Correlated information structures and optimal auctions

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Abstract

I study optimal information design in auctions under two assumptions: first, the auctioneer is constrained to provide any bidder with an information structure which is informative only about this bidder's, yet not the other bidders' valuations. Second, the auctioneer can correlate information structures across bidders. I show that correlating the information structures across bidders induces correlation among bidders' posterior valuations even if the underlying true valuations are independent, and, in fact, allows the auctioneer to virtually extract the public information first–best revenue.

Keywords: information design, mechanism design, correlation, rent extraction

JEL codes: D82, H57

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1 Introduction

A growing literature studies information design (aka Bayesian persuasion) as an instrument to influence the actions of one or a set of agents to achieve a certain goal of the designer. Applications include sender receiver games[1], price discrimination[2], auctions[3], voting[4], or belief concerns[5]. The literature adopts the approach that the designer can supply agents with any information structure which generates signals that are informative about an underlying payoff–relevant state of the world and can commit, often without observing herself, to disclosing the signal realization truthfully to the agents.

In this note, I study information design in auctions with two distinctive features. First, the auctioneer is constrained by the fact that he can provide an agent with an information structure only which is informative about this agent's, yet not other agents' valuations for the object. This captures that in many auction settings, an agent's valuation is ultimately this agent's private information not only vis-a-vis the auctioneer, but also vis-a-vis the other agents. For example, if agents' valuations reflect their idiosyncratic tastes (rather than common values), then by issuing more or less clear descriptions of the object, or by varying the time an agent is allowed to inspect the object, an auctioneer can influence how well an agent is informed about her valuation, yet in doing so, cannot affect what an agent believes about the other agents' (independent) valuations. Second, I allow the auctioneer to randomize over information structures and to correlate them across agents. For example, the auctioneer may provide the information through one of two "salesmen," one of whom informs agents truthfully about their valuations, whereas the other salesman has only praise for the object, irrespective of true valuations. The key observation of this note is that if the auctioneer randomizes between the two salesmen and bidders are not informed about the identity of the salesman chosen, then their posterior valuations become cor-

[1]Most notably, see Kamenica and Gentzkow (2011) and Rayo and Segal (2010). See also Kolotolin et al. (2015) when the receiver is informed, Deimen and Szalay (2016) when information is endogenous, Ely (2017) when information evolves dynamically, or Ivanov (2010) when the receiver can commit to an action.

[2]See Bergemann, Brooks, and Morris (2015), Rösler and Szentes (2017), Bergemann, Bonatti, and Smolin (2016).

[3]See Bergemann and Pesendorfer (2007), Bergemann and Wambach (2015), Esö and Szentes (2007), Bergemann, Brooks, and Morris (2016, 2017).

[4]See Alsonso and Câmara (2016), Heese and Lauermann (2016).

[5]See Ely, Fraenkel, and Kamenica (2015) and Rodina (2016) for models with suspense and career concerned agents, respectively.

related. This is so even if the bidders' underlying true valuations are stochastically independent. But, if posterior valuations are correlated, then it is well known from Crémer and McLean (1988, henceforth CM) that there are selling mechanisms with which the auctioneer can extract the full trading surplus. In fact, I show that, by correlating information structures, the auctioneer can obtain a revenue that is approximately equal to the first–best surplus which obtains if valuations are public information.[6]

On a more conceptual note, because I assume that the auctioneer can give an agent only information about his own valuation, she cannot affect an agent's (first-order) beliefs about the other agents' true valuations. She is therefore more constrained than a designer who can inform any agent about the entire profile of valuations (including those of other agents) and can hence directly influence an agent's first-order beliefs (about others' valuations).[7,8] However, by correlating information structures across bidders, the auctioneer affects their higher-order beliefs: In the auction, a bidder revises his beliefs about what a rival bidder believes about his (the rival's) valuation. Therefore, my auctioneer is less constrained than the auctioneer in the setting of Bergemann and Pesendorfer (2007) who, like me, assume that bidders can be only informed about their own valuations, but in addition restrict the auctioneer to only deterministically select an agent's information structure. Unlike in my setting with correlation, a bidder's higher-order beliefs are then unaffected because a signal is not informative about the signal received by the rival bidder, and this difference is precisely why full rent extraction can be achieved in my setting but not in Bergemann and Pesendorfer (2007). To the extent that higher-order beliefs matter in many incomplete information games, and that the restriction that the designer cannot affect first-

[6]More precisely, I consider an independent private values environment where bidders have finitely many possible valuations. The main question in the general environment is whether signals and a correlation device can be constructed so that agents' beliefs about each other satisfy CM's spanning condition. The argument extends beyond the auction setting to all mechanism design settings, including environments with interdependent values, where CM is applicable.

[7]In fact, if the auctioneer could affect agent's first-order beliefs without restriction, her problem would be almost trivial, because she could simply make agents' valuations common knowledge among them and then elicit them through a kind of shoot-the-liar mechanism.

[8]Settings in which a designer is entirely unrestricted in the choice of information structure are explored in various papers by Bergemann and Morris (2016) and Bergemann, Brooks, and Morris (2015, 2016, 2017) which characterize, in various contexts, entire equilibrium sets as a function of all possible information structures allowing signals to agents to condition on the underlying payoff–relevant state in arbitrary ways.

order beliefs is rather natural in many applications, the relevance of inducing correlation among information structures goes beyond the auction application studied in this note.[9]

The (almost) full surplus extraction result that I establish rests heavily on the use of CM mechanisms which are controversially discussed in the literature, because they abstract from limited liability or risk aversion.[10] However, if agents can sustain some losses, or are not too risk averse, there will still be benefits, though in a less stark form, from inducing correlation among agents' valuations.[11] Another objection raised against CM mechanisms is that they rest on demanding common knowledge assumptions and are rarely observed in practice. Addressing the question whether correlating information structures is beneficial in detail-free and practically observed mechanisms (such as first price auctions) would require a general analysis of how such mechanisms perform as a function of the correlation among agents' valuations. This, in itself, is an interesting question, but beyond the scope of this note.[12]

In independent work, Zhu (2017) pursues an idea similar to mine and considers an information plus mechanism design setting where agents have ex ante private information, and the designer can (commit to) disclose information about an initially unknown "shock" that affects all agents' preferences. Zhu (2017) establishes an equivalence between private and public disclosure by constructing an information structure so that the designer can achieve the same when she can and when she cannot observe the signal realizations disclosed to agents. This information structure has the feature that any individual signal released to an agent is entirely uninformative about the state, but when observed jointly, signals identify the state.[13] With two agents, Zhu

[9]In terms of notions of correlated equilibrium in incomplete information games (Bergemann and Morris, 2016, Forges, 2006), when one views the agents' messages available in the selling mechanism as corresponding to the actions of a basic game, then my construction corresponds to a correlated equilibrium where agents have no ex ante information beyond the prior and with the restriction that the mediator can condition the recommendation to an agent not on the entire state but only on an "aspect" of the state which only this agent can understand.

[10]In fact, in the case in which the first-best surplus almost fully extracted, there is almost no correlation between bidders' posterior valuation, and transfers by the bidders are nearly unbounded in some contingencies.

[11]In a similar vein, Dequiedt and Martimort (2015) propose a model of vertical contracting where correlation among the retailers' costs helps to extract more, yet not the full rent from retailers. This suggests that if the principal could design the retailers' information, even though she could not extract the full surplus, she would still benefit by correlating information structures across retailers.

[12]This question is somewhat different from Bergemann, Brooks, and Morris (2017) who derive worst (rather than best) case revenues over all possible correlations of bidder valuations in the first price auction.

[13]For a general characterization of such complementary signals, see also Börgers, Hernando-Veciana, and Krähmer

(2017)'s construction employs a CM mechanism to elicit private signal realizations without cost.[14] While not shown explicitly by Zhu (2017), his information structure can, as in my approach, be generated as the result of correlating information structures which provide agents with information only about their own yet not the other agents' preferences, but in contrast to my paper, Zhu (2017) does not focus on rent extraction.

That an auctioneer can benefit from endogenously creating correlation among bidder valuations has been observed in other contexts. Krähmer (2012) shows how an auctioneer can create correlation by randomizing over investments that improve valuations stochastically. Obara (2008) studies an auction model where buyers can take (hidden) actions that influence the joint distribution of their valuations. He demonstrates that almost full surplus extraction can be attained by a mechanism which implements a mixed action profile by buyers and has them report not only their valuation but also the realization of their actions.

Finally, Bergemann and Wambach (2015) also construct a disclosure policy and a mechanism which allows the seller to extract the first–best surplus. In contrast to my approach, which is static and relaxes incentive constraints through creating correlation, their disclosure rule relaxes incentive constraints by gradually releasing information to agents and screening them dynamically.

This article is organized as follows. The next section presents a simple example which conveys the main idea of the note. Sections 3 and 4 present the model and the main result. Section 5 contains concluding remarks. All proofs are in the appendix.

2 Example

Consider a seller who has one object for sale. There are two potential buyers who each have either a low valuation v_L or a high valuation v_H for the object ($v_L < v_H$). Valuations are equally likely and independent across buyers. (Thus, the underlying model displays independent private values.) Initially, no one knows the true valuations, but the seller, next to designing a selling mechanism, can provide buyers with a (private) signal about their valuation.

Specifically, the seller selects at random one of two (pre–programmed) "salesmen", T and

(2013).

[14] With more than two agents, Zhu (2017) more generally allows for settings without quasi-linear preferences and uses techniques from the implementation literature to construct mechanisms that elicit the agents' private signal realizations without cost.

N, who offer, say, descriptions of the object. When listening to the description of the "truthful" salesman T, each buyer can privately figure out his own valuation. The "non–truthful" salesman N, in contrast, always claims the object to be of high value. Formally, salesman T sends the message h to a buyer if this buyer has a high valuation, and he send the message ℓ to a buyer if this buyer has a low valuation. Salesman N always sends the message h, irrespective of a buyer's true valuation. Note that the information provided to a buyer depends on this buyer's valuation only, yet not on the other buyer's valuation. Therefore, a message informs a buyer about his own *true* valuation only, yet not about the other buyer's *true* valuation.

Crucially, buyers are *not* informed about the type of the salesman. Therefore, because both buyers face the same salesman, their posterior valuations will be correlated. In fact, receiving message ℓ reveals to a buyer that he has a low valuation; and, at the same time, makes it more likely that the other buyer received message ℓ, too. The reason is that observing message ℓ increases the likelihood (in fact, reveals) that salesman T has been selected, and thus, because salesman T more frequently than salesman N submits message ℓ (to the other buyer), increases the probability that the other buyer observed ℓ, too.

Formally, let $\lambda \in (0,1)$ be the probability with which salesman T is selected. Let $\hat{v}_{st} = E[v|s,t]$ be a buyer's posterior expected valuation, conditional on herself observing $s \in \{\ell, h\}$, and the other buyer observing $t \in \{\ell, h\}$. Denote by $b_s(t) = Pr(\text{other buyer observed } t|s)$ her belief that the other buyer observed $t \in \{\ell, h\}$. The following table depicts these values.

	$\hat{v}_{s\ell}$	$\hat{v}_{sh}$	$b_s(\ell)$
$s=\ell$	v_L	v_L	$1/2$
$s=h$	v_H	$\frac{2-2\lambda}{4-3\lambda}v_L + \frac{2-\lambda}{4-3\lambda}v_H$	$\frac{\lambda}{4-2\lambda}$

Table 1: posterior beliefs

Because beliefs are correlated (i.e., $b_\ell(\ell) \neq b_h(\ell)$), the seller can employ a CM type of mechanism to elicit the buyers' private information without leaving information rents.[15] Therefore, for a fixed $\lambda \in (0,1)$, the seller can extract the maximum trading surplus, given the buyers' posterior

[15] Observe that CM's requirement that beliefs be convexly or linearly independent is met, because a buyer's belief matrix

$$\begin{pmatrix} b_\ell(\ell) & b_h(\ell) \\ b_\ell(h) & b_h(h) \end{pmatrix} = \begin{pmatrix} 1/2 & \frac{\lambda}{4-2\lambda} \\ 1/2 & \frac{4-3\lambda}{4-2\lambda} \end{pmatrix}$$

has full rank for all $\lambda < 1$. (For $\lambda = 0$, message ℓ is never observed, and posterior beliefs are equal to prior beliefs.)

valuations:

$$\begin{aligned}
& Pr(\text{both buyers observe } \ell) \cdot \hat{v}_{\ell\ell} + Pr(\text{exactly one buyer observes } h) \cdot \hat{v}_{h\ell} \\
& + Pr(\text{both buyers observe } h) \cdot \hat{v}_{hh} \\
= \; & \lambda \frac{1}{4} \hat{v}_{\ell\ell} + \lambda \frac{1}{2} \hat{v}_{h\ell} + [\lambda \frac{1}{4} + (1-\lambda)] \hat{v}_{hh}.
\end{aligned}$$

As λ approaches 1, this converges to the full information first–best trading surplus $1/4 \cdot v_L + 3/4 \cdot v_H$. Hence, by choosing λ close to one, the seller obtains virtually the same revenue as if buyers' (true) valuations were public information. Notice that for $\lambda = 1$, the first–best is not attainable, because in this case, a buyer's observation is not informative about the other buyer's observation, and thus beliefs are independent.

In the next section, I show that the logic of the example generalizes to any auction environment with finitely many bidder valuations.

3 The model

There is one principal (seller) who has one object for sale, and there are two agents (bidders) $i, j \in \{1, 2\}$.[16] Agent i's true valuation for the object is $v_m^i > 0$ where m is drawn from a finite set of states $\mathscr{M}^i = \{1, \ldots, M^i\}$, $M^i \geq 2$, with probability $p_m^i > 0$. Let $p^i \in \Delta(\mathscr{M}^i)$ be the corresponding probability distribution (the prior).[17] To make the problem interesting, p^1 and p^2 are assumed to be stochastically independent.[18] Thus, the underlying model displays independent private values. Without loss of generality, I assume $M^1 \geq M^2$. Players are risk-neutral and have quasi-linear utilities. That is, if an agent i obtains the object with probability x^i and makes payments y^i to the principal, his utility is $v_m^i x^i - y^i$, and the principal's utility is $y^1 + y^2$.

At the outset, no one (including agents) has information about the valuations beyond the (commonly known) prior, but the principal can, without observing herself, disclose information to the agents. My objective is to study information disclosure under two distinctive assumptions: First, the principal can only disclose information to an agent which informs him about his, yet not the other agent's valuation. This captures the notion that valuations are private information not

[16] I consider two agents to simplify notation. The generalization to more than two agents is straightforward.

[17] Throughout, I denote by $\Delta(A)$ the set of probability distributions over A.

[18] If the underlying valuations are correlated, the problem is straightforward, because the principal could simply disclose valuations perfectly and then elicit them at no cost by a CM mechanism.

only vis-a-vis the principal, but also vis-a-vis the other agent. Second, the principal can correlate the disclosure of signals across the agents.

Specifically, an information structure for agent i consists of a set $\mathscr{S}^i = \{1, \dots, S^i\}$ of possible observations (or "signals") and for all $m \in M^i$, a conditional distribution $\pi^i_m \in \Delta(\mathscr{S}^i)$ over the set of signals. That is, $\pi^i_m(s) = Pr(s \mid m)$ is the probability that signal s occurs, conditional on agent i's valuation being v^i_m. Let $\Pi^i = (\pi^i_1, \dots, \pi^i_{M^i})$ be the corresponding $S^i \times M^i$–matrix.

To capture the idea that the principal can correlate the disclosure of signals across agents, I allow the principal to design a set $\{\Pi^i_1, \dots, \Pi^i_{K^i}\}$ of information structures for agent i[19] and to correlate information structures across agents by choosing the joint probability

$$\lambda_{k\ell} = Pr(\Pi^1 = \Pi^1_k, \Pi^2 = \Pi^2_\ell) \tag{1}$$

with which agent 1 is endowed with Π^1_k, and agent 2 is endowed with Π^2_ℓ. Let $\Lambda = (\lambda_{k\ell})_{k\ell}$ be the associated $K^1 \times K^2$–matrix. I refer to Λ as a correlation strategy.

I refer to a combination $((\mathscr{S}^i, \Pi^i_1, \dots, \Pi^i_{K^i})_{i=1,2}, \Lambda)$ of information structures for agent 1 and 2 and a correlation strategy as a (correlated) information structure. I say that a correlated information structure has "full support" if any information structure Π^i_k, $k = 1, \dots, K^i$, for agent i and any signal $s \in \mathscr{S}^i$ occurs with positive probability.

The timing is as follows. The principal publicly commits to a correlated information structure and a selling mechanism. Then a pair of information structures $(\Pi^1_{k^1}, \Pi^2_{k^2})$ is drawn according to the correlation strategy. Agents do not observe which information structures are drawn, but each agent i is privately informed about the signal $s \in \mathscr{S}^i$ that has been realized under $\Pi^i_{k^i}$. Agents then decide whether or not to participate in the selling mechanism. If an agent rejects, he gets his outside option normalized to zero, otherwise, the mechanism is implemented. By the revelation principle, I can focus on direct and incentive compatible mechanisms which require each agent i to report a signal and which induce truthful reporting. Moreover, because an agent's outside option can be replicated within the mechanism by never allocating the object to him and never make him pay, I can also restrict attention to individually rational mechanisms which supply each agent for all signals with at least 0 expected utility. Therefore, the objective of the principal is to design an information structure as well as an incentive compatible and individually rational mechanism to maximize her expected revenue.

[19]Note that I implicitly assume that the set of signals is independent of k and thus the same for all information structures. This is without loss of generality.

In what follows, posterior beliefs will be important. Let agent i's posterior mean valuation, conditional on herself observing s, and the other agent observing t be

$$\hat{v}^i_{st} = E[v^i_m | \text{ agent } i \text{ observed } s, \text{ agent } j \text{ observed } t], \tag{2}$$

where the expectation is taken over the states m. Let

$$b^i_s(t) = Pr(\text{agent } j \text{ observed } t \mid \text{ agent } i \text{ observed } s) \tag{3}$$

be the probability that agent i, when having observed s, assigns to agent j having observed t. Let $b^i_s \in \Delta(\mathscr{S}^j)$ be the corresponding distribution, and let $B^i = (b^i_1, \ldots, b^i_{S^i})$ be the associated $S^j \times S^i$–matrix. Note that conditional probabilities (conditional on signals) are well defined if the information structure has full support.

I conclude this section with showing how the belief matrix B^i is derived from the information structure using Bayes' rule. To do so, let $g^i_k(s) = Pr(s \mid k) = \sum_m \pi^i_{km}(s)p^i_m$ be the probability with which signal s is observed conditional on Π^i_k being selected for agent i. Denote by $g^i_k \in \Delta(\mathscr{S}^i)$ the associated distribution, and by

$$G^i = (g^i_1, \ldots, g^i_{K^i}) \tag{4}$$

the associated $S^i \times K^i$–matrix.

Moreover, let $\overline{g}^i(s)$ be the unconditional probability that agent i observes s (unconditional on the realization of the correlation strategy), and let $\overline{g}^i \in \Delta(\mathscr{S}^i)$ be the associated distribution.[20]

Lemma 1 *Let a full support information structure be given. Then*

$$B^i = G^j \Lambda^\top {G^i}^\top D(\overline{g}^i)^{-1}, \tag{5}$$

where $D(\overline{g}^i)$ is the $S^i \times S^i$–matrix with the coordinates of $\overline{g}^i$ on the diagonal and 0s elsewhere.

4 Optimal design

In this section, I show that the principal can design an information structure and a mechanism that yield her virtually the same revenue as if the agents' valuations were publicly known. When

[20]Formally, $\overline{g}^1 = \sum_k (\sum_\ell \lambda_{k\ell}) g^1_k$ and $\overline{g}^2 = \sum_k (\sum_\ell \lambda_{\ell k}) g^2_k$.

valuations are publicly known, the principal optimally offers the object to the agent with the highest valuation at a price equal to this valuation. Hence, she obtains the first-best revenue

$$R^{FB} = E[\max\{v_m^1, v_n^2\}], \tag{6}$$

where the expectation is taken over the distribution of states (m, n). Clearly, the first–best revenue is an upper bound on the principal's revenue when valuations are not publicly known.

Two other benchmarks are of interest. The first is the case when the principal can condition the information disclosed to an agent on the entire profile of valuations. In this case, the principal could simply fully disclose the valuations of both agents to each agent and elicit valuations at no cost by some "shoot-the-liar" mechanism, and therefore attain the first–best revenue.

In the second case, the principal is restricted to condition an agent's information on that agent's valuation and cannot correlate the information structures provided to agents. This case has been studied by Bergemann and Pesendorfer (2007) who show that the principal's revenue is then bounded away from the first–best. In contrast, the main result of this paper says that, by correlating information structures, the principal can virtually extract the first–best surplus:

Proposition 1 *The principal can virtually attain the first–best revenue. That is, for all $\varepsilon > 0$, there is an information structure and a selling mechanism under which the principal's revenue exceeds $R^{FB} - \varepsilon$.*

I establish the proposition through a sequence of lemmata. The basic idea is that by correlating information structures across agents, their posterior valuations get correlated which allows the extraction of all rents through a CM type of mechanism. Moreover, to obtain almost first–best, I will construct an information structure which almost perfectly informs agents about their valuations. More precisely, the conditions to apply CM in the present setting are as follows.

Lemma 2 *(Crémer and McLean, 1988) Let a full support information structure be given. Suppose that for all i, the matrix B^i satisfies the "spanning condition", that is, no b_s^i is in the convex hull of $b_1^i, \dots, b_{s-1}^i, b_{s+1}^i, \dots, b_{S^i}^i$. Then there is an incentive compatible and individually rational selling mechanism so that the principal's revenue $\hat{R}$ is equal to the first–best trading surplus, given the induced distribution of posterior valuations, i.e.,*

$$\hat{R} = E[\max\{\hat{v}_{st}^1, \hat{v}_{ts}^2\}], \tag{7}$$

where the expectation is taken over the joint distribution of signals (s, t) of agent 1 and agent 2.

While the spanning condition implies that agents' beliefs are correlated, it more specifically says that no signal (of agent i) is redundant in the sense that its informational content (about j's posterior valuations) could be replicated by randomizing over a set of other signals (of i). When the number of signals is the same for both agents, then B^i is a square matrix, and the spanning condition is equivalent to B^i being regular. I now construct an information structure which delivers such a regular B^i. I choose the set $\mathscr{S}^i$ of signals to be equal to set of states, $\mathscr{M}^1$, for agent 1 (recall that the number of states for agent 1, M^1, is weakly larger than that for agent 2). Intuitively, this is the minimal set of signals to include the possibility that one of the information structures Π^i_k for agent i is perfectly informative.

Lemma 3 *For $i = 1, 2$, let $\mathscr{S}^i = \mathscr{M}^1$ and $K^i = M^1$.*

(i) There are Π^i_k, $k = 1, \ldots, K^i$, so that whenever Λ is regular, then the correlated information structure $((\mathscr{S}^i, \Pi^i_1, \ldots, \Pi^i_{K^i})_{i=1,2}, \Lambda)$ has full support and B^i is regular (and thus satisfies the spanning condition).

(ii) Moreover, the information structure Π^i_1 for agent i can be chosen to be perfectly informative, that is, conditional on Π^i_1, agent i observes the realization $s = m$ if and only if his true valuation is v^i_m.

To show part (i), I construct an information structure with the feature that every signal $s \in \mathscr{S}^i = \mathscr{M}^1$ has positive probability under some Π^i_k. The full support property then follows because regularity of Λ implies that every Π^i_k occurs with positive probability. Moreover, under the information structure constructed, the matrices G^j and G^i have full rank. Regularity of B^i then follows from (5). It is noteworthy that there are various ways to construct an information structure with the desired properties, the proof presents merely one of them.

The role of part (ii) is that it permits the construction of an information structure that informs agents almost perfectly about their true valuations by choosing a correlation strategy Λ that places almost full probability weight on the perfectly informative information structure Π^1_1 and Π^2_1 for agents 1 and 2. This together with Lemma 2 implies that by adopting such a correlation strategy, the principal will obtain nearly the first–best revenue. The next lemma makes this more precise. To state it, let $\tilde{\Lambda}$ be the degenerate correlation strategy that selects the perfectly informative information structure Π^1_1 and Π^2_1 with probability 1. That is, $\tilde{\lambda}_{11} = 1$, and $\tilde{\lambda}_{k\ell} = 0$ otherwise.

Lemma 4 *Let the information structure for agent 1 and 2 constructed in Lemma 3 be given. Consider a sequence of regular correlation strategies which converges to* $\tilde{\Lambda}$.[21] *Then the principal's revenue* $\hat{R}$ *converges to the first–best revenue* R^{FB} *along the sequence.*

5 Concluding remarks

While I have presented my argument within the context of an auction setting, it equally applies more generally to all mechanism design settings such as public goods, bilateral trade, including ones with interdependent values, as long as CM arguments are applicable. Similarly, the argument carries over to the single agent case with the only difference that, instead of conditioning the agent's payments on the reports of the other agents, one needs to condition them on the realization of the correlation strategy ex post.[22]

One key assumption of my approach is that agents have no private information ex ante. A series of papers studies optimal disclosure of additional information when agents have some, yet imperfect ex ante information[23], but none of these papers considers the possibility of designing correlated ex post information structures.

Another avenue of future work that this note suggests is to explore more systematically the implications of constraints on the information design technology, such as the constraint that a designer cannot disclose to an agent what another agent privately values or thinks. While in the auction application studied in this note, such a constraint does (almost) not affect the designer's revenue, this will, of course, be different in general.

[21] As an example, consider the sequence $\Lambda^{(N)}$, $N = 1, 2, \ldots$, with $\lambda_{11}^{(N)} = 1 - 1/N$, $\lambda_{kk}^{(N)} = 1/(K^1 - 1)N$ for $k = 2, \ldots, K^1$, and $\lambda_{k\ell}^{(N)} = 0$ for $k \neq \ell$.

[22] See Riordan and Sappington (1988) for how to extract full surplus in the adverse selection principal agent model when there is correlation between the agent's private information and a public ex post signal.

[23] See Esö and Szentes (2007), Bergemann and Wambach (2015), Krähmer and Strausz (2015), Li and Shi (2017).

Appendix

Proof of Lemma 1 I omit the superindex i, and indicate by the superindex j the other agent. By Bayes' rule:

$$b_s(t) \;=\; \frac{\sum_{k,\ell} \lambda_{k\ell} g_k(s) g^j_\ell(t)}{\overline{g}(s)} = (g_1(s), \ldots, g_k(s))\,\Lambda \begin{pmatrix} g^j_1(t) \\ \vdots \\ g^j_{K^j}(t) \end{pmatrix} \cdot \frac{1}{\overline{g}(s)}. \tag{8}$$

Hence, $(b_s(1), \ldots, b_s(S)) = (g_1(s), \ldots, g_k(s))\Lambda G^{j\top} \cdot 1/\overline{g}(s)$, and we can write the column vector b_s as

$$b_s = G^j \Lambda^\top \begin{pmatrix} g_1(s) \\ \vdots \\ g_K(s) \end{pmatrix} \cdot \frac{1}{\overline{g}(s)}. \tag{9}$$

Basic matrix algebra now delivers

$$B = (b_1, \ldots, b_S) = G^j \Lambda^\top G^\top \begin{pmatrix} 1/\overline{g}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1/\overline{g}(S) \end{pmatrix} = G^j \Lambda^\top G^\top D(\overline{g})^{-1}, \tag{10}$$

as desired. Q.E.D.

Proof of Lemma 2 Let $x^i_s(t)$ be the probability that the mechanism allocates the object to agent i if agent i reports s and agent j reports t, and let $y^i_s(t)$ be the respective transfers to the principal. Denote by x^i_s and y^i_s the corresponding vectors in $\mathbb{R}^{S^j}$. Consider the first–best allocation rule, given posterior valuations: $x^i_s(t) = 1$ if $\hat{v}^i_{st} > \hat{v}^j_{ts}$, and, say, $x^i_s(t) = 1/2$ if $\hat{v}^i_{st} > \hat{v}^j_{ts}$. Let

$$w^i_{s,\tilde{s}} = \sum_t b^i_s(t) \hat{v}^i_{st} x^i_{\tilde{s}}(t) \tag{11}$$

be the gross utility of agent i, who observed s, from reporting $\tilde{s}$.

Transfers are constructed as follows. Under the spanning condition, the separating hyperplane theorem implies that for all $s \in \mathscr{S}^i$ there is a vector $a^i_s \in \mathbb{R}^{S^j}$ so that[24]

$$\langle b^i_s, a^i_s \rangle = 0, \quad and \quad \langle b^i_t, a^i_s \rangle > 0 \quad \forall t \neq s. \tag{12}$$

[24] $\langle \cdot, \cdot \rangle$ denotes the scalar product.

For numbers $\gamma_s^i > 0$, define the transfer $y_s^i = \gamma_s^i a_s^i + w_{s,s}^i \mathbb{1}$, where $\mathbb{1}$ is the vector in $\mathbb{R}^{S^j}$ consisting only of 1's. Then the mechanism is individually rational, because agents obtain zero expected utility from truth-telling, as

$$w_{s,s}^i - \langle b_s^i, y_s^i \rangle = w_{s,s}^i - w_{s,s}^i - \gamma_s^i \langle b_s^i, a_s^i \rangle = 0. \tag{13}$$

Moreover, truth-telling is optimal if for all $s, \tilde{s}$,

$$w_{s,s}^i - \langle b_s^i, y_s^i \rangle \geq w_{s,\tilde{s}}^i - \langle b_s^i, y_{\tilde{s}}^i \rangle \quad \Leftrightarrow \quad 0 \geq w_{s,\tilde{s}}^i - w_{\tilde{s},\tilde{s}}^i - \gamma_{\tilde{s}}^i \langle b_s^i, a_{\tilde{s}}^i \rangle, \tag{14}$$

where the right inequality obtains after inserting transfers. Now observe that because $\langle b_s^i, a_t^i \rangle > 0$ by (12), the right inequality can be met by choosing γ_t^i sufficiently large in which case the mechanism is incentive compatible.

Because the allocation rule is first–best, and agents receive no rents, the principal's revenue is the first–best trading surplus $\hat{R}$. And this completes the proof. Q.E.D.

Proof of Lemma 3 Let $K = K^1$, and let E_K be the identity matrix in $\mathbb{R}^{K,K}$ and let e_k be the k-th column vector of E_K. Define:

- $\Pi_1^1 = E_K, \quad \Pi_k^1 = (e_k, p^1, \ldots, p^1)$ for $k = 2, \ldots, K$.
- $\Pi_1^2 = (e_1, \ldots, e_{M^2}), \quad \Pi_k^2 = (e_k, p_0^2, \ldots, p_0^2)$ for $k = 2, \ldots, K$,

  where p_0^2 is the K-dimensional vector coinciding with p^2 in the first M^2 rows and being 0 in the last $K - M^2$ rows.

As to part (i). To see that the information structure has full support, observe that an information structure Π_k^i, for agent i, $i = 1, 2$, can only occur with probability zero if Λ has a row or a column consisting entirely of 0's. But this is impossible if Λ is regular. Moreover, by construction, for any signal s there is an information structure Pi_k^i for agent i and a state m so that $\pi_{km}^i(s) > 0$. Hence, any signal occurs with positive probability.

To show that B^i is regular, I show that G^1 and G^2 are regular. This implies the claim by (5). To see that G^1 is regular, recall that $G^1 = (g_1^1, \ldots, g_K^1)$. Now observe that by definition, $g_k^1 = \Pi_k^1 p^1$. Thus, I have to show that $\Pi_1^i p^1, \ldots, \Pi_K^1 p^1$ are linearly independent. Note that by construction,

$$\Pi_1^1 p^1 = p^1, \quad \Pi_k^1 p^1 = p_1^1 e_k + (1 - p_1^1) p^1. \tag{15}$$

To see linear independence, take $\alpha_1, \ldots, \alpha_K$ so that

$$\alpha_1 p^i + \sum_{k=2}^{K} \alpha_k [p_1^i e_k + (1-p_1^i)p^i] = 0 \quad \Leftrightarrow \quad \left(\alpha_1 + \sum_{k=2}^{K} \alpha_k^p (1-p_1^i)\right) p^i + \sum_{k=2}^{K} \alpha_k p_1^i e_k = 0. \quad (16)$$

Because $p^i, e_2, \ldots, e_K$ are linearly independent, the right equality implies:

$$\alpha_1 + \sum_{k=2}^{K} \alpha_k (1-p_1^i) = 0, \quad and \quad \alpha_k p_1^i = 0 \quad k = 2, \ldots K. \quad (17)$$

But this implies that $\alpha_k = 0$ for all k and establishes that G^1 is regular.

To see that G^2 is regular, a similar argument can be used to show the linear independence of

$$\Pi_1^2 p^2 = p_0^2, \quad \Pi_k^2 p^2 = p_1^2 e_k + (1-p_1^2) p_0^2, \quad k = 2, \ldots, K. \quad (18)$$

Finally, part (ii) follows from construction since $\Pi_1^1 = E_K$ and $\Pi_1^2 = (e_1, \ldots, e_{M^2})$. This completes the proof. Q.E.D.

Proof of Lemma 4 I have to show that as Λ converges to $\tilde{\Lambda}$, we have

$$\hat{R} = \sum_{(s,t) \in \mathscr{S}^1 \times \mathscr{S}^2} \max\{\hat{v}_{st}^1, \hat{v}_{ts}^2\} Pr(s,t) \quad \rightarrow \quad \sum_{(m,n) \in \mathscr{M}^1 \times \mathscr{M}^2} \max\{v_m^1, v_n^2\} p_m^1 p_n^2 = R^{FB}, \quad (19)$$

where $Pr(s,t) = \sum_{k,\ell} \lambda_{k\ell} g_k^1(s) g_\ell^2(t)$.

Indeed, recall that under the information structure defined in Lemma 3, $\mathscr{S}^1 = \mathscr{S}^2 = \mathscr{M}^1$, and that $\mathscr{M}^2 \subseteq \mathscr{M}^1$. I show that for all $(s,t) \in \mathscr{M}^1 \times \mathscr{M}^2$:

$$(\hat{v}_{st}^1, \hat{v}_{ts}^2) \quad \rightarrow \quad (v_s^1, v_t^2), \qquad and \qquad Pr(s,t) \quad \rightarrow \quad p_s^1 p_t^2. \quad (20)$$

This implies (19), because the right part also implies that $Pr(s,t) \rightarrow 0$ for all $(s,t) \notin \mathscr{M}^1 \times \mathscr{M}^2$, and since $\max\{\hat{v}_{st}^1, \hat{v}_{ts}^2\}$ is clearly bounded.

To establish the left part of (20), consider first agent 1. By Bayes' rule, the probability that agent 1's valuation is v_m^1 conditional on $(s,t) \in \mathscr{M}^1 \times \mathscr{M}^2$ is

$$Pr(v_m^1 \mid s,t) = \frac{\sum_n \sum_{k,\ell} \lambda_{k\ell} \pi_{km}^1(s) \pi_{\ell n}^2(t) p_m^1 p_n^2}{Pr(s,t)} = \frac{\sum_n \sum_{k,\ell} \lambda_{k\ell} \pi_{km}^1(s) \pi_{\ell n}^2(t) p_m^1 p_n^2}{\sum_{m,n} \sum_{k,\ell} \lambda_{k\ell} \pi_{km}^1(s) \pi_{\ell n}^2(t) p_m^1 p_n^2}. \quad (21)$$

Recall from the proof of Lemma 3 that $\Pi_1^1 = E_{M^1}$ and $\Pi_1^2 = (e_1, \ldots, e_{M^2})$. Because Λ converging to $\tilde{\Lambda}$ means that $\lambda_{11} \rightarrow 1$ and $\lambda_{k\ell} \rightarrow 0$ for $(k,\ell) \neq (1,1)$, this implies that

$$Pr(v_m^1 \mid s,t) \rightarrow 1 \text{ if } s = m, \quad \text{and} \quad Pr(v_m^1 \mid s,t) \rightarrow 0 \text{ otherwise}. \quad (22)$$

Hence,

$$\hat{v}^1_{st} = \sum_m v^1_m Pr(v^1_m \mid s,t) \quad \to \quad v^1_s, \tag{23}$$

as desired.

With similar steps it follows that $\hat{v}^2_{ts} \to v^2_t$ for all $(s,t) \in \mathcal{M}^1 \times \mathcal{M}^2$.

Finally, to see the right part in (19), note that again since $\Pi^1_1 = E_{M^1}$ and $\Pi^2_1 = (e_1, \ldots, e_{M^2})$, if $\Lambda \to \tilde{\Lambda}$, then

$$Pr(s,t) = \sum_{m,n}\sum_{k,\ell} \lambda_{k\ell}\pi^1_{km}(s)\pi^2_{\ell n}(t) p^1_m p^2_n \tag{24}$$

converges to $p^1_s p^2_t$ if $(s,t) \in \mathcal{M}^1 \times \mathcal{M}^2$, and to 0 otherwise. This implies the claim and completes the proof. Q.E.D.

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