Recursive equilibria in an Aiyagari-style economy with permanent income shocks*

Moritz Kuhn[1]

Abstract

We prove existence of a recursive competitive equilibrium (RCE) for an Aiyagari-style economy with permanent income shocks and derive important economic implications. We show that there exist equilibria where borrowing constraints are never binding and establish a non-trivial lower bound on the equilibrium interest rate. These results imply distinct consumption dynamics compared to existing studies. We present a new approach to solve the agent's problem that uses lattices of consumption functions to deal with permanent income shocks and an unbounded utility function. The approach provides a theoretical foundation for convergence of the time iteration algorithm widely used in applied work.

keywords : Permanent income shocks, incomplete markets, dynamic general equilibrium, heterogeneous agents

JEL codes : C62, C68, E21

*Manuscript received November 2010; revised May 2012

[1]I thank Wouter den Haan, Philip Jung, Dirk Krüger, Felix Kubler, Alexander Ludwig, Nicola Pavoni, Melanie Schienle and two anonymous referees for their comments and remarks, and in particular, my advisor Tom Krebs for his guidance and support. Furthermore, I am grateful for helpful comments and remarks from seminar participants at Bonn, Mannheim, and UCL and participants at the EEA in Barcelona, the ESWM in Budapest, the European Macro Workshop 2010, and the ESWC in Shanghai. The usual disclaimer applies.

1 Introduction

Over the last two decades, a large literature has studied the effects of income uncertainty on individual behavior in heterogeneous agents incomplete markets economies, a model class that is widely known as Aiyagari-style models.[2] Though the empirical literature has often found that labor income risk has a non-negligible random walk component[3] and some applied papers have worked with this type of income specification[4], theoretical work has often ruled out this case by assuming a compact state space[5]. Constantinides and Duffie (1996) and Krebs (2007) consider models with permanent income shocks, but these models are highly stylized in the sense that the structure of the endowment process allows the construction of no-trade equilibria. Heathcote et al. (2009) also consider a model with permanent income shocks, but to preserve tractability they have to assume a very specific market structure. This paper makes three tightly connected contributions on the existence, characterization, and computation of recursive equilibria in an Aiyagari-type model with permanent income shocks. Firstly, we prove the existence of stationary recursive equilibria under standard assumptions on preferences and technology. Secondly, in doing so we prove that for a large class of consumption-saving problems, the widely applied computational algorithm that only uses first-order conditions of the agent's problem (*time iteration*) converges. Thirdly, we prove that in equilibrium borrowing constraints never bind and derive a non-trivial lower bound on the equilibrium interest rate.

The equilibrium existence proof comprises three steps. The first step is to show the existence of an optimal solution to the agents' problem. The standard approach to this kind of problem uses the value function, the contraction property of the Bellman equation, and the principle of optimality to prove the existence of a solution.[6] In this paper, we

[2]For example, Aiyagari (1994), Huggett (1993), Telmer (1993) or the textbook by Ljungqvist and Sargent (2000)

[3]For example, Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Blundell et al. (2008)

[4]For example, Deaton (1991) and Carroll (1997)

[5]See Duffie et al. (1994) and Miao (2006).

[6]The classical reference is Stokey and Lucas (1989).

depart from this approach by relying only on first-order conditions of the agents' problem (*Euler equations*).[7] Similar approaches have been taken in Deaton and Laroque (1992), Coleman (1991), and Rabault (2002). However, all three papers deal with functions on a metric space and, in the case of Deaton and Laroque (1992) and Coleman (1991), apply only to problems with a compact state space and bounded utility. In contrast, in this paper we use a lattice of consumption functions and apply Tarski's fixed point theorem to prove the existence of a recursive policy function, which allows us to deal with permanent income shocks and an unbounded utility function.[8] Since the proof is constructive, it simultaneously establishes the convergence of the *time iteration* algorithm for consumption-saving problems, and thereby, provides a theoretical justification for its widespread use. To our knowledge, this proof has been missing from the literature.[9]

In the second step of the existence proof, we show that a unique stationary distribution exists, and in step three, we derive the existence of a market clearing interest rate. For this part of the existence proof, the presence of prudence (strictly convex marginal utility) is crucial in order to have precautionary savings in an equilibrium with permanent income shocks. As a corollary to the existence result, we show that borrowing constraints never bind.[10] This result is of interest because it shows that the non-existence result of Krebs (2004) does not extend to the case of a non-compact state space. In other words,

[7] Although the present paper focuses on the case of permanent income shocks, this step of the proof is presented for a general class of consumption-saving problems with Markovian income processes.

[8] The only other paper we are aware of that deals with a non-compact state space is Morand and Reffett (2003). They apply lattice theory to prove equilibrium existence for a representative agent stochastic growth economy but assume bounded utility and uncertainty that evolves according to a finite-state Markov process.

[9] The approach in Deaton and Laroque (1992) and Coleman (1991) covers only the case of a compact state space. Furthermore, the operator in Coleman (1991) and Morand and Reffett (2003) applies only to a representative agent economy. The approach by Rendahl (2006) assumes bounded utility and still relies on the convergence of the value function iteration.

[10] In contrast, Huggett and Ospina (2001) show that in models with mean-reverting shocks, prudence of agents is not needed to get precautionary savings because borrowing constraints are always binding for some agents.

in contrast to the model set-up usually analyzed by the theoretical literature, in the current model the two market imperfections "missing insurance markets" and "borrowing constraints in credit markets" can be disentangled.

Turning to our last result, we show that non-binding borrowing constraints imply a non-trivial lower bound on the equilibrium interest rate. This lower bound coincides with the equilibrium interest rate in no-trade economies as in Constantinides and Duffie (1996) and Krebs (2007). The reason for the higher interest rate in our model stems from the fact that in a production economy agents must hold on average assets in positive net supply.[11] The lower bound allows us to relate our results to existing partial equilibrium studies that examine consumption-saving decisions with permanent income shocks, like Deaton (1991) and Carroll (2004). In these studies, the authors restrict the interest rates to values that are below the lower bound we establish. This provides an explanation for why they find borrowing constraints to be always binding.[12] These models predict, therefore, long-run consumption dynamics that are similar to those of models with autarkic equilibria like in Constantinides and Duffie (1996) and Krebs (2007), where consumption tracks income one-to-one.[13] In contrast, the model in this paper features asset trade in equilibrium, so that income shocks will not affect consumption one-to-one, and as a consequence, the consumption dynamics will be quite distinct.

Recently, empirical researchers have studied the correlation between permanent income shocks and consumption (Blundell et al. (2008), Jappelli et al. (2008)). The predictions from the quantitative models could only poorly be reconciled with the empirical findings (Kaplan and Violante (2010)). However, the relaxation of the parameter restrictions that are possible based on our results, and the consequences for the implied saving dynamics

[11] In Krebs (2007), the bond is in zero net supply.

[12] Carroll (2004) allows for zero income and transitory shocks. These additional shocks induce assets in positive net supply in his model. If one drops these additional shocks, the model reduces to the Deaton (1991) case, and one finds again that borrowing constraints are always binding.

[13] However, the model by Carroll (2004) generates a reaction that is less than one-to-one if all sources of income risk (transitory and zero income shocks) as specified in the model are employed.

of the general equilibrium model might help to bring the model predictions closer to the empirical data. We provide a short quantitative example of the theoretical results in the last section of the paper.[14]

The rest of the paper is structured as follows: Section 2 presents the model. The existence of an optimal solution to the individual's problem is established in section 3. This section is more general and applies to a large class of Markovian income processes. In section 4, we prove the existence of a stationary distribution, and in section 5, we prove that a RCE exists. The discussion on borrowing constraints and the implications for the consumption-saving decision follows in section 6. Section 7 provides a short quantitative illustration of the theoretical results. Section 8 concludes. All proofs can be found in the appendix.

2 The model

We take time to be discrete and periods are labeled by an index $t \in \mathbb{N}$. The economy is populated by a continuum of mass 1 of ex ante identical agents.[15] Every agent has an infinite planning horizon, but faces a constant probability of death in every period. At the time of death, we normalize utility to zero. An agent who dies is replaced by a newborn agent. The initial endowment in assets and labor productivity $\{a_0, z_0\}$ of a newborn agent is drawn from a possibly degenerate distribution $\phi(a, z)$.[16]

At the beginning of her life, every agent chooses a recursive policy function that de-

[14]Kuhn (2010) investigates the quantitative implications for the consumption-saving decision and welfare in a model with permanent income shocks and finds significant self-insurance in equilibrium.

[15]We are aware of the technical issues regarding the measurability problem for models with a continuum of agents and i.i.d. income shocks. But we refer the interested reader to Green (1994) for detailed discussion of the appropriate construction of the set of agents to preserve measurability for all subset of agents. From now on we apply the law of large numbers without further discussion.

[16]Below, we also allow for a transitory component to productivity. In this case, we draw from an extended initial distribution with an independent transitory component that satisfies the assumptions on the transitory component in assumption 2 below. This extension is straightforward.

termines her behavior over time. We normalize the time endowment of every agent in every period to unity and assume an inelastic labor supply of this unit of time. Like in Aiyagari's (1994) model the agent decides only about consumption and savings. For preferences of agents we assume, as commonly done, that they can be represented by the expected discounted sum of constant relative risk aversion (CRRA) utility functions.

Assumption 1. *The period utility function is of the CRRA type*

$$u(c) = \begin{cases} \log(c) & \gamma = 1 \\ \frac{c^{1-\gamma}}{1-\gamma} & \textit{otherwise} \end{cases} \tag{1}$$

We denote the productivity state in period t by z_t.[17] As commonly assumed, labor productivity is stochastic over time but shocks to labor productivity are permanent. At the end of every period, every agent draws a survival shock η_{t+1} from a binomial distribution. We associate a draw of $\eta_{t+1} = 1$ with survival from period t to $t+1$. If an agent survives, her labor productivity in $t+1$ is determined by the following stochastic law of motion

$$z_{t+1} = z_t \varepsilon_{t+1}$$

where ε_{t+1} denotes the shock to labor productivity for which we make some mild distributional assumptions in assumption 2 below. If the agent dies, the labor productivity of her successor is drawn from $\phi(a, z)$. The decision problem of agents remains unaffected by all developments after death because utility in the case of death is zero. However, at the aggregate level of the economy the draw of newborn agents enters the law of motion for productivity. We describe the productivity process at the aggregate level by an augmented stochastic law of motion

$$z_{t+1} = \begin{cases} z_t \varepsilon_{t+1} & \eta_{t+1} = 1 \\ z_0 & \textit{otherwise} \end{cases} \tag{2}$$

[17] Throughout, we do not use subscripts for individuals because they only increase the notational burden and are not necessary for the proofs.

A straightforward interpretation of this structure of the economy is as an economy with different labor market cohorts. Every period a cohort enters the labor market with initial endowments in assets and productivity and over time workers randomly drop out of the labor market. Hence, death should not necessarily be associated with physical death.

Additionally to permanent shocks, we allow for transitory i.i.d. shocks ζ_t. Labor productivity in period t is then the product of a permanent and a transitory component. The initial endowments in this case include a transitory component ζ_0. Again, we only need some mild assumptions on the distribution of ζ. All assumptions on random variables can be summarized as follows

Assumption 2. *The shocks ε, ζ and η are i.i.d. and the distributions satisfy*

(i)	$\nexists e \in supp(\varepsilon) : Prob(e) = 1$	*(vi)*	$\mathbb{E}[\zeta] = 1$	
(ii)	$Prob(\varepsilon > 0) = 1$	*(vii)*	$Prob(\zeta > 0) = 1$	
(iii)	$Prob(\eta = 0) = \theta > 0$	*(viii)*	$\mathbb{E}[\zeta_t \varepsilon_s] = \mathbb{E}[\zeta_t]\,\mathbb{E}[\varepsilon_s]$	$\forall s, t \geq 0$
(iv)	$\mathbb{E}[\varepsilon] = 1$	*(ix)*	$\mathbb{E}[\zeta^{1-\gamma}] = M < \infty$	
(v)	$\tilde{\beta}(1-\theta)\mathbb{E}[\varepsilon^{1-\gamma}] < 1$			

where $\tilde{\beta}$ in (v) denotes the time discount factor of agents. These assumptions are little restrictive and have a straightforward interpretation. Assumption (i) rules out situations without permanent income shocks, (ii) and (vii) rule out zero income shocks, (iii) requires a positive probability of death, (iv), (vi) and $(viii)$ fix means of income risk and require independence. Finally, (v) and (ix) limit income uncertainty for cases $\gamma \neq 1$. Assumption (v) is equivalent to the assumption made in Krebs (2007) to satisfy the transversality condition.

At the aggregate level, we use $\mu_0(a, z, \zeta)$ to denote the initial distribution over assets and productivity levels of *all* agents. In a stationary equilibrium μ_0 coincides with the stationary equilibrium distribution. In equilibrium the distribution over all agents μ will be distinct from the distribution over newborn agents ϕ.

2.1 Agent's problem

The agent's objective is to maximize her discounted life-time utility from consumption. Consumption choices are made in every period contingent on the current state of the world. When making their consumption decision, agents take the interest rate and wage rate as given and constant over time.[18] To form expectations about future states, agents condition on their initial productivity and asset endowment. Once we condition on survival and use the information about initial endowments, the probability measure over future states can be constructed recursively in the usual way.[19] The probability of an agent to reach period t without dying is $(1-\theta)^t$ and utility in the case of death is normalized to zero.[20] The agent's objective function is

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\tilde{\beta}^t(1-\theta)^t u(c_t)\right] \tag{3}$$

where the subscript to the expectation operator indicates the information set available to the agent. The time discount factor is denoted by $\tilde{\beta}$ and $1-\theta$ is the period-to-period survival probability. From this formulation it is easy to see that the agent's problem with probability of death is equivalent to a problem with a higher time discount factor. We exploit this for notational convenience from now on and define $\beta := \tilde{\beta}(1-\theta)$.

The agent's choices are furthermore constraint by the fact that asset holdings have to satisfy the intertemporal budget constraint

$$c_t + a_{t+1} = (1+r)a_t + wz_t\zeta_t \tag{4}$$

where a_t denotes asset holdings at the beginning of period t and r and w denote the interest rate respectively wage rate. To rule out Ponzi schemes, we impose a no debt constraint $a_{t+1} \geq 0$ for all periods $t > 0$. We discuss this constraint extensively in section 6.

[18] We assume constant prices at this stage already because we focus on stationary equilibria below.

[19] In footnote 21 we restate the construction.

[20] From the individual's point of view, when forming expectations, death is an absorbing state. Constant utility can capture all situations where agent's utility is independent of the previous history.

When we collect all ingredients to the agent's decision problem, we get an optimal control problem under uncertainty

$$
\begin{aligned}
(5) \qquad \max_{\{c_t, a_{t+1}\}} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \, u(c_t) \right] \\
s.t. \quad & c_t + a_{t+1} = (1+r)a_t + w z_t \zeta_t \qquad \forall t \\
& z_{t+1} = z_t \varepsilon_{t+1} \qquad \forall t \\
& \{a_{t+1}, c_t\} \in [0, \infty) \times \mathbb{R}_+ \;\; \forall t \\
& \{a_0, z_0, \zeta_0\} \text{ given}
\end{aligned}
$$

We make the following standard assumption on the discount factor

Assumption 3. *θ and $\tilde{\beta}$ are such that $\beta < 1$.*

2.2 Firm's problem

Production in the model takes place in a perfectly competitive production sector. We model the production side of the economy as a representative firm producing at marginal costs. We assume that production takes place using a standard neoclassical production function.

Assumption 4. *The production function $Y_t = F(K_t, L_t)$ satisfies $F(K_t, L_t) = L_t f(k_t)$, $F(0, L_t) = F(K_t, 0) = 0$, and $f'(k_t) > 0$,$f''(k_t) < 0$.*

K_t denotes the aggregate capital stock, L_t labor in productivity units, i.e. labor supply times productivity aggregated over all individuals, and k_t denotes the capital to labor ratio $\frac{K_t}{L_t}$. The lower case letters for the production function denote the labor intensive form, i.e. $f(k) = F(K/L, 1)$. We construct the productivity process below such that aggregate effective labor supply is $L_t \equiv 1$ in all periods, so that it always holds that $k_t = K_t$.

We make the following assumption for the depreciation rate and the discount factor.

Assumption 5. *At $\bar{k}$ defined by $\delta \bar{k} = f(\bar{k})$ it holds that $\left(\beta(1 + f'(\bar{k}) - \delta)^{1-\gamma}\right)^{\frac{1}{\gamma}} < 1$.*

The assumption imposes joint restrictions on the preferences of individuals and the production technology. This is a sufficient condition only needed to make sure that for every possible aggregate capital stock there exists a strictly positive lower bound to the consumption function. It can be easily verified that for a risk aversion parameter $\gamma \leq 1$, which includes the important case of log utility, the assumption does not impose any additional restrictions on the choice of model parameters. In applications this condition should not impose any restriction on parameter choices.

2.3 Initial endowments and the probability of death

In line with the empirical evidence, we have incorporated a random walk component in the agent's income process. The random walk has the well-known property that the cross-sectional variance increases over time. However, introducing a constant probability of death allows us to have permanent income shocks in the model but nevertheless keep the equilibrium of the model stationary. This approach is not new and we take it from Constantinides and Duffie (1996). We prove below that the constant probability of death together with the fixed initial distribution for newborn agents guarantees the existence of a stationary equilibrium distribution. To make endowments from the initial distribution resource feasible, we require that in equilibrium initial asset endowments must be equal to asset holdings of agents who die.

Assumption 6. *For each given interest rate* $r \in (f'(\bar{k})-\delta, \beta^{-1}-1)$ *the initial endowments* $\{a_0, z_0, \zeta_0\}$ *of agents are drawn from initial distribution* $\phi(a, z, \zeta)$ *that satisfies*

$$\int z\zeta\phi(da, dz, d\zeta) = 1 \qquad \int a\phi(da, dz, d\zeta) = f'^{-1}(r+\delta)$$

where $f'^{-1}(r+\delta)$ denotes the inverse of marginal productivity, so that it maps interest rates to capital stocks. The assumptions on the means ensure that the average labor productivity in the population is always one and that the assets allocated to the entering cohort equal on average the asset holdings of the exiting agents in equilibrium. Transitory shocks are independent of the permanent component (assumption 2), so that means remain unaffected.

Remark 1. *It is important to notice that the initial endowments of agents are only resource feasible in equilibrium. If goods markets do not clear, then also the mean over assets of the exogenously fixed distribution does not coincide with the mean asset holdings of the agents that died.*

2.4 Equilibrium

We define a *recursive competitive equilibrium* (RCE) for this economy as a policy function $c_t = c(a_t, z_t, \zeta_t)$, a capital and labor demand K^d and L^d of the firm together with equilibrium prices r^* and w^* and a stationary equilibrium distribution $\mu(a, z, \zeta)$ over asset and productivity levels of agents such that

1. For every agent the policy function $c(a, z, \zeta)$ solves the agent's optimization problem in (5) given equilibrium prices w^* and r^*.

2. The firm's demand for capital K^d and labor L^d maximizes firm's profits given equilibrium prices w^* and r^*

$$\{K^d, L^d\} = \arg\max_{K,L} F(K, L) - w^*L - (r^* + \delta)K$$

3. The agent's policy function implies a supply of capital K^s and labor L^s and the firm's optimization imply capital and labor demand such that

$$K^s = \int a\mu(da, dz, d\zeta) = K^* = K^d \qquad L^s = \int z\zeta\mu(da, dz, d\zeta) = L^* = L^d \qquad \forall t$$

4. The stationary distribution $\mu(= \mu_0)$ satisfies

$$\mu = (1 - \theta)P\mu + \theta\phi$$

where the operator P uses the policy function $c(a,z,\zeta)$ and the distribution of the productivity shocks[21] to map a distribution over current states to a distribution over next period's states conditional on survival.

Below, in a slight abuse of notation we will use K^s and K^d as asset supply and demand at non-equilibrium prices, too.

3 Individual problem

In this section, we consider a more general consumption-saving problem where we allow for a larger class of Markovian labor productivity processes and looser ad hoc debt constraints. However, we still require that

$$Prob(wz_t\zeta_t - rD > 0) = 1$$

where $D \geq 0$ denotes the ad-hoc debt constraint, i.e. we require $a_{t+1} \geq -D$ for all periods. We reformulate the problem using cash-at-hand. We define

$$x_t := (1+r)a_t + wz_t\zeta_t + D$$

such that the generalized consumption-saving problem becomes

$$\begin{aligned} \max_{c_t} \quad & \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad & x_{t+1} = (1+r)(x_t - c_t) + wz_{t+1}\zeta_{t+1} - rD \\ & z_{t+1} = g(z_t, \varepsilon_{t+1}) \\ & x_t \geq c_t \geq 0 \\ & \{x_0, z_0\} \text{ given} \end{aligned} \tag{6}$$

where $g(z_t, \varepsilon_{t+1})$ is the (Markovian) law of motion for $\{z_t\}_{t=0}^{\infty}$.

[21] The operator P can be constructed as follows

$$Pr(a_{t+1}, z_{t+1}, \zeta_{t+1}) = \int \mathbb{1}(a_{t+1} = (1+r)a_t + wz_t\zeta_t - c(a_t, z_t, \zeta_t))Pr(z_{t+1} = z_t\varepsilon_{t+1})Pr(\zeta_{t+1})Pr(da_t, dz_t, d\zeta_t)$$

where $\mathbb{1}(\cdot)$ denotes the indicator function.

3.1 Characterization of the optimal solution

We know that every optimal solution to (6) must satisfy the first order conditions.

$$c_t^{-\gamma} - \kappa_t = \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] \qquad \forall t \tag{7}$$

$$\kappa_t(x_t - c_t) = 0 \qquad \forall t \tag{8}$$

where κ_t denotes the Lagrange multiplier on the debt constraint. In a RCE the optimal consumption plan must obey a recursive structure. In case of the generalized consumption-saving problem stated in (6), the state variables are the productivity state $z \in Z \subset \mathbb{R}_{++}$ and cash-at-hand $x \in X \subset \mathbb{R}$. We restrict attention to optimal solutions that have a recursive structure of the form

$$c_t = c(x_t, z_t)$$

where the dependence on z_t is necessary if the conditional distribution of income next period depends on the current state.[22]

Once we have restricted the optimal solution to obey a recursive structure, we drop time subscripts and use primes to indicate next period's values. The problem of finding a solution to the first-order conditions can now be formulated as finding a fixed point to the following equation

$$c(x,z) = \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[(c(x',z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \tag{9}$$

where the min-operator captures the complementary slackness condition in (8). This approach has been proposed by Deaton and Laroque (1992) and has been applied to consumption-saving problems in Deaton (1991) and Rabault (2002).[23] In the following, we establish the existence of a fixed point $c(x,z)$ to the modified Euler equation in (9). To establish the existence of a fixed point, we restrict the interest rate to a set

[22] It has been shown in Deaton (1991) that this dependence can be removed in the case when income shocks are permanent.

[23] Both authors iterate on the optimal marginal utility function whereas we iterate on the optimal consumption policy directly.

$\left[f'(\bar{k})-\delta, \beta^{-1}-1\right]$. As we show below, this is sufficient to establish the existence of a RCE.

3.2 Existence of an optimal solution

To prove the existence of a fixed point to this equation, we construct a lattice of consumption functions and an operator that is a self-map on this set of functions. We then apply a version of Tarski's fixed point theorem to establish the existence of a fixed point to this operator in a constructive way. In the first step, we construct a set of candidate consumption functions for the optimal solution to the consumption-saving problem. A consumption function is a map $c : X \times Z \to \mathbb{R}_+$. We restrict attention to the following set of consumption functions

$$
\begin{aligned}
C_0 := \{ & c : X \times Z \to \mathbb{R}_+ | \\
& \forall x_1, x_2 \in X : x_1 > x_2 \Rightarrow c(x_1, z) \geq c(x_2, z) \wedge x_1 - x_2 \geq c(x_1, z) - c(x_2, z)\}
\end{aligned}
$$

Hence, we only consider consumption functions that are increasing and Lipschitz continuous (with Lipschitz constant $L = 1$) in their first argument. This is a very weak condition because it only rules out that an increase in cash-at-hand leads to a more than one-for-one increase in consumption. For this class of functions, we apply the usual pointwise ordering

$$c_1(x, z) \geq c_2(x, z) \qquad \forall (x, z) \in X \times Z \Rightarrow c_1 \geq c_2$$

In the appendix, we show (lemma 6) that we can restrict the set of candidate solutions further by imposing an upper and a lower bound (c^u and c^l) on the set of consumption functions. The reason is that the operator we construct below is inward pointing[24] at the bounds. The restricted set of candidate solutions is the set C

$$C := \{c \in C_0 : c^l \leq c \leq c^u\}$$

[24]We call the operator T inward pointing if for the upper bound $\bar{x}$ it holds that $T\bar{x} \leq \bar{x}$ and respectively for the lower bound $\underline{x}$ it holds that $T\underline{x} \geq \underline{x}$.

The next step is to show that this set C together with the ordering just defined forms a complete lattice. To this end, we show that the supremum and the infimum for arbitrary sets always exist. In the appendix, we prove that we get the supremum (infimum) of two consumption functions as the upper (lower) envelope. Hence, we obtain the supremum $\bar{c}$ (infimum $\underline{c}$) by taking the pointwise maximum (minimum). Equivalently, we get the supremum $\bar{c}^{\infty}$ (infimum $\underline{c}^{\infty}$) of a possibly infinite subset of consumption functions $C' \subset C$ as the upper (lower) envelope. Since the set C has an upper bound c^u and a lower bound c^l the supremum and the infimum always exist, and it holds that $\bar{c}^{\infty} \leq c^u$ and $\underline{c}^{\infty} \geq c^l$. It follows that $(C, \leq)$ is a complete lattice. In the next step, we construct an operator on this set of functions. The operator T maps an element $c_i \in C$ to an element c_{i+1} by the following operation

$$\forall (x,z) \ : \quad c_{i+1}(x,z) = \lambda \ \text{ where } \lambda \text{ solves}$$
$$\lambda = \min\left\{ x, (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, z' \right) \right)^{-\gamma} \right] \right)^{-\frac{1}{\gamma}} \right\} \tag{10}$$

and we define the following function

(11)

$$G_i(x,z,\lambda) := \min\left\{ x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, z' \right) \right)^{-\gamma} \right] \right)^{-\frac{1}{\gamma}} \right\} - \lambda$$

such that we can represent the operator as $c_{i+1} = Tc_i$ with $c_{i+1}(x,z) = \lambda$ iff $G(x,z,\lambda) = 0$ for all (x,z).

In appendix A.2, we prove that the function $G(x,z,\lambda)$ is (i) increasing and continuous in x, (ii) strictly decreasing and continuous in λ, and (iii) for fixed (x,z) there is a unique solution λ^* that solves $G(x,z,\lambda^*) = 0$. It follows that the operator T maps every element $c_i \in C$ to a unique element c_{i+1}. In appendix A.3, we prove that the operator T has the properties of being (i) monotone increasing and (ii) a self-map, i.e. $T : C \to C$. Furthermore, we prove that imposing an upper bound and a lower bound on the possible set of consumption functions is valid because the operator is inward pointing at these bounds. Thus, we have constructed a monotone increasing operator that is a self-map on a complete lattice. This is already sufficient to prove the existence of a fixed point to the modified Euler equation in (9) using the fixed point theorem by Tarski (1955).

Tarski 1. *Every monotone increasing mapping $T : X \to X$ on a complete lattice X has a smallest and a greatest fixed point.*

As the theorem does not require a contraction property of the operator it also lacks the uniqueness result of a contracting operator. The proof is not constructive and establishes only the existence of a fixed point. However, constructiveness is certainly a desirable property. A constructive version of Tarski's theorem exists for continuous operators. The continuity of the operator T is established in the appendix so that a constructive version of Tarski's fixed point theorem applies.[25]

Tarski 2. *For $x^u := \sup(X)$, $x^l := \inf(X)$ and a continuous increasing mapping $T : X \to X$ on a complete lattice X we get that $\lim_{n\to\infty} T^n x^u$ and $\lim_{n\to\infty} T^n x^l$ converge to the largest respectively lowest fixed point $\bar{x}$ respectively $\underline{x}$ of $T : X \to X$.*

This constructive version of the iteration procedure proves the convergence of the standard numerical approach of *time iteration.* The *time iteration* algorithm starts with an initial guess for the policy function and applies the operator T repeatedly to this guess. If c^u is taken as initial guess, then iterating on the operator T will attain a fixed point to the modified Euler equation.[26]

First-order conditions are only necessary for an optimal solution. In the appendix, we show that under the maintained assumptions the transversality condition for the case of permanent income shocks is satisfied. We also state additional conditions for the case of general Markovian income processes and borrowing constraints with $D > 0$. Theorem 1 summarizes the results of this section.

Theorem 1. *Under the maintained assumptions there exists for every $r \in \left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$ an optimal recursive policy function to the agents' problem. It can be found as $\lim_{n\to\infty} T^n c^u$.*

[25]The constructive version of the theorem results from Kleene (1952) first recursion theorem. See Cousot and Cousot (1979) for discussion and further references.

[26]$c_0 = c^u$ is a common initial guess for this procedure.

4 Stationary distribution

For the existence of a stationary distribution, we again restrict attention to the case of permanent income shocks with a constant probability of death.[27] The joint stochastic process for asset holdings and productivity is

$$\begin{bmatrix} a_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} \eta_{t+1}((1+r)a_t + wz_t - c^*(x_t, z_t)) + (1-\eta_{t+1})a_0 \\ \eta_{t+1} z_t \varepsilon_{t+1} + (1-\eta_{t+1})z_0 \end{bmatrix}$$

where $c^*(x_t, z_t)$ denotes the optimal policy given r and w, and a_0 and z_0 are draws from $\phi(a, z)$. In the appendix, we prove that a unique stationary probability distribution for the process always exists. The idea of the proof is to exploit the renewal structure induced by the constant probability of death. With a positive probability of death the expected life-time of an agent is finite. Every time an agent dies there is a draw from a fixed distribution ϕ and the process starts from the support of ϕ. This implies that all sets with positive ϕ-mass must also have positive μ-mass. These two features of the stochastic process imply that the process is *recurrent* and *irreducible* such that a unique stationary distribution exists.[28] We also establish the continuity in the interest rate of the stationary distribution on the interval $\left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$. The proof relies on a result by Le Van and Stachurski (2007). Theorem 2 summarizes the results of the current section.

Theorem 2. *Under the maintained assumptions there exists for every* $r \in \left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$ *and* ϕ *a unique stationary distribution* μ_r *that is continuous in* r *on* $\left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$.

It is important to notice that the initial endowments of agents are only resource feasible in equilibrium. If goods markets do not clear, then also the mean over assets of the

[27]Since transitory shocks are an independent dimension of the stationary distribution, they do not affect stationary. The proofs also apply to the more general case of a Markovian process, if there is a positive probability of death, and an optimal recursive consumption policy exists.

[28]Further details and an extensive study of stability of Markovian processes can be found in the textbook by Meyn and Tweedie (1993).

exogenously fixed distribution does not coincide with the mean asset holdings of the agents' that died.

5 Equilibrium

To satisfy the equilibrium conditions of a RCE in section 2.4, we have to find a stationary distribution $\mu(a, z)$ such that all markets clear. The labor market is cleared by construction, and in the appendix, we show that the goods market clears for at least one interest rate in the set of interest rates for which an optimal solution to the agents' problem and a stationary distribution exist. The idea of the proof is to show that there is an interest rate low enough such that asset demand exceeds asset supply and an interest rate high enough such that the converse is true. Since asset demand and asset supply are continuous in the interest rate, it follows from the Intermediate Value Theorem that there must be at least one interest rate in between where asset markets clear. This proves the existence of a RCE for this model.[29] We now state the main theorem of this paper.

Theorem 3. *Under the maintained assumptions a recursive competitive equilibrium always exists.*

When we establish the existence of an interest rate for which there is aggregate excess supply of capital, we find that for sufficiently high interest rates and only permanent income shocks borrowing constraints are not binding. For this case, we need that consumers are prudent, i.e. have a positive third derivative of the utility function, to rule out equilibria without positive precautionary savings. This case provides an example where the argument by Huggett and Ospina (2001) for the existence of precautionary savings does

[29] Since a proof for the monotonicity of asset supply in the interest rate is lacking, we can not establish uniqueness of the equilibrium. The potential non-monotonicity of asset supply in the interest rate is also discussed in Aiyagari (1993, footnotes 25 and 26). In the appendix, we provide a numerical example for a non-monotone asset supply and discuss in a simple two period model under which condition a non-monotonicity in asset supply can arise.

not apply. Their result of the irrelevance of prudence relies on the fact that borrowing constraints must be binding in equilibrium. However, as we show below, there are equilibria with incomplete markets and idiosyncratic income risk where borrowing constraints are non-binding and precautionary savings arise only due to prudence of consumers.[30]

6 Borrowing constraints and consumption dynamics

We have established the existence of a RCE in a model with permanent and transitory income shocks. In this section, we remove transitory income risk. This allows us to prove some interesting properties of the equilibrium in this model. Especially, we prove that borrowing constraints *must* be non-binding. Theorem 4 states this result.

Theorem 4. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then borrowing constraints must be non-binding.*

To establish this result, it is important to recognize that the state space can be reduced to a single ratio variable[31]: *cash-at-hand to permanent labor income.* This variable is defined as

$$\tilde{x}_t := \frac{x_t}{wz_t} = (1+r)\frac{a_t}{wz_t} + 1$$

The reduction of the state space implies that the decision whether to save or not becomes independent of the current income level. However, the amount saved will still depend on the current level. This characteristic property[32] allows us to develop an intuitive understanding why borrowing constraints are non-binding.

Consider the case where asset holdings are zero ($\tilde{x}_t = 1$). At this point, the decision whether to save or not is the same for *all* agents. To understand this, recall that with

[30] The same bound for the interest rate at which borrowing constraints would be non-binding has been established in Rabault (2002) who studies the consumption-saving decision in a partial equilibrium framework. However, he puts it as an open question whether non-binding borrowing constraints can be sustained indefinitely if marginal utility at the optimal solution is bounded.

[31] This result is well-known and can be found in Deaton (1991).

[32] The state space reduction requires both permanent income shocks and CRRA utility.

permanent income shocks agents at every income level do neither expect future income growth nor income decline as it would be the case with mean-reverting shocks. With mean-reverting shocks the desire to borrow against expected future income growth leads to binding constraints. This motive does not exist with permanent income shocks. Now suppose that in the case with permanent shocks agents with no assets decided not to save. To sustain a positive aggregate capital stock in equilibrium, some agents with already higher cash-at-hand to permanent labor income ratios must save. However, as we prove in the appendix and as economic intuition suggests, this behavior is not optimal in equilibrium. Hence, an optimal policy that is compatible with an equilibrium must be a policy where agents with zero assets do save, and borrowing constraints are non-binding. This intuitive explanation leads us to associate the result of non-binding borrowing constraints rather with the existence of permanent income shocks than with the non-compactness of the state space although the two properties are inherently related.

It follows from the same line of reasoning and directly from the proof of theorem 4 that a unique *target insurance ratio* must exist.

Corollary 1. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then there is a unique $\bar{\tilde{x}}$ (target insurance ratio) such that the optimal policy implies $a_t = a_{t+1}$.*

The corollary formally defines the target insurance ratio as the state in the reduced state space at which the optimal decision of the agent is to keep assets constant between periods.[33] The uniqueness of the target insurance ratio implies that the dynamics induced by

[33] It is important to notice, that this does not coincide with the target insurance rate as defined in Carroll (2004) which is

$$\mathbb{E}_t[\tilde{x}_{t+1}] = \tilde{x}_t$$

To see this, plug $\tilde{c}_t = \frac{r}{1+r}\tilde{x}_t + \frac{1}{1+r}$ ($\Leftrightarrow c_t = ra_t + wz_t$) into the law of motion for the ratio variable, this yields

$$\mathbb{E}_t[\tilde{x}_{t+1}] = \mathbb{E}_t[\varepsilon_{t+1}^{-1}](\tilde{x}_t - 1) + 1 \neq \tilde{x}_t$$

the optimal consumption saving decision drive —apart from stochastic fluctuations— the agents' cash-at-hand ratio towards a target insurance ratio with positive asset holdings. As a further corollary to the result of non-binding borrowing constraints, we establish a non-trivial interval for the equilibrium interest rate.

Corollary 2. *If a RCE with non-binding borrowing constraints exists, then the equilibrium interest rate r lies in the interval* $[\underline{r}, \bar{r}] := \left((\beta \mathbb{E}[\varepsilon^{-\gamma}])^{-1} - 1; \beta^{-1} - 1 \right)$

The lower bound interest rate $\underline{r}$ separates three ranges for the interest rate that have all been independently studied in different strands of the literature with quite different implications for the consumption-saving decision. The corollary demonstrates nicely the contribution of this paper with respect to existing studies on consumption dynamics.

One strand of the literature has studied economies where the interest rate is exactly at the lower bound $\underline{r}$. These are the endowment economies as studied for example in Krebs (2007). In this model, assets are in zero net supply and the interest rate is chosen to balance the desire to accumulate and decumulate assets for all agents and there will be no trade in equilibrium. In this situation, the target insurance ratio is exactly at one ($\bar{\bar{x}} = 1$). This situation is not compatible with an equilibrium in a production economy when capital is an essential input in the production technology. Intuitively, the higher interest rate in the production economy is due to the fact that agents need an additional incentive to accumulate assets.

The interest rates below the lower bound, i.e. $r < \underline{r}$, have been extensively studied in partial equilibrium models developed by Deaton (1991) and Carroll (1997, 2004). Deaton (1991) conjectures that agents always run down assets to zero, become borrowing constrained, and stay borrowing constrained forever. We prove that his interest rate is never an equilibrium interest rate, once we impose equilibrium restrictions on prices. The bound on the interest rate in the models by Deaton and Carroll naturally arises in the proof for the existence of an optimal policy function. It can be shown that this condition can be relaxed without loosing existence of the optimal solution if the lower bound on the optimal consumption function $\underline{c}$ (lemma 6) is taken into account.

7 Quantitative example

To shed additional light on the relevance of the theoretical results, we provide in this section a short quantitative investigation of the model.[34] To make the model accessible to a quantitative analysis, we choose $\gamma = 1$ (log utility), $\tilde{\beta} = 0.9867$, and $\theta = 0.02857$ so that $\beta = 0.9585$. These parameters are taken from Kuhn (2010) and yield an equilibrium capital-to-output ratio of 3 and an average time in the labor market of 35 years if a period is one year. Firms use a Cobb-Douglas production function with capital share $\alpha = 0.33$ and a total factor productivity (TFP) so that equilibrium wages are normalized to 1. The depreciation rate is $\delta = 0.07$. We assume that innovations to the productivity process are log normally distributed with $\log(\varepsilon) \sim \mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma\right)$ and abstract from transitory shocks as in the last section of the theoretical analysis. We set $\sigma = 0.1$ for our analysis in line with the empirical evidence cited in the introduction. We specify the initial distribution $\phi(a, z)$ as a log-normal over assets and productivity levels with $(\log(a_0), \log(z_0)) \sim \mathcal{N}(\mu_{a0}, \mu_{z0}, \sigma_{a0}, \sigma_{z0}, \rho)$. We set $\sigma_{a0} = 0.5999$, $\sigma_{z0} = 0.3873$, $\rho = 0.6457$ and choose the means appropriately.[35] The value for σ_{z0} is set to match the same income inequality as in Kaplan and Violante (2010). The parameters for ρ and σ_{a0} are set to match mean equilibrium assets holdings[36] and the empirical counterpart of the standard deviation of the asset to labor income ratio for newborn agents. As empirical counterpart, we use the net worth to labor income ratio from the 1992 Survey of Consumer Finances in the group of households with household heads age $23 - 27$.

To highlight the effect of the interest rate on optimal policies and allocations, we do not impose general equilibrium restrictions so that we can abstract from the production side

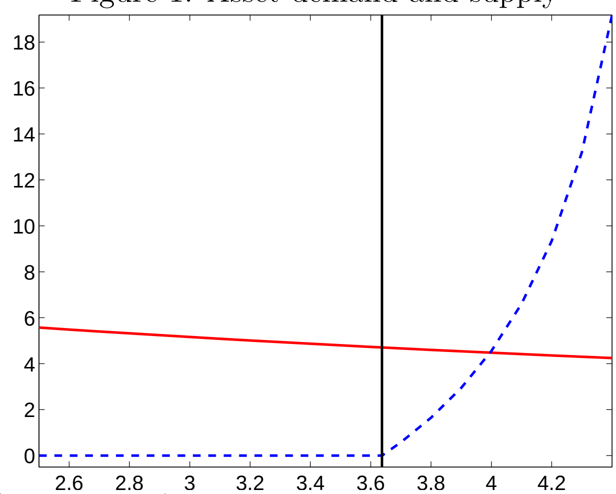
[34] Kuhn (2010) provides a fully calibrated version of the general equilibrium model together with a quantitative analysis and discusses the welfare effects of incomplete markets.

[35] In the simulation of the model the mean of initial productivity is chosen to keep average productivity at 1 and for the initial asset distribution to be consistent with the mean of the asset distribution of the agents who died.

[36] Note that given the targets for the capital to output ratio and the wage rate the equilibrium capital stock is determined.

of the model. This abstraction allows us to study interest rates below the theoretical lower bound $\underline{r} = (\beta \mathbb{E}[\varepsilon^{-\gamma}])^{-1} - 1$ (corollary 2) and to reproduce results from partial equilibrium studies like in Deaton (1991) and Carroll (2004). Figure 1 shows aggregate asset supply and asset demand given an exogenously chosen interest rate.[37] The figure shows two things: First, for all interest rates below the theoretically established lower bound (vertical black line) aggregate asset supply is zero. In this case the borrowing constraint is always binding, while for interest rates above the lower bound the borrowing constraint is never binding (lemma 14). Second, asset supply is monotone increasing in the interest rate so that the equilibrium of the economy is unique.[38]

Figure 1: Asset demand and supply



Notes: Asset demand (red solid line) derived from first-order conditions for profit maximization of firms. Asset supply (blue dashed line) derived numerically from the stationary distribution. The vertical black solid line shows the equilibrium lower bound for the interest rate. The interest rate in percentage points is given on the horizontal axis. The vertical axis gives capital in units of the numeraire.

Obviously, these results have strong implications for cross-sectional inequality. While income inequality is exogenously given to the model, the asset distribution is endogenous. In the case of interest rates below the lower bound $\underline{r}$, we have already seen that the distribution collapses to a single point and all agents hold zero assets. In case of interest rates above the lower bound, the model is able to generate a substantial amount of cross-

[37]The asset supply is derived as the mean of asset holdings of the stationary distribution. We also show asset demand derived analytically from the firm's problem.

[38]In the appendix, we provide an example where asset supply is non-monotone.

sectional inequality. For this step, we include production in the model and compare in table 1 the equilibria of a model with permanent income shocks to the model with fully transitory i.i.d. income shocks.[39] We use different specifications for the standard deviation of the income process in the transitory i.i.d. case and for the initial distribution.

Table 1: asset inequality

	Model 1	Model 2	Model 3	Model 4
transitory shocks	0.399	0.360	0.506	0.539
permanent shocks	0.307	0.307	0.507	0.593

Notes: Asset inequality as measured by the Gini coefficient for the model with permanent income shocks and transitory i.i.d. shocks. Model 1: Standard deviation of transitory shocks $\sigma_{iid} = 0.7296$ and initial distribution $\sigma_{a0} = 0.5999$. Model 2: Standard deviation of transitory shocks $\sigma_{iid} = 0.2$ and initial distribution $\sigma_{a0} = 0.5999$. Model 3: Standard deviation of transitory shocks $\sigma_{iid} = 0.7296$ and initial distribution $\sigma_{a0} = 1.2$. Model 4: Standard deviation of transitory shocks $\sigma_{iid} = 0.7296$ and initial distribution $\sigma_{a0} = 1.5$. Standard deviation for permanent shocks is $\sigma = 0.1$ for all models.

In model 1, we set the standard deviation for the transitory i.i.d. income shocks to reproduce income inequality of the model with permanent shocks. This results in a standard deviation $\sigma_{iid} = 0.7296$ compared to $\sigma = 0.1$ for the permanent shock case. All other parameters are set to the same values as in the case of permanent shocks. In model 2, we reduce the standard deviation to $\sigma_{iid} = 0.2$ so that income inequality reduces substantially.[40] We see that asset inequality compared to model 1 changes only little and that for models 1 and 2 asset inequality is larger in the case of transitory i.i.d. shocks than in the case of permanent shocks. In model 3, we increase asset inequality among newborn agents by setting $\sigma_{a0} = 1.2$ and set $\sigma_{iid} = 0.7296$ as in model 1. We see that asset inequality for the case of permanent and transitory shocks increases substantially. While asset inequality in the case of permanent shocks has been lower in models 1 and 2, it reaches the same level in model 3. Finally, in model 4 we increase σ_{a0} to 1.5. Asset inequality increases further and is now in the case of permanent shocks larger than in

[39] In this case the income process is $z_t = \varepsilon_t$ with $\log(\varepsilon_t) \sim \mathcal{N}(-\frac{\sigma_{iid}^2}{2}, \sigma_{iid})$.

[40] This is one of the cases studied in Aiyagari (1994).

the case of transitory i.i.d. shocks. In the calibration described above, we do not target asset inequality directly but only the dispersion of the asset to labor income ratio. The value of $\sigma_{a0} = 1.5$ implies a Gini coefficient for assets of newborn agents in the model of 0.71. This number is in line with the Gini coefficient of 0.69 for networth from the 1992 Survey of Consumer Finances in the group of households with household heads age $23 - 27$. When we set $\sigma_{a0} = 0.5999$ as in models 1 and 2, the Gini coefficient of newborn agents is 0.33. This shows that the model with permanent income shocks features unlike the model with transitory shocks a high persistence of initial asset inequality.

Finally, when we look at the effect of borrowing constraints, we find in line with the theoretical results that borrowing constraints are never binding in the case of permanent income shocks. In the case of transitory i.i.d. shocks in model 1, we find that in equilibrium 0.4 percent of agents face binding borrowing constraints. For model 2, the number decrease to less than 0.1 percent. For model 3 and model 4, we get a share of borrowing constrained agents in equilibrium of 0.9 percent respectively 1.2 percent. Although the results regarding borrowing constraints are qualitatively quite distinct, the quantitative results show that in the case of transitory i.i.d. shocks only a very small fraction of agents face binding borrowing constraints in equilibrium.

8 Conclusions

In this paper, we prove the existence of a recursive competitive equilibrium (RCE) for an Aiyagari-style economy with permanent income shocks and a perpetual youth structure. The proofs presented in the literature for the existence of an equilibrium do not apply to this economy because they require a compact state space. To prove that there exists an optimal recursive solution to the agent's problem in our economy, we present an approach based only on first-order conditions (*Euler equation*) and use lattices of consumption functions together with Tarski's fixed point theorem. This allows us to deal with the non-compact state space and an unbounded utility function. We present the approach for a general setting of Markovian income processes and show that it can be applied for a large class of consumption-saving problems. The fact that the proof is constructive serves as a theoretical foundation for the convergence of the *time iteration* algorithm that is popular in the quantitative literature.

In the second part of the paper, we prove that if an equilibrium exists where only permanent income shocks are present, then borrowing constraints must always be non-binding. This shows that the non-existence result of equilibria with non-binding borrowing constraints on compact state spaces by Krebs (2004) does not extend to the case of a non-compact state space. Importantly, this result is driven by the fact that income shocks are permanent rather than by the fact that the state space is non-compact.

From this result, we can establish the existence of a unique target insurance ratio and a non-trivial lower bound on the equilibrium interest rate. If we compare this lower bound to the interest rates in existing studies, we find that the interest rates in these studies are not compatible with the equilibrium interest rates. We show in a small quantitative example that the results in this paper have important implications for the quantitative predictions of the model and empirical tests of the theory.

Author: Moritz Kuhn, Department of Economics, University of Bonn, Adenauerallee 24-42, 53113 Bonn, Germany, email: mokuhn@uni-bonn.de, http://www.wiwi.uni-bonn.de/kuhn.

A Existence of an optimal consumption policy

Proof of theorem 1. For $r \in [f'(\bar{k}) - \delta, \beta^{-1} - 1]$ it follows from assumption 5 together with lemma 6 that a supersolution c^u and a subsolution c^l to the operator T, defined in (10), exist that form the infimum respectively supremum of the set C. Lemma 3 proves that $(C, \geq)$ is a complete lattice. Lemma 7 and lemma 10 prove that the operator T is a monotone increasing self-map on C. It follows from Tarski 1 that a recursive policy function that satisfies the first-order conditions exists and together with assumption 2 satisfies the transversality condition (section A.4). Hence, the solution is optimal. Finally, it follows from lemma 11 that T is continuous. Applying Tarski 2 shows that $\lim\limits_{n\to\infty} T^n c^u$ converges to the optimal solution. $\square$

A.1 Set of consumption functions as complete lattice

Define for all $(x, z) \in X \times Z$

$$\bar{c}(x,z) := \max\{c_1(x,z), c_2(x,z)\} \quad \underline{c}(x,z) := \min\{c_1(x,z), c_2(x,z)\}$$
$$\bar{c}^\infty(x,z) := \sup_{c\in C'}\{c(x,z)\} \qquad \underline{c}^\infty(x,z) := \inf_{c\in C'}\{c(x,z)\}$$

where c_1 and c_2 are two consumption functions from the set C defined in section 3.2 and C' denotes an arbitrary subset of C.

Lemma 1. *For every two consumption functions $c_1, c_2 \in C$, it holds that $\underline{c} = \inf\{c_1, c_2\}$ and $\bar{c} = \sup\{c_1, c_2\}$. Furthermore, it holds that $\underline{c}, \bar{c} \in C$.*

Proof. Suppose not. Suppose there is a $\hat{c}$ such that $\hat{c} \geq c_1$ and $\hat{c} \geq c_2$ but $\hat{c} < \bar{c}$. This yields immediately a contradiction because $\bar{c}(x,z) = \max\{c_1(x,z), c_2(x,z)\}$ and it holds that either $\hat{c} \ngeq c_1$ or $\hat{c} \ngeq c_2$ or $\hat{c} \leq c_1$ or $\hat{c} \leq c_2$. The argument for $\underline{c}$ is equivalent.
We have $c_1, c_2 \in C$, and therefore, it holds that $\bar{c} \in C$ because $\bar{c}$ is the piecewise continuous composition of parts of c_1 and c_2. $\square$

Lemma 2. *For every subset of consumption functions $C' \subset C$, it holds that $\underline{c}^\infty = \inf(C')$ and $\bar{c}^\infty = \sup(C')$. Furthermore, it holds that $\underline{c}^\infty, \bar{c}^\infty \in C$.*

Proof. Suppose not. Suppose there exists a $\hat{c} < \bar{c}^\infty$ such that $c \leq \hat{c}$ for all $c \in C'$. This implies that there exist $(x, z) \in X \times Z$ such that $\hat{c}(x, z) < \bar{c}^\infty(x, z)$. By definition, it holds that $\bar{c}^\infty(x, z) = \sup_{c \in C'}\{c(x, z)\}$, hence, $\hat{c}(x, z) \geq c(x, s)$ implies that $\hat{c}(x, z) \geq \sup_{c \in C'}\{c(x, z)\}$ which yields a contradiction because

$$\sup_{c \in C'}\{c(x, z)\} = \bar{c}^\infty(x, z) > \hat{c}(x, z) \geq \sup_{c \in C'}\{c(x, z)\}$$

It follows immediately from the fact that all $c \in C'$ are Lipschitz continuous that $\bar{c}^\infty(x, z)$ is also Lipschitz continuous so that $\bar{c}^\infty \in C$ holds. An equivalent argument applies for the infimum. $\square$

Remark 2. *The fact that $\bar{c}^\infty \in C$ holds follows directly from the Lipschitz property because for all (x_1, z) and (x_2, z) in $X \times Z$ with $x_1 \leq x_2$ it holds that*

$$\bar{c}^\infty(x_2, z) \leq \sup_{c \in C'}\{c(x_1, z) + x_2 - x_1\} = \bar{c}^\infty(x_1, z) + x_2 - x_1$$

and the same argument applies to the infimum.

Lemma 3. *$(C, \geq)$ is a complete lattice.*

Proof. From lemma 1 it follows that $(C, \geq)$ is a lattice, and from lemma 2 follows that it is complete. $\square$

A.2 Properties of $G_i(x, z, \lambda)$

Lemma 4. *$G_i(x, z, \lambda)$ is*

(a) increasing and continuous in x

(b) strictly decreasing and continuous in λ

Proof. We consider the two arguments of the min-operator separately

1. Suppose $G_i(x, z, \lambda) = x - \lambda$, (a) and (b) are obviously satisfied.

2. Suppose

$$G_i(x,z,\lambda) = \left(\beta(1+r)\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda)+wz'\zeta'-rD,z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} - \lambda \tag{12}$$

Since $u'(\cdot)$ is a strictly decreasing function, its inverse is strictly decreasing. By assumption, $c_i(x,z)$ is increasing and continuous in x. It follows that (12) must be increasing in x. The continuity of $c_i(x,z)$ together with the continuity of $u'(\cdot)$ and its inverse imply that (12) satisfies (a) because $c_i \geq c^l > 0$. We apply the same arguments for (b) and $\lambda \leq x$, and we get that (12) satisfies (b).

The min-operator forms the lower envelope of two continuous and increasing respectively strictly decreasing functions in x and λ. It preserves, therefore, the monotonicity and continuity of these functions. Hence, $G_i(x,z,\lambda)$ satisfies (a) and (b). □

Lemma 5. *For every (x,z), $G(x,z,\lambda)=0$ has a unique solution λ.*

Proof. It follows from the properties of $u'(\cdot)$ that for $\lambda=0$, $G(x,z,\lambda)\geq 0$ and for $\lambda \to x$, it follows from lemma 4 that $G(x,z,\lambda)$ is strictly decreasing with $G(x,z,\lambda)\leq 0$ if $\lambda = x$. Hence, the solution $G(x,z,\lambda)=0$ must be unique. □

A.3 Properties of T

Definition 1. *An operator T is called monotone increasing if for $x \geq y$ it holds that $Tx \geq Ty$.*

Definition 2. *An operator T is called continuous iff for every chain S $\sup T(S) = T(\sup(S))$.*

Other definitions and background reading can be found in Zeidler (1986).

Lemma 6. *For every r such that $\beta(1+r)\leq 1$ and $1-(\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}} > 0$ there exists a supersolution c^u and a subsolution c^l to the operator T.*

1. *For $c^u(x,z)=x$, it holds that $Tc^u \leq c^u$.*

2. *For* $c^l(x,z) = \iota x$ *with* $\iota := 1 - (\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}}$*, it holds that* $Tc^l > c^l$*.*

Proof. 1. By construction, we get that $c_1 = Tc^u \leq x$. Since $c_1(x,s) = \lambda \leq x$ where λ solves

$$\lambda = \min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c^u\left((1+r)(x-\lambda) + wz'\zeta', z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}$$

and it follows that $Tc^u \leq c^u$

2. Take $c^l(x,z) = \iota x$ and suppose that $G^l(x,z,\lambda) = 0$ for $\lambda \leq \iota x$ for some x. This implies that

$$\begin{aligned}
\iota x &\geq (\beta(1+r))^{-\frac{1}{\gamma}} \\
&\quad \left(\mathbb{E}\left[\left(c^l\left((1+r)(x-\iota x) + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
\iota x &\geq (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left(\iota\left((1+r)(1-\iota)x + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
x &> (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left((1+r)(1-\iota)x\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
1 &> (\beta(1+r))^{-\frac{1}{\gamma}} (1+r)(1-\iota) \\
(1-\iota) &> (1-\iota)
\end{aligned}$$

which yields a contradiction. Hence, it must be true that $\lambda > \iota x$ for all (x,z), and therefore, it holds that $Tc^l > c^l$.

□

Lemma 7. *The operator* T *is monotone increasing.*

Proof. Take c_i^1 and c_i^2 from C with $c_i^1 > c_i^2$. It follows from the fact that $u'(\cdot)$ and its inverse are strictly decreasing functions that

$$\begin{aligned}
&\min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i^1((1+r)(x-\lambda) + wz'\zeta' - rD, z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \geq \\
&\quad \min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i^2((1+r)(x-\lambda) + wz'\zeta' - rD, z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

From lemma 4, we know that $G_i(x,z,\lambda)$ is decreasing in λ. Since it holds that $G_i^1(x,z,\lambda) \geq G_i^2(x,z,\lambda)$, it follows that for all $(x,z) \in X \times Z$ we get $\lambda^1 \geq \lambda^2$. □

Lemma 8. *The operator T maps elements of C to continuous and increasing functions.*

Proof. Again, we proceed in two steps.

1. (*increasing*)

   (a) If $\lambda = x$, this is obvious.

   (b) If $\lambda = \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x-\lambda)+wz'\zeta'-rD,z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}$ pick $x_1 > x_2$. Lemma 4 implies that $G_i(x_1,z,\lambda) \geq G_i(x_2,z,\lambda)$ and it follows that $\lambda_1 \geq \lambda_2$ because $G_i(x,z,\lambda)$ is strictly decreasing in λ.

   From steps (1a) and (1b) it follows that $c_{i+1}(x,z)$ must be an increasing function.

2. (*continuous*) The continuity of the optimal solution follows directly from the implicit function theorem (Kumagai (1980))[41]. To see this, note that $G_i(\cdot,z,\cdot)$ is a continuous map $G_i : X \times X \subset \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. From lemma 5, we know that for all (x,z) there exists a unique solution $G_i(x,z,\lambda) = 0$, and from Kumagai (1980), it follows that $c_{i+1}(x,z)$ is continuous in a neighborhood of x if and only if there are open neighborhoods $B \subset X$ and $A \subset \mathbb{R}_+$ of x and λ, respectively, and $\forall x \in B : G_i(x,z,\cdot) : A \to \mathbb{R}$ is locally one-to-one (injective). From lemma 4, we know that $G(x,z,\lambda)$ is strictly decreasing in λ, and therefore, it is locally one-to-one. Hence, $c_{i+1}(x,z)$ will be continuous in x.

□

Lemma 9. *If $x_1 > x_2$ and $G(x_2,z,\lambda_2) = 0$ with $x_2 > \lambda_2$, then for $G(x_1,z,\lambda_1) = 0$ it holds that $x_1 > \lambda_1$.*

[41] Kumagai proves a theorem for the case of non-differentiable function.

Proof. Suppose not. It follows from lemma 4 that

$$\begin{aligned}
\lambda_1 &= x_1 \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i(wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_2-\lambda_2) + wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&= \lambda_2 \\
&< x_2
\end{aligned}$$

This yields a contradiction, and hence, it holds that if $x_1 > x_2$ and $x_2 > \lambda_2$, then also $x_1 > \lambda_1$. $\square$

Lemma 10. *The operator T is a self-map on C.*

Proof. From lemma 8, we know that T maps continuous and increasing functions to continuous and increasing functions. Consider the case where $x_1 > x_2$. We know from lemma 8 that $\lambda_1 \geq \lambda_2$. We consider now all possible combinations

I. $\lambda_1 = x_1$ and $\lambda_2 = x_2 \quad \Rightarrow \quad x_1 - x_2 = \lambda_1 - \lambda_2$.

II. $\lambda_1 < x_1$ and $\lambda_2 = x_2 \quad \Rightarrow \quad x_1 - x_2 > \lambda_1 - \lambda_2$.

III. $\lambda_1 = x_1$ and $\lambda_2 < x_2$. Not possible, see lemma 9.

IV. $\lambda_1 < x_1$ and $\lambda_2 < x_2$.

   (a) $\lambda_1 = \lambda_2 \Rightarrow x_1 - x_2 > \lambda_1 - \lambda_2$

   (b) $\lambda_1 > \lambda_2$: (Proof by contradiction) Suppose that $x_1 - x_2 < \lambda_1 - \lambda_2$. This implies $x_1 - \lambda_1 < x_2 - \lambda_2$.

$$\begin{aligned}
\lambda_1 &= \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_1-\lambda_1) + wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_2-\lambda_2) + wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&= \lambda_2
\end{aligned}$$

   but $\lambda_1 \leq \lambda_2$ yields a contradiction, because we started with the assumption that $\lambda_1 > \lambda_2$.

Hence, it must be true that $x_1 - x_2 \geq \lambda_1 - \lambda_2$ and the proof is complete. $\square$

Lemma 11. *The operator $T : C \to C$ is continuous.*

Proof. For finite chains the proof is obvious. For infinite chains, take a chain $C^S \subset C$. Define $\bar{c}^\infty = \sup(C^S)$. Denote the image set of C^S by $C^{S'} = \left\{c' \in C : c' = Tc \, \forall c \in C^S\right\}$ and $\bar{c}' = \sup(C^{S'})$. For all $(x, z) \in X \times Z$, we have $c'_i(x, z) = \lambda^*_i$ where λ^*_i solves $G_i(x, z, \lambda) = 0$. Again, $\bar{c}'$ is defined pointwise as $\bar{c}'(x, z) = \sup \lambda^* =: \bar{\lambda}^*$. Since T is monotone increasing and C^S is a chain, it holds that $\lambda^*_i \geq \lambda^*_j$ if $c_i \geq c_j$. It follows from the definition of a chain that for all $c_i, c_j \in C^S$ we either have $c_i \geq c_j$ or $c_i \leq c_j$. Now fix $(x, z, \bar{\lambda}^\infty)$ where $\bar{\lambda}^\infty = T\bar{c}^\infty(x, z)$. Put $c_i \in C^S$ in increasing order and define $\Delta_i := G_i(x, z, \bar{\lambda}^\infty)$. The $\{\Delta_i\}$ sequence is increasing and bounded because $\bar{\lambda}^\infty$ solve $G(x, z, \bar{\lambda}^\infty) = 0$ for $\bar{c}^\infty$. Since we have $\bar{c}^\infty = \sup(C^S)$, it follows from the proof of lemma 2 that for every c_i there exists a $c_{i+1} \in C^S$ such that $\bar{c}^\infty \geq c_{i+1} \geq c_i$ because otherwise $\bar{c}^\infty$ can not be the supremum of C^S. It follows that $\sup(\Delta_i) = 0$. Hence, $G_i(x, z, \bar{\lambda}^\infty) \to 0$ holds, and this implies that $\lambda^*_i \to \bar{\lambda}^\infty$ because λ^*_i solves $G_i(x, z, \lambda) = 0$ and $G_i(x, z, \cdot)$ is continuous in λ. Hence, we get $\bar{\lambda}^* = \bar{\lambda}^\infty$ for all (x, z) such that $T\bar{c}^\infty = \sup(Tc)$ holds. The equivalent argument applies to the infimum and the elements of the chain put in decreasing order. It follows that $T : C \to C$ is a continuous operator. $\square$

A.4 Transversality condition

The transversality condition reads

$$\lim_{t\to\infty} \beta^t \mathbb{E}_0 \left[c_t^{-\gamma}(1 + r)a_t\right] = 0 \tag{13}$$

We use the definition for cash-at-hand $x_t = (1 + r)a_t + wz_t\zeta_t + D$ and the result from lemma 6 that $c^*(x_t, z_t) > \iota x_t$ for all (x_t, z_t).

$$
\begin{aligned}
\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[c_t^{-\gamma}(1+r)a_t\right] &= \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\left(\frac{c_t}{x_t}\right)^{-\gamma}x_t^{-\gamma}(x_t-wz_t\zeta_t-D)\right] \\
&\le \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-\gamma}(x_t^{1-\gamma}-x_t^{-\gamma}wz_t\zeta_t-x_t^{-\gamma}D)\right] \\
&\le \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-\gamma}(x_t^{1-\gamma})\right]
\end{aligned}
$$

Consider first the case of log utility ($\gamma = 1$)

$$\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-1}x_t^0\right] = \lim_{t\to\infty}\beta^t\iota^{-1} = 0$$

For the $\gamma > 1$ case, we get

$$\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-\gamma}x_t^{1-\gamma}\right] \le \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-\gamma}(wz_t\zeta_t-rD)^{1-\gamma}\right]$$

We make the following additional assumption for the general case

Assumption 7. *If $\gamma \ge 1$, then it holds that* $\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[(wz_t\zeta_t-rD)^{1-\gamma}\right] = 0$.

From assumption 7, it follows that $\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[c_t^{-\gamma}(1+r)a_t\right] \le 0$ and for the case $D = 0$, the condition of assumption 7 simplifies to $\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[(wz_t\zeta_t)^{1-\gamma}\right] = 0$ and we get for the case of permanent income shocks the sufficient conditions $\beta\mathbb{E}\left[\varepsilon^{1-\gamma}\right] < 1$ and $\mathbb{E}[\zeta^{1-\gamma}] < M$. These conditions are satisfied by assumption 2.

Finally, consider the $\gamma < 1$ case

$$
\begin{aligned}
\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-\gamma}(x_t^{1-\gamma})\right] &\le \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\iota^{-\gamma}(1+(1-\gamma)(x_t-1))\right] \\
&\le \lim_{t\to\infty}\left(\beta^t(\iota^{-\gamma}-(1-\gamma))+\beta^t\mathbb{E}_0\left[\iota^{-\gamma}x_t\right]\right)
\end{aligned}
$$

We can determine an upper bound on $\mathbb{E}_0[x_t]$

$$\mathbb{E}_0[x_t] = \mathbb{E}_0\left[(1+r)a_t+wz_t\zeta_t+D\right] \le \mathbb{E}_0\left[(1+r)\bar{a}_t\right]+\mathbb{E}_0\left[wz_t\zeta_t\right]+D$$

where $\bar{a}_t$ is defined as follows

$$\begin{aligned}\bar{a}_1 &= (1+r)a_0 + wz_0\zeta_0 - \iota((1+r)a_0 + wz_0\zeta_0) \\ \bar{a}_2 &= ((1-\iota)(1+r))^2 a_0 + (1-\iota)^2(1+r)wz_0\zeta_0 + (1-\iota)wz_1\zeta_1 \\ &\vdots \\ \bar{a}_t &= ((1-\iota)(1+r))^t a_0 + (1-\iota)\sum_{s=0}^{t-1}((1-\iota)(1+r))^s wz_{t-1-s}\zeta_{t-1-s}\end{aligned}$$

We have $\beta(1+r) \leq 1$, and therefore, we get

$$\bar{a}_t \leq a_0 + \frac{1}{1+r}\sum_{s=0}^{t-1} wz_{t-1-s}\zeta_{t-1-s}$$

and

$$\mathbb{E}_0[x_t] \leq \mathbb{E}_0\left[\sum_{s=0}^{t} wz_{t-s}\zeta_{t-s}\right] + D + a_0(1+r) = x_0 + \mathbb{E}_0\left[\sum_{s=0}^{t-1} wz_{t-s}\right]$$

where the last equality holds because of assumption 2. For the general case we have to make an additional assumption

Assumption 8. *If $\gamma < 1$, then it holds that* $\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[\sum_{s=0}^{t-1} wz_{t-s}\right] = 0.$

For the case of permanent income shocks, the expression simplifies to

$$\lim_{t\to\infty}\beta^t wz_0\sum_{s=0}^{t-1}\left(\mathbb{E}[\varepsilon]\right)^{t-s} = 0$$

and is satisfied because of assumption 2. Hence, if for the general case assumption 7 respectively 8 holds, then there exists an upper bound for the transversality condition

$$\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0$$

Assumption 2 is sufficient for the existence of the upper bound.

To establish a lower bound, note that if $D = 0$, then the lower bound is trivially at zero. For the general case of $D > 0$ we need an additional assumption.

Assumption 9. *If $D > 0$, then it holds that* $\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[(wz_t\zeta_t - rD)^{-\gamma}\right] = 0.$

We have established an upper bound and an lower bound for the transversality condition

$$0 \leq \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0 \qquad \Longrightarrow \qquad \lim_{t\to\infty}\beta^t\mathbb{E}_0\left[c_t^{-\gamma}(1+r)a_t\right] = 0$$

and we can conclude that the transversality condition is satisfied.

B Existence of a unique stationary distribution

Let the state space for the stochastic process of labor productivity and asset holdings be S and the Borel σ-algebra on S be $\mathcal{B}(S)$. The stochastic process $\{a_t, z_t\}_{t=0}^{\infty}$ is denoted by Φ and the state in period t by $\Phi_t = \{a_t, z_t\}$.

Definition 3. *The return time probability from state Φ_0 to a set $A \in \mathcal{B}(S)$ is defined as*

$$L(\{a_0, z_0\}, A) := Prob(\Phi_t \text{ ever enters } A | \{a_0, z_0\})$$

Definition 4. *For any set $A \in \mathcal{B}(S)$, the occupation time η_A is the number of visits by Φ to A after time zero, and is given by*

$$\eta_A := \sum_{t=1}^{\infty} \mathbb{1}(\Phi_t \in A).$$

Definition 5. *We call a Markov chain φ-irreducible if there exists a measure φ on $\mathcal{B}(S)$ such that, whenever $\varphi(A) > 0$, we have $L(\{a, z\}, A) > 0$ for all $\{a, z\} \in S$.*

Definition 6. *The Markov chain is called ψ-irreducible if it is φ-irreducible for some φ and the measure ψ is a maximal irreducibility measure ($\psi \succ \varphi$).*

Definition 7. *The Markov chain Φ is called recurrent if it is ψ-irreducible and $\mathbb{E}[\eta_A] = \infty$ for every $\{a, z\} \in S$ and every $A \in \mathcal{B}(S)$ with $\psi(A) > 0$.*

Proof of theorem 2. From theorem 1 we know that an optimal policy function exists so that the stochastic process for assets and productivity states is defined. Pick any set $A_0 \in \mathcal{B}(S)$ so that $p := \phi(A_0) > 0$. A newborn agent has a positive probability to start from set A_0 and the probability to see a newborn agent is $\theta > 0$. It follows that $p\theta$ constitutes a lower bound for the process Φ to enter the set A_0, so that the return time probability $L(\{a, z\}, A_0)$ (definition 3) is strictly positive for each point $\{a, z\}$ in the support of ϕ. It follows that Φ is ϕ-irreducible (definition 5).

From proposition 4.2.2 in Meyn and Tweedie (1993) (MT) it follows that if Φ is ϕ-irreducible, it is ψ-irreducible (definition 6). From theorem 8.3.6 it follows that Φ is recurrent because Φ is ψ-irreducible, $L(\{a,z\}, A_0) = 1$, and that A_0 satisfies the conditions of the theorem follows from proposition 5.5.3 and theorem 5.2.2.

Every recurrent chain has a unique stationary measure (MT theorem 10.0.1). Every set A in the support of ϕ has a strictly positive lower bound for the return time probability, and hence, a finite expected hitting time so that the stationary measure can be normalized to be a probability measure (MT theorem 10.0.1).

The continuity follows from theorem 1 together with remark 1 in Le Van and Stachurski (2007) (LS). The optimal consumption policy is continuous in the interest rate and independent of the distribution of productivities and assets in the cross-section, assumptions 4 and 6 imply that the initial distribution ϕ varies continuously with the interest rate. Hence, assumption 1 in LS holds. Using as Lyapunov function $V(a,z) = a + (z - \mathbb{E}[z])^2 = a + (z-1)^2$ the positive probability of death (assumption 2) and the lower bound for the optimal consumption policy (lemma 6) yield boundedness, and therefore, assumption 2 in LS holds. That assumption 3 in LS holds follows from our assumption 4. This completes the proof. $\square$

Remark 3. *The proof for the existence and uniqueness of a stationary distribution does not require that initial endowments $\{a_0, z_0\}$ are uncorrelated with $\{a_t, z_t\}$. It only requires that the conditional distribution for $\{a_0, z_0\}$ has the same support as $\phi(a,z)$ and that the unconditional distribution over $\{a_0, z_0\}$ is $\phi(a,z)$. Hence, we can allow for correlation in assets and productivity levels of agents that leave and their successors.*

Remark 4. *Intuitively, the probability of death θ has at the level of the aggregate economy a stabilizing effect for the distribution as it induces some form of reversion to the mean. We exploit this to establish that the boundedness and compactness assumptions from Le Van and Stachurski (2007) are satisfied. For assets, the lower bound on the consumption share ι induces an additional effect of reversion to the mean so that together with the probability of death it keeps the asset process bounded in the sense of assumption 2. The highest sustainable capital stock keeps it compact in the sense of assumption 3.*

The mean of the productivity process is exogenously fixed, and again, the probability of death induces the reversion to the mean that bounds the variance of the process.

C Existence of a RCE

Proof of theorem 3. An optimal solution to the agent's problem exists (theorem 1). A stationary distribution exists (theorem 2). At $r = f'(\bar{k}) - \delta$ lemma 12 proves that $K^s < K^d$, for $r = \beta^{-1} - 1$ lemma 13 shows that $K^s > K^d$. From assumption 4 it follows that capital demand is continuous in the interest rate and if follows from theorem 2 that capital supply is continuous in the interest rate. It follows from the intermediate value theorem that there exists a K^* so that $K^* = K^s = K^d$. The labor market clears by construction. Hence, a recursive competitive equilibrium exists. $\square$

Lemma 12. *At $r = f'(\bar{k}) - \delta$ it holds that $K^s < K^d$.*

Proof. From assumption 5 it follows that capital supply $\bar{k}$ can only be sustained if $c^*(x, z) = 0$ for all $(x, z) \in X \times Z$. From lemma 6 it follows that $c^*(x, z) > 0$. Hence, it must be the case that $K^s < K^d$. $\square$

Lemma 13. *At $\beta(1 + r) = 1$ it holds that $K^s > K^d$.*

Proof. It follows from theorem 2 that a stationary distribution exists. Aggregate asset supply K^s is the sum of asset supply of newborn agents K^{new} and the asset holdings of agents that survived from the last period K^{old}, we get $K^s = \theta K^{new} + (1 - \theta)K^{old}$. The asset supply of the newborn generation K^{new} is determined by the initial distribution $\phi(a, z)$. The asset supply of the surviving generation K^{old} has been determined by a sequence of optimal consumption choices. We have to distinguish two cases.

(1) If borrowing constraints are binding for some agents, it follows from the first-order conditions (see Huggett and Ospina (2001)) that for $\beta(1 + r) = 1$ there is expected consumption growth in the cross-section conditional on survival

$$1 > \mathbb{E}_\mu \left[\left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma} \right] \Rightarrow \mathbb{E}_\mu \left[c^*_t \right] < \mathbb{E}_\mu \left[c^*_{t+1} \right]$$

where the μ subscript denotes the fact that the expectations are taken with respect to the stationary distribution μ.

(2) If $\beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \geq 1$, then lemma 14 applies and borrowing constraints are non-binding. The Euler equation holds as an equality, and the argument by Huggett and Ospina (2001) does not apply.

$$1 = \mathbb{E}\left[\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma}\right]$$

There is only one risk-less asset. Hence, $c_{t+1} = c_t$ is not an optimal choice for all realizations of ε_{t+1}. Hence, Jensen's inequality for strictly convex functions[42] applies, we get

$$1 = \mathbb{E}\left[\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma}\right] > \left(\mathbb{E}\left[\frac{c^*_{t+1}}{c^*_t}\right]\right)^{-\gamma} \Rightarrow 1 < \mathbb{E}\left[\frac{c^*_{t+1}}{c^*_t}\right] \Rightarrow \mathbb{E}_\mu\left[c^*_t\right] < \mathbb{E}_\mu\left[c^*_{t+1}\right]$$

and again, we get conditional on survival consumption growth in the cross-section.[43] Since expected labor income is constant, consumption growth can only be financed by accumulating assets on average. If assets grow for all surviving agents between periods, it follows that $K^{old} > K^{new}$ because the average capital of all generations at the beginning of the life has been K^{new}. As a consequence, we get $K^s > K^{new} = K^d$. □

Lemma 14. *If only permanent shocks are present, $D = 0$, and r is such that $\beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \geq 1$, then borrowing constraints are non-binding.*

Proof. The borrowing constraints are non-binding if for all (x,z) it holds that $G(x,z,x) < 0$. If only permanent income shocks are present, then the inequality always holds if $1 > \beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right]$. □

[42] Note that marginal utility is strictly convex if and only if $\frac{\partial^3 u(x)}{\partial x^3} > 0$.

[43] The same argument applies, if borrowing constraints were binding. The argument by Huggett and Ospina (2001) could therefore be replaced by this argument but to highlight the importance of prudence in the model with permanent shocks we decided to present the proof in two steps.

D Non-binding borrowing constraints

We build on the result by Carroll and Kimball (1996) that the optimal consumption function $c(\tilde{x})$ is concave.[44] Using this result, we prove that for the case when only permanent shocks are present borrowing constraints must be non-binding.

Proof of theorem 4. It follows from Lemma 6 that the optimal recursive policy c^* satisfies $c^* > c^l$. Carroll (2004) based on Carroll and Kimball (1996) proves that $\tilde{c}(\tilde{x})$ is concave. Together with $c^* > c^l$ concavity implies that ι is a lower bound for the slope of c^*. If an equilibrium exists, then K^* is such that $K^* = K^s = K^d > 0$. A positive equilibrium capital stock $K^s > 0$ requires that there exist x so that $c^*(x) < x$. Current income $ra_t + wz_t$ becomes in the reduced state space $\frac{r}{1+r}\tilde{x}_t + \frac{1}{1+r} = 1 + \frac{ra_t}{wz_t}$. Comparing $\frac{r}{1+r}$ and ι using $\beta(1+r) \leq 1$ yields

$$\iota = 1 - \frac{1}{1+r}(\beta(1+r))^{\frac{1}{\gamma}} \geq 1 - \frac{1}{1+r} = \frac{r}{1+r}$$

Since ι is a lower bound on the slope of c^*, it follows that the optimal solution has a slope larger than $\frac{r}{1+r}$. Suppose the optimal solution at $\tilde{x} = 1$ were $\tilde{c} = 1$, i.e. borrowing constraints were binding, then it follows that $\tilde{c}^*(\tilde{x}) \geq \tilde{x}$ for all $\tilde{x} > 1$. In this case, all agents run assets down to zero and stay there forever. This is not consistent with an equilibrium, hence, it must hold that $\tilde{c}^*(1) < 1$ and borrowing constraints must be non-binding in equilibrium. □

Proof of corollary 1. In equilibrium the optimal policy of the agent must such that optimal consumption is for some state smaller and for some states larger than current income. It follows directly from the continuity and concavity of the optimal policy function together with the lower bound c^l on the optimal policy that there must be a unique

[44]The result can also be used on the reduced state space as it is shown in Carroll (2004). The argument by Carroll and Kimball (1996) involves iteration on the Bellman equation but applies here as well because the sequences of consumption functions of the two approaches are equivalent. This can be easily verified because $G_i(x, z, \lambda) = 0$ is the necessary condition for updating the value function using the Bellman equation.

intersection of the optimal policy with current income. This intersection characterizes $\bar{\bar{x}}$. □

Proof of corollary 2. The upper bound follows from lemma 13. The lower bound can be derived from the fact that borrowing constraints are always non-binding. The Euler equation for the reduced state space variables and zero assets implies that if borrowing constraints are non-binding, then

$$1 < \beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \qquad \Longleftrightarrow \qquad r > \left(\beta\mathbb{E}[\varepsilon^{-\gamma}]\right)^{-1} - 1$$

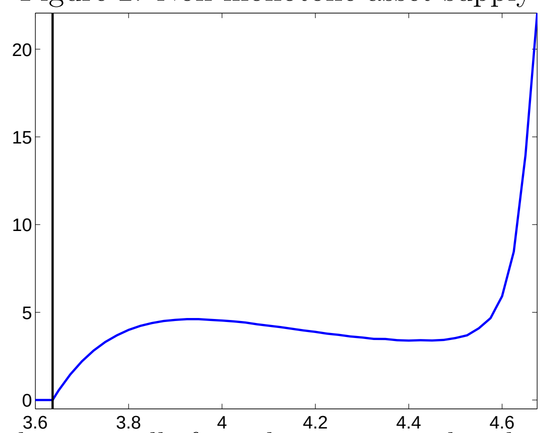
□

E Uniqueness of the equilibrium

The equilibrium existence proof (theorem 3) does not establish uniqueness of the equilibrium because monotonicity of asset supply in the interest rate is not established. In section 7, we show that in a calibrated model asset supply is increasing in the interest rate and the equilibrium is unique. We find that this monotonicity result holds for a large set of parameter combinations. However, it is possible to construct counterexamples where asset supply is no longer increasing in the interest rate. In this case, uniqueness of the equilibrium is no longer guaranteed and becomes a quantitative question.

To construct an example where asset supply is non-monotone, we change the calibration from section 7 and set the parameter α that governs the capital share in the Cobb-Douglas production function to $\alpha = 0.98$ and set the total factor productivity to match a wage rate of 1 at an interest rate of 4 percent. Since wages and interest rates are linked via the first-order condition of the firm, a high value for α yields a wage that reacts very strongly to changes in the interest rate. Figure 2 shows that for this case we get a non-monotone asset supply in interest rates over a relevant range, i.e. above the lower bound for the equilibrium interest rate (vertical black line), so that theoretically multiplicity of equilibria could arise. Quantitatively, however, asset demand exceeds asset supply by orders of magnitude so that no multiplicity of equilibria arises.[45] The problem that asset supply might be non-monotone in the interest rate and that multiplicity of equilibria can not be ruled out in general is also discussed in Aiyagari (1993)[46] and our numerical example builds on the argument from his paper. Our example shows that although a non-monotonicity in asset supply is possible, it seems to arise only under extreme parametric assumptions and no parameter combinations have been found that finally lead to multiplicity of equilibria.

Figure 2: Non-monotone asset supply



Notes: Asset supply derived numerically from the stationary distribution of the model described in the main part of the paper. The vertical black solid line shows the equilibrium lower bound for the interest rate. The interest rate in percentage points is given on the horizontal axis. The vertical axis gives capital in units of the numeraire.

To explore the mechanism that leads to the non-monotonicity, we use a simple two period example and illustrate that the interaction of changes in the wage with the precautionary savings motive can cause a non-monotonicity in asset supply. Consider the following problem of choosing optimal savings s for given initial assets a_0

$$\max_s \log((1+r)a_0+w-s)+\beta\left(p\log((1+r)s+(1-z)w)+(1-p)\log((1+r)s+(1+z)w)\right)$$

where p denotes the probability of the income event $(1-z)w$ and $1-p$ is the probability for income event $(1+z)w$, and w and r denote the wage and the interest rate. As in

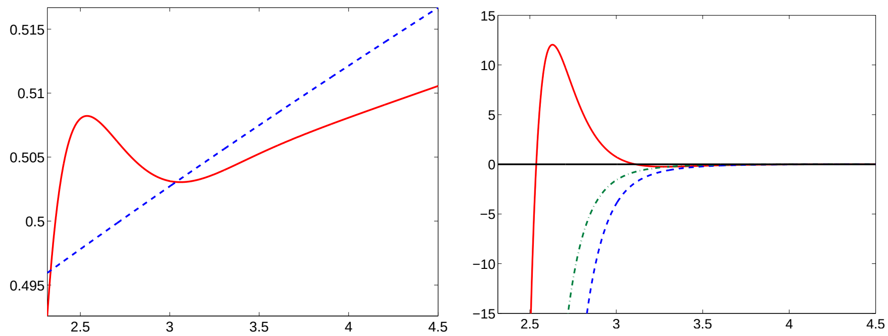
[45] We also tried a calibration with $\alpha = 0.995$. The non-monotonicity in this case becomes even stronger, but again no multiplicity of equilibria arises.

[46] We also construct examples with i.i.d. shocks and similar parameter combinations that yield a non-monotone asset supply. Results are available upon request.

section 7 and in the example above, we assume that the wage and the interest rates are linked via the first-order condition of a firm operating a Cobb-Douglas technology.[47]

Figure 3: Non-monotone savings in two period model

(a) Non-monotone savings (b) Non-monotonicity condition



Notes: Left panel: Optimal savings in the two period model. The red solid line shows the case with a strong wage reaction to interest rate changes ($\alpha = 0.98$) and the blue dashed line shows the case of modest wage reactions to interest rate changes ($\alpha = 0.33$). The interest rate in percentage points is given on the horizontal axis. The vertical axis gives optimal savings in units of the numeraire. Right panel: Non-monotonicity condition from (14). Positive values indicate decreasing savings in the interest rate. The red solid line is the condition including all terms, the blue dashed line is the case without the precautionary savings reaction, and the green dashed-dotted line is the case without wage reactions. The interest rate in percentage points is given on the horizontal axis.

In figure 3(a), we show optimal savings s as a function of the interest rate for two different specifications of the production function.[48] For a high capital share $\alpha = 0.98$ that implies a strong reaction of the wage to interest rate changes, we see that savings are a non-monotone function of the interest rate (red solid line). If we set $\alpha = 0.33$, so that wage reactions are modest, the saving function is monotonically increasing in the interest rate (blue dashed line). Intuitively, two effects counteract each other. On the one hand, the increase in the interest rate leads to a lower price for consumption in the second period, so that savings increase. On the other hand, the wage rate decreases and as a consequence income fluctuations decrease because shocks are multiplicative to the income level.[49] To

[47]For the Cobb-Doublas case $Y = AK^{\alpha}L^{1-\alpha}$, we get $w(r) = (1-\alpha)A\left(\frac{\alpha A}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$.

[48]The other parameters are $p = 0.5$, $z = 0.15$, $\beta = 0.9553$, $a_0 = 1$, and $\delta = 0.07$. In line with the calibration for the model in section 7.

[49]The multiplicative shocks are the standard assumption in this class of models because the productivity process is usually formulated in logs.

keep the analysis transparent, we restrict attention to the case $p = 0.5$, so that there is no expected income growth or decline. In this case, the condition for increasing savings[50] in the interest rate, i.e. $\frac{\partial s}{\partial r} > 0$, is

$$\begin{aligned} & -\left(\frac{(2+\beta)s}{(1+\beta)(1+r)}\frac{w}{1+r} + \frac{\beta}{1+\beta}\frac{w^2}{(1+r)^2}\left(2\frac{1-z^2}{\beta(1+r)} - 1\right)\right) \\ (14) \qquad & < \quad -\frac{\partial w}{\partial r}\left(\frac{\beta}{1+\beta}\frac{2w}{1+r}\left(\frac{1-z^2}{\beta(1+r)} - 1\right) + \frac{(2-\beta r)s - (1+r)a_0}{(1+r)(1+\beta)}\right) \end{aligned}$$

where s is the optimal amount of savings. Exploring the terms of this condition quantitatively shows that the term that generates the non-monotonicity in savings is the term on the right-hand side that can be associated with the precautionary savings reaction to wage changes

$$\frac{\beta}{1+\beta}\frac{2w}{1+r}\left(\frac{1-z^2}{\beta(1+r)} - 1\right) \tag{15}$$

If this term is negative and the wage reaction $\frac{\partial w}{\partial r}$ is sufficiently strong, then a non-monotonicity arises. To disentangle the different effects, we plot in figure 3(b) condition (14) for 3 cases. For each case, we plot the difference of the left-hand side and the right-hand side, so that the condition is satisfied if the value is negative. In the first case, we plot the condition as given (red solid line). In the second case, we shut down wage changes, i.e. we set $\frac{\partial w}{\partial r} = 0$ (green dashed-dotted line). In the third case, we shut down the precautionary savings reaction, i.e. we set the term in (15) to zero (blue dashed line). In the second and the third case, the condition is always satisfied, i.e. savings are always increasing in the interest rate. In particular, it suffices to set the term in (15), that we associate with the precautionary savings reaction, to zero (blue dashed line). In the case when the precautionary savings reaction is taken into account (red solid line) the condition is violated for some interest rates and a non-monotonicity in savings arises (see figures 3(a) and 3(b)). However, to generate this result we need a sufficiently strong

[50] The optimal amount of savings in this case is

$$s = -\frac{1}{2}Q + \sqrt{\frac{1}{4}Q^2 - R}$$

with $Q = \frac{2w}{\beta} - (1+r)a_0 - rw$ and $R = \frac{\beta}{(1+r)(1+\beta)}\left(\frac{1-z^2}{\beta(1+r)} - 1\right)w^2 - \frac{\beta}{1+\beta}a_0 w$.

reaction of the wage to interest rate changes and it remains a quantitative question how strong the reaction of the wage has to be, especially, in a more sophisticated model like the one considered in the main part of the paper.

Two remarks are in order. First, to get the non-monotonicity we need that the term in (15) is negative. Rearranging terms shows that this is equivalent to the following condition

$$\beta(1+r)(1-z^2) > 1 \qquad \Longleftrightarrow \qquad \beta(1+r)\mathbb{E}[\varepsilon^{-\gamma}] > 1$$

with $\varepsilon = \{1-z, 1+z\}$, equal probabilities ($p = 0.5$), and $\gamma = 1$. This condition is the same as the condition for the lower bound of the equilibrium interest rate from corollary 2. This suggests that the effect that generates the non-monotonicity in the two period example is also present over the relevant range of interest rates in the model in the main part of the paper. Indeed, the quantitative example shows that we generate a non-monotonicity for similar parameter choices in the two period model and in the model in the main part. Second, equation (14) suggests that also the second term on the left-hand side that comprises z^2 can generate a non-monotonicity in savings if z is sufficiently large. Although, we can generate this effect in the simple two period model, we could not generate a non-monotonicity from this effect, i.e. from a substantial increase in income risk, in the model in the main part of the paper.

To summarize, a non-monotonicity in asset supply can arise in a model with endogenous wage reactions to interest rate changes for certain parameter combinations and we argue that it is the interaction of a sufficiently strong wage reaction with the precautionary savings motive that is the likely cause for the non-monotonicity. Although such a non-monotonicity theoretically opens the possibility for multiplicity of equilibria, we have not found any parameter combination that finally lead to multiplicity of equilibria in the model in the main part of the paper.

References

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. Federal Reserve Bank of Minneapolis Working Paper, December 1993.

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659 – 684, 1994.

Richard Blundell, Luigi Pistaferri, and Ian Preston. Consumption inequality and partial insurance. *American Economic Review*, 98(5):1887–1921, 2008.

Christopher D. Carroll. Buffer-stock saving and the life cycle / permanent income hypothesis. *Quarterly Journal of Economics*, 112(1):1–55, 1997.

Christopher D. Carroll. Theoretical foundations of buffer stock saving. NBER Working Paper No. 10867, November 2004.

Christopher D. Carroll and Miles S. Kimball. On the concavity of the consumption function. *Econometrica*, 64(4):981 – 992, 1996.

Christopher D. Carroll and Andrew A. Samwick. The nature of precautionary wealth. *Journal of Monetary Economics*, 40(1):41 – 72, 1997.

II Coleman, Wilbur John. Equilibrium in a production economy with an income tax. *Econometrica*, 59(4):1091–1104, 1991.

George M. Constantinides and Darrell Duffie. Asset pricing with heterogeneous consumers. *Journal of Political Economy*, 104(2):219 – 240, 1996.

Patrick Cousot and Radhia Cousot. Constructive versions of tarski's fixed point theorems. *Pacific Journal of Mathematics*, 82(1):43–57, 1979.

Angus Deaton. Saving and liquidity constraints. *Econometrica*, 59(5):1221–1248, 1991.

Angus Deaton and Guy Laroque. On the behavior of commodity prices. *Review of Economic Studies*, 59(1):1–23, 1992.

Darrell Duffie, John Geanakoplos, Andreu Mas-Colell, and Andy McLennan. Stationary markov equilibria. *Econometrica*, 62:745 – 782, 1994.

Edward J. Green. Individual level randomness in a nonatomic population. 1994.

Jonathan Heathcote, Kjetil Storesletten, and Giovanni L. Violante. Consumption and labor supply with partial insurance: An analytical framework. NBER Working Paper No. 15257, August 2009.

Mark Huggett. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969, 1993.

Mark Huggett and Sandra Ospina. Aggregate precautionary savings: when is the third derivative irrelevant? *Journal of Monetary Economics*, 48:373 – 396, 2001.

Tullio Jappelli, Mario Padula, and Luigi Pistaferri. A direct test of the buffer stock model of saving. *Journal of the European Economic Association*, 6:1186–1210, 2008.

Greg Kaplan and Gianluca Violante. How much consumption insurance beyond self-insurance? *AEJ Macroeconomics*, 2(4):5387, 2010.

Stephen C. Kleene. *Introduction to Metamathematics*. North-Holland, Amsterdam, 1952.

Tom Krebs. Non-existence of recursive equilibria on compact state spaces when markets are incomplete. *Journal of Economic Theory*, 115:134 – 150, 2004.

Tom Krebs. Job displacement risk and the cost of business cycles. *American Economic Review*, 97(3):664 – 686, 2007.

Moritz Kuhn. Welfare analysis with permanent income shocks. Working paper, 2010, University of Mannheim, 2010.

Sadatoshi Kumagai. An implicit function theorem: Comment. *Journal of Optimization Theory and Applications*, 31(2):285–288, 1980.

Cuong Le Van and John Stachurski. Parametric continuity of stationary distributions. *Economic Theory*, 33(2):333 – 348, 2007.

Lars Ljungqvist and Thomas J. Sargent. *Recursive Macroeconomic Theory*. MIT Press, Cambridge, Massachusetts, 2000.

Costas Meghir and Luigi Pistaferri. Income variance dynamics and heterogeneity. *Econometrica*, 72(1):1–32, 2004.

Sean Meyn and Richard Tweedie. *Markov Chains and Stochastic Stability*. Springer London, 1993.

Jianjun Miao. Competitive equilibria of economies with a continuum of consumers and aggregate shocks. *Journal of Economic Theory*, 126:274 – 298, 2006.

Olivier F. Morand and Kevin L. Reffett. Existence and uniqueness of equilibrium in nonoptimal unbounded infinite horizon economies. *Journal of Monetary Economics*, 50:1351 – 1373, 2003.

Guillaume Rabault. When do borrowing constraints bind? some new results on the income fluctuation problem. *Journal of Economic Dynamics and Control*, 26:217–245, 2002.

Pontus Rendahl. Inequality constraints in recursive economies. EUI Working Paper 2006/6, 2006.

Nancy L. Stokey and Robert E. Lucas. *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, 1989.

Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.

Chris I. Telmer. Asset pricing puzzles and incomplete markets. *The Journal of Finance*, 48(5):1803 – 1832, 1993.

Eberhard Zeidler. *Nonlinear Functional Analysis and its Applications I, Fixed-Point Theorems*. Springer, Berlin Heidelberg New York, 1986.