output. We continue to report results for both the measurements of p_t^C and y_t^C developed from the HP filter and first-difference methods.[10] In this way we did B = 10,000 replications of this bootstrap procedure to compute (7) and (8). Doing this procedure we obtained

[10] The fitted regressions are:

$$p_t = 2.43 + 0.011 * t - 0.000005 * t^2 + p_t^C$$
$$y_t = 7.45 + 0.01 * t - 0.000008 * t^2 + y_t^C$$

The estimated coefficients are significant at 5 percent levels in every case. The results reported in this paper are qualitatively the same when we use the time-trend approach to measure the cyclical components of the price level and output.

$\hat{\mathrm{P}}\{\hat{\rho}\left(p^C y^C\right) < 0, \hat{\rho}\left[(\Delta p^C) y^C\right] > 0 \mid HP\} = 0.227$. Note that we added the "|HP" to indicate that we held the detrending fixed throughout the B replications.

Figures 1 and 2 present the histograms for the bootstrap based on detrending using the HP filter and the first-difference. In addition, we select the lag length for both cases using AIC. In the first-difference case, $\hat{\mathrm{P}}\{\hat{\rho}\left(p^C y^C\right) < 0, \hat{\rho}\left[(\Delta p^C) y^C\right] > 0 \mid \mathrm{DS}\} = 0.331$, where "|DS" designates the first-difference method of detrending.[11] Because the first-difference measure yields different patterns for the cross-correlation coefficients and because the methodology is flexible with respect to the questions we can ask, we compute $\hat{\mathrm{P}}\{-0.25 < \hat{\rho}\left(p^C y^C\right) < 0, -0.1 < \hat{\rho}\left[(\Delta p^C) y^C\right] < 0.1 \mid \mathrm{DS}\} = 0.462$. In other words, the single-equation approach indicates that the likelihood is 46.2 percent that the price-output correlation is between 0 and -0.25 and the inflation-output correlation coefficient is between -0.1 and 0.1.

Note that the histograms indicate that the events are massed around the combination that the two simulated correlation coefficients are equal to zero. The reason is straightforward. In both single-equation approaches, the residuals are uncorrelated. We observed in the simple AR processes used to derive the analytical results in Proposition 1, the two correlation coefficients depend on the residuals. Indeed, Equations (3) and (4) express the two correlation coefficients as linear functions of the correlation between two residuals, e_t and u_t. Based on Equations (3) and (4), if the two residuals are independent, both the price level and the inflation rate will be acyclical. Our numerical results are consistent with the Proposition 1; by adding a random term to the parameter estimates, our bootstrapping results indicate that the central tendency is consistent with the analytical results derived in Equations (3) and (4). In other words, the two correlation coefficients tend to be massed around zero.

3.2 VAR approach

[11] Brock, Durlauf, and West (2003), (2007) recommended a form of model averaging in order to reflect the "additional" uncertainty due to "model uncertainty". E.g. in view of arguments in the literature for both TS and DS detrending methods and in view of the difficulty of short time series data sets typical in macroeconomics in distinguishing between TS and DS methods, we might attach equal credibility of ½ to each of these two methods. In that case we might wish to bootstrap compute $\hat{\mathrm{P}}\{\hat{E}p^C y^C < 0, \hat{E}(\Delta p^C) y^C > 0 \mid DS\}$ and, perhaps, not only compare to evaluate the "sturdiness" of the joint fact to the method of detrending, but also report the average $(1/2)\hat{\mathrm{P}}\{\hat{E}p^C y^C < 0, \hat{E}(\Delta p^C) y^C > 0 \mid TS\} + (1/2)\hat{\mathrm{P}}\{\hat{E}p^C y^C < 0, \hat{E}(\Delta p^C) y^C > 0 \mid DS\}$ as a more appropriate statement of the joint fact. Or, better yet, the weights of ½ could be replaced with relative likelihoods as in Brock, Durlauf, and West (2003,2007).

In addition to the single-equation approach, we estimate a bivariate VAR. The VAR is expressed in terms of the cyclical components of the price level and output and the lag length is set at two. The coefficient estimates and the standards errors are reported in Table 3 for the HP Filter approach and Table 4 for the first-difference approach.

We then simulate the time series using the coefficient estimates, the standard errors of the coefficients and the standard errors of the each VAR equation. Based on the simulated time series, repeated 10,000 times, we compute the correlation coefficients for the price level and output and for the inflation rate and output. Since we simply estimated each row of the VAR as we did the AR(q)'s above, and since we estimated the covariances of the residuals the same way we did above, we repeated the bootstrapping procedure we did for the two AR(q)'s above, for our VAR. We present the histogram of the correlation coefficients using the HP filtered measured of the cyclical components in Figure 3. As Figure 3 shows, there is a dramatic change in the distribution of the correlation coefficients when compared with the histograms generated by the single equation approach. Specifically, the distribution presents events in the correlation coefficient between the price level and output is negative and the correlation coefficient between the inflation rate and output is positive. Thus, there are 10,000 cases that satisfy the joint condition in the VAR compared with cases reported in the single equation approach ranging from 2300 to 3400 cases.

For the DS-cyclic components, we use the VAR to create 10,000 simulated time series. We then compute the price level-output correlation and the inflation-output correlation for each of the 10,000 series and plot the resulting histogram in Figure 4. Based on this histogram, we find that for 52.6 percent of the time, the price level-output correlation is negative and the inflation-output correlation is positive. We also consider the case in which the price level-output correlation lies between 0 and -0.2 and the case in which the inflation-output correlation lies between -0.1 and 0.1. In the first-difference VAR approach, this occur 0.42 percent of the time. This is a sharp decline in the frequency of observations that fit into the "bin" corresponding to the observed correlations in the actual data. In Figure 4, we see that the frequency of correlations are massed at values of the price-output-correlation being greater than 0.3. It appears that the chief reason behind the low-frequency performance lies with the fitness of the output equation. The adjusted R-squared is 0.13 for the output equation. The R-squared values in the HP-Filter VARs is much higher with values of 0.77 for the output equation and 0.89 for the price equation. Simply put, the characterization of the frequency is a reflection of the goodness of fit.

Overall, we develop a methodology that allows us to construct simulated time series for the price level and output. Once, we have the price level series, it is straightforward to construct the inflation rate series. The key to the simulation process is to estimate linear models and then include uncertainty with respect to the fit of the equation and to the coefficient estimates, as well as uncertainty about which model to be fitted and which detrending method to be used. Once such model uncertainty is appropriately taken into account, we can construct histograms for any joint business cycle facts. This numerical approach then permits frequency questions regarding the joint statistics. In our view, the methodological development is a useful tool for studying multiple moments in a variety of economic applications, especially in business cycle contexts where people are interested in numerically assessing the likelihood of multiple stylized facts.

4. Theory to account for the pair of cross-correlations

We consider a dynamic, stochastic general equilibrium model in which money enters directly into the utility function. The MIUF approach does not take a stand on the friction that accounts for why fiat money is valued in the economy. Our aim here is to take an off-the-shelf model economy, modify it in a minimalist direction by inserting heterogeneous expectations, and derive the conditions in which we could account for the pair of observations presented in this paper.

Here is our main motive for using such a well-known off the shelf model. We wish to show why a purely rational expectations version cannot account for this particular joint fact. By using such a well-known strawman, it is straightforward to modify it one particular dimension to first see how far that modification can go towards "explaining" the joint fact. By introducing backwards looking beliefs into this standard model, we numerically show that we can generate a phase shift. Then, ultimately, we would like to move in the direction of Branch and Evans, De Grauwe, and Massaro by introducing an "ecology" of heterogeneous beliefs into this standard model where fully structural rational expectations are part of the ecology but are only available at a cost and where the fractions of each type of belief change over time depending upon relative performance histories. By "fully structural" here we mean the rational expectations believers take into account the effect of the other believers upon equilibrium. Simpler versions of this kind of model are treated in the references above and other papers they refer to. However, the infinite horizon together with the requirement that the rational expectations beliefs are "fully structural" in the sense we defined above make the model intractable. Hence we chose the simpler route of comparing two polar cases: (i) all believers are the same type of backwards believers, (ii) all believers are fully structural rational expectations believers. We will see that the model with beliefs of type (i) can account for the joint fact.

We think of this exercise is a useful complement to the received approaches, e.g. Rotemberg , Woodford, and numerous others who have introduced "sticky prices" (see Calvo (2003)) or by introducing "sticky prices" via adjustment costs as in Rotembeg . However, some of these methods of introducing "sticky prices" have been criticized by Robert Lucas in Aghion et al. (2003, page 138). More precisely he said firms (or consumers) should not be "…*locked into a pricing policy that is completely unsuited to the new policy regime. They understand this new regime fully, but they just cannot act on this knowledge*…" We are thinking along similar lines as Lucas where "pricing policy" is replaced by "belief dynamics" where the belief dynamics changes in response to relative past performance as in Branch and Evans, De Grauwe, and Massaro .

There are infinite number of discrete time periods with *t = 0,1,2,…* There is a single, perishable consumption good. The economy is populated by a measure-one continuum of representative agents. At each $t \geq 0$, each agent is endowed with income represented by $P_t y_t$ where P denotes the price level and y is income measured in units of the consumption good.

Formally, let the representative agent solve the following infinite-horizon, discounted problem:

$$\max \begin{array}{c} E_0\left\{\sum_{t=0}^{\infty}\beta^t\left[\xi_{ct}U(c_t)+\xi_{mt}V\left(\frac{M_t}{P_t}\right)\right]\right\} \\ s.t\ \ P_tc_t+M_t=P_ty_t+M_{t-1}+P_t\tau_t \\ M_0\ is\ given \end{array} \tag{9}$$

where ξ_{jt} for $j = c, m$ is a taste shock, following Woodford (2003, Chapter 2), for consumption and real balances, respectively. Further, let increases (decreases) in the money supply over time be distributed as a lump-sum payment (tax) represented by τ. Finally, *M* is the quantity of money balances held by each person. The functions, *U(.)* and *V(.)*, are twice continuously differentiable, strictly concave. The marginal utility of each good is nonnegative. Woodford (2003, Chapter 2) considers both the "cashless economy" case, V(.) = 0 and the "frictions" case where V(.) is non zero. He discusses the role of the taste shocks as well as various money supply and interest rate rules of the Central Bank in this framework for both cashless economies and monetary economies using this framework. We just take his model "off the shelf" here and use it to study the set of real output, money supply, taste shocks and the set of preferences that produce negative (positive) correlations between the cyclic components of the price level (inflation) with the cyclic component of real output. Woodford's model does not require that the utility function be "separable" in consumption and real balances as we do in (9). We impose separability for simplicity of the pricing formulas below. We leave it to future research to investigate non-separable utility functions as well as recursive and risk sensitive preferences.

In this economy, agents are price takers. The sequence of shocks to tastes and income are drawn from a distribution with positive supports. It is straightforward to derive the first-order necessary conditions for utility maximization. Formally, the Euler equation is

$$\xi_{ct}U'(c_t) = \xi_{mt}V'\left(\frac{M_t}{P_t}\right) + \beta E_t[\xi_{ct+1}U'(c_{t+1})(P_t/P_{t+1})] \qquad \forall t \geq 0 \qquad (10)$$

The money supply rule is: $M_t^S = M_{t-1}^S + T_t$, where $T_t = P_t\tau_t = F(y_{t-1}, P_{t-1}; s_t)$, where $s_t \in [\xi_{ct}, \xi_{mt}]$. In other words, the general setting is one in which the change in money supply depends on lagged output, the lagged price level and the exogenous tastes shocks. The function is written broadly enough to encompass money supply rules such as the Friedman k-percent rule or McCallum's base rule as well as more exotic versions. The numerical analysis will focus on a k-percent style rule, but these could be easily modified to be cyclically dependent or an elastic supply rule.

The goods market clears when the quantity of goods consumed equals the quantity of goods available. Thus, for equilibrium concepts that require market clearing, the goods market condition is represented as $c_t = y_t$.

To obtain some analytical results, we specify specific functional forms. For example, let $U(c_t) = ln(c_t), V\left(\frac{M_t}{P_t}\right) = a\ln\left(\frac{M_t}{P_t}\right)$. Further, let real income grow at rate g_t so that after taking logs, we get $\widetilde{y_t} = \widetilde{y_{t-1}} + g_t$ where $\{g_t\}_{t=1}^{\infty}$ is a stationary process and "tildes" are used to denote log transforms of the variables. Let $g_t = \phi_g g_{t-1} + n_t$ where g_0 is given and $n_t \sim IID(0, \sigma_n^2)$. Thus, real income is difference-stationary in this setup. We consider two polar cases of price expectations: (i) Backward looking and (ii) Forward looking, i.e. rational expectations. The second case is structural rational expectations. We treat the rational expectations, forward-looking case first.

4.1 Forward Looking Rational Expectations

The next step is to demonstrate what the correlations would look like in this model economy with forward-looking expectations. By examining this case, we can learn why the expectation formation plays a potentially important role in accounting for the phase shift present in the cyclical components of the price level and output. It is well known that this type of model has difficulty in accounting for the correlation patterns we are after in this article. This has been known for a long time and was a motive for researchers to move towards "sticky price" models like Rotemberg , Woodford , and others in the New Keynesian tradition. In the flexible-price versions, demand shocks could not account for countercyclical prices. More generally, money, being an asset, has a "price" $\Lambda_t \equiv 1/P_t$ that "jumps" too fast in response to new information, i.e. it is a "jump" variable. Hence modelers built models that introduced different methods of "clamping down" on this jump variable by forcing it to move in a "sticky" manner. But a criticism above and beyond criticisms like Robert Lucas's in Aghion et al. (2003) is that work on "design limits" like that of Brock, Durlauf, and Rondina (2013) suggests that there is a "waterbed" type effect, i.e. if a modeler or policy maker squashes down variance in one variable in a dynamical system, the variance is likely to "pop up" somewhere else. This notion can be made precise in the context of linear control dynamics in the frequency domain. But one might worry if this kind of thing can happen more generally. In order to show the jump variable phenomenon in, perhaps, a more transparent way than received to date, we deliberately write the standard model in asset price equation form and study the result below.

With forward- looking, rational expectations, the pricing equation becomes,

$$\Lambda_t = (\xi_{mt} / \xi_{ct})(aY_t / M_t) + \beta E_t\{(\xi_{c,t+1} / \xi_{ct})(Y_t / Y_{t+1})\Lambda_{t+1}\} \qquad (11)$$

Consider a case in which taste shocks are set equal to one for each date, $Y_t = Y_{t-1}e^{g_t}, M_t = M_{t-1}e^{m_t}$. We treat $\{aY_t / M_t\}$ as a dividend process, solving (11) recursively forward as in asset pricing theory. We obtain,

$$\Lambda_t = aY_t\{1 / M_t + E_t[\beta / M_{t+1} + \beta^2 / M_{t+2} + ...]\} \qquad (12)$$

assuming the "no bubble" condition so that the tail term goes to zero. Note the conditional expectation, E_t in (12), limits the money supply processes that we can consider and still maintain easy analytical tractability. If

$$M_{s+1} = e^{m_{s+1}} M_s,\ s = 1, 2, ... \qquad (13)$$

where the process, $\{m_s\}$ is IID with finite mean, $\tilde{m}$ and finite variance, we may write the solution (12) in the form,

$$\Lambda_t = (aY_t / M_t)\{1 / (1 - \beta E e^{-\tilde{m}})\} \qquad (14)$$

Note that no restriction has been made on the real output process in getting this solution. Here, the cyclical component is treated as difference-stationary log levels. Since this analysis applies to the difference-stationary representation, we use the DS-cyclic component. As such, we may now use (14) to compute the covariances of the DS-cyclic components of the price level and inflation with the DS-cyclic component of real output. We have,

$$\operatorname{cov}(p_t^C, y_t^C) = \operatorname{cov}(m_t - g_t, g_t) \qquad (15)$$

$$\operatorname{cov}(p_t^C - p_{t-1}^C, y_t^C) = \operatorname{cov}(m_t - g_t - (m_{t-1} - g_{t-1}), g_t) \qquad (16)$$

These two equations are the basis for the analysis of the correlations in a rational-expectations model.[12]

We start with the simplest case as a means to shed light on two correlations. Specifically, consider a case with a shock to output growth alone. We assume that the money supply is constant over time and output growth follows an AR(1) process: $g_t = \mu_g + \phi_g g_{t-1} + n_{gt}$, $t = 1, 2, \ldots$ where $\{n_{gt}\}$ is IID with zero mean and finite variance. By Equation (15), we know the price level is countercyclical in this model economy. Further,

$$\operatorname{cov}(p_t^C - p_{t-1}^C, y_t^C) = -\operatorname{cov}(n_{gt}, n_{gt}) / (1 + \phi_g) < 0. \qquad (17)$$

Equation (17) indicates that the inflation rate is countercyclical. With forward-looking agents, the price level and output are in phase because consumers anticipate the persistence in output. Forward-looking consumers, therefore, generate a cycle in the price level that occurs concomitant with the cycle in output.

With this setup in place, we can relax assumptions to see how they affect the two correlations. For example, consider a case in which the money growth is stochastic. With $\rho(m_t, n_{gt}) = 1$, we obtain

$$\operatorname{cov}(m_t, n_{gt}) = \rho(m_t, n_{gt})[\operatorname{var}(m_t)\operatorname{var}(n_{gt})]^{1/2}. \qquad (18)$$

If the variance of money growth is large enough relative to the variance of the DS-cyclic component of real output, i.e., $\operatorname{var}(m_t) > \operatorname{var}(n_{gt}) / (1 - \phi_g^2)^2$, then the inflation could be countercyclical, but the correlation with the price level will be "too large," equaling minus one.

Following Woodford (2003, Chapter 2, pages 102 and 103), we consider the role of taste shocks in the utility function in the MIUF model. If, for example, the Central Bank is trying to implement a specific policy, e.g. inflation targeting or price level targeting, then the real disturbances modeled by taste shocks could play an important role. We assume that the price level is countercyclical. Let

[12] The derivation of the equations below are in Brock and Haslag (2014)

$\xi_{m,t+1} = \xi_{m,t} e^{\varphi_{m,t+1}}$, $t = 1, 2, ...$ where $\{\varphi_{m,s}\}$ is IID with finite mean, $\tilde{\varphi}_m$, and variance. Following the solution procedure used above, we obtain

$$1/P_t = \Lambda_t = \{a\xi_{mt} / [u'(Y_t, \xi_{ct}) M_t]\}\{1/(1 - \beta E e^{\tilde{\varphi}_m - \tilde{m}})\}. \tag{19}$$

Because Equation (26) permits just about any specification of the utility function $u(Y_t, \xi_{ct})$ as well as just about any specification of the $\{\xi_{ct}\}, \{Y_t\}$ processes, there are several new channels that could lead to inflation being acyclical, given that the price level is countercyclical.

We consider with log utility represented as $u(Y_t, \xi_{ct}) = \xi_{ct} \ln(Y_t)$. The implication is that the equilibrium price level is represented as

$$P_t = (1 - \beta E e^{\tilde{\varphi}_m - \tilde{m}}) M_t \xi_{ct} / \{a\xi_{mt} / Y_t\} \tag{20}$$

From this, we obtain

$$\begin{aligned} &\mathrm{cov}(p_t^C, y_t^C) = \mathrm{cov}(\ln \xi_{ct} - \ln \xi_{c,t-1} + m_t - \varphi_{m,t} - g_t, g_t) \\ &\mathrm{cov}(p_t^C - p_{t-1}^C, y_t^C) \\ &= \mathrm{cov}(\ln \xi_{ct} - \ln \xi_{c,t-1} - (\ln \xi_{c,t-1} - \ln \xi_{c,t-2}) + m_t - m_{t-1} - (\varphi_{m,t} - \varphi_{m,t-1}) - (g_t - g_{t-1}), g_t) \end{aligned} \tag{21}$$

We may now use (21) to take logs and compute the covariance of the DS-cyclic component of the price level and inflation with the DS-cyclic component of real output. These covariances are given by,

$$\begin{aligned} &\mathrm{cov}(p_t^C, y_t^C) = \mathrm{cov}(\ln \xi_{ct} - \ln \xi_{c,t-1} + m_t - \varphi_{m,t} - g_t, g_t) \\ &\mathrm{cov}(p_t^C - p_{t-1}^C, y_t^C) = (1 - \phi_g)\, \mathrm{cov}(p_t^C, g_t). \end{aligned} \tag{22}$$

With $\{n_{gt}\}$ IID, we get $\mathrm{cov}(p_{t-1}^C, n_{gt}) = 0$. The last line follows by stationarity. Therefore, one implication is that by including the taste shock, the covariance between the DS-cyclic component of the price level and the growth rate of the DS-cyclic component of output determines whether the price level is countercyclical or not.

What do we learn from (22) and the AR(1) specification of the DS-cyclic component of real output? If the processes in the expression for $\mathrm{cov}(p_t^C, y_t^C)$ are such that this covariance is negative then the covariance, $\mathrm{cov}(p_t^C - p_{t-1}^C, y_t^C)$, is still negative, but is smaller when $0 < \phi_g < 1$. At first blush, we might also expect it to be close to zero when ϕ_g is close to one. But this would be wrong because for the special case where ξ_{ct} is constant over time, recall that $\mathrm{cov}(p_t^C - p_{t-1}^C, y_t^C) = -\mathrm{cov}(n_{gt}, n_{gt})/(1+\phi_g)$. Hence, even for $\phi_g = 1$ the covariance is bounded away from zero by half of the variance of n_{gt}. The problem is that $\mathrm{cov}(p_t^C, g_t)$ itself is a function of ϕ_g. Because we are interested in correlations, not covariances, we assume that the tastes shocks and money growth shocks are independent. Let $x_t \equiv \ln \xi_{c,t} - \ln \xi_{c,t-1} + m_t - \varphi_{m,t}$ so that $p_t^C = x_t - g_t$. Under the independence assumption, we can derive the following expression for the correlation between the price level and output

$$\rho(p_t^C, y_t^C) = -\sigma_n / [(1-\phi_g^2)\sigma_x^2 + \sigma_n^2]^{1/2} . \qquad (23)$$

Equation (23) suggests that not only is the $\rho(p_t^C, y_t^C)$ negative but it can be made quite small if $\mathrm{var}(x_t)$ is large enough, provided that when $g_t = \mu_g + \phi_g g_{t-1} + n_t$, $\{n_t\}$ IID with mean zero and finite variance, that ϕ_g^2 is not too close to one.

With independence, the variance of x_t is the sum of three variances: the variance of change of tastes for consumption, the variance of change in monetary policy and the variance of change of tastes for real balances. It remains to further investigation of data to determine how large these variances are relative to the variance of g_t. However, for reasonable data disciplined values of the persistence, ϕ_g of the DS-cyclic component of real output, the relative size of $\mathrm{var}(x)/\mathrm{var}(n)$ needed to obtain $\rho(p_t^C, y_t^C) = -0.13$ appears way too large to be consistent with independence of the $\{x_t\}$ process. This prompts search for sources of persistence of the $\{x_t\}$ process or sources of covariance of the $\{x_t\}$ process with the $\{g_t\}$ process. We investigate some possibilities in Section 5.3 below. However, even at this point, we think the theory has shown some value added by suggesting this particular line of further investigation into the data.

Turn now to $\rho(p_t^C - p_{t-1}^C, y_t^C)$. From the definition of x_t and (23), with $\{x_t\}$ IID and independent of $\{g_t\}$ and with $\sigma_g^2 = \sigma_n^2/(1-\phi_g^2)$, we obtain

$$\rho(p_t^C - p_{t-1}^C, y_t^C) = -\sigma_n(1-\phi_g)/[2(1-\phi_g^2)\sigma_x^2 + \phi_g^2\sigma_n^2]^{1/2} . \qquad (24)$$

Although $\rho(p_t^C - p_{t-1}^C, y_t^C)$ is negative, clearly one can make it as close to zero as one wishes by taking ϕ_g close to one. But taking ϕ_g close to one sends $\rho(p_t^C, y_t^C)$ to minus one which is too large in absolute value to be consistent with the data. Thus, we see that a tension between getting a small and barely positive $\rho(p_t^C - p_{t-1}^C, y_t^C)$ without getting too large a value of $|\rho(p_t^C, y_t^C)|$ remains.

Investigation into whether a plausible value of σ_x^2 can be found that breaks this tension suggests that values are too large to be plausible. One might then try for persistence in the process $z_t - z_{t-1} = \ln \xi_{ct} - \ln \xi_{c,t-1}$ but we are already at the limit by assuming $\{z_t - z_{t-1}\}$ is a random walk with IID innovations.

4.2 The Backwards Looking Case

In this part, we consider an economy composed of consumers who form expectations of next-period's price level by using last period's observed price level. Formally, suppose agents believe that the log of the price level after detrending is an AR(q) represented by

$$\ln P_s = \pi_0 + \pi_1 t + Q_s,\ Q_s = \sum_{i=1}^{q} a_i Q_{s-i} + \sigma_Q z_s \equiv a(L) Q_{s-1} + \sigma_Q z_s, s = 1, 2, ... \qquad (25)$$

At date *t*, our backward- looking consumer has estimated (25) using data available up to and including period *t-1*. Because consumers at *t-1* do not know what the market clearing price will be at date *t*, the Euler equation is given by

$$\begin{aligned} \xi_{c_t} u'(Y_t)(1/P_t) &= \xi_{m_t} v'(M_t / P_t)(1/P_t) + \beta E_t\{\xi_{c_{t+1}} u'(Y_{t+1})(1/P_{t+1}^e)\} \\ &= \xi_{m_t} v'(M_t / P_t)(1/P_t) + \beta E_t\{\xi_{c_{t+1}} u'(Y_{t+1})\} E_t\{(1/P_{t+1}^e)\} \end{aligned} \qquad (26)$$

Here we have placed a superscript "e" on the price level at date t+1 to denote beliefs about it that were formed before the market for money balances clears at date t. The R.H.S. of (26) follows because we assume conditional independence of the beliefs about the *t+1* price level and the taste shocks and real output per capita at *t+1*. We state this assumption formally here.

Assumption A4.1: $E_t\{\xi_{c_{t+1}} u'(Y_{t+1})(1/P_{t+1}^e)\} = E_t\{\xi_{c_{t+1}} u'(Y_{t+1})\} E_t\{(1/P_{t+1}^e)\},\ t = 1, 2, ...$

Assumption A4.1 is motivated our assumption that the DGP process for the exogenously given real output per capita process is known as well as the DGP process for the exogenously given taste shocks process.

For the case where the utility of consumption is logarithmic, and the utility of services from real balances is logarithmic,

$$u(c) = \ln c, v(M/P) = a\ln(M/P) \qquad (27)$$

With Equation (26), using A4.1, and assumptions on the growth process for real GDP per capita, may be rewritten as

$$\begin{aligned}\xi_{c_t}/P_t &= \xi_{m_t} a(1/M_t) + \beta E_t\{\xi_{c_{t+1}}(Y_t/Y_{t+1})E_t\{(1/P^e_{t+1})\} \\ &= \xi_{m_t} a(1/M_t) + \beta E_t\{\xi_{c_{t+1}} e^{-g_{t+1}}\}\exp[-\pi_0 - \pi_1(t+1) - a(L)Q_t + \sigma_Q^2/2]\end{aligned} \qquad (28)$$

Since $a(L)Q_t = a_1 Q_{t-1} + ... + a_q Q_{t-q}$ and the coefficients can be estimated by data available up to and including date *t-1* and the past Q's can be found from past P's once the trend parameters, π_0, π_1, have been estimated from data available up to and including date *t-1*, therefore estimated values of all the parameters needed about the price level predictor in (28) are available at date *t-1* before going into the money market at date *t*. Hence our backwards looking agent can be viewed as behaving like a sensible time series econometrician trying to form the best predictor given data available at date *t-1* in order to form its demand function for money balances going into the money market at date t.

We use equation (28) to generate a simulated time series for the price level. We initially consider an economy in which there are no tastes shocks; that is, $\xi_{mt} = \xi_{ct} = 1$. We initial the economy by setting $P_{-1} = 1$ and $y_{-1} = 100$. The initial growth rate for output is set at 1.019. For the baseline calibration, we let $n_t \sim N(0{,}0.009)$ and the growth rate follows the equation: $g_t = 0.1018 + 0.9 * g_{t-1}$. We assume the money growth rate is fixed at 4.5 percent so that $M_t = 1.044\, M_{t-1}$. Let $\beta = 0.99$. We start with taste parameters set equal to one. We simulated a time series for all the variables for 500 periods. After allowing for initial conditions, we use a time series of 248 observations. We take logs and first difference the price level and output, then take a first difference of the inflation rate. We do this simulation 1000 times.

For the sample of 1000 simulated model economies, we compute the sample means for the standard deviations and contemporaneous correlations. There are two versions of the backward-looking price forecasts. One sets $P^e_{t+1} = P_{t-1}$ while the other sets $P^e_{t+1} = \sum_{j=1}^{4} \alpha_j P_{t-j-1}$. Note that in the distributed lag forecast version of the model, the money supply follows: $M_t = 1.021\ M_{t-1}$. We report the results for both the simple and the distributed lag versions of the price forecast in Table 5. The results are qualitatively similar to the actual values reported. We included the standard deviations to indicate that there is some variability in the inflation rate.[13]

From the two set of simulated economies, the results highlight the role that "backward" looking price expectations play. In other words, the role that rational inattention could play in terms of accounting for the two observed correlations. Sims first characterized rational inattention as being governed by the information flow rate. Reis built the notion of rational inattentiveness in which it is costly to acquire, absorb and process information to update prices. The friction results in firms choosing to ignore new information for a segment of time. In this paper, we implement a Reis-style problem in the sense that it is costly to process information. We assume that the solution to this problem is to form price expectations by looking backward. In other words, it is costless to set next period's expected price level equal to the best linear predictor based upon an AR(q) model fitted to data available at the time which the prediction is being made. In this regard, our work is similar in spirit to Branch and Evans, De Grauwe , Massaro, and Brock and Hommes, in which the authors consider price expectations as a tradeoff between rational expectations, which are costly to form, and a simple rule that next period's expected price is what the price was last period.[14] What these authors argued is that the marginal gain from rational expectations must be enough to offset the marginal cost of resources needed to form rational expectations. Otherwise, agents will opt for expectations that are costless to form. The low-cost expectations are consistent with a kind of rational inattention. Branch and Evans, De Grauwe, Massaro, and Brock and Hommes demonstrate how modifying a model along these lines can quantitatively affect the dynamics in a model economy.

So, in equation (28), the simulated time series can quantitatively generate price dynamics that are consistent with the phase shift that explains why the price level is counter cyclical and the inflation rate is acyclical. The backward-looking expectation mechanism induces a stickiness to the price level that moves it out of phase with respect to the movements in output. The stickiness owes to the weight given to last

[13] One could imagine a case in the simulated economy in which prices and output are negatively correlated and the inflation rate is virtually constant. In such a case, the correlation between inflation and output would be zero. The results indicate there is enough variation in the inflation rate relative to the variation in the price level that correlation, or lack thereof, is caused by the absence of variation in the inflation rate.

[14] Burdett and Judd (1983) and Head, Liu, Menzio and Wright (2012) specify models in which equilibrium prices exhibit a stickiness owing to the search friction. Head, *et. al* refer to their price stickiness as owing to a form of rational inattention.

period's price level in computing the current price level. In contrast, what we observe in the rational-expectations equilibrium is the absence of a phase shift; in other words, the price level adjusts too quickly, resulting the price level and the inflation rate both being countercyclical in the model economy.

It is important to note that our reference to price stickiness differs in two important aspects from what people typically mean by sticky prices. First, prices are sticky in this model economy relative to what they would be in the rational expectations, forward looking agent version of the economy. By setting future price expectations as equal to the last observed price level, the expectations process imparts a backward-looking component that results in the price level not adjusting quantitatively by the same amount compared with the rational expectations equilibrium. The price level, however, is fully flexible. There is no Calvo clock nor menu costs that are imparting a stickiness to the price level. Rather, the idea is that price expectations are cheaper to form by looking at the most recent observed price level. Second, we analyze the equilibrium price level, ignoring commodity differentiation. In order to introduce sticky prices, most models specify economies with multiple goods with some of them subject to timing frictions or menu costs.

We introduced parameters pertaining to two different preference shocks. It is the relative size of the two shocks that really matter. We assume the distribution for the two shocks is uniform over the unit interval. For purposes of the experiment, we consider $\xi_{ct} \sim U(0.5, 0.1 * \frac{1}{12})$ and $\xi_{mt} \sim U(0.5, 0.025 * \frac{1}{12})$. For these settings, we see the mean correlation between the price level and output is 0.005 and the mean correlation between the inflation rate and output is 0.7e-04. In other words, both the price level and the inflation rate are acyclical. In this experiment, the consumption taste shock is more volatile than the taste shock for real balances.[15] The results indicate that with tastes shocks, the standard deviation for the price level is 0.08, roughly ten times greater than the volatility observed in the data. The implication is that taste shocks create much larger swings in the price level. Further, we see that the increase in volatility swamps any correlation between output and the price level.

Overall, the numerical analysis tells us that it is possible to construct a model economy that can account for the countercyclical price level and the acyclical inflation rate. The results are obtained with the contribution of one key assumption; namely that the price level expectations are determined by consumers exhibiting a form of rational inattention. In our particular version, the expected price level next period is set equal to last period's price level. With these expectations, price stickiness is incorporated into the model economy. We show that there exists a set of parameter values such that the price stickiness

[15] Though not reported here, if the money shock is more volatile than the consumption shock, the results are essentially the same.

is sufficient to generate countercyclical prices and acyclical inflation. The results are not terribly robust to changes in parameter settings, but constitute a valuable first step toward a more complete understanding of the relationship between the price level and output over the business cycle.

Thus, we are able to match the observed price level-output correlation and the inflation rate-output correlation in a model economy with a form of rational inattention. Our numerical analysis shows that the correlation of the DS-cyclic component of inflation with the DS-cyclic component of real output is very weakly positive while, at the same time, the correlation of the DS-cyclic component of the price level with the DS-cyclic component of real output is modestly negative. Backward-looking expectations induce the requisite phase shift in the model economy to generate the pattern in the model economy. Analytically, we're still not able to match the correlations very well in a forward-looking, rational expectations version of the model economy. In view of the somewhat disappointing failure to locate sufficient conditions above for the rational expectations solution to give us values of $\text{cov}(p_t^C, y_t^C)$ and $\text{cov}(p_t^C - p_{t-1}^C, y_t^C)$ that are closer to the values found in the data, we turn to Section 5.3 below.

In a series of recent papers, researchers have examined the role of heterogeneous expectations on business cycle facts. More specifically, building on Brock and Hommes, these researchers are building model economies with cognitive limitations that are manifested in expectations formations. DeGrauwe (2011) builds a model populated with optimists and pessimists. In his model economy, the correlation of beliefs produce waves of optimism and pessimism. Indeed, these waves cause business cycle fluctuations akin to Keynes' animal spirits. Branch and Evans study economies in which agents select best-performing statistical models to compute expected values. Armed with the perceived laws of motion, Branch and Evans use the model to study volatility in inflation and output growth. Massaro studies a model economy in which there is a combination of agents with cognitive limitations and others with rational expectations.

It is crucial to make that we differentiate our paper from Rotemberg and den Haan. In those two papers, the authors were interested in studying the relationship between output and the price level at business cycle frequencies. Den Haan focused on deriving what the facts were. Rotemberg was interested in developing a Calvo-style sticky price model in which monetary policy shocks alone could match the facts. Rotemberg can account for the negative correlation between predictable output and predictable price movements over long horizons. Throughout his analysis, Rotemberg focuses exclusively on the correlation of expected and unexpected movements in prices, output, and hours. Our contribution is clear in that our aim is to account for the (unconditional) correlation between prices and output and between inflation and output.

V. Robust Control

In this section, we consider the robust control problem for our representative agent's demands for consumption and money. Since we are backing off from fully rational expectations, some care must be taken in stating the problem. For example, the price level, output level, and transfers are known at date *t*, but expectations must be formed at date *t* for dates *t+s*, for *s = 1, 2,…* Then the problem must be "re-solved" at date *t+1* for dates *t+s+1*, for *s = 1,2,…* At date *t+1* the true values of the output level, transfers, and the price level are revealed to the agent, and so on it goes.[16]

We state the zero-sum dynamic game problem as follows:

$$
\begin{aligned}
&\max\min E_0\{\sum_{t=1}^{\infty}\beta^{t-1}(\xi_{ct}u(C_t)+(1/2\theta_Y)\beta G_{Yt}^2)+(1/2\theta_{\Lambda^e})\beta G_{\Lambda^e t}^2)+\xi_{mt}v(M_t/P_t))\} \\
&s.t.,\ P_tC_t+M_t=P_tY_t+M_{t-1}+Tr_t, M_0\ given \\
&Y_{t+1}=e^{g_{t+1}}Y_t \\
&g_{t+1}=\mu_g+\sigma(-G_{Yt}+n_{t+1}) \\
&\Lambda_{t+1}^e=\Lambda_{t-1}+\sigma_\Lambda(-G_{\Lambda^e t}+n_{\Lambda^e,t+1})
\end{aligned}
\tag{29}
$$

where, $\{n_t\}$ is $IID(0,1)$, $\{n_{\Lambda^e t}\}$ is $IID(0,1)$.

In this dynamic game, at date *t*, the representative agent is assumed to have observed the true values of all past quantities when forming its demands for money balances for dates *t+1,t+2,…* The timing and market equilibration work as follows. At each date *t*, the representative agent forms expectations about the future price levels and future level of output. Based on these expectations, the representative agent computes demand for money balances when going into the date-*t* money market. In each period, the money market clears and the actual price level is determined as well as the demand for consumption equal to output, Y_t in a Nash dynamic equilibrium determined by the repeated zero-sum game . At dates *t+1, t+2, …*this computation by the agent and the market equilibration process is repeated. We ignore any issues raised by the potential time inconsistency of dynamic zero-sum Nash equilibria. As the reader will see, any specification that leads to the first-order necessary conditions (FONC) in equations (30) and (31) will satisfy time consistency.

[16] Robustness and concerns about misspecification were popularized by Hansen and Sargent (2008). In part, the motive for present models with preferences that are robust to misspecification comes from Kasa (2002) in which he derives the duality between models with rational inattention and models with robust control. Kasa's derivations are within the Linear Quadratic Regulator problem. His findings suggest that models with robust control are a viable option for matching the pair of observed correlations since a model with a form of rational inattention accomplishes such a match.

The idea here is to represent low-frequency movements in output and the money supply by the "trend" terms $\left\{\mu_{gs}, \mu_{ms}, s=1,2,...\right\}$. We assume the agent has rational expectations on these low-frequency components, which are given exogenously. The components at frequencies relevant for business cycles and policy making will be denoted by *X's*. More will said about this as we proceed.

We derive the first-order necessary conditions (FONC) for the minimizing agent, expressed as

$$G_{\Lambda t} = \sigma_{\Lambda}\theta_{\Lambda}E_t\left(M_{t+1}^e / Y_{t+1}^e\right) \tag{30}$$

And the maximizing agents, expressed as

$$\begin{aligned}\Lambda_t &= aY_t / M_t + \beta E_t\left[\left(Y_t / Y_{t+1}^e\right)\Lambda_{t+1}^{e*}\right] \\ &= aY_t / M_t + \beta E_t\left\{\left(Y_t / Y_{t+1}^e\right)\left[\Lambda_{t+1}^e + \sigma_{\Lambda}\left(-G_{\Lambda t} + n_{\Lambda,t+1}\right)\right]\right\} \\ &= aY_t / M_t + \beta E_t\left\{\left(Y_t / Y_{t+1}^e\right)\left[\Lambda_{t+1}^e - \theta_{\Lambda}\sigma_{\Lambda}^2 E_t\left(M_{t+1}^e / Y_{t+1}^e\right)\right]\right\}\end{aligned} \tag{31}$$

where Λ is the price of money and Λ_{t+1}^{e*} stands for the next-period expected value of the price of money consistent with robust control. We assume that the robustness shocks, $\left\{n_{\Lambda,t+1}\right\}$ are mean zero and independent of other shocks present in the economy. For each date *t*, the demand for money is determined by Equations (30) and (31). The money supply processes are exogenously given in (30) and (31). Market clearing is defined the usual way, where the demand for money is equal to the supply of money, thus pinning down the price of money.

Note that variables with "*e*" superscripts are not necessarily rational expectations. The actual processes are given by

$$\begin{aligned}Y_{s+1} &= e^{\mu_{g,t+1}+\varepsilon X_{g,t+1}}Y_s \\ M_{s+1} &= e^{\mu_{\mathrm{m},t+1}+\varepsilon X_{\mathrm{m},t+1}}M_s\end{aligned} \tag{32}$$

for *s= 0, 1,2,...* where $\left\{X_{gs}\right\}_{s=0}^{\infty}$ and $\left\{X_{ms}\right\}_{s=0}^{\infty}$ are stationary processes with mean zero and finite variance.

Our strategy is to expand $Cov\left(\Lambda_t, Y_t\right)$ and $Cov\left(\Lambda_t - \Lambda_{t-1}, Y_t\right)$ in ε evaluated at ε = 0. With these expressions, we derive sufficient conditions such that $Cov\left(\Lambda_t, Y_t\right) > 0$ and $Cov\left(\Lambda_t - \Lambda_{t-1}, Y_t\right) \approx 0$

up to and including second-order terms. It will turn out that zero-order and first-order terms will make zero contribution to the covariances. Ultimately we will concentrate on second-order terms of the expansions of the covariances.

Lemma 3: Let $\left\{X_{it}(\varepsilon)\right\}_{s=0}^{\infty}$ and i=1,2 be two stationary stochastic processes such that $\left\{X_{it}(0)\right\}_{s=0}^{\infty}$ for i = 1,2 are a pair of constants. Assume these processes have the expansions

$$\left\{X_{it}(\varepsilon)=X_{i}(0)+X_{it}^{'}(0)\varepsilon+X_{it}^{''}(0)\left(\varepsilon^{2}/2\right)+\tilde{o}_{it}\left(\varepsilon^{2}\right)\right\}_{t=0}^{\infty} \tag{33}$$

for i = 1,2. Here $\tilde{o}_{it}\left(\varepsilon^{2}\right)$ denotes terms of higher order than ε^{2} . It follows that

$$Cov\left(X_{1t}(\varepsilon),X_{2t}(\varepsilon)\right)=\varepsilon^{2}Cov\left(X_{1t}^{'}(0),X_{2t}^{'}(\varepsilon)\right)+o_{t}\left(\varepsilon^{2}\right) \tag{34}$$

for t = 0, 1,2,…

Proof:

$$\begin{aligned}
&Cov\left(X_{1t}(\varepsilon),X_{2t}(\varepsilon)\right)\\
&=Cov\left(X_{1}(0)+X_{1t}^{'}(0)\varepsilon+X_{1t}^{''}(0)\left(\varepsilon^{2}/2\right)+\tilde{o}_{1t}(\varepsilon),X_{2}(0)+X_{2t}^{'}(0)\varepsilon+X_{2t}^{''}(0)\left(\varepsilon^{2}/2\right)+\tilde{o}_{2t}(\varepsilon)\right)\\
&=Cov\left(X_{1t}^{'}(0)\varepsilon+X_{1t}^{''}(0)\left(\varepsilon^{2}/2\right)+\tilde{o}_{1t}(\varepsilon),X_{2t}^{'}(0)\varepsilon+X_{2t}^{''}(0)\left(\varepsilon^{2}/2\right)+\tilde{o}_{2t}(\varepsilon)\right)\\
&=Cov\left(X_{1t}^{'}(0)\varepsilon,X_{2t}^{'}(0)\varepsilon\right)+o_{i}\left(\varepsilon^{2}\right)\\
&=\varepsilon^{2}Cov\left(X_{1t}^{'}(0),X_{2t}^{'}(0)\right)+o_{i}\left(\varepsilon^{2}\right)
\end{aligned}$$

for *t = 0, 1,2,…*■

Because we assume our representative agent faces an environment in which low-frequency components are determined by trend values that are independent of short-term policy actions, a natural way to express the time path for output as follows

$$Y_{t}=Y_{0}\exp\left(\sum_{s=1}^{t}\left(\mu_{gs}+\varepsilon X_{gs}\right)\right), \tag{35}$$

for *t = 1,2,…*Note that in Equation (35), the stochastic process, $\left\{X_{gs}\right\}_{s=1}^{\infty}$ represents the total effect of short-term fluctuations from previous period up to date *t*. The term $\varepsilon>0$ is intended to capture the idea that the fluctuations at business-cycle frequencies are small relative to the low-frequency components,

μ_{gs} , that are driven by institutions, technological progress, demographics, etc. We specify the dynamics of the money supply process similarly.

We summarize the specifications of the stochastic processes in the following assumption.

Assumption 5.1:

$$
\begin{aligned}
&Y_t = Y_0 e^{\mu_g t + \varepsilon X_{gt}} \\
&M_t = M_0 e^{\mu_m t + \varepsilon X_{mt}} \\
&Y_{t+1}^e = Y_0 e^{\mu_g (t+1) + \varepsilon X_{g,t+1}^e} \\
&M_{t+1}^e = M_0 e^{\mu_m (t+1) + \varepsilon X_{\mathrm{m},t+1}^e} \\
&\Lambda_{t+1}^e = \Lambda_0 (\varepsilon) e^{\lambda_{t+1} + \varepsilon X_{\Lambda,t+1}^e} = \Lambda_0 (\varepsilon) e^{(t+1)(\mu_g - \mu_m) + \varepsilon X_{\Lambda,t+1}^e} \\
&\mu_m - \mu_g = \pi = 0
\end{aligned}
$$

for $t = 0,1,2,\ldots$ More will be said about the actual specifications of the stochastic processes in Assumption 5.1 as we work through special cases below. Note that Assumption 5.1 imposes rational expectations on long-term trend in the middle two lines. The last line of Assumption 5.1 assumes the inflation target is set equal to zero.

In the following expression, we consider a general inflation target at a constant rate and also at the constant zero rate so that one can see the modification necessary. Under Assumption 5.1,

$$
\begin{aligned}
&\Lambda_t(\varepsilon) = a(Y_0 / \mathrm{M}_0) e^{(\mu_g - \mu_m)t + \varepsilon(X_{gt} - X_{mt})} \\
&+\beta E_t \left\{ \left(e^{-\mu_g + \varepsilon(X_{gt} - X_{g,t+1}^e)} \left[\Lambda_0(\varepsilon) e^{(t+1)(\mu_g - \mu_m) + \varepsilon X_{\Lambda,t+1}^e} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) E_t \left(e^{(\mu_m - \mu_g)(t+1) + \varepsilon(X_{m,t+1}^e - X_{g,t+1}^e)} \right) \right] \right) \right\} \\
&\Lambda_t(\varepsilon) = a(Y_0 / \mathrm{M}_0) e^{\varepsilon(X_{gt} - X_{mt})} \\
&+\beta E_t \left\{ \left(e^{-\mu_g + \varepsilon(X_{gt} - X_{g,t+1}^e)} \left[\Lambda_0(\varepsilon) e^{\varepsilon X_{\Lambda,t+1}^e} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) E_t \left(e^{\varepsilon(X_{m,t+1}^e - X_{g,t+1}^e)} \right) \right] \right) \right\} \\
&\Lambda_t^{'}(0) = a(Y_0 / \mathrm{M}_0)(X_{gt} - X_{mt}) + \beta e^{-\mu_g} \Lambda_0^{'}(0) e^{(t+1)(\mu_g - \mu_m)} \\
&+\beta E_t \left\{ \left(e^{-\mu_g} \left(X_{gt} - X_{g,t+1}^e \right) \left[\Lambda_0 e^{\lambda_{t+1}} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) \left(e^{(\mu_m - \mu_g)(t+1)} \right) \right] \right) \right\} \\
&+\beta E_t \left\{ \left(e^{-\mu_g} \left[\Lambda_0 X_{\Lambda,t+1}^e - \theta_\Lambda \sigma_\Lambda^2 E_t (M_0 / Y_0) \left(e^{(\mu_m - \mu_g)(t+1)} \right) \left(X_{m,t+1}^e - X_{g,t+1}^e \right) \right] \right) \right\} \\
&\Lambda_t^{'}(0) = a(Y_0 / \mathrm{M}_0)(X_{gt} - X_{mt}) + \beta e^{-\mu_g} \Lambda_0^{'}(0) \\
&+\beta E_t \left\{ \left(e^{-\mu_g} \left(X_{gt} - X_{g,t+1}^e \right) \left[\Lambda_0 - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) \right] \right) \right\} \\
&+\beta E_t \left\{ \left(e^{-\mu_g} \left[\Lambda_0 X_{\Lambda,t+1}^e - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) E_t \left(X_{m,t+1}^e - X_{g,t+1}^e \right) \right] \right) \right\}
\end{aligned} \tag{36}
$$

From here, our strategy is locate the sufficient conditions for

$$
\begin{aligned}
&Cov\left(e^{t(\mu_m - \mu_g)} \Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0) \right) > 0 \\
&and \\
&Cov\left(e^{t(\mu_m - \mu_g)} \Lambda_t^{'}(0) - e^{(t-1)(\mu_m - \mu_g)} \Lambda_{t-1}^{'}(0), e^{-t\mu_g} Y_t^{'}(0) \right) \le 0
\end{aligned} \tag{37}
$$

where Equation (37) holds for stationary components.

With $\Lambda_t(\varepsilon) = 1 / P_t(\varepsilon)$, it follows that $\Lambda_t^{'}(0) = -\left[1 / P_t(0)\right]^2 P_t^{'}(0) = -\Lambda_t(0)^2 P_t^{'}(0).$ By either line 1 or line 2 in Equation (36), we can see that $\Lambda_t(0)^2$ is deterministic.. While we will work is the price of money for convenience, keep in mind that the sign of the derivative of the price of money is opposite to the sign of the derivative of the price level.

Assumption 5.2: (Inflation targeting at rate π) Assume (i) $\mu_m = \mu_g + \pi;$ (ii) $\lambda_{t+1} = -(t+1)\pi.$

Assumption 5.2.ii imposes long-run neutrality on the trend rate in the price of money when monetary policy is targeting the trend inflation rate. Of course, at business-cycle frequencies, beliefs need to made also on the short-run fluctuations about trend in the price of money; specifically, $X^e_{\Lambda.t+1}$.Beliefs over short horizons need not be rational expectations beliefs. Next, we make specific assumptions regarding beliefs about the deterministic trends.

Assumption 5.3: $E_t X^e_{g,t+1} = X_{g,t-1}, E_t X^e_{\mathrm{m},t+1} = X_{m,t-1}$.

Assumption 5.4: $\begin{array}{l} E_t X^e_{g,t+1} = 0 = E_t X_{m,t+1} = E_t X^e_{\Lambda,t+1}, t = 0,1,2,... \\ EX_{gt} = 0 = EX_{mt}, t = 1,2,... \end{array}$

Together Assumptions 5.2, 5.3, and 5.4 with $\pi = 0$ allow us to compute the covariance as follows (recall that the constant term, $\beta e^{-\mu_g} \Lambda_0^{'}(0)$ cancels out in the calculating the covariance):

$$
\begin{aligned}
&Cov\left(\Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) = \\
&E\left\{\begin{array}{l}
\left\{X_{gt}\left[a\left(Y_0 / M_0\right)\left(X_{gt} - X_{mt}\right)\right]\right\} \\
+\beta\left\{e^{-\mu_g}\left[\Lambda_0 - \theta_\Lambda \sigma_\Lambda^2\left(M_0 / Y_0\right)\right]\right\}\left(X_{gt} - X_{g,t-1}\right) \\
+\beta\left\{e^{-\mu_g}\left[\Lambda_0 X^e_{\Lambda,t+1} - \theta_\Lambda \sigma_\Lambda^2\left(M_0 / Y_0\right)\left(X_{m,t-1} - X_{\mathrm{g},t-1}\right)\right]\right\}
\end{array}\right\}
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
&Cov\left(\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) = \\
&E\left\{\begin{array}{l}
\left\{X_{gt}\left[a\left(Y_0 / M_0\right)\left(dX_{gt} - dX_{mt}\right)\right]\right\} \\
+\beta\left\{e^{-\mu_g}\left[\Lambda_0 - \theta_\Lambda \sigma_\Lambda^2\left(M_0 / Y_0\right)\right]\right\}\left(dX_{gt} - dX_{g,t-1}\right) \\
+\beta\left\{e^{-\mu_g}\left[\Lambda_0 dX^e_{\Lambda,t+1} - \theta_\Lambda \sigma_\Lambda^2\left(M_0 / Y_0\right)\left(dX_{m,t-1} - dX_{\mathrm{g},t-1}\right)\right]\right\}
\end{array}\right\}
\end{aligned} \tag{39}
$$

where $dZ_t \equiv Z_t - Z_{t-1}$ for any random variable.

With this derivation, we can restate our goal to identify conditions sufficient to obtain

$$
\begin{aligned}
&Cov\left(\Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) > 0 \\
&and \\
&Cov\left(\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}, e^{-t\mu_g} Y_t^{'}(0)\right) \le 0
\end{aligned}
\tag{40}
$$

We proceed by specifying the *X* process in the VARs. For convenience, we adopt the following labelling conventions: *1 = g*, *2 = m*, and *3 = Λ*. Thus,

$$
\begin{aligned}
&X_{t,t+1} = \sum_{j=1}^{3} a_{ij} X_{jt} + n_{i,t+1}, i = 1,2,3; t = 1,2,... \\
&X_{t+1} = AX_t + n_{t+1} \\
&where \{n_t\}_{t=1}^{\infty}, IID(0, \Sigma_n) \\
&E\left(X_{t+1} X_{t+1}^T\right) = AE\left(X_t X_t^T\right) A^T + \Sigma_n, t = 1,2,... \\
&E\left(X_0 X_0^T\right) given
\end{aligned}
\tag{41}
$$

With (41), there is considerable flexibility to specify stochastic process for the business-cycle fluctuations about trend that will satisfy the objective in (40). Note that data are roughly consistent with the rate of change in the price of money having a small or zero covariance. The real issue is whether our the sufficient conditions that satisfy (40) are plausible from a macroeconomic perspective.

We recognize that the reader may object to our specification regarding the expectations of the price of money. The concern is that Λ_{t+1}^e is not rational enough. Next, we consider some self-consistency restrictions that do not go all the way to imposing rational expectations. Therefore, we assume the following:

Assumption 5.5: $\lambda_{t+1} = (t+1)\left(\mu_m - \mu_g\right), \Lambda_0^e = \Lambda_0$

By Assumption 5.5, we affirm that expectations on the trend component and on the constant term are rational. Recall that we have already assumed rational expectations on the constant term, Λ_0 in Assumption 5.1, using it in Equation (36). In Assumption 5.5, we formally assume that the agent has rational expectations with respect to the constant term.

What other properties might be plausible to impose on the shorter-term components of inflation expectations? Haubrich, Pennacchi, and Ritchken (2012) describe the methodological approach that has yielded detailed estimates of the public's inflation expectations. In other words, the work by Haubrich, et al. will stand as reference point for constructing plausible measures of $\left\{E_t X^e_{\Lambda,t+1}\right\}_{t=0}^{\infty}$.

We start with a very special case in which short-term movements in inflation expectations may produce the sign pattern that we seek for the two covariance expressions. To illustrate, we adopt the following shortened notation:

$$E_t X^e_{i,t+1} \equiv X^e_{i,t+1|t}, \quad \text{for } I = g,\ m,\ \Lambda \tag{42}$$

Next, we assume

Assumption 5.6: $\mu_m - \mu_g = 0, X^e_{g,t+1|t} = X_{gt}, \theta_\Lambda = 0$

So that Assumption 5.6 sets long-run monetary policy so that the targeted inflation rate is equal to zero, output dynamics are "quasi-rational and zero robustness.

By Assumption 5.6, we obtain

$$\begin{aligned}
&\Lambda_t(\varepsilon) = a\left(Y_0 / M_0\right)e^{\varepsilon\left(X_{gt}-X_{mt}\right)} + \beta e^{-\mu_g}\Lambda_0(\varepsilon)E_t e^{\varepsilon X^e_{\Lambda,t+1}} \\
&\Lambda_t^{'}(\varepsilon) = a\left(Y_0 / M_0\right)\left(X_{gt}-X_{mt}\right) + \beta e^{-\mu_g}\Lambda_0(0)X^e_{\Lambda,t+1|t} + \beta e^{-\mu_g}\Lambda_0^{'}(0) \\
&\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}(0) = a\left(Y_0 / M_0\right)\left(X_{gt}-X_{g,t-1}\right) - \left(X_{mt}-X_{\mathrm{m},t-1}\right) \\
&+\beta e^{-\mu_g}\Lambda_0(0)\left(X^e_{\Lambda,t+1|t}-X^e_{\Lambda,t|t-1}\right)
\end{aligned} \tag{43}$$

From (43), we can show

$$\begin{aligned}
&Cov\left(\Lambda_t^{'}(0), X_{gt}\right) = a\left(Y_0 / M_0\right)E\left[\left(X_{gt}-X_{mt}\right)X_{gt}\right] + \beta e^{-\mu_g}\Lambda_0(0)E\left[X^e_{\Lambda,t+1|t}X_{gt}\right] \\
&Cov\left(\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}(0), X_{gt}\right) = a\left(Y_0 / M_0\right)E\left[\left(dX_{gt}-dX_{mt}\right)X_{gt}\right] \\
&+\beta e^{-\mu_g}\Lambda_0(0)E\left[dX^e_{\Lambda,t+1|t}X_{gt}\right]
\end{aligned} \tag{44}$$

where we use the "d" notation to denote first-differences.

Here, we return to Webb and his quotes lifted from Wesley Claire Mitchell. In this discussion, Webb represents it as plausible that the covariance of contemporaneous inflation expectations and output is non-negative; that is, formally, $E\left(X^e_{\Lambda,t+1|t}X_{gt}\right) \geq 0.$ Add to the mix the argument that

$E\left[\left(X_{gt}-X_{mt}\right)X_{gt}\right]>0$ is a plausible assumption, then we can show that the fluctuations in the price of money and output are positive.

Next, we turn to the covariance between the change in the price of money and output. Specifically, are there conditions in which the $E\left[dX^{e}_{\Lambda,t+1|t}X_{gt}\right]<0$ make sense? Using Webb citation, Mitchell supported the notion that as output rises above trend, the rate of change of the inflation rate will tend to rise above trend. This is directly what we are interested in. The implication is that the rate of change in the price of money will tend to fall. Hence, $E\left[dX^{e}_{\Lambda,t+1|t}X_{gt}\right]<0$ is plausible.

We specify the following Lemma when computing the covariances form the VARs.

<u>Lemma</u> <u>4</u>: Consider the general VAR represented by (41) with constant coefficients. In the second line, the VAR is written in matrix form and the dynamics of the NxN covariance matrix are presented in the third line. Assume the initial condition, $E\left(X_0X_0^T\right)$ for the variance matrix is at the stationary solution in the last line of (41). The linear combination with constant coefficients is $Z_t=\sum_{j=1}^{l}\alpha_jX_{jt}$ for t = 1,2,… then the unconditional covariances of Z and its first-differences with say X_{1t} is given by

$$
\begin{aligned}
&Cov\left(Z_tX_{1t}\right)=\sum_{j=1}^{N}\alpha_jE\left(X_{it}X_{1t}\right)\\
&Cov\left(\left(Z_t-Z_{t-1}\right)X_{1t}\right)=\sum_{j=1}^{N}\alpha_jE\left(X_{it}X_{1t}\right)-\sum_{j=1}^{N}\alpha_jE\left[X_{i,t-1}\sum_{i=1}^{N}a_{1i}\left(X_{\mathrm{i},t-1}+n_{1t}\right)\right]\\
&Cov\left(Z_tX_{1t}\right)-\sum_{j=1}^{N}a_{1j}Cov\left(Z_{t-1}X_{j,t-1}\right) \qquad (45)\\
&=Cov\left(Z_tX_{1t}\right)-\sum_{j=1}^{N}a_{1j}Cov\left(Z_{t-1}X_{j,t}\right)\\
&=\left(1-a_{11}\right)Cov\left(Z_tX_{1t}\right)-\sum_{j=2}^{N}a_{1j}Cov\left(Z_{t-1}X_{j,t}\right)
\end{aligned}
$$

Proof: The first line of Lemma 4 follows from the fact that Z and the X's all have mean zero and we start with initial conditions have mean zero. The second and third lines follow from computation. The fourth line follows from the stationarity assumption. The fifth line follows immediately from there.■

Suppose the VAR is diagonal. Lemma 4 implies that it is not possible for the signs of $Cov(Z_t X_{1t})$ and $Cov[(Z_t - Z_{t-1}) X_{1t}]$ to be different. Therefore, we must work with a non-diagonal VAR if we want to qualitatively match the signs of the two covariances.

We apply Lemma 4 to different cases of Equations (38) and (39). We start with the simplest case with no robustness—that is, $\theta_\Lambda = 0$ --and deterministic money supply. With $X_{mt} = 0$, and after dropping the irrelevant constants, we get

$$
\begin{aligned}
&Cov\left(\Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) = a(Y_0 / M_0) E\left[X_{gt}\left(X_{gt} - X_{mt}\right)\right] + \beta e^{-\mu_g} \Lambda_0(0) E\left(X_{gt} X_{\Lambda,t+1|t}^{e}\right) \\
&= \beta_1 E\left[X_{1t}\left(X_{1t} - X_{2t}\right)\right] + \beta_3 E\left(X_{1t} X_{3t}\right) \\
&= \beta_1 E X_{1t}^2 + \beta_3 E\left(X_{1t} X_{3t}\right)
\end{aligned} \tag{46}
$$

where $\beta_1 \equiv a(Y_0 / M_0), \beta_3 \equiv \beta e^{-\mu_g} \Lambda_0(0) = \beta e^{-\mu_g} a(Y_0 / M_0) / \left(1 - \beta e^{-\mu_g}\right)$. Note that the third line of (46) follows from the rationality of inflation expectations in the constant term, $\Lambda_0(0) = a(Y_0 / M_0) / \left(1 - \beta e^{-\mu_g}\right)$. The second covariance is computed as follows:

$$
\begin{aligned}
&Cov\left(\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) = \\
&a(Y_0 / M_0) E\left[X_{gt}\left(X_{gt} - X_{g,t-1}\right)\right] + \beta e^{-\mu_g} \Lambda_0(0) E\left[X_{gt}\left(X_{\Lambda,t+1|t}^{e} - X_{\Lambda,t|t-1}^{e}\right)\right] \\
&= \beta_1 E\left[X_{1t}\left(X_{1t} - X_{1,t-1}\right)\right] + \beta_3 E\left[X_{1t}\left(X_{3t} - X_{3,t-1}\right)\right] \\
&= \beta_1 (1 - a_{11}) E X_{1t}^2 - \beta_1 a_{13} E\left(X_{1t} X_{3t}\right) + \beta_3 E\left(X_{1t} X_{3t}\right) \\
&- \beta_3 a_{11} E\left(X_{1t} X_{3,t-1}\right) - \beta_3 a_{13} E X_{3,t-1}^2 \\
&= (1 - a_{11})\left[\beta_1 E X_{1t}^2 + \beta_3 E\left(X_{1t} X_{3t}\right)\right] - \beta_1 a_{13} E\left(X_{1t} X_{3t}\right) - \beta_3 a_{13} E X_{3,t}^2
\end{aligned} \tag{47}
$$

In both (46) and (47), we use stationarity in computing the covariances. In (47), note that the first term in the last line has the same sign as $Cov\left(\Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right)$. It is easy to identify the sufficient conditions so that $Cov\left(\Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) < 0$, or that the price level is countercyclical. With $a_{13} > 0$, then $-\beta a_{13} E X_{3t}^2 < 0$. In this case, the algebraic value of $Cov\left(\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right)$ falls. Indeed, it is possible to show that the value is positive when $Cov\left(\Lambda_t^{'}(0), e^{-t\mu_g} Y_t^{'}(0)\right) < 0$. The other "extra" term,

$-\beta a_{13}E\left(X_{1t},X_{3t}\right)$ helps our case when $E\left(X_{1t},X_{3t}\right)>0$. If we modify Webb's argument, substituting inflation expectations for the inflation rate, then this condition is plausible.

Next, consider the relationship between β_1 and β_3. We know that $\beta_3>\beta_1$ if and only if $2\beta e^{-\mu_g}>1$. With μ_g small, the inequality will likely be satisfied since β is close to one. Hence, the terms weighted by β_3 will play a bigger role in determining whether the sufficiency conditions will be satisfied or not. For example, if we ignore the terms weighted by β_1, we get

$$\begin{aligned}&Cov\left(\Lambda_t^{'}\left(0\right),e^{-t\mu_g}Y_t^{'}\left(0\right)\right)\simeq\beta_3 E\left(X_{1t},X_{3t}\right)\\&Cov\left(\left(\Lambda_t^{'}\left(0\right)-\Lambda_{t-1}^{'}\left(0\right)\right),e^{-t\mu_g}Y_t^{'}\left(0\right)\right)\simeq\beta_3\left[\left(1-a_{11}\right)E\left(X_{1t},X_{3t}\right)-a_{13}EX_{3t}^2\right]\end{aligned}\ . \quad (48)$$

With $\beta_1/\beta_3\to 0$ and with $\beta e^{-\mu_g}\to 1$, the sign pattern of the two covariances is determined by the terms on the RHS. Note that the VAR and the covariances between the *X's* are independent of β_1,β_3. Equation (48) indicates that if $E\left(X_{1t},X_{3t}\right)>0$ and if $a_{13}EX_{3t}^2$ is large enough, then the covariance between the price of money and output will be positive and the covariance between the change in the price of money and output will be positive. In other words, the price level will be countercyclical and the inflation rate will be procyclical.

The implication is that the closed-form solutions for $E\left(X_{1t},X_{3t}\right)>0$ and if $a_{13}EX_{3t}^2$ can satisfy the sufficiency conditions resulting in countercyclical price level and procyclical inflation rate. By using Lemma 4 and Equations (41), (45), $Z_t=\sum_{j=1}^{l}\alpha_j X_{jt}$ and by identifying a set of VAR coefficients and conditions on the covariance matrix of the shocks, Σ_n, we derive a set of sufficient conditions. The simplifying key is that monetary policy is deterministic so that Lemma 4 and Equations (41), (45), and definition of Z reduces the calculations to a bivariate VAR.

It is important to emphasize that our route to matching the two covariances does not depend on the assumptions that prices are sticky. The channel here is the interlinked dynamic path regarding the relationship between fluctuations in output about trend and fluctuations in inflation expectations about trend.

There is an important distinction between the problem we are studying and the standard robust control problem in engineering mathematics. In our setup, the representative agent uses robust control to determine the intertemporal demands for consumption and real balances. These demands feed into an equilibration process that determines the price of money overtime. The equilibrium feedback is not present in robust control against "Nature."

Next, we extend the analysis to consider how robustness would affect our results. Recall that we have imposed rational expectations on the constant term of inflation expectations; that is, Λ_0. With $\Lambda_t^{'}(0) = -\Lambda_t(0)^2 P_t^{'}(0)$, at $t = 0$, $\varepsilon=0$, and $\lambda_{t+1} = (t+1)(\mu_m - \mu_g)$, we get

$$\begin{aligned}
&\Lambda_0(0) = \Lambda_0 = a(Y_0 / M_0) + \beta e^{-\mu_g} \Lambda_0(0) e^{\mu_m - \mu_g} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) e^{\mu_m - \mu_g} \\
&\Lambda_0(0) = \left(1 - \beta e^{-2\mu_g + \mu_m}\right)^{-1} \left[a(Y_0 / M_0)\right] - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) e^{\mu_m - \mu_g}
\end{aligned} \quad . \tag{49}$$

To save some space, we make the basic points by treating the robust control case of (36) for the special case of inflation targeting evaluated at the targeted rate equal to zero. We rewrite (36) for this special case, noting that under rationality of expectations for trends, $e^{\lambda_{t+1}} = 1,$ yielding

$$\begin{aligned}
&\Lambda_t(\varepsilon) = a(Y_0 / \mathrm{M}_0) e^{\varepsilon(X_{gt} - X_{mt})} \\
&+\beta E_t\left\{\left(e^{-\mu_g + \varepsilon\left(X_{gt} - X_{g,t+1}^e\right)}\left[\Lambda_0(\varepsilon) e^{\varepsilon X_{\Lambda,t+1}^e} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) E_t\left(e^{\varepsilon\left(X_{m,t+1}^e - X_{g,t+1}^e\right)}\right)\right]\right)\right\} \\
&\Lambda_t(\varepsilon) = a(Y_0 / \mathrm{M}_0) e^{\varepsilon(X_{gt} - X_{mt})} \\
&+\beta E_t\left\{\left(e^{-\mu_g + \varepsilon\left(X_{gt} - X_{g,t+1}^e\right)}\left[\Lambda_0(\varepsilon) e^{\varepsilon X_{\Lambda,t+1}^e} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) E_t\left(e^{\varepsilon\left(X_{m,t+1}^e - X_{g,t+1}^e\right)}\right)\right]\right)\right\} \\
&\Lambda_t^{'}(0) = a(Y_0 / \mathrm{M}_0)\left(X_{gt} - X_{mt}\right) + \beta e^{-\mu_g} \Lambda_0^{'}(0) \\
&+\beta E_t\left\{\left(e^{-\mu_g}\left(X_{gt} - X_{g,t+1}^e\right)\left[\Lambda_0 e^{\lambda_{t+1}} - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0)\right]\right)\right\} \\
&+\beta E_t\left\{\left(e^{-\mu_g}\left[\Lambda_0 X_{\Lambda,t+1}^e e^{\lambda_{t+1}} - \theta_\Lambda \sigma_\Lambda^2 E_t (M_0 / Y_0)\left(X_{m,t+1}^e - X_{g,t+1}^e\right)\right]\right)\right\} \\
&\Lambda_t^{'}(0) = a(Y_0 / \mathrm{M}_0)\left(X_{gt} - X_{mt}\right) + \beta e^{-\mu_g} \Lambda_0^{'}(0) \\
&+\beta E_t\left\{\left(e^{-\mu_g}\left(X_{gt} - X_{g,t+1}^e\right)\left[\Lambda_0 - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0)\right]\right)\right\} \\
&+\beta E_t\left\{\left(e^{-\mu_g}\left[\Lambda_0 X_{\Lambda,t+1}^e - \theta_\Lambda \sigma_\Lambda^2 (M_0 / Y_0) E_t\left(X_{m,t+1}^e - X_{g,t+1}^e\right)\right]\right)\right\}
\end{aligned} \tag{50}$$

We can see from (50) that the simplest possible case where robustness still matters is captured by the following assumption.

Assumption 5.7: $X^{e}_{g,t+1|t} = X_{gt}, X^{e}_{\mathrm{m},t+1|t} = X_{mt}$.

Assumption 5.7 implies that the representative agent believes that the business-cycle component of the stochastic processes for the growth rate and money supply about trend will be at date *t+1* the same as it was at date *t.* With Assumption 5.7, we can write a simplified version of (50) as

$$\begin{aligned}&\Lambda_t(\varepsilon) = a(Y_0/\mathrm{M}_0)e^{\varepsilon(X_{gt}-X_{mt})}\\&+\beta E_t\left\{\left(e^{-\mu_g}\left[\Lambda_0(\varepsilon)e^{\varepsilon X^e_{\Lambda,t+1}} - \theta_\Lambda\sigma^2_\Lambda(M_0/Y_0)E_t\left(e^{\varepsilon\left(X^e_{m,t+1}-X^e_{g,t+1}\right)}\right)\right]\right)\right\}\\&\Lambda'_t(0) = \left[a(Y_0/\mathrm{M}_0)+\beta e^{-\mu_g}\theta_\Lambda\sigma^2_\Lambda(M_0/Y_0)\right](X_{gt}-X_{mt})+\beta e^{-\mu_g}\Lambda'_t(0)\\&+\beta e^{-\mu_g}\Lambda'_t(0)E_t\left(X^e_{\Lambda,t+1}\right)\end{aligned}\quad. \tag{51}$$

Now, impose the constraint on $\Lambda_0(0)$ that is consistent with rationality and follows from the first two lines in (51) with $t = 0$ and $\varepsilon = 0$, so that we get

$$\Lambda_0(0) = \left(1-\beta e^{-\mu_g}\right)\left[a(Y_0/M_0)-\beta e^{-\mu_g}\theta_\Lambda\sigma^2_\Lambda(M_0/\mathrm{Y}_0)\right]. \tag{52}$$

Now, we can write the last two lines of (51) as

$$\begin{aligned}&\Lambda'_t(0) = \left[a(Y_0/\mathrm{M}_0)+\beta e^{-\mu_g}\theta_\Lambda\sigma^2_\Lambda(M_0/Y_0)\right](X_{gt}-X_{mt})+\beta e^{-\mu_g}\Lambda'_t(0)\\&+\beta e^{-\mu_g}\left(1-\beta e^{-\mu_g}\right)^{-1}\left[a(Y_0/\mathrm{M}_0)+\beta e^{-\mu_g}\theta_\Lambda\sigma^2_\Lambda(M_0/Y_0)\right]E_t\left(X^e_{\Lambda,t+1}\right)\end{aligned}\quad. \tag{53}$$

The following assumption is necessary for the price level to be positive.

Assumption 5.8: $\left[a(Y_0/M_0)-\beta e^{-\mu_g}\theta_\Lambda\sigma^2_\Lambda(M_0/\mathrm{Y}_0)\right] > 0.$

The impact of robustness on the price of money at date zero is much like a “risk” adjustment effect. Thus, the larger is the lack of confidence in the mind of the representative agent about its specification of the dynamics of expectations on the future value of money; in other words, an increase in θ_Λ , for example, then the larger is the adjustment to “earnings” of money to reflect this increased “specification risk”. We need the value of money to be positive, so this restraint puts restraints on the size of the parameters in Assumption 5.8.

We have a system with a 3x3 VAR tht is manageable. We use that system to compute our two covariances. We use the following code for the VAR order: $X_{gt} = X_{1t}, X_{mt} = X_{2t}, X^{e}_{\Lambda,t+1|t} = X_{3t}$. Next, we write out the equations for the stationary variance matrix, $EXX^T = S.$ Thus,

$$S = ASA^T + Enn^T \equiv ASA^T + \Sigma \tag{54}$$

With this representation of *S* we can identify forces likely to qualitatively match the observation that the price level is countercyclical and the inflation rate is either acyclical or procyclical.

We assume, for tractability that the matrix *A* is upper triangular, so that the rows of (54) can be written as

$$\begin{aligned}
&S_{11} = \sum_{s=1}^{3} a_{1s} \sum_{j=1}^{3} a_{1j} S_{js} + \sigma_{11}, S_{12} = a_{22} \sum_{j=1}^{3} a_{1j} S_{j2} + \sigma_{12}, \mathrm{S}_{13} = a_{33} \sum_{j=1}^{3} a_{1j} S_{j3} + \sigma_{13} \\
&\mathrm{S}_{21} = \sum_{j=1}^{3} a_{22} S_{2j} a_{1j} + \sigma_{21,}\ \mathrm{S}_{22} = a_{22}^2 S_{22} + \sigma_{22}, \mathrm{S}_{23} = a_{22} S_{23} a_{33} + \sigma_{23} \qquad (55) \\
&\mathrm{S}_{33} = \sum_{j=1}^{3} a_{33} S_{3j} a_{1j} + \sigma_{31,}\ \mathrm{S}_{23} = a_{22}^3 S_{32} + \sigma_{32}, \mathrm{S}_{33} = a_{33}^2 S_{33} + \sigma_{33}
\end{aligned}$$

The advantage of the upper triangular system is this. We can solve for $S_{33} \equiv EX_3^2$ from the last element of the last row of (55). Similarly, we can obtain $S_{32} \equiv EX_3 X_2$, which is the covariance between the fluctuation in the expectations of the price of money at business-cycle frequency and fluctuation in output about trend. In this manner, we may solve for $S_{21}, S_{22}, \mathrm{S}_{23}, S_{12}$ and S_{13}. With more work and with symmetries, $S_{ij} = S_{ji}$, we can solve for S_{11}.

The next step is to compute the covariances up to the dominant term *s* by using (53). We focus on cases in which $\beta e^{-\mu_g}$ is close enough to one. In these cases, $Cov\left(\Lambda_t^{'}(0), \mathrm{e}^{-t\mu_g} Y_t^{'}(0)\right)$ is determined by the sign of

$$S_{31} \equiv EX_3 X_1 = \left(1 - a_{33} a_{11}\right)^{-1} \left[a_{33} a_{12} \sigma_{32} / \left(1 - a_{22}^2\right) + a_{13} a_{33} \sigma_{33} / \left(1 - a_{33}^2\right) + \sigma_{31} \right] . \tag{55}$$

The sign of $Cov(\Lambda_t^{'}(0) - \Lambda_{t-1}^{'}(0), e^{-t\mu_t} Y_t^{'}(0))$ will be determined by the sign of

$$(1 - a_{11}) S_{31} - (a_{12} S_{32} + a_{13} S_{33}) = (1 - a_{11}) S_{31} - [a_{12} \sigma_{32} / (1 - a_{22} a_{33}) + a_{13} \sigma_{33} / (1 - a_{33}^2)] . \tag{56}$$

We break Equation (56) up to look at the contributions from various components. In Equation (56), the sign of a_{12} captures the effect that an increase in money supply above trend last period, for example, is associated with a change in output relative to trend this period. Next, the sign of a_{13} captures the effect that an increase in expectations on the price of money above trend last period, for example, has on output relative to trend this period. Finally, σ_{13} captures the correlation between shocks to output and shocks to expectations on the price of money.

Suppose a_{12} is approximately zero and $a_{13} > 0$. In addition, a plausible case can be made that a positive (negative) shock to output to be associated with a positive (negative) shock to expectations on the price of money. Hence, $\sigma_{13} > 0$. From here, we get the following proposition.

Proposition 3: With $a_{12} \approx 0$, $a_{13} > 0$, and $\sigma_{13} > 0$, then $S_{31} > 0$. More importantly, the price level is countercyclical and the inflation rate is procyclical.

From Equation (56), note that if a_{11} is close to one, but $a_{11}a_{33}$ is bounded away from one then it is plausible that the RHS of (56) is negative, which is the sign pattern that we seek. However, we caution that serious quantitative investigation must be done before one can have much confidence in this conclusion. At this point it seems safe to say that we have gone far enough to show that this approach is potentially useful for matching the two correlation's sign pattern and no "sticky prices" needed to be introduced to do it. Turning to robustness, we see from (107) that the main effect of robustness is to make the weights on the potentially dominant terms in the two covariances smaller.

Of course we hasten to add that we have explored only one type of robustness. There may be doubts in the mind of the representative agent about the specification of the dynamics of the VAR, about the specification of the dynamics of output, even about trends, and so on. We plan to do much more exploration of robustness effects in future research.

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VI. Summary, Conclusions, and Suggestions for Future Research.

In this paper, we focus on the relationship between the price level and output at business cycle frequencies. In particular, we are interested in characterizing two contemporaneous correlations: one is the relationship between the price level and output and the other is the relationship between the inflation rate and output. Because the inflation rate is the rate of change in the price level, the two facts convey some underlying feature of the economy. In our case, the evidence consistently points to the price level being countercyclical while the inflation rate is either procyclical or acyclical, depending on what method we use to filter out the low-frequency component.

One thing we want to do is to develop a methodology to assess how much confidence we should put in a "stylized fact" in a data disciplined way. In particular, our methodology is proposed as a data disciplined way to assess the amount of uncertainty surrounding a "stylized fact" that takes into account, not only estimation uncertainty but also model uncertainty. We illustrated our methodology with assessing the uncertainty surrounding the stylized fact that the price level is countercyclical and the rate of inflation is procyclical using U.S. data under two methods of filtering, e.g. difference stationary and trend stationary (Nelson and Plosser (1982)). We found a rather high level of uncertainty in our illustrative example. This should give pause to any theorist who wants to take a strong stand on which model or theory to work with based on this particular "stylized fact."

We see a promising and useful strand of future research that extends our methodology to combined evidence from more than one country's data. For example, the recent paper by Konstantakopoulou et al. (2009), reports results on the correlation of filtered real output and filtered price level and the correlation of real output and rate of inflation for many countries and several methods of filtering out the low-frequency component. The countercylicality of the price level and the procyclicality of the rate of inflation appear to be quite robust. However, one must be wary of thinking that that each country's evidence represents independent support of this particular "stylized fact". This is so because the business cycles of these 9 OECD countries are quite likely to be cross correlated and this cross correlation must be taken into account when applying the bootstrap (Efron (1982), Efron and Tibshirani (1986)) to compute the probability of the cyclicality pattern of price level and inflation under scrutiny here.

The main contribution of this paper is to specify a model economy in which rational inattention is present and one in which robust control theory is applied. One goal is to quantitatively assess whether the model with a form of rational inattention can account for the phase shift in the price that is embodied in the two unconditional correlation coefficients. The idea is pretty straightforward. The quantitative results indicate how important price flexibility is to accounting for the negative correlation between the price level and output. In many model economies, consumers are forward-looking, and when combined with persistent output shocks and central bank operating rules, the quantitative analysis indicate that both the price level and the inflation rate are countercyclical. Following the price expectations approach formulated in work by Branch and Evans, de Grauwe, and Massaro, we propose a model economy that can account for the pair of correlations observed; namely, that in a difference-stationary setting, the price level is countercyclical and the inflation rate is acyclical. The key is that price expectations are not rational, thereby imparting a price stickiness. Our numerical results demonstrate that by putting enough weight on backward-looking price expectations, we can induce the phase shift in the price level that is consistent with the observed pattern of correlations. In a forward-looking rational expectations model economy, we find that movements in the price level adjust quantitatively so that the correlation between the price level and output and the inflation rate and output are of the same sign. In an unabridged version of this paper (Brock and Haslag (2014)), we considered a few modifications to the model economy, including money supply rules and persistence in the output growth.

In addition, we apply robust control theory, deriving sufficient conditions such that the price level is countercyclical and the inflation rate is procyclical. In this perturbation approach, we focus on fluctuations about trend where we assume rational expectations on the longer term trend dynamics but back off from rational expectations on the shorter term fluctuations about trends. Not only does this approach introduce some novelty in the study of a well-known "textbook" received MIUF model, it also helps us uncover forces that are not "sticky price" forces that are consistent with stylized facts where it seems that a consensus has formed that "sticky prices" need to be introduced in order to match these stylized facts.

Other directions of future research that we see as potentially fruitful are (i) developing extensions of the simple methods of our paper towards something like a "sturdy" reporting of stylized facts rather in the style of the "sturdy econometrics" advocated by Leamer (1994) and extension and adaptations of the simple methods illustrated in our paper towards Bayesian Model Uncertainty methods of reporting better measures of uncertainty when reporting a "stylized fact" than the more usual reporting of parameter estimates with their corresponding standard errors.

Appendix

Proof Proposition 2:

Assume that the expected value of the product of the price level and output is constant over time. We derive the following condition: $E[p_t y_t] = \{[b_1 p_{t-1} + e_t][a_1 y_{t-1} + u_t]\}$. After collecting terms and simplifying, we can write this as $[py] = \frac{E[eu]}{1-a_1 b_1}$. With $E[eu] < 0$, it follows that $E[py] < 0$ iff $a_1 b_1 < 1$.

Similarly, $E[\pi y] = \{[b_1 p_{t-1} + e_t - p_{t-1}][a_1 y_{t-1} + u_t]\}$. After collecting terms and simplifying, we get $[\pi y] = \frac{E[eu](1-a_1)}{1-a_1 b_1}$. With $E[eu] < 0$, and with $a_1 b_1 < 1$, it follows that $a_1 > 1$ for $E[\pi y] > 0$.■

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Table 1

Summary statistics for Cyclical Components

Variable	Mean	Std Dev	Cross-correlation with y
HP y	2.65e-04	0.017	1.0
HP p	3.2e-05	0.01	-0.224
HP π	1.22e-04	0.004	0.364
DS y		0.0098	1.0
DS p		0.0067	-0.145
DS π		0.0043	-0.013

Table 2

Coefficient Estimates HP Filter—Single Equation Approach

	Coefficient Estimates	Stnd Errors
const	**-3.46 e-06**	**2.07 e-04**
p_{t-1}	**1.2457*** **	**0.0634**
p_{t-2}	**-0.2975*** **	**0.1015**
p_{t-3}	**0.0942**	**0.1008**
p_{t-4}	**-0.2393*** **	**0.061**

	Coefficient Estimates	Stnd Errors
const	**5.37 e-05**	**5.31 e-04**
y_{t-1}	**1.0825*** **	**0.0636**
y_{t-2}	**-0.1161**	**0.0947**
y_{t-3}	**-0.2243*** **	**0.0638**

Table 3

Estimated Coefficients for VAR—HP Filtered Data

Price Level Equation:		
Variable	Estimated Coefficient	Standard Error
constant	1.4692e-05	0.0002
p_{t-1}^{C}	1.4192***	0.0546
p_{t-2}^{C}	-0.5337***	0.0559
y_{t-1}^{C}	0.0471*	0.0257
y_{t-2}^{C}	-0.0166	0.0253

Output Equation:		
Variable	Estimated Coefficient	Standard Error
constant	0.0001	0.0005
p_{t-1}^{C}	0.0241	0.1288
p_{t-2}^{C}	-0.2735**	0.1318
y_{t-1}^{C}	1.0762***	0.0606
y_{t-2}^{C}	-0.3372***	0.0597

Table 4

Estimated Coefficients for VAR—1st Differenced Data

Price Level Equation:		
Variable	Estimated Coefficient	Standard Error
constant	4.29 e-04	5.2 e-04
p_{t-1}^{C}	0.611***	0.0627
p_{t-2}^{C}	0.2314***	0.0628
y_{t-1}^{C}	0.0566**	0.0273
y_{t-2}^{C}	0.0414	0.0275

Output Equation:		
Variable	Estimated Coefficient	Standard Error
constant	0.0072***	0.0012
p_{t-1}^{C}	0.0356	0.1482
p_{t-2}^{C}	-0.2615*	0.1485
y_{t-1}^{C}	0.2904***	0.0645
y_{t-2}^{C}	0.0666	0.0649

Legend: *** indicates significant at 1% level

** indicates significant at 5% level

* indicates significant at 10% levels

Table 5

Simulated and actual values for

difference-stationary economies

(Baseline parameter settings)

	Actual	Simulated -- Simple	Simulated – Distributed Lag
Standard deviation			
Price level	0.0067	0.0085	0.0081
Output	0.0098	0.0194	0.0194
Inflation rate	0.0043	0.0108	0.0108
Cross-correlation			
$\boldsymbol{\rho}(p, y)$	-0.13	-0.145	-0.123
$\boldsymbol{\rho}(\pi, y)$	0.03	-0.013	-0.022

Figure 1

Frequency distribution of

$\hat{\rho}\left(py\right)<0,\hat{\rho}\left[(\Delta p)y\right]>0\,|\,HP$ **Single Equation Approach**

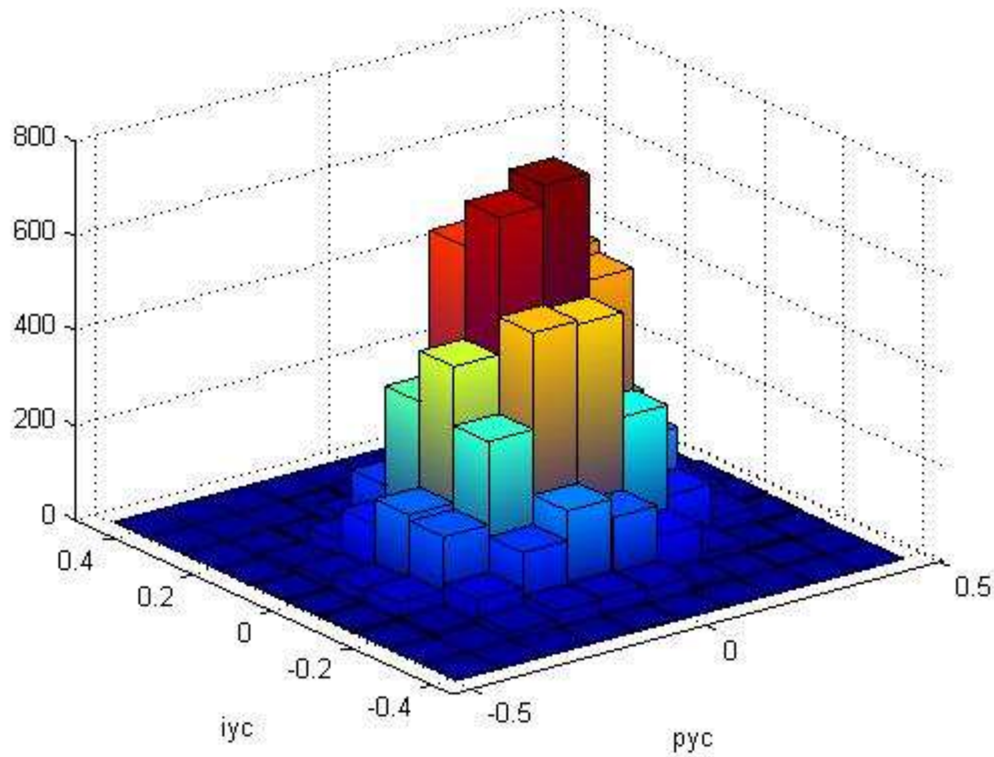

Figure 2

Frequency distribution of

$\hat{\rho}\left(py\right) < 0, \hat{\rho}\left[(\Delta p)y\right] > 0 \,|\, DS$ **: Single Equation Approach**

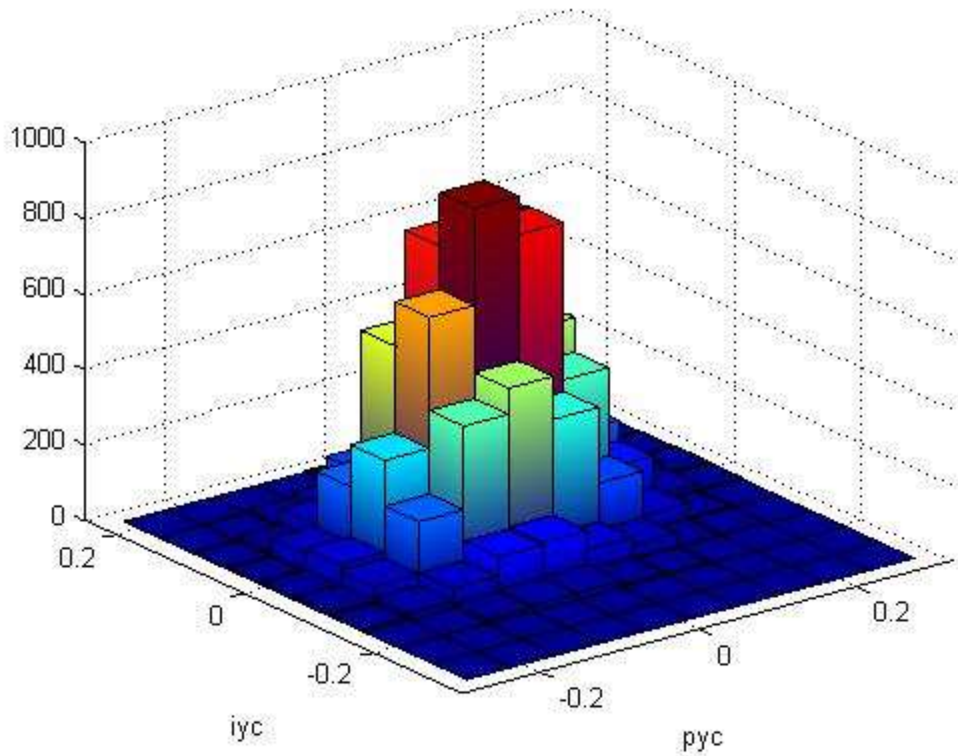

Figure 3

Frequency distribution of

$\hat{E}py < 0, \hat{E}(\Delta p)y > 0 \mid HP$ **VAR Approach**

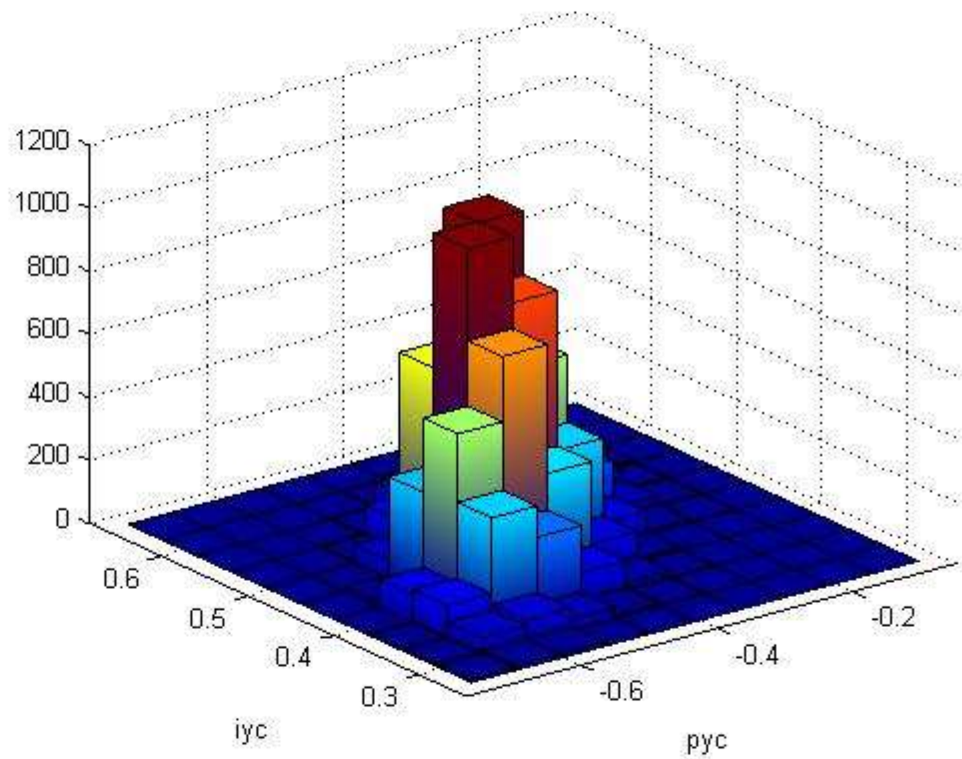

Figure 4

Frequency distribution of

$\hat{E}py < 0, \hat{E}(\Delta p)y > 0 \mid DS$ **VAR Approach**