Under-Insurance in Human Capital Models with Limited Enforcement*

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Abstract

This paper uses a macroeconomic model calibrated to U.S. data to show that limited contract enforcement leads to substantial under-insurance against human capital risk. The model economy is populated by a large number of risk-averse households who can invest in risk-free physical capital and risky human capital. Expected human capital returns are age-dependent and calibrated to match the observed life-cycle profile of median labor income. Households have access to a complete set of credit and insurance contracts, but their ability to use the available financial instruments is limited by the possibility of default (limited contract enforcement). According to the baseline calibration, young households are severely under-insured against human capital (labor income) risk and the welfare losses due to the lack of insurance are substantial. These results are robust to realistic variations in parameter values.

Keywords: Human Capital Risk, Limited Enforcement, Insurance

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1. Introduction

A large empirical literature has documented that returns to human capital investment are high on average.[1] In other words, human capital is an asset with high expected returns. Human capital is also a risky asset in the sense that ex-post investment returns vary greatly across ex-ante identical individuals. This suggests substantial gains from insurance, but there is abundant evidence that many households are not well insured against human capital risk. For example, a large empirical literature has found that consumption insurance against individual labor income risk is far from perfect.[2] This begs the question why households are not better insured against human capital risk, even though human capital risk is large and has important economic consequences.[3]

In this paper, we argue that limited contract enforcement explains the under-insurance of many young and middle-aged households in the US.[4] Intuitively, young households have access to a risky investment opportunity with high expected returns, but have little financial wealth to buy insurance or invest in human capital. In a world with a complete set of competitive markets and full enforcement of contracts, these households would borrow money

[1]See Blundell et al. (1999) for a survey of the literature estimating returns to on-the-job training and Card (1999) for a survey of the literature on schooling (education) returns.

[2]Meghir and Pistaferri (2011) summarize the voluminous empirical literature on individual labor income risk and the consumption response to labor income shocks. See, for example, Blundell, Pistaferri, and Preston (2008) for a recent empirical contribution regarding the consumption response to labor income shocks.

[3]Krebs (2003) shows that human capital risk has important consequences for economic growth, Huggett, Ventura, and Yaron (2011) argue that human capital risk is an important determinant of inequality in (lifetime) earnings, and Guvenen, Kuruscu, and Ozkan (2014) explain cross-country differences in earnings inequality through the interaction of human capital risk and taxation.

[4]The literature on limited enforcement/commitment is large. See, for example, Alvarez and Jermann (2000), Kehoe and Levine (1993), Kocherlakota (1996), and Thomas and Worrall (1988) for seminal theoretical contributions and Krueger and Perri (2006) and Ligon, Thomas, and Worrall (2002) for influential quantitative work. This literature has commonly used an exogenous process of labor income. Andolfatto and Gervais (2006) and Lochner and Monge (2011) provide an insightful analysis of models with human capital accumulation and endogenous borrowing constraints due to enforcement problems, but do not consider human capital risk.

to buy full insurance and invest more in human capital. In contrast, in a world with limited enforcement of credit contracts due to the possibility of default, these households cannot borrow enough money to achieve the full-insurance outcome even if insurance is available and priced in an actuarially fair manner. Thus, limited contract enforcement leads to under-insurance against human capital risk and under-investment in human capital.

In a recent paper (Krebs, Kuhn, and Wright, 2015), we use a calibrated macroeconomic model with human capital to show that limited contract enforcement explains a substantial amount of under-insurance against labor income risk for young and middle-aged households in the US.[5] This finding stands in stark contrast to Krueger and Perri (2006), who find a high level of insurance in the equilibrium of a calibrated macroeconomic model with limited contract enforcement.[6] In this paper, we discuss the reasons for these apparently conflicting results, and argue that the model used in Krueger and Perri (2006) misses two dimensions of the data that are important drivers of under-insurance. In addition, we argue that our finding of substantial under-insurance for young households in the US is robust to realistic variations in parameter values. Finally, we generalize a number of theoretical results shown in Krebs, Kuhn, and Wright (2015).

The model analyzed in this paper is a production economy with an aggregate constant-returns-to-scale production function using physical and human capital as input factors. There are a large number (a continuum) of individual households with CRRA-preferences who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to shocks to the stock of human capital that follow a stationary Markov

[5]Krebs, Kuhn, and Wright (2015) also provide empirical evidence drawn from life insurance data in support of the model's main economic channel generating under-insurance.

[6]Krueger and Perri (2006) match the cross-sectional distribution of consumption fairly well, but the implied volatility of individual consumption is small in their model. See also Cordoba (2008) for a discussion of this point.

process with finite support (a Markov chain). In the main part of the paper, we assume that all shocks are idiosyncratic, but we also discuss how our theoretical characterization result can be extended to the case in which idiosyncratic and aggregate shocks co-exists. Households have access to a complete set of credit and insurance contracts, but their ability to use the available financial instruments is limited by the possibility of default, which produces endogenous borrowing, or short-sale, constraints. Defaulting households continue to participate in the labor market, but part of their labor income might be garnished and they are excluded from financial markets until a stochastically determined future date.

The tractability of the model derives from two equilibrium characterization results. First, the consumption-investment choice of households is linear in total wealth (financial wealth plus human capital) and the portfolio choice of households is independent of wealth. Further, the solution to the household decision problem can be obtained solving a static maximization problem, and this maximization problem is convex so that a simple FOC-approach is applicable. Second, recursive equilibria can be found by solving a fixed-point problem that is independent of the wealth distribution. Thus, a rather complex, infinite-dimensional fixed-point problem has been transformed into a much simpler, finite-dimensional fixed-point problem.

For the quantitative analysis we consider a version of the model with i.i.d. human capital shocks and stochastically aging households divided into 9 age groups. Household age affects expected human capital returns and younger households have higher returns than older households. The model is calibrated to be consistent with the U.S. evidence on labor market risk and life-cycle earnings. Specifically, we choose the model parameters determining the life-cycle profile of expected human capital returns so that the implied life-cycle profile of median earnings the data. Further, in our model, i.i.d. shocks to the stock of human capital translate into a labor income process that follows a logarithmic random walk; that is, labor income shocks are permanent. The random-walk specification has often been used in the empirical literature to model the permanent component of labor income risk, and we use the estimates obtained by this literature to calibrate our model economy. Finally, for the baseline calibration we use a degree of relative risk aversion of 1 (log-utility) and a level of contract enforcement (exclusion from financial markets in case of default) in line with the US bankruptcy code.

The quantitative analysis shows that young households are borrowing constrained and substantially under-insured, where we measure the degree of consumption insurance by the insurance coefficient defined as one minus the ratio of the volatility of consumption growth to the volatility of income growth. For example, this insurance coefficient is only 0.40 for households of age group $26-30$ even though insurance markets exist and are perfectly competitive. Further, the welfare consequences of the lack of consumption insurance are severe. For households of age group $26-30$, welfare would increase by 6 percent of lifetime consumption if they had unlimited access to financial markets.[7] The quantitative results are robust to realistic variations in the model parameters describing human capital risk, risk aversion, and contract enforcement.[8]

In contrast to our under-insurance result, Krueger and Perri (2006) find that households are well insured against labor income risk in the equilibrium of their calibrated model economy. In this paper, we show that this difference is largely driven by two assumptions. First, Krueger and Perri (2006) disregard the life-cycle component of earnings and the corresponding heterogeneity in financial wealth. In other words, in the data most young households

[7]We further show that the calibrated model is in line with the observed life-cycle pattern of household portfolio choices (mix between financial capital and human capital) even though the model is not calibrated to match this dimension of the data. This finding provides additional evidence supporting the model.

[8]However, such parameter variations have non-negligible effects on the extent of equilibrium insurance, which suggest that the model presented here has the potential to account for substantial differences in consumption insurance over time and across countries.

have high earnings growth and hold little financial wealth, but the calibrated model economy of Krueger and Perri (2006) misses this feature of the data. Second, Krueger and Perri (2006) assume permanent exclusion from credit markets for defaulting households, whereas in the US the bankruptcy code is considerably more lenient. To show the importance of these two assumptions, we consider a version of our human capital model without a life-cycle component and permanent exclusion of defaulting households. In this case, we find an insurance coefficient close to one, which is much higher than the insurance coefficient of young households in our baseline model and in line with the finding in Krueger and Perri (2006).

In sum, in this paper we make two contributions to the literature. First, we develop a framework with human capital risk and limited contract enforcement that is more general than the one discussed in Krebs, Kuhn, and Wright (2015) and show that the tractability of the model is still preserved. Second, we provide an extensive quantitative analysis that illuminates why, contrary to the results obtained by the previous literature, the work by Krebs, Kuhn, and Wright (2015) finds that a large group of US households is under-insured due to limited contract enforcement.

2. Model

In this section, we develop the model and define the relevant equilibrium concept. The model is a generalization of Krebs, Kuhn, and Wright (2015), which in turn is based on a combination of the human capital model developed in Krebs (2003) and the limited commitment model with linear technology presented in Wright (2001).[9]

[9] Angeletos (2007) and Moll (2014) develop tractable models of entrepreneurial activity in which individual consumption/saving policies are linear in wealth. In all these approaches, tractability is achieved through the assumption that individual investment returns are independent of household wealth (financial and/or human).

2.1 Human Capital Production

Time is discrete, open ended, and indexed by $t = 0, 1, \ldots$. There is a continuum of households who live for a stochastic length of time. A household who dies is replaced by a new-born household so that the mass of all households alive is normalized to one. We denote the cohort of a household (the period of birth) by n, but will suppress the cohort-index for notational ease until we discuss the aggregate market clearing conditions. The exogenous state of an individual household is denoted by s_t and has several components $s_t = (s_{1t}, \ldots, s_{mt})$. In our quantitative application, s_t has two components, one denoting the age of the household and a second representing human capital risk. Depending on the application, additional components can be used to model either ex-ante heterogeneity or ex-post heterogeneity (risk). For example, Krebs, Kuhn, and Wright (2015) use additional components to model the family structure of households in detail. For simplicity, we assume that s_t can only take on a finite number of values. We assume that for each household of cohort n, the process $\{s_t\}_{t=n}^{\infty}$ is Markov with a stationary transition function and denote the transition probabilities by $\pi(s_{t+1}|s_t)$. Note that household variables should in principle have a cohort index n in addition to the time index t, but to ease the notation we suppress the cohort index whenever possible.

There is one good that can be consumed or used as physical capital in production (see below). Each household can transform one unit of the good into $\phi(s_t)$ units of human capital. The accumulation equation for human capital, h, of an individual household is given by

$$h_{t+1} \;=\; (1 + \epsilon(s_{t-1}, s_t))\, h_t + \phi(s_t) x_{ht} \;, \tag{1}$$

where x_{ht} is human capital investment of the individual household in period t and ϵ is an idiosyncratic human capital shock.

Equation (1) assumes that human capital production is linear in goods investment, x_h, and this assumption is essential for our tractability result. Formulation (1) further assumes that human capital production only uses market goods, but this assumption is not essential for our result. Indeed, in Section 3.6 we present an extension of the model to the case in which individual households buy market goods and use their own time to produce human capital, and show that our tractability result still holds if human capital production exhibits constant returns to scale with respect to the quantity of goods and time invested.[10]

The term ϵ in (1) captures deterministic and random changes in human capital that are due to depreciation, learning-by-doing, and various shocks to human capital (skills) of households. For example, a negative human capital shock could can occur when a household member loses firm- or sector-specific human capital subsequent to job termination (worker displacement). A decline in health (disability) or death of a household member provide further examples of negative human capital shocks. In this case, both general and specific human capital are lost. Internal promotions and upward movement in the labor market provide two examples of positive human capital shocks.

We impose the restriction that the stock of human capital must be non-negative, or $h \geq 0$. This creates no technical difficulty and our general characterization of the household decision rule (proposition 1) holds with this constraint imposed, regardless of whether or not it binds. In our quantitative analysis, this constraint never binds (does not bind for all households types and uncertainty states). We do not impose the requirement that gross human capital investment be non-negative, or $x_h \geq 0$. This is necessary for tractability which, in turn, is essential for the theoretical and quantitative analysis conducted in this paper. In the calibrated model economy used for our quantitative analysis, human capital investment is

[10]In line with equation (1), Jones and Manuelli (1990) and Rebelo (1991) focus on the goods cost of human capital production and constant human capital returns. Heckman, Lochner, and Taber (1998) and Huggett et al. (2011) consider models of post-school human capital formation (on-the-job training) with only time cost of human capital investment and diminishing returns. Lochner and Monge (2011) consider education choices with only goods cost of investment and diminishing education returns. As suggested by Ben-Porath (1967) and Trostel (1993), in many applications both goods cost and time cost are non-negligible components of the total cost of human capital production.

non-negative in equilibrium for working-age households. Thus, imposing these restrictions would not change the conclusions drawn in the quantitative analysis.

2.1 Household Budget Constraint

An individual household born in period n of type s_n begins life with an initial endowment of human capital, h_n and an initial endowment of financial assets, a_n. The initial state of an individual household is therefore a vector (a_n, h_n, s_n). In each period $t \geq n$, households can buy and sell a (sequentially) complete set of financial contracts (assets) with state-contingent payoffs, and we assume that for each state s there is one contract or Arrow security. We denote by $a_{t+1}(s_{t+1})$ the quantity bought (or sold, if negative) in period t of the contract that pays off one unit of the good in period $t+1$ if s_{t+1} occurs, and denote the price of this contract by $q_t(s_{t+1})$. A budget-feasible plan has to satisfy the sequential budget constraint

$$
\begin{aligned}
\tilde{r}_{ht} z(s_t) h_t + a_t(s_t) &= c_t + x_{ht} + \sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1}) \\
\sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1}) + \frac{h_{t+1}}{\phi(s_t)} &\geq 0 \\
a_{t+1}(s_{t+1}) &\geq -\bar{D} \left[\sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1}) + \frac{h_{t+1}}{\phi(s_t)} \right] \\
c_t \geq 0 \ , \quad h_{t+1} &\geq 0.
\end{aligned}
\tag{2}
$$

The variable z denotes an idiosyncratic shock to the productivity of human capital while $\tilde{r}_{ht}$ denotes the (common) rental rate per efficiency unit of human capital. Note that $h_{t+1}/\phi(s_t)$ is the value of human capital in units of the consumption good and the expression $\sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1}) + \frac{h_{t+1}}{\phi(s_t)}$ is the value of total wealth at the beginning of period $t+1$, which is chosen in period t (before the realization of s_{t+1} is known). Thus, the second inequality in (2) states that total wealth is non-negative. In combination with the non-negativity of human capital this constraint rules out Ponzi-schemes. The third inequality in (2) bounds the share of financial wealth in total wealth for all possible states s_{t+1} and ensures that the household decision problem has a solution.[11] In our quantitative application, we choose the bound, $\bar{D}$, large enough so that the third inequality in (2) does not bind in equilibrium.

Given the initial state (a_n, h_n, s_n), a household of cohort n chooses a plan $\{c_t, a_t, h_t\}_{t=n}^{\infty}$, where each plan is a sequence of functions mapping histories, $s^{n,t}$, into actions, $c_t(s^{n,t})$, $a_{t+1}(s^{n,t}, .)$, and $h_{t+1}(s^{n,t})$, where for given $s^{n,t}$ the variable $a_{t+1}(s^{n,t}, .)$ is a vector with components s_{t+1}. Here $s^{n,t} = (s_n, \ldots, s_t)$ denotes the history of individual states s_t from period n up to period t. Note that the household level equations (1) and (2) have to hold in realizations; that is, they have to hold for all histories, $s^{n,t}$.

2.3 Preferences

Households have identical preferences over consumption plans. Households are risk-averse and their preferences allow for a time-additive expected utility representation:

$$U\left(\{c_t\}_{t=n}^{\infty} | s_n\right) \doteq \sum_{t=n}^{\infty} \beta^{t-n} E[\nu_{n,t} u(c_t) | s_n] \ , \tag{3}$$

where $\nu_{n,t}$ is the probability that a household born in period n is alive in period t and the expectations is taken over all individual histories

$$E[\nu_{n,t} u(c_t) | s_n] \ \doteq \ \sum_{s^{n,t} | s_n} \nu_{n,t}(s^{n,t-1}) u(c_t(s^{n,t})) \pi(s^{n,t} | s_n) \ .$$

Here $\pi(s^{n,t}|s_n)$ stands for the history that $s^{n,t}$ occurs given s_n, which is given by $\pi(s^{n,t}|s_n) = \pi(s_{n+1}|s_n) \times \ldots (s_t|s_{t-1})$. We assume that $\nu_{n,t}(s^{n,t-1}) = \prod_{k=n}^{t-1} \rho(s_k)$, where $\rho(s_k)$ is the survival probability in period $k+1$ of a household who in period k is in state s_k. Note that survival probabilities may depend on age, as encoded in s_t, but do not depend on cohort. We assume that the one-period utility function exhibits constant relative risk aversion: $u(c) =$

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[11] We thank our discussant, Andrew Glover, for pointing out that the second inequality in (2) is not sufficient to ensure existence of a solution. Note that we cannot simply impose a lower bound on asset holdings, $a_{t+1} \geq -\bar{D}$, since in equilibrium the extensive variables are generally unbounded (endogenous growth) so that this type of constraint would necessarily bind at some point in time.

$\frac{c^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ and $u(c) = \ln c$ otherwise. In other words, preferences are homothetic in consumption.

2.4 Enforcement/Participation Constraint

We confine attention to equilibria in which households have no incentive to default. Thus, household choices are required to satisfy the sequential enforcement (or participation) constraints. That is, for all $t \geq n$ and all $s^{n,t}$ we have:

$$\sum_{m=t}^{\infty} \beta^{m-t} E[\nu_{n,m} u(c_m) | s^{n,t}] \geq V_d(h_t(s^{n,t-1}), s_t) \tag{4}$$

where V_d is the continuation value of a household who decides to default in period t. Alvarez and Jermann (2000) provide a general argument how to re-write (4) as a system of short-sale (debt) constraints in exchange economies. For the class of tractable production models considered here, we can derive an explicit formula for the short-sale (debt) constraints implicitly defined by (4) – see Proposition 2.

The default value function, V_d, is determined as follows. We assume that upon default all debts of the household are canceled and all financial assets seized so that $a_t(s_t) = 0$. While in the default state, households are excluded from purchasing insurance contracts and borrowing (going short). Further, households in default retain their human capital, can invest in human capital, and earn a wage rate $(1-\tau)\tilde{r}_h$ per efficiency unit of human capital, where $0 \leq \tau \leq 1$ is a parameter that measures the fraction of labor income that is garnished. Thus, the punishment for default is exclusion from financial markets and possible garnishment of labor income. We assume that households remain in the default state until a stochastically determined future date that occurs with probability $(1-p)$ in each period; that is, the probability of remaining in default is p. After moving out of the default state, the household's expected continuation value is V^e, which depends on h and s at the time of exiting default ($a = 0$ at that point in time). For the individual household the function V^e is taken as given, but we close the model and determine this function endogenously by requiring that $V^e = V$, where V is the equilibrium value function associated with the maximization problem of a household who participates in financial markets.[12]

In sum, V_d is the value function associated with the following household maximization problem

$$
\begin{aligned}
V_d(h_t(s^{n,t-1}), s_t) \doteq & \max_{\{c_m, h_m\}_{m=t}^{\infty}} \sum_{m=t}^{\infty} (p\beta)^{m-t} E[\nu_{n,m} u(c_m) | s^{n,t}] \\
& + (1-p) \sum_{m=t+1}^{\infty} \beta^{m-t} p^{m-t-1} E[\nu_{n,m} V_m^e(h_m(s^{n,m-1}), s_m) | s^{n,t}]
\end{aligned}
$$

where the continuation plan $\{c_m, h_m\}_{m=t}^{\infty}$ has to satisfy the sequential budget constraint

$$
\begin{aligned}
(1-\tau)\tilde{r}_{h,m} z(s_m) h_m &= c_m + x_{h,m} \\
h_{m+1} &= (1 + \epsilon(s_{m-1}, s_m)) \, h_m + \phi(s_m) x_{h,m} \qquad (5) \\
c_m \geq 0 \;\; , \;\; h_{m+1} \geq 0 &
\end{aligned}
$$

2.5 Household Decision Problem

For given initial state (a_n, h_n, s_n), a household of cohort n chooses a plan $\{c_t, a_{t+1}, h_{t+1}\}_{t=n}^{\infty}$. The set of budget feasible household plans is defined as

$$
B(a_n, h_n, s_n) \doteq \{ \{c_t, a_{t+1}, h_{t+1}\}_{t=n}^{\infty} \,|\, \{c_t, a_{t+1}, h_{t+1}\}_{t=n}^{\infty} \; satisfies \; (1), (2), \; and \; (4) \}
$$

The decision problem of a household of initial type (a_n, h_n, s_n) is

$$
\begin{aligned}
& \max_{\{c_t, a_{t+1}, h_{t+1}\}_{t=n}^{\infty}} U(\{c_t\}_{t=n}^{\infty} | s_n) \qquad (6) \\
& s.t. \;\; \{c_t, a_{t+1}, h_{t+1}\}_{t=n}^{\infty} \in B(a_n, h_n, s_n)
\end{aligned}
$$

[12] The previous literature has usually assumed $p = 1$ (permanent autarky). See, however, Krueger and Uhlig (2006) for a model with $p = 0$ following a similar approach to ours. Note also that the credit (default) history of an individual household is not a state variable affecting the expected value function, V^e; we assume that the credit (default) history of households is information that cannot be used for contracting purposes. This is in line with the U.S. bankruptcy code, which limits the history of past behavior that can be retained in credit reports.

where the lifetime utility function, U, is defined in (3).

2.6 Goods Production and Physical Capital Accumulation

There is one good that can be consumed or used as physical capital in production. Production of this good is undertaken by a representative firm that rents capital and labor in competitive markets and uses these input factors to produce output, Y_t, according to the aggregate production function $Y_t = F(K_t, H_t)$. Here K_t is the aggregate stock of physical capital and H_t is the aggregate level of efficiency-weighted human capital employed by the firm.

The aggregate production function, F, is a standard neoclassical production function, that is, it has constant-returns-to-scale, satisfies a Inada condition, and is continuous, concave, and strictly increasing in each argument. Given these assumptions on F, the implied intensive-form production function, $f(\tilde{K}) = F(\tilde{K}, 1)$, is continuous, strictly increasing, strictly concave, and satisfies a corresponding Inada condition, where we introduced the "capital-to-labor ratio" $\tilde{K} = K/H$. Given the assumption of perfectly competitive labor and capital markets, profit maximization implies

$$\begin{aligned} \tilde{r}_{kt} &= f'(\tilde{K}_t) \\ \tilde{r}_{ht} &= f(\tilde{K}_t) - f'(\tilde{K}_t)\tilde{K}_t \ , \end{aligned} \tag{7}$$

where $\tilde{r}_k$ is the rental rate of physical capital and $\tilde{r}_h$ is the rental rate of human capital. Note that $\tilde{r}_h$ is simply the wage rate per unit of human capital. Clearly, (7) defines rental rates as functions of the capital-to-labor ratio: $\tilde{r}_k = \tilde{r}_k(\tilde{K})$ and $\tilde{r}_h = \tilde{r}_h(\tilde{K})$.

The accumulation equation for the aggregate stock of physical capital is

$$K_{t+1} = (1 - \delta_k)K_t + X_{kt} \ , \tag{8}$$

where δ_k is the depreciation rate of physical capital and X_{kt} is investment in physical capital.

2.6 Equilibrium

We confine attention to equilibria in which financial contracts are priced in a risk-neutral manner,

$$q_t(s_{t+1}) \;=\; \frac{\pi(s_{t+1}|s_t)}{1+r_{ft}} \;, \tag{9}$$

where r_f is the interest rate on financial transactions, which is equal to the return on physical capital investment, $r_{ft} = \tilde{r}_{kt} - \delta_k$. The pricing equation (9) can be interpreted as a zero-profit condition. More precisely, consider financial intermediaries that sell insurance contracts to individual households and invest the proceeds in the risk-free asset that can be created from the complete set of financial contracts and yields a certain return r_f. Given that financial intermediaries face linear investment opportunities and assuming no quantity restrictions on the trading of financial contracts for financial intermediaries, equilibrium requires that financial intermediaries make zero profit, namely condition (9).

Capital market clearing requires that the aggregate stock of physical capital employed by the representative firm is equal to the value of financial wealth held by households. Similarly, labor market clearing requires that the firm's demand for labor equals the aggregate amount of efficiency-weighted human capital supplied by households. More precisely, in equilibrium we have

$$\begin{aligned} K_{t+1} &= \sum_{s_{t+1}} \sum_{n=0}^{t} E[\nu_{n,t+1} q_t(s_{t+1}) a_{n,t+1}(s_{t+1})|s_{t+1}] \;+\; \int_{a_{t+1}} a_{t+1} d\mu_{new,t+1}(a_{t+1}) \qquad (10) \\ H_{t+1} &= \sum_{n=0}^{t} E[\nu_{n,t+1} z(s_{t+1}) h_{n,t+1}] \;+\; \int_{h_{t+1},s_{t+1}} z(s_{t+1}) h_{t+1} d\mu_{new,t+1}(h_{t+1}, s_{t+1}) \;, \end{aligned}$$

where $\mu_{new,t+1}$ is the distribution of new-born households in period $t+1$ over initial states, which is an exogenous object. Note that the expectations in (10) is taken over all individual histories and all possible initial states. That is, we define

$$E[\nu_{n,t+1} q_t(s_{t+1}) a_{n,t+1}(s_{t+1})|s_{t+1}] \;\doteq$$

$$\int_{a_n,h_n,s_n} \sum_{s^{n,t+1}|s_n} \nu_{n,t+1}(s^{n,t}) q_t(s_{t+1};s_t) a_{n,t+1}(s_{t+1};s^{n,t},a_n,h_n,s_n)\pi(s^{n,t}|s_n) d\mu_{new,n}(a_n,h_n,s_n)$$

and

$$E[\nu_{n,t+1} z(s_{t+1}) h_{n,t+1}] \doteq$$

$$\int_{a_n,h_n,s_n} \sum_{s^{n,t+1}|s_n} \nu_{n,t+1}(s^{n,t}) z(s_{t+1}) h_{n,t+1}(s^{n,t},a_n,h_n,s_n)\pi(s^{n,t+1}|s_n) d\mu_{new,n}(a_n,h_n,s_n)$$

Note that we allow the distributions of new-born households, $\mu_{new,n}$, to depend on the cohort n in order to be permit an endogenous growth path.

The distribution $\mu_{new,n}$ has to satisfy an aggregate resource restriction. Specifically, we assume that the aggregate stock of total capital (physical plus human capital) of new-born households is proportional to the aggregate stock of total capital of households who have died:

$$\int_{a_{n'+1}} a_{n'+1} d\mu_{new,n'+1}(a_{n'+1}) + \int_{h_{n'+1}} \frac{h_{n'+1}}{\phi(s_{n'})} d\mu_{new,n'+1}(h_{n'+1},s_{n'}) = \tag{11}$$

$$\lambda \sum_{n=0}^{n'} \int_{a_n,h_n,s_n} \sum_{s^{n,n'}} (1-\rho(s_{n'}))\nu_{n,n'}(s^{n,n'-1}) a_{n,n'+1}(s^{n,n'},a_n,h_n,s_n)\pi(s^{n,n'}|s_n) d\mu_{new,n}(a_n,h_n,s_n)$$

$$+ \lambda \sum_{n=0}^{n'} \int_{a_n,h_n,s_n} \sum_{s^{n,n'}} (1-\rho(s_{n'}))\nu_{n,n'}(s^{n,n'-1}) \frac{h_{n'+1}(s^{n,n'},a_n,h_n,s_n)}{\phi(s_{n'})} \pi(s^{n,n'}|s_n) d\mu_{new,n}(a_n,h_n,s_n)$$

where λ is a parameter that measures the relationship between the aggregate stock of total capital of households born in period $n'+1$ relative to the aggregate stock of total capital of households who leave the model in period $n'+1$. This parameter summarizes to what extent capital is passed on to the next generation and to what extent new-born generation start with additional capital.[13]

The aggregate resource constraint states that total output produced is equal to aggregate

[13] The specification (11) implies that the model exhibits endogenous growth in equilibrium. In contrast, if we assume that the human capital of new-born households is independent of the aggregate stock of existing capital, then the model admits for a steady state equilibrium with zero growth.

consumption plus aggregate investment

$$Y_t \;=\; C_t + X_{kt} + X_{ht} \tag{12}$$

where X_{kt} is aggregate investment in physical capital and X_{ht} is aggregate investment in human capital. As in (10), we compute aggregate variables from the respective household-level variables by summing over cohorts and averaging over individual histories and possible initial states. It is straightforward to show that the capital and labor market clearing conditions (10) in conjunction with the household budget constraint (2) and the capital accumulation equations (1) and (8) imply the goods market clearing condition (12) using the asset pricing formula (9). In our equilibrium analysis we will use focus on the two market clearing conditions in (10), which can be subsumed to one market clearing condition because of the constant-returns-to-scale assumption (see below).

Our definition of a sequential equilibrium is standard:

Definition 1 A sequential equilibrium is a sequence of aggregate stocks of physical capital and (productivity weighted) human capital, $\{K_t, H_t\}$, rental rates, $\{\tilde{r}_{kt}, \tilde{r}_{ht}\}$, and a family of household plans, $\{c_t, a_t, h_t\}_{t=n}^{\infty}$, one for each cohort n and initial household type (a_n, h_n, s_n), so that

i) Utility maximization of households: for each initial state, (a_n, h_n, s_n), the plan $\{c_t, a_t, h_t\}_{t=n}^{\infty}$ solves the household problem (6).

ii) Profit maximization of firms: (K_t, H_t) maximizes profit for all t, that is, the aggregate capital-to-labor ratio, $\tilde{K}_t$, and rental rates, $\tilde{r}_{kt}$ and $\tilde{r}_{ht}$ satisfy the first-order conditions (7) for all t.

iii) Profit maximization of financial intermediaries: financial contracts are priced according to (9).

iv) Market clearing in capital and labor markets: equation (10) holds.

v) Rational expectations: expected continuation value functions are equal to actual continuation value functions: $V^e = V$.

Before we turn to the characterization of equilibria in the next section, we briefly discuss the basic properties that define human capital in this paper.

2.7 Nature of Human Capital

In this paper, we define human capital as an asset with two properties, risk and non-pledgeability, and argue that the non-pledgeability of human capital (limited contract enforcement) leads to under-insurance against human capital risk through endogenous borrowing constraints. To make this point as sharply as possible, we introduce human capital as an asset that differs from the risk-free asset (physical capital) only along the two dimensions "risk" and "pledgeability". In other words, we assume that human capital production exhibits constant-returns-to-scale at the household level. Note, however, that both physical capital and human capital display diminishing returns at the aggregate level, which is in line with Jones and Manuelli (1990) and Rebelo (1991). In contrast, papers by Heckman et al. (1998), Huggett et al. (2011), and Lochner and Monge (2011) assume that human capital production faces diminishing returns to scale at the household level.

Given that there are two modeling approaches to human capital production (i.e. constant returns versus diminishing returns), it is natural to ask which one should be used? We argue that the answer depends on the question at hand. In this paper, the focus of the analysis is on two, and only two, properties of human capital, namely riskiness and non-pledgeability. Thus, in this case our approach to modeling human capital production seems appropriate from a theoretical point of view. The tractability of the model provides an additional argument in favor of the approach taken here. From an empirical point of view, the two approaches produce observationally equivalent implications for the dimensions of the data that we emphasize as long as human capital returns are type-dependent (age-dependent), as they are in this paper.

3. Theoretical Results

In this section, we state the two main theoretical results. First, the solution to the individual household maximization problem is linear in total individual wealth (financial and human). This partial equilibrium result is stated in proposition 2 and the proof is based on a monotone operator argument (proposition 1). Second, the distribution of total wealth (financial plus human), Ω, over household types, s, is a sufficient aggregate state variable. This general equilibrium result is stated in proposition 3. We begin this section with a discussion of a convenient change of variables and a definition of recursive equilibria with aggregate state Ω.[14]

3.1 Change of Variables

For the characterization of equilibria, it is convenient to introduce new variables that emphasize the fact that individual households solve a standard inter-temporal portfolio choice problem (with additional participation constraints). To this end, introduce the following variables:

$$
\begin{aligned}
\tilde{w}_t &\doteq \frac{h_t}{\phi(s_{t-1})} + \sum_{s_t} q_{t-1}(s_t) a_t(s_t) \\
\theta_{ht} &\doteq \frac{h_t}{\phi(s_{t-1}) w_t} \quad , \quad \theta_{at}(s_t) = \frac{a_t(s_t)}{w_t}
\end{aligned}
$$

[14]In this paper, we do not analyze the efficiency properties of equilibria. Gottardi, Kajii, and Nakajima (2015) and Toda (2015) study the (constrained) efficiency properties of equilibria of an incomplete-market version of the human capital model used in this paper. Alvarez and Jermann (2000) and Kehoe and Levine (1993) show the constrained efficiency of competitive equilibria of exchange models with limited enforcement. In production models, including the current human capital model, the default value often depends on endogenous prices (rental rates), which renders competitive equilibria in general constrained inefficient (Abraham and Carceles-Poveda, 2006).

$$1 + r(\theta_t, s_{t-1}, s_t) \doteq \begin{cases} (1 + r_{ht}(s_{t-1}, s_t))\theta_{ht} + \theta_{at}(s_t) & \text{if no default} \\ (1 + r_{hd,t}(s_{t-1}, s_t))\theta_{ht} & \text{if default} \end{cases} \quad (13)$$

where $r_{ht}(s_{t-1}, s_t) \doteq z(s_t)\phi(s_t)\tilde{r}_{ht} + \epsilon(s_{t-1}, s_t)$ is the return on human capital investment if the household does not default and $r_{hd,t}(s_{t-1}, s_t) \doteq (1-\tau)z(s_t)\phi(s_t)\tilde{r}_{ht} + \epsilon(s_{t-1}, s_t)$ is the return on human capital investment in case of default. In (13) the expression $\frac{h_t}{\phi(s_{t-1})}$ is the value of human capital measured in units of the consumption good and the variable $\tilde{w}_t$ stands for beginning-of-period total wealth, which is equal to the value of financial wealth, $\sum_{s_t} q_{t-1}(s_t)a_t(s_t)$, plus the value of human wealth, $\frac{h_t}{\phi(s_{t-1})}$. The variable $\theta_t = (\theta_{ht}, \theta_{at})$ denotes the vector of portfolio shares and $(1+r)$ is the total return to investment. Recall that for given history of shocks, θ_{ht} is a number, but θ_{at} is a vector with components $\theta_{at}(s_t)$. Using the new notation and substituting out the investment variables, x_{kt} and x_{ht}, the budget constraint (2) and human capital accumulation equation (1) read

$$\begin{aligned} \tilde{w}_{t+1} &= (1 + r(\theta_t, s_{t-1}, s_t))\,\tilde{w}_t - c_t \\ 1 &= \theta_{h,t+1} + \sum_{s_{t+1}} q_t(s_{t+1})\theta_{a,t+1}(s_{t+1}) \\ c_t &\geq 0 \;, \quad \tilde{w}_{t+1} \geq 0 \;, \quad \theta_{h,t+1} \geq 0 \;, \quad \theta_{a,t+1}(s_{t+1}) \geq -\bar{D} \;. \end{aligned} \quad (14)$$

Clearly, (14) is the budget constraint corresponding to an inter-temporal portfolio choice problem with linear investment opportunities and no exogenous source of income.

It is convenient to use as individual state variable wealth including current asset payoffs ("cash at hand") defined as $w_t \doteq (1+r_t)\tilde{w}_t$. Using this concept of total wealth, the budget constraint (14) can be written as

$$\begin{aligned} w_{t+1} &= (1 + r(\theta_{t+1}, s_t, s_{t+1}))\,(w_t - c_t) \\ 1 &= \theta_{h,t+1} + \sum_{s_{t+1}} q_t(s_{t+1})\theta_{a,t+1}(s_{t+1}) \\ c_t &\geq 0 \;, \quad w_{t+1} \geq 0 \;, \quad \theta_{h,t+1} \geq 0 \;, \quad \theta_{a,t+1}(s_{t+1}) \geq -\bar{D} \;. \end{aligned} \quad (15)$$

Further, the default value function, V_d, can be written as a function of w, and (w, s) is therefore a sufficient state for the enforcement constraint (4). Thus, the household maximization problem (6) is equivalent to the household maximization problem

$$\max_{\{c_t, w_{t+1}, \theta_{t+1}\}_{t=n}^{\infty}} U\left(\{c_t\}_{t=n}^{\infty} | s_n\right) \qquad (16)$$
$$s.t. \ \{c_t, w_{t+1}, \theta_{t+1}\}_{t=n}^{\infty} \in B(w_n, s_n)$$

where the budget set is now defined as

$$B(w_n, s_n) \doteq \{\{c_t, w_{t+1}, \theta_{t+1}\}_{t=n}^{\infty} \,|\, \{c_t, w_{t+1}, \theta_{t+1}\}_{t=n}^{\infty} \text{ } satisfies \text{ } (4) \text{ } and \text{ } (15)\}$$

3.2 Recursive Equilibrium: Definition

We next define a recursive equilibrium. To this end, we first note that the market clearing condition (10) can be reduced to the condition

$$\tilde{K}_{t+1} = \frac{\sum_{s_{t+1}} \sum_{n=0}^{t} E[\nu_{n,t+1} q_t(s_{t+1}) a_{n,t+1}(s_{t+1}) | s_{t+1}] + \int_{a_{t+1}} a_{t+1} d\mu_{new,t+1}(a_{t+1})}{\sum_{n=0}^{t} E[\nu_{n,t+1} z(s_{t+1}) h_{n,t+1}] + \int_{h_{t+1}, s_{t+1}} z(s_{t+1}) h_{t+1} d\mu_{new,t+1}(h_{t+1}, s_{t+1})} \qquad (17)$$

because of the constant-returns-to-scale assumption. In a sequential equilibrium, the expectations in (17) is taken over all individual histories and all initial states, and it depends in general explicitly on time t. In a recursive equilibrium, the expectations is taken over individual states conditional on the aggregate state, and it is time-independent. Note we can replace the plans $\{a_t, h_t\}$ in (17) by plans $\{w_t, \theta_t\}$ using the definition of portfolio choices and total wealth (13).

The household maximization problem (16) suggests that we can use (w, s) as the individual state variable. For the aggregate state, in general the distribution, μ, over individual states, (w, s), is the minimal state variable. However, for the current model, the type-dependent wealth distribution, $\Omega \in \mathbb{R}^n$, defined as

$$\Omega_t(s_t) \doteq \frac{E\left[\sum_{n=0}^{t} \nu_{n,t} w_{n,t} | s_t\right]}{E\left[\sum_{n=0}^{t} \nu_{n,t} w_{n,t}\right]} .$$

turns out to be sufficient (see below). Here $\Omega_t(s_t)$ is the share of aggregate total wealth owned by all households of type s_t. Note that Ω is a distribution since $E[\Omega_t] = \sum_{s_t} \Omega_t(s_t) = 1$. Note further that the distribution μ is an infinite-dimensional object, whereas the distribution Ω is finite-dimensional.

Below we construct a recursive equilibrium with aggregate state variable Ω that evolves according to an endogenous law of motion $\Omega' = \Phi(\Omega)$, where the prime denotes next period's variable. We further show that next period's optimal portfolio choice is independent of w, which implies that the market clearing condition (17) becomes a condition that defines a function $\tilde{K}' = \tilde{K}'(\Omega)$. Together with the first-order conditions (7) this defines rental rate functions $\tilde{r}'_k = \tilde{r}'_k(\Omega)$ and $\tilde{r}'_h = \tilde{r}'_h(\Omega)$. Note that in this case total investment returns, r, become a function $r = (\theta, s, s', \Omega)$, where the dependence on Ω represents general-equilibrium effects. Given our definition of sequential equilibrium and the variables defined so far, our definition of recursive equilibrium is standard:

Definition 2 A recursive equilibrium is a law of motion, Φ, for the aggregate state variable, Ω, a function $\tilde{K}' = \tilde{K}'(\Omega)$, rental rate functions $\tilde{r}'_k = \tilde{r}'_k(\Omega)$ and $\tilde{r}'_h = \tilde{r}'_h(\Omega)$, a value function, V, an expected value function, V^e, and a household policy function, g,[15] such that
i) Utility maximization of households: for all household cohorts, n, and household types, (w_n, s_n), the household policy function, g, in conjunction with the law of motion, Φ, generates a plan, $\{c_t, w_{t+1}, \theta_{t+1}\}_{t=n}^{\infty}$, that solves the household maximization problem (16). Further, V is the associated value function.
ii) Profit maximization of firms: for any sequence $\{\tilde{K}\}_{t=0}^{\infty}$, the rental rate sequences $\{\tilde{r}_{kt}\}_{t=0}^{\infty}$ and $\{\tilde{r}_{ht}\}_{t=0}^{\infty}$ are defined by the first-order conditions (7).
iii) Profit maximization of financial intermediaries: financial contracts are priced according to (9)
iv) Market clearing: for any initial state Ω, the law of motion Φ in conjunction with the

[15]The function g defines next period's endogenous state as a function of this period's endogenous state and this period's exogenous shock, $w_{t+1} = g(w_t, s_t)$, as well as current consumption as a function of the state: $c_t = c(w_t, s_t)$.

function $\tilde{K}'$ generate a sequence $\{\tilde{K}\}_{t=0}^{\infty}$ that satisfies the market clearing condition (17)
v) Rational expectations: $V^e = V$ and Φ is the law of motion induced by g.

3.3 Characterization of Household Problem (Partial Equilibrium)

The principle of optimality in conjunction with our discussion in the previous section regarding the appropriate aggregate state suggest that the household maximization problem (16) is equivalent to the Bellman equation

$$\begin{aligned} V(w,s,\Omega) \quad &= \quad \max_{c,w',\theta'} \left\{ u\left(c\right) + \beta\rho(s) \sum_{s'} V\left(w',s',\Omega'\right) \pi(s'|s) \right\} \\ s.t. \quad & w' \;=\; (1 + r(\theta', s, s', \Omega))(w - c) \\ & 1 \;=\; \theta'_h + \sum_{s'} \frac{\pi(s'|s)\theta'_a(s')}{1 + r_f(\Omega)} \\ & w' \geq 0 \;\;,\;\; \theta'_h \geq 0 \;\;,\;\; \theta'_a(s') \geq -\bar{D} \\ & V(w',s',\Omega') \;\geq\; V_d(w',s',\Omega') \\ & \Omega' \;=\; \Phi(\Omega) \end{aligned} \tag{18}$$

where the default value function is given by

$$\begin{aligned} V_d(w,s,\Omega) \quad = \quad \max_{c,w'} & \left\{ u\left(c\right) + \beta\rho(s)p \sum_{s'} \rho(s') V_d\left(w',s',\Omega'\right) \pi(s'|s) \right. \\ & \left. +\beta\rho(s)(1-p) \sum_{s'} V^e\left(w',s',\Omega'\right) \pi(s'|s) \right\} \\ & w' \;=\; (1 + r_{hd}(s,s',\Omega))(w-c) \\ & \Omega' \;=\; \Phi(\Omega) \end{aligned}$$

and the dependence of the returns r_f, r, and r_d on Ω represents general-equilibrium effects.

Let T be the operator associated with the Bellman equation (18). In contrast to the standard case without a participation constraint, the Bellman operator, T, defined by equation (18) is in general not a contraction. However, it is still a monotone operator. Monotone operators might have multiple fixed points, but under certain conditions we can construct a sequence that converges to the maximal element of the set of fixed points. This maximal solution is also the value function (principle of optimality). More precisely, if the condition that for all s

$$\begin{array}{llll} \forall\theta' : & \beta\rho(s)\sum_{s'}(1+r(\theta',s,s',\Omega'))^{1-\gamma}\,\pi(s'|s) & < 1 & if\ \ 0<\gamma<1 \\ \exists\theta' : & \beta\,\rho(s)\sum_{s'}(1+r(\theta',s,s',\Omega'))^{1-\gamma}\,\pi(s'|s) & < 1 & if\ \ \gamma>1 \end{array} \tag{19}$$

holds,[16] then we have the following results:

Proposition 1. Suppose that condition (19) is satisfied and that the law of motion, Φ, and the value function of a household in financial autarky, V_d, are continuous. Let T stand for the operator associated with the Bellman equation (18). Then

i) There is a unique continuous solution, V_0, to the Bellman equation (18) without participation constraint.

ii) $\lim_{k\to\infty} T^k V_0 = V_\infty$ exists and is the maximal solution to the Bellman equation (18)

iii) V_∞ is the value function, V, of the sequential household maximization problem.

Proof. See Appendix.

Consider the case $V^e = V$. Using proposition 2 and an induction argument, we can then show that the value function, V, has the functional form

$$V(w,s,\Omega)=\begin{cases}\tilde{V}(s,\Omega)w^{1-\gamma} & if\ \gamma\neq 1\\ \tilde{V}_0(s,\Omega)\,+\,\tilde{V}_1(s)\ln w & otherwise\end{cases} \tag{20}$$

and that the corresponding optimal policy function, g, is

$$c(w,s) \;=\; \tilde{c}(s,\Omega)\,w$$

[16]Note that for the log-utility case, no condition of the type (19) is required.

$$
\begin{aligned}
w'(w,s,s',\Omega) &= (1+r(\theta',s,s',\Omega))(1-\tilde{c}(s,\Omega))\,w \\
\theta'(w,s,\Omega) &= \theta'(s,\Omega)\,.
\end{aligned}
$$

In other words, the value function has the functional form of the underlying utility function, consumption and next-period wealth are linear functions of current period wealth, and next-period portfolio choices are independent of wealth. Moreover, we also show that the intensive-form value function, $\tilde{V}$, together with the optimal consumption and portfolio choices, $\tilde{c}$ and θ, can be found by solving an intensive-form Bellman equation that reads

$$
\begin{aligned}
\tilde{V}(s,\Omega) &= \max_{\tilde{c},\theta'}\left\{\frac{\tilde{c}^{1-\gamma}}{1-\gamma}+\beta\rho(s)(1-\tilde{c})^{1-\gamma}\sum_{s'}(1+r(\theta',s,s',\Omega))^{1-\gamma}\,\tilde{V}(s',\Omega')\,\pi(s'|s)\right\} \\
s.t.\quad 1 &= \theta'_h+\sum_{s'}\frac{\theta'_a(s')\pi(s'|s)}{1+r_f(\Omega)} \qquad (21) \\
&\theta'_h\geq 0\ ,\ \ \theta'_a(s')\geq -\bar{D}\ ,\ \ 0\ \leq\ \tilde{c}\ \leq\ 1 \\
&\left(\frac{\tilde{V}(s',\Omega')}{\tilde{V}_d(s',\Omega')}\right)^{\frac{1}{1-\gamma}}(1+r(\theta',s,s',\Omega))\ \geq\ (1+r_{hd}(s,s',\Omega))\theta'_h \\
&\Omega'\ =\ \Phi(\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{V}_d(s,\Omega) = \max_{\tilde{c}_d}\Bigg\{&\frac{\tilde{c}_d^{1-\gamma}}{1-\gamma}+p\beta\rho(s)(1-\tilde{c}_d)^{1-\gamma}\sum_{s'}(1+r_{hd}(s,s',\Omega))^{1-\gamma}\,\tilde{V}_d(s',\Omega')\pi(s'|s) \\
&(1-p)\beta\rho(s)(1-\tilde{c}_d)^{1-\gamma}\sum_{s'}(1+r_{hd}(s,s',\Omega))^{1-\gamma}\,\tilde{V}(s',\Omega')\pi(s'|s)\Bigg\}
\end{aligned}
$$

for $\gamma \neq 1$. In the case of log-utility, the intensive-form Bellman equation reads

$$
\tilde{V}_0(s,\Omega) = \max_{\tilde{c},\theta'}\Bigg\{\ln\tilde{c}\ +\ \beta\rho(s)\sum_{s'}\tilde{V}_0(s')\pi(s'|s)\ +\ \beta\rho(s)\left[\ln(1-\tilde{c})\right]\sum_{s'}\tilde{V}_1(s')\pi(s'|s)
$$

$$
\begin{aligned}
&+ \beta\rho(s) \sum_{s'} \tilde{V}_1(s') \ln(1 + r(\theta', s, s', \Omega))\pi(s'|s) \Bigg\} \\
s.t. \quad 1 \;&=\; \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1 + r_f(\Omega)} \\
&\theta'_h \geq 0 \;\;,\;\; \theta'_a(s') \geq -\bar{D} \\
&e^{(1-\beta)\left(\tilde{V}_0(s',\Omega') - \tilde{V}_{d0}(s',\Omega')\right)} \, (1 + r(\theta', s, s', \Omega)) \;\geq\; (1 + r_{hd}(s, s', \Omega))\theta'_h \\
&\Omega' \;=\; \Phi(\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{V}_{0d}(s, \Omega) \;\;=\;\; \max_{\tilde{c}_d} \Bigg\{ &\ln \tilde{c}_d \;+\; \beta \left[\ln(1 - \tilde{c}_d)\right] \sum_{s'} \tilde{V}_1(s')\rho(s')\pi(s'|s) \\
&+ \beta\rho(s) \sum_{s'} \tilde{V}_1(s') \ln(1 + r_{hd}(s, s', \Omega))\pi(s'|s) \\
&+ p\beta\rho(s) \sum_{s'} \tilde{V}_{0d}(s')\pi(s'|s) \;+\; (1-p)\beta\rho(s) \sum_{s'} \tilde{V}_0(s')\pi(s'|s) \Bigg\}
\end{aligned}
$$

where the coefficients $\tilde{V}_1$ are the solution to

$$
\tilde{V}_1(s) \;=\; 1 \;+\; \beta\rho(s) \sum_{s'} \tilde{V}_1(s')\pi(s'|s)
$$

Proposition 2. Suppose that condition (19) is satisfied, the law of motion, Φ, is continuous, and $V^e = V$. Then value function, V, and optimal policy function, g, have the functional form (20). Moreover, the intensive-form value function, $\tilde{V}$, and the corresponding optimal consumption and portfolio choices, $\tilde{c}$ and θ', are the maximal solution to the intensive-form Bellman equation (21). This maximal solution is obtained by iteratively applying $\tilde{T}$, the operator associated with the intensive-form Bellman equation (21), starting from $\tilde{V}_0$, the solution of the intensive-form Bellman equation (22) without participation constraint:

$$
\tilde{V} \;=\; \lim_{k \to \infty} \tilde{T}^k \tilde{V}_0 \;.
$$

Proof. See Appendix.

Note that proposition 2 cannot simply be proved by the guess-and-verify method since multiple solutions to the Bellman equation (21) may exist. Specifically, the operator associated with the Bellman equation is monotone, but not a contraction, and hence multiple fixed points may exist. However, proposition 2 ensures that we have indeed found the value function associated with the original utility maximization problem, and also provides us with a iterative method to compute this solution. Note further that the constraint set in (21) is defined by an equation system involving only linear functions (the return functions are linear in θ). Thus, the constraint set is convex and we have transformed the original utility maximization problem into a convex problem. In other words, the non-convexity problem alluded to in the introduction has been resolved.

3.4 Characterization of Recursive Equilibria

Proposition 2 shows how to rewrite the maximization problem of individual households as a recursive problem that is wealth-independent. One implication of the intensive-form representation of the individual maximization problem is that optimal portfolio choices are independent of wealth, w. This result in turn implies that the market clearing condition (17) can be re-written as

$$\tilde{K}' = \frac{\sum_s \left[\rho(s) + \lambda(1-\rho(s))\sum_{s'}(1-\theta_h(s'))\mu_{new}(s')\right](1-\theta_h(s,\Omega))\,(1-\tilde{c}(s,\Omega))\Omega(s)}{\bar{z}\sum_s \left[\rho(s) + \lambda(1-\rho(s))\sum_{s'}\phi(s')\theta_h(s')\mu_{new}(s')\right]\phi(s)\theta_h(s,\Omega)(1-\tilde{c}\sum(s,\Omega))\Omega(s)} \tag{22}$$

where we have already incorporated restriction (11) and $\bar{z}$ stands for the mean of z. In addition, in (22) we assume that new-born households begin life with portfolio shares identical to the portfolio chosen by households alive in the previous period. Note that $\mu_{new}(s')$ stands for the share of aggregate total capital of new-born households that is received by new-born households of type s', where we interpret $\mu_{new}(s') = 0$ as the case that no new-born household is of type s'.

Equation (22) defines a function $\tilde{K}' = \tilde{K}'(\Omega)$, which in turn defines rental rate functions $\tilde{r}'_k = \tilde{r}'_k(\Omega)$ and $\tilde{r}'_h = \tilde{r}'_h(\Omega)$ using the first-order conditions (7). A second implication of proposition 2 is that the equilibrium law of motion, Φ, can be explicitly derived:

$$\Omega'(s') \;=\; \frac{\sum_s \left[\rho(s) + \lambda(1-\rho(s))\mu_{new}(s')\right](1-\tilde{c}(s,\Omega))(1+r(\theta'(s,\Omega),s',\Omega))\pi(s'|s)\Omega(s)}{\sum_{s,s'} \left[\rho(s) + \lambda(1-\rho(s))\mu_{new}(s')\right](1-\tilde{c}(s,\Omega))(1+r(\theta'(s,\Omega),s',\Omega))\pi(s'|s)\Omega(s)}\,. \tag{23}$$

Note that the expression in the denominator of (23) ensures that $\sum_{s'} \Omega'(s') = 1$.

In sum, a recursive equilibrium can be found by solving (21) using (22) and (23) as equilibrium function and equilibrium law of motion:

Proposition 3. Suppose that $(\theta, \tilde{c}, \tilde{V})$ is an intensive-form equilibrium, that is, $(\theta, \tilde{c}, \tilde{V})$ solves the intensive-form household decision problem (21) using the function $\tilde{K}' = \tilde{K}'(\Omega)$ defined by (22) and a law of motion $\Omega' = \Phi(\Omega)$ defined by $(\theta, \tilde{c})$ and (23). Then $(g, V, \tilde{K}', \Phi)$ is a recursive equilibrium, where g is the individual policy function associated with $(\theta, \tilde{c})$ and V is the value function associated with $(\theta, \tilde{c}, \tilde{V})$.

Proof. See Appendix.

Proposition 3 simplifies the computation of recursive equilibria. In our framework, the infinite-dimensional wealth distribution is not a relevant state variable. Instead, the distribution of wealth shares over household types, Ω, becomes a relevant state variable. Note that Ω is in many applications a low-dimensional object. For example, suppose that $s_t = (s_{1t}, s_{2t})$, where $\{s_{1t}\}$ and $\{s_{2t}\}$ are two independent processes and $\{s_{2t}\}$ is an i.i.d process. In this case neither $\tilde{c}$ nor θ depend on s_2 and the relevant aggregate state is $\Omega(s_1)$ only.

3.5 Extension: Aggregate Shocks

So far, we have considered economies with only idiosyncratic risk, but it is straightforward to introduce aggregate risk into the framework. To this end, suppose that there are idiosyncratic shocks, s, and aggregate shocks, S, and that uncertainty is described by a stationary joint Markov process $\{s_t, S_t\}$ with transition probabilities denoted by $\pi(s_{t+1}, S_{t+1}|s_t, S_t)$. The relevant aggregate state then becomes (Ω_t, S_t), where Ω_t is defined as before. In a recursive equilibrium, the evolution of the endogenous aggregate state variable is given by an endogenous law of motion $\Omega_{t+1} = \Phi(\Omega_t, S_t, S_{t+1})$. Further, the aggregate capital-to-labor ratio is a function $\tilde{K}_{t+1}(\Omega_t, S_t)$ and the rentals rates are function $\tilde{r}_{k,t+1} = \tilde{r}_k(\Omega_t, S_t)$ and $\tilde{r}_{h,t+1} = \tilde{r}_h(\Omega_t, S_t)$. The definition of a recursive equilibrium is, *mutatis mutandis*, as before.

A straightforward (though lengthy) extension of the subsequent theoretical analysis shows that a modified version of our general characterization results still hold. In particular, recursive equilibria can be computed by solving a convex problem that is independent of the wealth distribution, though clearly the finite-dimensional distribution of relative wealth, Ω, still enters into the equilibrium conditions.

3.6 Further Extensions

There a several further extensions of the model that can be incorporated without sacrificing the tractability of the model. First, we can introduce a time cost in human capital production if we replace the term $\phi(s_t)x_{ht}$ in (1) by $\tilde{\phi}(s_t)(h_t l_{ht})^\alpha x_{ht}^{1-\alpha}$, where l_{ht} is the time spent in human capital production. In the simplest extension, the household allocates time between working and producing human capital (learning). However, we can also add a labor-leisure choice as long as preferences remain homothetic in consumption. It is straightforward to show that the human capital production function $\tilde{\phi}(s_t)(h_t l_{ht})^\alpha x_{ht}^{1-\alpha}$ gives rise to a human capital accumulation equation (1) that is still linear in x_{ht} after the optimal choice of l_{ht} has been substituted out.

A second extension is shocks to preferences (taste shocks, health shocks, change in family structure). These can easily be incorporated by replacing the one-period utility function by one that depends on the state s_t. Third, the tractability is preserved in a model with taxes and transfers as long as these payments are proportional to either financial capital (capital income) or human capital (labor income). To see this, simply re-define the returns in (13) as returns after taxes and transfers have been taken into account. Note that taxes and transfers can be arbitrary non-linear functions of the state s_t.

4. Model Specification and Computation

In this section, we present the version of the model that is used for the quantitative analysis and discuss our approach to computing equilibria.

4.1 Parametric Specification

We set the period length to one year. We assume that the economy is in stationary equilibrium and drop the time index t. We further assume that the exogenous individual state has two components, $s = (s_1, s_2)$. The first component, s_1, denotes the age of a household, which can take on 9 values, $s_1 = 1, \ldots, 9$, corresponding to the following 9 age groups: 25 and younger, 26 - 30, 31 - 36, ..., 56 - 60, and older than 60. We assume that households stochastically age with the transitions from one age group to another age group governed by transition probabilities $\pi(s_1'|s_1)$. We assume that households cannot move up more than one age group at a time, and choose $\pi(s_1 + 1|s_1)$ so that so that households spend on average 5 years in the first 8 age groups. That is, for $s_1 \leq 8$ we have $\pi(s_1|s_1) = 4/5$ and $\pi(s_1+1|s_1) = 1/5$. Households in the oldest age group die stochastically and the probability of death is chosen so that these households live on average a further 25 years. Old households who leaves the model are replaced by households in the youngest age group.

The first component, s_1, determines expected human capital returns and the second component, s_2, represents human capital risk. Specifically, s_1 and s_2 affect human capital accumulation through the ϵ-function appearing in the human capital equation (1) as follows:

$\epsilon(s_1, s_2) = \varphi(s_1) - \delta_h + \eta(s_2)$. We interpret φ as a learning-by-doing parameter which depends on age and which, in our calibration below, is stronger for younger households so that $\varphi(s_1) > \varphi(s_1 + 1)$. The parameter δ_h is the average depreciation rate of human capital in the economy. We have set the labor productivity parameter $z = 1$ so that all labor income risk is generated through the human capital shock η, which is assumed to be i.i.d. over time and across households and independent of household age s_1.[17] Assuming that the cost of human capital in terms of consumption goods ϕ is constant, the return to human capital is given by $r_h(s_1, s_2) = \phi\tilde{r}_h + \varphi(s_1) - \delta_h + \eta(s_2)$. Normalizing the mean of the human capital shocks to zero, or $\sum_{s_2} \eta(s_2)\pi(s_2) = 0$, we find that the expected human capital returns for a household of age s_1 are $\bar{r}_h(s_1) = \sum_{s_2} r_h(s_1, s_2)\pi(s_2) = \phi\tilde{r}_h + \varphi(s_1) - \delta_h$. For the oldest household group, $s_1 = 9$, we assume that human capital returns are low enough so that they only invest in financial capital yielding a portfolio return equal to the risk-free r_f (retirement).

With this specification in hand, we can verify that human capital accumulation decisions satisfy various non-negativity constraints on human capital investment. For example, in equilibrium the restriction holds that total human capital investment inclusive of learning-by-doing is always non-negative for all working-age households: $\varphi(s_1)h_t + \phi x_{ht} \geq 0$ for $s_1 \leq 8$.

4.2 Deriving the Equilibrium Conditions

We begin with a discussion of the equilibrium conditions (21), (22), and (23) for the model version specified in this section. Given the assumption made so far, the intensive-form Bellman equation (21) for households of age $s_1 \leq 8$ becomes

$$\tilde{V}(s_1) \quad = \quad \max_{\tilde{c},\theta'} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + \beta(1-\tilde{c})^{1-\gamma} \sum_{s'_1, s'_2} \left(1 + r(\theta', s'_1, s'_2)\right)^{1-\gamma} \tilde{V}(s'_1)\pi(s'_2)\pi(s'_1|s_1) \right\}$$

[17] In Krebs, Kuhn, and Wright (2015) we consider the more general version with additional shocks to z.

$$(24)$$

$$s.t. \quad 1 \;=\; \theta'_h + \sum_{s'_1,s'_2} \frac{\theta'_a(s'_1,s'_2)\pi(s'_2)\pi(s'_1|s_1)}{1+r_f} \quad , \quad 0 \;\le\; \tilde{c} \;\le\; 1 \quad , \quad \theta'_h \ge 0$$

$$\left(\frac{\tilde{V}(s'_1)}{\tilde{V}_d(s'_1)}\right)^{\frac{1}{1-\gamma}} (1+r(\theta',s'_1,s'_2)) \;\ge\; (1+r_h(s'_1,s'_2))\theta'_h \qquad \forall\,(s'_1,s'_2)$$

$$\theta'_a(s'_1 s'_2) \;\ge\; -\bar{D} \qquad \forall\,(s'_1,s'_2)$$

with

$$\begin{aligned}\tilde{V}_d(s_1) \;&=\; \max_{\tilde{c}_d}\Bigg\{\frac{\tilde{c}_d^{1-\gamma}}{1-\gamma} + p\beta(1-\tilde{c}_d)^{1-\gamma}\sum_{s'_1,s'_2}\left(1+r_{hd}(s'_1,s'_2)\right)^{1-\gamma}\tilde{V}_d(s'_1)\pi(s'_2)\pi(s'_1|s_1)\\ &\qquad (1-p)\beta(1-\tilde{c}_d)^{1-\gamma}\sum_{s'_1,s'_1}\left(1+r_{hd}(s'_1,s'_2)\right)^{1-\gamma}\pi(s'_2)\tilde{V}(s_1)\pi(s'_1|s_1)\Bigg\}\end{aligned}$$

for $\gamma \neq 1$. In the case of log-utility, the intensive-form Bellman equation (21) becomes

$$\begin{aligned}\tilde{V}(s_1) \;&=\; \max_{\theta'}\Bigg\{\ln(1-\beta) \;+\; \frac{\beta}{1-\beta}\ln\beta \;+\; \frac{\beta}{1-\beta}\sum_{s'_1,s'_2}\ln(1+r(\theta',s'_1,s'_2)\pi(s'_2)\pi(s'_1|s_1)\\ &\qquad +\beta\sum_{s'_1}\tilde{V}(s'_1)\pi(s'_1|s_1)\Bigg\}\end{aligned}$$

$$s.t. \quad 1 \;=\; \theta'_h + \sum_{s'_1,s'_2} \frac{\theta'_a(s'_1,s'_2)\pi(s'_2)\pi(s'_1|s_1)}{1+r_f} \quad , \quad \theta'_h \ge 0$$

$$e^{(1-\beta)\left(\tilde{V}(s'_1)-\tilde{V}_d(s'_1)\right)}(1+r(\theta',s'_1,s'_2) \;\ge\; (1+r_{hd}(s'_1,s'_2))\theta'_h \qquad \forall\,(s'_1,s'_2)$$

$$\theta'_a(s'_1 s'_2) \;\ge\; -\bar{D} \qquad \forall\,(s'_1,s'_2)$$

with

$$\tilde{V}_d(s_1) \;=\; \ln(1-\beta) \;+\; \frac{\beta}{1-\beta}\ln\beta \;+\; \frac{\beta}{1-\beta}\sum_{s'_1,s'_2}\ln(1+r_{hd}(s'_1,s'_2)\pi(s'_2)\pi(s'_1|s_1)$$

$$+p\,\beta\sum_{s_1'}\tilde{V}_d(s_1')\pi(s_1'|s_1)\;+\;(1-p)\beta\sum_{s_1'}\tilde{V}(s_1')\pi(s_1'|s_1)$$

From (24) it immediately follows that the optimal portfolio choice, θ, and the optimal consumption-saving choice, $\tilde{c}$, only depend on age s_1 but not on human capital shocks s_2. In other words, household consumption and portfolio choices are independent of i.i.d. shocks. This in turn implies that the relevant aggregate state, Ω, only depends on age, s_1. The stationary Ω is then determined by the following set of equations, defined first for $s_1 = 1$:

$$\Omega(1) \;=\; N\left[\frac{4}{5}\sum_{s_2'}(1-\tilde{c}(1))(1+r(\theta'(1),1,s_2'))\pi(s_2')\Omega(1)\;+\;\frac{\lambda}{25}(1+r_f)(1-\tilde{c}(9))\Omega(9)\right]$$

and then $\forall s_1 \; with \; 2 \leq s_1 \leq 8$:

$$\begin{aligned}\Omega(s_1) \;=\; N\Bigg[&\frac{4}{5}\sum_{s_2'}(1-\tilde{c}(s_1))(1+r(\theta'(s_1),s_1,s_2'))\pi(s_2')\Omega(s_1)\;+ \\ &\frac{1}{5}\sum_{s_2'}(1-\tilde{c}(s_1-1))(1+r(\theta'(s_1-1),s_2'))\pi(s_2')\Omega(s_1-1)\Bigg]\end{aligned} \tag{25}$$

and then lastly:

$$\Omega(9) \;=\; N\left[\frac{24}{25}(1-\tilde{c}(9))(1+r_f)\Omega(9)\;+\;\frac{1}{5}(1-\tilde{c}(8))(1-\theta_h(8))(1+r_f)\Omega(8)\right]$$

where λ measures the ratio of total capital (human plus financial) of newborn households relative to the physical capital of retired households who die in a given period and N is a normalization constant chosen to ensure $\sum_{s_1}\Omega(s_1) = 1$. Note that (25) is the stationary version of (23) for the current model set-up, where we have already used the assumption that new-born households begin life in age group (state) $s_1 = 1$.

Taking into account that $\rho(s_1) = 1$ for $s_1 = 1, \ldots, 8$ and $\theta_h(9) = 0$, we find that the market clearing condition (22) becomes

$$\tilde{K} \;=$$

$$\frac{\sum_{s_1\neq 9}(1-\tilde{c}(s_1))(1-\theta_h(s_1))\Omega(s_1)+\frac{24}{25}(1-\tilde{c}(9))(1-\theta_h(9))\Omega(9)+(1-\theta_h(1))\frac{\lambda}{25}(1+r_f)(1-\tilde{c}(9))\Omega(9)}{\phi\left[\sum_{s_1\neq 9}(1-\tilde{c}(s_1))\theta_h(s_1)\Omega(s_1)+\frac{24}{25}(1-\tilde{c}(9))\theta_h(9)\Omega(9)+\theta_h(1)\frac{\lambda}{25}(1+r_f)(1-\tilde{c}(9))\Omega(9)\right]} \tag{26}$$

Equations (24), (25), and (26) in conjunction with the corresponding rental rate functions determine a stationary recursive equilibrium for this specification of the model.

4.3 Solving the Equilibrium Conditions

For the general equilibrium analysis, one needs to solve the three equations (24), (25), and (26). The solution algorithm works as follows. First, pick an aggregate capital-to-labor ratio, $\tilde{K}$, which determines the rental rates $\tilde{r}_k$ and $\tilde{r}_h$ and therefore also the investment return function r. Second, given the values for the investment returns, solve the intensive-form household decision problem (24) and recover the stationary state Ω. Third, use the values for θ, $\tilde{c}$, and Ω, to determine a new value for $\tilde{K}$ using (26). Finally, iterate until convergence. Note that in the log-utility case with $\rho(s_1)=1$ there is no need to solve for $\tilde{c}$ since we have $\tilde{c}=1-\beta$.

We solve the partial equilibrium problem (24) by iteration. More precisely, consider the case $\gamma\neq 1$ and define the values $\tilde{V}^k(s_1)$ and $\tilde{V}^k_d(s_1)$, recursively by

$$\begin{aligned}\tilde{V}^{k+1}(s_1) &= \Bigg\{\frac{\left(\tilde{c}^k(s_1)\right)^{1-\gamma}}{1-\gamma}+ \\ &\quad \beta\left(1-\tilde{c}^k(s_1)\right)^{1-\gamma}\sum_{s_1',s_2'}\left(\theta_h^k(s_1)(1+r_h(s_1',s_2'))+\theta_a^k(s_1,s_1',s_2')\right)^{1-\gamma}\tilde{v}^k(s_1)\pi(s_2')\pi(s_1'|s_1)\Bigg\} \\ \tilde{c}^k(s_1) &= 1-\left(\beta\sum_{s_1',s_2'}\left(\theta_h^k(s_1)(1+r_h(s_1',s_2'))+\theta_a^k(s_1,s_1',s_2')\right)^{1-\gamma}\pi(s_2')\pi(s_1'|s_1)\right)^{\frac{1}{\gamma}}\end{aligned} \tag{27}$$

and

$$\tilde{V}_d^{k+1}(s_1) = \frac{(\tilde{c}_d(s_1))^{1-\gamma}}{1-\gamma}+\beta(1-\tilde{c}_d(s_1))^{1-\gamma}p\sum_{s_1',s_2'}(1+r_{hd}(s_1',s_2'))^{1-\gamma}\tilde{v}_d^k(s_1')\pi(s_2')\pi(s_1'|s_1)$$

$$+\beta(1-\tilde{c}_d(s_1))^{1-\gamma}(1-p)\sum_{s_1',s_2'}(1+r_{hd}(s_1',s_2'))^{1-\gamma}\,\pi(s_2')\tilde{V}^k(s_1')\pi(s_1'|s_1)$$

$$\tilde{c}_d(s_1) \;=\; 1-\left(\beta\sum_{s_1',s_2'}(1+r_{hd}(s_1',s_2'))^{1-\gamma}\,\pi(s_2')\pi(s_1'|s_1')\right)^{\frac{1}{\gamma}}$$

where the portfolio choices $(\theta_h^k(s_1), \theta_a^k(s_1))$ for each s_1 are the solution to

$$\begin{aligned}
&\max_{\theta_h,\theta_a}\sum_{s_1',s_2'}(\theta_h(1+r_h(s_1',s_2'))+\theta_a(s_1',s_2'))^{1-\gamma}\,\pi(s')\\
s.t.\quad &\theta_h+\sum_{s_1',s_2'}\frac{\theta_a(s_1',s_2')\pi(s_2')\pi(s_1'|s_1)}{1+r_f} \;=\; 1\\
&\theta_h(1+r_h(s_1',s_2'))+\theta_a(s_1',s_2') \;\geq\; \theta_h(1+r_{hd}(s_1',s_2'))\left(\frac{\tilde{V}_d^k(s_1')}{\tilde{V}^k(s_1')}\right)^{\frac{1}{1-\gamma}}\\
&\theta_a'(s_1's_2')\geq -\bar{D}
\end{aligned} \tag{28}$$

The intensive-form value function and the corresponding optimal portfolio choice are obtained by taking the limit $\tilde{V} = \lim_{k\to\infty}\tilde{V}^k$, $\tilde{V}_d = \lim_{k\to\infty}\tilde{V}^k$, and $\theta = \lim_{k\to\infty}\theta^k$. The solution procedure for the case $\gamma = 1$ works accordingly.

To solve the portfolio problem (28) for given k and s_1, we first fix $\theta_h^k(s_1) = \bar{\theta}_h^k(s_1)$ and find $\theta_a^k(s_1)$ solving (28) for given $\bar{\theta}_h^k(s_1)$. To this end, for each s_1', order the pairs (s_1', s_2') so that $r_h(s_1, 1) > r_h(s_1, 2) > \ldots > r_h(s_1, S)$. Given s_1 suppose that the participation constraint is binding for the first $J(s_1)$ states. Then for the first $J(s_1)$ states $\theta_a^k(s_1)$ is given by

$$\theta_a^k(s_1,s_1',s_2') = \bar{\theta}_h^k(s_1)\left[(1+r_{hd}(s_1',s_2'))\left(\frac{\tilde{V}_d^k(s_1')}{\tilde{V}^k(s_1')}\right)^{\frac{1}{1-\gamma}} - (1+r_h(s_1',s_2'))\right] \quad for \;\; (s_1',s_2') = 1,\ldots,J(s_1) \tag{29}$$

while for the remaining states, we have

$$\theta_a^k(s_1,s_1',s_2') \;=\; \bar{r}^k(s_1) - \bar{\theta}_h^k(s_1)(1+r_h(s_1',s_2'))$$

where $\bar{r}^k(s_1)$ is determined by the portfolio constraint in (28). Using the corresponding first-order conditions it is easy to see that, for given $\bar{\theta}_h^k(s_1)$, the solution $\theta_a^k(s_1)$ to (28) is determined by (29), where $J(s_1)$ is the smallest number for which the portfolio choice satisfies the participation constraint. Finally, we find optimal $\theta_h^k(s_1)$ using a standard one-dimensional optimization routine.

5. Model Calibration

In this section, we describe our approach to calibrating the model. We begin with a brief discussion of the micro-level data used to calibrate the model. We then present the calibration of the model using the partial equilibrium restrictions (exogenous returns). Finally, we discuss how to close the model using the general equilibrium restrictions.

5.1 Data

For the calibration and the results discussed below, we use data on earnings and financial wealth drawn from the Survey of Consumer Finances (SCF). The SCF is a triennial survey of U.S. households and we use data from 1989 to 2013. For most steps, we follow Krebs, Kuhn, and Wright (2015) with our construction and treatment of the data. Our measure of earnings (labor income) is wages and salaries plus two-thirds of the farm and business income (if applicable). Our measure of financial wealth is net worth, defined as the sum of the consolidated household balance sheet (including net housing wealth). All data has been deflated using the BLS consumer price index for urban consumers (CPI-U-RS)

We follow Heathcote et al. (2010) for the sample selection and confine attention to households with household head age 23 years and older. Specifically, we drop the wealthiest 1.47 % of households in each calender year, which makes the sample more comparable to that of the Panel Study of Income Dynamics (PSID) and the Consumer Expenditure Survey (CEX). Further, we drop all households that report negative labor income or that report positive hours worked but have missing labor income or that report positive labor income but zero or negative hours worked. We also drop in each year households with a wage rate that is below half the minimum wage of the respective year, where we compute the wage rate by dividing labor income by total hours worked.

We construct separately for each survey year life-cycle profiles of median log earnings and financial wealth to earnings ratios. We smooth life-cycle profiles separately for each survey year using linear least squares on a cubic polynomial in age and average the smoothed profiles across survey years to remove time effects.[18] We compute earnings growth rates from age differences in earnings of the smoothed, cross-sectional earnings profiles.

5.2 Calibration: Partial Equilibrium

In this section, we calibrate the partial equilibrium version of the economy, that is, we find values for the expected investment returns r_f and $\bar{r}_h(s_1) = \phi\tilde{r}_h + \varphi(s_1) - \delta_h$ without specifying the production function that generates these returns. We calibrate an annual risk-free rate of $r_f = 3\%$, in line with Kaplan and Violante (2010) and roughly in line with Huggett et al. (2011) and Krueger and Perri (2006) who use a 4% annual risk-free rate, but also deduct capital income taxes.

We choose the age-dependent expected human capital returns, $\bar{r}_h(s_1)$, to match life-cycle profile of earnings growth of the median household in the data for the first 8 age groups. Specifically, we first construct a life-cycle profile of annual median household earnings and earnings growth as described in the previous section, and then construct a corresponding life-cycle profile of earnings growth for the relevant age groups. The result is depicted in Figure 1 and shows the expected life-cycle pattern. Earnings growth rate are very high for young households, monotonically decreasing in age, and turn negative for households older than 50.

FIGURE 1 HERE

[18] We only use observations until age 60 for the regression.

We assume that human capital shocks, η, are approximately normally distributed, that is, we choose the probabilities $\pi(s_2)$ and the realizations $\eta(s_2)$ to approximate a normal distribution with mean 0 and standard deviation $\sigma_\eta = 0.15$. The parameter σ_η measures human capital risk and our choice of $\sigma_\eta = 0.15$ is motivated by the following considerations.

In the model economy, labor income of an individual household in period t is given by $y_{ht} = \tilde{r}_h h_t$, so that the growth rate of labor income is equal to the growth rate of human capital: $y_{h,t+1}/y_{ht} = h_{h,t+1}/h_t$. We can use the equilibrium solution to compute the human capital growth between year t and year $t+1$. Neglecting transitions across age groups s_1, this yields:

$$\frac{h_{t+1}}{h_t} \; = \; (1-\tilde{c})\left(\theta_h(s_{1,t-1})(1+\phi\tilde{r}_h+\varphi(s_{1,t})-\delta_h+\eta(s_{2t}))+\theta_a(s_{1,t-1},s_{1t},s_{2t})\right) \tag{30}$$

Equation (30) can be written as

$$\ln y_{h,t+1} = \ln y_{ht} + d(s_1) + \tilde{\eta}_t, \tag{31}$$

where $d(s_1)$ is a constant and $\{\tilde{\eta}_t\}$ is a sequence of i.i.d. random variables with

$$\sigma^2_{\tilde{\eta}}(s_1) \; = \; \theta^2_h(s_1)\sigma^2_\eta + var[\theta_a|s_1] \tag{32}$$

Hence, the logarithm of labor income follows a random walk with drift d and innovation term $\tilde{\eta}_t$.[19] The random walk specification is often used by the empirical literature to model the permanent component of labor income risk (Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Storesletten et al. (2004)). Thus, their estimate of the standard deviation of the error term for the random walk component of annual labor income can be used to find a value for $\sigma^2_{\tilde{\eta}}$ for given portfolio choices θ_h and θ_a. For young households, we

[19]We have $\tilde{\eta}_t$ instead of $\tilde{\eta}_{t+1}$ in equation (31), and the latter is the common specification for a random walk. However, this is not a problem if the econometrician observes the idiosyncratic depreciation shocks with a one-period lag. In this case, (31) is the correct equation from the household's point of view, but a modified version of (31) with $\tilde{\eta}_{t+1}$ replacing $\tilde{\eta}_t$ is the specification estimated by the econometrician.

will see below that θ_h is close to one and insurance payments, $\theta_a(s_2)$, are small, so that we have $\sigma^2_{\tilde{\eta}} \approx \sigma^2_\eta$. In our baseline calibration, we use $\sigma_\eta = .15$, which lies on the lower end of the spectrum of estimates found by the empirical literature. For example, Carroll and Samwick (1997) find .15, Meghir and Pistaferri (2004) estimate .19, and Storesletten et al. (2004) have .25 (averaged over age-groups and, if applicable, over business cycle conditions). All these studies use labor income before transfer payments, which is the relevant variable from our point of view.

For the baseline calibration, we assume that households who default regain access to financial markets after 7 years: $(1-p) = 1/7$. We further assume no garnishment of labor income, $\tau = 0$, in the baseline calibration. Finally, we assume a degree of relative risk aversion of $\gamma = 1$ (log-utility) and set the annual discount factor to $\beta = 0.95$.[20] We choose the human capital rental rate, $\phi \tilde{r}_h$, to match the average value of the financial wealth to earnings ratio for households age 23-60 (see Figure 2 below).

5.3 Calibration: General Equilibrium

We now close the model by specifying a production function. We use a Cobb-Douglas production function $f(\tilde{K}) = A\tilde{K}^\alpha$, where $0 < \alpha < 1$ is capital's share in output and A is a productivity parameter. In this case, the rental rates of physical capital and human capital are given by

$$\begin{aligned} \tilde{r}_k &= \alpha A \tilde{K}^{\alpha-1} \\ \tilde{r}_h &= (1-\alpha) A \tilde{K}^\alpha \end{aligned} \tag{33}$$

As in Krebs, Kuhn, and Wright (2015), we target an aggregate share of capital income, $\tilde{r}_k K/Y$, of 0.32 so that $\alpha = 0.32$. We also follow Krebs, Kuhn, and Wright (2015) and target

[20] An alternative calibration approach is to require the model to match a given expected human capital return for the young and then use β to match the observed earnings growth rate of the young.

an aggregate capital-to-output ratio of 2.94. This target in conjunction with the target $r_f = \tilde{r}_k - \delta_k = 0.03$ yields $\tilde{r}_k = 0.1085$ and $\delta_k = 0.0785$.

Recall that in the partial equilibrium calibration we have chosen the value of $\phi\tilde{r}_h$ to match the average financial-wealth-to-earnings ratio from the SCF for households age 23-60. Given that the values for capital's share in output/income and $\tilde{r}_k$ and δ_k are also pinned down, the only way to have the general equilibrium model match a particular target for the aggregate capital-to-output ratio, K/Y, is to vary the parameter λ measuring the human capital of newborn households relative to the financial capital of retired household who die in a given period. This is also the approach taken in Krebs, Kuhn, and Wright (2015). Following this approach, we find that we need a value of $\lambda = 0.811$ to match the aggregate capital-to-output ratio of 2.94. This means that aggregate total capital of new-born households (physical plus human) is equal to 81 percent of the physical capital of retired households who die in a given period, which in our setting means that the total capital (physical plus human) of a new-born household is equal to 81 percent of the physical capital of a retired household (recall that retired households do not hold human capital).

The calibration approach discussed so far determines $\phi\tilde{r}_h$, the rental rate of human capital in consumption units, but does not determine separately ϕ and $\tilde{r}_h$. To resolve this indeterminacy, Krebs, Kuhn, and Wright (2015) impose the (somewhat arbitrary) condition that $\tilde{K} = 0.4$, where the value 0.4 for the capital-to-labor ratio is in line with the results obtained in Krebs (2003) using a model with $\phi = 1$ (one unit of the consumption/capital good can be transformed into one unit of human capital).

Finally, we note that the life-cycle profile of human capital returns, $\bar{r}_h(s_1)$, which is pinned down by the partial equilibrium calibration, and the human capital rental rate, $\phi\tilde{r}_h$ (see above), imply a life-cycle profile for the difference $\varphi(s_1) - \delta_h$. However, the values for the learning-by-doing parameters and the human capital depreciation rate are still not separately identified. Krebs, Kuhn, and Wright (2015) choose $\delta_h = 0.04$, which then pins down the life-cycle profile of learning-by-doing parameters.

6. Quantitative Results

In this section, we discuss the main quantitative results, that is, we present the implications of the baseline model with respect to the life-cycle profiles of portfolio choices, under-insurance, and welfare losses associated with under-insurance.

6.1 Portfolio Choice and Human Capital Returns

We begin with an examination of household's portfolio allocation between financial assets and human capital. To this end, we first use current earnings as a proxy for human capital and construct the life-cycle profile of the ratio of financial wealth to earnings in the model and in the data.[21] Figure 2 depicts the result and shows that the model does an excellent job of matching this life-cycle profile, though it somewhat over-predicts the financial wealth holdings of the oldest households. Note that the model matches the life-cycle average of the financial wealth to earnings ratio by construction since we choose the human capital rental rate, $\phi\tilde{r}_h$, accordingly. However, we have no additional parameter to match the shape of the life-cycle profile depicted in Figure 2.

FIGURE 2 HERE

The advantage of using the financial wealth to earnings ratio as a measure of portfolio choice, as we have done in Figure 2, is the ease with which this variable can constructed from the data without imposing additional assumptions. The disadvantage is that current earnings is a very crude proxy of human capital. We therefore construct an alternative measure of portfolio choice from the data that uses the present value of future lifetime

[21] In the model, this ratio is computed as $\frac{1-\theta_h(s_1)}{\phi\tilde{r}_h\theta_h(s_1)}$.

earnings as a proxy for human capital, where future earnings are discounted at the risk-free rate implied by the calibrated model. Figure 3 depicts the result and shows that the model lines up reasonably well with the data. However, the model over-predicts the human capital share, and this over-prediction becomes more severe with age. The explanation for this is that in the model older households are almost fully insured against the human capital loss upon retirement, which means that the model tends to overstate the value of human capital at older age.[22]

FIGURE 3 HERE

In order to help understand the portfolio allocation decisions of households, Figure 4 presents the excess return to investing in human capital as a function of age. As shown in the figure, young households face an excess return of almost ten percent, explaining why the young hold very little financial wealth. Moreover, excess human capital returns in the vicinity of 10 percent are in line with estimated rates of return to on-the-job-training (Blundell et al. 1999 and Mincer 1994). The excess return available to the oldest working households is less than one-half of one percent, which explains why they hold so much more financial wealth than the young.

FIGURE 4 HERE

Our assumption that expected human capital returns are constant but age-dependent means that our approach matches by construction the observed life-cycle profile of earnings growth depicted in Figure 1. Thus, the fact that the calibrated model economy matches the

[22]Note that the value of human capital in the model is always equal to the expected present discounted value of lifetime earnings if future earnings are discounted using the relevant intertemporal marginal rate of substitution and the model earnings process is used to computed expected lifetime earnings. The difference between the model implication and the data depicted in Figure 3 arises because i) different discount rates are used and ii) the model earnings process (in conjunction with the almost full-insurance result for older households) does not capture the data well after age 50.

observed life-cycle profile earnings growth should not be interpreted as evidence in favor of the theory. In contrast, the model generates endogenously the life-cycle profile of portfolio choices depicted in Figures 2 and 3. The fact that the theoretical life-cycle profile of portfolio choices lines up reasonably well with the empirical profile is therefore evidence in support of the theory.

6.2 Consumption Insurance and Welfare

The youngest households not only hold little financial wealth, but they are also dramatically under-insured. Figure 5 plots a measure of consumption insurance, the insurance coefficient, defined as one minus the ratio of the standard deviation of household consumption growth to the standard deviation of household income growth. As shown in the figure, households of the youngest age group are insured against only one third of their income risk, whereas older households are insured against roughly 90 percent of their income risk.

FIGURE 5 HERE

Figure 6 examines the welfare consequences of this underinsurance, depicting the equivalent variation of moving to full insurance measured in units of lifetime consumption.[23] As shown in the figure, the youngest households would require an increase of almost 7.5 percent in their annual consumption to be as well off as if they had access to full insurance. Thus, for young households the welfare loss due to lack of insurance against labor market risk are quite substantial. Further, even for households age 40 these welfare losses amount to 3 percent of lifetime consumption. For the older working households, however, this equivalent variations has fallen to less than one-half of one percent.

FIGURE 6 HERE

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[23]We determine the equivalent variation by removing all enforcement problems for fixed human capital choice, that is, we compute the welfare gain from achieving full insurance given the earnings process.

7. Robustness

In this final section, we analyze the robustness of our main results with respect to changes in the bankruptcy regime, human capital risk, and risk aversion. Clearly, in all these examples borrowing constraints adjust endogenously to changes in the economic environment, which differentiates the approach taken here from an analysis based on incomplete-market models with human capital as in Krebs (2003) and Guvenen et al. (2014). We also compare our under-insurance result to the "almost full-insurance result" of Krueger and Perri (2006) and argue that the finding of Krueger and Perri (2006) is driven by two (counter-factual) assumptions.

7.1 The Effect of Changing Personal Bankruptcy Regimes

Figures 7 through 10 explore the consequences of changing the details of the personal bankruptcy regime either by increasing the time for which a household is considered bankrupt, or by allowing for wage garnishment during bankruptcy. Figures 7 and 8 focus on the effects of changing the duration of financial market exclusion of defaulting households from an average of 7 years to an average of 10 years. As shown in Figure 7, the human capital portfolio share of the youngest households rises from almost one, denoting no financial wealth, to a value greater than one, denoting negative net financial assets. This increase is reflected throughout the age distribution, although the increases are quite modest in size and decline with age. Figure 8 shows that, although human capital investment increases, insurance against human capital risk is almost unchanged, with the blue and red lines almost atop one another. The reason is that households prefer, on the margin, to borrow more in order to invest in human capital and not buy any further insurance. This is also confirmed by the green line in Figure 8 which shows the effect on risk sharing if the household is constrained from increasing their human capital portfolio. In this case, risk sharing is increased across all age groups, with the largest effect on the young, who are most likely to be constrained, and whose consumption insurance rises from about a third of income risk to roughly 40 percent of income risk.

FIGURES 7 AND 8 HERE

Figures 9 and 10 repeat the above analysis, this time by introducing garnishment of 20 percent of wages while bankrupt. Figure 9 shows that this results in a qualitatively similar increase in human capital portfolio shares with the portfolio share of the very youngest working households exceeding one by roughly 7 percent. Borrowing levels are also positive (the human capital share remains larger then one) for households throughout their 30's and into their 40's. Figure 10 also shows that there is no significant increase in risk sharing. If households are constrained from investing more in human capital (the green line), consumption insurance increases dramatically with the young now insured against almost 60 percent of their income risk.

FIGURES 9 AND 10 HERE

7.2 The Effect of Changing Human Capital Risk

We now consider an increase in the standard deviation of labor income shocks from $\sigma_\eta = 0.15$ to $\sigma_\eta = 0.20$. Figure 11 shows that the effect of this increase on human capital investment is not monotone: young households increase, while older households decrease, their investments in human capital. This is the result of two offsetting forces. On the one hand, an increase in labor income risk makes investments in human capital less attractive for a given mean return. On the other hand, increases in labor income risk make the prospect of declaring bankruptcy less attractive as the household must bear the full cost of this risk while bankrupt. This effect improves enforcement of debt contracts and leads to better insurance. For young households, who desire to hold more human capital, the latter effect dominates, while for older households the former effect dominates.

FIGURE 11 HERE

Figure 12 shows the implications of these choices for consumption insurance. Whereas the blue line shows that consumption insurance is increased for all household ages, the green line depicts what would have happened to consumption insurance if the households had been unable to adjust their human capital holdings. As shown in the figure, the youngest households would have increased their consumption insurance even further, while the oldest households would have had less consumption insurance.

FIGURE 12 HERE

7.3 The Effect of Changing Risk Aversion

Lastly, Figures 13 and 14 illustrate the effects of changing the coefficient of relative risk aversion from $\gamma = 1$ to $\gamma = 2$ keeping all factor returns constant at the levels calibrated in the baseline. Figure 13 shows that greater risk aversion, everything else equal, results in greater investments in human capital. This result stems from the fact that declaring bankruptcy is now more expensive, and hence households are able to borrow more for the purpose of investing in human capital. This offsets the fact that more risk averse individuals are otherwise less inclined to invest in risky assets.

FIGURE 13 HERE

Figure 14 shows that consumption insurance is also improved for households with higher risk aversion reflecting both a greater demand for insurance and greater possibilities for insurance resulting from the fact that default is less desirable for such households. The figure also shows, as depicted by the green line, that consumption insurance would have increased still further if households had been unable to increase their investments in risky human capital (fixed portfolio choice).

FIGURE 14 HERE

7.4 Comparison to Krueger and Perri (2006)

Finally, we turn to a comparison of our under-insurance result depicted in Figure 5 and the result reported in Krueger and Perri (2006) that households are well insured against idiosyncratic labor income shocks in the equilibrium of a calibrated macro model with limited enforcement and physical capital. Clearly, one difference between the two approaches is that Krueger and Perri (2006) consider a model with an exogenous labor income process, whereas in the current paper the labor income process is endogenous since human capital is endogenous. However, for fixed human capital choice, the household's optimality conditions derived in this paper are comparable to the optimality conditions used by Krueger and Perri (2006) and the two solutions should therefore be similar if the two earnings processes are similar.[24] This is indeed the case: in this paper earnings follow a random walk, see equation (31), and Krueger and Perri (2006) use an earnings process that has an AR(1) component with a persistence parameter of $\rho = 0.9898$. This suggests that the model analyzed in this paper should replicate the almost-full-insurance result of Krueger and Perri (2006) once parameter values are chosen in line with the Krueger and Perri (2006) model.

Figure 15 shows that the human capital model analyzed here generates insurance results comparable to Krueger and Perri (2006) once we change the model calibration, counterfactually, along two dimensions that make it comparable to Krueger and Perri (2006). Specifically, the red line with circles in Figure 15 depicts our baseline calibration and blue line

[24]Clearly, even for fixed human capital choice the budget constraint (2) differs from the budget constraint of the model analyzed in Krueger and Perri (2006) because the resource cost of producing human capital, x_h, enters into (2). The human capital channel is also essential for our result that earnings and financial capital are proportional to each other, a property that does not hold for the model with exogenous earnings process considered by Krueger and Perri (2006). However, the results shown in Figures 15 and 16 suggest that this model feature is not an important driver of the difference in equilibrium insurance.

with squares shows the model without a life-cycle (no age-dependence of earnings growth).[25] In the case without a life-cycle, the insurance coefficient is 0.70, which is roughly the insurance coefficient of households age 50 in the corresponding life-cycle economy and much larger than the insurance coefficient of young households in the life-cycle economy. If we further follow Krueger and Perri (2006) and assume that, contrary to the US bankruptcy code, defaulting households are permanently excluded from access to credit markets, then the insurance coefficient increases to a value of 0.95 (i.e. 95 percent of labor income risk is insured). Thus, by introducing two realistic features into a model similar to Krueger and Perri (2006), we move from a situation in which all households insure 95 percent of their labor income risk to a situation in which many households insure only 40 percent of their labor income risk.

FIGURE 15 HERE

In Figure 15 we do not re-calibrate the model to match a target value of the capital-to-output ratio, K/Y. The capital-to-output ratio of the calibrated model corresponding to Figure 15 is $K/Y = 1.74$. This is substantially lower than our baseline value of 2.94 and the target value chosen in Krueger and Perri (2006), which is 2.6. In Figure 16 we show the effect of removing the life-cycle on equilibrium insurance if we also adjust the interest rate so that the model generates a capital-to-output ratio of 2.6. In this case, the insurance coefficient drops to 0.85. If we further assume that defaulting households are permanently excluded from access to credit markets, then the insurance coefficient becomes 1 – households are fully insured against labor income risk. Thus, endogenous borrowing constraints never bind in this version of the model without a life-cycle and permanent exclusion. Figure 16 provides a

[25] In the model without a life-cycle, the learning-by-doing parameter φ, which determines expected human capital returns, is set so that the model generates zero earnings growth. When we change the enforcement parameter, p, in Figures 15 and 16 we keep the human capital choice, and therefore the earnings process, fixed.

stark demonstration that the almost-full-insurance result of Krueger and Perri (2006) heavily dependence on two counter-factual assumptions, namely the absence of a life-cycle profile of earnings (growth) and the permanent exclusion of defaulting households.

FIGURE 16 HERE

8. Conclusion

We develop a tractable macroeconomic model with human capital risk and limited contract enforcement, and provide an equilibrium characterization result that is useful for theoretical and quantitative analysis. We use a calibrated version of the model to show that, contrary to previous results in the literature, limited contract enforcement leads to substantial under-insurance against human capital risk.

In this paper, we show that private insurance markets lead to the inefficient provision of insurance if the enforcement of credit contracts is limited by the possibility of default. We also show that the under-insurance against human capital risk causes under-investment in human capital, but we do not explore the macroeconomic consequences of this under-investment in human capital. The analysis in Krebs (2003) suggests that the lack of insurance against human capital risk and the corresponding under-investment in human capital can severely harm economic growth. Consequently, government policy improving the provision of private insurance can lead to substantial gains in economic growth. The analysis of this issue is an important topic for future research.

Appendix

Proof of Proposition 1

To simplify the notation, suppress the dependence on the aggregate state, Ω, and consider a household of cohort $n = 0$. Further, define an action variable $x_t = (c_t, \theta_{t+1})$ and a feasibility correspondence, Γ, that for every (w_t, s_t) restricts the choice of (w_{t+1}, x_t) according to (16). Using this notation, the household maximization problem reads

$$\begin{aligned} max \quad & E\left[\sum_{t=0}^{\infty} \beta^t \nu_t u(x_t|w_0\right] & (A1) \\ s.t. \quad & (w_{t+1}, x_t) \in \Gamma(w_t, s_t) \\ & E\left[\sum_{m=0}^{\infty} \beta^m \nu_m u(x_{t+m})|w_0, s^t\right] \geq V_d(w_t, s_t) \end{aligned}$$

The corresponding Bellman equation reads:

$$\begin{aligned} V(w,s) \quad = \quad & \max_{x,w'} \left\{ u(x) + \beta\rho(s) \sum_{s'} V(w',s')\pi(s'|s) \right\} & (A2) \\ s.t. \quad & (x, w') \in \Gamma(w,s) \\ & V(w',s') \geq V_d(w',s') \end{aligned}$$

Define an operator, T, that maps semi-continuous functions into semi-continuous functions as

$$\begin{aligned} TV(w,s) \quad = \quad & \max_{x,w'} \left\{ u(x) + \beta E[V(w',s')|s] \right\} & (A3) \\ s.t. \quad & (x,w') \in \Gamma(w,s) \\ & V(w',s') \geq V_d(w',s') \; . \end{aligned}$$

A standard contraction mapping argument shows that there is a unique continuous solution, V_0, to the Bellman equation (A2) without participation constraint if i) u is continuous, ii) Γ is compact-valued and continuous, and (19) holds. Extending the argument of Rustichini

(1998),[26] it can be shown that $V_\infty = \lim_{k\to\infty} T^k V_0$ exists, is equal to the maximal solution of the Bellman equation (A2), and is the value function of the sequential maximization problem (A1) if the following four conditions hold: i) u is continuous, ii) Γ is compact-valued and continuous, iii) for all states, (w, s), there exists a feasible plan for the sequential problem (A1) so that the corresponding expected lifetime utility (payoff) is greater than $-\infty$, and iv) for any given state, (w, s), the value function of the max-problem without participation constraints satisfies $V_0^*(w, s) < +\infty$. Thus, to prove proposition 1 it suffices to show that conditions i)-iv) hold.

The continuity of the payoff function, u, is obvious. The correspondence, Γ, is compact-valued since portfolio-choices, θ', are elements of a closed and bounded subset of $\mathbb{R}^m$. Closedness follows from the fact that the set is defined by equalities and weak inequalities. Restricting attention to a bounded set can be shown to be without loss of generality. Continuity of the correspondence Γ is also straightforward to show. A standard argument shows that conditions iii) and iv) hold if condition (19) is satisfied. This proves proposition 1.

Proof of Proposition 2

As before, let V_0 be the solution of the Bellman equation (A2) without the participation constraint. To save space, we only conduct the prove for the case $\gamma \neq 1$. Simple guess-and-verify shows that V_0 has the following functional form:

$$V_0(w, s) \;=\; \tilde{V}_0(s)\, w^{1-\gamma} \qquad (A4)$$

where $\tilde{V}_0$ is the solution to the intensive-form Bellman equation (21) without participation constraint. Let the operator T be defined as in (A3). We show by induction that if $V_k = T^k V_0$

[26] Rustichini (1998) consider a class of dynamic programming problems with participation constraint (incentive compatability constraint) and possibly unbounded utility. However, he requires bi-convergence, which is always satisfied if lifetime-utility is bounded for all feasible paths (Streufert, 1990). Unfortunately, in our problem with $\gamma \geq 1$ the requirement of lower convergence is not satisfied, so that Rustichini (1998) is not directly applicable.

has the functional form, then $V_{k+1} = T^{k+1}V_0$ has the functional form. For $k = 0$ the claim is true because V_0 has the functional form. Suppose now V_k has the functional form. We then have

$$
\begin{aligned}
V_{k+1}(w, s) &= T\, V_k(w, s) \\
&= \max_{w', c, \theta'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} \;+\; \rho(s) \sum_{s'} \tilde{V}_k(s')(1 + r(\theta', s, s'))^{1-\gamma}(w')^{1-\gamma}\pi(s'|s) \right\} \\
s.t. \quad & w' \;=\; (1 + r(\theta', s, s'))(w - c) \\
& 1 \;=\; \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1 + r_f} \qquad\qquad (A5) \\
& \theta'_a(s') \;\geq\; -\bar{D} \;\;,\;\; \theta'_h \geq 0 \;\;,\;\; w' \geq 0 \\
& \tilde{V}_k(s')\,(1 + r(\theta'_h, \theta_a(s'), s, s'))^{1-\gamma}\,(w')^{1-\gamma} \\
& \geq\; \tilde{V}_d(s')(1 + r_{hd}(s, s'))\theta'_h \;.
\end{aligned}
$$

Clearly, the solution to the maximization problem defined by the right-hand-side of (A5) has the form

$$
\begin{aligned}
w'_{k+1} &= (1 - \tilde{c}_{k+1}(s))w \qquad\qquad (A6) \\
\theta'_{k+1} &= \theta'_{k+1}(s) \;,
\end{aligned}
$$

where the subscript $k + 1$ indicates step $k + 1$ in the iteration. Thus, we have $V_{k+1}(w, s) = \tilde{V}_{k+1}(s)w^{1-\gamma}$ where $\tilde{V}_{k+1}$ is defined accordingly.

From proposition 1 we know that $V_\infty = \lim_{k\to\infty} T^k V_0$ exists and that it is the maximal solution to the Bellman equation (A2) as well as the value function of the corresponding sequential maximization problem (A1). Since the set of functions with this functional form is a closed subset of the set of semi-continuous functions, we know that V_∞ has the functional form. This prove proposition 2.

Proof of Proposition 3

From proposition 2 we know that individual households maximize utility subject to the budget constraint and participation constraint if condition (19) is satisfied. Thus, it remains to show that the market clearing condition can be written as (22) and that the law of motion (23) describes the equilibrium evolution of the relative wealth distribution.

For simplicity, we only consider the case of infinitely-live households (one cohort $n = 0$). For the aggregate value of financial asset holdings we find:

$$\begin{aligned} K_{t+1} &= E\left[\frac{\theta_{a,t+1} w_{t+1}}{1+r_{t+1}}\right] \qquad (A7) \\ &= E[\theta_{a,t+1}(1-\tilde{c}_t)w_t] \\ &= E[E[\theta_{a,t+1}(1-\tilde{c}_t)w_t|s_t]] \\ &= E[\theta_{a,t+1}(1-\tilde{c}_t)E[w_t|s_t]] \\ &= E\left[w_t\right] \frac{E[\theta_{a,t+1}(1-\tilde{c}_t)E[w_t|s_t]]}{E\left[w_t\right]} \\ &= E\left[w_t\right] \ E\left[\theta_{a,t+1}(1-\tilde{c}_t)\Omega(s_t)\right] \ . \end{aligned}$$

where the first line follows from the budget constraint, the second line from the law of iterated expectations, the third line from the fact that $\theta_{a,t+1}$ and $\tilde{c}_t$ are independent of wealth and s^{t-1}, and the last line from the definition of Ω. A similar argument shows that

$$H_{t+1} = E\left[w_t\right] \bar{z} E\left[\theta_{h,t+1}(1-\tilde{c}_t)\Omega(s_t)\right]. \qquad (A8)$$

Dividing the two expressions shows that $\tilde{K}'$ is given by (22).

Finally, the law of motion for Ω can be found as:

$$\begin{aligned} \Omega_{t+1}(s_{t+1}) &= \frac{E\left[w_{t+1}|s_{t+1}\right]}{E\left[w_{t+1}\right]} \qquad (A9) \\ &= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)w_t|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)w_t\right]} \\ &= \frac{E\left[E\left[(1+r_{t+1})(1-\tilde{c}_t)w_t|s_t\right]|s_{t+1}\right]}{E\left[E\left[(1+r_{t+1})(1-\tilde{c}_t)w_t|s_t\right]\right]} \end{aligned}$$

$$
\begin{aligned}
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[w_t|s_t\right]|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[w_t|s_t\right]\right]} \\
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[w_t|s_t\right]|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[w_t|s_t\right]\right]} \times \frac{E\left[w_t\right]}{E\left[w_t\right]} \\
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)\Omega_t(s_t)|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)\Omega_t(s_t)\right]},
\end{aligned}
$$

where the second line follows from the budget constraint, the third line from the law of iterated expectations, the fourth line from the fact that θ_{t+1} and $\tilde{c}_t$ are independent of wealth and s^{t-1}, and the last line from the definition of Ω. This completes the proof of proposition 3.

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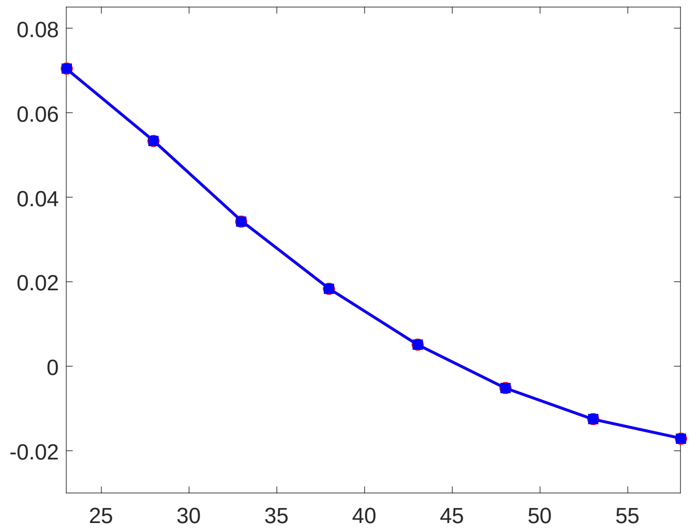
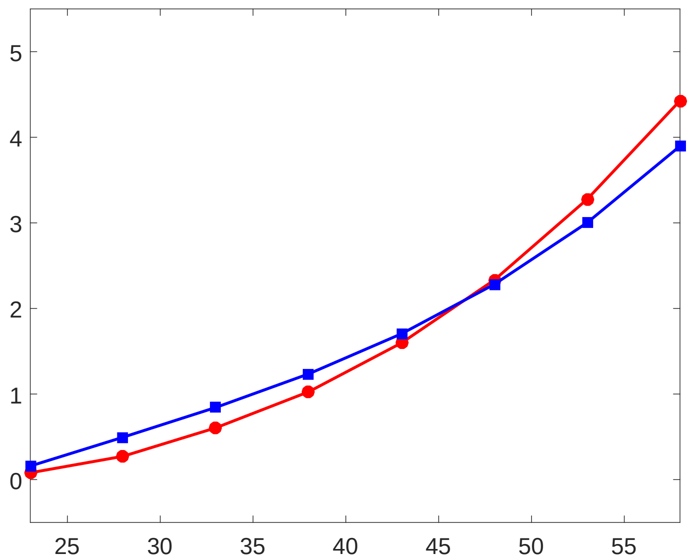
Figure 1: Earnings growth rates



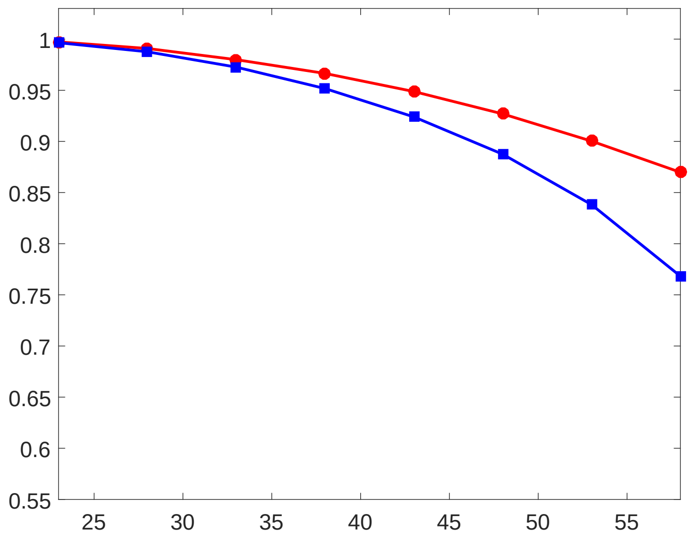
Notes: Earnings growth rates from the model and the data. The red line with circles shows the model, the blue line with squares the data. Horizontal axis shows average age within each age group and vertical axis shows annual earnings growth in percent.

Figure 2: Financial wealth to earnings ratio



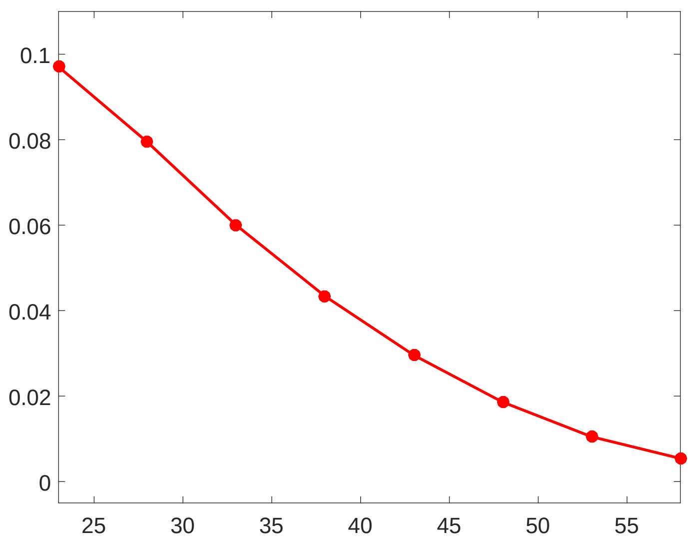
Notes: Wealth-to-income from the model and the data. The red line with circles shows the model, the blue line with squares the data. Horizontal axis shows average age within each age group.

Figure 3: Human capital portfolio share



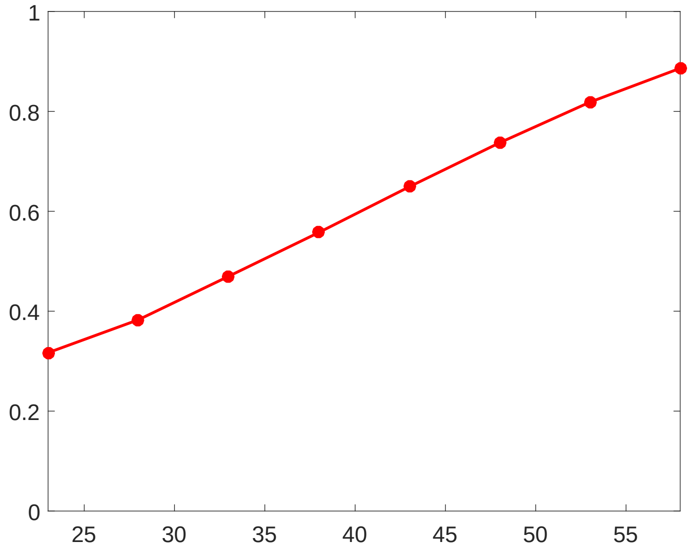
Notes: Human capital portfolio share in total wealth from the model and the data. The red line with circles shows the model, the blue line with squares the data. Horizontal axis shows average age within each age group.

Figure 4: Excess human capital returns



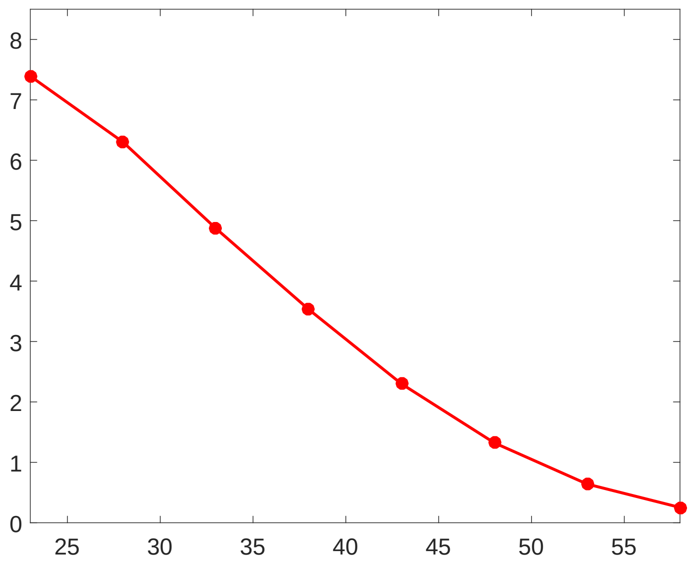
Notes: Excess human capital returns over the life-cycle. Horizontal axis shows average age within each age group.

Figure 5: Consumption insurance



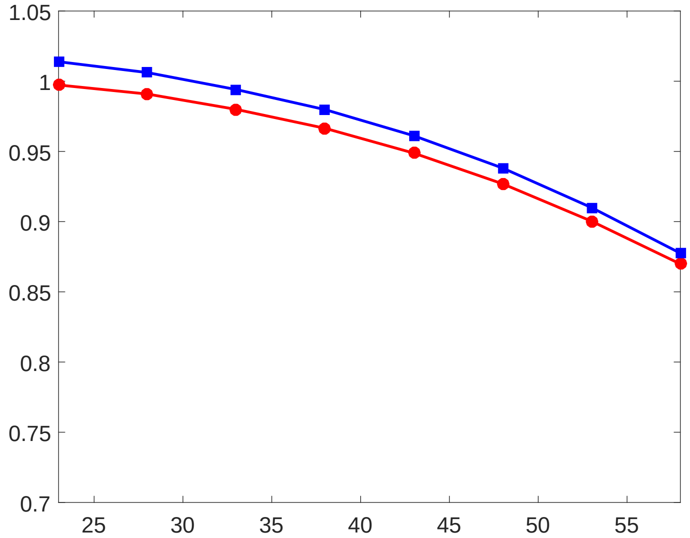
Notes: Consumption insurance of households over the life-cycle. Consumption insurance is measured by the insurance coefficient. The insurance coefficient is constructed as one minus the ratio of the standard deviation of consumption growth over the standard deviation of income growth. Full insurance yields an insurance coefficient of 1 and no insurance (autarky) an insurance coefficient of 0. Horizontal axis shows average age within each age group.

Figure 6: Welfare cost of underinsurance



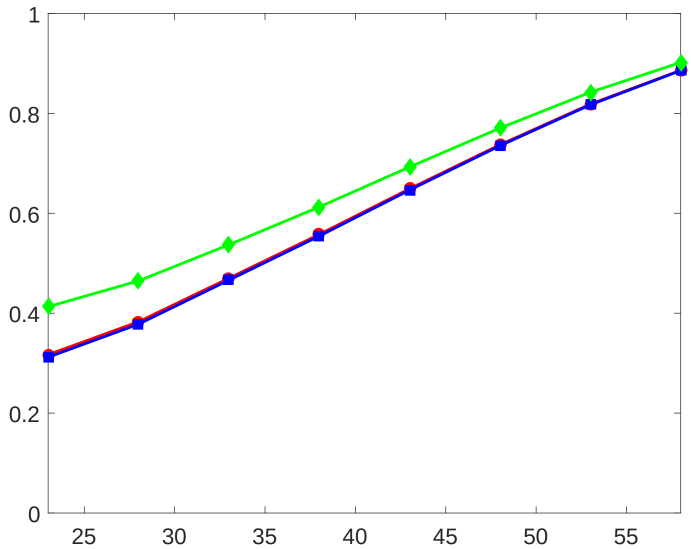
Notes: Welfare costs of limited contract enforcement over the life-cycle. Welfare costs are shown as consumption equivalent variation in percentage points. Horizontal axis shows average age within each age group.

Figure 7: Human capital portfolio share and exclusion after default



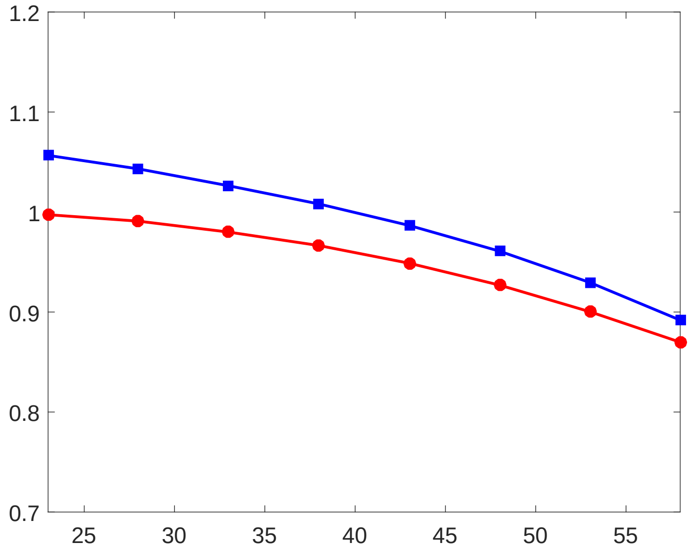
Notes: Human capital portfolio share in total wealth. The red line with circles shows the baseline model with $p = 1 - \frac{1}{7}$ and the blue line with squares shows the model with $p = 1 - \frac{1}{10}$. Horizontal axis shows average age within each age group.

Figure 8: Consumption insurance and exclusion after default



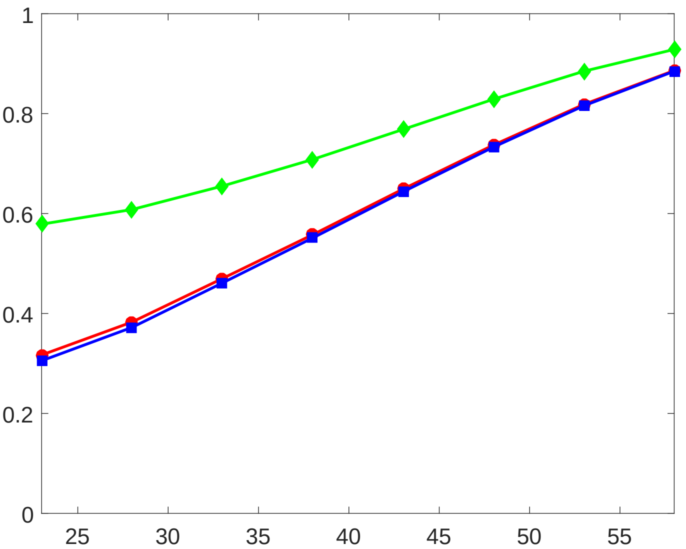
Notes: Consumption insurance of households over the life-cycle measured by the insurance coefficient. The red line with circles shows the baseline model with $p = 1 - \frac{1}{7}$, the blue line with squares shows the model with $p = 1 - \frac{1}{10}$, and the green line with diamonds shows the model with $p = 1 - \frac{1}{10}$ and human capital allocation fixed to the baseline. Horizontal axis shows average age within each age group.

Figure 9: Human capital portfolio share and wage garnishment in default



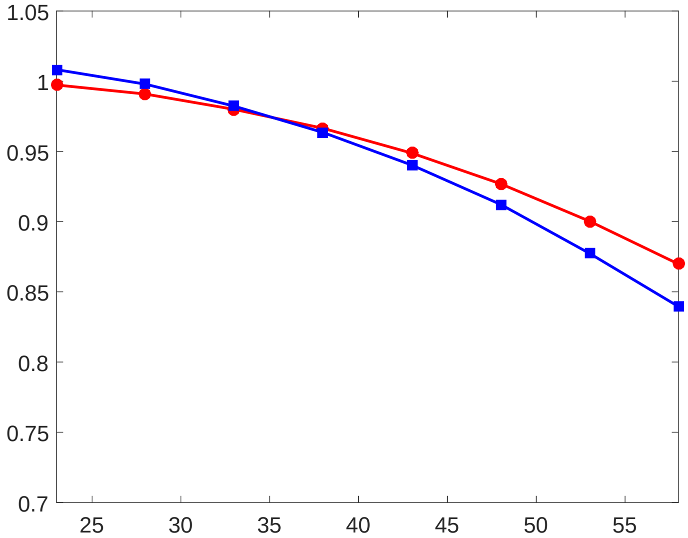
Notes: Human capital portfolio share in total wealth. The red line with circles shows the baseline model and the blue line with squares shows the model with 20% wage garnishment during default. Horizontal axis shows average age within each age group.

Figure 10: Consumption insurance and wage garnishment in default



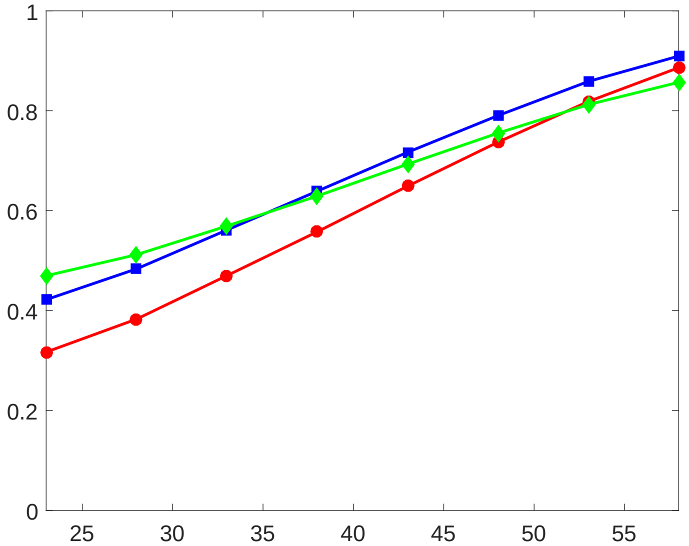
Notes: Consumption insurance of households over the life-cycle measured by the insurance coefficient. The red line with circles shows the baseline model, the blue line with squares shows the model with 20% wage garnishment during default, and the green line with diamonds shows the model with 20% wage garnishment during default and human capital allocation fixed to the baseline. Horizontal axis shows average age within each age group.

Figure 11: Human capital portfolio share and human capital risk



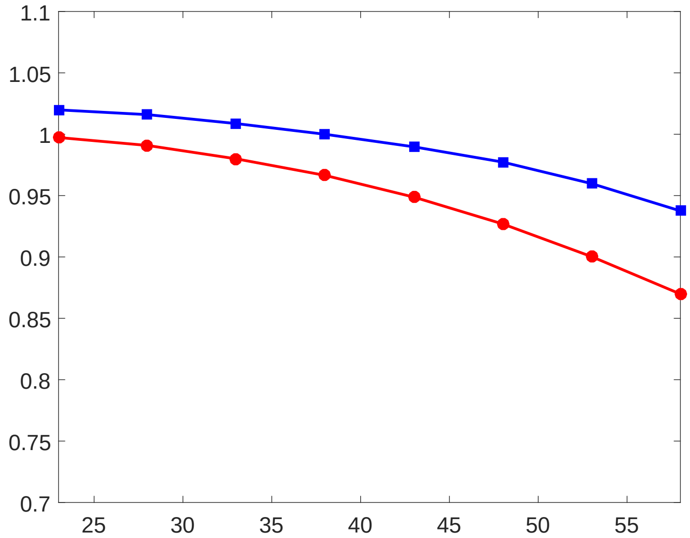
Notes: Human capital portfolio share in total wealth. The red line with circles shows the baseline model with a standard deviation of human capital risk of 0.15 and the blue line with squares shows the model with a standard deviation of human capital risk of 0.2. Horizontal axis shows average age within each age group.

Figure 12: Consumption insurance and human capital risk



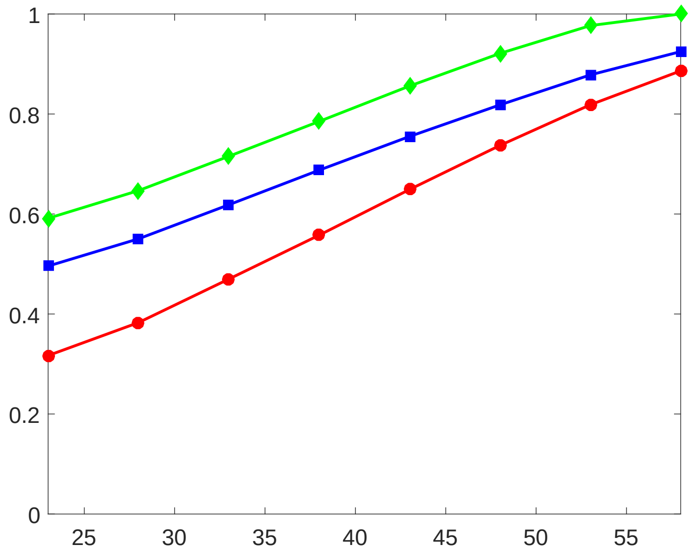
Notes: Consumption insurance of households over the life-cycle measured by the insurance coefficient. The red line with circles shows the baseline model with a standard deviation of human capital risk of 0.15, the blue line with squares shows the model with a standard deviation of human capital risk of 0.20, and the green line with diamonds shows the model with a standard deviation of human capital risk of 0.20 and human capital allocation fixed to the baseline. Horizontal axis shows average age within each age group.

Figure 13: Human capital portfolio share and risk aversion



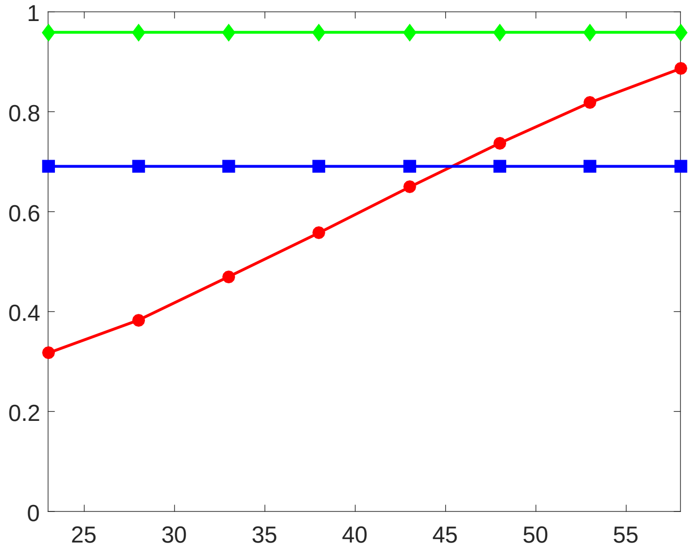
Notes: Human capital portfolio share in total wealth. The red line with circles shows the baseline model with the degree of relative risk aversion $\gamma = 1$ and the blue line with squares shows the model with the degree of relative risk aversion $\gamma = 2$. Horizontal axis shows average age within each age group.

Figure 14: Consumption insurance and risk aversion



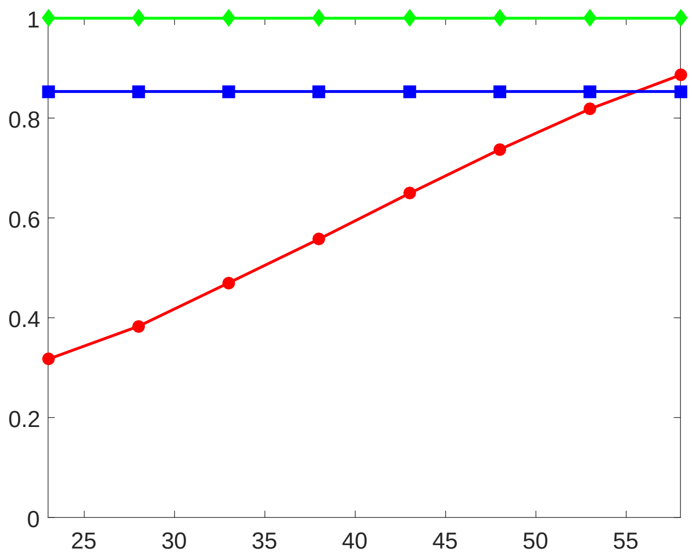
Notes: Consumption insurance of households over the life-cycle measured by the insurance coefficient. The red line with circles shows the baseline model with the degree of relative risk aversion $\gamma = 1$, the blue line with squares shows the model with the degree of relative risk aversion $\gamma = 2$, and the green line with diamonds shows the model with the degree of relative risk aversion $\gamma = 2$ and human capital allocation fixed to the baseline. Horizontal axis shows average age within each age group.

Figure 15: Comparison to Krueger and Perri (2006) I



Notes: Consumption insurance of households measured by the insurance coefficient. The red line with circles shows the baseline model with life-cycle variation in earnings growth and enforcement parameter $p = 1 - \frac{1}{7}$. The blue line with squares depicts the model without life-cycle variation in earnings growth. The green line with diamonds shows the model without life-cycle variation in earnings growth and enforcement parameter $p = 1$ (infinite exclusion of defaulting households). Horizontal axis shows average age within each age group. The baseline model implies a capital-to-output ratio of 2.94 and the two models without a life-cycle generate a capital-to-output ratio of 1.74.

Figure 16: Comparison to Krueger and Perri (2006) II



Notes: Consumption insurance of households measured by the insurance coefficient. The red line with circles shows the baseline model with life-cycle variation in earnings growth and enforcement parameter $p = 1 - \frac{1}{7}$. The blue line with squares depicts the model without life-cycle variation in earnings growth. The green line with diamonds shows the model without life-cycle variation in earnings growth and enforcement parameter $p = 1$ (infinite exclusion of defaulting households). Horizontal axis shows average age within each age group. The baseline model implies a capital-to-output ratio of 2.94 and the two models without a life-cycle generate a capital-to-output ratio of 2.6 (the value used in Krueger and Perri, 2006)