Commitment contracts*

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Abstract

We analyze a consumption-saving problem in which time-inconsistent preferences generate demand for commitment, but uncertainty about future consumption needs generates demand for flexibility. We characterize in a standard contracting framework the circumstances under which this combination is possible, in the sense that a *commitment contract* exists that implements the desired state-contingent consumption plan, thus offering both commitment and flexibility. The key condition that we identify is a *preference reversal* condition: high desired consumption today should be associated with low marginal utility at future dates. We argue that there are conditions under which this preference reversal condition is naturally satisfied. The key insight of our paper is that time-inconsistent preferences effectively turn a single-agent contracting problem into a multi-agent mechanism design problem, because the agent's different selves have different preferences but share some information.

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1 Introduction

Preferences with hyperbolic time discounting, introduced by Strotz (1956), are widely used to model individual behavior in a variety of settings.[1] In his original article, Strotz observed that hyperbolic discounting generates demand for *commitment*.[2] But in addition to commitment, individuals value the *flexibility* to respond to economic shocks. For example, an individual is likely to be uncertain about his future consumption needs. In such cases, an individual will be reluctant to commit to future consumption levels that are state-independent. In other words, there is a tension between commitment and flexibility (Amador et al 2006). In this paper, we analyze the extent to which commitment and flexibility can be successfully combined.

In our setting, an individual would like to commit at date 0 to a consumption plan that may depend on unverifiable shocks that are realized in the future. To this end, the individual can enter into a *commitment contract* with the aim of implementing self 0's[3] desired consumption plan. The key contracting difficulty is that the shocks are realized only after the contract is signed, and since they are unverifiable, the contract cannot directly condition the individual's consumption on their realization. Rather, a commitment contract must provide the individual both with flexibility to respond to these shocks, and with incentives to adhere to self 0's desired consumption plan.

Our results characterize conditions under which the tension between commitment and flexibility can be at least partially resolved. Our key condition is a *preference reversal* condition, which loosely speaking states that desired consumption at date 1 is negatively correlated with marginal utility (MU) at date 2. When this condition is satisfied, it is possible to design a commitment contract in which an individual is deterred from overconsumption at date 1 by the prospect that future selves will engage in more costly forms of overconsumption at subsequent dates.

The key insight of our paper is that time-inconsistent preferences are not only the source of the individual's commitment problem, but also allow its possible solution. With time-inconsistent preferences, the individual's different selves have different preferences but still share knowledge of the shock realizations. This opens up the possibility of later selves punishing prior selves for

[1] See Frederick, Loewenstein and O'Donoghue (2002) for a review of models of time discounting. Applications of hyperbolic discounting include consumer finance (e.g., Laibson 1996 on savings behavior in general; Laibson, Repetto, and Tobacman 1998 on retirement planning; DellaVigna and Malmendier 2004 and Shui and Ausubel 2004 on credit card usage; Skiba and Tobacman 2008 on payday lending; and Jackson 1986 on bankruptcy law), asset pricing (e.g., Luttmer and Mariotti 2003), and procrastination (e.g., O'Donoghue and Rabin 1999a, 1999b, 2001).

[2] See Ariely and Wertenbroch (2002) for direct evidence of demand for commitment.

[3] We follow the literature and refer to the individual at date t as *self* t.

deviating from self 0's desired consumption plan, which would be impossible if their preferences were the same. In essence, time-inconsistent preferences turn a single-agent contracting problem into a multi-agent mechanism design problem. As is well known from the implementation theory literature,[4] this can dramatically expand the set of outcomes that are attainable in equilibrium.

1.1 Illustrative examples

To illustrate our main results, consider the following examples. In all examples, we consider an individual with logarithmic utility; three consumption dates; and quasi-hyperbolic time preferences over these dates, with a hyperbolic discount factor of $\beta = \frac{1}{2}$ and no regular time discounting.

Example 1: An individual sequentially encounters consumption opportunities. Consumption opportunities may be good or bad. If consumption opportunities are good at date t, the individual's instantaneous utility from consumption c_t at date t is $\frac{3}{2} \log c_t$, whereas if they are bad, instantaneous utility is simply $\log c_t$. There are a limited number of good consumption opportunities, so the conditional probability of finding a good opportunity at date 2 is lowered by finding a good opportunity at date 1. For simplicity, we focus here on the extreme case in which the individual encounters a good consumption opportunity at exactly one of dates 1 and 2. The quality of consumption opportunities is observed by the individual, but is unverifiable.

The individual's total endowment is $3\frac{1}{2}$. So in this example, the individual would like to commit at date 0 to a consumption path $(c_1, c_2, c_3) = \left(\frac{3}{2}, 1, 1\right)$ if he finds a good opportunity at date 1, and to a consumption path $(c_1, c_2, c_3) = \left(1, \frac{3}{2}, 1\right)$ otherwise. However, hyperbolic discounting means that, at date 1, the individual prefers consumption $\left(\frac{3}{2}, 1, 1\right)$ even if the date 1 opportunity is bad.[5]

Suppose, however, that the individual arranges his financial affairs so that self 1 picks between consumption levels 1 and $\frac{3}{2}$; and that following the higher consumption choice $\frac{3}{2}$, self 2 picks between $(c_2, c_3) = (1, 1)$ and $\left(\frac{3}{2}, \frac{1}{3}\right)$, while following the lower consumption choice 1, subsequent consumption is simply $(c_2, c_3) = \left(\frac{3}{2}, 1\right)$.[6]

This arrangement allows the individual to commit to self 0's desired consumption plan, as

[4] See Maskin and Sjöström (2002), Palfrey (2002), and Serrano (2004) for surveys of implementation theory.

[5] Formally, $\log \frac{3}{2} + \beta \frac{3}{2} \log 1 + \beta \log 1 > \log 1 + \beta \frac{3}{2} \log \frac{3}{2} + \beta \log 1$.

[6] There are a variety of ways to implement these consumption paths. For example, at date 0, the individual can deposit 1 each in one- and three-period savings accounts and $\frac{3}{2}$ in a two-period savings account. An amount $\frac{1}{2}$ can be withdrawn early from the date 2 account, without penalty. Furthermore, if this date 1 early withdrawal is made, at date 2 an additional $\frac{1}{2}$ can be withdrawn early from the date 3 account, but this second withdrawal carries a penalty of $\frac{1}{6}$, so that the date 3 account is reduced by $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$. (The feature that the possibility of the second early withdrawal is triggered by the first early withdrawal is not essential, and can be dispensed with.)

follows. Working backwards, we first consider self 2. If self 1 chose high consumption $\frac{3}{2}$ because the date 1 consumption opportunity was good, then the date 2 opportunity is bad, and self 2 chooses $(1,1)$. If instead self 1 overconsumed and chose consumption $\frac{3}{2}$ when the date 1 opportunity was bad, then the date 2 opportunity is good, and self 2 chooses $\left(\frac{3}{2},\frac{1}{3}\right)$, with less total consumption.[7]

Next consider self 1. If the date 1 opportunity is bad, self 1 understands that if he chooses high consumption, then self 2 will pick $\left(\frac{3}{2},\frac{1}{3}\right)$, leaving self 3 with very little consumption. This outcome is unattractive enough to deter self 1 from choosing high consumption if the date 1 opportunity is bad.[8] If, however, the date 1 opportunity is good, then self 1 can choose high consumption, secure in the knowledge that self 2 will face a bad opportunity at date 2 and so choose $(1,1)$.[9]

Example 2: An individual may learn at date 1 that he will face an essential expenditure of $\frac{1}{2}$ at date 2 (for example, a major home repair). This expenditure does not contribute towards utility: formally, date 2 utility is $\ln\left(c_2 - \frac{1}{2}\right)$. The individual's total endowment is 3. So in this example, the individual would like to commit at date 0 to a consumption path $(1,1,1)$ if the expenditure is not required; but if the expenditure is required, he would like to preemptively start saving at date 1 to partially fund the date 2 expenditure, and consume $\left(\frac{5}{6},\frac{4}{3},\frac{5}{6}\right)$. However, and as in Example 1, hyperbolic discounting means that, at date 1, the individual prefers not to save, even though he knows he faces the expenditure at date 2.[10]

Suppose, however, that the individual arranges his financial affairs so that self 1 picks between consumption levels $\frac{5}{6}$ and 1; and that following the higher consumption choice 1, self 2 picks between $(c_2,c_3)=(1,1)$ and $\left(\frac{4}{3},\frac{1}{2}\right)$, while following the lower consumption choice $\frac{5}{6}$, subsequent consumption is simply $(c_2,c_3)=\left(\frac{4}{3},\frac{5}{6}\right)$.[11]

By a parallel argument to Example 1, this arrangement allows the individual to commit to self 0's desired consumption plan. If self 1 chooses high consumption 1, then self 2 chooses $(1,1)$ if there is no date 2 expenditure, but chooses $\left(\frac{4}{3},\frac{1}{2}\right)$, with less total consumption, if there is a date

[7]Formally, $\frac{3}{2}\log\frac{3}{2}+\beta\log\frac{1}{3}=\beta\log\left(\frac{3^3}{2^3}\frac{1}{3}\right)>\log 1=\frac{3}{2}\log 1+\beta\log 1$ (date 2 consumption opportunity is good) and $\log 1+\beta\log 1>\log\frac{3}{2}+\beta\log\frac{1}{3}=\log\frac{3}{2\sqrt{3}}$ (date 2 consumption opportunity is bad).

[8]Formally, $\log 1+\beta\frac{3}{2}\log\frac{3}{2}+\beta\log 1>\log\frac{3}{2}+\beta\frac{3}{2}\log\frac{3}{2}+\beta\log\frac{1}{3}$ since $1>\frac{3}{2\sqrt{3}}$.

[9]Formally, $\frac{3}{2}\log\frac{3}{2}+\beta\log 1+\beta\log 1>\frac{3}{2}\log 1+\beta\log\frac{3}{2}+\beta\log 1$.

[10]Formally, $\log 1+\beta\log\left(1-\frac{1}{2}\right)+\beta\log 1>\log\frac{5}{6}+\beta\log\left(\frac{4}{3}-\frac{1}{2}\right)+\beta\log\frac{5}{6}$ since $\left(\frac{1}{2}\right)^{\frac{1}{2}}>\frac{5}{6}\frac{5}{6}$.

[11]There are a variety of ways to implement these consumption paths. For example, at date 0, the individual can deposit $\frac{5}{6}$, $\frac{7}{6}$ and 1 into one-, two- and three-period savings accounts respectively. An amount $\frac{1}{6}$ can be withdrawn one period early, and without penalty, against either the date 2 account, or the date 3 account, but not both. If the early withdrawal is made at date 1, a second early withdrawal of $\frac{1}{3}$ can be made at date 2, but this carries a penalty of $\frac{1}{6}$, so that the date 3 account is reduced by $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$.

2 expenditure.[12] Given this, when self 1 foresees the need for the date 2 expenditure, he chooses low consumption $\frac{5}{6}$, but when he foresees that there is no date 2 expenditure, he chooses high consumption 1.[13]

Example 3: In the two examples above the individual is able to commit to self 0's desired consumption plan. We now consider an example where this is not the case. We return to the framework of Example 1, but assume now that if the individual finds a good consumption opportunity at date 1, the opportunity will also be available at date 2; but if the individual encounters a bad opportunity at date 1, the opportunity at date 2 is also bad. The individual's total endowment is now 4. Consequently, self 0 would like to commit to a consumption path $\left(\frac{3}{2}, \frac{3}{2}, 1\right)$ if he finds the good opportunity, but to $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ if he finds the bad opportunity. As in the previous examples, hyperbolic discounting causes self 1 to prefer the high date-1 consumption path $\left(\frac{3}{2}, \frac{3}{2}, 1\right)$ regardless of the quality of consumption opportunities.[14]

The key to attaining commitment in the previous examples is that it was possible to offer self 2 an alternative consumption path after self 1 picks high consumption (here, $\frac{3}{2}$) that self 2 prefers to $\left(\frac{3}{2}, 1\right)$ if and only if self 1 overconsumed, and that also hurts self 1. But this is impossible in this example, as follows. First, the only consumption paths that raise self 2's utility while lowering self 1's utility are those that increase date 2 consumption and reduce date 3 consumption. But second, if self 2 prefers such a path when the date 2 opportunity is bad—which is when self 1 has overconsumed—then he also prefers such a path when the date 2 opportunity is good. Consequently, it is impossible to impose a state-contingent punishment on self 1 for choosing high consumption. This point is formalized in Lemma 2 below; this simple result is in many ways the key to our analysis.

1.2 Discussion

The above examples illustrate the importance of a condition we term *preference reversal*. In Examples 1 and 2, when the individual desires high consumption at date 1, his subsequent MU at

[12]Formally, $\log\left(\frac{4}{3} - \frac{1}{2}\right) + \beta \log \frac{1}{2} > \log\left(1 - \frac{1}{2}\right) + \beta \log 1$ since $\frac{5}{6}\left(\frac{1}{2}\right)^{\frac{1}{2}} > \frac{1}{2}$ (expenditure of $\frac{1}{2}$ at date 2), and $\log 1 + \beta \log 1 > \log \frac{4}{3} + \beta \log \frac{1}{2}$ since $1 > \frac{4}{3}\left(\frac{1}{2}\right)^{\frac{1}{2}}$ (no expenditure at date 2).

[13]Formally, $\log \frac{5}{6} + \beta \log\left(\frac{4}{3} - \frac{1}{2}\right) + \beta \log \frac{5}{6} > \log 1 + \beta \log\left(\frac{4}{3} - \frac{1}{2}\right) + \beta \log \frac{1}{2}$ since $\left(\frac{5}{6}\right)^{\frac{3}{2}} > \left(\frac{1}{2}\right)^{\frac{1}{2}}$ (date 2 expenditure anticipated), and $\log 1 + \beta \log 1 + \beta \log 1 > \log \frac{5}{6} + \beta \log \frac{4}{3} + \beta \log \frac{5}{6}$ since $1 > \left(\frac{5}{6}\right)^{\frac{3}{2}}\left(\frac{4}{3}\right)^{\frac{1}{2}}$ (no date 2 expenditure anticipated).

[14]Formally, $\log \frac{3}{2} + \beta \log \frac{3}{2} + \beta \log 1 > \log \frac{4}{3} + \beta \log \frac{4}{3} + \beta \log \frac{4}{3}$ since $\left(\frac{3}{2}\right)^{\frac{3}{2}} > \left(\frac{4}{3}\right)^{2}$.

date 2 is low. In contrast, this is not the case in Example 3; there, high desired consumption at date 1 is followed by high MU at date 2. This distinction is at the heart of our formal analysis below, which explores the importance of preference reversal for an individual's ability to resolve the tension between commitment and flexibility.

As illustrated by Examples 1 and 2, there are at least two economic forces that lead the preference reversal condition to be naturally satisfied. The force operating in Example 2 is that if the individual can forecast that future MU will be low, this leads him to increase consumption in advance. Hence, ceteris paribus, low MU at date 2 implies high desired consumption at date 1. Of course, the reverse implication does not hold, and accordingly, there are certainly many environments where preference reversal is not satisfied. The force operating in Example 1 is that if good consumption opportunities are limited, encountering a good opportunity at date 1 simultaneously increases desired consumption at date 1, and reduces the probability of encountering another good opportunity at date 2.

In the above examples, we focus directly on consumption paths, rather than on the specific contractual arrangements that implement these consumption paths. However, under many circumstances the consumption paths we characterize as optimal can be implemented using standard consumer financial products;[15] see, for example, footnotes 6 and 11 above. Qualitatively, the key feature of contracts that allow self 0 to commit to his preferred consumption path is *excess flexibility*: in Examples 1 and 2, self 2 is given the flexibility to increase his consumption beyond self 0's desired consumption, and this flexibility is not used in equilibrium.

In this paper, we focus on one particular form of time-inconsistent preferences, namely the present-bias generated by hyperbolic discounting. However, our key insight—that time-inconsistent preferences turn a single-agent contracting problem into a multi-agent mechanism design problem—is more widely applicable. In particular, consider any source of time-inconsistent preferences that an individual is self-aware enough to anticipate. For example, an individual may understand today that, in the future, he will misinterpret the relevance of a small number of data points. Just as in the current setting, he can potentially commit to a course of action that avoids this bias, while at the same time maintaining flexibility to respond to shocks.

[15] Formal results are contained in an earlier version of this paper.

2 Related literature

Central to our analysis is the idea that the commitment contract sets up a game between selves. O'Donoghue and Rabin (1999a) demonstrate that this inter-self game has some surprising properties; for example, "sophistication" may worsen self-control problems relative to "naïveté."[16] This previous paper focuses on a setting in which an individual must take an action exactly once, and takes the costs and rewards of this action as exogenously given. The basic commitment problem confronted by an individual in our paper is covered by their analysis: for instance, in Example 1, the individual can take an immediate reward of $\ln \frac{3}{2} - \ln 1$ at date 1, with the cost of this reward deferred until the future. Our main results explore whether it is possible to design a contract (which determines costs and rewards) that deters the individual from taking the immediate reward at date 1. When such a contract exists, it gives the individual the possibility of taking two rewards. Although such a contract falls outside O'Donoghue and Rabin's framework, because the number of actions is not fixed, the basic flavor of our contract is related to their Example 4, in which self 1 is deterred from taking the immediate reward by the knowledge that, if he does so, self 2 will then also take an early reward.[17]

Our paper is closely related to Amador et al (2006). Like us, they study a hyperbolic individual who is hit by unverifiable taste shocks, but consider only a two-period version of the problem. This restriction immediately rules out the possibility of self 2 imposing a state-contingent punishment on self 1 for deviating—a key feature of our setting—because with two periods self 1 is effectively the only strategic agent.[18] Consequently, the only way to deter self 1 from deviating is to distort consumption in at least some states; the authors characterize the least costly way to do so.

Like Amador et al (2006), DellaVigna and Malmendier (2004) restrict attention to two periods, again ruling out the possibility of self 2 punishing self 1. Moreover, in their setting self 1 faces a binary choice (e.g., whether or not to go to the gym) and consequently a contract exists under which self 1 acts exactly as self 0 desires. The authors characterize the contract that maximizes the

[16] Following the literature, *sophistication* refers to the case in which self t correctly understands that selves $s > t$ have present-biased preferences. In contrast, *naïveté* refers to the case in which self t incorrectly believes that selves $s > t$ are not present-biased. See subsection 6.2 for a discussion of partial naïveté.

[17] In addition, O'Donoghue and Rabin observe that present-biased preferences often violate independence of irrelevant alternatives (their Proposition 5), a point they refer to as a "smoking gun." This point—that actions never taken on the equilibrium path may nonetheless affect equilibrium decisions—is central to the design of contracts in our paper.

[18] Amador et al (2003) extend the analysis to three or more periods. They assume that shocks are independent across dates; see subsection 4.2 below.

profits of a monopolist counterparty facing a partially naïve agent (subsection 6.2 discusses partial naïveté). In particular, they characterize the combination of flat upfront fees and per-usage fees in the profit-maximizing contract.[19]

O'Donoghue and Rabin (1999b) analyze optimal contracts for procrastinators in a multi-period environment, where the socially efficient date at which a task should be performed is random. They explicitly rule out the use of contracts that induce an agent to reveal his type, which are the focus of our paper. As they observe, this restriction is without loss of generality in the main case they study, that of agents who are completely naïve about their future preferences. By contrast, we study sophisticated agents (again, see subsection 6.2 for a discussion of partial naïveté).

While we examine the use of *external* commitment devices, such as contracts, other research considers what might be termed *internal* commitment devices. Bernheim, Ray, and Yeltekin (2013) and Krusell and Smith (2003) consider deterministic models in which an individual is infinitely lived, and show that Markov-perfect equilibria exist in which he gains some commitment ability from the fact that deviations will cause future selves to punish him. Carrillo and Mariotti (2000) and Benabou and Tirole (e.g., 2002, 2004) consider models in which an individual can commit his future selves to some action by manipulating their beliefs, respectively, through the extent of his own information acquisition, through direct distortion of beliefs, or through self-signalling.

3 Model

At each of dates $t = 1, 2, 3$, a single agent consumes c_t. At dates 1 and 2 his contemporaneous utility depends on state variables $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$, realized at dates 1 and 2 respectively, and is given by $u_1(c_1; \theta_1)$ and $u_2(c_2; \theta_2)$. Without loss, we assume $\Pr(\theta_t) > 0$ for all $\theta_t \in \Theta_t$. We write $\Theta \equiv \Theta_1 \times \Theta_2$, and assume Θ is compact. At date 3 his contemporaneous utility is $u_3(c_3)$. We note that in a setting with more than 3 dates, $u_3(c_3)$ can be interpreted more generally as the expected future discounted utility of an agent inheriting wealth c_3.

The agent discounts the future quasi-hyperbolically: for $t = 0, 1, 2$, self t's intertemporal utility function is $U^t \equiv u_t + \beta \sum_{s=t+1}^{3} u_s$, so that $\beta \in (0, 1)$ is the hyperbolic discount factor. Note that we normalize the regular, i.e., non-hyperbolic, discount rate to zero; likewise, the risk-free interest rate is zero. Finally, the agent is self-aware (i.e., sophisticated), in the sense that at each date,

[19]Similarly, Eliaz and Spiegler (2006) analyze profit maximization by a monopolist who deals with a population of time-inconsistent individuals who differ in their degree of sophistication (see Section 6.2 below).

he correctly anticipates his preferences at future dates (see the discussion in subsection 6.2). The contemporaneous utility functions u_t are strictly increasing and strictly concave in c_t. Finally, we write $V^t = \sum_{s=t}^{3} u_s$ for the agent's utility under exponential discounting.

We write $C(\theta_1, \theta_2) = (C_1(\theta_1), C_2(\theta_1, \theta_2), C_3(\theta_1, \theta_2))$ for a contract, consisting of a sequence of state-contingent consumption levels. Clearly date 1 consumption cannot depend on the date 2 state θ_2; in contrast, date 2 and 3 consumption may depend on both the states θ_1 and θ_2.

The total resources available for the agent to consume across the three dates is W, and is state-independent.[20] This could either represent an initial endowment of the agent, or the present value of future income. Since our main focus is on the effect of hyperbolic discounting on intertemporal efficiency, not its effect on insurance across states, we rule out transfers across states, so that the following resource constraint must hold: for all $(\theta_1, \theta_2) \in \Theta$,

$$C_1(\theta_1) + C_2(\theta_1, \theta_2) + C_3(\theta_1, \theta_2) \leq W. \tag{RC}$$

This assumption also facilitates comparison with the existing literature, which like us focuses on intertemporal efficiency.[21] Moreover, it would be hard—and sometimes impossible—to insure the agent if self 0 had private information about the relative probability of different states.[22] Note that RC covers even zero-probability state realizations (θ_1, θ_2), a point we discuss below.

3.1 Incentive constraints

The central friction in our framework is that the states θ_1 and θ_2 are unverifiable. Unverifiability of the state induces a potential trade-off between commitment and flexibility for the agent, as discussed in Amador et al (2006). Unverifiability means that a contract must satisfy the following incentive compatibility (IC) constraints, which ensure that the agent does not gain by misrepresenting the

[20] However, the additive shock parameterization of our environment that we introduce below is equivalent to allowing W to vary in an unverifiable way across states.

[21] Amador, Werning, and Angeletos (2006) rule out transfers across states. O'Donoghue and Rabin (1999b) and DellaVigna and Malmendier (2004) study risk-neutral agents, and so insurance across states is not a concern.

[22] Note that private information about the distribution of θ_1 would not affect our analysis, which characterizes when intertemporal efficiency is possible.

state. At date 2, the IC constraints are: for all $\theta_1 \in \Theta_1$ and $\theta_2, \tilde{\theta}_2 \in \Theta_2$,[23]

$$U^2 \left(C \left(\theta_1, \theta_2 \right); \theta_2 \right) \geq U^2 \left(C \left(\theta_1, \tilde{\theta}_2 \right); \theta_2 \right). \tag{IC$_2$}$$

At date 1, the IC constraints are: for all $\theta_1, \tilde{\theta}_1 \in \Theta_1$,

$$E_{\theta_2} \left[U^1 \left(C \left(\theta_1, \theta_2 \right); \theta_1, \theta_2 \right) | \theta_1 \right] \geq E_{\theta_2} \left[U^1 \left(C \left(\tilde{\theta}_1, \theta_2 \right); \theta_1, \theta_2 \right) | \theta_1 \right]. \tag{IC$_1$}$$

We say that a contract C is *feasible* if it satisfies IC$_1$, IC$_2$ and RC.

3.2 Self 0's problem

Self 0's problem is to choose a contract C to solve

$$\max_{C \text{ s.t. RC, IC}_1\text{, IC}_2} E_{\theta_1, \theta_2} \left[U^0 \left(C \left(\theta_1, \theta_2 \right); \theta_1, \theta_2 \right) \right].$$

To characterize the agent's ability to successfully combine commitment with flexibility, our results compare the solution to this problem to two more relaxed problems, namely:

Problem I, in which neither IC$_1$ nor IC$_2$ is imposed.

Problem II, in which IC$_1$ is not imposed, but IC$_2$ is imposed.

3.3 Preference assumptions and a preliminary result

Before formally stating assumptions on how the state (θ_1, θ_2) affects preferences, it is useful to give two leading examples:

Example, multiplicative shocks: $u_t (c_t; \theta_t) = \theta_t u (c_t)$ for $t = 1, 2$ and $\theta_t \in \Theta_t$, where u has non-increasing absolute risk aversion (NIARA).[24]

[23] In principle, a contract could also condition on self 2's report of the date 1 state, say θ_{21}, so that the contract would take the form $C(\theta_1, \theta_2, \theta_{21})$. However, given that self 2's preferences depend only on θ_2, and are independent of state θ_1, the only way in which a contract with $C(\theta_1, \theta_2, \theta_{21}) \neq C\left(\theta_1, \theta_2, \tilde{\theta}_{21}\right)$ could be incentive compatible is if self 2 is indifferent between $C(\theta_1, \theta_2, \theta_{21})$ and $C\left(\theta_1, \theta_2, \tilde{\theta}_{21}\right)$, and resolves the indifference differently depending on the true realization of the date 1 state. We assume throughout that self 2 resolves indifference in the same way in all states, and accordingly, write the contract and ICs as in the main text. Note that this assumption only affects the analysis of subsection 4.2, which deals with zero correlation between states θ_1 and θ_2. Moreover, in a discussion of the same issue, Amador et al (2003) show that indifference is only possible in a finite number of states, so that if there are a continuum of states, as in Section 4.4, this assumption is without loss.

[24] Amador et al (2006) focus completely on this case.

Example, additive shocks: $u_t(c_t;\theta_t) = u(c_t+\theta_t)$ for $t=1,2$ and $\theta_t \in \Theta_t$, where u has NIARA. This shock specification has a natural interpretation as either essential expenditure shocks (e.g., the agent must take his child to the doctor), or income shocks (e.g., the individual receives a bonus).

Motivated by these examples, we make Assumptions 1-4:

Assumption 1 *For any* $s,t \in \{1,2\}$, $\theta \in \Theta_t$ *and* $\tilde{\theta} \in \Theta_s$, *sign* $\left(u_t'(x;\theta) - u_s'\left(x;\tilde{\theta}\right)\right)$ *is independent of* x.

Assumption 1 simply says that states can be unambiguously ordered in terms of their impact on MU. Given the assumption, we write $u_t'(\cdot;\theta) \geq u_s'\left(\cdot;\tilde{\theta}\right)$ for the case in which $u_t'(x;\theta) \geq u_s'\left(x;\tilde{\theta}\right)$ for all x. Without loss, we assume $u_2'\left(\cdot;\tilde{\theta}_2\right) \neq u_2'(\cdot;\theta_2)$ whenever $\tilde{\theta}_2 \neq \theta_2$, and order Θ_2 so that $\tilde{\theta}_2 > \theta_2$ if and only if $u_2'\left(\cdot;\tilde{\theta}_2\right) > u_2'(\cdot;\theta_2)$. We write $\underline{\theta}_2$ and $\bar{\theta}_2$ for the minimal and maximal elements of Θ_2 under this ordering.

Assumption 2 *For* $t=1,2$ *and* $\theta_t \in \Theta_t$, *there exists* $\underline{c}_t$ *such that* $u_t'(c_t;\theta_t) \to \infty$ *as* $c_t \to \underline{c}_t$.

Assumption 2 is simply the standard Inada condition, and ensures solutions are always interior. The minimum consumption level $\underline{c}_t$ is potentially different from 0, reflecting the example of additive shocks given above.

Assumption 3 $u_3(c_3) \to -\infty$ *and* $u_3'(c_3) \to \infty$ *as* $c_3 \to 0$.

Assumption 3 guarantees that variation in contemporaneous date 3 utility is not overwhelmed by variation in contemporaneous utility at earlier dates; its main use is in Part (B) of Proposition 1. It is satisfied in both the multiplicative and additive examples above, if, for example, u_3 has a constant coefficient of relative risk aversion of 1 or above.

Assumption 4 *If* $u_t'(c_t;\theta) \geq \gamma u_s'\left(c_s;\tilde{\theta}\right)$ *for some* $s,t \in \{1,2\}$, c_s, $c_t \geq c_s$, $\theta \in \Theta_t$, $\tilde{\theta} \in \Theta_s$ *and* $\gamma \leq 1$, *then* $u_t'(c_t+x;\theta) \geq \gamma u_s'\left(c_s+x;\tilde{\theta}\right)$ *for all* $x>0$.

Assumption 4 is a relatively mild regularity condition, which is satisfied by both the multiplicative and additive shock examples (see appendix).

In addition to Assumptions 1-4, Part (B) of Proposition 1 and the analysis of economy B in Proposition 3 also make use of the following pair of relatively mild assumptions, which we state here for completeness. Both assumptions relate to the size of the shocks θ_1 and θ_2.

Assumption 5 *There exists* $\theta_1 \in \Theta_1$ *and* $\theta_2 \in \Theta_2$ *such that* $u_2'(\cdot;\theta_2) \geq u_1'(\cdot;\theta_1)$.

Assumption 6 *If* C *solves either Problem I or II,*

$$\max_{\theta_1,\tilde{\theta}_1 \in \Theta_1} \left| C_1(\theta_1) - C_1\left(\tilde{\theta}_1\right) \right| < \min_{\theta_1,\theta_2 \in \Theta_1 \times \Theta_2} C_3(\theta_1,\theta_2).$$

Assumption 5 simply says that MU does not strictly decrease from date 1 to 2. Assumption 6 is a relatively mild restriction on the strength of shocks relative to consumption-smoothing motives: ignoring incentive constraints, in self 0's preferred contract the variation in date 1 consumption is less than the minimum date 3 consumption.

Finally, we note the following preliminary result: the date 2 incentive constraints immediately imply that date 2 consumption $C_2(\theta_1,\cdot)$ must increase in date 2 MU. This monotonicity result is standard to mechanism design problems.[25]

Lemma 1 *If* $\tilde{\theta}_2 > \theta_2$ *and* C *satisfies* IC_2 *then* $C_2\left(\theta_1,\tilde{\theta}_2\right) \geq C_2(\theta_1,\theta_2)$ *for all* $\theta_1 \in \Theta_1$.

4 Analysis

In subsections 4.1-4.3 we assume that both Θ_1 and Θ_2 are binary; this assumption is relaxed in subsection 4.4.

4.1 Perfect correlation

We start by considering the case of perfect correlation, so that the date 2 state θ_2 is a deterministic function of θ_1. Proposition 1 below characterizes when there is a feasible contract that solves Problem I, i.e., when the agent can fully resolve the tension between commitment and flexibility.

The key condition identified by Proposition 1 is a *preference reversal* condition. Specifically, when hyperbolic discounting is severe (β low), there is a feasible contract that solves Problem I if and only if a desire for high date 1 consumption is followed by low date 2 MU, and vice versa.[26]

[25] See Lemma 2 of Myerson (1981); or Chapter 2.3 of Bolton and Dewatripont (2005).

[26] The preference reversal condition may remind readers of Maskin's (1999) monotonicity condition. However, while preference reversal may fail in our setting, monotonicity is trivially satisfied as long as some self's preferences differ across the two states. In our setting, the social choice rule of interest is $F(\theta_1,\theta_2) = C(\theta_1,\theta_2)$. This social choice rule is monotonic if and only if for all (θ_1,θ_2) and $(\theta_1',\theta_2') \neq (\theta_1,\theta_2)$, $U^t(C(\theta_1,\theta_2);\theta_1,\theta_2) \geq U^t(x;\theta_1,\theta_2)$ and $U^t(C(\theta_1,\theta_2);\theta_1',\theta_2') < U^t(x;\theta_1',\theta_2')$ for some self $t \in \{1,2,3\}$ (self 0 is non-strategic) and some $x \in \mathbb{R}^3$. As long as some self's preferences differ across the two states, this condition is satisfied

To state the formal result, let C^* be a solution of Problem I, and write $\bar{\theta}_1$ and $\underline{\theta}_1$ for the elements of Θ_1 such that $C_1^*\left(\bar{\theta}_1\right) \geq C_1^*\left(\underline{\theta}_1\right)$; note that Problem I is independent of β, and that date 1 consumption is the same in any solution. Since Θ_2 is binary, $\Theta_2 = \left\{\underline{\theta}_2, \bar{\theta}_2\right\}$.

Proposition 1 *If $C_1^*\left(\bar{\theta}_1\right) > C_1^*\left(\underline{\theta}_1\right)$ then:*
(A, No Preference Reversal): If $\Pr\left(\bar{\theta}_2|\bar{\theta}_1\right) = \Pr\left(\underline{\theta}_2|\underline{\theta}_1\right) = 1$, then for all β sufficiently small, no feasible contract solves Problem I.
(B, Preference reversal): If $\Pr\left(\bar{\theta}_2|\underline{\theta}_1\right) = \Pr\left(\underline{\theta}_2|\bar{\theta}_1\right) = 1$, then for all β sufficiently small,[27] there exists a feasible contract that solves Problem I.

In Part (A) of Proposition 1, by IC_1 self 1 is dissuaded from claiming high consumption $C_1\left(\bar{\theta}_1\right)$ in the low consumption state $\underline{\theta}_1$ if and only if

$$u_1\left(C_1\left(\underline{\theta}_1\right); \underline{\theta}_1\right) + \beta V^2\left(C\left(\underline{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) \geq u_1\left(C_1\left(\bar{\theta}_1\right); \underline{\theta}_1\right) + \beta V^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_2\right).$$

Consumption $C_1\left(\underline{\theta}_1\right)$, $C_1\left(\bar{\theta}_1\right)$ and $C\left(\underline{\theta}_1, \underline{\theta}_2\right)$ all lie on the equilibrium path, while consumption $C_2\left(\bar{\theta}_1, \underline{\theta}_2\right)$ and $C_3\left(\bar{\theta}_1, \underline{\theta}_2\right)$ lie off the equilibrium path. Consequently, when hyperbolic discounting is strong (β small), the only way for a solution to Problem I to satisfy the above incentive constraint is if $C_2\left(\bar{\theta}_1, \underline{\theta}_2\right)$ and $C_3\left(\bar{\theta}_1, \underline{\theta}_2\right)$ are such that $V^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_2\right)$ is very low. However, this is impossible. From Lemma 1, IC_2 imply that $C\left(\bar{\theta}_1, \underline{\theta}_2\right)$ must deliver less date 2 consumption than $C\left(\bar{\theta}_1, \bar{\theta}_2\right)$. Because self 2 needs to have the incentive to pick $C\left(\bar{\theta}_1, \underline{\theta}_2\right)$ over $C\left(\bar{\theta}_1, \bar{\theta}_2\right)$, this implies that $V^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_2\right)$ must be greater than $V^2\left(C\left(\bar{\theta}_1, \bar{\theta}_2\right); \underline{\theta}_2\right)$. Because $C\left(\bar{\theta}_1, \bar{\theta}_2\right)$ is on the equilibrium path, this establishes a lower bound on $V^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_2\right)$, and establishes Part (A).[28]

Summarizing, the above argument says that self 2 cannot be induced to impose an effective punishment on self 1 in state $\underline{\theta}_2$. This is the key observation behind our preference reversal condition, because when the low date-1 consumption state is followed by low date 2 MU-state $\underline{\theta}_2$, it is precisely in state $\underline{\theta}_2$ that a punishment is needed. Because of the centrality of this argument to our analysis, we state the following Lemma for use in subsequent results:

Lemma 2 *If C satisfies IC_2 then $V^2\left(C\left(\theta_1, \underline{\theta}_2\right); \underline{\theta}_2\right) \geq V^2\left(C\left(\theta_1, \tilde{\theta}_2\right); \underline{\theta}_2\right)$ for any $\tilde{\theta}_2 \in \Theta_2$.*

[27] Under additional preference assumptions, Part (B) can be established for all β. Details are available upon request.
[28] As an aside, note that the above argument also establishes that Part (A) holds whenever β is low enough that

$$u_1\left(C_1^*\left(\underline{\theta}_1\right); \underline{\theta}_1\right) + \beta V^2\left(C^*\left(\underline{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) < u_1\left(C_1^*\left(\bar{\theta}_1\right); \underline{\theta}_1\right) + \beta V^2\left(C^*\left(\bar{\theta}_1, \bar{\theta}_2\right); \underline{\theta}_2\right).$$

Turning to Part (B), by IC_1 self 1 is dissuaded from claiming high consumption $C_1\left(\bar{\theta}_1\right)$ in the low consumption state $\underline{\theta}_1$ if and only if

$$u_1\left(C_1\left(\underline{\theta}_1\right);\underline{\theta}_1\right)+\beta V^2\left(C\left(\underline{\theta}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)\geq u_1\left(C_1\left(\bar{\theta}_1\right);\underline{\theta}_1\right)+\beta V^2\left(C\left(\bar{\theta}_1,\bar{\theta}_2\right);\bar{\theta}_2\right).$$

Now, it is consumption $C\left(\bar{\theta}_1,\bar{\theta}_2\right)$ that lies off the equilibrium path.

In this case, it is possible to exploit the agent's time-inconsistency to punish self 1 for over-consumption. To do so, one can make $C_2\left(\bar{\theta}_1,\bar{\theta}_2\right)$ very high relative to $C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)$, while at the same time making $C_3\left(\bar{\theta}_1,\bar{\theta}_2\right)$ low relative to $C_3\left(\bar{\theta}_1,\underline{\theta}_2\right)$; note that this is consistent with Lemma 1. Precisely because of hyperbolic discounting, self 2 will choose $C\left(\bar{\theta}_1,\bar{\theta}_2\right)$ over $C\left(\underline{\theta}_1,\bar{\theta}_2\right)$ in the high MU state $\bar{\theta}_2$. By making $C_3\left(\bar{\theta}_1,\bar{\theta}_2\right)$ low, it is possible to make $V^2\left(C\left(\bar{\theta}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)$ low, and thus satisfy the date 1 incentive constraint above.

A perhaps surprising aspect of Proposition 1 is that the continuation utility $V^2\left(C\left(\bar{\theta}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)$ can be made sufficiently low even when hyperbolic discounting is severe (β low), since the more severe discounting becomes, the greater the punishment that is needed. The reason this is possible is that there is an offsetting effect: as hyperbolic discounting becomes more severe, self 2 cares less and less about date 3 consumption, and so the size of punishment that self 2 can be induced to inflict on self 1 grows.[29]

The proof of part (B) of Proposition 1 is constructive, and we conclude this subsection with a brief sketch. Observe first that self 1's utility U^1 from consumption c is related to V^1 by

$$U^1\left(c;\theta_1,\theta_2\right)=\left(1-\beta\right)u_1\left(c;\theta_1\right)+\beta V^1\left(c;\theta_1,\theta_2\right). \tag{1}$$

Consequently, the gain to self 1 in state $\left(\underline{\theta}_1,\bar{\theta}_2\right)$ of obtaining consumption $C^*\left(\bar{\theta}_1,\underline{\theta}_2\right)$ instead of $C^*\left(\underline{\theta}_1,\bar{\theta}_2\right)$ is strictly less than

$$\left(1-\beta\right)\left(u_1\left(C_1^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)-u_1\left(C_1^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)\right), \tag{2}$$

since by definition $C^*\left(\underline{\theta}_1,\bar{\theta}_2\right)$ maximizes $V^1\left(\cdot;\underline{\theta}_1,\bar{\theta}_2\right)$. Second, observe that $C\left(\bar{\theta}_1,\bar{\theta}_2\right)$ defined by

[29] Part (B) of Proposition 1 makes use of Assumptions 5 and 6. Assumption 5 is used to show that there is enough variation in contemporaneous utility at date 2; but for some important classes of shocks, notably additive shocks, it is not needed. (A proof is available from the authors upon request.) Assumption 6 is used to establish that the punishment $C\left(\bar{\theta}_1,\bar{\theta}_2\right)$ satisfies the resource constraint RC. Note that this consumption is chosen only off the equilibrium path, so if RC is not required off the equilibrium path, it too can be dispensed with.

$C_1(\bar{\theta}_1) = C_1^*(\bar{\theta}_1)$ and

$$u_2(C_2(\bar{\theta}_1, \bar{\theta}_2); \bar{\theta}_2) = u_2(C_2^*(\bar{\theta}_1, \underline{\theta}_2); \bar{\theta}_2) + u_1(C_1^*(\bar{\theta}_1); \underline{\theta}_1) - u_1(C_1^*(\underline{\theta}_1); \underline{\theta}_1) \tag{3}$$

$$u_3(C_3(\bar{\theta}_1, \bar{\theta}_2)) = u_3(C_3^*(\bar{\theta}_1, \underline{\theta}_2)) - \frac{1}{\beta}(u_1(C_1^*(\bar{\theta}_1); \underline{\theta}_1) - u_1(C_1^*(\underline{\theta}_1); \underline{\theta}_1)) \tag{4}$$

delivers self 1 utility in state $(\underline{\theta}_1, \bar{\theta}_2)$ that is exactly an amount (2) below $U^1(C^*(\bar{\theta}_1, \underline{\theta}_2); \underline{\theta}_1, \bar{\theta}_2)$. Consequently, $C(\bar{\theta}_1, \bar{\theta}_2)$ delivers enough punishment to induce self 1 to pick low consumption in state $\underline{\theta}_1$, i.e., $\text{IC}_1(\underline{\theta}_1, \bar{\theta}_1)$ is satisfied. Observe that $C(\bar{\theta}_1, \bar{\theta}_2)$ increases date 2 consumption relative to $C^*(\bar{\theta}_1, \underline{\theta}_2)$, while simultaneously decreasing date 3 consumption; and it does so in such a way that self 2 actually chooses $C(\bar{\theta}_1, \bar{\theta}_2)$ in state $\bar{\theta}_2$, since $\text{IC}_2(\bar{\theta}_1, \bar{\theta}_2, \underline{\theta}_2)$ holds at equality.[30]

4.2 Zero correlation

Next, consider the case of zero correlation, in which the conditional probability $\Pr(\theta_2|\theta_1)$ of the date 2 state θ_2 is independent of the date 1 state θ_1. For example, this is the case if date 2 utility u_2 is independent of the state of the world, as in Amador et al (2006). This is also the case if θ_1 and θ_2 are distributed identically and independently, as in Amador et al (2003). Given zero correlation, IC_1 simplifies to

$$u_1(C_1(\theta_1); \theta_1) + \beta E_{\theta_2}[V^2(C(\theta_1, \theta_2); \theta_2)] \geq u_1\left(C_1\left(\tilde{\theta}_1\right); \theta_1\right) + \beta E_{\theta_2}\left[V^2\left(C\left(\tilde{\theta}_1, \theta_2\right); \theta_2\right)\right]. \tag{IC$_1$}$$

Because of zero correlation, there is a strictly positive probability of all state-combinations (θ_1, θ_2), and hence $C(\theta_1, \theta_2)$ is fully determined by the solution to Problem I. Consequently, if the solution to Problem I entails date 1 consumption that differs across states, then for β sufficiently low no solution to Problem I is feasible.[31] In this case, consumption must be distorted relative to what self 0 would desire, as analyzed by Amador et al (2006).

4.3 Imperfect correlation

Under perfect correlation, θ_2 is perfectly forecastable at date 1, and so the only reason to give self 2 a consumption choice is so he can punish self 1 for overconsumption. In contrast, under imperfect

[30] The remaining details of the construction are handled in the Appendix. The main remaining steps are to show that the definition (3), (4) is actually feasible, and to show that RC is satisfied.

[31] By exactly the same argument, no solution to Problem II is feasible either.

correlation, state θ_2 is unknown at date 1. In this case, self 2 is required to play two roles. First, he must punish self 1 for overconsumption, as before. Second, he must himself choose appropriate consumption at date 2. Because self 3 simply consumes whatever is left, it is impossible to punish self 2 for overconsumption at date 2 (the subproblem starting at date 2 is covered by Amador et al 2006). Consequently, when hyperbolic discounting is severe enough self 2 cannot be deterred from deviating from the solution to Problem I.

Since no solution to Problem I is feasible for β small, we instead consider Problem II, in which IC_2 but not IC_1 is imposed. In words, we ask whether unverifiability of state θ_1 imposes a cost over and above unverifiability of state θ_2.

A useful preliminary result, which helps in the statement of results below, is:[32]

Lemma 3 *The solution to Problem II is independent of β for all β sufficiently small.*

Given Lemma 3, let C^{**} be a solution to Problem II for small β, and let $\bar{\bar{\theta}}_1$ and $\underline{\underline{\theta}}_1$ denote the elements of Θ_1 such that $C_1^{**}\left(\bar{\bar{\theta}}_1\right) \geq C_1^{**}\left(\underline{\underline{\theta}}_1\right)$.

Parallel to the perfect correlation case, self 1 is dissuaded from claiming high consumption in the low consumption state $\underline{\underline{\theta}}_1$ if and only if

$$\begin{aligned} & u_1\left(C_1\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right) + \beta \Pr\left(\bar{\theta}_2|\underline{\underline{\theta}}_1\right) V^2\left(C\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) + \beta \Pr\left(\underline{\theta}_2|\underline{\underline{\theta}}_1\right) V^2\left(C\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) \\ \geq\ & u_1\left(C_1\left(\bar{\bar{\theta}}_1\right);\underline{\underline{\theta}}_1\right) + \beta \Pr\left(\bar{\theta}_2|\underline{\underline{\theta}}_1\right) V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) + \beta \Pr\left(\underline{\theta}_2|\underline{\underline{\theta}}_1\right) V^2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right). \end{aligned}$$

As before, this constraint is violated for severe hyperbolic discounting (β small) unless the expected continuation utility when self 1 claims $\bar{\bar{\theta}}_1$ when the actual state is $\underline{\underline{\theta}}_1$ is very low, i.e., the following quantity is very low:

$$\Pr\left(\bar{\theta}_2|\underline{\underline{\theta}}_1\right) V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) + \Pr\left(\underline{\theta}_2|\underline{\underline{\theta}}_1\right) V^2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right).$$

To delineate the consequences of imperfect correlation, we examine two distinct cases. First, we consider the case in which the date 2 state following the high-consumption date 1 state $\bar{\bar{\theta}}_1$ is deterministic. In this case, one of $C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)$ and $C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right)$ lies off the equilibrium path. Consequently, there is some hope of being able to punish self 1 for overconsumption without distorting

[32]The key argument in Lemma 3 is that, for β small, any solution to Problem II makes consumption at dates 2 and 3 independent of state θ_2. This result is formally stated in Lemma A-3 in the Appendix, and generalizes Proposition 1 of Amador et al (2006) beyond the case of multiplicative shocks.

consumption along the equilibrium path. Our result for this case is:

Proposition 2 *If* $\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right) \in \{0,1\}$ *and* $C_1^{**}\left(\bar{\bar{\theta}}_1\right) > C_1^{**}\left(\underline{\underline{\theta}}_1\right)$ *then:*
(A, No Preference Reversal) If $\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right) = 1$*, then for all* β *sufficiently small, no feasible contract solves Problem II.*
(B, Preference Reversal) If $\Pr\left(\underline{\theta}_2|\bar{\bar{\theta}}_1\right) = 1$ *and*

$$u_2\left(C_2^{**}\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right)+C_3^{**}\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\bar{\theta}_2\right) > u_2\left(C_2^{**}\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\bar{\theta}_2\right)+\frac{u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\underline{\underline{\theta}}_1\right)-u_1\left(C_1^{**}\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right)}{\Pr\left(\bar{\theta}_2|\underline{\underline{\theta}}_1\right)} \tag{5}$$

then for all β *sufficiently small, there exists a feasible contract that solves Problem II.*

Proposition 2 extends Proposition 1 beyond perfect correlation. Again, if high date 1 consumption is followed by high date 2 MU (no preference reversal), there is no feasible solution to Problem II for strong hyperbolic discounting. In contrast, if high date 1 consumption is followed by low date 2 MU (preference reversal), then under many cases there is a feasible solution to Problem II. In this case, hyperbolic discounting has no effect on outcomes beyond the fact that self 2 cannot be prevented from overconsuming.

The economics behind Proposition 2 is the same as behind Proposition 1. If the high consumption date 1 state $\bar{\bar{\theta}}_1$ is followed by the high MU date 2 state $\bar{\theta}_2$, then consumption $C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right)$ lies off the equilibrium path, and one would like to punish self 1 for overconsumption in $\underline{\underline{\theta}}_1$ by making the continuation utility $V^2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right)$ low. But by Lemma 2, this is impossible, leading to the conclusion of Part (A). If instead the high consumption date 1 state $\bar{\bar{\theta}}_1$ is followed by the low MU date 2 state $\underline{\theta}_2$, then consumption $C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)$ lies off the equilibrium path, and one would like to punish self 1 for overconsumption in $\underline{\underline{\theta}}_1$ by making the continuation utility $V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)$ low. This is potentially achievable, exactly as in the case of perfect correlation. Condition (5) is needed because $\Pr\left(\bar{\theta}_2|\underline{\underline{\theta}}_1\right) < 1$, so that self 1 has some chance of being able to falsely report $\bar{\bar{\theta}}_1$ in state $\underline{\underline{\theta}}_1$ and then escape the punishment that occurs when the date 2 state is $\bar{\theta}_2$. Consequently, because the punishment is not always imposed, it needs to be larger than in the case of perfect correlation. Condition (5) is enough to guarantee that this larger punishment is feasible; note that it is in the same spirit as Assumption 6, which says that cross-state variation in date 1 consumption is smaller than the minimum level of date 3 consumption.

Second, we consider the case in which both date 2 states are possible after $\bar{\bar{\theta}}_1$, while the date 2 state following the low-consumption date 1 state $\underline{\underline{\theta}}_1$ is deterministic. In this case, it is impossible to punish self 1 without distorting consumption along the equilibrium path.[33] Consequently, there is no feasible solution to Problem II when hyperbolic discounting is severe. Accordingly, we instead characterize the minimum utility cost of making date 1 consumption $C_1^{**}(\cdot)$ feasible. Formally, this cost is

$$\begin{aligned} \kappa \quad &\equiv \quad E_{\theta_1,\theta_2}\left[U^0\left(C^{**}\left(\theta_1,\theta_2\right);\theta_1,\theta_2\right)\right] \\ &\quad - \max_{C \text{ s.t. RC, IC}_1\text{, IC}_2\text{, } C_1(\cdot,\cdot)\equiv C_1^{**}(\cdot,\cdot)} E_{\theta_1,\theta_2}\left[U^0\left(C\left(\theta_1,\theta_2\right);\theta_1,\theta_2\right)\right]. \end{aligned} \tag{6}$$

Proposition 3 *Let A and B be two different economies such that $\Pr\left(\underline{\theta}_2|\underline{\underline{\theta}}_1\right) = 1$ in economy A but $\Pr\left(\bar{\theta}_2|\underline{\underline{\theta}}_1\right) = 1$ in economy B. Both the date 1 probabilities $\Pr\left(\theta_1\right)$ and utility difference $u_1\left(C^{**}\left(\bar{\bar{\theta}}_1\right);\underline{\underline{\theta}}_1\right) - u_1\left(C^{**}\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right)$ are constant across the two economies. Moreover, $\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right) \notin \{0,1\}$ in both economies. Then for all β sufficiently small, the cost of making $C_1^{**}(\cdot)$ feasible is greater in economy A than economy B, i.e., $\kappa_A > \kappa_B$.*

Proposition 3 is in keeping with the conclusions of Propositions 1 and 2: ceteris paribus, the impact of hyperbolic discounting is smaller when low date 1 consumption is followed by high date 2 MU, i.e., when there is preference reversal. The conditions in Proposition 3 formalize the ceteris paribus condition: the two economies being compared have the same difference in date 1 contemporaneous utility.

Although the proof of Proposition 3 is quite long, the basic economics is the same as for previous results. The state in which self 1 potentially overconsumes is $\underline{\underline{\theta}}_1$. In Economy A, this state is followed by the low MU state $\underline{\theta}_2$, and so we need self 2 to impose a punishment in this state. But Lemma 2 bounds the size of the punishment that is possible when date 2 MU is low: specifically, $V^2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right)$ cannot fall below $V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right)$. Consequently, the only way to effectively punish self 1 is to distort $C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)$, which is costly in utility terms.

In contrast, in Economy B, we need self 2 to impose a punishment in the high MU state $\bar{\theta}_2$. The proof of Proposition 3 bounds the utility cost of this punishment by constructing a specific contract. In particular, the construction of the punishment $\left(C_2\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right),C_3\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)\right)$ is based on the same punishment as used in Proposition 1 above (see (3) and (4)), which we know provides

[33] On the other hand, there is no longer a problem of failing to punish self 1 when he overconsumes: in this case, if self 1 overconsumes in state $\underline{\underline{\theta}}_1$, there is no uncertainty about his continuation utility.

more than enough punishment.

4.4 Continuum of states

We next extend our analysis, and assume that both θ_1 and θ_2 may take a continuum of different realizations. We focus on the case of perfect correlation. To avoid economically uninteresting mathematical complications, we assume that $C_1^*(\theta_1) \neq C_1^*\left(\tilde{\theta}_1\right)$ whenever $\theta_1 \neq \tilde{\theta}_1$, and accordingly order Θ_1 so that $C_1^*(\theta_1)$ is strictly increasing in θ_1.

Our main result is the following generalization of Proposition 1. To state the result, let $\phi(\theta_1)$ be the date 2 state that deterministically follows the date 1 state θ_1:

Proposition 4 *(A, No Preference Reversal) If ϕ is strictly increasing, there is no feasible solution to Problem I for β small.*
(B, Preference Reversal) If ϕ is strictly decreasing and differentiable, then for any β, there exists a feasible solution to Problem I provided that $\max_{\theta_1,\tilde{\theta}_1\in\Theta_1}\left|C^*(\theta_1,\phi(\theta_1)) - C^*\left(\tilde{\theta}_1,\phi\left(\tilde{\theta}_1\right)\right)\right|$ is sufficiently small.

Proposition 4 extends the conclusion of prior results to the case of a continuous state space. Again, hyperbolic discounting affects outcomes if low date 1 consumption is associated with low date 2 MU (Part (A), no preference reversal), but not if it is associated with high date 2 MU (Part (B), preference reversal). The economic forces are the same as previously identified. In Part (A), it is impossible to punish self 1 for overconsuming, because the punishment needs to be inflicted in a date 2 state with low MU, which by Lemma 2 is impossible.

In contrast, in Part (B), such punishment is possible. The new complication relative to previous results is that now self 1 can overconsume to various degrees. In general, greater overconsumption necessitates a more severe punishment. In particular, a greater punishment is typically needed if self 1 lies and reports $\tilde{\theta}_1$ when the true state is θ_1 rather than the true state being $\theta_1' > \theta_1$. The challenge is to design the contract C so self 2 picks the punishment appropriate to the degree of overconsumption, i.e., picks different punishments in states θ_1' and θ_1.

The proof of Part (A) is immediate from the analysis of the binary case. If $\phi(\theta_1)$ is increasing, then in particular there exist θ_1 and $\tilde{\theta}_1 > \theta_1$ such that $\phi\left(\tilde{\theta}_1\right) > \phi(\theta_1)$. Then from Proposition 1, there is no feasible solution to Problem I when $\Theta_1 = \left\{\theta_1, \tilde{\theta}_1\right\}$ and $\Theta_2 = \left\{\phi(\theta_1), \phi\left(\tilde{\theta}_1\right)\right\}$. *A fortiori*, there is no feasible solution to Problem I in the continuum state case under consideration.

The proof of Part (B) is constructive, and we sketch some of the elements of the construction here. A useful starting point is the following observation, which gives a sufficient condition for IC_2:

Lemma 4 *$IC_2(\theta_1, \cdot, \cdot)$ is satisfied if*[34] *$C_2(\theta_1, \cdot)$ is increasing in θ_2 and for all $\theta_2 \in \Theta_2$,*

$$u_2'\left(C_2\left(\theta_1, \theta_2\right); \theta_2\right) \frac{\partial C_2\left(\theta_1, \theta_2\right)}{\partial \theta_2} = -\beta u_3'\left(C_3\left(\theta_1, \theta_2\right)\right) \frac{\partial C_3\left(\theta_1, \theta_2\right)}{\partial \theta_2}. \tag{7}$$

Next, consider any self 1 report $\tilde{\theta}_1$. Provided IC_2 is satisfied, then self 2 reports truthfully. Consequently, if self 2 reports $\tilde{\theta}_2 = \phi\left(\tilde{\theta}_1\right)$, he is confirming that self 1 reported truthfully. Accordingly, we set $C\left(\tilde{\theta}_1, \phi\left(\tilde{\theta}_1\right)\right) = C^*\left(\tilde{\theta}_1, \phi\left(\tilde{\theta}_1\right)\right)$. In contrast, if self 2 reports $\tilde{\theta}_2 > \phi\left(\tilde{\theta}_1\right)$, then since ϕ is decreasing he is reporting that self 1 reported too high a state, i.e., $\tilde{\theta}_1 > \phi^{-1}\left(\tilde{\theta}_2\right)$, meaning that self 1 overconsumed. To deter such overconsumption, we essentially[35] define $C\left(\tilde{\theta}_1, \theta_2\right)$ for $\theta_2 \geq \phi\left(\tilde{\theta}_1\right)$ to satisfy the pair of differential equations differential equations (7)—so that, by Lemma 4, IC_2 is satisfied—and

$$\frac{dU^1\left(C\left(\tilde{\theta}_1, \theta_2\right); \phi^{-1}\left(\theta_2\right), \theta_2\right)}{d\theta_2} = \frac{dU^1\left(C^*\left(\phi^{-1}\left(\theta_2\right), \theta_2\right); \phi^{-1}\left(\theta_2\right), \theta_2\right)}{d\theta_2},$$

subject to the boundary condition $C\left(\tilde{\theta}_1, \phi\left(\tilde{\theta}_1\right)\right) = C^*\left(\tilde{\theta}_1, \phi\left(\tilde{\theta}_1\right)\right)$.[36] This definition implies

$$U^1\left(C\left(\tilde{\theta}_1, \theta_2\right); \phi^{-1}\left(\theta_2\right), \theta_2\right) = U^1\left(C^*\left(\phi^{-1}\left(\theta_2\right), \theta_2\right); \phi^{-1}\left(\theta_2\right), \theta_2\right),$$

so that self 1 is indifferent between reporting $\tilde{\theta}_1$ and the true realization of θ_1, namely $\phi^{-1}\left(\theta_2\right)$. Hence IC_1 is satisfied.

Finally, for part (B), the condition that $\max_{\theta_1, \tilde{\theta}_1 \in \Theta_1}\left|C^*\left(\theta_1, \phi\left(\theta_1\right)\right) - C^*\left(\tilde{\theta}_1, \phi\left(\tilde{\theta}_1\right)\right)\right|$ is not too large ensures that RC can be satisfied. In common with the role of Assumptions 5 and 6 in Proposition 1, this condition is not needed if RC is allowed to be violated off the equilibrium path.

[34] As an aside, it is worth noting that condition (7) is a *necessary* as well as a sufficient condition for IC_2 at any point at which $C\left(\theta_1, \theta_2\right)$ is differentiable with respect to θ_2; a proof is available upon request.

[35] Full details are in the proof of Lemma 4. The complication relative to the main text is that, by Lemma 1, $C_2\left(\tilde{\theta}_1, \theta_2\right)$ must be weakly increasing in θ_2.

[36] Note that the boundary condition also implies $C_1\left(\tilde{\theta}_1\right) = C_1^*\left(\tilde{\theta}_1\right)$.

5 Private savings

So far, we have assumed that the agent has no ability to save outside the contract. This assumption fits some applications well; for example, if consumption is leisure, this is indeed the case. This assumption also approximates the case in which private saving is possible, but only at a very disadvantageous interest rate. Nonetheless, in other cases this assumption is unrealistically strong. In this section we relax this assumption, and instead allow the agent to privately save at the economy's risk-free interest rate, which recall we assume to be zero.[37]

Given the possibility of private savings, a contract is now specified by $X(\theta_1, \theta_2, s_1)$; here, θ_1 is the report of self 1, while self 2 reports θ_2 and also the level of savings s_1 he has inherited from self 1. We use the notation X rather than C because, off the equilibrium path, consumption is impacted by hidden savings. In particular, if self 1 saves s_1 and self 2 saves s_2, then consumption is

$$(c_1, c_2, c_3) = \left(-s_1 + X_1(\theta_1, \theta_2, s_1), s_1 + X_2(\theta_1, \theta_2, s_1) - s_2, s_2 + X_3(\theta_1, \theta_2, s_1)\right).$$

We denote this vector by $s_1 + X(\theta_1, \theta_2, s_1) - s_2$; note that s_1 and s_2 here enter with the signs relevant for self 2.

By standard arguments (see, e.g., Cole and Kocherlakota 2001), without loss we restrict attention to contracts that induce self 1 to choose zero private savings, and self 2 to likewise choose zero private savings after every self 1 report θ_1 and savings decision s_1. The IC constraints are hence as follows.[38] At date 2, for all $\theta_1 \in \Theta_1$ and $\theta_2, \tilde{\theta}_2 \in \Theta_2$, and all $s_1, \tilde{s}_1, s_2 \geq 0$,

$$U^2(s_1 + X(\theta_1, \theta_2, s_1); \theta_2) \geq U^2\left(s_1 + X\left(\theta_1, \tilde{\theta}_2, \tilde{s}_1\right) - s_2; \theta_2\right), \quad (\text{IC}_2)$$

and at date 1, for all $\theta_1, \tilde{\theta}_1 \in \Theta_1$ and $s_1 \geq 0$,

$$E_{\theta_2}\left[U^1(X(\theta_1, \theta_2, 0); \theta_1, \theta_2) | \theta_1\right] \geq E_{\theta_2}\left[U^1\left(s_1 + X\left(\tilde{\theta}_1, \theta_2, s_1\right); \theta_1, \theta_2\right) | \theta_1\right]. \quad (\text{IC}_1)$$

We consider only the most basic version of our economy in which Θ_1 and Θ_2 are both binary,

[37] Recent papers in the growing literature on contracting with hidden savings include Kocherlakota (2004), Doepke and Townsend (2006), and He (2009).

[38] We also impose the mild regularity condition that, for all $(\theta_1, \theta_2) \in \Theta$, $X(\theta_1, \theta_2, s_1)$ is a finite function of s_1, in the sense of having at most finitely many points of discontinuity. This regularity condition is used only in proving the necessity half of Proposition 6. It can be relaxed, though only at the cost of introducing economically uninteresting mathematical complexity.

and there is perfect correlation across the two dates. Moreover, we assume preferences satisfy the following additional assumption:

Assumption 7 *For any s_1, sign $\left(u_2'(s_1 + x; \theta_2) - u_2'\left(x; \tilde{\theta}_2\right)\right)$ is independent of x.*

Assumption 7 guarantees that self 2's indifference curves in states θ_2 and $\tilde{\theta}_2$ cross only once, even when he inherits a different level of savings in the two states. It is straightforward to see that the assumption is satisfied in the case of additive date 2 shocks. Moreover, our results can be generalized to dispense with this assumption; an earlier version of the paper contains full details.

As before, let $\bar{\theta}_1$ and $\underline{\theta}_1$ be such that $C_1^*\left(\bar{\theta}_1\right) \geq C_1^*\left(\underline{\theta}_1\right)$. Given perfect correlation, for conciseness we write u_2, U^2 and V^2 directly in terms of θ_1, rather than in terms of the date 2 state that deterministically follows θ_1. Likewise, we write C^* and U^1 as functions of θ_1 only.

As before, hyperbolic discounting makes it tempting for self 1 to falsely claim that he is in the high consumption state $\bar{\theta}_1$. The new complication is that he may also privately save $s_1 > 0$. However, it is clearly not tempting for self 1 to both falsely claim high consumption and privately save a very large amount. Consequently, there is an upper bound on the level of private saving that is relevant. This upper bound is an important object in our analysis, and we denote it by s_1^*. It is formally defined as follows. First, define $\hat{s}_2(s_1) = \arg\max_{s_2 \geq 0} U^2\left(s_1 + C^*\left(\bar{\theta}_1\right) - s_2; \underline{\theta}_1\right)$, i.e., self 2's private savings decision in state $\underline{\theta}_1$ given baseline consumption $s_1 + C^*\left(\bar{\theta}_1\right)$. Then s_1^* itself is given by

$$s_1^* = \sup\left\{s_1 \geq 0 : U^1\left(C^*\left(\underline{\theta}_1\right); \underline{\theta}_1\right) < U^1\left(s_1 + C^*\left(\bar{\theta}_1\right) - \hat{s}_2(s_1); \underline{\theta}_1\right)\right\},$$

where $s_1^* = 0$ if the above set is empty. Note that $s_1^* > 0$ whenever hyperbolic discounting is sufficiently severe, i.e., β low enough.

Proposition 5 *For any β low enough such that $s_1^* > 0$, there exists a feasible solution to Problem I only if*

$$u_2'\left(x; \bar{\theta}_1\right) \leq u_2'\left(s_1^* + x; \underline{\theta}_1\right). \tag{SPR}$$

Proposition 5 extends Part (A) of Proposition 1 to the case of private savings. This previous result stated that for β sufficiently small, preference reversal is a necessary condition for there to exist a feasible solution to Problem I: high date 1 consumption C^* must be followed by low date

2 MU, i.e., if $u_2'\left(\cdot;\bar{\theta}_1\right) < u_2'\left(\cdot;\underline{\theta}_1\right)$. With private savings, this condition is replaced with a strictly stronger condition, which we label *strong preference reversal* (SPR).

The economics of this result is exactly as earlier. When $u_2'\left(s_1^* + x;\underline{\theta}_1\right) < u_2'\left(x;\bar{\theta}_1\right)$, the combination of inherited savings s_1^* and state $\underline{\theta}_1$ gives lower MU at date 2 than the combination of no inherited savings and state $\bar{\theta}_1$. By Lemma 2,[39] this makes it impossible to lower the continuation utility $V^2\left(s_1^* + X\left(\bar{\theta}_1, \underline{\theta}_1, s_1^*\right);\underline{\theta}_1\right)$ below $V^2\left(s_1^* + X\left(\bar{\theta}_1, \bar{\theta}_1, 0\right);\underline{\theta}_1\right)$. Because $X\left(\bar{\theta}_1, \bar{\theta}_1, 0\right)$ lies on the equilibrium path, and in particular equals $C^*\left(\bar{\theta}_1\right)$ in any solution to Problem I, this bounds the punishment that can be imposed on self 1 for claiming $\bar{\theta}_1$ when the true state is $\underline{\theta}_1$ and self 1 has passed savings s_1^* onto self 2.

Proposition 5 leaves open the issue of whether there is a feasible solution to Problem I when SPR is satisfied. We address this next. The main issue relative to previous results is that the possibility of private saving by self 2 places a limit on how much date 2 consumption can be raised at the expense of date 3 consumption. This is turn limits the extent to which self 2 is able to punish self 1 for overconsumption. We next derive a condition that, combined with SPR, is both necessary and sufficient for there to exist a feasible solution to Problem I.

In our analysis of the continuous state space case, we saw that different punishments are potentially required if self 1 falsely reports $\tilde{\theta}_1$ depending on whether the true state is θ_1 or $\theta_1' \neq \theta_1$. Likewise, when the state space is binary but private saving is possible, different punishments are required if self 1 falsely reports $\bar{\theta}_1$ depending on whether self 1 saves s_1 or $s_1' \neq s_1$. Moreover, and again as in the continuous state space case, self 2 must be induced to choose the appropriate punishment. Just as Lemma 4 provided the key tool for ensuring IC_2 was satisfied in the continuous state space case, the following analogous result serves the same role here:

Lemma 5 *The subset of IC_2,*

$$U^2\left(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right);\underline{\theta}_1\right) \geq U^2\left(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - s_2;\underline{\theta}_1\right) \text{ for all } s_1, \tilde{s}_1, s_2 \geq 0 \quad (8)$$

is satisfied if $X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)$ is weakly decreasing in s_1, and satisfies $u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right);\underline{\theta}_1\right) \geq \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)\right)$ and

$$u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right);\underline{\theta}_1\right)\frac{\partial X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)}{\partial s_1} = -\beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)\right)\frac{\partial X_3\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)}{\partial s_1}. \quad (9)$$

[39] The proof of Proposition 5 establishes an analogue of Lemma 2 for private savings.

Conversely, if X satisfies (8) then $U^2\left(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right)$ is continuous in s_1; and at any point at which $X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)$ is continuous in s_1, (9) holds.

To give a sufficient condition for a feasible solution to Problem I to exist, we use the differential equation (9) to construct a specific contract, X^*. The construction begins at the private savings level s_1^*. By definition, if self 1 falsely claims the high consumption state $\bar{\theta}_1$ and then saves s_1^*, no punishment is needed: in state $\underline{\theta}_1$ he is indifferent between $C^*\left(\bar{\theta}_1\right)$ and saving s_1^*, and $C^*\left(\underline{\theta}_1\right)$. Accordingly, we define $X^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1^*\right) = C^*\left(\bar{\theta}_1\right)$, which also implies $X_1^*\left(\bar{\theta}_1, \cdot, \cdot\right) = C_1^*\left(\bar{\theta}_1\right)$. For lower levels of private savings, we aim to provide the minimum punishment that still deters self 1 from claiming too much consumption. The basic idea is to define $X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, \cdot\right)$ and $X_3^*\left(\bar{\theta}_1, \underline{\theta}_1, \cdot\right)$ to satisfy the differential equations (9) and $\frac{d}{ds_1}U^1\left(s_1 + X_1^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right) = 0$. However, we also need to ensure that $X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, \cdot\right)$ is decreasing in savings s_1 (this is analogous to the issue noted in footnote 35). This is essentially an implication of Lemma 1: date 2 consumption must increase in date 2 MU, and hence decrease in inherited savings. Accordingly, $X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, \cdot\right)$ and $X_3^*\left(\bar{\theta}_1, \underline{\theta}_1, \cdot\right)$ are defined as the solutions to the pair of differential equations (9) and

$$\frac{\partial X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)}{\partial s_1} = \min\left\{0, \frac{-u_1'\left(-s_1 + X_1^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right) + \beta u_2'\left(s_1 + X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right)}{\left(1-\beta\right) u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right)}\right\}, \tag{10}$$

subject to the boundary condition $X^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1^*\right) = C^*\left(\bar{\theta}_1\right)$. The idea behind (10) is that—as is readily verified—it ensures that $\frac{d}{ds_1}U^1\left(s_1 + X_1^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right) = 0$ whenever $\frac{\partial X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)}{\partial s_1} < 0$.

Given the definition of $X^*\left(\bar{\theta}_1, \underline{\theta}_1, \cdot\right)$ over $[0, s_1^*]$, we can state:

Proposition 6 *There exists a feasible solution to Problem I if and only if both SPR and the following condition hold:*

$$u_2'\left(s_1 + X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right) \geq \beta u_3'\left(X_3^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)\right) \text{ for all } s_1 \in [0, s_1^*] \tag{NS}$$

Proposition 6 is the analogue of Part (B) of previous results. If there is preference reversal—i.e., high date 1 consumption is associated with low date 2 MU—then hyperbolic discounting may have little or no effect on equilibrium consumption. Condition NS is a *no savings* conditions, and simply says that self 2 has no incentive to privately save in state $\underline{\theta}_1$, given $X^*\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)$. Numerical simulations, available upon request, suggest that condition NS is satisfied for a large fraction of the cases in which SPR holds.

Finally, it can be shown that condition SPR is equivalent to the following simple condition on consumption profiles,

$$C_1^*\left(\bar{\theta}_1\right)+C_2^*\left(\bar{\theta}_1\right)\leq C_1^*\left(\underline{\theta}_1\right)+C_2^*\left(\underline{\theta}_1\right). \tag{11}$$

(A proof of this equivalence is contained in an earlier version of this paper.) In words, SPR is equivalent to the condition that the state $\bar{\theta}_1$ that has higher desired consumption at date 1 also has weakly lower desired consumption across dates 1 and 2 together.

6 Discussion

Our analysis characterizes circumstances under which an agent with hyperbolic discounting is able to resolve the tension between commitment and flexibility. The key condition we identify is preference reversal: high desired consumption at date 1 is associated with low MU at date 2.

As illustrated by Examples 1 and 2 in the introduction, there are at least two economic forces that lead the preference reversal condition to be naturally satisfied. The force operating in Example 2 is that if the individual can forecast that future MU will be low, this leads him to increase consumption in advance. Hence, ceteris paribus, low MU at date 2 implies high desired consumption at date 1. Of course, the reverse implication does not hold, and accordingly, there are certainly many environments where preference reversal is not satisfied. The force operating in Example 1 is that if good consumption opportunities are limited, encountering a good opportunity at date 1 simultaneously increases desired consumption at date 1, and reduces the probability of encountering another good opportunity at date 2.

We have focused throughout on how unverifiability affects the individual's ability to combine commitment with flexibility. In doing so, we have abstracted from other possible impediments, such as a lack of exclusivity in contracting, a lack of commitment by contract counterparties, or other frictions in the contracting process. In this sense, our analysis provides an upper bound on an individual's ability to combine commitment with flexibility.[40]

We conclude with a discussion of two remaining points: other interpretations of our model, and the effect of partial naïveté.

[40] An earlier version of this paper contains a more detailed discussion of some of these issues. One point worth noting here is that the analysis of private savings already deals with one particular way in which exclusivity may be violated. The earlier version also contains results on how the contracts we characterize can be constructed using simple financial instruments, at least in the case of a binary state space with perfect correlation (subsection 4.1).

6.1 Applications

We have described the agent's problem in terms of deciding what fraction of a fixed endowment W to consume at each date. However, there are at least two other interpretations of our environment that are worth discussing.

First, one can interpret our environment in terms of when an agent chooses to work on a task. Consequently, our model can be used to analyze the provision of incentives to procrastinators, which has been a leading application of hyperbolic discounting. In this interpretation, the agent must complete a task, which will take h hours in total.[41] His total time endowment across the three dates is $W + h$. The agent must decide how much leisure c_t to enjoy at each of dates 1,2,3, subject to the constraint that he completes the task, $\sum_{t=1}^{3} c_t = W$. When a feasible contract exists that solves Problem I or II, it has the qualitative feature that self 2 can choose to miss a deadline, but that doing so increases the work required of him in the future.[42]

Second, Amador et al (2006) discuss an interpretation in which society wishes to constrain government spending, while recognizing that in some circumstances higher government spending is socially desirable. Our analysis permits an extension of this interpretation to the case of federal and local government. In this interpretation, the federal government chooses spending c_1; taking federal spending as given, the local government chooses spending c_2; and the private sector consumes c_3. Both federal and local governments want to spend more than is socially optimal, corresponding to hyperbolic discounting. Our analysis suggests that under some circumstances, a constitution can be designed that controls government spending. In particular, this is possible if date 0 uncertainty centers on whether the efficient provision of government services is at the federal or local level, since in this case the preference reversal assumption is naturally satisfied.[43]

6.2 Naïveté

Thus far, we have assumed that the agent is fully self-aware (sophisticated), in the sense that at any date, he correctly anticipates his future selves' preferences. In this section we show that

[41] In the additive shock parameterization, different states can interpreted as changes in the amount of time required to complete the task.

[42] It is also worth noting that if work is publicly observable, this is a case in which the issue of private savings is not a concern.

[43] This case in analogous to Example 1 in the introduction. In general, note that the government spending intepretation of the model is one in which the restriction against cross-state insurance is natural, as noted by Amador et al.

commitment contracts of the kind we analyze above often enable an agent who is partially (but not completely) naïve about his future selves' preferences to commit. We follow the literature and use the specification introduced by O'Donoghue and Rabin (2001): at each date, the agent's true hyperbolic discount rate is β, but he incorrectly believes that his future selves' rate is $\tilde{\beta} > \beta$. We continue to write U^t for self t's true preferences, and use $\tilde{U}^t$ to denote the preferences incorrectly attributed to self t by prior selves. We focus on the simplest version of our environment, in which the state space is binary, θ_1 and θ_2 are perfectly correlated, and private saving is impossible. We also assume that the preference reversal condition identified in Proposition 1 is satisfied, i.e., $\bar{\theta}_1$ is deterministically followed by $\underline{\theta}_2$ and $\underline{\theta}_1$ is deterministically followed by $\bar{\theta}_2$.[44]

Partial naïveté changes the conditions under which a feasible solution to Problem I exists. Recall from the discussion following Proposition 1 that the contract component $C\left(\bar{\theta}_1, \bar{\theta}_2\right)$ acts as a punishment that is inflicted on self 1 if he claims the high consumption state $\bar{\theta}_1$ when the true state is $\underline{\theta}_1$ (and hence the true date 2 state is $\bar{\theta}_2$). Under partial naïveté, a contract C must satisfy the modified date 2 IC

$$\tilde{U}^2\left(C\left(\bar{\theta}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) \geq \tilde{U}^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right); \bar{\theta}_2\right), \tag{12}$$

which says that self 1 *believes* that self 2 will impose this punishment.

Whenever hyperbolic discounting is sufficiently severe that self 1 is tempted to claim high consumption at date 1, i.e., $U^1\left(C^*\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_1, \bar{\theta}_2\right) > \tilde{U}^1\left(C^*\left(\underline{\theta}_1, \bar{\theta}_2\right); \underline{\theta}_1, \bar{\theta}_2\right)$, then prior arguments imply that a contract C is a feasible solution to Problem I only if $C\left(\bar{\theta}_1, \bar{\theta}_2\right)$ raises date 2 consumption relative to $C\left(\bar{\theta}_1, \underline{\theta}_2\right)$, exactly as in the case of full sophistication. The key impact of partial naïveté is that condition (12) is now more demanding: self 1 underestimates the present-bias of self 2, and hence underestimates self 2's willingness to increase date 2 consumption at the expense of date 3 consumption.

Given these observations, it is straightforward to show that if a feasible solution to Problem I exists for a partially naïve agent, then a feasible solution also exists for a sophisticated agent, while the reverse implication does not hold. In this sense, the combination of flexibility with commitment becomes harder as the agent's naïveté increases. We stress, however, that by continuity slight naïveté has only a small impact on the range of circumstances under which commitment and flexibility can be combined, i.e., there is a feasible solution to Problem I.[45]

[44] Here, $\underline{\theta}_1$, $\bar{\theta}_1$ and C^* are all defined as in Section 4.

[45] At the extreme of complete naïveté (i.e., $\tilde{\beta} = 1$), no feasible contract solves Problem I.

While the combination of commitment and flexibility is often possible when the agent is partially naïve, naïveté does have a significant impact an agent's incentive to choose an appropriate contract. Under full sophistication, self 0 has every incentive to sign up to a contract that reconciles commitment and flexibility. In contrast, this is not the case when the agent is partially naïve.

There are two related issues here. First, as in Heidhues and Kőszegi (2010), an agent's naïveté means that self 0 may agree to a contract that increases date 1 consumption relative to the solution to Problem I, but that distorts consumption at dates 2 and 3. In brief, self 0 finds the contract attractive because he incorrectly believes that he can increase both date 1 and total consumption by borrowing at a below-market rate; while the counterparty is happy to agree to the contract because he correctly understands that self 2 will choose repayment terms that correspond to the market rate.

Second, under partial naïveté self 0 may incorrectly believe that he does not have a commitment problem. In these circumstances, there is scope for a benevolent government to improve welfare (at least for self 0) by imposing a commitment contract.

However, it also important to note that while a government-mandated commitment contract can improve the welfare of a partially naïve agent, it can actually hurt a very naïve agent, relative to the alternative of simply allowing self 1 to choose freely between self 0's desired consumption paths $C^*\left(\bar{\theta}_1, \underline{\theta}_2\right)$ and $C^*\left(\underline{\theta}_1, \bar{\theta}_2\right)$. First, note that the punishment component of the contract must satisfy

$$u_2\left(C_2\left(\bar{\theta}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) + u_3\left(C_3\left(\bar{\theta}_1, \bar{\theta}_2\right)\right) < u_2(C_2^*\left(\bar{\theta}_1, \underline{\theta}_2\right); \bar{\theta}_2) + u_3(C_3^*\left(\bar{\theta}_1, \underline{\theta}_2\right)), \tag{13}$$

since otherwise the punishment would not deter self 1 from overconsuming in state θ.[46] Consequently, at date 1 a completely naïve agent (i.e., $\tilde{\beta} = 1$) will claim the high consumption state $\bar{\theta}_1$ when the true state is $\underline{\theta}_1$, believing that self 2 will then report $\underline{\theta}_2$, delivering consumption $C^*\left(\bar{\theta}_1, \underline{\theta}_2\right)$. However, after self 1 claims the high consumption state $\bar{\theta}_1$, self 2 in fact reports $\bar{\theta}_2$, delivering consumption $C\left(\bar{\theta}_1, \bar{\theta}_2\right)$, so that self 0's equilibrium utility in $\left(\underline{\theta}_1, \bar{\theta}_2\right)$ is $U^0\left(C\left(\bar{\theta}_1, \bar{\theta}_2\right); \underline{\theta}_1, \bar{\theta}_2\right)$. But by (13), this is strictly less than the utility self 0 would get from a contract allowing self 1 to choose freely between $C^*\left(\bar{\theta}_1, \underline{\theta}_2\right)$ and $C^*\left(\underline{\theta}_1, \bar{\theta}_2\right)$, namely $U^0\left(C^*\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_1, \bar{\theta}_2\right)$.[47] Consequently,

[46]Formally, if $u_2\left(C_2\left(\bar{\theta}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) + u_3\left(C_3\left(\bar{\theta}_1, \bar{\theta}_2\right)\right) \geq u_2(C_2^*\left(\bar{\theta}_1, \underline{\theta}_2\right); \bar{\theta}_2) + u_3(C_3^*\left(\bar{\theta}_1, \underline{\theta}_2\right))$ then $U^1\left(C\left(\bar{\theta}_1, \bar{\theta}_2\right); \underline{\theta}_1, \bar{\theta}_2\right) > U^1\left(C^*\left(\bar{\theta}_1, \underline{\theta}_2\right); \underline{\theta}_1, \bar{\theta}_2\right)$.

[47]The argument here is closely related to Heidhues and Kőszegi (2010). Self 2 effectively borrows on expensive terms that self 1 naïvely believed he would not agree to.

although there is scope for government paternalism to improve welfare if the government has a reasonably precise estimate of the degree of naïveté, such paternalism is dangerous if agents are instead much more naïve than the government believes.[48]

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[48]Eliaz and Spiegler (2006) analyze profit maximization by a monopolist who deals with a population of time-inconsistent individuals who differ in their degree of sophistication. The problem noted in the main text suggests that the parallel question of welfare maximization for a population of differentially sophisticated time-inconsistent individuals would also be interesting. We leave this topic for future research. Also related is the problem of designing a contract for a population of partially naïve agents who differ in the strength of their hyperbolic discounting, e.g., β varies across agents while $\hat{\beta}/\beta$ is constant. Again, we leave this interesting question for future research.

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Results omitted from main text

Lemma A-1 *If $\tilde{c}_2 \geq c_2$ and $V^2(\tilde{c};\theta_2) \geq V^2(c;\theta_2)$, then $U^2(\tilde{c};\theta_2) \geq U^2(c;\theta_2)$. The inequality is strict if either $\tilde{c}_2 > c_2$ or $V(\tilde{c};\theta_2) > V(c;\theta_2)$.*

Proof of Lemma A-1: Rewriting $V^2(\tilde{c};\theta_2) \geq V^2(c;\theta_2)$ and $U^2(\tilde{c};\theta_2) \geq U^2(c;\theta_2)$ gives, respectively, $u_2(\tilde{c}_2;\theta_2) - u_2(c_2;\theta_2) \geq u_3(c_3;\theta_2) - u_3(\tilde{c}_3;\theta_2)$ and $u_2(\tilde{c}_2;\theta_2) - u_2(c_2;\theta_2) \geq \beta(u_3(c_3;\theta_2) - u_3(\tilde{c}_3;\theta_2))$. The result is then immediate. **QED**

Lemma A-2 *If C satisfies $IC_2\left(\theta_1, \theta_2, \tilde{\theta}_2\right)$ with equality and sign* $\left(C_2(\theta_1, \theta_2) - C_2\left(\theta_1, \tilde{\theta}_2\right)\right) =$ *sign* $\left(\theta_2 - \tilde{\theta}_2\right)$ *then C satisfies $IC_2\left(\theta_1, \tilde{\theta}_2, \theta_2\right)$.*

Proof: Since $\text{IC}_2\left(\theta_1, \theta_2, \tilde{\theta}_2\right)$ holds with equality, $u_2(C_2(\theta_1, \theta_2); \theta_2) - u_2\left(C_2\left(\theta_1, \tilde{\theta}_2\right); \theta_2\right) = \beta\left(u_3\left(C_3\left(\theta_1, \tilde{\theta}_2\right)\right) -\right.$ If either $C_2(\theta_1, \theta_2) \geq C_2\left(\theta_1, \tilde{\theta}_2\right)$ and $\theta_2 > \tilde{\theta}_2$, or $C_2(\theta_1, \theta_2) \leq C_2\left(\theta_1, \tilde{\theta}_2\right)$ and $\theta_2 < \tilde{\theta}_2$, then $u_2\left(C_2(\theta_1, \theta_2); \tilde{\theta}_2\right) - u_2\left(C_2\left(\theta_1, \tilde{\theta}_2\right); \tilde{\theta}_2\right) \leq \beta\left(u_3\left(C_3\left(\theta_1, \tilde{\theta}_2\right)\right) - u_3(C_3(\theta_1, \theta_2))\right)$, which is equivalent to $\text{IC}_2\left(\theta_1, \tilde{\theta}_2, \theta_2\right)$. **QED**

Lemma A-3 *Fix $C_1(\theta_1)$, and define $(\check{c}_2, \check{c}_3) = \arg\max_{(\tilde{c}_2, \tilde{c}_3)\ st\ \tilde{c}_2 + \tilde{c}_3 \leq W - C_1(\theta_1)} V^2(\tilde{c}_2, \tilde{c}_3; \underline{\theta}_2)$. If $\beta < \frac{u_2'(\check{c}_2; \underline{\theta}_2)}{u_2'\left(\check{c}_2; \bar{\theta}_2\right)}$, then for any $p \in (0,1)$ the solution to*

$$\max_{C(\theta_1, \cdot)\ s.t. IC_2\left(\theta_1, \underline{\theta}_2, \bar{\theta}_2\right),\ IC_2\left(\theta_1, \bar{\theta}_2, \underline{\theta}_2\right),\ RC} pV^2(C(\theta_1, \underline{\theta}_2); \underline{\theta}_2) + (1-p) V^2\left(C\left(\theta_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$$

has $C(\theta_1, \underline{\theta}_2) = C\left(\theta_1, \bar{\theta}_2\right)$.

Proof of Lemma A-3: The proof is by contradiction. Suppose to the contrary that there is a solution with $C_2\left(\theta_1, \bar{\theta}_2\right) \neq C_2(\theta_1, \underline{\theta}_2)$. By Lemma 1, $C_2\left(\theta_1, \bar{\theta}_2\right) > C_2(\theta_1, \underline{\theta}_2)$. There are three cases.

Case, $C_2(\theta_1, \underline{\theta}_2) \geq \check{c}_2$: By Assumption 4, $\beta < \frac{u_2'(x; \underline{\theta}_2)}{u_2'\left(x; \bar{\theta}_2\right)}$ for all $x \geq \check{c}_2$. So

$$\beta\left(u_2\left(C_2\left(\theta_1, \bar{\theta}_2\right); \bar{\theta}_2\right) - u_2\left(C_2(\theta_1, \underline{\theta}_2); \bar{\theta}_2\right)\right) < u_2\left(C_2\left(\theta_1, \bar{\theta}_2\right); \underline{\theta}_2\right) - u_2(C_2(\theta_1, \underline{\theta}_2); \underline{\theta}_2).$$

By $\text{IC}_2\left(\theta_1, \underline{\theta}_2, \bar{\theta}_2\right)$, the RHS is bounded above by $\beta\left(u_3(C_3(\theta_1, \underline{\theta}_2)) - u_3\left(C_3\left(\theta_1, \bar{\theta}_2\right)\right)\right)$. Hence

$$u_2\left(C_2\left(\theta_1, \bar{\theta}_2\right); \bar{\theta}_2\right) + u_3\left(C_3\left(\theta_1, \bar{\theta}_2\right)\right) < u_2\left(C_2(\theta_1, \underline{\theta}_2); \bar{\theta}_2\right) + u_3(C_3(\theta_1, \underline{\theta}_2)),$$

so that C is strictly dominated by the alternative contract in which $C\left(\theta_1, \bar{\theta}_2\right)$ is set to $C(\theta_1, \underline{\theta}_2)$, contradicting the hypothesis.

Case, $C_2(\theta_1, \underline{\theta}_2) < \check{c}_2 \leq C_2\left(\theta_1, \bar{\theta}_2\right)$: Consider the perturbation $\tilde{C}$ of C in which $\tilde{C}(\theta_1, \underline{\theta}_2) = (c_1(\theta_1), \check{c}_2, \check{c}_3)$ and $\tilde{C}\left(\theta_1, \bar{\theta}_2\right)$ is set to whichever of $C\left(\theta_1, \bar{\theta}_2\right)$ and $\tilde{C}(\theta_1, \underline{\theta}_2)$ maximizes $U^2\left(\cdot; \bar{\theta}_2\right)$. By definition, $V^2\left(\tilde{C}(\theta_1, \underline{\theta}_2); \underline{\theta}_2\right) > V^2(C(\theta_1, \underline{\theta}_2); \underline{\theta}_2)$, and so by Lemma A-1, $U^2\left(\tilde{C}(\theta_1, \underline{\theta}_2); \underline{\theta}_2\right) > U^2(C(\theta_1, \underline{\theta}_2); \underline{\theta}_2)$.

In the subcase in which $\tilde{C}\left(\theta_1, \bar{\theta}_2\right) = C\left(\theta_1, \bar{\theta}_2\right)$, it is immediate that $\tilde{C}$ strictly dominates C, giving a contradiction if $\tilde{C}$ satisfies $\text{IC}_2\left(\theta_1, \underline{\theta}_2, \bar{\theta}_2\right)$ and $\text{IC}_2\left(\theta_1, \bar{\theta}_2, \underline{\theta}_2\right)$. $\text{IC}_2\left(\theta_1, \underline{\theta}_2, \bar{\theta}_2\right)$ is satisfied because, since the original contract C is feasible, $U^2\left(C\left(\theta_1,\underline{\theta}_2\right);\underline{\theta}_2\right) \geq U^2\left(C\left(\theta_1,\bar{\theta}_2\right);\underline{\theta}_2\right)$. $\mathrm{IC}_2\left(\theta_1,\bar{\theta}_2,\underline{\theta}_2\right)$ holds because in this subcase $U^2\left(C\left(\theta_1,\bar{\theta}_2\right);\bar{\theta}_2\right) \geq U^2\left(\tilde{C}\left(\theta_1,\underline{\theta}_2\right);\bar{\theta}_2\right)$.

In the subcase in which $\tilde{C}\left(\theta_1,\bar{\theta}_2\right)=\tilde{C}\left(\theta_1,\underline{\theta}_2\right)$, it is immediate that $\mathrm{IC}_2\left(\theta_1,\underline{\theta}_2,\bar{\theta}_2\right)$ and $\mathrm{IC}_2\left(\theta_1,\bar{\theta}_2,\underline{\theta}_2\right)$ hold. In this subcase, $U^2\left(\tilde{C}\left(\theta_1,\underline{\theta}_2\right);\bar{\theta}_2\right) \geq U^2\left(C\left(\theta_1,\bar{\theta}_2\right);\bar{\theta}_2\right)$ and since $\tilde{C}_2\left(\theta_1,\underline{\theta}_2\right)=\check{c}_2 \leq C_2\left(\theta_1,\bar{\theta}_2\right)$, Lemma A-1 implies $V^2\left(\tilde{C}\left(\theta_1,\underline{\theta}_2\right);\bar{\theta}_2\right) \geq V^2\left(C\left(\theta_1,\bar{\theta}_2\right);\bar{\theta}_2\right)$, so that the perturbation $\tilde{C}$ strictly dominates C, a contradiction.

Case, $C_2\left(\theta_1,\bar{\theta}_2\right)<\check{c}_2$: Since date 2 MU is higher in $\bar{\theta}_2$ than $\underline{\theta}_2$, the perturbation $\tilde{C}$ of C in which $\tilde{C}\left(\theta_1,\underline{\theta}_2\right)=\tilde{C}\left(\theta_1,\bar{\theta}_2\right)=\left(c_1\left(\theta_1\right),\check{c}_2,\check{c}_3\right)$ strictly dominates C, giving an immediate contradiction.
QED

Lemma A-4 *If SPR holds then* $u_2'\left(s_1^*+C_2^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right) > \beta u_3'\left(C_3^*\left(\bar{\theta}_1\right)\right)$ *and* $U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right) = U^1\left(s_1^*+C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)$.

Proof of Lemma A-4: By definition, $u_2'\left(C_2^*\left(\bar{\theta}_1\right);\bar{\theta}_1\right)=u_3'\left(C_3^*\left(\bar{\theta}_1\right)\right)$, and SPR then implies $u_2'\left(s_1^*+C_2^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)>\beta u_3'\left(C_3^*\left(\bar{\theta}_1\right)\right)$. The definition of s_1^* then implies $U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)=U^1\left(s_1^*+C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)$.
QED

A Proofs of results stated in main text (excluding Section 5)

Proof of Lemma 1: From IC_2, $U^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\tilde{\theta}_2\right) \geq U^2\left(C\left(\theta_1,\theta_2\right);\tilde{\theta}_2\right)$ and $U^2\left(C\left(\theta_1,\theta_2\right);\theta_2\right) \geq U^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)$. Expanding, this implies $u_2\left(C_2\left(\theta_1,\tilde{\theta}_2\right);\tilde{\theta}_2\right)-u_2\left(C_2\left(\theta_1,\theta_2\right);\tilde{\theta}_2\right) \geq u_2\left(C_2\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)-u_2\left(C_2\left(\theta_1,\theta_2\right);\theta_2\right)$ which by $u_2'\left(\cdot;\tilde{\theta}_2\right)>u_2'\left(\cdot;\theta_2\right)$ implies $C_2\left(\theta_1,\tilde{\theta}_2\right) \geq C_2\left(\theta_1,\theta_2\right)$. **QED**

Proof that Assumption 4 is satisfied by multiplicative and additive shocks:

Case, multiplicative shocks: By supposition, $\theta u'\left(c_t\right) \geq \gamma\tilde{\theta}u'\left(c_s\right)$. Hence it suffices to show that $u'\left(c_t+x\right)/u'\left(c_s+x\right)$ is increasing in x, or equivalently, $-\frac{u''\left(c_s+x\right)}{u'\left(c_s+x\right)} \geq -\frac{u''\left(c_t+x\right)}{u'\left(c_t+x\right)}$. This is indeed the case since $c_t \geq c_s$ and u has NIARA.

Case, additive shocks: If $c_t-\theta<c_s-\tilde{\theta}$ then $u'\left(c_t+x-\theta\right)>u'\left(c_s+x-\tilde{\theta}\right) \geq \gamma u'\left(c_s+x-\tilde{\theta}\right)$ for all x. If instead $c_t-\theta \geq c_s-\tilde{\theta}$ then the proof follows exactly as in the multiplicative case.
QED

Proof of Proposition 1, Part (B): The proof is constructive. Define a contract C by $C\left(\bar{\theta}_1,\underline{\theta}_2\right)=C^*\left(\bar{\theta}_1,\underline{\theta}_2\right)$ and $C\left(\underline{\theta}_1,\bar{\theta}_2\right)=C^*\left(\underline{\theta}_1,\bar{\theta}_2\right)$, since these quantities are uniquely determined in any solution to Problem I. Moreover, define $C\left(\underline{\theta}_1,\underline{\theta}_2\right)=C^*\left(\underline{\theta}_1,\bar{\theta}_2\right)$. Finally, define $C\left(\bar{\theta}_1,\bar{\theta}_2\right)$ as in the main text, i.e., by (3), (4).

So-defined, C satisfies IC_1 and IC_2, as follows. The main text establishes that $\text{IC}_1(\underline{\theta}_1, \bar{\theta}_1)$ holds. $\text{IC}_1(\bar{\theta}_1, \underline{\theta}_1)$ holds because C solves Problem I, and so

$$u_1\left(C_1\left(\bar{\theta}_1\right);\bar{\theta}_1\right) + V^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right);\underline{\theta}_2\right) \geq u_1\left(C_1\left(\underline{\theta}_1\right);\bar{\theta}_1\right) + V^2\left(C\left(\underline{\theta}_1, \underline{\theta}_2\right);\underline{\theta}_2\right),$$

which since $C_1\left(\bar{\theta}_1\right) \geq C_1\left(\underline{\theta}_1\right)$ implies

$$u_1\left(C_1\left(\bar{\theta}_1\right);\bar{\theta}_1\right) + \beta V^2\left(C\left(\bar{\theta}_1, \underline{\theta}_2\right);\underline{\theta}_2\right) \geq u_1\left(C_1\left(\underline{\theta}_1\right);\bar{\theta}_1\right) + \beta V^2\left(C\left(\underline{\theta}_1, \underline{\theta}_2\right);\underline{\theta}_2\right).$$

As noted in the main text, $\text{IC}_2(\bar{\theta}_1, \bar{\theta}_2, \underline{\theta}_2)$ holds at equality; Lemma A-2 and $C_2\left(\bar{\theta}_1, \bar{\theta}_2\right) > C_2\left(\bar{\theta}_1, \underline{\theta}_2\right)$ then imply that $\text{IC}_2(\bar{\theta}_1, \underline{\theta}_2, \bar{\theta}_2)$ is satisfied. Finally, $\text{IC}_2(\underline{\theta}_1, \cdot, \cdot)$ hold trivially, since $C\left(\underline{\theta}_1, \cdot\right)$ is constant.

The remainder of the proof establishes that the pair of equations (3) and (4) have a solution, and that RC is satisfied for β small enough.

Solution to (3) and (4): Assumption 3 ensures that $C_3\left(\bar{\theta}_1, \bar{\theta}_2\right)$ can be chosen to satisfy (4). To show that $C_2\left(\bar{\theta}_1, \bar{\theta}_2\right)$ can be chosen to satisfy (3), by the mean value theorem it is sufficient to prove that

$$\begin{aligned} & u_1\left(C_1\left(\underline{\theta}_1\right);\underline{\theta}_1\right) + u_2\left(\max\left\{C_2\left(\underline{\theta}_1, \bar{\theta}_2\right), C_1\left(\bar{\theta}_1\right) + C_2\left(\bar{\theta}_1, \underline{\theta}_2\right) - C_1\left(\underline{\theta}_1\right)\right\};\bar{\theta}_2\right) \\ \geq\ & u_1\left(C_1\left(\bar{\theta}_1\right);\underline{\theta}_1\right) + u_2\left(C_2\left(\bar{\theta}_1, \underline{\theta}_2\right);\bar{\theta}_2\right). \end{aligned} \quad \text{(A-1)}$$

If $C_2\left(\underline{\theta}_1, \bar{\theta}_2\right) \geq C_1\left(\bar{\theta}_1\right) + C_2\left(\bar{\theta}_1, \underline{\theta}_2\right) - C_1\left(\underline{\theta}_1\right)$ then it is immediate that (A-1) holds, since $C\left(\underline{\theta}_1, \bar{\theta}_2\right)$ is part of a solution to Problem I and so, in particular, $C_1\left(\underline{\theta}_1\right)$ and $C_2\left(\underline{\theta}_1, \bar{\theta}_2\right)$ solve

$$\max_{c_1, c_2 \text{ s.t. } c_1 + c_2 \leq C_1\left(\underline{\theta}_1\right) + C_2\left(\underline{\theta}_1, \bar{\theta}_2\right)} u_1\left(c_1;\underline{\theta}_1\right) + u_2\left(c_2;\bar{\theta}_2\right).$$

Next, consider the opposite case, which is equivalent to $C_3\left(\underline{\theta}_1, \bar{\theta}_2\right) > C_3\left(\bar{\theta}_1, \underline{\theta}_2\right)$. Since C^* solves Problem I,[49]

$$\begin{aligned} u_1'\left(C_1\left(\underline{\theta}_1\right);\underline{\theta}_1\right) &= u_2'\left(C_2\left(\underline{\theta}_1, \bar{\theta}_2\right);\bar{\theta}_2\right) = u_3'\left(C_3\left(\underline{\theta}_1, \bar{\theta}_2\right)\right) \\ &< u_3'\left(C_3\left(\bar{\theta}_1, \underline{\theta}_2\right)\right) = u_2'\left(C_2\left(\bar{\theta}_1, \underline{\theta}_2\right);\underline{\theta}_2\right) \leq u_1'\left(C_1\left(\bar{\theta}_1\right);\bar{\theta}_1\right). \end{aligned} \quad \text{(A-2)}$$

[49]Note that the final inequality in (A-2) holds with equality; however, the proof of Proposition 3 also uses (A-2).

Since $C_1\left(\bar{\theta}_1\right) \geq C_1\left(\underline{\theta}_1\right)$, it then follows that $u_1'\left(\cdot;\bar{\theta}_1\right) > u_1'\left(\cdot;\underline{\theta}_1\right)$. Inequality (A-1) simplifies to

$$u_2\left(C_1\left(\bar{\theta}_1\right)+C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)-C_1\left(\underline{\theta}_1\right);\bar{\theta}_2\right)-u_2\left(C_2\left(\bar{\theta}_1,\underline{\theta}_2\right);\bar{\theta}_2\right) \geq u_1\left(C_1\left(\bar{\theta}_1\right);\underline{\theta}_1\right)-u_1\left(C_1\left(\underline{\theta}_1\right);\underline{\theta}_1\right).$$

If $C_2\left(\bar{\theta}_1,\underline{\theta}_2\right) \geq C_1\left(\underline{\theta}_1\right)$, this follows from $u_2'\left(C_2\left(\bar{\theta}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) > u_1'\left(C_1\left(\underline{\theta}_1\right);\underline{\theta}_1\right)$ (see (A-2) above), which by Assumption 4 implies $u_2'\left(C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)+x;\underline{\theta}_2\right) > u_1'\left(C_1\left(\underline{\theta}_1\right)+x;\underline{\theta}_1\right)$ for $x \geq 0$, and hence

$$u_2\left(C_1\left(\bar{\theta}_1\right)+C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)-C_1\left(\underline{\theta}_1\right);\underline{\theta}_2\right)-u_2\left(C_2\left(\bar{\theta}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) \geq u_1\left(C_1\left(\bar{\theta}_1\right);\underline{\theta}_1\right)-u_1\left(C_1\left(\underline{\theta}_1\right);\underline{\theta}_1\right),$$

which since $u_2'\left(\cdot;\bar{\theta}_2\right) \geq u_2'\left(\cdot;\underline{\theta}_2\right)$ implies the result. If instead $C_2\left(\bar{\theta}_1,\underline{\theta}_2\right) < C_1\left(\underline{\theta}_1\right)$, the result follows from the fact that $u_2'\left(\cdot;\bar{\theta}_2\right) \geq u_1'\left(\cdot;\underline{\theta}_1\right)$ (since given $u_1'\left(\cdot;\bar{\theta}_1\right) > u_1'\left(\cdot;\underline{\theta}_1\right)$, $u_1'\left(\cdot;\underline{\theta}_1\right) > u_2'\left(\cdot;\bar{\theta}_2\right)$ would violate Assumption 5).

RC satisfied for β sufficiently small: By construction, $C\left(\theta_1,\theta_2\right)$ satisfies RC for $\left(\theta_1,\theta_2\right) \neq \left(\bar{\theta}_1,\bar{\theta}_2\right)$. To show that $C\left(\bar{\theta}_1,\bar{\theta}_2\right)$ satisfies RC, note that (A-1) above implies

$$C_2\left(\bar{\theta}_1,\bar{\theta}_2\right) \leq \max\left\{C_2\left(\underline{\theta}_1,\bar{\theta}_2\right), C_1\left(\bar{\theta}_1\right)+C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)-C_1\left(\underline{\theta}_1\right)\right\}.$$

Observe that $C_2\left(\underline{\theta}_1,\bar{\theta}_2\right) < C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)+C_3\left(\bar{\theta}_1,\underline{\theta}_2\right)$, since this inequality is equivalent to $C_1\left(\underline{\theta}_1\right)+C_3\left(\underline{\theta}_1,\bar{\theta}_2\right) > C_1\left(\bar{\theta}_1\right)$, which is true by Assumption 6. Likewise, Assumption 6 implies $C_1\left(\bar{\theta}_1\right)+C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)-C_1\left(\underline{\theta}_1\right) < C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)+C_3\left(\bar{\theta}_1,\underline{\theta}_2\right)$. Hence

$$C_1\left(\bar{\theta}_1\right)+C_2\left(\bar{\theta}_1,\bar{\theta}_2\right) < C_1\left(\bar{\theta}_1\right)+C_2\left(\bar{\theta}_1,\underline{\theta}_2\right)+C_3\left(\bar{\theta}_1,\underline{\theta}_2\right)=W. \tag{A-3}$$

Since $C_3\left(\bar{\theta}_1,\bar{\theta}_2\right) \to 0$ as $\beta \to 0$, it follows that RC satisfied for β sufficiently small. **QED**

Proof of Lemma 2: Suppose to the contrary that $V^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\underline{\theta}_2\right) > V^2\left(C\left(\theta_1,\underline{\theta}_2\right);\underline{\theta}_2\right)$. By Lemma 1, $C_2\left(\theta_1,\tilde{\theta}_2\right) \geq C_2\left(\theta_1,\underline{\theta}_2\right)$. But then Lemma A-1 implies $U^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\underline{\theta}_2\right) > U^2\left(C\left(\theta_1,\underline{\theta}_2\right);\underline{\theta}_2\right)$, contradicting IC_2. **QED**

Proof of Lemma 3: Given Assumption 3, it is straightforward to show that there exists $\bar{c}_1$ such that, for any $\beta \in [0,1]$ and any solution to Problem II, $C_1\left(\theta_1\right) \leq \bar{c}_1$ for $\theta_1 = \underline{\theta}_1, \bar{\theta}_1$. Consequently, the solution to Problem II coincides with the solution to a more constrained problem—Problem II+, say—in which the constraint $C_1\left(\theta_1\right) \leq \bar{c}_1$ is added to the existing constraints in Problem II.

Let $\check{c}_2$ be as defined in Lemma A-3. Define $\bar{\beta} = \frac{u_2'\left(\check{c}_2;\underline{\theta}_2\right)}{u_2'\left(\check{c}_2;\bar{\theta}_2\right)}$ where $\check{c}_2$ is the value associated with

$C_1(\theta_1) = \bar{c}_1$. Given that $\check{c}_2$ is decreasing in $C_1(\theta_1)$, Assumption 4 implies that $\bar{\beta} \leq \frac{u_2'(\check{c}_2;\underline{\theta}_2)}{u_2'(\check{c}_2;\bar{\theta}_2)}$ for $\check{c}_2$ associated with $C_1(\theta_1) < \bar{c}_1$.

Given Lemma A-3, it then follows that for $\beta < \bar{\beta}$, any solution to Problem II+ features $C(\theta_1, \underline{\theta}_2) = C(\theta_1, \bar{\theta}_2)$. Consequently, the solution to Problem II+ is independent of β when $\beta < \bar{\beta}$. Hence the solution to Problem II is also independent of β for $\beta < \bar{\beta}$. **QED**

Proof of Proposition 2, Part (B): The proof is parallel to Proposition 1. Define $C = C^{**}$ everywhere except $C_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ and $C_3\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, which are defined by

$$\begin{aligned}
u_2\left(C_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) &= u_2\left(C_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \bar{\theta}_2\right) + \frac{u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)}{\Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right)} \\
u_3\left(C_3\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)\right) &= u_3\left(C_3\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)\right) - \frac{u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)}{\beta \Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right)}.
\end{aligned}$$

The contract C satisfies IC_1 and IC_2 as follows. $\mathrm{IC}_1\left(\bar{\bar{\theta}}_1, \underline{\underline{\theta}}_1\right)$ holds as in the proof of Proposition 1 because, given $\Pr\left(\underline{\theta}_2 | \bar{\bar{\theta}}_1\right) = 1$, $C^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) = C^{*}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$. By construction, $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2, \underline{\theta}_2\right)$ holds at equality, and Lemma A-2 then implies $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$ holds also. $\mathrm{IC}_2\left(\underline{\underline{\theta}}_1, \cdot, \cdot\right)$ hold because $C\left(\underline{\underline{\theta}}_1, \cdot\right) \equiv C^{**}\left(\underline{\underline{\theta}}_1, \cdot\right)$. Finally, $\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$ holds by a generalization of the proof for Proposition 1. First, using the utility decomposition (1), along with the fact that $C\left(\underline{\underline{\theta}}_1, \cdot\right) = C^{**}\left(\underline{\underline{\theta}}_1, \cdot\right)$, the expected gain to self 1 in state $\underline{\underline{\theta}}_1$ of getting consumption $C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$ for sure instead of the intended consumption is

$$\begin{aligned}
&\Pr\left(\underline{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\underline{\theta}}_1, \underline{\theta}_2\right) + \Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\underline{\theta}}_1, \bar{\theta}_2\right) \\
&- \left(\Pr\left(\underline{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\underline{\underline{\theta}}_1, \underline{\theta}_2\right); \underline{\underline{\theta}}_1, \underline{\theta}_2\right) + \Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\underline{\underline{\theta}}_1, \bar{\theta}_2\right); \underline{\underline{\theta}}_1, \bar{\theta}_2\right)\right) \\
\leq\ & (1-\beta)\left(u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right).
\end{aligned}$$

Second, by the construction of $C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, if self 1 reports $\bar{\bar{\theta}}_1$ in state $\underline{\underline{\theta}}_1$ his expected utility is

$$\begin{aligned}
&\Pr\left(\underline{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\underline{\theta}}_1, \underline{\theta}_2\right) + \Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \underline{\underline{\theta}}_1, \bar{\theta}_2\right) \\
=\ & \Pr\left(\underline{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\underline{\theta}}_1, \underline{\theta}_2\right) + \Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right) U^1\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\underline{\theta}}_1, \bar{\theta}_2\right) \\
&- (1-\beta)\left(u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right).
\end{aligned}$$

Combining these two observations delivers exactly $\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$.

As in the proof of Proposition 1, it remains to establish that C can be defined in this way, and that RC holds. The fact that C can be defined in this way is immediate from (5). RC holds by the same argument as in the proof of Proposition 1, given that (5) implies $C_2\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right) \leq C_2^{**}\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right) + C_3^{**}\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right)$. **QED**

Proof of Proposition 3: For use throughout, note that since the date 2 state following $\underline{\underline{\theta}}_1$ is deterministic, $C^{**}\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right) = C^{*}\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right)$ in economy A, and $C^{**}\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right) = C^{*}\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)$ in economy B.

Economy A: Let C be a solution to the problem defined by (6). Simplifying,

$$\begin{aligned}\kappa_A &= \Pr\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right)\left(V^2\left(C^{**}\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right)\right)\\ &\quad + \Pr\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right)\left(V^2\left(C^{**}\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right)\right)\\ &\quad + \Pr\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)\left(V^2\left(C^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)\right).\end{aligned}$$

Since $C^{**}\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right) = C^{*}\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right)$, $V^2\left(C\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) \leq V^2\left(C^{**}\left(\underline{\underline{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right)$. By Lemma 2,

$$-V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) \geq V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right).$$

Moreover,

$$\begin{aligned}V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) &= V^2\left(C^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)\\ &\quad + \int_{C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)}^{C_2\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)}\left(u_2'\left(c_2;\underline{\theta}_2\right) - u_2'\left(c_2;\bar{\theta}_2\right)\right)dc_2,\end{aligned}$$

which, since $u_2'\left(c_2;\underline{\theta}_2\right) - u_2'\left(c_2;\bar{\theta}_2\right) \leq 0$, implies that

$$\begin{aligned}V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right) &\geq V^2\left(C^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right) - V^2\left(C^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)\\ &\quad + \int_{C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)}^{W-C_1^{**}\left(\bar{\bar{\theta}}_1\right)}\left(u_2'\left(c_2;\underline{\theta}_2\right) - u_2'\left(c_2;\bar{\theta}_2\right)\right)dc_2.\end{aligned}$$

Substituting in these inequalities yields

$$\begin{aligned}\kappa_A \geq\ & \Pr\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) + \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \underline{\theta}_2\right) \\ & - \Pr\left(\bar{\bar{\theta}}_1\right) V^2\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) + \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) \int_{C_2^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)}^{W - C_1^{**}\left(\bar{\bar{\theta}}_1\right)} \left(u_2'\left(c_2; \underline{\theta}_2\right) - u_2'\left(c_2; \bar{\theta}_2\right)\right) dc_2.\end{aligned}$$

Finally, substituting in $\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$, i.e.,

$$V^2\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) \leq V^2\left(C\left(\underline{\underline{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) - \frac{1}{\beta}\left(u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right),$$

delivers the following lower bound, entirely in terms of C^{**}:

$$\begin{aligned}\kappa_A \geq\ & \Pr\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) + \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \underline{\theta}_2\right) \\ & + \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) \int_{C_2^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)}^{W - C_1^{**}\left(\bar{\bar{\theta}}_1\right)} \left(u_2'\left(c_2; \underline{\theta}_2\right) - u_2'\left(c_2; \bar{\theta}_2\right)\right) dc_2 \\ & - \Pr\left(\bar{\bar{\theta}}_1\right)\left(V^2\left(C^{**}\left(\underline{\underline{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) - \frac{1}{\beta}\left(u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right)\right).\end{aligned}$$

Economy B: For all β sufficiently small we construct a feasible contract C such that $C\left(\underline{\underline{\theta}}_1, \cdot\right) \equiv C_1^*\left(\underline{\underline{\theta}}_1, \cdot\right)$, $C_1\left(\bar{\bar{\theta}}_1\right) \equiv C_1^{**}\left(\bar{\bar{\theta}}_1\right)$, $V^2\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) \geq V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right)$ and the incentive constraint $\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$ holds with equality. The existence of such a contract implies the following upper bound on the cost κ_B:

$$\kappa_B \leq \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)\left(V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)\right).$$

$\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$ at equality and $C\left(\underline{\underline{\theta}}_1, \cdot\right) \equiv C_1^*\left(\underline{\underline{\theta}}_1, \cdot\right)$ imply (using $\Pr\left(\bar{\theta}_2 | \underline{\underline{\theta}}_1\right) = 1$) that

$$V^2\left(C^*\left(\underline{\underline{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) - V^2\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) = \frac{1}{\beta}\left(u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right).$$

Consequently,

$$\begin{aligned}\kappa_B \leq\ & \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)\left(V^2\left(C^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) - V^2\left(C^*\left(\underline{\underline{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)\right) \\ & + \Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) \frac{1}{\beta}\left(u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right).\end{aligned}$$

Since $\Pr\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) < \Pr\left(\bar{\bar{\theta}}_1\right)$, the result that $\kappa_B < \kappa_A$ when β is sufficiently small follows.

The remainder of the proof details the construction of the contract C.

Step 1: We first define a preliminary contract C^- by $C^-\left(\underline{\theta}_1, \bar{\theta}_2\right) = C^-\left(\underline{\theta}_1, \underline{\theta}_2\right) = C^*\left(\underline{\theta}_1, \bar{\theta}_2\right)$; $C_1^-\left(\bar{\bar{\theta}}_1, \cdot\right) \equiv C_1^{**}\left(\bar{\bar{\theta}}_1, \cdot\right)$; $C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$ maximizes $V^2\left(c; \underline{\theta}_2\right)$ given $C_1^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$, and subject to RC; and given $C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$, $C^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ is determined by the analogues of (3) and (4), with $C_t^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$ replacing $C_t^*\left(\bar{\theta}_1, \underline{\theta}_2\right)$ for $t = 2, 3$, and C_1^{**} replacing C_1^*.

$\mathrm{IC}_1\left(\underline{\theta}_1, \bar{\bar{\theta}}_1\right)$ holds exactly as in Step 1 of the proof of Proposition 1.

The definition of $C^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ is possible by the same proof as in Step 2 of the proof of Proposition 1. The one condition we have to check is that the analogue of condition (A-2) holds. This is the case, as follows. By Lemma A-3, for β low, $C^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) = C^{**}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, so $u_3'\left(C_3^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)\right) = u_1'\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right); \bar{\bar{\theta}}_1\right)$. Moreover, the definition of $C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$ and $\underline{\theta}_2 < \bar{\theta}_2$ imply $C_2^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) \leq C_2^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$, or equivalently, $C_3^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) \geq C_3^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$. Hence

$$u_2'\left(C_2^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) = u_3'\left(C_3^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)\right) \leq u_1'\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right); \bar{\bar{\theta}}_1\right) = u_1'\left(C_1^-\left(\bar{\bar{\theta}}_1\right); \bar{\bar{\theta}}_1\right),$$

so the analogue of condition (A-2) holds.

Finally, for β small enough, C^- satisfies RC by the same proof as in Step 3 of the proof of Proposition 1; the only slight complication is to note that, as observed above, $C_3^{**}\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) \leq C_3^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$, which ensures that the analogue of (A-3) holds.

Step 2: We next construct the actual contract C by perturbing the elements $C_2^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ and $C_3^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ of C^-; everywhere else, C equals C^-. To construct this perturbation, we make use of the following definitions. Let $\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ be the perturbation that maximizes date 2 consumption subject to satisfying RC and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2, \underline{\theta}_2\right)$; observe that this perturbation $\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ satisfies both RC and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2, \underline{\theta}_2\right)$ with equality. Let $\hat{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ be the perturbation that maximizes $V^2\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$ subject to RC and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$. Finally, for any x, let $C^x\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ be the perturbation that maximizes date 3 consumption subject to $C_2^x\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) = x$, RC and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$. Because RC and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$ both give upper bounds for $C_3^x\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, it follows from Assumption 3 that C^x exists, and is continuous in x.

Observe that $\bar{\theta}_2 > \underline{\theta}_2$ implies $C_2^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) \leq \hat{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, while trivially, $C_2^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right) \leq C_2^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) \leq \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$. By Lemma A-2, $\check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ satisfies $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$. Hence $C^x\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) = \check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ at $x = \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, while trivially $C^x\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) = \hat{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ at $x = \hat{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$. By the definition of $\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ and Lemma A-1, if $c_2 \geq \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ and $c_2 + c_3 \leq \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) + \check{C}_3\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ then $V^2\left(c; \bar{\theta}_2\right) \leq V^2\left(\check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$. Since $\check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ satisfies $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$, it follows that $\hat{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right) \leq \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$.

Recall that C^- satisfies $\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$, which is equivalent to $V^2\left(C^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$ being below

$$V^2\left(C^*\left(\underline{\underline{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) - \frac{1}{\beta}\left(u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right); \underline{\underline{\theta}}_1\right) - u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right); \underline{\underline{\theta}}_1\right)\right). \tag{A-4}$$

By the definitions of $\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ and $C^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, $U^2\left(\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) = U^2\left(C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \bar{\theta}_2\right) = U^2\left(C^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$. By Lemma A-1, $V^2\left(\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) \leq V^2\left(C^-\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$. Hence $V^2\left(\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$ is below expression (A-4).

Consumption $C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$ is independent of β, and hence for all β sufficiently small, $V^2\left(C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \bar{\theta}_2\right)$ exceeds expression (A-4). Trivially (since C^- satisfies RC) $V^2\left(C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \bar{\theta}_2\right) \leq V^2\left(\hat{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$. Hence $V^2\left(\hat{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$ exceeds expression (A-4) for β sufficiently small.

Define C to be the perturbation C^x such that $V^2\left(C^x\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$ equals expression (A-4) and $x \in \left[\hat{C}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right), \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)\right]$; existence of such a perturbation follows by continuity when β is sufficiently small.

Step: The contract C is feasible: By construction, C satisfies RC, $\mathrm{IC}_1\left(\underline{\underline{\theta}}_1, \bar{\bar{\theta}}_1\right)$ with equality, $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$, and $\mathrm{IC}_2\left(\underline{\underline{\theta}}_1, \cdot, \cdot\right)$. It remains to show it satisfies $\mathrm{IC}_1\left(\bar{\bar{\theta}}_1, \underline{\underline{\theta}}_1\right)$ and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2, \underline{\theta}_2\right)$.

$\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2, \underline{\theta}_2\right)$ is satisfied, as follows. Because $C = C^x$ for some $x \in \left[\hat{C}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right), \check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)\right]$, at least one of RC and $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$ must hold with equality. If RC holds with equality, then because $C_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$ lies between $C_2^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right)$ and $\check{C}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right)$, and because $U^2\left(C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \bar{\theta}_2\right) = U^2\left(\check{C}\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$, then $U^2\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) \geq U^2\left(C^-\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \bar{\theta}_2\right)$, so that $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \bar{\theta}_2, \underline{\theta}_2\right)$ is satisfied. If instead $\mathrm{IC}_2\left(\bar{\bar{\theta}}_1, \underline{\theta}_2, \bar{\theta}_2\right)$ holds with equality, the implication follows from Lemma A-2.

$\mathrm{IC}_1\left(\bar{\bar{\theta}}_1, \underline{\underline{\theta}}_1\right)$ is satisfied, as follows. Expanding, the constraint is

$$\begin{aligned} & u_1\left(C_1\left(\bar{\bar{\theta}}_1\right); \bar{\bar{\theta}}_1\right) + \beta \Pr\left(\bar{\theta}_2 | \bar{\bar{\theta}}_1\right) V_2\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) + \beta \Pr\left(\underline{\theta}_2 | \bar{\bar{\theta}}_1\right) V_2\left(C\left(\bar{\bar{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right) \\ \geq \; & u_1\left(C_1\left(\underline{\underline{\theta}}_1\right); \bar{\bar{\theta}}_1\right) + \beta \Pr\left(\bar{\theta}_2 | \bar{\bar{\theta}}_1\right) V_2\left(C\left(\underline{\underline{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right) + \beta \Pr\left(\underline{\theta}_2 | \bar{\bar{\theta}}_1\right) V_2\left(C\left(\underline{\underline{\theta}}_1, \underline{\theta}_2\right); \underline{\theta}_2\right). \end{aligned}$$

Substituting in, and in particular using the fact that $V_2\left(C\left(\bar{\bar{\theta}}_1, \bar{\theta}_2\right); \bar{\theta}_2\right)$is equal to (A-4), the constraint is equivalent to

$$\begin{aligned} & u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\bar{\bar{\theta}}_1\right)-\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right)\left(u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\underline{\underline{\theta}}_1\right)-u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right)\right)+\beta\Pr\left(\underline{\theta}_2|\bar{\bar{\theta}}_1\right)V_2\left(C\left(\bar{\bar{\theta}}_1,\underline{\theta}_2\right);\underline{\theta}_2\right) \\ \geq\; & u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right);\bar{\bar{\theta}}_1\right)+\beta\Pr\left(\underline{\theta}_2|\bar{\bar{\theta}}_1\right)V_2\left(C^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right). \end{aligned}$$

By Lemma 3, for β small every quantity in this inequality, other than β itself, is independent of β, and hence $\text{IC}_1\left(\bar{\bar{\theta}}_1,\underline{\underline{\theta}}_1\right)$ holds provided

$$u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\bar{\bar{\theta}}_1\right)-u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right);\bar{\bar{\theta}}_1\right)>\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right)\left(u_1\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\underline{\underline{\theta}}_1\right)-u_1\left(C_1^*\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right)\right).$$

Consequently, it is sufficient to prove

$$u_1'\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\bar{\bar{\theta}}_1\right)\geq\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right)u_1'\left(C_1^*\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right). \tag{A-5}$$

By Lemma A-3, C^{**} is independent of the realization of θ_2, and so

$$\begin{aligned} u_1'\left(C_1^{**}\left(\bar{\bar{\theta}}_1\right);\bar{\bar{\theta}}_1\right) &= \Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right)u_2'\left(C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)+\Pr\left(\underline{\theta}_2|\bar{\bar{\theta}}_1\right)u_2'\left(C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right)=u_3'\left(C_3^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)\right) \\ u_1'\left(C_1^*\left(\underline{\underline{\theta}}_1\right);\underline{\underline{\theta}}_1\right) &= u_2'\left(C_2^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)=u_3'\left(C_3^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)\right). \end{aligned}$$

Consequently, (A-5) is equivalent to

$$\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right)u_2'\left(C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right)+\Pr\left(\underline{\theta}_2|\bar{\bar{\theta}}_1\right)u_2'\left(C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right);\underline{\theta}_2\right)\geq\Pr\left(\bar{\theta}_2|\bar{\bar{\theta}}_1\right)u_2'\left(C_2^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right);\bar{\theta}_2\right).$$

To complete the proof of the step, we show that $C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)\leq C_2^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)$. Suppose to the contrary that $C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)>C_2^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)$. Combined with $\bar{\theta}_2>\underline{\theta}_2$, this implies $u_3'\left(C_3^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)\right)<u_3'\left(C_3^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)\right)$, or equivalently $C_3^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)>C_3^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)$, which combined with $C_1^{**}\left(\bar{\bar{\theta}}_1\right)>C_1^*\left(\underline{\underline{\theta}}_1\right)$ implies $C_2^{**}\left(\bar{\bar{\theta}}_1,\bar{\theta}_2\right)<C_2^*\left(\underline{\underline{\theta}}_1,\bar{\theta}_2\right)$, a contradiction. **QED**

Proof of Lemma 4: Differentiating,

$$\frac{dU^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)}{d\tilde{\theta}_2}=u_2'\left(C_2\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)\frac{\partial C_2\left(\theta_1,\tilde{\theta}_2\right)}{\partial\tilde{\theta}_2}+\beta u_3'\left(C_3\left(\theta_1,\tilde{\theta}_2\right)\right)\frac{\partial C_3\left(\theta_1,\tilde{\theta}_2\right)}{\partial\tilde{\theta}_2}.$$

Substituting in (7),

$$\frac{dU^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)}{d\tilde{\theta}_2}=\left(u_2'\left(C_2\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)-u_2'\left(C_2\left(\theta_1,\tilde{\theta}_2\right);\tilde{\theta}_2\right)\right)\frac{\partial C_2\left(\theta_1,\tilde{\theta}_2\right)}{\partial\tilde{\theta}_2}.$$

Since $C_2\left(\theta_1,\tilde{\theta}_2\right)$ is increasing in $\tilde{\theta}_2$, $sign\left(\frac{dU^2\left(C\left(\theta_1,\tilde{\theta}_2\right);\theta_2\right)}{d\tilde{\theta}_2}\right)=-sign\left(\tilde{\theta}_2-\theta_2\right)$, implying the result. **QED**

Proof of Proposition 4: As noted in the main text, Part (A) is immediate from prior results. The proof of Part (B) is constructive. For any $\theta_1\in\Theta_1$, define the contract C by $C(\theta_1,\theta_2)=C^*(\theta_1,\phi(\theta_1))$ if $\phi^{-1}(\theta_2)>\theta_1$; while for $\phi^{-1}(\theta_2)\leq\theta_1$, or equivalently $\theta_2\geq\phi(\theta_1)$, define C by $C_1(\theta_1)=C_1^*(\theta_1)$ and the pair of differential equations (7) and

$$\begin{aligned}(1-\beta)\,u_2'\left(C_2(\theta_1,\theta_2);\theta_2\right)\frac{\partial C_2(\theta_1,\theta_2)}{\partial\theta_2} &= \max\left\{0,\frac{\partial}{\partial\theta_2}u_1\left(C_1(\theta_1);\phi^{-1}(\theta_2)\right)+\frac{\partial}{\partial\theta_2}u_2\left(C_2(\theta_1,\theta_2);\theta_2\right)\right.\\ &\quad\left.-\frac{d}{d\theta_2}U^1\left(C^*\left(\phi^{-1}(\theta_2),\theta_2\right);\phi^{-1}(\theta_2),\theta_2\right)\right\},\end{aligned}\qquad\text{(A-6)}$$

subject to the boundary condition that $C(\theta_1,\theta_2)=C^*(\theta_1,\phi(\theta_1))$ at $\theta_2=\phi(\theta_1)$.

The differential equations (7) and (A-6) imply that, for any $\tilde{\theta}_1$ and $\theta_2\geq\phi\left(\tilde{\theta}_1\right)$,

$$\frac{d}{d\theta_2}U^1\left(C\left(\tilde{\theta}_1,\theta_2\right);\phi^{-1}(\theta_2),\theta_2\right)\leq\frac{d}{d\theta_2}U^1\left(C^*\left(\phi^{-1}(\theta_2),\theta_2\right);\phi^{-1}(\theta_2),\theta_2\right).$$

Given the boundary condition, it follows that, for any $\tilde{\theta}_1$ and $\theta_2\geq\phi\left(\tilde{\theta}_1\right)$,

$$U^1\left(C^*\left(\tilde{\theta}_1,\theta_2\right);\phi^{-1}(\theta_2),\theta_2\right)\leq U^1\left(C^*\left(\phi^{-1}(\theta_2),\theta_2\right);\phi^{-1}(\theta_2),\theta_2\right).$$

Changing variables $\theta_1=\phi^{-1}(\theta_2)$, for any $\tilde{\theta}_1$ and $\theta_1\leq\tilde{\theta}_1$,

$$U^1\left(C^*\left(\tilde{\theta}_1,\phi(\theta_1)\right);\theta_1,\phi(\theta_1)\right)\leq U^1\left(C^*(\theta_1,\phi(\theta_1));\theta_1,\phi(\theta_1)\right).$$

Hence $IC_1\left(\theta_1,\tilde{\theta}_1\right)$ holds for any θ_1 and $\tilde{\theta}_1>\theta_1$.

Next, we show that $IC_1\left(\theta_1,\tilde{\theta}_1\right)$ holds for any θ_1 and $\tilde{\theta}_1<\theta_1$. In this case, $C\left(\tilde{\theta}_1,\phi(\theta_1)\right)=C^*\left(\tilde{\theta}_1,\phi\left(\tilde{\theta}_1\right)\right)$ since $\phi^{-1}(\phi(\theta_1))>\tilde{\theta}_1$, while $C(\theta_1,\phi(\theta_1))=C^*(\theta_1,\phi(\theta_1))$. Hence we must show that $U^1\left(C^*(\theta_1,\phi(\theta_1));\theta_1,\phi(\theta_1)\right)\geq U^1\left(C^*\left(\tilde{\theta}_1,\phi\left(\tilde{\theta}_1\right)\right);\theta_1,\phi(\theta_1)\right)$. By the ordering of Θ_1,

$C_1^*\left(\tilde{\theta}_1\right) \leq C_1^*(\theta_1)$, and by the optimality of C^*, $V^1(C^*(\theta_1,\phi(\theta_1));\theta_1,\phi(\theta_1)) \geq V^1\left(C^*\left(\tilde{\theta}_1,\phi\left(\tilde{\theta}_1\right)\right);\theta_1,\phi(\theta_1)\right)$. By the obvious analogue of Lemma A-1 for self 1's preferences, the result then follows.

Finally, RC is certainly satisfied for $\phi^{-1}(\theta_2) \geq \theta_1$, or equivalently, $\theta_2 \leq \phi(\theta_1)$. For $\theta_2 > \phi(\theta_1)$, observe that, by (7),

$$\frac{\partial C_2(\theta_1,\theta_2)}{\partial\theta_2} + \frac{\partial C_3(\theta_1,\theta_2)}{\partial\theta_2} = \left(1 - \frac{1}{\beta}\frac{u_2'(C_2(\theta_1,\theta_2);\theta_2)}{u_3'(C_3(\theta_1,\theta_2))}\right)\frac{\partial C_2(\theta_1,\theta_2)}{\partial\theta_2}.$$

At $\theta_2 = \phi(\theta_1)$, $u_2'(C_2(\theta_1,\theta_2);\theta_2) = u_3'(C_3(\theta_1,\theta_2))$ by the definition of C^*, so the term $1 - \frac{1}{\beta}\frac{u_2'(C_2(\theta_1,\theta_2);\theta_2)}{u_3'(C_3(\theta_1,\theta_2))}$ is strictly negative. The condition stated in Part (B) ensures that this expression remains negative for all $\theta_2 \in \left(\phi(\theta_1),\bar{\theta}_2\right)$. By construction, $\frac{\partial C_2(\theta_1,\theta_2)}{\partial\theta_2} \geq 0$. Hence $\frac{\partial C_2(\theta_1,\theta_2)}{\partial\theta_2} + \frac{\partial C_3(\theta_1,\theta_2)}{\partial\theta_2} \leq 0$, which implies RC is satisfied for all $\theta_2 \in (\phi(\theta_1),\bar{\theta}_2]$. **QED**

B Proofs for Section 5

Proof of Proposition 5: Suppose to the contrary that $u_2'\left(x;\bar{\theta}_1\right) > u_2'(s_1^* + x;\underline{\theta}_1)$ but there exists a feasible contract X with $X(\theta_1,\theta_1,0) = C^*(\theta_1)$ for $\theta_1 = \bar{\theta}_1,\underline{\theta}_1$.

Step 1, $V^2\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) \geq V^2\left(s_1 + X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\underline{\theta}_1\right)$ for all s_1 sufficiently close to s_1^*: (Note that this is the analogue of Lemma 2.) Among IC_2 are the constraints $U^2\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) \geq U^2\left(s_1 + X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\underline{\theta}_1\right)$ and $U^2\left(X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\bar{\theta}_1\right) \geq U^2\left(X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\bar{\theta}_1\right)$. Because of the supposition that $u_2'\left(x;\bar{\theta}_1\right) > u_2'(s_1 + x;\underline{\theta}_1)$ for s_1 close to s_1^*, exactly the same argument as in Lemma 1 implies that $X_2\left(\bar{\theta}_1,\bar{\theta}_1,0\right) \geq X_2\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right)$. The claimed inequality then follows from Lemma A-1, $X_2\left(\bar{\theta}_1,\bar{\theta}_1,0\right) \geq X_2\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right)$ and $U^2\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) \geq U^2\left(s_1 + X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\underline{\theta}_1\right)$.

Step 2, $u_2'\left(s_1^* + C_2^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right) \geq \beta u_3'\left(C_3^*\left(\bar{\theta}_1\right)\right)$: Suppose to the contrary that $u_2'\left(s_1^* + C_2^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right) < \beta u_3'\left(C_3^*\left(\bar{\theta}_1\right)\right)$. Consequently, by the definition of s_1^* there exists $s_1 < s_1^*$ and $s_2 > 0$ such that $U^1(C^*(\underline{\theta}_1);\underline{\theta}_1) < U^1\left(s_1 + C^*\left(\bar{\theta}_1\right) - s_2;\underline{\theta}_1\right)$ and $s_2 \in \arg\max_{\tilde{s}_2 \geq 0} U^2\left(s_1 + C_2^*\left(\bar{\theta}_1\right) - s_2;\underline{\theta}_1\right)$. Because $X\left(\bar{\theta}_1,\bar{\theta}_1,0\right) = C^*\left(\bar{\theta}_1\right)$, the constraints RC and IC_2 imply that the only possible value for $s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right)$ is $s_1 + C^*\left(\bar{\theta}_1\right) - s_2$. But then $U^1(X(\underline{\theta}_1,\underline{\theta}_1,0);\underline{\theta}_1) < U^1\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$, i.e., IC_1 is violated, giving a contradiction.

Step 3, completing the proof: Since $s_1^* > 0$, the definition of s_1^* and step 2 imply that $U^1\left(s_1 + C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right) > U^1(C^*(\underline{\theta}_1);\underline{\theta}_1)$ for all $s_1 < s_1^*$ sufficiently close to s_1^*, or equivalently, $U^1\left(s_1 + X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\underline{\theta}_1\right) > U^1(X(\underline{\theta}_1,\underline{\theta}_1,0);\underline{\theta}_1)$. Among IC_1 is the constraint $U^1(X(\underline{\theta}_1,\underline{\theta}_1,0);\underline{\theta}_1) \geq U^1\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$. Hence $U^1\left(s_1 + X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\underline{\theta}_1\right) > U^1\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$. But this contradicts step 1, completing the proof. **QED**

Proof of Lemma 5:

Step, sufficient conditions for (8): Let X be any contract such that $X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)$ is weakly decreasing in s_1, $u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1\right) \geq \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)\right)$, and (9) is satisfied. Define $f\left(\tilde{s}_1\right) = \max_{s_2 \geq 0} U^2\left(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - s_2; \underline{\theta}_1\right)$. We show that f has a global maximum at $\tilde{s}_1 = s_1$. By the envelope theorem,

$$f'\left(\tilde{s}_1\right) = u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2; \underline{\theta}_1\right) \frac{\partial X_2^*\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)}{\partial \tilde{s}_1} + \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right) \frac{\partial X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)}{\partial \tilde{s}_1}$$

where $\hat{s}_2 = \arg\max_{s_2 \geq 0} U^2\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - s_2; \underline{\theta}_1\right)$. Substituting in (9), $f'\left(\tilde{s}_1\right)$ equals

$$\beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right) \left(\frac{u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2; \underline{\theta}_1\right)}{\beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right)} - \frac{u_2'\left(\tilde{s}_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1\right)}{\beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)\right)} \right) \frac{\partial X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)}{\partial \tilde{s}_1}.$$

Since $X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)$ is weakly decreasing in $\tilde{s}_1$, it suffices to show that

$$\frac{u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2; \underline{\theta}_1\right)}{\beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right)} \geq (\leq) \frac{u_2'\left(\tilde{s}_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1\right)}{\beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)\right)} \text{ if } \tilde{s}_1 > (<)\, s_1.$$

Consider first the case $\tilde{s}_1 > s_1$. If $u_2'\left(\tilde{s}_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1\right) = \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)\right)$ then the required inequality holds since $u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2; \underline{\theta}_1\right) \geq \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right)$. If instead $u_2'\left(\tilde{s}_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1\right) > \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)\right)$ then $u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1\right) > \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)\right)$, which implies $\hat{s}_2 = 0$, and the inequality follows. Second, consider the case $\tilde{s}_1 < s_1$. If $u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2; \underline{\theta}_1\right) = \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right)$ then the required inequality holds since $u_2'\left(\tilde{s}_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1\right) \geq \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right)\right)$. If instead $u_2'\left(s_1 + X_2\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2; \underline{\theta}_1\right) > \beta u_3'\left(X_3\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right) - \hat{s}_2\right)$ then $\hat{s}_2 = 0$ and the inequality again follows.

Step, necessary conditions for (8): From *(8)*, for any s_1 and $\tilde{s}_1$,

$$\begin{aligned} & U^2(\tilde{s}_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1) - U^2(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1) \\ \geq \; & U^2(\tilde{s}_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1\right); \underline{\theta}_1) - U^2(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1) \\ \geq \; & U^2(\tilde{s}_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1) - U^2(s_1 + X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right); \underline{\theta}_1). \end{aligned} \tag{A-7}$$

Taking the limit as $|\tilde{s}_1 - s_1| \to 0$ establishes continuity. To establish (9), consider any $\tilde{s}_1 > s_1$, and divide inequality (A-7) by $\tilde{s}_1 - s_1$. If $X\left(\bar{\theta}_1, \underline{\theta}_1, s_1\right)$ is continuous at s_1, then the upper bound and the lower bound both converge to $u_2'(s_1 + X(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1)$ as $\tilde{s}_1 \to s_1$, implying $\frac{d}{ds_1} U^2(s_1 + X(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1) = u_2'(s_1 + X(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1)$, which is equivalent to (9). **QED**

Proof of Proposition 6 (sufficiency): The main text defines X^* only at $(\bar{\theta}_1, \underline{\theta}_1)$ and over the interval $[0, s_1^*]$. We start by defining X^* elsewhere. Define $X^*(\theta_1, \theta_2, s_1)$ for $(\theta_1, \theta_2) \neq (\bar{\theta}_1, \underline{\theta}_1)$, or for $(\theta_1, \theta_2) = (\bar{\theta}_1, \underline{\theta}_1)$ but $s_1 > s_1^*$, as the consumption profile resulting from $C^*(\theta_1)$ when self 2 freely privately saves, given the state is θ_2; formally

$$X^*(\theta_1, \theta_2, s_1) = s_1 + C^*(\theta_1) - \arg\max_{s_2 \geq 0} U^2(s_1 + C^*(\theta_1) - s_2; \theta_1).$$

Step1, RC is satisfied: This is immediate from the construction of X^* everywhere except for $X^*(\bar{\theta}_1, \underline{\theta}_1, s_1)$ where $s_1 \in [0, s_1^*]$. Over this range, it follows because by NS and (9), $X_2^*(\bar{\theta}_1, \underline{\theta}_1, s_1) + X_3^*(\bar{\theta}_1, \underline{\theta}_1, s_1)$ is increasing in s_1 over $[0, s_1^*]$. Hence if $s_1 \in [0, s_1^*]$ then $X_2^*(\bar{\theta}_1, \underline{\theta}_1, s_1) + X_3^*(\bar{\theta}_1, \underline{\theta}_1, s_1) \leq X_2^*(\bar{\theta}_1, \underline{\theta}_1, s_1^*) + X_3^*(\bar{\theta}_1, \underline{\theta}_1, s_1^*) = C_2^*(\bar{\theta}_1) + C_3^*(\bar{\theta}_1)$.

Step 2: $U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1) \geq U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \underline{\theta}_1)$: This is an immediate implication of Lemma 5 provided that $X^*(\bar{\theta}_1, \underline{\theta}_1, s_1)$ satisfies (9). By construction (9) holds for $s_1 \in [0, s_1^*]$. Moreover, condition (9) holds in the neighborhood to the right of s_1^* because, by Lemma A-4, here $X^*(\bar{\theta}_1, \underline{\theta}_1, s_1) \equiv C^*(\bar{\theta}_1)$. Finally, once $s_1 > s_1^*$ is large enough that $X^*(\bar{\theta}_1, \underline{\theta}_1, s_1) \neq C^*(\bar{\theta}_1)$, (9) holds because $u_2'(s_1 + X_2^*(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1) = \beta u_3'(X_3^*(\bar{\theta}_1, \underline{\theta}_1, s_1))$ and $\frac{d}{ds_1}(X_2^*(\bar{\theta}_1, \underline{\theta}_1, s_1) + X_3^*(\bar{\theta}_1, \underline{\theta}_1, s_1)) = 0$.

Step 3: $U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1) \geq U^2(s_1 + X^*(\bar{\theta}_1, \bar{\theta}_1, \tilde{s}_1) - s_2; \underline{\theta}_1)$: Because of the definition of $X^*(\bar{\theta}_1, \bar{\theta}_1, \cdot)$, it is sufficient to establish that $U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, s_1); \underline{\theta}_1) \geq U^2(s_1 + X^*(\bar{\theta}_1, \bar{\theta}_1, 0) - s_2; \underline{\theta}_1)$. Because $X^*(\bar{\theta}_1, \bar{\theta}_1, 0) = C^*(\bar{\theta}_1) = X^*(\bar{\theta}_1, \underline{\theta}_1, s_1^*)$, this is implied by the prior step.

Step 4: $U^2(s_1 + X^*(\bar{\theta}_1, \bar{\theta}_1, s_1); \bar{\theta}_1) \geq U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \bar{\theta}_1)$: Suppose to the contrary that $U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \bar{\theta}_1) > U^2(s_1 + X^*(\bar{\theta}_1, \bar{\theta}_1, s_1); \bar{\theta}_1)$, or equivalently (given the definition of $X^*(\bar{\theta}_1, \bar{\theta}_1, s_1)$), $U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \bar{\theta}_1) > \max_{\tilde{s}_2 \geq 0} U^2(s_1 + C^*(\bar{\theta}_1) - \tilde{s}_2; \bar{\theta}_1)$. By step 1, $X_2^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) + X_3^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) \leq C_2^*(\bar{\theta}_1) + C_3^*(\bar{\theta}_1)$, and so it follows that $X_2^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2 > C_2^*(\bar{\theta}_1)$. By step 2, we know $U^2(s_1^* + X^*(\bar{\theta}_1, \underline{\theta}_1, s_1^*); \underline{\theta}_1) \geq U^2(s_1^* + X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \underline{\theta}_1)$. Since $X^*(\bar{\theta}_1, \underline{\theta}_1, s_1^*) = C_2^*(\bar{\theta}_1) < X_2^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2$, SPR then implies $U^2(X^*(\bar{\theta}_1, \underline{\theta}_1, s_1^*); \bar{\theta}_1) \geq U^2(X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \bar{\theta}_1)$, which in turn implies $U^2(s_1 + C^*(\bar{\theta}_1); \bar{\theta}_1) \geq U^2(s_1 + X^*(\bar{\theta}_1, \underline{\theta}_1, \tilde{s}_1) - s_2; \bar{\theta}_1)$. This last inequality contradicts the original supposition, completing the proof.

Step 5: IC_2 holds: The fact that IC_2 holds after self 1 reports $\bar{\theta}_1$ follows from steps 2, 3 and 4, together with the fact that the definition of X^* immediately implies $U^2\left(s_1+X^*\left(\bar{\theta}_1,\bar{\theta}_1,s_1\right);\bar{\theta}_1\right)\geq U^2\left(s_1+X^*\left(\bar{\theta}_1,\bar{\theta}_1,\tilde{s}_1\right)-s_2;\bar{\theta}_1\right)$. The fact that IC_2 holds after self 1 reports $\underline{\theta}_1$ is likewise immediate from the definition of X^*.

Step 6: IC_1 holds: The definition of X^* implies that $\frac{d}{ds_1}U^1\left(s_1+X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)\geq 0$ over $[0,s_1^*]$. Hence for any $s_1\in[0,s_1^*]$, Lemma A-4 implies that

$$\begin{aligned}U^1\left(X^*\left(\underline{\theta}_1,\underline{\theta}_1,0\right);\underline{\theta}_1\right) &= U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)=U^1\left(s_1^*+C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)\\ &= U^1\left(s_1^*+X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right)\geq U^1\left(s_1+X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right).\end{aligned}$$

It is straightforward to show $U^1\left(X^*\left(\underline{\theta}_1,\underline{\theta}_1,0\right);\underline{\theta}_1\right)\geq U^1\left(s_1+X^*\left(\underline{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$, so that $\mathrm{IC}_1\left(\underline{\theta}_1,\bar{\theta}_1\right)$ holds. Similarly, its is straightforward to show that $U^1\left(X^*\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\bar{\theta}_1\right)\geq U^1\left(s_1+X^*\left(\theta_1,\bar{\theta}_1,s_1\right);\bar{\theta}_1\right)$ for $\theta_1=\bar{\theta}_1,\underline{\theta}_1$, so that $\mathrm{IC}_1\left(\bar{\theta}_1,\underline{\theta}_1\right)$ holds. **QED**

Proof of Proposition 6 (necessity): From Proposition 5, if SPR is violated then there is no feasible solution to Problem I. Here, we show that if SPR is satisfied but NS fails, again there is no feasible solution to Problem I. The proof is by contradiction: suppose to the contrary that SPR holds, NS fails, but there exists a feasible X such that $X(\theta,\theta,0)=C^*(\theta)$ for $\theta=\bar{\theta}_1,\underline{\theta}_1$.

Step 1: There exists $s_1^+\in[0,s_1^*]$ such that $u_2'\left(s_1^++X_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)<\beta u_3'\left(X_3^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right)\right)$ and $U^1\left(s_1^++X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)=U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)$.

For any x_1 and x_2, the expression $-u_1'\left(-s_1+x_1;\underline{\theta}_1\right)+\beta u_2'\left(s_1+x_2;\underline{\theta}_1\right)$ is strictly decreasing in s_1. Hence from the definition of X^*, either, (A),

$$\frac{\partial X_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right)}{\partial s_1}=\frac{-u_1'\left(-s_1+X_1^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)+\beta u_2'\left(s_1+X_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)}{(1-\beta)\,u_2'\left(s_1+X_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)} \tag{A-8}$$

for all $s_1\in(0,s_1^*)$, or else (B) there exists some $\hat{s}_1\in(0,s_1^*)$ such that (A-8) holds for $s_1\in(\hat{s}_1,s_1^*)$, and X^* is constant over $[0,\hat{s}_1]$. Equality (A-8) is equivalent to $\frac{d}{ds_1}U^1\left(s_1+X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)=0$. Using Lemma A-4,

$$U^1\left(s_1+X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)=U^1\left(s_1^*+X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right)=U^1\left(s_1^*+C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)=U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)$$

for all $s_1\in[0,s_1^*]$ in case (A), or for all $s_1\in[\hat{s}_1,s_1^*]$ in case (B). By the supposition that NS is violated, it then follows that there exists $s_1^+\in[0,s_1^*]$ such that both $u_2'\left(s_1^++X_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)<\beta u_3'\left(X_3^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right)\right)$ and $U^1\left(s_1^++X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)=U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)$.

Step 2: $U^2\left(s_1^+ + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right) < U^2\left(s_1^+ + X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)$.

Suppose to the contrary that $U^2\left(s_1^+ + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right) \geq U^2\left(s_1^+ + X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)$. By the supposition that X is IC, and step 1,

$$U^1\left(s_1^+ + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right) \leq U^1\left(X\left(\underline{\theta}_1,\underline{\theta}_1,0\right);\underline{\theta}_1\right) = U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right) = U^1\left(s_1^+ + X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right).$$

Since $X_1\left(\bar{\theta}_1,\cdot,\cdot\right) = X_1^*\left(\bar{\theta}_1,\cdot,\cdot\right)$, $V^2\left(s_1^+ + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right) \leq V^2\left(s_1^+ + X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)$. By Lemma A-1, $X_2\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right) \geq X_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right)$, and hence $X_3\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right) \leq X_3^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right)$. But then $u_2'\left(s_1^+ + X_2\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right) < \beta u_3'\left(X_3\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right)\right)$, violating IC_2.

Completing the proof: Define $\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,\cdot\right)$, a perturbation of $X^*\left(\bar{\theta}_1,\underline{\theta}_1,\cdot\right)$, as follows. At s_1^*, define $\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right)$ so that $\tilde{X}_1^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right) = C_1^*\left(\bar{\theta}_1\right)$ and $\tilde{X}_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right) < C_2^*\left(\bar{\theta}_1\right)$ and $U^1\left(s_1^* + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right) = U^1\left(s_1^* + C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)$. It follows from Lemma A-1 that $U^2\left(s_1^* + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right) < U^2\left(s_1^* + C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right)$. Given the boundary condition, for all $s_1 \in [0,s_1^*)$ define $\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right)$ by the pair of differential equations (9) and $\frac{d}{ds_1}U^1\left(s_1 + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) = 0$; hence $\tilde{X}^*$ is defined analogously to X^*, but without the imposition that $\tilde{X}_2^*$ be increasing. Hence $U^1\left(s_1 + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) = U^1\left(s_1^* + C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right) = U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)$ for all $s_1 \in [0,s_1^*]$, where the second equality follows from Lemma A-4.

Provided $\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right)$ is chosen sufficiently close to $X^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right)$, the inequality $U^2\left(s_1^+ + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right) < U^2\left(s_1^+ + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^+\right);\underline{\theta}_1\right)$ is inherited from step 2. Moreover, by IC_2, $U^2\left(s_1^* + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right) \geq U^2\left(s_1^* + X\left(\bar{\theta}_1,\bar{\theta}_1,0\right);\underline{\theta}_1\right) = U^2\left(s_1^* + C^*\left(\bar{\theta}_1\right);\underline{\theta}_1\right) > U^2\left(s_1^* + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right)$.

By Lemma 5, $U^2\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$, is continuous. Define

$$s_1^{++} = \sup\left\{s_1 \in \left[s_1^+,s_1^*\right] : U^2\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) < U^2\left(s_1 + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)\right\}.$$

By continuity, $s_1^{++} < s_1^*$.

We next claim that there exists some $s_1 \in \left[s_1^{++},s_1^*\right]$ such that $\tilde{X}_2^*\left(s_1\right) > X_2\left(s_1\right)$. To prove this, suppose to the contrary that $X_2\left(s_1\right) \geq \tilde{X}_2^*\left(s_1\right)$ for all $s_1 \in \left[s_1^{++},s_1^*\right]$. Because $X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right)$ is continuous at all but finitely many points, $U^2\left(s_1 + X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) - U^2\left(s_1 + \tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$ is differentiable with respect to s_1 at all but finitely many points, with a derivative of (by Lemma 5 and (9))

$$u_2'\left(s_1 + X_2\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) - u_2'\left(s_1 + \tilde{X}_2^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right) \leq 0.$$

Because $U^2\left(s_1+X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)-U^2\left(s_1+\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$ is continuous (again by Lemma 5), it follows that $U^2\left(s_1+X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)-U^2\left(s_1+\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)\leq 0$ for all $s_1\in\left[s_1^{++},s_1^*\right]$. But this contradicts the inequality $U^2\left(s_1^*+X\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right)>U^2\left(s_1^*+\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1^*\right);\underline{\theta}_1\right)$.

Finally, take $s_1\in\left[s_1^{++},s_1^*\right]$ such that $\tilde{X}_2^*\left(s_1\right)>X_2\left(s_1\right)$. We know $U^2\left(s_1+X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)\geq U^2\left(s_1+\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)$. So by Lemma A-1, $U^1\left(s_1+X\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)>U^1\left(s_1+\tilde{X}^*\left(\bar{\theta}_1,\underline{\theta}_1,s_1\right);\underline{\theta}_1\right)=U^1\left(C^*\left(\underline{\theta}_1\right);\underline{\theta}_1\right)=U^1\left(X\left(\underline{\theta}_1,\underline{\theta}_1,0\right);\underline{\theta}_1\right)$. But this violates IC_1, giving a contradiction and completing the proof. **QED**