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THE DIRECT APPROACH TO DEBT OPTION PRICING

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ABSTRACT. We review the continuous–time literature on the so–called direct approach to bond option pricing. Going back to Ball and Torous (1983), this approach models bond price processes directly (i.e. without reference to interest rates or state variable processes) and applies methods that Black and Scholes (1973) and Merton (1973) had originally developed for stock options. We describe the principal modelling problems of the direct approach and compare in detail the solutions proposed in the literature.

1. INTRODUCTION

The valuation of debt options, i.e. options written on bonds, has occupied a central place in the literature on contingent claim pricing and the term structure of interest rates. Despite the fact that commonly traded debt options are written on coupon bearing bonds, many papers propose pricing formulae for European options on zero coupon (i.e. pure discount) bonds. There is a specific reason for the amount of research done on these derivatives. Among all interest rate dependent claims, options on zero coupon bonds most closely resemble options on non–divident paying stocks for which Black and Scholes (1973) derived their famous pricing formula. Applying the techniques that had been successful with stock options, several authors were able to obtain solutions as tractable and elegant as the Black–Scholes formula. On the other hand, discount bond options gain theoretical significance from their role as building blocks for other derivatives. It is well known, for instance, that caps and floors can be decomposed into strings of options on zero coupon bonds.[1] In some circumstances, it is possible to write the price of a coupon bond option as a sum of prices of discount bond options.[2]

There are essentially two approaches to the valuation of bond options: a term structure approach and a price–based approach.We shall only consider examples where the time parameter is continuous. Within the term structure approach, analytic formulae for option prices were obtained for instance by Cox, Ingersoll and Ross (1985), Jamshidian (1989) [in the term structure model of Vasicek (1977)], Heath, Jarrow and Morton (1992) and Longstaff and Schwartz (1992). See below for papers using the price–based approach. A term structure model aims at describing the price processes of all traded discount bonds. Typically, these prices are determined as functions of one or more state variables like the short term interest rate.In Vascicek (1977) and Cox, Ingersoll and Ross (1985), for example, the short rate is the single state variable. Longstaff and Schwartz (1992) have two state variables, the short rate and its instantaneous variance. In such a framework, pricing bond options takes two steps. First, prices of discount bonds must be calculated. This step usually relies on a no–arbitrage argument that guarantees

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[1]See for instance Sandmann (1991) or Briys, Crouhy and Schöbel (1991).

[2]See Jamshidian (1989) and El Karoui and Rochet (1989).

the existence of so–called market prices of risk for the state variables. These parameters incorporate investor characteristics such as their attitudes towards risk and must be specified exogenously to close the model. Once this is done, the prices of bond options are calculated. One constructs a dynamic trading strategy in bonds that replicates the payoff of the option. According to the law of one price, the value of the duplicating portfolio and the option price must coincide.

The price–based approach, by contrast, is a one–step procedure starting straight from a model of bond prices. More precisely, the continuous–time price–based models proposed in the literature specify the price process of just two bonds: the underlying bond, i.e. the bond on which the option to be valued is written, and a reference bond, a discount bond of the same maturity as the option. These two bonds are used to construct a duplicating strategy. As this approach avoids the calculation of bond prices from state variable processes, it has also been called the direct approach.

In this paper, we review the continuous–time literature on the direct approach. The development of this literature can be outlined as follows. It starts with Ball and Torous (1983) where the stock option pricing model of Merton (1973), an extension of Black and Scholes' work that allows for stochastic interest rates, is adapted to debt options. The main contribution of Ball and Torous consists in replacing the Brownian motion which drives the Black–Scholes or Merton stock price model by a Brownian bridge process. Thus, they succeed in modelling the principal difference between stocks and bonds: under absence of default risk, bonds reach a predetermined face value at their maturity whereas stocks have no such target value. In the Ball–Torous model, the volatility[3] of bond prices is constant. As Kemna, de Munnik and Vorst (1989) note, this implies that the instantaneous variance of a bond's yield grows without bound as the maturity date approaches. Introducing bond price processes with time dependent volatility, these authors are able to keep the instantaneous variance of bond yields bounded.

However, the models of Ball and Torous (1983) and Kemna, de Munnik and Vorst (1989) both have a serious drawback: due to lognormality of bond prices, they assign positive probability to negative bond yields and negative forward yields. This problem has been addressed by Schöbel (1986). He derives boundary conditions for discount bond options under the assumption that yields do not become negative. Then he proposes a method to modify option price formulae like that of Ball and Torous (1983) in accordance with these boundary conditions. Yet Schöbel leaves the underlying bond price model unchanged; he develops no model in which yields would indeed remain non–negative. Bühler and Käsler (1989) were the first to achieve this within the direct approach.[4] With a very ingenious formulation of bond prices, their model guarantees positive bond yields as well as positive forward yields and still has the advantage of providing analytic solutions for option prices.

While the papers mentioned so far deal exclusively with discount bonds, Schaefer and Schwartz (1987) and Bühler (1988) use the direct approach to price options on coupon bearing bonds. Both papers let the volatility of the underlying bond depend on the bond's duration. In such a setting, bond option prices must be calculated numerically. Unfortunately, both papers make strong assumptions about the reference bond in order to keep the numerical complexity of the valuation problem at a reasonable level. Schaefer and Schwartz assume a constant rate of return on the reference bond. Bühler models this rate as the underlying bond's yield multiplied by a time dependent factor. It is an advantage of Bühler's model that this bond yield always remains positive.

The preceding paragraphs mentioned two of the main modelling problems encountered by the direct approach: first, the problem of specifying bond price processes that reach par value at maturity with probability one; second, the problem of modelling bond prices in a way that precludes negative yields. A

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[3] Practitioners as well as academic researchers have used the term "*volatility*" to denote various quantities that measure the riskiness of an asset. We adopt the following convention: "*volatility*" is synonymous with "*instantaneous standard deviation of returns*".

[4] As for the term structure approach, Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992) are models with positive yields.

third problem has to do with the internal consistency of models: bond price processes must be specified such that no arbitrage opportunities between the bonds arise. A sufficient condition for the absence of arbitrage opportunities is the existence of a so–called martingale measure, i.e. a new probability measure under which all asset prices, expressed in units of a numeraire asset, can simply be calculated as expected values of future prices.[5] Due to the technical complexity of this question, the existence of a martingale measure has rarely been investigated within the direct approach.[6] Cheng (1991) shows that there is no such measure for the Ball–Torous model. Reacting to Cheng's work, de Munnik (1990) proves the existence of a martingale measure for the model of Kemna, de Munnik and Vorst (1989). Bühler and Käsler (1989) provide the most elegant solution. While de Munnik's work is technically rather intricate, Bühler and Käsler are able to give a straightforward proof that their model admits a martingale measure.
The aim of our paper is to emphasize the above modelling problems and to discuss in detail the different solutions proposed in the literature. As the problems of the direct approach arise already with the modelling of zero coupon bonds, we will focus on papers where options on zero coupon bonds are studied.
The remaining sections of the paper are organized as follows. Next, we give a short introduction to the principal features of discount bonds and discount bond options. The third section reviews the general framework of the direct approach. We show that the construction of duplicating strategies on forward rather than spot markets simplifies the technique and makes the structure of pricing formulae more transparent. Section 4 and 5 discuss the lognormal models of Ball and Torous (1983) and Kemna, de Munnik and Vorst (1989), respectively. In section 6, we analyse the modified pricing formulae proposed by Schöbel (1987). The model of Bühler and Käsler (1989) is presented in section 7. Section 8 contains concluding remarks. Some proofs and technical details are given in an appendix.

2. Principal features of bond prices and debt options

A zero coupon bond pays its owner a predetermined amount of money, the face value, at a predetermined calendar date in the future, the expiration date.[7] The face value is usually normalized to one. Our notation for the time t price of a zero coupon bond which expires at $T \geq t$ is $B(t,T)$. As the price of the bond at maturity has to equal its face value, we get the following terminal value condition:

$$B(T,T) = 1 \qquad \forall T \; . \tag{1}$$

It is this condition that makes bond price modelling more intricate than stock price modelling.
We may classify bond price models according to whether they generate negative yields or not. Defining the *yield to maturity* $Y(t,T)$ and the *forward yield* $Y(t,T_1,T_2)$ as usual by

$$B(t,T) = \exp\{-(T-t)\cdot Y(t,T)\} \qquad \forall\, t < T$$

and

$$\frac{B(t,T_2)}{B(t,T_1)} = \exp\{-(T_2-T_1)\cdot Y(t,T_1,T_2)\} \qquad \forall\, t \leq T_1 < T_2 \; ,$$

we get

$$Y(t,T) < 0 \Leftrightarrow B(t,T) > 1$$

and

$$Y(t,T_1,T_2) < 0 \Leftrightarrow \frac{B(t,T_2)}{B(t,T_1)} > 1 \Leftrightarrow B(t,T_2) > B(t,T_1) \; .$$

A simple argument shows that under absence of arbitrage the condition

$$B(t,T) \leq 1 \qquad \forall t,T : t < T \tag{2}$$

[5] See Harrison and Pliska (1981) or Müller (1985). For models using the direct approach, it is convenient to choose the reference bond as numeraire asset. Asset prices expressed in units of this numeraire are just forward prices.

[6] By contrast, this question has had great influence on the term structure literature; see in particular Heath, Jarrow and Morton (1992).

[7] We consider only bonds without any default risk, e.g. treasury bills.

is equivalent to

$$B(t,T_2) \le B(t,T_1) \qquad \forall t, T_1, T_2 : t \le T_1 < T_2 \ . \tag{3}$$

Thus, an arbitrage–free bond price model generates negative yields to maturity if and only if it generates negative forward yields. In the following, a model that violates (2) and (3) will simply be said to *generate negative yields*. We cannot a priori exclude negative yields when considering a bond market without the possibility of holding cash: Some agents may wish to transfer so much of their wealth into the future that they are willing to accept negative yields. On the other hand, rational agents who prefer more to less and are able to hold cash will never engage in a (forward) loan contract with negative yield.[8] So a "realistic" bond model should fulfil (2) and (3).

A *European call* option on a zero coupon bond expiring at T is the right to buy the bond at some specified date $\tau < T$ for some fixed amount K. If the price of the bond at the exercise date $\tau < T$ is higher than the exercise price K the net cashflow of the call will be the difference $B(\tau,T) - K$, otherwise the net cashflow is zero. Therefore, at the exercise date τ, the call is worth

$$[B(\tau,T) - K]^+ := \max\{0, B(\tau,T) - K\} \ .$$

A *European put* option is the right to sell a bond for some fixed amount K. The net cashflow of this option can be written as

$$[K - B(\tau,T)]^+ := \max\{0, K - B(\tau,T)\} \ .$$

The so–called *put–call parity*[9] describes the relation between today's prices of *European* call and put options:

$$\mathrm{Put}[t, B(t,T), B(t,\tau), \tau, K] = \mathrm{Call}[t, B(t,T), B(t,\tau), \tau, K] - B(t,T) + K \cdot B(t,\tau) \ .$$

As the net cashflow of an option is always non–negative, put–call parity gives us lower bounds for option prices, i.e.

$$\mathrm{Call}[t, B(t,T), B(t,\tau), \tau, K] \quad \ge \quad \max\{0, B(t,T) - K \cdot B(t,\tau)\} \ . \tag{4}$$

An upper bound[10] for the price of a call is the price of the underlying security itself, so

$$\mathrm{Call}[t, B(t,T), B(t,\tau), \tau, K] \le B(t,T) \ . \tag{5}$$

An additional upper bound holds when there are no negative yields. In this case interesting exercise prices K lie between 0 and 1 and the maximal payoff of a call is $1 - K$. The call price is therefore bounded from above by the present value of $1 - K$:

$$\mathrm{Call}[t, B(t,T), B(t,\tau), \tau, K] \le B(t,\tau) \cdot (1 - K) \ . \tag{6}$$

This was first observed by Schöbel (1987). Furthermore, combining (4) and (6), Schöbel obtains the following condition for time t call prices whenever $B(t,T) = B(t,\tau)$:

$$\mathrm{Call}[t, B(t,T) = B(t,\tau), B(t,\tau), \tau, K] = B(t,\tau) \cdot (1 - K) \ . \tag{7}$$

Similar results for put options may be derived using put–call parity.

So far no assumptions have been made on the stochastic behaviour of bond prices. But it is already obvious that the price of an option will not only depend on its underlying bond, but also on the price $B(t,\tau)$ of a zero coupon bond with exactly the same maturity τ as the option.[11] The bond price based approach to option pricing studies models in which these two bonds are indeed all that is needed to determine the option price. This will be discussed in the following section.

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[8] Remember that we are dealing with *nominal* securities and therefore *nominal* yields.

[9] See for example Stoll (1968).

[10] Boundary conditions (4) and (5) were derived in Merton (1973). For condition (5) see also Gleit (1978).

[11] This bond will be called the *reference bond* and its price denoted by $R(t)$ instead of $B(t,\tau)$.

3. Option pricing by portfolio duplication

We repeat in this section the standard portfolio duplication argument which is the basis of derivative asset pricing. We shall only treat the special case of a European call option written on a zero coupon bond with face value 1 and maturity T. The option is assumed to have exercise date $\tau < T$ and strike price K. For $0 \leq t \leq T$, $B(t)$ denotes the time t price of the bond on which the option is written (the "underlying bond"). The price of the reference bond at time $t \in [0, \tau]$ is denoted by $R(t)$.

Suppose today's bond prices, $B(0)$ and $R(0)$, are known. In order to evaluate the option, we have to specify the uncertainty governing future price changes, i.e. the nature of the stochastic processes $\{B(t)\}_{t\in[0,T]}$ and $\{R(t)\}_{t\in[0,\tau]}$. For the moment, we only assume that these processes are continuous Itô processes.[12]

Now, the main idea is to construct a dynamically adjusted portfolio in the two bonds that yields the same cashflow as the option. To make this more precise, we need some definitions:[13]

A *portfolio strategy* is represented by a two–dimensional predictable stochastic process $\phi = (\phi^1, \phi^2)$ on the time interval $[0, \tau]$ such that the stochastic integrals $\int \phi^1 dB$ and $\int \phi^2 dR$ exist. Think of $\phi^1(t)$ and $\phi^2(t)$ as the number of underlying resp. reference bonds held at time t. Predictability means that the decision how many bonds to hold at t is based only on information available before t. The stochastic integrals above can be interpreted as the gains or losses from bond trade according to the strategy ϕ.

The *value process* of a strategy ϕ is given by

$$V_\phi := \phi^1 B + \phi^2 R \ .$$

A strategy ϕ is called *self–financing* if V_ϕ has the stochastic differential

$$dV_\phi = \phi^1 dB + \phi^2 dR \ .$$

This means that after the initial investment $V_\phi(0)$ is made, the adjustment of the portfolio is financed without injecting or taking out any money. Changes in the portfolio value are exclusively due to gains or losses from bond trade.

We say a self–financing strategy *generates the option* if the terminal portfolio value equals the cashflow of the option, i.e.

$$V_\phi(\tau) = [B(\tau) - K]^+ \ ,$$

and $V_\phi(t)$ respects at any time the lower and upper bounds mentioned in section 2, i.e. for all $t \leq \tau$

$$[B(t) - K \cdot R(t)]^+ \leq V_\phi(t) \leq B(t) \tag{8}$$

resp.

$$[B(t) - K \cdot R(t)]^+ \leq V_\phi(t) \leq \min\{B(t), R(t)(1-K)\} \tag{9}$$

if the bond price model precludes negative yields. Then, if there are to be no arbitrage opportunities,[14] the option price must indeed coincide with the portfolio value, i.e.

$$C(t) = V_\phi(t)$$

for all $t \in [0, \tau]$. $V_\phi(t)$ is called the *arbitrage price* of the option.

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[12] An introduction to the theory of such processes and their use in finance models can be found in Duffie (1992). For the sake of simplicity, technical requirements such as integrability conditions will not be made explicit here.

[13] We shall not define a space of admissible portfolio strategies as in Harrison–Pliska (1981) or Müller (1985). But the strategies we shall deal with can be checked to have the relevant properties.

[14] Formally, an arbitrage opportunity can be defined as an admissible self–financing portfolio strategy with negative initial investment, but non–negative final value; see for instance Duffie (1992). Thus, an arbitrage opportunity is a trading strategy that provides a gain today without creating future liabilities.

The construction of a generating strategy can be simplified in the following way. Instead of the two–dimensional bond price process (B, R), we consider the normalized process $(\hat{B}, 1)$ where

$$\hat{B} := \frac{B}{R} \ .$$

We assume that R^{-1} is also an Itô process. In the same way, we set

$$\hat{V}_\phi := \frac{V_\phi}{R} = \phi^1 \hat{B} + \phi^2$$

for the value process of a strategy ϕ. This may look as a purely formal definition, but there is an interesting interpretation. The (B, R)–model is a model of the spot markets, so a portfolio strategy ϕ requires continuous spot trading, i.e. continuous adjustments of long and short positions on the spot markets for the two bonds, and V_ϕ is the spot value process of the strategy. Suppose now that there exist forward markets at the same time. Then $\hat{B}(t)$ is just the time t forward price of the underlying bond for delivery at τ (obviously, the corresponding forward price of the reference bond is always 1). If we now implement our strategy ϕ on the forward markets, the resulting forward value process is just $\hat{V}_\phi$. We can define properties of a strategy ϕ in terms of forward markets: We call ϕ *self–financing on the forward markets* if

$$d\hat{V}_\phi = \phi^1 d\hat{B} \ ,$$

and we say such a self–financing strategy ϕ *generates the option on the forward markets* if

$$\hat{V}_\phi(\tau) = [\hat{B}(\tau) - K]^+$$

and $\hat{V}_\phi(t)$ respects the bounds resulting from division of (8) resp. (9) by $R(t)$. Now a generating strategy ϕ determines the arbitrage forward price of the option:

$$\hat{C}(t) = \hat{V}_\phi(t) \ .$$

The following lemma says that we are free to choose the market we want to work in.

Lemma 1: *A portfolio strategy is self–financing on the spot markets if and only if it is self–financing on the forward markets. Furthermore, a strategy generates the option on the spot markets if and only if it generates the option on the forward markets.*

The proof of the first part consists essentially of an application of Itô's formula and is given in Müller (1985) for a more general framework. The second part then follows trivially.

For the construction of a generating strategy in the $(\hat{B}, 1)$–model of the forward markets, we need an explicit description of the forward price process $\{\hat{B}(t)\}_{t \in [0,\tau]}$. We assume that this continuous Itô process can be described by

$$d\hat{B}(t) = \alpha(t) \cdot \hat{B}(t)dt + v\Big(\hat{B}(t), t\Big) \cdot \hat{B}(t)dW(t) \tag{10}$$

where α is some stochastic process, $v(x, t)$ is a continuous function and W denotes a standard Wiener process. We call α the *drift rate process* and v the *volatility function* of the forward bond price, interpreting them as instantaneous expectation and standard deviation, respectively, of the infinitesimal rate of return $\frac{d\hat{B}}{\hat{B}}$. Thus, (10) restricts the volatility of the forward bond to be a deterministic function of the current forward bond price and time. This restriction, which rules out more complicated dependence of the forward bond volatility on current or past bond prices $B(t)$ and $R(t)$, will enable us to determine the arbitrage price of the option.[15]

Our second lemma shows how to construct generating strategies. Here, the interval I is the state space of the forward price process $\hat{B}$; $\overline{I}$ is its closure. We assume that either $I =]0, \infty[$ or $I =]0, 1[$.[16] In view of Lemma 1, we do not specify the market where we use the strategy.

[15] See Jamshidian (1990) for a formulation of the same result in a term structure model.

[16] This covers all the models we shall deal with except Schöbel (1987).

Lemma 2: *Let $u(x,t)$ be continuous on $\overline{I} \times [0,\tau]$ and a solution of the partial differential equation*

$$u_t(x,t) + \frac{1}{2}v^2(x,t)x^2u_{xx}(x,t) = 0 \tag{11}$$

on $I \times [0,\tau[$. Then, the strategy ϕ defined by

$$\phi^1(t) = u_x\Big(\hat{B}(t),t\Big) \ ; \quad \phi^2(t) = u\Big(\hat{B}(t),t\Big) - u_x\Big(\hat{B}(t),t\Big) \cdot \hat{B}(t) \tag{12}$$

is self–financing. Moreover, suppose u has the terminal value $u(x,\tau) = [x-K]^+$ and satisfies

$$[x-K]^+ \leq u(x,t) \leq x \qquad \text{if } I =]0,\infty[$$

or

$$[x-K]^+ \leq u(x,t) \leq \min\{x, 1-K\} \qquad \text{if } I =]0,1[\ .$$

Then ϕ generates the call option.

This can be seen as follows. (12) implies $\hat{V}_\phi(t) = u\Big(\hat{B}(t),t\Big)$. By Itô's formula and (10),

$$d\hat{V}_\phi(t) = \left[u_t\Big(\hat{B}(t),t\Big) + \frac{1}{2}v^2\Big(\hat{B}(t),t\Big) \cdot \hat{B}(t)^2 \cdot u_{xx}\Big(\hat{B}(t),t\Big)\right]dt + u_x\Big(\hat{B}(t),t\Big)d\hat{B}(t).$$

By (11) and (12), this reduces to $d\hat{V}_\phi(t) = \phi^1 d\hat{B}(t)$, so ϕ is self–financing. The rest is easy to check.

A generating strategy as in lemma 2 yields the arbitrage forward price

$$\hat{C}(t) = u\Big(\hat{B}(t),t\Big)$$

and the arbitrage spot price

$$C(t) = R(t) \cdot \hat{C}(t) = R(t) \cdot u\Big(\hat{B}(t),t\Big)$$

for the European call. In accordance with Merton's theory of rational option pricing (1973), the spot price is homogeneous of degree one in the price B of the underlying security and the discount factor R. Furthermore, since only the volatility function v appears in the partial differential equation (11), the drift term $\alpha(t)\hat{B}(t)$ in (10) does not enter the functional relationship between the arbitrage price of the option and the bond prices B and R.

However, it would be wrong to conclude that the drift is irrelevant for option pricing. When deriving the above option price, we simply *postulated* that there are no arbitrage opportunities between traded securities. The drift of the forward bond price emerges as an important factor when we start to look for conditions that *guarantee* the internal consistency of the bond price model (B,R).[17] A sufficient condition for the absence of arbitrage opportunities is the existence of a so–called martingale measure for the forward bond price. This is a new probability measure that has the same zero probability events as the original measure and makes the forward price a martingale, which means that at any time the current forward price is the best estimate of future forward prices. Under such a measure, the forward value processes of self–financing portfolio strategies are martingales as well. In particular, the initial investment required by a self–financing portfolio strategy equals the expectation of the strategy's terminal value under the martingale measure. As taking expectations preserves non–negativity, a trading strategy with non–negative final value must have a non–negative initial investment. In other words, if there exists a martingale measure, arbitrage opportunities are precluded.

In the setting described by equation (10), a martingale measure exists if and only if the quotient of the drift and the volatility of $\hat{B}$,

$$\frac{\alpha(t)}{v(\hat{B}(t),t)}$$

[17] In the following, we only try to convey the main ideas. For a thorough discussion including technical details, see for instance Müller (1985) or Duffie (1992).

satisfies certain integrability conditions.[18] Thus, the internal consistency of a bond price model depends indeed on both the drift and the volatility of the forward bond price.

There is a second important reason why the drift term matters in option pricing. When applying an option pricing model, we need estimates for the volatility parameters which enter the valuation formula. It is in general impossible to estimate these parameters from historical price data without taking into account the drift as well.[19]

Let us conclude this section with an example of how the above lemmas are applied. Consider bond price processes that have the stochastic differentials

$$\begin{aligned} dB(t) &= \alpha_B(t) \cdot B(t)dt + \beta_B(t) \cdot B(t)dW_B(t) \\ dR(t) &= \alpha_R(t) \cdot R(t)dt + \beta_R(t) \cdot R(t)dW_R(t) \end{aligned} \tag{13}$$

with stochastic drift rate processes α_B resp. α_R but with volatility functions β_B and β_R depending only on time t.[20] W_B and W_R are assumed to be Wiener processes having infinitesimal correlation

$$dW_B(t)dW_R(t) = \rho dt$$

with constant $\rho \in [-1, 1]$.[21] After applying Itô's formula to calculate $d\hat{B}(t)$, it is easy to verify that there exists a Brownian motion W such that (10) holds with volatility function $v : [0, \tau] \to \mathbb{R}_+$ given by

$$v(t)^2 = \beta_B(t)^2 - 2 \cdot \rho \cdot \beta_B(t) \cdot \beta_R(t) + \beta_R(t)^2 \ .$$

In fact, W can be defined by

$$dW(t) = \frac{\beta_B(t)}{v(t)} dW_B(t) - \frac{\beta_R(t)}{v(t)} dW_R(t) \ .$$

The state space is $I =]0, \infty[$. The unique solution of (11) satisfying the terminal value condition and the bounds specified in lemma 2 is well known:[22]

$$u(x,t) = x \cdot N\left(\frac{1}{\sqrt{s}}\left(\ln\frac{x}{K} + \frac{s}{2}\right)\right) - K \cdot N\left(\frac{1}{\sqrt{s}}\left(\ln\frac{x}{K} - \frac{s}{2}\right)\right)$$

where N denotes the standard normal distribution function and

$$s = s(t) = \int_t^\tau v(\theta)^2 d\theta \ .$$

This yields the familiar formula

$$C(t) = B(t) \cdot N(d_1) - K \cdot R(t) \cdot N(d_2)$$

with

$$d_{1/2} = \frac{1}{\sqrt{s}}\left(\ln\frac{B(t)}{KR(t)} \pm \frac{s}{2}\right)$$

for the arbitrage spot price of a call. It is easy to verify that the generating strategy for the option is

$$\phi^1 = N(d_1) \quad ; \quad \phi^2 = -K \cdot N(d_2).$$

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[18] See Harrison and Kreps (1979), Harrison and Pliska (1983) and Müller (1985).

[19] A treatment of this estimation problem is beyond the scope of this paper. We therefore refer the reader to Lo (1986,1988) and references given there. De Munnik (1992) applies Lo's methodology to the model of Kemna, de Munnik and Vorst (1989). Practitioners often use an "implied volatility approach" to avoid the estimation problem altogether; inverting the option price formula, they calculate volatility parameters from observed option prices.

[20] This is the framework common to Ball and Torous (1983) and Kemna, de Munnik and Vorst (1989). The models that Black and Scholes (1973) and Merton (1973) used for stock option pricing can also be seen as special cases of (13).

[21] The correlation coefficient ρ could of course be made time dependent as well.

[22] The growth condition $0 \le u(x,t) \le x$ guarantees uniqueness of the solution; see Gleit (1978).

4. Constant volatility: The Brownian bridge

The first approach to price call and put options on zero coupon bonds is due to Ball and Torous (1983). The starting point of their analysis is the following observation: The Black–Scholes (1973) model of stock price movements, a geometric Brownian motion

$$S(t) = S(0) \cdot \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) \cdot t + \sigma \cdot W(t) \right\} , \qquad \mu, \sigma \ \text{ constant} , \tag{14}$$

cannot be reinterpreted as a model for bond prices since this process specification is incompatible with the face value condition (1). In fact, the variance of the process is strictly increasing with time: $V[S(t)] = S(0)^2 \cdot \exp\{2\mu \cdot t\}(\exp\{\sigma^2 \cdot t\} - 1)$.
Ball and Torous for the first time incorporated the important face value condition. They model the zero coupon bond price process $\{B(t)\}_{t\in[0,T]}$ with maturity T and face value 1 by

$$B(t) = B(0) \cdot \exp\{\mu_B \cdot t + \sigma_B \cdot \eta(t,T)\} \tag{15}$$

where $\{\eta(t,T)\}_{t\in[0,T]}$ is a standard Brownian bridge,[23] i.e. a continuous Gaussian process with

$$\begin{aligned} \eta(0,T) &= \eta(T,T) = 0 \qquad \text{a.s,} \\ E[\eta(t,T)] &= 0 \qquad \forall t , \\ E[\eta(s,T)\eta(t,T)] &= \frac{s(T-t)}{T} \qquad \forall s < t . \end{aligned}$$

In particular, the variance of the normally distributed random variable $\eta(t,T)$ is $\frac{t(T-t)}{T}$ which increases on $[0, \frac{T}{2}]$ and decreases on $[\frac{T}{2}, T]$. This bridge process can be constructed as the solution of the stochastic differential equation

$$d\eta(t,T) = \frac{-\eta(t,T)}{T-t} dt + dW(t)$$

where W is a Brownian motion. Note how the drift pulls the process back to zero. The pull–back force, $-\frac{1}{T-t}$, becomes stronger as time goes by and eventually pulls the process towards its fixed endpoint. The parameter μ_B in (15) is now adjusted to fulfil the terminal value condition:

$$1 = B(T) = B(0) \cdot \exp\{\mu_B \cdot T + \sigma_B \cdot \eta(T,T)\} \quad \Rightarrow \quad \mu_B = -\frac{\ln B(0)}{T} .$$

μ_B is just the yield to maturity of the bond at the initial time $t = 0$. With (15) the bond price process consists of two parts: the price path that would occur if there were no uncertainty,

$$B(0) \cdot \exp\{\mu_B t\} = B(0)^{\frac{T-t}{T}}$$

and a stochastic drift term driven by $\eta(t,T)$ that characterises the random fluctuations around this path. As the distribution of $\eta(t,T)$ is symmetric around 0, the deterministic path describes the time t median of the bond price distribution. $B(t)$ is lognormally distributed with time dependent mean and variance:

$$E[B(t)] = B(0) \cdot \exp\left\{ \mu_B \cdot t + \frac{1}{2}\sigma_B^2 \cdot \frac{t(T-t)}{T} \right\} \xrightarrow{t \to T} 1$$

$$V[B(t)] = B(0)^2 \cdot \exp\left\{ 2\mu_B \cdot t + \sigma_B^2 \cdot \frac{t(T-t)}{T} \right\} \cdot \left(\exp\left\{ \sigma_B^2 \cdot \frac{t(T-t)}{T} \right\} - 1 \right) \xrightarrow{t \to T} 0 .$$

Lognormality obviously implies that at any time $0 < t < T$, the price of the zero coupon bond has a positive probability of exceeding its face value. Thus, negative yields to maturity are generated. But the Ball–Torous model satisfies the important face value condition. In contrast to the Black–Scholes model, the density function of the price of the underlying security degenerates at maturity ($t = T$). Figures 1a and 1b show this difference very clearly. The geometric Brownian motion implies an increasing variance with respect to t; i.e. the density becomes flatter. By contrast, the Brownian bridge process implies a first increasing and then decreasing variance for the price process of a zero coupon bond.

[23] For details on the Brownian bridge process see for example Karlin, Taylor (1981).

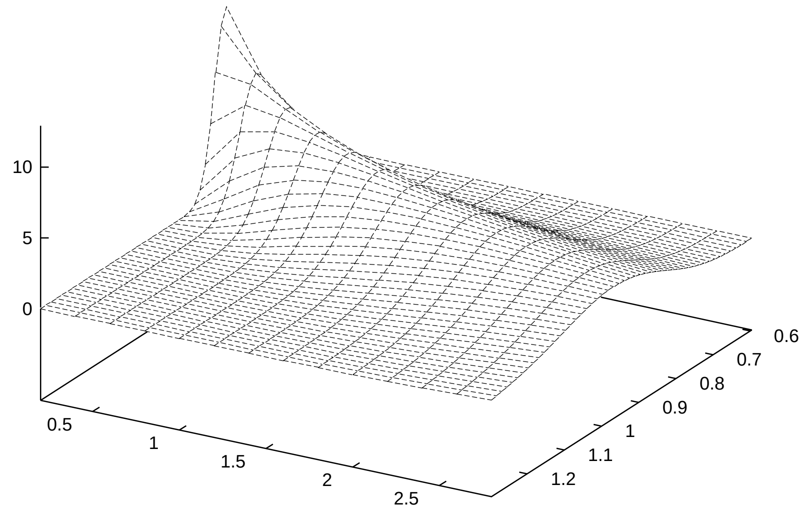

Figure 1a: Density functions for geometric Brownian motion price process $S(t)$ with $T = 3,\ \sigma = 0.15$, $S(0) = 0.785$ and $\mu = -\frac{\ln S(0)}{T}$.

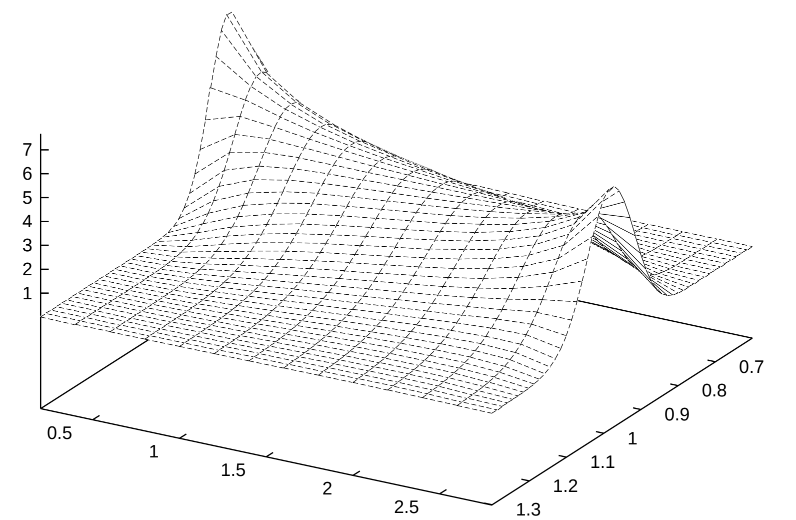

Figure 1b: Density functions for a zero coupon bond price process $B(t)$ as implied by the Ball–Torous model with $T = 3,\ \sigma_B = 0.15$ and $B(0) = 0.785$.

For option pricing, a reference bond with maturity τ equal to the exercise date of the option is needed. Ball and Torous suppose that the price process of the reference bond is of type (15) as well. This leads to the following model:

$$\begin{aligned} B(t) &= B(0)\exp\{\mu_B \cdot t + \sigma_B \cdot \eta(t,T)\} = B(0)^{\frac{T-t}{T}} \cdot \exp\{\sigma_B \cdot \eta(t,T)\} \\ R(t) &= R(0)\exp\{\mu_R \cdot t + \sigma_R \cdot \eta(t,\tau)\} = R(0)^{\frac{\tau-t}{\tau}} \cdot \exp\{\sigma_R \cdot \eta(t,\tau)\} \end{aligned}$$

with

$$\begin{aligned} d\eta(t,T) &= \frac{-\eta(t,T)}{T-t}dt + dW_B(t) \\ d\eta(t,\tau) &= \frac{-\eta(t,\tau)}{\tau-t}dt + dW_R(t) \end{aligned}$$

The instantaneous correlation coefficient between the Brownian motions W_B and W_R is assumed to be constant, i.e.

$$dW_B(t)dW_R(t) = \rho dt \ .$$

The forward price $\hat{B}(t) = \frac{B(t)}{R(t)}$ is given by

$$\hat{B}(t) = \hat{B}(0)\exp\{(\mu_B - \mu_R)t + \sigma_B\eta(t,T) - \sigma_R\eta(t,\tau)\} \ .$$

Being the quotient of two lognormally distributed variables, it is itself lognormal. Therefore, negative forward yields have a positive probability at any time $t \in]0,\tau]$.

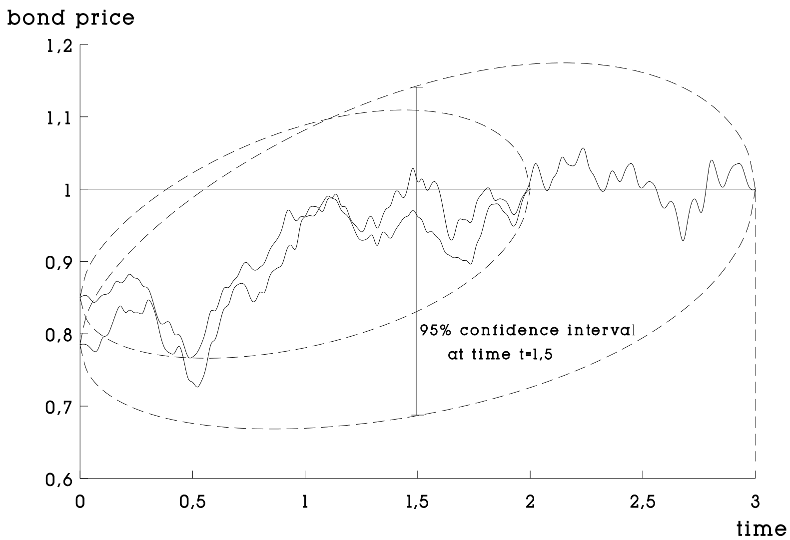

Figure 2: Sample paths of Ball-Torous bond price processes $R(t)$ and $B(t)$ for $B(0) = 0.785$, $R(0) = 0.85$, $\tau = 2$, $T = 3$, $\sigma_R = 0.12$, $\sigma_B = 0.15$ and $\rho = 0.75$ with unconditional 95% band.

To illustrate this, consider a symmetric $1-\alpha$ band for the Brownian bridge process $\eta(t,\tau)$. Since $\eta(t,\tau)$ is normally distributed with variance $\frac{t(\tau-t)}{\tau}$ the frontiers of the $1-\alpha$ band are given by $\pm\mu_{1-\alpha/2}\sqrt{\frac{t(\tau-t)}{\tau}}$ for $t \in]0,\tau[$; i.e. with probability $1-\alpha$ the realisation of $\eta(t,\tau)$ at time t is contained in this interval.[24] Using the relationship between $R(t)$ and $\eta(t,\tau)$, we obtain an unconditional $1-\alpha$ band for the price of the reference bond:

$$a_{1/2}(t) = R(0)\exp\left\{\mu_R \cdot t \mp \mu_{1-\alpha/2} \cdot \sigma_R \cdot \sqrt{\frac{(\tau-t)\cdot t}{\tau}}\right\} \tag{16}$$

That is, $\text{prob}[a_1(t) < R(t) \leq a_2(t)] = 1-\alpha$ for all $t \in]0,\tau[$. The unconditional $1-\alpha$ band for $B(t)$ can be calculated in the same way. An example of bond price paths together with unconditional 95% bands is shown in Figure 2. The paths go above 1, generating negative yields to maturity, and they cross, generating negative forward yields. The unconditional 95% bands reach also above 1. The same idea can be used to calculate a conditional $1-\alpha$ price band for the underlying bond $B(t)$ conditioned on the price of the reference bond $R(t)$:

$$\begin{aligned} b_{1/2}(t) = \hat{B}(0)\cdot R(t)\exp\Bigg\{&\mu_B\cdot t + \rho\frac{\sigma_B}{\sigma_R}\cdot\sqrt{\frac{(T-t)\tau}{(\tau-t)T}}\cdot\Big(\ln R(t) - \mu_R t\Big) \\ &\mp \mu_{1-\alpha/2}\cdot\sigma_B\cdot\sqrt{\frac{t(T-t)}{T}\ (1-\rho^2)}\Bigg\} \end{aligned} \tag{17}$$

[24] $\mu_{1-\alpha/2}$ is the $1-\alpha/2$ fractile of the standard normal distribution.

Thus, $\text{prob}[b_1(t) < B(t) \le b_2(t) \mid R(t)] = 1 - \alpha$ for all $t \in]0, \tau[$.
Returning to option pricing, we calculate the stochastic differentials of the bond price processes. By Itô's Lemma,

$$
\begin{aligned}
dB(t) &= \left(\frac{\sigma_B^2}{2} - \frac{\ln B(t)}{T-t}\right) B(t)dt + \sigma_B B(t) dW_B(t) \\
dR(t) &= \left(\frac{\sigma_R^2}{2} - \frac{\ln R(t)}{\tau-t}\right) R(t)dt + \sigma_R R(t) dW_R(t) \ .
\end{aligned}
$$

This is an example of the general specification (13). We can apply the results of section 3 with a constant volatility function for the forward price,

$$v(t) = \sqrt{\sigma_B^2 - 2\rho\sigma_B\sigma_R + \sigma_R^2} \quad \forall t \in [0, \tau] \ ,$$

and obtain the arbitrage price of a European call in the Ball–Torous setting:

$$\text{Call}[t, B(t), R(t), \tau, K] = B(t) \cdot N(d_1) - K \cdot R(t) \cdot N(d_2) \tag{18}$$

where

$$d_{1/2} = \frac{1}{\sqrt{s(t)}} \left(\ln \frac{B(t)}{KR(t)} \pm \frac{s(t)}{2}\right)$$

with

$$s(t) = \int_t^\tau (\sigma_B^2 - 2\rho\sigma_R\sigma_B + \sigma_R^2) d\theta = (\sigma_B^2 - 2\rho\sigma_R\sigma_B + \sigma_R^2)(\tau - t) \ .$$

The arbitrage price of the European put option is determined by put–call parity:

$$\text{Put}[t, B(t), R(t), \tau, K] = -B(t) \cdot N(-d_1) + K \cdot R(t) \cdot N(-d_2). \tag{19}$$

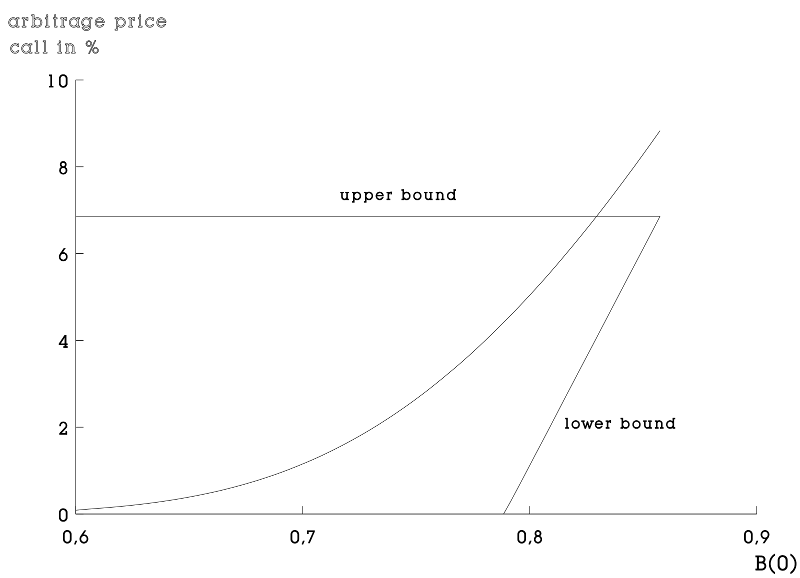

Figure 3: Arbitrage price of a European call option as given by the Ball–Torous model for different bond prices $B(0)$. $T = 3$, $\tau = 2$, $K = 0.92$, $R(0) = (1.08)^{-2}$, $\sigma_B = 0.15$, $\sigma_R = 0.12$, $\rho = 0.75$, "lower bound" refers to (4), "upper bound" to (6).

These closed form solutions for European bond options have exactly the same structure as the well–known Black–Scholes (1973) formulae. This may be surprising at first sight: after all, within the common framework of (13), the above bond price model differs considerably from the stock price model of Black and Scholes. Yet we saw that only the volatility of the forward price of the underlying asset enters the option price formula. As both models assume that this volatility is constant, the similarity of the resulting pricing relationships is easily explained. Formally, the Black–Scholes call price formula is obtained from

(18) by setting $\sigma_R = 0$, i.e. by assuming the reference bond to have a constant yield, and by replacing $B(t)$ with the stock price.

Despite their formal similarity, the option price formulae derived in the Ball–Torous model and those calculated in the Black–Scholes model have a fundamentally different theoretical status. While the latter model possesses a martingale measure[25] and hence satisfies sufficient conditions for the absence of arbitrage opportunities, the Ball–Torous model admits no martingale measure. Cheng (1991) shows that the drift term of the Brownian bridge which forces the process towards a fixed endpoint is incompatible with the requirements for the existence of a martingale measure. However, this does not necessarily imply that there are arbitrage opportunities in the Ball–Torous model: the existence of a martingale measure is sufficient, but in general not necessary for the absence of arbitrage opportunities.[26] To stress the difference between the Black–Scholes and the Ball–Torous model, we might say that pricing in the former model proceeds safely from sufficient conditions for no arbitrage, whereas pricing in the latter model is merely based on necessary conditions for no arbitrage: all we have shown is that if the Ball–Torous model is arbitrage–free, option prices must be given by equations (18) and (19).

On a less theoretical level, one can criticise the Ball–Torous bond price model for the unrealistic yield behaviour that it implies. This problem, together with a possible solution, will be addressed in the following section.

5. Time dependent volatility

Using a Brownian bridge, Ball and Torous succeed in specifying a bond price process that satisfies the terminal value condition, i.e. that reaches par value at maturity. It is instructive to examine the resulting yield process. (15) implies

$$Y(t,T) = -\frac{1}{T-t} \cdot \ln B(t) \;=\; \mu_B \;-\; \frac{\sigma_B}{T-t} \cdot \eta(t,T)\ . \tag{20}$$

This yield to maturity is normally distributed with mean μ_B and variance

$$V[Y(t,T)] = \frac{\sigma_B^2}{(T-t)^2} \cdot V[\eta(t,T)] = \frac{\sigma_B^2}{(T-t)^2} \cdot \frac{t(T-t)}{T} = \frac{\sigma_B^2 t}{(T-t)T} \tag{21}$$

which increases without bounds as t tends to T. We can analyse this further by looking at yield changes over infinitesimal time periods. The stochastic differential of $Y(t,T)$ is

$$dY(t,T) = -\frac{\sigma_B}{(T-t)^2} \cdot \eta(t,T)dt - \frac{\sigma_B}{T-t} d\eta(t,T) = -\frac{\sigma_B}{T-t} dW_B \tag{22}$$

by Itô's lemma and the expression for $d\eta(t,T)$ given in the previous section. Thus, the diffusion coefficient (instantaneous standard deviation) of the yield process explodes as t tends to T. This makes yield movements over very short time intervals ever more variable and, by adding up, leads to the unbounded growth of the variance $V[Y(t,T)]$. Moreover, Kemna, de Munnik and Vorst (1989) point out that the unbounded diffusion coefficient causes almost every yield path $\{Y(t,T) : 0 \le t \le T\}$ to reach negative values. Hence negative yields to maturity are generated with probability 1!

This highlights the serious drawbacks of the Ball–Torous model. One possible way to avoid them is to replace the Brownian bridge $\eta(t,T)$ by a process of the form

$$\tilde{\eta}(t,T) := k(t,T) \cdot W_B(t) \sim N\left(0, k^2(t,T) \cdot t\right) \tag{23}$$

---

[25] See for example Müller (1985).

[26] Existence of a martingale measure and absence of arbitrage are equivalent if the state space of the asset price model is finite; see Harrison and Pliska (1981). This equivalence breaks down if the state space is infinite. Back and Pliska (1991) give an example of a securities market which is arbitrage–free, but has no martingale measure.

where $k(t,T)$, a differentiable function defined for $t \in [0,T]$, is positive for $t < T$ and zero for $t = T$. Defining μ_B as before and setting

$$B(t) = B(0) \cdot \exp\{\mu_B \cdot t + \sigma_B \cdot \tilde{\eta}(t,T)\} \tag{24}$$

one obtains a bond price model that satisfies the terminal value condition. As in Ball and Torous (1983), the distributions of $B(t)$ and $Y(t,T)$ are lognormal and normal, respectively. More precisely,

$$\begin{aligned} \ln B(t) &\sim N\left(-\mu_B \cdot (T-t), \sigma_B^2 \cdot k^2(t,T)\right) , \\ Y(t,T) = \mu_B - \frac{\sigma_B}{T-t} \cdot \tilde{\eta}(t,T) &\sim N\left(\mu_B, \frac{\sigma_B^2 \cdot k^2(t,T)t}{(T-t)^2}\right) . \end{aligned} \tag{25}$$

The variance of the yield remains bounded as t tends to T if and only if $\frac{k(t,T)}{T-t}$ does so. This is also the condition for the diffusion coefficient of $Y(t,T)$ to stay bounded, as we can see by applying Itô's lemma twice:

$$d\tilde{\eta}(t,T) = \frac{k'(t,T)}{k(t,T)} \cdot \tilde{\eta}(t,T)dt + k(t,T)dW_B \tag{26}$$

and

$$\begin{aligned} dY(t,T) &= -\frac{\sigma_B}{(T-t)^2} \cdot \tilde{\eta}(t,T)dt - \frac{\sigma_B}{T-t} d\tilde{\eta}(t,T) \\ &= \left[\frac{1}{T-t} + \frac{k'(t,T)}{k(t,T)}\right] \cdot [Y(t,T) - \mu_B]dt - \frac{\sigma_B \cdot k(t,T)}{T-t} dW_B(t) . \end{aligned} \tag{27}$$

A model of this type, with $k(t,T) = \frac{T-t}{T}$, was proposed by Kemna, de Munnik and Vorst (1989). The resulting yield process is simply a Brownian motion starting at μ_B. This model succeeds where the Ball–Torous model fails. First, yields to maturity have bounded variance. Second, while negative yields occur with positive probability, as is the case in any model with lognormal bond prices, this probability is far smaller than 1 for reasonable parameter values. Third, de Munnik (1992) shows that this model admits a martingale measure and hence precludes arbitrage opportunities.

Turning to the valuation of bond options in a model where bond prices are of the form (24), we use Itô's formula once more to calculate the stochastic differential $dB(t,T)$. The result is

$$dB(t) = \alpha_B(t) \cdot B(t)dt + \sigma_B \cdot k(t,T) \cdot B(t)dW_B(t) \tag{28}$$

with drift rate process

$$\alpha_B(t) = \mu_B + \sigma_B \cdot k'(t,T) \cdot W_B(t) + \frac{1}{2}\sigma_B^2 \cdot k^2(t,T) . \tag{29}$$

Let $R(t)$, the price of the reference bond, also be of type (24), i.e

$$R(t) = R(0) \cdot \exp\{\mu_R \cdot t + \sigma_R \cdot \tilde{\eta}(t,\tau)\} \tag{30}$$

with $\tilde{\eta}(t,\tau) = k(t,\tau)W_R(t)$, and assume, as usual, that the instantaneous correlation coefficient ρ of the Wiener processes W_B and W_R is constant. This is again a special case of (13), and the results of section 3 apply. The volatility of the forward bond price is time dependent:

$$v(t) = \sqrt{\sigma_B^2 k^2(t,T) - 2\rho\sigma_B\sigma_R k(t,T)k(t,\tau) + \sigma_R^2 k^2(t,\tau)} . \tag{31}$$

The arbitrage price for a European call in this situation is again of the familiar form

$$\mathrm{Call}[t, B(t), R(t), \tau, K] = B(t) \cdot N(d_1) - K \cdot R(t) \cdot N(d_2) \tag{32}$$

where d_1 and d_2 are defined as in section 3, with the function $s(t)$ now given by

$$s(t) = \sigma_B^2 \int_t^\tau k^2(\theta,T)d\theta - 2\rho\sigma_B\sigma_R \int_t^\tau k(\theta,T)k(\theta,\tau)d\theta + \sigma_R^2 \int_t^\tau k^2(\theta,\tau)d\theta . \tag{33}$$

We have no empirical argument for a special form of the function $k(t,T)$. On the other hand, it would be at least of some theoretical interest to compare for example the option prices given by the Ball–Torous and Kemna–de Munnik–Vorst models. The pricing formulae obtained in these models differ only in the definition of the function $s(t)$. For a theoretical comparison of option prices, we have to relate the parameters σ_B, σ_R and ρ of one model to the corresponding parameters of the other model. There are many equally plausible (and equally arbitrary) ways to do this. For example, one might impose the condition that the integral of $V[\ln B(t)]$ over the life–time of the bond be the same in both models. For the Ball–Torous model with volatility parameter σ_{BT} for the underlying bond, this integral is

$$\int_0^T V[\ln B(t)]dt = \int_0^T \sigma_{BT}^2 \left(\frac{T-t}{T}\right) t dt = \sigma_{BT}^2 \frac{T^2}{6} .$$

For the Kemna–de Munnik–Vorst bond price process with parameter σ_{KMV}, one calculates

$$\int_0^T V[\ln B(t)]dt = \int_0^T \sigma_{KMV}^2 \left(\frac{T-t}{T}\right)^2 t dt = \sigma_{KMV}^2 \frac{T^2}{12} .$$

Requiring these quantities to be equal therefore amounts to imposing the relation

$$\sigma_{KMV} = \sqrt{2}\, \sigma_{BT} .$$

As shown in Figure 4, this implies that for small t the unconditional variance of $\ln B(t)$ (and hence of $B(t)$ as well) is larger in the Kemna–de Munnik–Vorst model than in the Ball–Torous model, whereas the reverse holds for t close to the maturity of the bond.

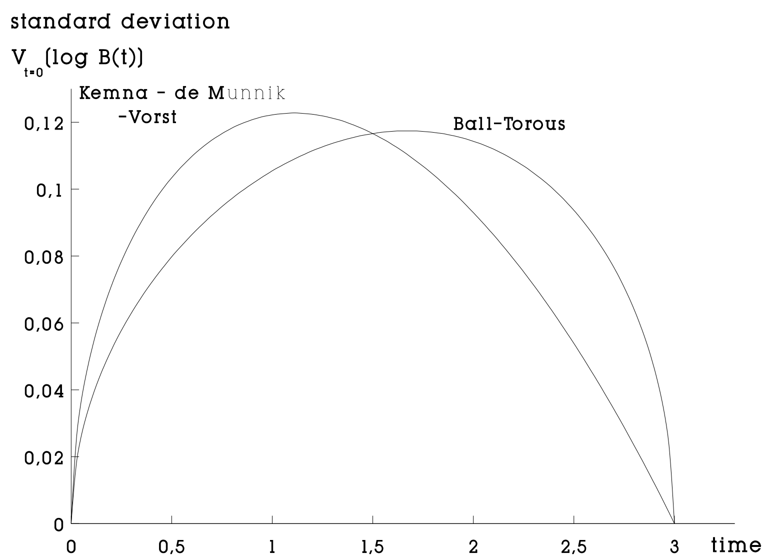

Figure 4: Variance of $\ln B(t)$ in the Ball–Torous and Kemna–de Munnik–Vorst model for $B(0) = (1.084)^{-3}$, $T = 3$, $\sigma_{BT} = 0.15, \sigma_{KMV} = \sqrt{2}\sigma_{BT}$.

Assuming the analogous relationship for the volatility parameter of the reference bond and using the same correlation coefficient ρ in both models, one can now convince oneself that the Kemna–de Munnik–Vorst price of a European option is higher than the Ball–Torous price if the time difference $T-\tau$ is relatively large, and smaller than the Ball–Torous price if $T-\tau$ is relatively small.

While two flaws of the Ball–Torous model, namely the exploding variance of the yield to maturity and the non–existence of a martingale measure, can be remedied by specifying bond price processes with time dependent volatility, a major problem remains unsolved. In all the models considered so far, yields to maturity and forward yields can take negative values. This in turn distorts option prices: for example, a call option written on a zero coupon bond with exercise price equal to the bond's face value has a positive price in these models. Schöbel (1987), Bühler and Käsler (1989) and Käsler (1991) proposed solutions to this problem. We shall analyse them in the following two sections.

6. Correcting for negative yields: An absorbing boundary for the forward bond price

We have seen in section 2 that non–negativity of forward yields implies property (7) which states that the price of a call with strike price $K \in [0,1]$ is $(1-K)R(t)$ whenever $B(t) = R(t)$. In terms of forward prices, (7) says that the forward call price is $1-K$ whenever $\hat{B}(t) = 1$.
The pricing formulae derived in lognormal models such as Ball and Torous (1983) or Kemna, de Munnik and Vorst (1989) do not fulfil (7), which reflects the fact that these models generate negative yields. Indeed, the last part of section 3 shows that the call prices calculated in any model which satisfies (10) with at most time dependent volatility function will violate (7). In this situation, Schöbel (1987) and Briys, Crouhy and Schöbel (1991) propose the alternative call price formula

$$\mathrm{Call}[t, B(t), R(t), \tau, K] = R(t) \cdot u^*\Big(t, \hat{B}(t)\Big)$$

where $u^* : [0,\tau] \times [0,1] \longrightarrow \mathbb{R}_+$ solves (11) with time dependent volatility $v : [0,\tau] \longrightarrow \mathbb{R}_+$ and satisfies

$$\begin{aligned} u^*(\tau, x) &= [x-K]^+ \ , \\ u^*(t, 0) &= 0 \ , \\ u^*(t, 1) &= 1-K \ . \end{aligned}$$

The first equation is the usual terminal value condition. The second equation is a boundary condition derived from property (5). The third condition is new; it imposes property (7).
Schöbel solves this problem by transforming it into a heat conduction problem on the non–negative real half–axis. This transformation is rather complicated. Fortunately, it can be avoided given our knowledge of the standard case studied in the last part of section 3. Let us write $u(t,x;K)$ for the solution calculated in section 3 corresponding to exercise price K. For $K > 0$ and all t, we have

$$\begin{aligned} u(t,1;K) &= N\left(\frac{1}{\sqrt{s}}(-\ln K + \frac{s}{2})\right) - K \cdot N\left(\frac{1}{\sqrt{s}}(-\ln K - \frac{s}{2})\right) \\ u(t,1;\frac{1}{K}) &= N\left(\frac{1}{\sqrt{s}}(\ln K + \frac{s}{2})\right) - \frac{1}{K} \cdot N\left(\frac{1}{\sqrt{s}}(\ln K - \frac{s}{2})\right) \end{aligned}$$

where

$$s = s(t) = \int_t^\tau v^2(\theta) d\theta$$

as usual. This implies

$$\begin{aligned} u(t,1;K) - K \cdot u(t,1;\frac{1}{K}) &= N\left(\frac{1}{\sqrt{s}}(-\ln K + \frac{s}{2})\right) - K \cdot N\left(\frac{1}{\sqrt{s}}(-\ln K - \frac{s}{2})\right) \\ &\quad -K \cdot N\left(\frac{1}{\sqrt{s}}(\ln K + \frac{s}{2})\right) + N\left(\frac{1}{\sqrt{s}}(\ln K - \frac{s}{2})\right) \\ &= 1-K \end{aligned}$$

since $N(-z) + N(z) = 1$. Therefore, if we set

$$u^*(t,x;K) = u(t,x;K) - K \cdot u(t,x;\frac{1}{K})$$

we clearly get a solution of (11) satisfying the above conditions. More explicitly, we can write

$$\begin{aligned} u^*(t,x;K) &= x \cdot N\Big(d_1(t,x;K)\Big) - K \cdot N\Big(d_2(t,x;K)\Big) \\ &- \quad K \cdot x \cdot N\Big(d_1(t,x;\frac{1}{K})\Big) + N\Big(d_2(t,x;\frac{1}{K})\Big) \ . \end{aligned}$$

This leads to the call price formula

$$\begin{aligned} \text{Call}[t, B(t), R(t), \tau, K] &= B(t) \cdot N(d_1) - K \cdot R(t) \cdot N(d_2) \\ &- \Big(K \cdot B(t) \cdot N(d_3) - R(t) \cdot N(d_4)\Big) \end{aligned} \tag{34}$$

where d_1 and d_2 are the same as in section 3 and

$$d_{3/4} = \frac{1}{\sqrt{s(t)}} \left(\ln \frac{KB(t)}{R(t)} \pm \frac{s(t)}{2} \right) .$$

The first part of (34) coincides of course with the formula of section 3. Schöbel (1987), dealing with the case of constant volatility, calls the second part $B(t)KN(d_3) - R(t)N(d_4)$ the *antioption*, interpreting it as the Ball–Torous price of a European call with exercise price 1 written on a discount bond with face value K. Our derivation of the pricing formula suggests a slightly different interpretation. In the case of constant volatility, for example, the second part of (34) is simply the Ball–Torous price of K calls with exercise price $\frac{1}{K}$ written on the original underlying bond.
For the European put option, put–call parity yields

$$\begin{aligned} \text{Put}[t, B(t), R(t), \tau, K] &= K \cdot R(t) \cdot N(-d_2) - B(t) \cdot N(-d_1) \\ &- \Big(K \cdot B(t) \cdot N(d_3) - R(t) \cdot N(d_4)\Big). \end{aligned} \tag{35}$$

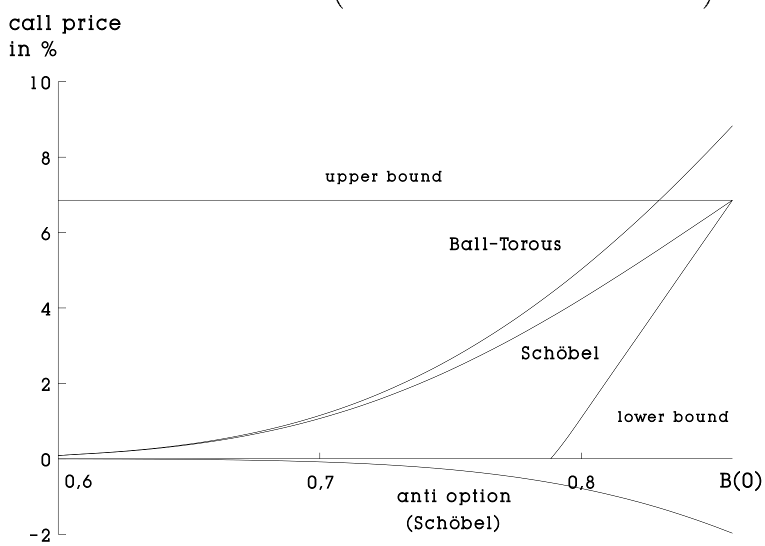

Figure 5: European call option and anti option of the Ball–Torous and Schöbel models. $T = 3, \tau = 2, K = 0.92, R(0) = (1.08)^{-2}, \sigma_T = 0.15, \sigma_\tau = 0.12, \rho = 0.75$, "lower bound" refers to (4), "upper bound" to (6).

Neither Schöbel (1987) nor Briys, Crouhy and Schöbel (1991) describe bond price processes such that portfolio duplication would lead to formula (34) or (35). As a first step in this direction, we follow Breeden and Litzenberger (1978) and calculate Arrow–Debreu or state prices implied by (34) and (35). Assume that time 0 bond prices are $B(0) = B$ and $R(0) = R$ with $B \leq R$. Now suppose that there are no negative yields. The states of the world at time τ are given by the possible values of $B(\tau)$, i.e. we have the continuum of states $]0, 1]$. We look for a distribution function F with $F(0) = 0$ and $F(1) = 1$ such that time 0 bond and option prices are discounted expected values of time τ payoffs with respect to F. For the underlying bond, this means

$$B = R \int_0^1 x dF(x) , \tag{36}$$

and for put options with exercise prices $0 \leq K \leq 1$,

$$P(K) = R\int_0^1 [K-x]^+ dF(x) = R\int_0^K (K-x)dF(x) \tag{37}$$

where we have chosen the simple notation $P(K)$ for the put price $\text{Put}[0, B, R, \tau, K]$ given by (35). The number $RF(x)$ can be interpreted as the price of an Arrow–Debreu security $I_{\{B(\tau)\leq x\}}$ paying 1 if $B(\tau) \leq x$ and 0 else. (36) and (37) express the consistency of these Arrow–Debreu prices with actual prices of bonds and options.

Integration by parts yields

$$\int_0^K (K-x)dF(x) = [(K-x)F(x)]_0^K + \int_0^K F(x)dx = \int_0^K F(x)dx\ .$$

Therefore,

$$P(K) = R\int_0^K F(x)dx\ .$$

P has continuous derivatives of all orders on $]0,1[$. In particular, F is continuous on $]0,1[$ and satisfies

$$F(K) = \frac{1}{R}\frac{\partial P}{\partial K}(K) \quad for\ 0 < K < 1\ .$$

We calculate the derivative of P:

$$\begin{aligned}\frac{\partial P}{\partial K} &= -\frac{Bn(-d_1)}{K\sqrt{s(0)}} + R\cdot N(-d_2) + \frac{Rn(-d_2)}{\sqrt{s(0)}} + \frac{Rn(d_4)}{K\sqrt{s(0)}} - B\cdot N(d_3) - \frac{Bn(d_3)}{\sqrt{s(0)}} \\ &= R\cdot N(-d_2) - B\cdot N(d_3)\end{aligned}$$

where n denotes the standard normal density function.[27] Thus,

$$F(K) = N(-d_2) - \frac{B}{R}\cdot N(d_3)\ . \tag{38}$$

Note that F is continuous at 0: $F(K) \to 0$ for $K \downarrow 0$. But for $K \uparrow 1$,

$$F(K) \to N\left(-\frac{\ln\frac{B}{R} + \frac{1}{2}s(0)}{\sqrt{s(0)}}\right) - \frac{B}{R}\cdot N\left(\frac{\ln\frac{B}{R} + \frac{1}{2}s(0)}{\sqrt{s(0)}}\right) < 1$$

so F has a jump at 1. On $]0,1[$, F is continuously differentiable. We denote its derivative $]0,1[$ by f and calculate[28]

$$\begin{aligned}f(K) = \frac{\partial F}{\partial K}(K) &= \frac{1}{K\sqrt{s(0)}}\cdot\left[n(-d_2) - \frac{B}{R}\cdot n(d_3)\right] \\ &= \frac{B}{R\cdot K\sqrt{s(0)}}\cdot n(d_3)\cdot\left[\left(\frac{B}{R}\right)^{\frac{2\ln K}{s(0)}} - 1\right]\end{aligned} \tag{39}$$

Note that f is positive on $]0,1[$ since $B < R$ and $\ln K < 0$. Therefore, F is indeed increasing on $]0,1[$. It can be shown that $f(K) \to 0$ as $K \downarrow 0$, and the formula for f clearly implies $f(K) \to 0$ as $K \uparrow 1$.

The fact that F has a single jump at the boundary 1 of the state space implies that the Arrow–Debreu security $I_{\{B(\tau)=1\}}$ has a positive price, in contrast to all the other securities $I_{\{B(\tau)=x\}}$ with $x < 1$ having price zero. Imposing the boundary condition (7) means that probability mass which the original bond price model places on outcomes $B(\tau) \geq 1$ has been concentrated in the state $B(\tau) = 1$, so this state occurs with positive probability. In particular, any bond price model consistent with formulae (34) and (35) necessarily assigns positive probability to the event that the yield $Y(\tau, T)$ becomes zero. Note that if this happens, there is no reward for holding the underlying bond from τ to T.

Using a different method of investigation, Rady (1992) shows that any arbitrage-free bond price model which does not generate negative yields and supports the option price formulae (34) and (35) necessarily

---

[27] Note that $\frac{n(-d_2)}{n(-d_1)} = \frac{B}{KR}$ and $\frac{n(d_4)}{n(d_3)} = \frac{R}{KB}$.

[28] We use $\frac{n(-d_2)}{n(d_3)} = \left(\frac{B}{R}\right)^{\frac{2\ln K}{s(0)}+1}$.

has a forward price process $\hat{B}$ with an absorbing boundary at 1. This boundary is reached with positive probability.[29] In other words, at each time $0 < t \leq \tau$, there is a positive probability for $B(t) = R(t)$, and once this has happened, the bond prices coincide until τ. Therefore, while satisfying condition (7), the proposed pricing formulae imply rather implausible bond price behaviour. A more satisfactory model will be presented in the following section.

7. Beside lognormality: Time and state dependent volatility

In section 5, we have considered models of the type

$$R(t) = h_R(t) \cdot \exp\{g_R(t) \cdot W_R(t)\}$$

$$B(t) = h_B(t) \cdot \exp\{g_B(t) \cdot W_B(t)\}$$

with at most time dependent functions h_R, h_B, g_R and g_B. $h_R(t)$ and $h_B(t)$, the median values of $R(t)$ and $B(t)$, can be interpreted as describing price paths under certainty, whereas the exponential factors characterise the random movement around these median paths. Such a model postulates that after taking the logarithm of bond prices, i.e. after applying the bijective mapping

$$\Lambda : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}^2 \; ; \; \begin{pmatrix} r \\ b \end{pmatrix} \mapsto \begin{pmatrix} \ln r \\ \ln b \end{pmatrix} , \tag{40}$$

we are dealing with Gaussian processes. More precisely, the image of the bond prices under Λ is equal to the image of the medians plus a Wiener process term with time dependent coefficients:

$$\Lambda \begin{pmatrix} R(t) \\ B(t) \end{pmatrix} = \Lambda \begin{pmatrix} h_R(t) \\ h_B(t) \end{pmatrix} + \begin{pmatrix} g_R(t) \cdot W_R(t) \\ g_B(t) \cdot W_B(t) \end{pmatrix} \tag{41}$$

The main argument against this approach is that such a model generates negative yields. Indeed, to ensure positive yields to maturity and forward yields, the bond price vector $\binom{R(t)}{B(t)}$ ought to take values in the triangle

$$D := \left\{ \begin{pmatrix} r \\ b \end{pmatrix} \in \,]0,1[^{\,2} \; : \; r > b \right\} . \tag{42}$$

Given a bijective mapping $\psi : D \mapsto \mathbb{R}^2$, we can construct a model that has positive yields by rewriting (41) with ψ rather than Λ, i.e. by postulating that bond prices satisfy

$$\psi \begin{pmatrix} R(t) \\ B(t) \end{pmatrix} = \psi \begin{pmatrix} h_R(t) \\ h_B(t) \end{pmatrix} + \begin{pmatrix} g_R(t) \cdot W_R(t) \\ g_B(t) \cdot W_B(t) \end{pmatrix} . \tag{43}$$

The bond prices themselves can be recovered by means of the inverse mapping $\psi^{-1} : \mathbb{R}^2 \longrightarrow D$. As before, $h_R(t)$ and $h_B(t)$ are the median values of $R(t)$ and $B(t)$.

However, which transformation ψ should we use? There is no obvious choice. Ideally, it would be a simple mapping that leads to a tractable bond price distribution and closed form solutions for option pricing. In fact, these goals are achievable, as Bühler and Käsler (1989) prove with the very ingenuous choice of the mapping[30]

$$\psi : D \longrightarrow \mathbb{R}^2 : \begin{pmatrix} r \\ b \end{pmatrix} \mapsto \begin{pmatrix} \ln \frac{r}{1-r} \\ \ln \frac{b}{r-b} \end{pmatrix} . \tag{44}$$

Its inverse is given by

$$\psi^{-1} : \mathbb{R}^2 \longrightarrow D : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{1+e^{-w_1}} \\ \frac{1}{(1+e^{-w_1})\cdot(1+e^{-w_2})} \end{pmatrix} . \tag{45}$$

---

[29] (38) can be interpreted as the transition probability of the forward bond price under a martingale measure. It turns out that under such a measure, $\hat{B}$ is a geometric Brownian motion absorbed at 1.

[30] See also Käsler (1991). A one–dimensional variant of this mapping was first used by Bühler (1988) to model the price process of a coupon bond. See below for a brief discussion of this model.

The resulting bond prices are

$$\begin{aligned} R(t) &= \frac{1}{1+\frac{1-h_R(t)}{h_R(t)}\exp\{-g_R(t)\cdot W_R(t)\}}\,, \\ B(t) &= R(t)\cdot\frac{1}{1+\frac{h_R(t)-h_B(t)}{h_B(t)}\exp\{-g_B(t)\cdot W_B(t)\}}\,. \end{aligned} \tag{46}$$

Note that the price of the underlying bond depends explicitly on the price of the reference bond. In particular, both sources of uncertainty, W_B and W_R, have an impact on the price process of the underlying bond. By contrast, as $B(t)$ is just a multiple of $R(t)$, the forward bond price has a relatively simple representation, involving only the Wiener process W_B:

$$\hat{B}(t)=\frac{B(t)}{R(t)}=\frac{1}{1+\frac{h_R(t)-h_B(t)}{h_B(t)}\cdot\exp\{-g_B(t)\cdot W_B(t)\}} \tag{47}$$

Bühler and Käsler(1989) develop this model for constant g_R and g_B; the generalisation to time dependent parameters presented here is trivial. Rather than specifying a functional form of h_R and h_B, they suggest estimating these functions from the current term structure, but do not go into details. If one wishes to fix a functional form for h_R and h_B a priori, one can, for example, proceed in analogy with the models discussed in previous sections and specify the median paths as

$$\begin{aligned} h_R(t) &:= R(0)^{\frac{\tau-t}{\tau}}, \\ h_B(t) &:= B(0)^{\frac{T-t}{T}}. \end{aligned} \tag{48}$$

The Bühler–Käsler model (46) fulfils all the natural requirements discussed in section 2: the terminal value condition (1); equation (2) which precludes negative yields to maturity; and equation (3) which rules out negative forward yields. This is visualised in figure 6, where trajectories of the bond price processes and the upper boundaries of the conditional resp. unconditional 95% bands are drawn.[31]

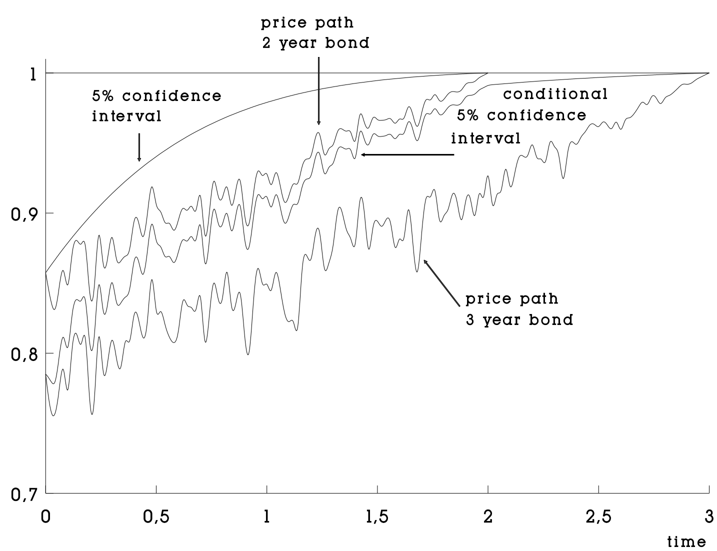

Figure 6: Bond price sample paths in the Bühler–Käsler model with the upper boundaries of the unconditional and conditional 95% band, where $R(0)=(1.08)^{-2}$, $B(0)=(1.084)^{-3}$, $\tau=2, T=3$, $\rho=0.75$, $g_R=1.62$ $g_B=1.23$.

[31] See the appendix for the calculation of these unconditional resp. conditional $1-\alpha$ bands, and compare with the discussion in section 4.

The distributions of $R(t)$ and $\hat{B}(t)$ and the conditional distribution of $B(t)$ given $R(t)$ belong to a class of distributions studied already by Johnson (1946, 1949).[32] It is easy to calculate their density functions. The bond price $R(t)$, for example, has the density function

$$\xi(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{g_R(t)\sqrt{t}} \frac{1}{x \cdot (1-x)} \exp\left\{ -\frac{\left(\ln \frac{x}{1-x} - \ln \frac{h_R(t)}{1-h_R(t)}\right)^2}{2t g_R(t)^2} \right\}, \quad x \in]0,1[. \tag{49}$$

Johnson has shown that random variables with density functions of this type have finite moments, but there are no closed form expressions for them. In addition, one can show (see the appendix) that the expected value of $R(t)$ is bounded by

$$\frac{1}{1 + \frac{1-h_R(t)}{h_R(t)} \cdot \exp\left\{\frac{1}{2} g_R(t)^2 t\right\}} \leq E[R(t)] \leq \frac{1}{1 + \frac{1-h_R(t)}{h_R(t)} \cdot \exp\left\{-\frac{1}{2} g_R(t)^2 t\right\}}. \tag{50}$$

For option pricing we need to calculate the stochastic differential of the forward price process of the underlying bond. Itô's formula yields[33]

$$d\hat{B} = \left[\frac{h_B' h_R - h_B h_R'}{h_B(h_R - h_B)} + g_B' W_B + g_B^2 \left(\frac{1}{2} - \hat{B}\right) \right] \hat{B}(1-\hat{B})dt + g_B \hat{B}(1-\hat{B}) dW_B. \tag{51}$$

The volatility of the forward bond price is time and state dependent:

$$v(x,t) = g_B(t) \cdot (1-x)$$

in the notation of section 3. The state space of $\hat{B}$ is $]0,1[$. In view of lemma 2, we therefore want to solve

$$u_t(x,t) + \frac{1}{2} g_B^2(t) x^2 (1-x)^2 u_{xx}(x,t) = 0$$

on $[0,1] \times [0,\tau]$ with the terminal value condition

$$u(x,\tau) = [x-K]^+$$

and the bounds

$$[x-K]^+ \leq u(x,t) \leq \min\{x, 1-K\}$$

in order to determine the arbitrage price for the European call option. It is shown in the appendix how to solve this problem by transforming it into a heat conduction problem on the real axis. The solution is

$$\begin{aligned} u(x,t) &= (1-K) \cdot x \cdot N\left(\frac{1}{\sqrt{s(t)}} \left[\ln \frac{x(1-K)}{(1-x)K} + \frac{s(t)}{2} \right] \right) \\ &\quad - K \cdot (1-x) \cdot N\left(\frac{1}{\sqrt{s(t)}} \left[\ln \frac{x(1-K)}{(1-x)K} - \frac{s(t)}{2} \right] \right) \end{aligned} \tag{52}$$

where N denotes the standard normal distribution function and

$$s(t) = \int_t^\tau g_B(\theta)^2 d\theta \tag{53}$$

Consequently, the Bühler–Käsler arbitrage price of the European call option is given by:

$$\begin{aligned} \text{Call}[t, B(t), R(t), \tau, K] &= R(t) \cdot u\left(\frac{B(t)}{R(t)}, t\right) \\ &= (1-K) \cdot B(t) \cdot N(e_1) - K \cdot \Big(R(t) - B(t)\Big) \cdot N(e_2) \end{aligned} \tag{54}$$

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[32] Johnson constructs classes of distributions by applying the "method of translation" to a standard normal variable Z. The class of lognormal distributions, for instance, is obtained by means of the exponential transformation $Z \mapsto \exp\{\gamma + \delta Z\}$. The transformation $Z \mapsto (\gamma + \delta \exp\{\theta Z\})^{-1}$ defines a class which Johnson denotes by S_B. This is the type of distributions we are dealing with in the Bühler–Käsler model.

[33] The time variable t has been omitted to simplify the notation.

with

$$e_{1/2} = \frac{1}{\sqrt{s(t)}} \left(\ln \frac{B(t) \cdot (1-K)}{(R(t) - B(t)) \cdot K} \pm \frac{s(t)}{2} \right) .$$

The generating strategy for the option is, in the notation of section 3,

$$\phi^1 = (1-K) \cdot N(e_1) + K \cdot N(e_2); \quad \phi^2 = -K \cdot N(e_2).$$

The Bühler–Käsler model (46) is unique within the direct approach in as much as it guarantees positive yields to maturity as well as positive forward yields and still produces a closed form solution for the arbitrage price of European debt options. Moreover, Bühler and Käsler point out that the existence of a martingale measure is easily demonstrated for the model with constant g_R and g_B.[34]

By construction, the pricing formulae of Bühler and Käsler (1989) and Schöbel (1987) both satisfy condition (7):

$$\lim_{B(t)\to R(t)} \mathrm{Call}[t, B(t), R(t), \tau, K] = (1-K)R(t).$$

A theoretical comparison of the prices given by these formulae for $B(t) < R(t)$ must, as in section 5, be based on a hypothetical relationship between the relevant model parameters, i.e. g_B on the one hand and σ_B, σ_R and ρ on the other hand. We assume that these parameters are constant and choose the simplest approach, postulating that the volatility of the forward bond price at time 0 is the same in both models. This leads to the relation

$$g_B = \frac{\sqrt{\sigma_B^2 - 2\rho\sigma_B\sigma_R + \sigma_R^2}}{1 - \hat{B}(0)}. \tag{55}$$

The curve labelled “Bühler-Käsler I” in Figure 7 has been calculated under this assumption. Thus, the parameter g_B has been adjusted to different initial forward prices. By contrast, the curve labelled “Bühler-Käsler II” is based on a single value of g_B regardless of $\hat{B}(0)$.

[34] It was said in section 3 that a martingale measure exists if and only if the process defined as the quotient of the drift process and the volatility of $\hat{B}$ satisfies certain integrability conditions. In the model of Bühler and Käsler (1989), this process is bounded and hence fulfils those conditions trivially.

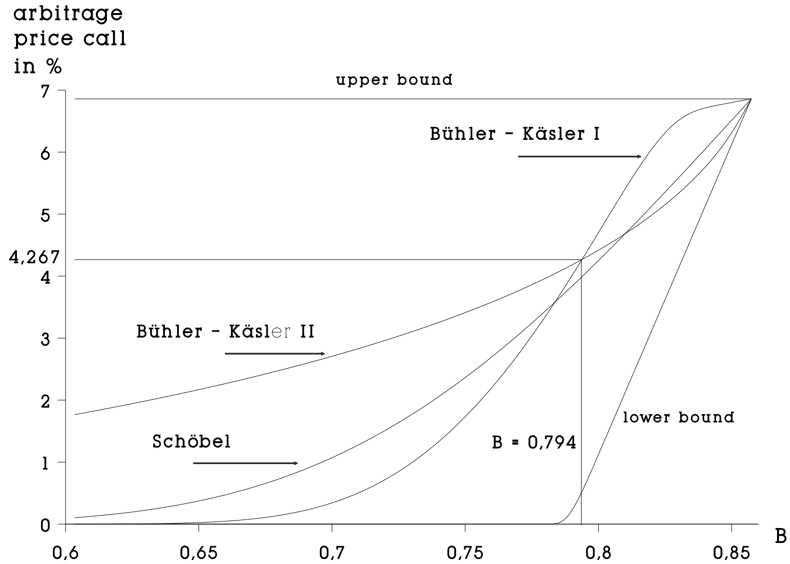

Figure 7: Arbitrage price of a European call for different initial bond prices $B(0)$ as given by the Schöbel and Bühler–Käsler formulae with $R(0) = (1.08)^{-2}$, $T = 3$, $\tau = 2$, $\sigma_B = 0.15$, $\sigma_R = 0.12$, $\rho = 0.75$, and $K = 0.92$. The curve labelled "Bühler-Käsler I" is based on relation (55) with g_B adjusting as $\hat{B}(0)$ varies. The curve labelled "Bühler-Käsler II" is calculated for the fixed parameter value $g_B = 1.343$, which corresponds to (55) with $\hat{B}(0) = (1.08)^{-3} = 0.794$.

Bühler and Käsler (1989) were not the first to develop a bond price model with state dependent volatility. Using the direct approach to evaluate options on coupon bearing bonds, Schaefer and Schwartz (1987) assume that the price process of the underlying coupon bond satisfies

$$dB(t) = \alpha_B(t) \cdot B(t)dt + k \cdot B(t)^l \cdot D(B(t), t)dW$$

where k and l are constants and $D(B(t), t)$ is the duration of the bond.[35] This specification of volatility reflects the fact that bond returns become less variable as the maturity date approaches. The authors leave the drift rate process α_B unspecified because they are mainly interested in (and provide empirical evidence on) the connection between duration and the variability of bond returns, and because the drift rate does not enter the valuation equation (11). Neither the terminal value condition nor the question of negative yields are addressed in this paper. Due to the complicated volatility function, there are in general no analytic solutions for option prices. Schaefer and Schwartz assume that the reference bond has a constant rate of return r: $dR(t) = rR(t)dt$. This assumption is of course hard to justify, it merely serves to keep the numerical valuation procedure as simple as possible.

Bühler (1988), also using duration to describe bond volatility, proposes a more sophisticated alternative to the model of Schaefer and Schwartz (1987). In his model, the price process of the underlying coupon bond fulfils the terminal value condition, and the bond yield remains always positive. He starts from the following observation. Define $B_{\max}(t)$ as the par value plus the undiscounted coupon payments from t on. Then the yield of the bond at time t is positive if and only if $B(t) < B_{\max}(t)$. Bühler goes on to

[35] For a discussion of the duration concept see for example Cox, Ingersoll and Ross (1979). The duration of a zero coupon bond is just its time to maturity. Thus, if $l = 1$ and the underlying bond pays no coupons, one obtains the same bond price volatility as in Kemna, de Munnik and Vorst (1989).

construct a bond price process with this property[36] and derives the following bond price dynamics:

$$dB(t) = -\frac{\ln B(t)}{T-t} \cdot B(t)dt + k \cdot B(t) \cdot \frac{B_{max}(t) - B(t)}{B_{max}(t) - 1 + \delta} \cdot D(B(t), t)dW(t) \tag{56}$$

with constants k and δ. The drift term pulls the process towards the par value (which we have normalised to one) and away from the boundaries of the state space, 0 and $B_{max}(t)$. Again, option prices must be calculated numerically. Rather than imposing a constant rate of return for the reference bond, Bühler simplifies the numerical procedure by specifying

$$dR(t) = r(B(t)) \cdot R(t)dt \tag{57}$$

where $r(B(t))$ is the yield of the underlying bond multiplied by a time dependent factor. This supposes perfect positive correlation between the bond yields, which, though far less restrictive than the assumption made by Schaefer and Schwartz, is still a problematic hypothesis.

It may well be that by relaxing the restrictive assumptions made by Schaefer and Schwartz or Bühler, the direct approach could eventually provide a satisfactory valuation model for options on coupon bonds; the Bühler model in particular indicates that this would involve considerable technical complications. The term structure approach seems more appropriate for the pricing of coupon bond options. Modelling simultaneously the discount bonds of all maturities, this approach can treat coupon bonds simply as linear combinations of discount bonds. Thus, one encounters no particular modelling difficulties when moving from discount bonds to coupon bonds. Moreover, there are term stucture models that ensure positive yields and possess a martingale measure.[37] Finally, Jamshidian (1989) and El Karoui and Rochet (1989) showed that certain term structure models provide tractable formulae for the prices of European options on coupon bonds: in these models, the price of a coupon bond option can be written as the sum of the prices of discount bond options. For these reasons, the use of term structure models is generally seen as the natural approach to the valuation of options on coupon bearing bonds.

8. Conclusion

In this paper, we have given a detailed survey of the direct or price–based approach to debt option pricing. This approach specifies bond price processes directly, without relating them to state variables such as the short term interest rate. The presentation of the portfolio duplication technique in section 3 stresses the fact that the volatility of the forward bond price is the crucial model characteristic for the calculation of option prices. Therefore, we have structured the paper according to the specification of volatility, reaching from constant volatility (Ball and Torous (1983)) over time dependent volatility (Kemna, de Munnik and Vorst (1989)) to time and state dependent volatility (Bühler and Käsler (1989)). Focusing on zero coupon bonds, we have emphasized the main modelling problems encountered by the direct approach: first, the problem of specifying bond prices that fulfil the terminal value condition, i.e. that reach par value at maturity; second, the problem of precluding negative yields to maturity and negative forward yields; third, the problem of ensuring an arbitrage–free bond price model.

The model of Bühler and Käsler (1989) is the only one to solve all three problems. Lognormal models such as Ball and Torous (1983) and Kemna, de Munnik and Vorst (1989) have the advantage of leading to analytic solutions for bond option prices which are of the same type as the well–known stock option pricing formulae of Black and Scholes (1973) and Merton (1973). The common weakness of lognormal models, however, is that negative yields to maturity and negative forward yields occur with positive probability. As this distorts option prices, Schöbel (1986) proposes modified pricing formulae. We have analysed his approach in some detail: imposing an additional constraint on option prices, he implicitely

[36] This is the first example of the transformation method described at the beginning of this section. Bühler uses a monotonic mapping to transform a process with values in $\mathbb{R}$ in such a way that the resulting process has the desired properties.

[37] See for example Cox, Ingersoll and Ross (1985) or Heath, Jarrow and Morton (1992).

assumes that the forward yield has an absorbing boundary at zero. Bühler and Käsler (1989), by contrast, construct a bond price model with positive yields to maturity and positive forward yields that avoids the implausible assumption of an absorbing boundary and still provides closed form solutions for option prices.

We have not studied the problems of model testing and parameter estimation. These issues are of course crucial for the choice of a model and its implementation. For example, a practitioner will prefer a simple model with some weaknesses to a theoretically more satisfactory model if the parameters of the latter are much harder to estimate, or if the theoretical weakness of the simple model is negligible for realistic parameter values.[38]

Of course, the above models deal only with options on zero coupon bonds and hence are of limited practical use. As for the valuation of options on coupon bonds using the direct approach, we discussed the models of Schaefer and Schwartz (1987) and Bühler (1988). The latter model in particular indicates that direct modelling of the price process of a coupon bond involves considerable technical complications. In a term structure model, by contrast, one can easily exploit the fact that a coupon bond is just a portfolio of discount bonds. We concluded that for the valuation of coupon bond options, the natural approach is to use a term structure model.

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Appendix

Unconditional $1-\alpha$ band of the Bühler–Käsler model:

$$
\begin{aligned}
1-\alpha &= \text{prob}\left[\; a_1(t) < R(t) \le a_2(t) \;\right] \\
&= \text{prob}\left[\; a_1(t) < \frac{1}{1+\frac{1-h_R(t)}{h_R(t)}\exp\{-g_R(t)W_R(t)\}} \le a_2(t) \;\right] \\
&= \text{prob}\left[\; \ln\frac{a_1}{1-a_1} - \ln\frac{h_R(t)}{1-h_R(t)} < g_R(t)\cdot W_R(t) \le \ln\frac{a_2(t)}{1-a_2(t)} - \ln\frac{h_R(t)}{1-h_R(t)} \;\right]
\end{aligned}
$$

Since $g_R(t)W_R(t) \sim N(0, g_R(t)^2 t)$ the symmetric $1-\alpha$ intervall for fixed $t \in]0,\tau[$ is equal to

$$
\begin{aligned}
& \ln\frac{a_1}{1-a_1} - \ln\frac{h_R(t)}{1-h_R(t)} = \sqrt{t}\, g_R(t)\cdot\mu_{\frac{\alpha}{2}} = -\sqrt{t}\, g_R(t)\cdot\mu_{1-\frac{\alpha}{2}} \\
\Leftrightarrow \quad & a_1(t) = \frac{1}{1+\frac{1-h_R(t)}{h_R(t)}\cdot\exp\{\sqrt{t}\, g_R(t)\mu_{1-\frac{\alpha}{2}}\}} \\
\text{resp.} \quad & a_2(t) = \frac{1}{1+\frac{1-h_R(t)}{h_R(t)}\cdot\exp\{-\sqrt{t}\, g_R(t)\mu_{1-\frac{\alpha}{2}}\}}
\end{aligned}
$$

where μ_α is the α–fractil of the standard normal distribution.

Conditional $1-\alpha$ band of the Bühler–Käsler model:

$$\begin{aligned}
1-\alpha &= \text{prob}[b_1(t) < B(t) \le b_2(t) | R(t)] \\
&= \text{prob}\left[\frac{b_1(t)}{R(t)} =: \hat{b}_1(t) < \hat{B}(t) \le \hat{b}_2(t) := \frac{b_2(t)}{R(t)} \right| R(t) \Bigg] \\
&= \text{prob}\Bigg[\ln \frac{\hat{b}_1(t)}{1-\hat{b}_1(t)} - \ln \frac{h_B(t)}{h_R(t)-h_B(t)} < g_B(t) W_R(t) \\
&\qquad \le \left. \frac{\ln \hat{b}_2(t)}{1-\hat{b}_2(t)} - \frac{\ln h_B(t)}{h_R(t)-h_B(t)} \right| R(t) \Bigg]
\end{aligned}$$

The conditional distribution of $g_B(t)W_R(t)$ is $N\left(0, g_B(t)^2(1-\rho^2)\right)$ and therefore

$$\begin{aligned}
b_1(t) &= R(t)\, \frac{1}{1+\frac{h_R(t)-h_B(t)}{h_R(t)} \cdot \exp\{\sqrt{t}\; g_B(t)\sqrt{1-\rho^2}\mu_{1-\frac{\alpha}{2}}\}} \\
b_2(t) &= R(t)\, \frac{1}{1+\frac{h_R(t)-h_B(t)}{h_R(t)} \cdot \exp\{-\sqrt{t}\; g_B(t)\sqrt{1-\rho^2}\mu_{1-\frac{\alpha}{2}}\}}
\end{aligned}$$

Upper and lower bounds for the expected value of the zero coupon bond $R(t)$ (Bühler–Käsler model):

Define $\overline{\mu} := \ln \frac{h_R(t)}{1-h_R(t)}$
$\overline{\sigma} := g_R(t)\sqrt{t}$

$$\begin{aligned}
\Rightarrow E[R(t)] &= \int_0^1 \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{x}{x(1-x)} \exp\left\{ -\frac{\left(\ln\frac{x}{1-x} - \overline{\mu}\right)^2}{2\overline{\sigma}^2} \right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{1}{1+e^x} \exp\left\{ -\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2} \right\} dx \\
&= 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{e^x}{1+e^x} \exp\left\{ -\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2} \right\} dx \\
&= 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{1}{1+e^x} \exp\left\{ -\frac{\left(x-(\overline{\mu}+\overline{\sigma}^2)\right)^2}{2\overline{\sigma}^2} + \overline{\mu} + \frac{1}{2}\overline{\sigma}^2 \right\} dx \\
&= 1 - e^{\overline{\mu}+\frac{1}{2}\overline{\sigma}^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{1}{1+e^x} \exp\left\{ -\frac{(x-\overline{\mu}-\overline{\sigma}^2)^2}{2\overline{\sigma}^2} \right\} dx \\
&= 1 - e^{\overline{\mu}-\frac{1}{2}\overline{\sigma}^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{e^{\overline{\sigma}^2}}{1+e^{x+\overline{\sigma}^2}} \exp\left\{ -\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2} \right\} dx \\
&\le 1 - e^{\overline{\mu}-\frac{1}{2}\overline{\sigma}^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\;\overline{\sigma}}\; \frac{1}{1+e^{x}} \exp\left\{ -\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2} \right\} dx \\
\Rightarrow E[R(t)] &\le \frac{1}{1+\exp\left\{\overline{\mu}-\frac{1}{2}\overline{\sigma}^2\right\}} = \frac{1}{1+\frac{h_R(t)}{1-h_R(t)}\exp\left\{-\frac{1}{2}g_R(t)^2 t\right\}}
\end{aligned}$$

Since the function $\frac{1}{1+r}$ is convex on $[0,+\infty[$ we know from Jensen's inequality

$$\begin{aligned}
E\left[\frac{1}{1+\frac{h_R(t)}{1-h_R(t)}\exp\{-g_R(t)W(t)\}}\right] &\ge \frac{1}{1+\frac{h_R(t)}{1-h_R(t)} E[\exp\{-g_R(t)W_R(t)\}]} \\
&= \frac{1}{1+\frac{h_R(t)}{1-h_R(t)}\exp\{\frac{1}{2}g_R^2(t)t\}}
\end{aligned}$$

Solution of the Bühler – Käsler terminal value problem:

$$\begin{aligned}
&u_t(x,t) + \frac{1}{2}g_B(t)^2x^2(1-x)^2u_{xx}(x,t) = 0 \\
&u(x,\tau) = f(x)
\end{aligned}$$

This terminal value problem on $]0,1[\times[0,\tau]$ is transformed by introducing the new time variable

$$s = \int_t^\tau g_B(\theta)^2 d\theta \ ,$$

the new space variable

$$z = \ln \frac{x}{1-x} \qquad \text{or} \quad x = \frac{1}{1+e^{-z}}$$

and finally setting

$$u(x,t) = a(z)b(s)h(z,s) \ .$$

The differentiable functions a and b are to be chosen in such a way that any solution h of the heat conduction equation yields a solution u of the original partial differential equation.
One easily calculates the derivatives

$$\begin{aligned}
z_x &= \frac{1}{x(1-x)} \\
z_{xx} &= \frac{2x-1}{x^2(1-x)^2} \\
u_x &= [a_z h + a h_z] b z_x \\
u_{xx} &= \{a_{zz}h + 2a_z h_z + a h_{zz} + (2x-1)[a_z h + a h_z]\} b \frac{1}{x^2(1-x)^2} \\
u_t &= a[b_s h + b h_s]\Big(-g_B^2(t)\Big)
\end{aligned}$$

As

$$2x - 1 = \frac{e^z - 1}{e^z + 1} = \frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} = \tanh\frac{z}{2}$$

we get by inserting and dividing by g_B^2:

$$ab\left[\frac{1}{2}h_{zz} - h_s\right] + \left(a_z + \frac{1}{2}a\cdot\tanh\frac{z}{2}\right) b h_z + \left[\frac{1}{2}\left(a_{zz} + \frac{1}{2}a_z \cdot \tanh\frac{z}{2}\right) b - a b_s\right] h = 0 \ .$$

In order to make the h_z–term vanish, a has to solve the linear differential equation

$$a_z + \frac{1}{2}a \cdot \tanh\frac{z}{2} = 0 \ .$$

Separation of variables leads to the solutions

$$a(z) = \frac{c}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} \ ;$$

we choose $c = 1$.
Using the equation for a, one obtains

$$a_{zz} + a_z \cdot \tanh\frac{z}{2} = -\frac{1}{4}a \ .$$

Therefore, the h–term vanishes if

$$\frac{1}{8}b + b_s = 0 \ .$$

We choose the solution

$$b(s) = e^{-\frac{s}{8}} \ .$$

<u>Result:</u> Setting

$$u(x,t) = \frac{1}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} e^{-\frac{s}{8}} h(z,s) \ ,$$

we obtain the transformed problem on $\mathbb{R} \times [0,\tau]$

$$\frac{1}{2}h_{zz} - h_s = 0,$$

$$h(z,0) = (e^{\frac{z}{2}} + e^{-\frac{z}{2}}) f\left(\frac{1}{1+e^{-z}}\right).$$

The solution is:

$$h(z,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(z + \rho\sqrt{s}, 0) e^{-\frac{\rho^2}{2}} d\rho \ .$$

We omit the explicit formula for $u(x,t)$.

Valuation of a call option in the Bühler-Käsler model:

For the initial value condition

$$h(z,0) = (e^{\frac{z}{2}} + e^{-\frac{z}{2}}) \left[\frac{1}{1+e^{-z}} - K \right]^+$$

the solution is given by

$$\begin{aligned} h(z,s) &= \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{s}}\left[\ln \frac{K}{1-K} - z\right]}^{\infty} \left(e^{\frac{1}{2}[z+\rho\sqrt{s}]} + e^{-\frac{1}{2}[z+\rho\sqrt{s}]} \right) \left(\frac{1}{1+e^{-[z+\rho\sqrt{s}]}} - K \right) e^{-\frac{\rho^2}{2}} d\rho \\ &= (1-K) I_1 - K I_2 \end{aligned}$$

with

$$\begin{aligned} I_1 = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{s}}\left[\ln \frac{K}{1-K} - z\right]}^{\infty} e^{\frac{1}{2}[z+\rho\sqrt{s}]} e^{-\frac{\rho^2}{2}} d\rho &= e^{\frac{z}{2}} \cdot e^{\frac{s}{8}} N \left(\frac{1}{\sqrt{s}} \left[z + \ln \frac{1-K}{K} + \frac{s}{2} \right] \right) \\ &:= e^{\frac{z}{2}} \cdot e^{\frac{s}{8}} N_1 \\ I_2 = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{s}}\left[\ln \frac{K}{1-K} - z\right]}^{\infty} e^{-\frac{1}{2}[z+\rho\sqrt{s}]} e^{-\frac{\rho^2}{2}} d\rho &= e^{-\frac{z}{2}} \cdot e^{\frac{s}{8}} N \left(\frac{1}{\sqrt{s}} \left[z + \ln \frac{1-K}{K} - \frac{s}{2} \right] \right) \\ &:= e^{-\frac{z}{2}} \cdot e^{\frac{s}{8}} N_2 . \end{aligned}$$

Therefore:

$$\begin{aligned} u(x,t) = \frac{e^{-\frac{s}{8}}}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} h(z,s) &= (1-K) \underbrace{\frac{e^{\frac{z}{2}}}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}}}_{=x} N_1 - K \underbrace{\frac{e^{-\frac{z}{2}}}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}}}_{=1-x} N_2 \\ &= (1-K) x N \left(\frac{1}{\sqrt{s}} \left[\ln \frac{x(1-K)}{(1-x)K} + \frac{s}{2} \right] \right) \\ &\quad - K(1-x) N \left(\frac{1}{\sqrt{s}} \left[\ln \frac{x(1-K)}{(1-x)K} - \frac{s}{2} \right] \right) \end{aligned}$$

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