MODELING MARKET RISK IN A JUMP-DIFFUSION SETTING
A GENERALIZED HOFMANN-PLATEN-SCHWEIZER-MODEL

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Abstract. We generalize the paper of *Hofmann*, *Platen* and *Schweizer* [HPS92] to jump-diffusion models. First we introduce securities which are replicable in a self-financing way. Then we characterize market risks which are in a special way 'orthogonal' to these securities. Moreover we prove, that every general arbitrage-free security has a unique decomposition into a self-financing replicable security and such a market risk.

Then we discuss the martingale measures for our jump-diffusion model. In particular we examine the minimal equivalent martingale measure and show that in our model the minimal martingale measure is characterized by preserving the market risk processes under a change of measure. But we state also that unlike in the continuous case it does not preserve the orthogonality to the martingale part of the underlyings.

1. Introduction

In 1973 *Black* and *Scholes* [BS73] and *Merton* [Mer73] were the first to give a formula for pricing options and corporate liabilities by arbitrage arguments alone. Since then, this formula is used in practice to approximate the prices of financial products. But only for approximation, because the assumptions underlying the model of *Black*, *Scholes* and *Merton* are very restrictive. In the last decade, the assumption of constant volatility has been relaxed by several authors, for example *Hull* and *White* [HW87], *Wiggins* [Wig87] and others. Also some authors have studied models with market imperfections like transaction costs, different interest rates for borrowing and lending or the impact of taxes and other market constraints. *Merton* [Mer76] first generalized the *Black-Scholes-Merton*-formula to models allowing the stock prices to jump. This generalization is motivated in the literature by the arrival of new information outside the usual trading-information. These models are also used to explain fat-tails and kurtosis in asset-price distributions. After *Merton* [Mer76] only few authors have dealed with so called jump- or jump-diffusion-models. Some of these are *Aase* [Aas88], who examined the completeness of such markets, or *Bardhan* and *Chao* [BC93, BC95], who studied those models with respect to portfolio optimization problems.

Stochastic volatility models as well as jump-diffusion-models in most cases lead to incomplete markets. In such models it is no longer possible to price a contingent claim, like an option, by arbitrage arguments only. This results in the problem of finding on the one hand a replicating strategy with minimal 'risk' (leaving open the question of what 'risk' is) and on the other hand among the infinitely many pricing measures (called the martingale measures) we have to find an 'optimal' one, where again the term 'optimal' needs to be defined. The question of strategies with minimal 'risk' was answered in a very intuitive way by *Föllmer* and *Sondermann* [FS86] and *Schweizer* [Sch91]. Corresponding with their approach of so called '(*locally*) *risk-minimizing strategies*' *Föllmer* and *Schweizer* [FS91] introduced the '*minimal martingale measure*'. This is characterized by the condition of

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preserving the martingale property of martingales orthogonal to the underlyings. This rather technical concept can be interpreted in finance as leaving the risks which cannot be hedged by trading strategies in the underlying securities unpriced under the pricing measure. It can be proved that the minimal martingale measure does not only preserve the martingale property of orthogonal martingales but also the orthogonality if - and that is the decisive restriction - the underlying securities have continuous paths. In such models, it is possible to calculate the locally risk-minimizing strategy by changing the measure to the minimal martingale measure, then calculating the unique and always existing risk-minimizing strategy. This strategy equals the locally risk-minimizing strategy since it is characterized by a *cost process* which is an orthogonal martingale (see *Schweizer* [Sch91]). Motivated by this characterization, *Hofmann*, *Platen* and *Schweizer* [HPS92] generalized this concept in a multidimensional diffusion-risk-setting. They characterize general market risks which are not tradable, as so called '*totally untradable assets*'. These assets or securities are in a special way orthogonal to the underlying securities. Furthermore, they characterize all securities which are replicable by a self-financing strategy, as '*purely tradable assets*'. Then they proved that the minimal martingale measure is characterized by the condition of leaving the processes of totally untradable assets invariant under a change of measure.

By a simple counterexample *Schweizer* [Sch89] pointed out that the orthogonality-preserving property no longer holds if the underlying processes are discontinuous. We will show that it is still possible to generalize the notation of *Hofmann*, *Platen* and *Schweizer* [HPS92] to jump-diffusion-models. We characterize the minimal martingale measure as the only martingale measure leaving the processes of totally untradable assets unchanged after the change of measure. Nevertheless these securities are no longer orthogonal in the usual martingale-sense. To get our characterization of the minimal martingale measure in a jump-diffusion setting, we show that the *Radon-Nikodym*-density of every martingale measure can be decomposed into the sum of the density of the minimal equivalent martingale measure and a density orthogonal (in the martingale sense) to the minimal martingale density. A similar result was already proved in a very general setting by *Christopeit*, *Musiela* [CM92], *Schweizer* [Sch92a, Sch94a], *Ansel*, *Stricker* [AS92] or by *Bardhan*, *Chao* [BC96] for marked point processes.

In section 2 we will define the market model and (for clarification) discuss the jump parameters in more detail. Section 3 introduces the notion of strategies and the equivalent concept of a portfolio-process. So we can define securities which are replicable in a self-financing manner. In section 4 we discuss the equivalent martingale measures of our model and study more general securities than those mentioned above. Section 5 will contain the proof of our main result, after we study the minimal equivalent martingale measure. Section 6 concludes and in the appendix we give a detailed description of some formulas.

2. The Model

Like an example in *Schweizer* [Sch93a, Sch94a], we consider a multi-dimensional jump-diffusion-model. Let S be a d-dimensional stochastic process, describing the price-processes of d nonredundant securities, called the *stocks* of our market model. These processes are described by the stochastic differential equations

$$dS_t^{(i)} = S_{t-}^{(i)}\left(\mu_t^{(i)}dt + \sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}dW_t^{(j)} + \sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}d\bar{N}_t^{(k)}\right), \quad \text{for } t\in[0,T] \text{ and } i=1,...,d, \tag{2.1}$$

with $S_0^{(i)}>0$, $i=1,...,d$, P-a.s. Here W is a m-dimensional *Brownian* motion and $\bar{N}$ the compensated process of a k-variate point process with (deterministic) intensity ν, i. e.

$$\bar{N}_t^{(k)} = N_t^{(k)} - \int_0^t \nu_u^{(k)} du, \quad \text{for } t \in [0,T] \text{ and } k = 1,...,n. \tag{2.2}$$

All processes are defined on a probability space $(\Omega, \mathfrak{F}, P)$ equipped with the augmented filtration $(\mathfrak{F}_t)_{t\in[0,T]}$ generated by W and N. It is known that W and N are independent (*Ikeda*, *Watanabe* [IW81], Theorem II.6.3). The market coefficients μ, σ and ρ are assumed to be $(\mathfrak{F}_t)$-predictable and (for simplicity) P-a.s. bounded uniformly in t and ω. Furthermore we need a *money market account* (also called a *bond*) B as numeraire which we define by

$$dB_t = B_t r_t dt, \quad \text{for } t \in [0,T], \tag{2.3}$$

where r is the deterministic interest rate. Since we only want to consider discounted securities, we set

$$B_t = 1, \quad \text{for } t \in [0,T]. \tag{2.4}$$

Before we study the model in detail, let us briefly examine the jump part of the underlying securities:
Like the the instantaneous standard deviation of the continuous part of the return of S is σ (called the volatility), ρ describes the instantaneous standard deviation of the jump part of the return. But it is also natural to interpret ρ as the jump size of the stock, if a jump event occurs.

For simplicity take $\mu = 0$, $m = 0$ and $n = 1$. Then we can write S as

$$dS_t = S_{t-}\rho_t d\bar{N}_t, \quad \text{for } t \in [0,T]. \tag{2.5}$$

We know that the jump size of N is 1, if a jump event occurs. Otherwise N is constant. Let ρ_t' be the relative jump size of S, if a jump event occurs, i. e. if $t = T_n$. Here $\{T_n \,|n \in I\!N\}$ is the sequence of jump times. Then the price of S in t can be written as

$$S_t = S_{t-}\rho_t', \quad \text{for } t \in \{T_n \,|n \in I\!N\}. \tag{2.6}$$

The jump size of S at a jump-time t is given by

$$\Delta S_t = (S_t - S_{t-})\,\Delta N_t = S_{t-}\left(\rho_t' - 1\right)\Delta N_t, \quad \text{for } t \in [0,T]. \tag{2.7}$$

If we compare (2.5) and (2.7), we obtain

$$\rho_t = \rho_t' - 1, \quad \text{for } t \in [0,T]. \tag{2.8}$$

To avoid negative prices in our model (2.1), we must assume that ρ' is always positive (see (2.6)). So we make the additional (and usual) assumption

$$\rho_t^{(i,k)} > -1, \quad P\text{-a.s. for } t \in [0,T] \text{ and } i = 1,...,d; k = 1,...,n. \tag{2.9}$$

We also need σ to be strictly positive, i. e.

$$\sigma_t^{(i,j)} > 0, \quad P\text{-a.s. for } t \in [0,T] \text{ and } i = 1,...,d; j = 1,...,m. \tag{2.10}$$

Furthermore we assume the components of the $(P, \mathfrak{F}_t)$-intensity ν of N to be strictly positive and uniformly bounded in t.

To guarantee the absence of arbitrage in our model, we define the $d \times m + n$-valued matrix-process Σ by

$$\Sigma_t := \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \right), \quad \text{for } t \in [0,T], \tag{2.11}$$

where the product $(\cdot|\cdot)$ will be explained and motivated in section 3 (see (3.19)). Then, P-a.s. there exists the inverse Σ^{-1} and the process $\Sigma^{-1}\mu$ is P-a.s. uniformly bounded in t and ω. As we will see in section 4, this condition guarantees the existence of an equivalent martingale measure in our setting.

We define $x \cdot y$ as the coordinatewise product of vectors (respectively matrices) x and y, i. e.

$$(x \cdot y)^{(k)} = x^{(k)} y^{(k)}, \quad \text{for } k = 1, ..., n. \tag{2.12}$$

Furthermore we let $*$ denote transposition of a transposed vector or matrix. We assume Σ to be nondegenerate. Later we also need the notation 1_n, denoting the n-dimensional vector with components 1. Now we can write (2.1) in a simplifying vector-style[(1)]:

$$dS_t = S_{t-} \cdot \left(\mu_t dt + \sigma_t dW_t + \rho_t d\bar{N}_t \right), \quad \text{for } t \in [0,T]. \tag{2.13}$$

Next we need S to be a square-integrable special P-semimartingale. To avoid arbitrage, it is necessary for S to be a semimartingale (see *Delbaen*, *Schachermayer* [DS94], Theorem 7.2, p.504). Obviously the finite variation part A of S, defined by

$$dA_t^{(i)} = S_{t-}^{(i)} \mu_t^{(i)} dt, \quad \text{for } t \in [0,T] \text{ and } i = 1, ..., d, \tag{2.14}$$

is a predictable (in particular continuous) square-integrable process of finite variation, if we assume $S_-^{(i)} \mu^{(i)}$ to be *Lebesgue*-integrable with

$$E\left(\left(\int_0^t S_{u-}^{(i)} \mu_u^{(i)} du \right)^2 \right) < \infty, \quad \text{for } t \in [0,T] \text{ and } i = 1, ..., d. \tag{2.15}$$

Furthermore we define the (local) martingale part M of S by its continuous part M^c and its purely discontinuous part M^d :

$$\begin{aligned} M_t^{(i)} &= \left(M_t^{(i)} \right)^c + \left(M_t^{(i)} \right)^d \\ &= \int_0^t S_{u-}^{(i)} \sum\nolimits_{j=1}^m \sigma_u^{(i,j)} dW_u^{(j)} + \int_0^t S_{u-}^{(i)} \sum\nolimits_{k=1}^n \rho_u^{(i,k)} d\bar{N}_u^{(k)}, \\ &\quad \text{for } t \in [0,T] \text{ and } i = 1, ..., d. \end{aligned} \tag{2.16}$$

Therefore we assume σ and ρ to satisfy

$$E\left(\int_0^t \sum\nolimits_{j=1}^m \left(S_{u-}^{(i)} \sigma_u^{(i,j)} \right)^2 du \right) < \infty \tag{2.17}$$

and

$$E\left(\int_0^t \sum\nolimits_{k=1}^n \left(S_{u-}^{(i)} \rho_u^{(i,k)} \right)^2 \nu_u^{(k)} du \right) < \infty,$$

[(1)]Note that we only consider componentwise defined vector-valued *Itô*-Integrals in the sense of *Chatelain*, *Stricker* [CS94].

$$\text{for } t \in [0,T] \text{ and } i = 1,...,d^{(2)}. \tag{2.18}$$

Then because of *Protter* [Pro90], Chapter IV, Lemma, p. 142, M^c and M^d are (locally) square-integrable (local) martingales. In section 5 we need A to be absolutely continuous with respect to $\langle M, M\rangle$, i. e. there exists a predictable process α with

$$dA_t^{(i)} = \alpha_t^{(i)} d\left\langle M^{(i)}, M^{(i)}\right\rangle_t, \quad \text{for } t \in [0,T] \text{ and } i = 1,...,d. \tag{2.19}$$

In our model α is easily determined:

$$\begin{aligned} dA_t^{(i)} &= \alpha_t^{(i)} d\left\langle M^{(i)}, M^{(i)}\right\rangle_t \\ &= S_{t-}^{(i)}\alpha_t^{(i)}\left(S_{t-}^{(i)}\left(\sum\nolimits_{j=1}^m \left(\sigma_t^{(i,j)}\right)^2 + \sum\nolimits_{k=1}^n \left(\rho_t^{(i,k)}\right)^2 \nu_t^{(k)}\right)\right) dt \\ &= S_{t-}^{(i)}\mu_t^{(i)} dt, \quad \text{for } t \in [0,T] \text{ and } i = 1,...,d. \end{aligned} \tag{2.20}$$

Thus α must satisfy the equations

$$\alpha_t^{(i)}\left(S_{t-}^{(i)}\left(\sum\nolimits_{j=1}^m \left(\sigma_t^{(i,j)}\right)^2 + \sum\nolimits_{k=1}^n \left(\rho_t^{(i,k)}\right)^2 \nu_t^{(k)}\right)\right) = \mu_t^{(i)}, \qquad \text{for } t \in [0,T] \text{ and } i = 1,...,d, \tag{2.21}$$

and so it is determined by

$$\alpha_t^{(i)} = \mu_t^{(i)}\left(S_{t-}^{(i)}\left(\sum\nolimits_{j=1}^m \left(\sigma_t^{(i,j)}\right)^2 + \sum\nolimits_{k=1}^n \left(\rho_t^{(i,k)}\right)^2 \nu_t^{(k)}\right)\right)^{-1}, \qquad \text{for } t \in [0,T] \text{ and } i = 1,...,d, \tag{2.22}$$

where we use the nondegeneracy of the matrix Σ and the positivity (of every component) of S and ν for every $t \in [0,T]$.

Next we determine a solution to (2.1) (or equivalently (2.13)). Therefore let X be the vector valued process defined by

$$dX_t^{(i)} = \mu_t^{(i)} dt + \sum\nolimits_{j=1}^m \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^n \rho_t^{(i,k)} d\bar{N}_t^{(k}, \qquad \text{for } t \in [0,T] \text{ and } i = 1,...,d, \tag{2.23}$$

with $X_{0-}^{(i)} = 0$ for $i = 1,...,d$. From *Elliott* [Ell82], Theorem 13.5, p. 156 a solution of (2.1) is given by

$$\begin{aligned} S_t^{(i)} =& S_{0-}^{(i)} \exp\left(X_t^{(i)} - \frac{1}{2}\left\langle X^{c(i)}, X^{c(i)}\right\rangle_t\right) \prod\nolimits_{0\le u\le t}\left(1 + \Delta X_u^{(i)}\right) \exp\left(-\Delta X_u^{(i)}\right) \\ =& S_{0-}^{(i)} \exp\left(\int_0^t \mu_u^{(i)} du + \sum\nolimits_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} - \frac{1}{2}\sum\nolimits_{j=1}^m \int_0^t \left(\sigma_u^{(i,j)}\right)^2 du \right. \\ & \left. - \sum\nolimits_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du\right) \prod\nolimits_{0\le u\le t}\left(\sum\nolimits_{k=1}^n \rho_u'^{(i,k)} \Delta N_u^{(k)}\right), \end{aligned}$$

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(2) Note that an *Itô*-Integral $\int \xi dM$ with respect to a square-integrable martingale M is also a square-integrable martingale, if ξ is predictable and satisfies $E\left(\int_0^T \xi_u^2 d\langle M, M\rangle_u\right) < \infty$.

Let M be a local martingale, then it is sufficient to assume that ξ is predictable and locally bounded for $\int \xi dM$ to be a local martingale.

For $\int \xi dM$ to be a locally square-integrable local martingale, where M is a locally square-integrable local martingale, it is sufficient to assume that ξ is predictable and satisfies $P\left(\int_0^T \xi_u^2 d\langle M, M\rangle_u < \infty\right) = 1$.

for $t \in [0,T]$ and $i = 1, ..., d$. (2.24)

In the last equation we use the fact that the $N^{(k)}$ have no common jumps P-a.s.

As we see, this is a generalization of what is known to be a geometric *Brownian* motion like in the *Black-Scholes* model or a geometric *Poisson* process (see for instance *Müller* [Mül85], p.28 and p.72ff).

At the end of this section, we briefly denote some special cases of our model.

For $n = 0$ we get the model of *Hofmann*, *Platen* and *Schweizer* [HPS92]. If $n = 0$, $d = m = 1$, and the market coefficients μ and σ are constant we get the *Black-Scholes* model. For $d = n = m = 1$ and constant market coefficients, our model coincides with the model of *Merton* [Mer76]. Furthermore see *Schweizer* [Sch93a, Sch94a], where other special cases are discussed.

As a rule, pure diffusion models are *complete*, if the number of sources of uncertainty coincide with the number of nonredundant securities, i. e. $d = m + n$. In the case of jump-diffusion-models this is only true if the jump sizes are predictable (see *Bardhan*, *Chao* [BC96]). Otherwise, for nonlinear payoff structures, there is no possibility to replicate a contingent claim in a self-financing way. But our model assumes predictable jump sizes, and so we have a complete market model if $d = m + n$.

If $d > m+n$ then, because of the nondegeneracy of Σ, the market model has to contain redundant securities which can be omitted without losing information about the market structure. Otherwise, because of the informal equivalence of the existence of an equivalent martingale measure and the no-arbitrage condition (see *Delbaen*, *Schachermayer* [DS94] [DS97]), there would be arbitrage opportunities in our model.

If we have more sources of uncertainty than nonredundant securities ($d < m + n$), our market is *incomplete*. One way to prove this is via the existence of infinitely many equivalent martingale measures, as we will see in section 4. In such a model, we have the problem to choose an intuitive and/or 'efficient' equivalent martingale measure. We solve this problem in section 5, where we discuss the minimal equivalent martingale measure as one intuitive solution to this problem.

3. Strategies and self-financing replicable securities

This section introduces the notation necessary to describe the trading of agents in our market model. First we introduce hedging strategies as usually done in the literature of option pricing. Then we define portfolio strategies which are used in the literature of portfolio-optimization-problems.

Definition 3.1.
A trading *(or* hedging*)* strategy *is a pair* (ξ, η) *of processes with*

1. ξ *is* $\mathbb{R}^d$*-valued and* $(\mathfrak{F}_t)$*-predictable,*
2. η *is* $\mathbb{R}$*-valued and* $(\mathfrak{F}_t)$*-adapted,*
3. $E\left(\int_0^T \left(\xi_u^{(i)}\right)^2 d\left\langle M^{(i)}, M^{(i)}\right\rangle_u\right) < \infty,$ *for* $i = 1, ..., d$,
4. $E\left(\left(\int_0^T \left|\xi_u^{(i)}\mu_u^{(i)}\right| du\right)^2\right) < \infty,$ *for* $i = 1, ..., d$,
5. *The value process* $V(\varphi)$*, defined by*

$$V_t(\varphi) := \sum\nolimits_{i=1}^d \xi_t^{(i)} S_t^{(i)} + \eta_t = \xi_t^* S_t + \eta_t, \quad \text{for } t \in [0,T], \tag{3.1}$$

has càdlàg-paths and is in $\mathfrak{L}^2(P)$*, i. e.* $V_t(\varphi) \in \mathfrak{L}^2(P)$ *for all* $t \in [0,T]$.

The above definition of a hedging strategy was first introduced by *Schweizer* [Sch91] for semimartingales S. We take this special case from *Colwell*, *Elliott* [CE93], who examine the case of a jump-diffusion model, where the jumps are described by a general marked point process. Here ξ_t is the amount of stocks the agent holds at t, whereas η_t is the amount of money in the money market accounts. The agent's gains and losses from using a strategy are described by the so-called *gains process* $G(\varphi)$,

$$G_t(\varphi) := \int_0^t \xi_u^* dS_u, \quad \text{for } t \in [0,T]. \tag{3.2}$$

In general, the value process and the gains process of a strategy are different, i. e.

$$V_t(\varphi) = G_t(\varphi) + C_t(\varphi), \quad \text{for } t \in [0,T]. \tag{3.3}$$

Here $C(\varphi)$ is the *cost process* of the strategy φ, i. e. the money the agent needs to invest or to consume, such that the strategy has a certain value which can be specified by the value of a contingent claim. A *contingent claim* is a (positive) random variable $H \in \mathfrak{L}^2(P)$, describing the payoff of a contingent security. Usually we consider a European call option H, i. e. the payoff $(S_T - K)^+$ at time T, but it is also possible to consider other types of derivative securities.

In the following context a strategy is always used to hedge a contingent claim H. We say a hedging strategy *replicates* (or *duplicates*) the contingent claim H, if $H = V_T(\varphi)$, P-a.s. So it is always possible to find a replicating strategy for every contingent claim H: take any strategy up to time $T-$ and then pay the difference between the contingent claim H and $V_T(\varphi)$ (or take the money, if the difference is negative).

If an agent, investor or a company (like a bank) use strategies to hedge against the risk of selling or buying a contingent claim H, a strategy as described above is clearly not very useful. It is well known that in a complete market model it is possible to replicate every contingent claim H with a so called *self-financing strategy*. For such a strategy the cost process $C(\varphi)$ is constant, i. e.

$$C_t(\varphi) = C_0(\varphi), \quad \text{for } t \in [0,T]. \tag{3.4}$$

In this case

$$V_t(\varphi) = C_0(\varphi) + G_t(\varphi) = V_0(\varphi) + \int_0^t \xi_u^* dS_u, \quad \text{for } t \in [0,T], \tag{3.5}$$

and

$$H = V_T(\varphi) = C_0(\varphi) + G_T(\varphi) = V_0(\varphi) + \int_0^T \xi_u^* dS_u. \tag{3.6}$$

If we discount this and take the expectation under a certain martingale measure, the value of H at $t = 0$ is $V_0(\varphi)$ (see section 4). As noted, such a strategy is called self-financing, because it has neither inflows not outflows before the trading horizont.

In incomplete market models it is not possible to find a self-financing replicating strategy for every contingent claim H. In these models the cost process is generally nonconstant. Therefore *Föllmer* and *Sondermann* [FS86] introduce a broader class of strategies useful for hedging. These strategies are called *mean-self-financing*, because their cost process is a P-martingale, implying

$$E(C_t(\varphi)) = C_0(\varphi), \quad \text{for } t \in [0,T]. \tag{3.7}$$

Since there are many (mean-self-financing) strategies replicating a contingent claim, one is interested in finding an 'optimal' strategy. Therefore *Föllmer* and *Sondermann* [FS86] define the optimization problem:

Minimize $E\left((C_T(\varphi) - C_t(\varphi))^2 \mid \mathfrak{F}_t\right)$ for all $t \in [0,T]$ over all replicating strategies for H.

They show that in the case of S being a P-martingale, such a strategy always exists and is uniquely determined. Such a strategy is called a *risk-minimizing strategy* and the process $R(\varphi)$ defined by

$$R_t(\varphi) := E\left((C_T(\varphi) - C_t(\varphi))^2 \mid \mathfrak{F}_t\right), \quad \text{for } t \in [0,T], \tag{3.8}$$

is denoted the *risk process* of the strategy φ. In general, when S is only a P-semimartingale, such a strategy does not need to exist. In this case *Schweizer* [Sch91] introduces the notion of a *locally risk-minimizing strategy.* The exact definition is rather technical, but *Schweizer* [Sch91] proves that such strategies can be characterized by a cost process which is orthogonal (in the *Hilbert* space sense of the space of square-integrable P-martingales) to the martingale part M of S.

Another optimization problem also first noted by *Föllmer* and *Sondermann* [FS86] (see also *Schweizer* and others [Sch92b, Sch93a, Sch93b, Sch94a, Sch94c, Sch94b, Sch95, PRS97, DMS+97], *Duffie* and *Richardson* [DR91], *Monat* and *Stricker* [MC95]), is the so called *mean-variance-hedging* approach. Here the problem is:

Minimize $E\left((H - V_0(\varphi) - G_T(\varphi))^2\right)$ over all self-financing trading strategies φ.

Another variation of this approach is the problem:

Minimize $E\left((H - c - G_T(\varphi))^2\right)$ over all self-financing trading strategies φ with fixed initial value c.

Such strategies are called *variance-minimizing.*

As proved by *Schweizer* [Sch91] [Sch93a], a strategy solving one of these problems is necessarily mean-self-financing. In the last two problems mentioned above this means $E(H - V_0(\varphi) - G_T(\varphi)) = 0$ (resp. $E(H - c - G_T(\varphi)) = 0$). One should also note that a variance-minimizing strategy cannot be risk-minimizing.

Now we introduce another notion of hedging strategies, usually used in the literature of portfolio optimization problems. Here one is not interested in hedging against a contingent claim but to maximize the consumption and the payoff of a strategy.

Definition 3.2.
A portfolio process π *is a* $\mathbb{R}^d$*-valued* $(\mathfrak{F}_t)$*-predictable process with*

1. $E\left(\int_0^T \left(\pi_u^{(i)}\right)^2 \sum_{j=1}^m \left(\sigma_u^{(i,j)}\right)^2 du\right) < \infty$, *for* $i = 1, ..., d$,
2. $E\left(\int_0^T \left(\pi_u^{(i)}\right)^2 \sum_{k=1}^n \left(\rho_u^{(i,k)}\right)^2 \nu_u^{(k)} du\right) < \infty$, *for* $i = 1, ..., d$,
3. $E\left(\left(\int_0^T \left|\pi_u^{(i)} \mu_u^{(i)}\right| du\right)^2\right) < \infty$, *for* $i = 1, ..., d$.

By $V(\pi)$ we denote the *value process* of a portfolio process π. This shall be a right-continuous square-integrable adapted process. As above, this is the value of the portfolio of an agent following π, where $\pi_t^{(i)}$ is the money invested in the i-th stock at time t. Then the value invested in the money account B at time t is $V_t(\pi) - \sum_{i=1}^d \pi_t^{(i)}$.

Lemma 3.1.
The definitions 3.1 and 3.2 are equivalent if one sets

$$\pi_t^{(i)} = S_{t-}^{(i)} \xi_t^{(i)}, \quad \text{for } t \in [0,T]. \tag{3.9}$$

Proof.
Clearly π defined by (3.9) is predictable, if and only if ξ is predictable.
Next we show that (1) and (2) of definition 3.1 is equivalent to definition 3.2, (3):

$$\begin{aligned} & E\left(\int_0^T \left(\pi_u^{(i)}\right)^2 \sum_{j=1}^m \left(\sigma_u^{(i,j)}\right)^2 du\right) \\ &= \left(\int_0^T \left(\xi_u^{(i)} S_{u-}^{(i)}\right)^2 \sum_{j=1}^m \left(\sigma_u^{(i,j)}\right)^2 du\right) \\ &= E\left(\int_0^T \left(\xi_u^{(i)}\right)^2 \sum_{j=1}^m \left(S_{u-}^{(i)} \sigma_u^{(i,j)}\right)^2 du\right) \\ &= E\left(\int_0^T \left(\xi_u^{(i)}\right)^2 d\left\langle \left(M^{(i)}\right)^c, \left(M^{(i)}\right)^c \right\rangle\right), \quad \text{for } i = 1, ..., d \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} & E\left(\int_0^T \left(\pi_u^{(i)}\right)^2 \sum_{k=1}^n \left(\rho_u^{(i,k)}\right)^2 \nu_u^{(k)} du\right) \\ &= E\left(\int_0^T \left(\xi_u^{(i)} S_{u-}^{(i)}\right)^2 \sum_{k=1}^n \left(\rho_u^{(i,k)}\right)^2 \nu_u^{(k)} du\right) \\ &= E\left(\int_0^T \left(\xi_u^{(i)}\right)^2 \sum_{k=1}^n \left(S_{u-}^{(i)} \rho_u^{(i,k)}\right)^2 \nu_u^{(k)} du\right) \\ &= E\left(\int_0^T \left(\xi_u^{(i)}\right)^2 d\left\langle \left(M^{(i)}\right)^d, \left(M^{(i)}\right)^d \right\rangle\right), \quad \text{for } i = 1, ..., d. \end{aligned} \tag{3.11}$$

Then (3.10) and (3.11) are finite, if and only if definition 3.1, (3) holds.
Next we compare definition 3.1, (4) and definition 3.2, (3):

$$E\left(\left(\int_0^T \left|\pi_u^{(i)} \mu_u^{(i)}\right| du\right)^2\right) = E\left(\left(\int_0^T \left|\xi_u^{(i)} S_{u-}^{(i)} \mu_u^{(i)}\right| du\right)^2\right) < \infty, \quad \text{for } i = 1, ..., d. \tag{3.12}$$

Since S is square-integrable and finite on $[0, T]$, (3.12) holds if and only if

$$E\left(\left(\int_0^T \left|\xi_u^{(i)} \mu_u^{(i)}\right| du\right)^2\right) < \infty, \quad \text{for } i = 1, ..., d. \tag{3.13}$$

By definition the value processes of both kinds are right-continuous, square-integrable and adapted, in particular if we note that η is defined by

$$\eta_t = V_t(\pi) - \sum_{i=1}^d \pi_t^{(i)}, \quad \text{for } i = 1, ..., d, \tag{3.14}$$

and so it is adapted. □

So we can use the terms portfolio process and hedging strategy synonymously. A portfolio process is self-financing, if its value process fulfils

$$\begin{aligned} V_t(\varphi) &= V_0(\xi) + \int_0^t \xi_u^* dS_u \\ &= V_0(\xi) + \int_0^t \xi_u^* S_{u-} \cdot \mu_u du + \int_0^t \xi_u^* S_{u-} \cdot (\sigma_u dW_u) + \int_0^t \xi_u^* S_{u-} \cdot \left(\rho_u d\bar{N}_u\right) \end{aligned}$$

$$= V_0(\pi) + \int_0^t \pi_u^* \mu_u du + \int_0^t \pi_u^* \sigma_u dW_u + \int_0^t \pi_u^* \rho_u d\bar{N}_u,$$
$$\text{for } t \in [0,T]. \tag{3.15}$$

Motivated by the work of *Hofmann*, *Platen* and *Schweizer* [HPS92] and (3.15), we define securities which can be replicated by a self-financing portfolio process (or hedging strategy). *Hofmann*, *Platen* and *Schweizer* [HPS92] called these securities '*purely tradable assets*'. One could also call them *redundant* in the market model or, as we will do, *self-financing replicable* which is more precise in an economic sense.

Definition 3.3.
A security F is self-financing replicable, *if there exists a self-financing portfolio process π with*

$$F_t = V_0(\pi) + \int_0^t \pi_u^* \mu_u du + \int_0^t \pi_u^* \sigma_u dW_u + \int_0^t \pi_u^* \rho_u d\bar{N}_u, \quad \text{for } t \in [0,T]. \tag{3.16}$$

It is well known that in complete markets every security (including the underlyings) is self-financing replicable. But we are interested in considering incomplete markets, where we also have securities which are not self-financing replicable. To characterize these, we need the following discussion:
Let Π be the space of $\mathbb{R}^{m+n}$-valued predictable processes ζ with

$$E\left(\int_0^t \sum_{j=1}^m \left(\zeta_u^{(j)}\right)^2 d\left\langle W^{(j)}, W^{(j)}\right\rangle_u\right) + E\left(\int_0^t \sum_{k=1}^n \left(\zeta_u^{(m+k)}\right)^2 d\left\langle \bar{N}^{(k)}, \bar{N}^{(k)}\right\rangle_u\right)$$
$$= E\left(\int_0^t \sum_{j=1}^m \left(\zeta_u^{(j)}\right)^2 du\right) + E\left(\int_0^t \sum_{k=1}^n \left(\zeta_u^{(m+k)}\right)^2 \nu_u^{(k)} du\right) < \infty,$$
$$\text{for } t \in [0,T], \tag{3.17}$$

and define the following product on Π:

$$\widetilde{(\zeta_t \,|\vartheta_t)} = \left\langle \int_0^\cdot \sum_{j=1}^m \zeta_u^{(j)} dW_u^{(j)}, \int_0^\cdot \sum_{j=1}^m \vartheta_u^{(j)} dW_u^{(j)} \right\rangle_t$$
$$+ \left\langle \int_0^\cdot \sum_{k=1}^n \zeta_u^{(m+k)} d\bar{N}_u^{(k)}, \int_0^\cdot \sum_{k=1}^n \vartheta_u^{(m+k)} d\bar{N}_u^{(k)} \right\rangle_t$$
$$= \int_0^t \sum_{j=1}^m \zeta_u^{(j)} \vartheta_u^{(j)} du + \int_0^t \sum_{k=1}^n \zeta_u^{(m+k)} \vartheta_u^{(m+k)} \nu_u^{(k)} du,$$
$$\text{for } t \in [0,T] \text{ and } \zeta, \vartheta \in \Pi. \tag{3.18}$$

Here we use explicitly the orthogonality of W and $\bar{N}$, allowing us to drop terms involving the covariation of W with $\bar{N}$.

Motivated by the last definition, we define the following product,

$$(\zeta_t \,|\vartheta_t) = \sum_{j=1}^m \zeta_u^{(j)} \vartheta_u^{(j)} + \sum_{k=1}^n \zeta_u^{(m+k)} \vartheta_u^{(m+k)} \nu_u^{(k)}, \quad \text{for } t \in [0,T] \text{ and } \zeta, \vartheta \in \Pi. \tag{3.19}$$

This product is a modified vector-multiplication - modified in the sense that the product of the the last k-components of ζ and ϑ, are extended by the intensity ν. Using this last notation, we can prove the following

Lemma 3.2.
$(\Pi, (\cdot\,|\cdot))$ *is a Hilbert-space.*

Proof.
First let us observe that Π is a $\mathbb{R}$-vector-space. Since $\mathbb{R}^{m+n}$ is a $\mathbb{R}$-vector-space, and the

sum of predictable processes is predictable, it is sufficient to check only the integrability-condition.
Let $\zeta, \vartheta \in \Pi$, then

$$
\begin{aligned}
& E\left(\int_0^t \sum_{j=1}^m \left((\zeta+\vartheta)_u^{(j)}\right)^2 du\right) + E\left(\int_0^t \sum_{k=1}^n \left((\zeta+\vartheta)_u^{(m+k)}\right)^2 \nu_u^{(k)} du\right) \\
= & E\left(\int_0^t \sum_{j=1}^m \left(\zeta_u^{(j)}\right)^2 + 2\zeta_u^{(j)}\vartheta_u^{(j)} + \left(\vartheta_u^{(j)}\right)^2 du\right) \\
& + E\left(\int_0^t \sum_{k=1}^n \left(\left(\zeta_u^{(m+k)}\right)^2 + 2\zeta_u^{(m+k)}\vartheta_u^{(m+k)} + \left(\vartheta_u^{(m+k)}\right)^2\right) \nu_u^{(k)} du\right) \\
= & E\left(\int_0^t \sum_{j=1}^m \left(\zeta_u^{(j)}\right)^2 du\right) + E\left(\int_0^t \sum_{k=1}^n \left(\zeta_u^{(m+k)}\right)^2 \nu_u^{(k)} du\right) \\
& + E\left(\int_0^t \sum_{j=1}^m \left(\vartheta_u^{(j)}\right)^2 du\right) + E\left(\int_0^t \sum_{k=1}^n \left(\vartheta_u^{(m+k)}\right)^2 \nu_u^{(k)} du\right) \\
& + 2E\left(\int_0^t \sum_{j=1}^m \zeta_u^{(j)}\vartheta_u^{(j)} du\right) + 2E\left(\int_0^t \sum_{k=1}^n \zeta_u^{(m+k)}\vartheta_u^{(m+k)}\nu_u^{(k)} du\right), \\
& \text{for } t \in [0,T]. \qquad (3.20)
\end{aligned}
$$

The first two lines in this sum are finite, since $\zeta, \vartheta \in \Pi$. The third line is also finite, because of the *Kunita-Watanabe-* inequality:

$$
\begin{aligned}
& \int_0^t \left|\zeta_u^{(j)}\vartheta_u^{(j)}\right| du \\
= & \int_0^t \left|\zeta_u^{(j)}\vartheta_u^{(j)}\right| d\left\langle W^{(j)}, W^{(j)}\right\rangle_u \\
\leq & \left(\int_0^t \left(\zeta_u^{(j)}\right)^2 d\left\langle W^{(j)}, W^{(j)}\right\rangle_u\right)^{\frac{1}{2}} \left(\int_0^t \left(\vartheta_u^{(j)}\right)^2 d\left\langle W^{(j)}, W^{(j)}\right\rangle_u\right)^{\frac{1}{2}} \\
= & \left(\int_0^t \left(\zeta_u^{(j)}\right)^2 du\right)^{\frac{1}{2}} \left(\int_0^t \left(\vartheta_u^{(j)}\right)^2 du\right)^{\frac{1}{2}}, \\
& \text{for } t \in [0,T] \text{ and } j = 1, ..., m; P\text{-a.s.}, \qquad (3.21)
\end{aligned}
$$

respectively

$$
\begin{aligned}
& \int_0^t \left|\zeta_u^{(k)}\vartheta_u^{(k)}\right| \nu_u^{(k)} du \\
= & \int_0^t \left|\zeta_u^{(k)}\vartheta_u^{(k)}\right| d\left\langle \bar{N}^{(k)}, \bar{N}^{(k)}\right\rangle_u \\
\leq & \left(\int_0^t \left(\zeta_u^{(k)}\right)^2 d\left\langle \bar{N}^{(k)}, \bar{N}^{(k)}\right\rangle_u\right)^{\frac{1}{2}} \left(\int_0^t \left(\vartheta_u^{(k)}\right)^2 d\left\langle \bar{N}^{(k)}, \bar{N}^{(k)}\right\rangle_u\right)^{\frac{1}{2}} \\
= & \left(\int_0^t \left(\zeta_u^{(k)}\right)^2 \nu_u^{(k)} du\right)^{\frac{1}{2}} \left(\int_0^t \left(\vartheta_u^{(k)}\right)^2 \nu_u^{(k)} du\right)^{\frac{1}{2}}, \\
& \text{for } t \in [0,T] \text{ and } k = 1, ..., n; P\text{-a.s.} \qquad (3.22)
\end{aligned}
$$

So we get $\zeta + \vartheta \in \Pi$.
Clearly the 0-process is the 0-vector in Π and the additive inverse of $\zeta \in \Pi$ is $-\zeta$ and is also in Π. Now let $c \in R$ and $\zeta \in \Pi$, then

$$
\begin{aligned}
&E\left(\int_0^t \sum_{j=1}^m \left((c\zeta)_u^{(j)}\right)^2 du\right) + E\left(\int_0^t \sum_{k=1}^n \left((c\zeta)_u^{(m+k)}\right)^2 \nu_u^{(k)} du\right) \\
&= c^2 E\left(\int_0^t \sum_{j=1}^m \left(\zeta_u^{(j)}\right)^2 du\right) + c^2 E\left(\int_0^t \sum_{k=1}^n \left(\zeta_u^{(m+k)}\right)^2 \nu_u^{(k)} du\right) < \infty, \\
&\text{for } t \in [0,T], \qquad (3.23)
\end{aligned}
$$

so $c\zeta \in \Pi$. The other conditions are easy to check, since we just need the conditions of the vector-space $I\!R^{m+n}$.
Since $(\cdot\,|\cdot)$ is just a modified vector-multiplication in $I\!R^{m+n}$, it is easy to check that $(\cdot\,|\cdot)$ is a scalar-product on Π. For example consider

$$
\begin{aligned}
&(\zeta_t + \vartheta_t \,|\chi_t) \\
&= \sum_{j=1}^m \left(\zeta_u^{(j)} + \vartheta_u^{(j)}\right) \chi_u^{(j)} + \sum_{k=1}^n \left(\zeta_u^{(m+k)} + \vartheta_u^{(m+k)}\right) \chi_u^{(m+k)} \nu_u^{(k)} \\
&= \sum_{j=1}^m \zeta_u^{(j)} \chi_u^{(j)} + \sum_{j=1}^m \vartheta_u^{(j)} \chi_u^{(j)} \\
&\quad + \sum_{k=1}^n \zeta_u^{(m+k)} \chi_u^{(m+k)} \nu_u^{(k)} + \sum_{k=1}^n \vartheta_u^{(m+k)} \chi_u^{(m+k)} \nu_u^{(k)} \\
&= (\zeta_t \,|\chi_t) + (\vartheta_t \,|\chi_t), \quad \text{for } t \in [0,T] \text{ and } \zeta, \vartheta, \chi \in \Pi. \qquad (3.24)
\end{aligned}
$$

□

If ζ is a matrix-valued process with rows in Π, we define in analogy to (3.19):

$$
(\zeta_t \,|\vartheta_t) = \begin{pmatrix} \left(\zeta_t^{(1)} \,|\vartheta_t\right) \\ \vdots \\ \left(\zeta_t^{(d)} \,|\vartheta_t\right) \end{pmatrix} \qquad (3.25)
$$

Let ζ and ϑ be matrices with rows in Π, then we define

$$
(\zeta_t \,|\vartheta_t) = \begin{pmatrix} \left(\zeta_t^{(1)} \middle| \vartheta_t^{(1)}\right) & \cdots & \left(\zeta_t^{(1)} \middle| \vartheta_t^{(d)}\right) \\ \vdots & \ddots & \vdots \\ \left(\zeta_t^{(d)} \middle| \vartheta_t^{(1)}\right) & \cdots & \left(\zeta_t^{(d)} \middle| \vartheta_t^{(d)}\right) \end{pmatrix} \qquad (3.26)
$$

Next we define the 'kernel' and the 'range' of the matrix $\begin{pmatrix} \sigma & \rho \end{pmatrix}$ resp. $\begin{pmatrix} \sigma & \rho \end{pmatrix}^*$:

$$
K\begin{pmatrix} \sigma & \rho \end{pmatrix} := \left\{\vartheta \,\middle|\, \vartheta \in \Pi, \left(\begin{matrix} \sigma_t & \rho_t \end{matrix} \,\middle|\vartheta_t\right) = 0,\, t \in [0,T],\, P-\text{a.s.}\right\} \qquad (3.27)
$$

$$
R\left(\begin{pmatrix} \sigma & \rho \end{pmatrix}^*\right) := \left\{\chi \,\middle|\, \chi \in \Pi, \exists\theta \text{ portfolio process}, \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}^* \theta_t = \chi_t, t \in [0,T],\, P-\text{a.s.}\right\} \qquad (3.28)
$$

Then $K\begin{pmatrix} \sigma & \rho \end{pmatrix}$ and $R\left(\begin{pmatrix} \sigma & \rho \end{pmatrix}^*\right)$ yield an orthogonal decomposition of Π with respect to $(\cdot\,|\cdot)$. Let $\vartheta \in K\begin{pmatrix} \sigma & \rho \end{pmatrix}$ and $\chi \in R\left(\begin{pmatrix} \sigma & \rho \end{pmatrix}^*\right)$. Since Π is a *Hilbert*-space, every element has a orthogonal decomposition and it holds

$$
\begin{aligned}
(\vartheta_t \,|\chi_t) &= \sum_{j=1}^m \vartheta_u^{(j)} \chi_u^{(j)} + \sum_{k=1}^n \vartheta_u^{(m+k)} \chi_u^{(m+k)} \nu_t^k \\
&= \sum_{j=1}^m \vartheta_u^{(j)} \left(\sum_{i=1}^d \sigma_u^{(i,j)} \theta_u^{(i)}\right) + \sum_{k=1}^n \vartheta_u^{(m+k)} \nu_u^{(k)} \left(\sum_{i=1}^d \rho_u^{(i,k)} \theta_u^{(i)}\right)
\end{aligned}
$$

$$= \sum_{i=1}^{d} \theta_u^{(i)} \sum_{j=1}^{m} \sigma_u^{(i,j)} \vartheta_u^{(j)} + \sum_{i=1}^{d} \theta_u^{(i)} \sum_{k=1}^{n} \rho_u^{(i,k)} \vartheta_u^{(m+k)} \nu_u^{(k)}$$
$$= \sum_{i=1}^{d} \theta_u^{(i)} \underbrace{\left(\sum_{j=1}^{m} \sigma_u^{(i,j)} \vartheta_u^{(j)} + \sum_{k=1}^{n} \rho_u^{(i,k)} \vartheta_u^{(m+k)} \nu_u^{(k)} \right)}_{=0} = 0,$$
$$\text{for } t \in [0,T], \tag{3.29}$$

where θ is a $I\!R^d$-valued predictable process, such that $\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}^* \theta_t = \chi_t$, $t \in [0,T]$.

As *Karatzas et. al.* [KLSX91] note (in a continuous setting), it is possible to complete a market model, by adding sufficiently many nonredundant 'securities' to the market model. This means we extend the model such that the volatility follows a $(m+n)\times(m+n)$-matrix-valued process, invertible for every $t \in [0,T]$. These nonredundant additional 'securities' have volatility processes in $K\begin{pmatrix} \sigma & \rho \end{pmatrix}$, while all self-financing replicable securities have volatility processes in $R\left(\begin{pmatrix} \sigma & \rho \end{pmatrix}^*\right)$. So we know how to define the volatility of 'securities' orthogonal to the self-financing replicable securities. But it is not clear, how to define the drift of theses processes. To get an idea of this, we need the notation of a martingale measure for S.

4. Martingale measures, arbitrage-free securities, and nonhedgable market risks

By now the equivalence between the existence of a martingale measure and the absence of arbitrage opportunities is well studied, see for instance *Delbaen*, *Schachermayer* [DS94, DS97] and the references therein. An *equivalent martingale measure* is a probability measure $\tilde{P}$ equivalent to P, such that S (i. e. every component of S) is a $\tilde{P}$-martingale (remember that we only consider discounted securities). We distinguish between an equivalent martingale measure and an *equivalent local martingale measure*, under which S is only a local $\tilde{P}$-martingale. *Schweizer* [Sch92a] generalizes these terms and considers only the densities of such measures, and calls them *martingale densities.* In general these are no longer strictly positive, i. e. they define signed measures (see also *Bardhan*, *Chao* [BC96], *Schweizer* [Sch93a, Sch94d, Sch94a, Sch94b, Sch95]).

Before we continue in characterizing securities like in section 3, we study the existence of equivalent martingale measures. Therefore we need a *Girsanov*-type theorem suited for our context:

Theorem 4.1.
Let $\tilde{\lambda}$ be a $I\!R^m$-valued predictable process satisfying

$$E\left(\int_0^T \sum_{j=1}^{m} \left(\tilde{\lambda}_u^{(j)}\right)^2 du\right) < \infty \tag{4.1}$$

and $\tilde{\kappa}$ be a $I\!R^n$-valued predictable process satisfying

$$E\left(\int_0^T \sum_{k=1}^{n} \left(\tilde{\kappa}_u^{(k)} + 1\right)^2 \nu_u^{(k)} du\right) < \infty \tag{4.2}$$

and

$$\tilde{\kappa}_t^{(k)} > -1, \quad \text{for } t \in [0,T] \text{ and } k = 1, ..., n. \tag{4.3}$$

Define $\tilde{Z}$ by

$$d\tilde{Z}_t := \tilde{Z}_{t-} \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix}^* d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T], \tag{4.4}$$

with $\tilde{Z}_0 = 1$. *Suppose furthermore that*

$$E\left(\tilde{Z}_t\right) = 1, \quad \text{for } t \in [0,T]. \tag{4.5}$$

Then $\tilde{Z}$ *is a nonnegative* P*-martingale and* $\tilde{P}$ *defined by*

$$\frac{d\tilde{P}}{dP}|_{\mathfrak{F}_t} = \tilde{Z}_t, \quad \text{for } t \in [0,T], \tag{4.6}$$

is an equivalent probability measure, such that $\tilde{W}$, *given by*

$$\tilde{W}_t = W_t - \int_0^t \tilde{\lambda}_u du, \quad \text{for } t \in [0,T], \tag{4.7}$$

is a $\tilde{P}$*-Brownian motion and* N *has* $\left(\tilde{P}, \mathfrak{F}_t\right)$*-Intensity* $(\tilde{\kappa} + 1_n)$.

Proof.
Use *Karatzas, Shreve* [KS81], Chapter 3, Theorem 5.1, p. 191 and *Brémaud* [Bré81], Chapter VI, Theorem T3, p. 166[(3)] and the orthogonality of W and N[(4)]. □

Now we determine $\tilde{\lambda}$ and $\tilde{\kappa}$, such that S is a (local) $\tilde{P}$-martingale. For this note that if $\tilde{\kappa}'^{(k)}$ is the relative jump size of $\tilde{Z}$ of the k-th type, we can write

$$\tilde{Z}_t = \tilde{Z}_{t-}\left(\sum_{k=1}^n \tilde{\kappa}_t'^{(k)}\right), \quad \text{for } t \in [0,T], \tag{4.8}$$

if a jump occurs, where $\sum_{k=1}^n \tilde{\kappa}^{(k)} = \sum_{k=1}^n \tilde{\kappa}'^{(k)} - 1$ (compare (2.7) and (2.8)). Then we get

$$\begin{aligned} S_t^{(i)}\tilde{Z}_t - S_{t-}^{(i)}\tilde{Z}_{t-} &= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\left(1 + \sum_{k=1}^n \rho_t^{(i,k)}\right)\left(1 + \sum_{k=1}^n \tilde{\kappa}_t^{(k)}\right) - 1\right) \\ &= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\sum_{k=1}^n \rho_t^{(i,k)} + \tilde{\kappa}_t^{(k)} + \rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right), \quad \text{for } t \in [0,T], \end{aligned} \tag{4.9}$$

if a jump occurs. Here we make use of the fact that the different jump-types have no common jumps. In general (to omit the supplement 'if a jump occurs') we write

$$S_t^{(i)}\tilde{Z}_t - S_{t-}^{(i)}\tilde{Z}_{t-} = S_{t-}^{(i)}\tilde{Z}_{t-}\left(\sum_{k=1}^n \left(\rho_t^{(i,k)} + \tilde{\kappa}_t^{(k)} + \rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)\Delta N_t^{(k)}\right), \quad \text{for } t \in [0,T], \tag{4.10}$$

and note:

$$\Delta N^{(k)} = \Delta \bar{N}^{(k)}, \quad \text{for } k = 1, ..., n. \tag{4.11}$$

Theorem 4.2.
The security process S *is a (local)* $\tilde{P}$*-martingale if* $\tilde{\lambda}$ *and* $\tilde{\kappa}$ *solve the linear system of equations*

$$\left(\left(\begin{array}{cc} \sigma_t & \rho_t \end{array}\right) \middle| - \left(\begin{array}{c} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{array}\right)\right) = \mu_t, \quad \text{for } t \in [0,T]. \tag{4.12}$$

---

(3) Note that we use another notation than Brémaud[Bré81]. There the intensity under the new measure is $\tilde{\kappa} \cdot \nu$, while in our model the new intensity is $(\tilde{\kappa} + 1_n) \cdot \nu$. To ensure the positivity of the intensity, we need (4.3). We use this notation, to use $\tilde{\kappa}$ in a similar way to $\tilde{\lambda}$.

(4) Note that we use stronger assumptions on $\tilde{\lambda}$ and $\tilde{\kappa}$ than necessary, since we need them to be in Π.

Proof.
Using *Elliott* [Ell82], Chapter 13, Lemma 13.10, p. 161 it is sufficient and necessary to show that $S\tilde{Z}$ is a (local) P-martingale. Therefore we use the *Itô*-rule with $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \longmapsto xy$ and consider:

$$\begin{aligned} df\left(\tilde{Z}_t, S_t\right) &= S_{t-}d\tilde{Z}_t + \tilde{Z}_{t-}dS_t + d\left\langle \tilde{Z}^c, S^c\right\rangle_t + \left(\tilde{Z}_tS_t - \tilde{Z}_{t-}S_{t-} - S_{t-}\Delta\tilde{Z}_t - \tilde{Z}_{t-}\Delta S_t\right) \\ &= \tilde{Z}_{t-}S_{t-}\cdot\left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} + \rho_t\tilde{\kappa}_t d\bar{N}_t \right. \\ &\quad \left. + \left(\mu_t dt + \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix}\right) dt\right), \quad \text{for } t \in [0,T]. \end{aligned} \tag{4.13}$$

It is well known that under the regularity conditions of the integrands above, $S\tilde{Z}$ is a (local) P-martingale if the last term (the 'dt'-term) vanishes, i. e. if (4.12) holds. □

Since Σ has an inverse (compare (2.11)), (4.12) has at least one solution, given by

$$-\begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} = \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}^* \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}\right)^{-1} \mu_t, \quad \text{for } t \in [0,T]. \tag{4.14}$$

Note that this solution is in $R\left(\begin{pmatrix} \sigma & \rho \end{pmatrix}^*\right)$. If d is less than $m+n$, this solution is not unique, i. e. the space of equivalent (local) martingale measures is infinite.

Remark 4.1.

1. *We call* $-\begin{pmatrix} \tilde{\lambda} \\ \tilde{\kappa} \end{pmatrix}$ *the* market price of risk process *of our market model.*
2. *For* $\rho \equiv 0$ (4.14) *defines the market price of risk process* $-\tilde{\lambda}$ *associated with the minimal (local) martingale measure in a diffusion-model. As we will see* (4.14) *defines also the minimal (local) martingale measure in our jump-diffusion model (see section 5).*

After we have characterized (local) martingale measures in our model and assured their existence, we continue by characterizing general securities.

For the absence of arbitrage, it is necessary for S to be a (special) semimartingale (see *Delbaen* and *Schachermayer* [DS94], Theorem 7.2, p. 504). This suggests the following definition taken from *Hofmann*, *Platen* and *Schweizer* [HPS92].

Definition 4.1.

1. *A* general security *is a security F, with the price process being a P-semimartingale.*
2. *An* arbitrage-free security *is a general security, such that the (discounted) price process is a (local) martingale for some equivalent (local) martingale measure for S.*

In *Hofmann*, *Platen* and *Schweizer* [HPS92] arbitrage-free securities are called *compatible*. Like them, we show in our more general model that every arbitrage-free security has a decomposition into a self-financing replicable security and a part 'orthogonal' to the self-financing replicable part. For this we note that the market price of risk processes defined by (4.12) are elements of Π. Thus every market price of risk process has a representation

$$-\begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} = \chi_t + \vartheta_t, \quad \text{for } t \in [0,T], \tag{4.15}$$

with $\chi \in R\left(\left(\begin{array}{cc}\sigma & \rho\end{array}\right)^*\right)$ and $\vartheta \in K\left(\begin{array}{cc}\sigma & \rho\end{array}\right)$. Since a portfolio process π is a $\mathbb{R}^d$-valued process, such that $\left(\begin{array}{cc}\sigma & \rho\end{array}\right)^* \pi \in \Pi$, we have for every $\chi \in R\left(\left(\begin{array}{cc}\sigma & \rho\end{array}\right)^*\right)$ a portfolio process θ, with $\left(\begin{array}{cc}\sigma & \rho\end{array}\right)^* \theta = \chi$. So we get

$$-\left(\begin{array}{c}\tilde{\lambda}_t\\ \tilde{\kappa}_t\end{array}\right) = \left(\begin{array}{cc}\sigma_t & \rho_t\end{array}\right)^* \theta_t + \vartheta_t, \quad \text{for } t \in [0,T]. \tag{4.16}$$

Multiplying both sides with $\left(\begin{array}{cc}\sigma & \rho\end{array}\right)$ which has full rank d, we get

$$\left(\left(\begin{array}{cc}\sigma_t & \rho_t\end{array}\right) \Big| -\left(\begin{array}{c}\tilde{\lambda}_t\\ \tilde{\kappa}_t\end{array}\right)\right) = \Sigma_t\theta_t = \mu_t, \quad \text{for } t \in [0,T]. \tag{4.17}$$

Since Σ has an inverse, we get

$$\theta_t = \Sigma_t^{-1}\mu_t, \quad \text{for } t \in [0,T]. \tag{4.18}$$

Using these equations, we can prove the following result.

Theorem 4.3.
Every arbitrage-free security F has a representation

$$dF_t = \pi_t^*\mu_t + \pi_t^*\left(\begin{array}{cc}\sigma_t & \rho_t\end{array}\right) d\left(\begin{array}{c}W_t\\ \bar{N}_t\end{array}\right) + (\zeta_t\,|\vartheta_t)\,dt + \zeta_t d\left(\begin{array}{c}W_t\\ \bar{N}_t\end{array}\right), \quad \text{for } t \in [0,T], \tag{4.19}$$

with a portfolio process π and $\zeta, \vartheta \in K\left(\begin{array}{cc}\sigma & \rho\end{array}\right)$. This representation is unique up to additive constants.

Proof.
Since F is a (special) semimartingale, there is a (predictable) process of finite variation $\tilde{A}$ and a (local) P-Martingale $\tilde{M}$, such that

$$dF_t = d\tilde{A}_t + d\tilde{M}_t, \quad \text{for } t \in [0,T]. \tag{4.20}$$

Using the martingale-representation theorem (in the general version for W and $\bar{N}$), there exists a predictable processes $\gamma \in \Pi$, such that $\tilde{M}$ has a representation

$$d\tilde{M}_t = \gamma_t^* d\left(\begin{array}{c}W_t\\ \bar{N}_t\end{array}\right), \quad \text{for } t \in [0,T]. \tag{4.21}$$

Since $\gamma \in \Pi$, there are processes $\chi \in R\left(\left(\begin{array}{cc}\sigma & \rho\end{array}\right)^*\right)$ and $\vartheta \in K\left(\begin{array}{cc}\sigma & \rho\end{array}\right)$ with

$$\gamma_t = \chi_t + \vartheta_t, \quad \text{for } t \in [0,T]. \tag{4.22}$$

Then a portfolio process π exists, such that we can write (4.22) as

$$\gamma_t = \left(\begin{array}{cc}\sigma_t & \rho_t\end{array}\right)^* \pi_t + \vartheta_t, \quad \text{for } t \in [0,T]. \tag{4.23}$$

Then we can rewrite (4.20):

$$dF_t = d\tilde{A}_t + \gamma_t d\left(\begin{array}{c}W_t\\ \bar{N}_t\end{array}\right)$$

$$= d\tilde{A}_t + \pi_t^* \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} + \vartheta_t d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} \quad \text{for } t \in [0,T]. \tag{4.24}$$

Since F is arbitrage-free, there exists a (local) martingale measure $\tilde{P}$ with corresponding market price of risk process $-\begin{pmatrix} \tilde{\lambda} \\ \tilde{\kappa} \end{pmatrix}$, such that F is a (local) $\tilde{P}$-martingale. Thus we can rewrite (4.24):

$$\begin{aligned} dF_t &= d\tilde{A} + \pi_t^* \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} \right) dt + \pi_t^* \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d \begin{pmatrix} W_t - \int_0^t \tilde{\lambda}_u du \\ N_t - \int_0^t (1_k + \tilde{\kappa}_u) \cdot \nu_t du \end{pmatrix} \\ &\quad + \left(\zeta_t \middle| \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} \right) dt + \zeta_t d \begin{pmatrix} W_t - \int_0^t \tilde{\lambda}_u du \\ N_t - \int_0^t (1_k + \tilde{\kappa}_u) \cdot \nu_t du \end{pmatrix} \\ &= d\tilde{A} + \pi_t^*(-\pi) dt + \left(\zeta_t \middle| \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} \right) dt + \pi_t^* \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d \begin{pmatrix} \tilde{W}_t \\ \tilde{\bar{N}}_t \end{pmatrix} + \zeta_t d \begin{pmatrix} \tilde{W}_t \\ \tilde{\bar{N}}_t \end{pmatrix} \\ &\quad \text{for } t \in [0,T]. \end{aligned} \tag{4.25}$$

Since F is a (local) martingale under $\tilde{P}$, the finite variation part of F must vanish. Thus $\tilde{A}$ must satisfy

$$d\tilde{A}_t = \pi_t^* \mu_t dt - \left(\vartheta_t \middle| \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} \right) dt, \quad \text{for } t \in [0,T]. \tag{4.26}$$

Using the decomposition (4.16) of $-\begin{pmatrix} \tilde{\lambda} \\ \tilde{\kappa} \end{pmatrix}$, we get

$$d\tilde{A}_t = \pi_t^* \mu_t dt - (\vartheta_t \,|\zeta_t)\, dt, \quad \text{for } t \in [0,T]. \tag{4.27}$$

Thus using (4.20) and (4.24) we get (4.19).
To prove the uniqueness, assume another representation (4.19) of F with coefficients $\breve{\pi}$, $\breve{\vartheta}$, $\breve{\zeta}$. Then we have

$$\begin{aligned} &\pi_t^* \mu_t + \pi_t^* \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} + (\zeta_t \,|\vartheta_t)\, dt + \zeta_t d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} \\ &= \breve{\pi}_t^* \mu_t + \breve{\pi}_t^* \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} + \left(\breve{\zeta}_t \middle| \breve{\vartheta}_t \right) dt + \breve{\zeta}_t d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T]. \end{aligned} \tag{4.28}$$

which we can simplify to

$$\begin{aligned} &(\pi_t^* - \breve{\pi}_t)\, \mu_t dt + \left((\zeta_t \,|\vartheta_t) - \left(\breve{\zeta}_t \middle| \breve{\vartheta}_t \right) \right) dt \\ &= (\breve{\pi}_t^* - \pi_t)^* \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} + \left(\breve{\zeta}_t - \zeta_t \right)^* d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T]. \end{aligned} \tag{4.29}$$

Since the first line is a process of finite variation and the second line is a (local) P-martingale, the processes must be constant, i. e. the integrands must be zero. Since $\begin{pmatrix} \sigma & \rho \end{pmatrix}$ is not the zero-matrix and has full rank d, we must have

$$(\pi_t - \breve{\pi}_t) = 0, \quad \text{for } t \in [0,T]. \tag{4.30}$$

Then we get immediately

$$\left(\breve{\zeta}_t - \zeta_t \right) = 0, \quad \text{for } t \in [0,T], \tag{4.31}$$

respectively

$$(\zeta_t \,|\vartheta_t\,) - \left(\breve{\zeta}_t \middle| \breve{\vartheta}_t\right) = 0, \quad \text{for } t \in [0,T]. \tag{4.32}$$

So the representations are equal apart from a additive constant. □

Comparing the representation (4.19) with definition 3.3, we see that every arbitrage-free security F has an unique decomposition into a self-financing replicable security, and an additional 'risk-part'. This risk part can be characterized by the fact that it is not possible to find a self-financing portfolio to replicate it with the stocks S and the money market account B. *Hofmann*, *Platen* and *Schweizer* [HPS92] call these risks *totally untradable assets.* Since these risks are more precisely not hedgable in our market model, we call them *unhedgable market risks.*

Definition 4.2.
An unhedgable market risk *is a process R^M, satisfying*

$$dR_t^M = (\zeta_t \,|\vartheta_t\,)\, dt + \zeta_t d \begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T], \tag{4.33}$$

with $\zeta, \vartheta \in K \begin{pmatrix} \sigma & \rho \end{pmatrix}$.

Remark 4.2.

1. *In a continuous setting, an unhedgable market risk is a generalization of the cost process of a (locally) risk-minimizing strategy (see Föllmer and Sondermann* [FS86] *and Schweizer* [Sch91]*), where the cost process is just a P-martingale.*
2. *Theorem 4.3 states that every arbitrage-free security has a unique decomposition into a self-financing replicable security and an unhedgable market risk. In a continuous setting this generalizes the Föllmer-Schweizer-decomposition (see Föllmer and Schweizer* [FS91]*).*

5. The minimal martingale measure and the risk-preserving martingale measure

Like *Hofmann*, *Platen* and *Schweizer* [HPS92] we will characterize the minimal equivalent (local) martingale measure, first defined by *Föllmer* and *Schweizer* [FS91], with respect to unhedgable market risks. An equivalent (local) martingale measure $\widehat{P}$ is minimal, if all (locally) square-integrable (local) P-martingales orthogonal to every component of M (where M is the (local) martingale part of S) are also (local) $\widehat{P}$-martingales. This definition is motivated by the characterization of locally risk-minimizing strategies. Under some regularity conditions, a locally risk-minimizing strategy is characterized by a cost-process which is orthogonal to M (see *Schweizer* [Sch91]). To find a locally risk-minimizing strategy, it is sufficient to find a risk-minimizing strategy under the minimal equivalent martingale measure. The latter strategy always exists, as shown in *Föllmer* and *Sondermann* [FS86] in a continuous setting. Then this strategy equals the locally risk-minimizing strategy under the 'subjective' measure P. *Föllmer* and *Schweizer* [FS91] prove in a continuous setting that the minimal equivalent martingale measure also preserves orthogonality, in the sense that every P-martingale orthogonal to M is a $\widehat{P}$-martingale orthogonal to M under $\widehat{P}$. But *Schweizer* [Sch89] gives an (easy to understand) example, showing that this no longer holds for discontinuous securities-markets. *Hofmann*, *Platen* and *Schweizer* [HPS92] generalize the orthogonality preserving condition to unhedgable market risks (or totally untradable assets). They prove that the minimal equivalent (local) martingale measure is the only martingale measure under which the unhedgable market risks follow the same stochastic differential equation as under the 'subjective' measure. As we will see, in our jump-diffusion-model we can prove a similar result. But in our setting the unhedgable market risks are no longer orthogonal to M.

The existence of the minimal equivalent (local) martingale measure is discussed in the literature in comprehensive manner. *Ansel* and *Stricker* [AS93] show that the minimal equivalent local martingale measure exists, if (2.19) holds with

$$P\left(\alpha_t^{(i)}\Delta M_t^{(i)} > -1\right) = 1 \quad \text{for } t \in [0,T] \text{ and } i = 1, ..., d. \tag{5.1}$$

Otherwise the martingale density $\widehat{Z} = \frac{d\widehat{P}}{dP}$ is no longer positive and the minimal equivalent local martingale measure does not exist. *Schweizer* [Sch89] proves that the minimal equivalent martingale measure exists, if and only if (2.19) holds and the process $\widehat{Z}$, defined by

$$d\widehat{Z}_t = -\widehat{Z}_{t-}\alpha_t^* dM_t, \quad \text{for } t \in [0,T], \tag{5.2}$$

is a square-integrable positive P-martingale where α must perform

$$E\left(\int_0^T \left|\alpha_u^{(i)}\right| \left(\ln\left|\alpha_u^{(i)}\right|\right)^+ d\left\langle M^{(i)}, M^{(i)}\right\rangle_u\right) < \infty \tag{5.3}$$

In all these cases, the minimal equivalent (local) martingale measure $\widehat{P}$ is unique, whenever it exists.

As we have seen in section 4, with the stated assumptions of our model, an equivalent (local) martingale measure exists. It is now easy to prove that also the minimal equivalent martingale measure exists.

Theorem 5.1.
The minimal equivalent (local) martingale measure exists and is defined by its density $\hat{Z}$ with respect to P, following

$$d\widehat{Z}_t = \widehat{Z}_{t-}\begin{pmatrix} \widehat{\lambda}_t \\ \widehat{\kappa}_t \end{pmatrix}^* d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T], \tag{5.4}$$

where $-\begin{pmatrix} \widehat{\lambda}_t \\ \widehat{\kappa}_t \end{pmatrix}$ *is given by* (4.14).

Proof.
Let L be a square-integrable P-martingale orthogonal to every component of M. Then there exists a process $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \Pi$, such that L has the representation

$$dL_t = \begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix}^* d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T]. \tag{5.5}$$

Then we get

$$\begin{aligned} d\langle M, L\rangle_t &= d\left\langle \int_0^\cdot S_{u-}\cdot\begin{pmatrix} \sigma_u & \rho_u \end{pmatrix} d\begin{pmatrix} W_u \\ \bar{N}_u \end{pmatrix}, \int_0^\cdot \begin{pmatrix} \phi_u \\ \psi_u \end{pmatrix}^* d\begin{pmatrix} W_u \\ \bar{N}_u \end{pmatrix}\right\rangle_t \\ &= S_{t-}\cdot\left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}\middle|\begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix}\right) dt \\ &= 0, \quad \text{for } t \in [0,T], \end{aligned} \tag{5.6}$$

i. e. L is orthogonal if $\begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix} \in K\begin{pmatrix} \sigma & \rho \end{pmatrix}$.

Now let $\widehat{Z}$ be defined by (4.14) and consider

$$\begin{aligned} d\left(L_t\widehat{Z}_t\right) &= L_{t-}d\widehat{Z}_t + \widehat{Z}_{t-}dL_t + d\left\langle L^c, \widehat{Z}^c\right\rangle_t \\ &\quad + \left(L_t\widehat{Z}_t - L_{t-}\widehat{Z}_{t-} - L_{t-}\Delta\widehat{Z}_t - \widehat{Z}_{t-}\Delta L_t\right) \\ &= \widehat{Z}_{t-}\left(L_{t-}\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix}^* d\begin{pmatrix}W_t \\ \bar{N}_t\end{pmatrix} + \begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix}^* d\begin{pmatrix}W_t \\ \bar{N}_t\end{pmatrix}\right. \\ &\quad \left. + (\psi_t \cdot \widehat{\kappa}_t)^* d\bar{N}_t + \left(\begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix}\middle|\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix}\right)dt\right), \quad \text{for } t \in [0,T], \end{aligned} \tag{5.7}$$

Since L is orthogonal to every component of M, i. e. (5.6) holds, and since $-\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix} \in R\left(\begin{pmatrix}\sigma & \rho\end{pmatrix}^*\right)$, the last part of (5.7) vanishes, and so L is also a $\widehat{P}$-martingale.
The uniqueness of $\widehat{P}$ follows from the uniqueness of the decomposition (4.15) for every market price of risk process $-\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix}$. □

In the case of continuous price-processes, the minimal equivalent (local) martingale measure preserves the orthogonality of P-martingales like L. In our model this is no longer true. For this consider

$$\begin{aligned} dL_t &= \begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix}^* d\begin{pmatrix}W_t \\ \bar{N}_t\end{pmatrix} \\ &= \begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix}^* d\begin{pmatrix}\widehat{W}_t \\ \widehat{\bar{N}}_t\end{pmatrix} + \left(\begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix}\middle|\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix}\right)dt, \quad \text{for } t \in [0,T], \end{aligned} \tag{5.8}$$

where we use again (5.5) and $\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix} \in R\left(\begin{pmatrix}\sigma & \rho\end{pmatrix}^*\right)$. Then we get under $\widehat{P}$

$$\begin{aligned} d\langle M, L\rangle_t &= d\left\langle \int_0^\cdot S_{u-}\cdot\begin{pmatrix}\sigma_u & \rho_u\end{pmatrix} d\begin{pmatrix}\widehat{W}_u \\ \widehat{\bar{N}}_u\end{pmatrix}, \int_0^\cdot \begin{pmatrix}\phi_u \\ \psi_u\end{pmatrix}^* d\begin{pmatrix}\widehat{W}_u \\ \widehat{\bar{N}}_u\end{pmatrix}\right\rangle_t \\ &= S_{t-}\cdot\left(\begin{pmatrix}\sigma_t & \rho_t\end{pmatrix}\middle|\begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix}\right)dt + (\rho_t\psi_t)\cdot\widehat{\kappa}_t\cdot\nu_t dt, \quad \text{for } t \in [0,T], \end{aligned} \tag{5.9}$$

where we use again $\begin{pmatrix}\widehat{\lambda}_t \\ \widehat{\kappa}_t\end{pmatrix} \in R\left(\begin{pmatrix}\sigma & \rho\end{pmatrix}^*\right)$. So L is under $\widehat{P}$ orthogonal to M, if the last term vanishes. Since we assume $(\rho_t\psi_t)\cdot\nu_t = 0$, (5.9) generally does not vanish. It would vanish for example for constant $\widehat{\kappa}_t$, but this is a rather uninteresting case.

Remark 5.1.

1. *We conclude that there exists no equivalent (local) martingale measure* $\widetilde{P}$*, such that any orthogonal* P*-martingale* L *defined by* (5.5) *and* (5.6) *is also an orthogonal* $\widetilde{P}$*-martingale, because the minimal equivalent (local) martingale measure is the only equivalent (local) martingale measure preserving the martingale-property.*
2. *Note that* L *defined by* (5.6) *is also an unhedgable market risk in the sense of definition 4.2, since* $\begin{pmatrix}\phi_t \\ \psi_t\end{pmatrix} \in K\begin{pmatrix}\sigma & \rho\end{pmatrix}$.

Now we prove our main result which generalizes a similar result in the diffusion-model of *Hofmann*, *Platen* and *Schweizer* [HPS92].

Theorem 5.2.
The minimal equivalent (local) martingale measure $\widehat{P}$ is the only martingale measure, leaving the processes of unhedgable market risks invariant under a measure-transformation.

Proof.
Let $\bar{P}$ be an equivalent (local) martingale measure, such that an unhedgable market risk R^M follows the same stochastic differential equation under P and $\bar{P}$. Then there exist ζ, $\vartheta \in K\begin{pmatrix} \sigma & \rho \end{pmatrix}$, such that

$$dR_t^M = (\zeta_t \,|\vartheta_t)\, dt + \zeta_t^* d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix}, \quad \text{for } t \in [0,T], \tag{5.10}$$

respectively

$$dR_t^M = (\zeta_t \,|\vartheta_t)\, dt + \zeta_t^* d\begin{pmatrix} \bar{W}_t \\ \bar{\bar{N}}_t \end{pmatrix}, \quad \text{for } t \in [0,T], \tag{5.11}$$

where $\bar{W}$ is a *Brownian motion* under $\bar{P}$, defined by

$$\bar{W}_t = W_t - \int_0^t \bar{\lambda}_u du, \quad \text{for } t \in [0,T], \tag{5.12}$$

$\bar{\bar{N}}$ is the compensated *Poisson* process defined by

$$\bar{\bar{N}}_t = N_t - \int_0^t (1_k + \bar{\kappa}_u) \cdot \nu_u du, \quad \text{for } t \in [0,T], \tag{5.13}$$

and $-\begin{pmatrix} \bar{\lambda}_t \\ \bar{\kappa}_t \end{pmatrix}$ is the associated market price of risk process of $\bar{P}$.
Substituting (5.12) and (5.13) in (5.11), we get

$$dR_t^M = (\zeta_t \,|\vartheta_t)\, dt + \zeta_t^* d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} - \left(\zeta_t \middle| \begin{pmatrix} \bar{\lambda}_t \\ \bar{\kappa}_t \end{pmatrix}\right) dt, \quad \text{for } t \in [0,T]. \tag{5.14}$$

Recall that the market price of risk process has a decomposition

$$-\begin{pmatrix} \bar{\lambda}_t \\ \bar{\kappa}_t \end{pmatrix} = \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}^* \bar{\theta}_t + \bar{\vartheta}_t, \quad \text{for } t \in [0,T], \tag{5.15}$$

with a portfolio process $\bar{\theta}$ and $\bar{\vartheta} \in K\begin{pmatrix} \sigma & \rho \end{pmatrix}$. Then (5.14) equals

$$dR_t^M = (\zeta_t \,|\vartheta_t)\, dt + \zeta_t^* d\begin{pmatrix} W_t \\ \bar{N}_t \end{pmatrix} + \left(\zeta_t \,|\bar{\vartheta}_t\right) dt, \quad \text{for } t \in [0,T]. \tag{5.16}$$

Since we have $\bar{\vartheta}$, $\zeta \in K\begin{pmatrix} \sigma & \rho \end{pmatrix}$, the only (local) martingale measure $\bar{P}$ which preserves (5.10) and (5.11), has a market price of risk process with $\bar{\vartheta} = 0$, P-a.s, i. e. $\bar{P} = \widehat{P}$. □

The last result seems to be a contradiction to the condition that in jump-diffusion models the orthogonality of martingales is not preserved by a measure transformation. But the next result will clear this point.

Theorem 5.3.
Under the minimal martingale measure, the unhedgable market risks are in general not orthogonal to M.

Proof.
Let R^M be an unhedgable market risk under $\widehat{P}$ and let $M^{(i)}$ be the martingale part of $S^{(i)}$ under $\widehat{P}$. Then

$$\begin{aligned} d\left\langle M, R^M\right\rangle_t &= d\left\langle \int_0^\cdot S_{u-}\cdot\begin{pmatrix}\sigma_u & \rho_u\end{pmatrix} d\begin{pmatrix}\widehat{W}_u\\ \widehat{N}_u\end{pmatrix}, \int_0^\cdot (\zeta_u\,|\,\vartheta_u)\,du + \int_0^\cdot \zeta_u^* d\begin{pmatrix}\widehat{W}_u\\ \widehat{N}_u\end{pmatrix}\right\rangle_t \\ &= S_{t-}\cdot\begin{pmatrix}0 & \rho_t\end{pmatrix}\zeta_t\cdot\widehat{\kappa}_t\cdot\nu_t, \quad \text{for } t\in[0,T]. \end{aligned} \tag{5.17}$$

Again the last term does not vanish in general, and so in general the unhedgable market risk R^M is not orthogonal to M under $\widehat{P}$. □

6. Conclusion

We show that it is possible, to generalize the paper of *Hofmann*, *Platen* and *Schweizer* [HPS92] to jump-diffusion-models. Similar to them, we model securities which are replicable in a self-financing way and market risks which can not hedged by the underlying securities. Then we define arbitrage-free securities and prove that they can be decomposed into a self-financing replicable security and an unhedgable market risk.

Furthermore we study the equivalent (local) martingale measures in our model, and characterize them by the their associated market price of risk process. By doing this, we introduce the notion of the minimal martingale measure. We prove that this is characterized by leaving the unhedgable market risks invariant under a change of measure. So we generalize a similar result in *Hofmann*, *Platen* and *Schweizer* [HPS92]. Furthermore we state that this is no contradiction to the well known property of the minimal martingale measure that it does not preserve orthogonality in discontinuous models. We prove that the unhedgable market risks are in general not orthogonal to the (local) martingale part of S under the minimal (local) martingale measure.

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Appendix A. Formulas

Equation (2.20):

$$\begin{aligned} dA_t^{(i)} &= \alpha_t^{(i)} d\left\langle M^{(i)}, M^{(i)}\right\rangle_t \\ &= \alpha_t^{(i)}\left(d\left\langle \left(M^{(i)}\right)^c, \left(M^{(i)}\right)^c\right\rangle_t + d\left\langle \left(M^{(i)}\right)^d, \left(M^{(i)}\right)^d\right\rangle_t\right) \\ &= \alpha_t^{(i)}\left(\left(S_{t-}^{(i)}\right)^2 \sum_{j=1}^m \left(\sigma_t^{(i,j)}\right)^2 dt + \left(S_{t-}^{(i)}\right)^2 \sum_{k=1}^n \left(\rho_t^{(i,k)}\right)^2 \nu_t^{(k)} dt\right) \\ &= S_{t-}^{(i)}\alpha_t^{(i)}\left(S_{t-}^{(i)}\left(\sum_{j=1}^m \left(\sigma_t^{(i,j)}\right)^2 + \sum_{k=1}^n \left(\rho_t^{(i,k)}\right)^2 \nu_t^{(k)}\right)\right) dt \\ &= S_{t-}^{(i)}\mu_t^{(i)} dt, \quad \text{for } t \in [0,T] \text{ and } i = 1, ..., d. \end{aligned} \tag{A.1}$$

Equation (2.24):

$$\begin{aligned} S_t^{(i)} = \quad & S_{0-}^{(i)} \exp\left(X_t^{(i)} - \frac{1}{2}\left\langle X^{c(i)}, X^{c(i)}\right\rangle_t^{(i,i)}\right) \prod_{0\le u\le t}\left(1+\Delta X_u^{(i)}\right)\exp\left(-\Delta X_u^{(i)}\right) \\ = \quad & S_{0-}^{(i)} \exp\left(X_{0-}^{(i)} + \int_0^t \mu_u^{(i)} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} + \sum_{k=1}^n \int_0^t \rho_u^{(i,k)} d\bar{N}_u^{(k)}\right. \\ & - \frac{1}{2}\left\langle \int_0^t \mu_u^{(i)} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} - \sum_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du,\right. \\ & \left.\left. \int_0^t \mu_u^{(i} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} - \sum_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du\right\rangle_t\right) \\ & \prod_{0\le u\le t}\left(1+\sum_{k=1}^n \rho_u^{(i,k)}\Delta\bar{N}_u^{(k)}\right)\exp\left(-\sum_{k=1}^n \rho_u^{(i,k)}\Delta\bar{N}_u^{(k)}\right) \\ = \quad & S_{0-}^{(i)} \exp\left(\int_0^t \mu_u^{(i)} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} + \sum_{k=1}^n \int_0^t \rho_u^{(i,k)} dN_u^{(k)}\right. \\ & \left. - \sum_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du - \frac{1}{2}\sum_{j=1}^m \int_0^t \left(\sigma_u^{(i,j)}\right)^2 du\right) \\ & \prod_{0\le u\le t}\left(1+\sum_{k=1}^n \rho_u^{(i,k)}\Delta N_u^{(k)}\right)\prod_{0\le u\le t}\exp\left(-\sum_{k=1}^n \rho_u^{(i,k)}\Delta N_u^{(k)}\right) \\ = \quad & S_{0-}^{(i)} \exp\left(\int_0^t \mu_u^{(i)} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} - \frac{1}{2}\sum_{j=1}^m \int_0^t \left(\sigma_u^{(i,j)}\right)^2 du\right. \\ & \left. - \sum_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du\right)\prod_{0\le u\le t}\left(1+\sum_{k=1}^n \rho_u^{(i,k)}\Delta N_u^{(k)}\right) \\ = \quad & S_{0-}^{(i)} \exp\left(\int_0^t \mu_u^{(i)} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} - \frac{1}{2}\sum_{j=1}^m \int_0^t \left(\sigma_u^{(i,j)}\right)^2 du\right. \\ & \left. - \sum_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du\right)\prod_{0\le u\le t}\left(1+\sum_{k=1}^n \left(\rho_u'^{(i,k)}-1\right)\Delta N_u^{(k)}\right) \\ = \quad & S_{0-}^{(i)} \exp\left(\int_0^t \mu_u^{(i)} du + \sum_{j=1}^m \int_0^t \sigma_u^{(i,j)} dW_u^{(j)} - \frac{1}{2}\sum_{j=1}^m \int_0^t \left(\sigma_u^{(i,j)}\right)^2 du\right. \\ & \left. - \sum_{k=1}^n \int_0^t \rho_u^{(i,k)}\nu_u^{(k)} du\right)\prod_{0\le u\le t}\left(\sum_{k=1}^n \rho_u'^{(i,k)}\Delta N_u^{(k)}\right), \\ & \text{for } t \in [0,T] \text{ and } i = 1, ..., d. \end{aligned} \tag{A.2}$$

Equation (4.9):

$$
\begin{aligned}
S_t^{(i)}\tilde{Z}_t - S_{t-}^{(i)}\tilde{Z}_{t-} &= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\left(1+\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}\right)\left(1+\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}\right)-1\right)\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\left(1+\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}+\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}+\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}-1\right)\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}+\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}+\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}+\tilde{\kappa}_t^{(k)}+\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right), \quad \text{for } t\in[0,T], \qquad \text{(A.3)}
\end{aligned}
$$

Equation (4.13):

$$
\begin{aligned}
df\left(\tilde{Z}_t, S_t^{(i)}\right) &= S_{t-}^{(i)}d\tilde{Z}_t + \tilde{Z}_{t-}dS_t^{(i)} + d\left\langle \tilde{Z}^c, S^{c(i)}\right\rangle_t\\
&\quad + \left(\tilde{Z}_tS_t^{(i)} - \tilde{Z}_{t-}S_{t-}^{(i)} - S_{t-}^{(i)}\Delta\tilde{Z}_t - \tilde{Z}_{t-}\Delta S_t^{(i)}\right)\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\sum\nolimits_{j=1}^{m}\tilde{\lambda}_t^{(j)}dW_t^{(j)} + S_{t-}^{(i)}\tilde{Z}_{t-}\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}d\bar{N}_t^{(k)}\\
&\quad +\tilde{Z}_{t-}S_{t-}^{(i)}\mu_t^{(i)}dt + \tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}dW_t^{(j)}\\
&\quad +\tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}d\bar{N}_t^{(k)} + \tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}\tilde{\lambda}_t^{(j)}dt\\
&\quad +\tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{k=1}^{n}\left(\rho_t^{(i,k)}+\tilde{\kappa}_t^{(k)}+\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)dN_t^{(k)}\\
&\quad -S_t^{(i)}\tilde{Z}_{t-}\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}\Delta N_t^{(k)} - \tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}\Delta N_t^{(k)}\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\sum\nolimits_{j=1}^{m}\tilde{\lambda}_t^{(j)}dW_t^{(j)} - S_{t-}^{(i)}\tilde{Z}_{t-}\sum\nolimits_{k=1}^{n}\tilde{\kappa}_t^{(k)}\nu_t^{(k)}dt\\
&\quad +\tilde{Z}_{t-}S_{t-}^{(i)}\mu_t^{(i)}dt + \tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}dW_t^{(j)}\\
&\quad -\tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}\nu_t^{(k)}dt + \tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}\tilde{\lambda}_t^{(j)}dt\\
&\quad +\tilde{Z}_{t-}S_{t-}^{(i)}\sum\nolimits_{k=1}^{n}\left(\rho_t^{(i,k)}+\tilde{\kappa}_t^{(k)}+\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)dN_t^{(k)}\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\sum\nolimits_{j=1}^{m}\left(\sigma_t^{(i,j)}+\tilde{\lambda}_t^{(j)}\right)dW_t^{(j)}\right.\\
&\quad +S_{t-}^{(i)}\tilde{Z}_{t-}\sum\nolimits_{k=1}^{n}\left(\rho_t^{(i,k)}+\tilde{\kappa}_t^{(k)}+\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)d\bar{N}_t^{(j)}\\
&\quad +\left(\mu_t^{(i)}+\sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}\tilde{\lambda}_t^{(j)}+\right.\\
&\quad \left.\left.+\sum\nolimits_{k=1}^{n}\left(-\rho_t^{(i,k)}\nu_t^{(k)}-\tilde{\kappa}_t^{(k)}\nu_t^{(k)}+\left(\rho_t^{(i,k)}+\tilde{\kappa}_t^{(k)}+\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)\nu_t^{(k)}\right)\right)dt\right)\\
&= S_{t-}^{(i)}\tilde{Z}_{t-}\left(\sum\nolimits_{j=1}^{m}\left(\sigma_t^{(i,j)}+\tilde{\lambda}_t^{(j)}\right)dW_t^{(j)}\right.\\
&\quad +S_{t-}^{(i)}\tilde{Z}_{t-}\sum\nolimits_{k=1}^{n}\left(\rho_t^{(i,k)}+\tilde{\kappa}_t^{(k)}+\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\right)d\bar{N}_t^{(j)}\\
&\quad \left.+\left(\mu_t^{(i)}+\sum\nolimits_{j=1}^{m}\sigma_t^{(i,j)}\tilde{\lambda}_t^{(j)}+\sum\nolimits_{k=1}^{n}\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\nu_t^{(k)}\right)dt\right),\\
&\quad \text{for } t\in[0,T]. \qquad \text{(A.4)}
\end{aligned}
$$

Equation (4.15) equals the following equations:

$$
-\tilde{\lambda}_t^{(j)} = \chi_t^{(j)} + \vartheta_t^{(j)}, \quad \text{for } t\in[0,T] \text{ and } j=1,...,m, \qquad \text{(A.5)}
$$

and

$$-\tilde{\kappa}_t^{(k)} = \chi_t^{(m+k)} + \vartheta_t^{(m+k)}, \quad \text{for } t \in [0,T] \text{ and } k = 1,...,n, \tag{A.6}$$

Equation (4.16) equals the following equations:

$$-\tilde{\lambda}_t^{(j)} = \sum\nolimits_{i=1}^{d} \sigma_t^{(i,j)} \theta_t^{(i)} + \vartheta_t^{(j)}, \quad \text{for } t \in [0,T] \text{ and } j = 1,...,m \tag{A.7}$$

and

$$-\tilde{\kappa}_t^{(j)} = \sum\nolimits_{i=1}^{d} \rho_t^{(i,k)} \theta_t^{(i)} + \vartheta_t^{(m+k)} \quad \text{for } t \in [0,T] \text{ and } k = 1,...,n. \tag{A.8}$$

Equation (4.17):

$$\begin{aligned}
&\left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| - \begin{pmatrix} \tilde{\lambda}_t \\ \tilde{\kappa}_t \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}^* \theta_t \right) + \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \zeta_t \right) \\
&= \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix}^* \theta_t \right) \\
&= \left(\begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \middle| \begin{pmatrix} \sigma_t & \rho_t \end{pmatrix} \right) \theta_t \\
&= \Sigma_t \theta_t = \mu_t, \quad \text{for } t \in [0,T].
\end{aligned} \tag{A.9}$$

Equation (4.19):

$$\begin{aligned}
dF_t = &\sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} dt + \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} d\bar{N}_t^{(k)} \right) \\
&+ \sum\nolimits_{j=1}^{m} \zeta_t^{(j)} \vartheta_t^{(j)} dt + \sum\nolimits_{k=1}^{n} \zeta_t^{(m+k)} \vartheta_t^{(m+k)} \upsilon_t^{(k)} dt \\
&+ \sum\nolimits_{j=1}^{m} \zeta_t^{(j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \zeta_t^{(m+k)} d\bar{N}_t^{(k)}, \quad \text{for } t \in [0,T],
\end{aligned} \tag{A.10}$$

Equation (4.21):

$$d\tilde{M}_t = \sum\nolimits_{j=1}^{m} \gamma_t^{(j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \gamma_t^{(m+k)} d\bar{N}_t^{(k)}, \quad \text{for } t \in [0,T]. \tag{A.11}$$

Equation (4.23):

$$\gamma_t^{(j)} = \sum\nolimits_{i=1}^{d} \sigma_t^{(i,j)} \pi_t^{(i)} + \vartheta_t^{(j)}, \quad \text{for } t \in [0,T] \text{ and } j = 1,...,m \tag{A.12}$$

Equation (4.24):

$$\begin{aligned}
dF_t &= d\tilde{A}_t + \sum\nolimits_{j=1}^{m} \gamma_t^{(j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \gamma_t^{(m+k)} d\bar{N}_t^{(k)} \\
&= d\tilde{A}_t + \sum\nolimits_{j=1}^{m} \left(\sum\nolimits_{i=1}^{d} \sigma_t^{(i,j)} \pi_t^{(i)} + \vartheta_t^{(j)} \right) dW_t^{(j)} \\
&\quad + \sum\nolimits_{k=1}^{n} \left(\sum\nolimits_{i=1}^{d} \rho_t^{(i,k)} \pi_t^{(i)} + \vartheta_t^{(m+kj)} \right) d\bar{N}_t^{(k)} \\
&= d\tilde{A}_t + \sum\nolimits_{j=1}^{m} \sum\nolimits_{i=1}^{d} \sigma_t^{(i,j)} \pi_t^{(i)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \sum\nolimits_{i=1}^{d} \rho_t^{(i,k)} \pi_t^{(i)} d\bar{N}_t^{(k)}
\end{aligned}$$

$$+\sum_{j=1}^{m}\vartheta_t^{(j)}dW_t^{(j)}+\sum_{k=1}^{n}\vartheta_t^{(m+k)}d\bar{N}_t^{(k)}, \quad \text{for } t\in[0,T]. \tag{A.13}$$

Equation (4.25)

$$\begin{aligned}
dF_t &= d\tilde{A}_t+\sum_{j=1}^{m}\sum_{i=1}^{d}\sigma_t^{(i,j)}\pi_t^{(i)}\tilde{\lambda}_t^{(j)}dt\\
&\quad+\sum_{j=1}^{m}\sum_{i=1}^{d}\sigma_t^{(i,j)}\pi_t^{(i)}d\left(W_t^{(j)}-\int_0^t\tilde{\lambda}_u^{(j)}du\right)\\
&\quad+\sum_{k=1}^{n}\sum_{i=1}^{d}\rho_t^{(i,k)}\pi_t^{(i)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt\\
&\quad+\sum_{k=1}^{n}\sum_{i=1}^{d}\rho_t^{(i,k)}\pi_t^{(i)}d\left(\bar{N}_t^{(k)}-\int_0^t\left(\tilde{\kappa}_u^{(k)}+1\right)\upsilon_u^{(k)}du\right)\\
&\quad+\sum_{j=1}^{m}\vartheta_t^{(j)}\widetilde{\lambda}_t^{(j)}dt+\sum_{j=1}^{m}\vartheta_t^{(j)}d\left(W_t^{(j)}-\int_0^t\widetilde{\lambda}_u^{(j)}du\right)\\
&\quad+\sum_{k=1}^{n}\vartheta_t^{(m+k)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt\\
&\quad+\sum_{k=1}^{n}\vartheta_t^{(m+k)}d\left(\bar{N}_t^{(k)}-\int_0^t\left(\tilde{\kappa}_u^{(k)}+1\right)\upsilon_u^{(k)}du\right)\\
&= d\tilde{A}_t+\sum_{j=1}^{m}\sum_{i=1}^{d}\sigma_t^{(i,j)}\pi_t^{(i)}\tilde{\lambda}_t^{(j)}dt\\
&\quad+\sum_{k=1}^{n}\sum_{i=1}^{d}\rho_t^{(i,k)}\pi_t^{(i)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt\\
&\quad+\sum_{k=1}^{n}\vartheta_t^{(m+k)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt+\sum_{j=1}^{m}\vartheta_t^{(j)}\widetilde{\lambda}_t^{(j)}dt\\
&\quad+\sum_{k=1}^{n}\sum_{i=1}^{d}\rho_t^{(i,k)}\pi_t^{(i)}d\tilde{\bar{N}}_t^{(k)}+\sum_{k=1}^{n}\vartheta_t^{(m+k)}d\tilde{\bar{N}}_t^{(k)}\\
&\quad+\sum_{j=1}^{m}\sum_{i=1}^{d}\sigma_t^{(i,j)}\pi_t^{(i)}d\widetilde{W}_t^{(j)}+\sum_{j=1}^{m}\vartheta_t^{(j)}d\widetilde{W}_t^{(j)}\\
&= d\tilde{A}_t+\sum_{i=1}^{d}\pi_t^{(i)}\left(\sum_{j=1}^{m}\sigma_t^{(i,j)}\tilde{\lambda}_t^{(j)}+\sum_{k=1}^{n}\rho_t^{(i,k)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}\right)dt\\
&\quad+\sum_{k=1}^{n}\vartheta_t^{(m+k)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt+\sum_{j=1}^{m}\vartheta_t^{(j)}\widetilde{\lambda}_t^{(j)}dt\\
&\quad+\sum_{k=1}^{n}\sum_{i=1}^{d}\rho_t^{(i,k)}\pi_t^{(i)}d\tilde{\bar{N}}_t^{(k)}+\sum_{k=1}^{n}\vartheta_t^{(m+k)}d\tilde{\bar{N}}_t^{(k)}\\
&\quad+\sum_{j=1}^{m}\sum_{i=1}^{d}\sigma_t^{(i,j)}\pi_t^{(i)}d\widetilde{W}_t^{(j)}+\sum_{j=1}^{m}\vartheta_t^{(j)}d\widetilde{W}_t^{(j)}\\
&= d\tilde{A}_t+\sum_{i=1}^{d}\pi_t^{(i)}\left(-\mu_t^{(i)}\right)dt\\
&\quad+\sum_{k=1}^{n}\vartheta_t^{(m+k)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt+\sum_{j=1}^{m}\vartheta_t^{(j)}\widetilde{\lambda}_t^{(j)}dt\\
&\quad+\sum_{k=1}^{n}\sum_{i=1}^{d}\rho_t^{(i,k)}\pi_t^{(i)}d\tilde{\bar{N}}_t^{(k)}+\sum_{k=1}^{n}\vartheta_t^{(m+k)}d\tilde{\bar{N}}_t^{(k)}\\
&\quad+\sum_{j=1}^{m}\sum_{i=1}^{d}\sigma_t^{(i,j)}\pi_t^{(i)}d\widetilde{W}_t^{(j)}+\sum_{j=1}^{m}\vartheta_t^{(j)}d\widetilde{W}_t^{(j)},\\
&\quad\text{for } t\in[0,T].
\end{aligned} \tag{A.14}$$

Equation (4.26)

$$d\tilde{A}_t=\sum_{i=1}^{d}\pi_t^{(i)}\mu_t^{(i)}dt-\sum_{j=1}^{m}\vartheta_t^{(j)}\tilde{\lambda}_t^{(j)}dt-\sum_{k=1}^{n}\vartheta_t^{(m+k)}\tilde{\kappa}_t^{(k)}\upsilon_t^{(k)}dt, \quad \text{for } t\in[0,T]. \tag{A.15}$$

Equation (4.27):

$$\begin{aligned}
d\tilde{A}_t &= \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} dt + \sum\nolimits_{j=1}^{m} \vartheta_t^{(j)} \sum\nolimits_{i=1}^{d} \sigma_t^{(i,j)} \theta_t^{(i)} dt \\
&\quad + \sum\nolimits_{k=1}^{n} \vartheta_t^{(m+k)} \sum\nolimits_{i=1}^{d} \rho_t^{(i,k)} \theta_t^{(i)} \upsilon_t^{(k)} dt \\
&\quad + \sum\nolimits_{j=1}^{m} \vartheta_t^{(j)} \zeta_t^{(j)} dt + \sum\nolimits_{k=1}^{n} \vartheta_t^{(m+k)} \zeta_t^{(m+k)} \upsilon_t^{(k)} dt \\
&= \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} dt \\
&\quad + \sum\nolimits_{i=1}^{d} \theta_t^{(i)} \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} \vartheta_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} \vartheta_t^{(m+k)} \upsilon_t^{(k)} \right) dt \\
&\quad + \sum\nolimits_{j=1}^{m} \vartheta_t^{(j)} \zeta_t^{(j)} dt + \sum\nolimits_{k=1}^{n} \vartheta_t^{(m+k)} \zeta_t^{(m+k)} \upsilon_t^{(k)} dt \\
&= \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} \\
&= \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} dt + \sum\nolimits_{j=1}^{m} \vartheta_t^{(j)} \zeta_t^{(j)} dt + \sum\nolimits_{k=1}^{n} \vartheta_t^{(m+k)} \zeta_t^{(m+k)} \upsilon_t^{(k)} dt, \\
&\quad \text{for } t \in [0,T]. \qquad \text{(A.16)}
\end{aligned}$$

Equation (4.28):

$$\begin{aligned}
&\sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} dt + \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} d\bar{N}_t^{(k)} \right) \\
&+ \sum\nolimits_{j=1}^{m} \zeta_t^{(j)} \vartheta_t^{(j)} dt + \sum\nolimits_{k=1}^{n} \zeta_t^{(m+k)} \vartheta_t^{(m+k)} \upsilon_t^{(k)} dt + \sum\nolimits_{j=1}^{m} \zeta_t^{(j)} dW_t^{(j)} \\
&+ \sum\nolimits_{k=1}^{n} \zeta_t^{(m+k)} d\bar{N}_t^{(k)} \\
= &\sum\nolimits_{i=1}^{d} \breve{\pi}_t^{(i)} \mu_t^{(i)} dt + \sum\nolimits_{i=1}^{d} \breve{\pi}_t^{(i)} \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} d\bar{N}_t^{(k)} \right) \\
&+ \sum\nolimits_{j=1}^{m} \breve{\zeta}_t^{(j)} \breve{\vartheta}_t^{(j)} dt + \sum\nolimits_{k=1}^{n} \breve{\zeta}_t^{(m+k)} \breve{\vartheta}_t^{(m+k)} \upsilon_t^{(k)} dt + \sum\nolimits_{j=1}^{m} \breve{\zeta}_t^{(j)} dW_t^{(j)} \\
&+ \sum\nolimits_{k=1}^{n} \breve{\zeta}_t^{(m+k)} d\bar{N}_t^{(k)}, \quad \text{for } t \in [0,T]. \qquad \text{(A.17)}
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\sum\nolimits_{i=1}^{d} \pi_t^{(i)} \mu_t^{(i)} dt - \sum\nolimits_{i=1}^{d} \breve{\pi}_t^{(i)} \mu_t^{(i)} dt + \sum\nolimits_{j=1}^{m} \zeta_t^{(j)} \vartheta_t^{(j)} dt - \sum\nolimits_{j=1}^{m} \breve{\zeta}_t^{(j)} \breve{\vartheta}_t^{(j)} dt \\
&+ \sum\nolimits_{k=1}^{n} \zeta_t^{(m+k)} \vartheta_t^{(m+k)} \upsilon_t^{(k)} dt - \sum\nolimits_{k=1}^{n} \breve{\zeta}_t^{(m+k)} \breve{\vartheta}_t^{(m+k)} \upsilon_t^{(k)} dt \\
= &\sum\nolimits_{i=1}^{d} \breve{\pi}_t^{(i)} \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} d\bar{N}_t^{(k)} \right) \\
&- \sum\nolimits_{i=1}^{d} \pi_t^{(i)} \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} d\bar{N}_t^{(k)} \right) \\
&+ \sum\nolimits_{j=1}^{m} \breve{\zeta}_t^{(j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \breve{\zeta}_t^{(m+k)} d\bar{N}_t^{(k)} \\
&- \sum\nolimits_{j=1}^{m} \zeta_t^{(j)} dW_t^{(j)} - \sum\nolimits_{k=1}^{n} \zeta_t^{(m+k)} d\bar{N}_t^{(k)}, \quad \text{for } t \in [0,T], \qquad \text{(A.18)}
\end{aligned}$$

which we can simplify to equation (4.29)

$$\begin{aligned}
&\sum\nolimits_{i=1}^{d} \left(\pi_t^{(i)} - \breve{\pi}_t^{(i)} \right) \mu_t^{(i)} dt + \sum\nolimits_{j=1}^{m} \left(\zeta_t^{(j)} \vartheta_t^{(j)} - \breve{\zeta}_t^{(j)} \breve{\vartheta}_t^{(j)} \right) dt \\
&+ \sum\nolimits_{k=1}^{n} \left(\zeta_t^{(m+k)} \vartheta_t^{(m+k)} - \breve{\zeta}_t^{(m+k)} \breve{\vartheta}_t^{(m+k)} \right) \upsilon_t^{(k)} dt \\
= &\sum\nolimits_{i=1}^{d} \left(\breve{\pi}_t^{(i)} - \pi_t^{(i)} \right) \left(\sum\nolimits_{j=1}^{m} \sigma_t^{(i,j)} dW_t^{(j)} + \sum\nolimits_{k=1}^{n} \rho_t^{(i,k)} d\bar{N}_t^{(k)} \right)
\end{aligned}$$

$$+\sum_{j=1}^{m}\left(\breve{\zeta}_t^{(j)}-\zeta_t^{(j)}\right)dW_t^{(j)}+\sum_{k=1}^{n}\left(\breve{\zeta}_t^{(m+k)}-\zeta_t^{(m+k)}\right)d\bar{N}_t^{(k)},\quad \text{for } t\in[0,T]\tag{A.19}$$

Equation (4.33):

$$\begin{aligned}dR_t^M=&\sum_{j=1}^{m}\zeta_t^{(j)}\vartheta_t^{(j)}dt+\sum_{k=1}^{n}\zeta_t^{(m+k)}\vartheta_t^{(m+k)}\upsilon_t^{(k)}dt\\&+\sum_{j=1}^{m}\zeta_t^{(j)}dW_t^{(j)}+\sum_{k=1}^{n}\zeta_t^{(m+k)}d\bar{N}_t^{(k)},\quad \text{for } t\in[0,T],\end{aligned}\tag{A.20}$$

Equation (5.5):

$$dL_t=\sum_{j=1}^{m}\phi_t^{(j)}dW_t^{(j)}+\sum_{k=1}^{n}\psi_t^{(k)}d\bar{N}_t^{(k)},\quad \text{for } t\in[0,T].\tag{A.21}$$

Equation (5.6):

$$\begin{aligned}&d\left\langle M^{(i)},L\right\rangle_t\\&=d\left\langle\int_0^{\cdot}\sum_{j=1}^{m}S_{u-}^{(i)}\sigma_u^{(i,j)}dW_u^{(j)}+\int_0^{\cdot}\sum_{k=1}^{n}S_{u-}^{(i)}\rho_u^{(i,k)}d\bar{N}_u^{(k)},\right.\\&\qquad\left.\int_0^{\cdot}\sum_{j=1}^{m}\phi_u^{(j)}dW_u^{(j)}+\int_0^{\cdot}\sum_{k=1}^{n}\psi_u^{(k)}d\bar{N}_u^{(k)}\right\rangle_t\\&=d\left\langle\int_0^{\cdot}\sum_{j=1}^{m}S_{u-}^{(i)}\sigma_u^{(i,j)}dW_u^{(j)},\int_0^{\cdot}\sum_{j=1}^{m}S_{u-}^{(i)}\phi_u^{(j)}dW_u^{(j)}\right\rangle_t\\&\qquad+d\left\langle\int_0^{\cdot}\sum_{k=1}^{n}\rho_u^{(i,k)}d\bar{N}_u^{(k)},\int_0^{\cdot}\sum_{k=1}^{n}\psi^{(k)}d\bar{N}_u^{(k)}\right\rangle_t\\&=S_{t-}^{(i)}\left(\sum_{j=1}^{m}\sigma_t^{(i,j)}\phi_t^{(j)}dt+\sum_{k=1}^{n}\rho_t^{(i,k)}\psi_t^{(k)}\nu_t^{(k)}dt\right)\\&=0,\quad\text{for } t\in[0,T],\end{aligned}\tag{A.22}$$

Equation (5.7):

$$\begin{aligned}d\left(L_t\widehat{Z}_t\right)&=L_{t-}d\widehat{Z}_t+\widehat{Z}_{t-}dL_t+d\left\langle L^c,\widehat{Z}^c\right\rangle_t\\&\quad+\left(L_t\widehat{Z}_t-L_{t-}\widehat{Z}_{t-}-L_{t-}\Delta\widehat{Z}_t-\widehat{Z}_{t-}\Delta L_t\right)\\&=L_{t-}\widehat{Z}_{t-}\sum_{j=1}^{m}\widehat{\lambda}_t^{(j)}dW_t^{(j)}+L_{t-}\widehat{Z}_{t-}\sum_{i=1}^{k}\widehat{\kappa}_t^{(k)}d\bar{N}_t^{(k)}\\&\quad+\widehat{Z}_{t-}\sum_{j=1}^{m}\phi_t^{(j)}dW_t^{(j)}+\widehat{Z}_{t-}\sum_{i=1}^{k}\psi_t^{(k)}d\bar{N}_t^{(k)}\\&\quad+\widehat{Z}_{t-}\sum_{j=1}^{m}\phi_t^{(j)}\widehat{\lambda}_t^{(j)}dt\\&\quad+\widehat{Z}_{t-}\left(\sum_{i=1}^{k}\left(\psi_t^{(k)}+L_{t-}\widehat{\kappa}_t^{(k)}+\psi_t^{(k)}\widehat{\kappa}_t^{(k)}\right)\Delta N_t^{(k)}\right)\\&\quad-L_{t-}\widehat{Z}_{t-}\sum_{i=1}^{k}\widehat{\kappa}_t^{(k)}\Delta N_t^{(k)}-\widehat{Z}_{t-}\sum_{i=1}^{k}\psi_t^{(k)}\Delta N_t^{(k)}\\&=L_{t-}\widehat{Z}_{t-}\sum_{j=1}^{m}\widehat{\lambda}_t^{(j)}dW_t^{(j)}-L_{t-}\widehat{Z}_{t-}\sum_{i=1}^{k}\widehat{\kappa}_t^{(k)}\nu_t^{(k)}dt\\&\quad+\widehat{Z}_{t-}\sum_{j=1}^{m}\phi_t^{(j)}dW_t^{(j)}\\&\quad-\widehat{Z}_{t-}\sum_{i=1}^{k}\psi_t^{(k)}\nu_t^{(k)}dt+\widehat{Z}_{t-}\sum_{j=1}^{m}\phi_t^{(j)}\widehat{\lambda}_t^{(j)}dt\\&\quad+\widehat{Z}_{t-}\left(\sum_{i=1}^{k}\left(\psi_t^{(k)}+L_{t-}\widehat{\kappa}_t^{(k)}+\psi_t^{(k)}\widehat{\kappa}_t^{(k)}\right)dN_t^{(k)}\right)\end{aligned}$$

$$
\begin{aligned}
&= \widehat{Z}_{t-} \sum_{j=1}^{m} \left(L_{t-} \widehat{\lambda}_t^{(j)} + \phi_t^{(j)} \right) dW_t^{(j)} \\
&\quad + \widehat{Z}_{t-} \left(\sum_{i=1}^{k} \left(\psi_t^{(k)} + L_{t-} \widehat{\kappa}_t^{(k)} + \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \right) d\bar{N}_t^{(k)} \right) \\
&\quad + \widehat{Z}_{t-} \sum_{j=1}^{m} \phi_t^{(j)} \widehat{\lambda}_t^{(j)} dt - \widehat{Z}_{t-} \sum_{i=1}^{k} \psi_t^{(k)} \nu_t^{(k)} dt \\
&\quad - L_{t-} \widehat{Z}_{t-} \sum_{i=1}^{k} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} dt \\
&\quad + \widehat{Z}_{t-} \left(\sum_{i=1}^{k} \left(\psi_t^{(k)} + L_{t-} \widehat{\kappa}_t^{(k)} + \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \right) \nu_t^{(k)} dt \right) \\
&= \widehat{Z}_{t-} \sum_{j=1}^{m} \left(L_{t-} \widehat{\lambda}_t^{(j)} + \phi_t^{(j)} \right) dW_t^{(j)} \\
&\quad + \widehat{Z}_{t-} \left(\sum_{i=1}^{k} \left(\psi_t^{(k)} + L_{t-} \widehat{\kappa}_t^{(k)} + \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \right) d\bar{N}_t^{(k)} \right) \\
&\quad + \widehat{Z}_{t-} \sum_{j=1}^{m} \phi_t^{(j)} \widehat{\lambda}_t^{(j)} dt \\
&\quad + \widehat{Z}_{t-} \sum_{i=1}^{k} \left(-\psi_t^{(k)} \nu_t^{(k)} - L_{t-} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} + \psi_t^{(k)} \nu_t^{(k)} \right. \\
&\quad \left. + L_{t-} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} + \zeta_t^{(k)} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} \right) dt \\
&= \widehat{Z}_{t-} \sum_{j=1}^{m} \left(L_{t-} \widehat{\lambda}_t^{(j)} + \phi_t^{(j)} \right) dW_t^{(j)} \\
&\quad + \widehat{Z}_{t-} \left(\sum_{i=1}^{k} \left(\psi_t^{(k)} + L_{t-} \widehat{\kappa}_t^{(k)} + \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \right) d\bar{N}_t^{(k)} \right) \\
&\quad + \widehat{Z}_{t-} \sum_{j=1}^{m} \phi_t^{(j)} \widehat{\lambda}_t^{(j)} dt + \widehat{Z}_{t-} \sum_{i=1}^{k} \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} dt, \quad \text{for } t \in [0,T], \qquad \text{(A.23)}
\end{aligned}
$$

where we use

$$
\begin{aligned}
& L_t \widehat{Z}_t - L_{t-} \widehat{Z}_{t-} \\
&= \left(L_{t-} + \sum_{k=1}^{n} \psi_t^{(k)} \Delta N_t^{(k)} \right) \hat{Z}_{t-} \left(1 + \sum_{k=1}^{n} \widehat{\kappa}_t^{(k)} \Delta N_t^{(k)} \right) - L_{t-} \widehat{Z}_{t-} \\
&= \left(L_{t-} \widehat{Z}_{t-} + \widehat{Z}_{t-} \sum_{k=1}^{n} \psi_t^{(k)} \Delta N_t^{(k)} \right) \left(1 + \sum_{k=1}^{n} \widehat{\kappa}_t^{(k)} \Delta N_t^{(k)} \right) - L_{t-} \widehat{Z}_{t-} \\
&= \left(L_{t-} \widehat{Z}_{t-} + \widehat{Z}_{t-} \sum_{k=1}^{n} \psi_t^{(k)} \Delta N_t^{(k)} + L_{t-} \widehat{Z}_{t-} \sum_{k=1}^{n} \widehat{\kappa}_t^{(k)} \Delta N_t^{(k)} \right. \\
&\quad \left. + \widehat{Z}_{t-} \sum_{k=1}^{n} \psi_t^{(k)} \Delta N_t^{(k)} \sum_{k=1}^{n} \widehat{\kappa}_t^{(k)} \Delta N_t^{(k)} \right) - L_{t-} \widehat{Z}_{t-} \\
&= \widehat{Z}_{t-} \sum_{k=1}^{n} \psi_t^{(k)} \Delta N_t^{(k)} + L_{t-} \widehat{Z}_{t-} \sum_{k=1}^{n} \widehat{\kappa}_t^{(k)} \Delta N_t^{(k)} \\
&\quad + \widehat{Z}_{t-} \sum_{k=1}^{n} \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \Delta N_t^{(k)} \\
&= \widehat{Z}_{t-} \sum_{k=1}^{n} \left(\psi_t^{(k)} + L_{t-} \widehat{\kappa}_t^{(k)} + \psi_t^{(k)} \widehat{\kappa}_t^{(k)} \right) \Delta N_t^{(k)}, \quad \text{for } t \in [0,T]. \qquad \text{(A.24)}
\end{aligned}
$$

Equation (5.8):

$$
\begin{aligned}
dL_t &= \sum_{j=1}^{m} \phi_t^{(j)} dW_t^{(j)} + \sum_{k=1}^{n} \psi_t^{(k)} d\bar{N}_t^{(k)} \\
&= \sum_{j=1}^{m} \phi_t^{(j)} dW_t^{(j)} + \sum_{k=1}^{n} \psi_t^{(k)} d \left(N_t^{(k)} - \int_0^t \nu_t^{(k)} du \right) \\
&= \sum_{j=1}^{m} \phi_t^{(j)} d \left(W_t^{(j)} - \int_0^t \widehat{\lambda}_u^{(j)} du \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\phi_t^{(k)}d\left(N_t^{(k)}-\int_0^t\left(1+\widehat{\kappa}_u^{(k)}\right)\nu_t^{(k)}du\right)\\
&+\sum_{j=1}^{m}\phi_t^{(j)}\widehat{\lambda}_t^{(j)}dt+\sum_{k=1}^{n}\psi_t^{(k)}\widehat{\kappa}_t^{(k)}\nu_t^{(k)}dt\\
=&\sum_{j=1}^{m}\phi_t^{(j)}d\widehat{W}_t^{(j)}+\sum_{k=1}^{n}\psi_t^{(k)}d\widehat{\bar{N}}_t^{(k)}\\
&+\sum_{j=1}^{m}\phi_t^{(j)}\widehat{\lambda}_t^{(j)}dt+\sum_{k=1}^{n}\psi_t^{(k)}\widehat{\kappa}_t^{(k)}\nu_t^{(k)}dt\\
=&\sum_{j=1}^{m}\phi_t^{(j)}d\widehat{W}_t^{(j)}+\sum_{k=1}^{n}\psi_t^{(k)}d\widehat{\bar{N}}_t^{(k)},\quad\text{for } t\in[0,T],
\end{aligned}
\tag{A.25}
$$

Equation (5.9):

$$
\begin{aligned}
&d\left\langle M^{(i)},L\right\rangle_t\\
=&\,d\left\langle\int_0^{\cdot}\sum_{j=1}^{m}S_{u-}^{(i)}\sigma_u^{(i,j)}d\widehat{W}_u^{(j)}+\int_0^{\cdot}\sum_{k=1}^{n}S_{u-}^{(i)}\rho_u^{(i,k)}d\widehat{\bar{N}}_u^{(k)},\right.\\
&\left.\int_0^{\cdot}\sum_{j=1}^{m}\phi_u^{(j)}d\widehat{W}_u^{(j)}+\int_0^{\cdot}\sum_{k=1}^{n}\psi_u^{(k)}d\widehat{\bar{N}}_u^{(k)}\right\rangle_t\\
=&\,d\left\langle\int_0^{\cdot}\sum_{j=1}^{m}S_{u-}^{(i)}\sigma_u^{(i,j)}d\widehat{W}_u^{(j)},\int_0^{\cdot}\sum_{j=1}^{m}S_{u-}^{(i)}\phi_u^{(j)}d\widehat{W}_u^{(j)}\right\rangle_t\\
&+d\left\langle\int_0^{\cdot}\sum_{k=1}^{n}\rho_u^{(i,k)}d\widehat{\bar{N}}_u^{(k)},\int_0^{\cdot}\sum_{k=1}^{n}\psi_u^{(k)}d\widehat{\bar{N}}_u^{(k)}\right\rangle_t\\
=&\,S_{t-}^{(i)}\left(\sum_{j=1}^{m}\sigma_t^{(i,j)}\phi_t^{(j)}dt+\sum_{k=1}^{n}\rho_t^{(i,k)}\psi_t^{(k)}\left(1+\widehat{\kappa}_t^{(k)}\right)\nu_t^{(k)}dt\right)\\
=&\,S_{t-}^{(i)}\left(\sum_{j=1}^{m}\sigma_t^{(i,j)}\phi_t^{(j)}+\sum_{k=1}^{n}\rho_t^{(i,k)}\psi_t^{(k)}\nu_t^{(k)}\right.\\
&\left.+\sum_{k=1}^{n}\rho_t^{(i,k)}\psi_t^{(k)}\widehat{\kappa}_t^{(k)}\nu_t^{(k)}\right)dt\\
=&\,S_{t-}^{(i)}\left(\sum_{k=1}^{n}\rho_t^{(i,k)}\psi_t^{(k)}\widehat{\kappa}_t^{(k)}\nu_t^{(k)}\right)dt,\quad\text{for } t\in[0,T],
\end{aligned}
\tag{A.26}
$$

Equation (5.10):

$$
\begin{aligned}
dR_t^M=&\sum_{j=1}^{m}\zeta_t^{(j)}\vartheta_t^{(j)}dt+\sum_{k=1}^{n}\zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt\\
&+\sum_{j=1}^{m}\zeta_t^{(j)}dW_t^{(j)}+\sum_{k=1}^{n}\zeta_t^{(m+k)}d\bar{N}_t^{(k)},\quad\text{for } t\in[0,T],
\end{aligned}
\tag{A.27}
$$

Equation (5.11):

$$
\begin{aligned}
dR_t^M=&\sum_{j=1}^{m}\zeta_t^{(j)}\vartheta_t^{(j)}dt+\sum_{k=1}^{n}\zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}\\
&+\sum_{j=1}^{m}\zeta_t^{(j)}d\bar{W}_t^{(j)}+\sum_{k=1}^{n}\zeta_t^{(m+k)}d\bar{\bar{N}}_t^{(k)},\quad\text{for } t\in[0,T],
\end{aligned}
\tag{A.28}
$$

Equation (5.12):

$$
\bar{W}_t^{(j)}=W_t^{(j)}-\int_0^t\bar{\lambda}_u^{(j)}du,\quad\text{for } t\in[0,T],
\tag{A.29}
$$

Equation (5.13):

$$
\bar{\bar{N}}_t^{(k)}=N_t^{(k)}-\int_0^t\left(1+\bar{\kappa}_u^{(k)}\right)\nu_u^{(k)}du,\quad\text{for } t\in[0,T],
\tag{A.30}
$$

Equation (5.14):

$$\begin{aligned}
dR_t^M &= \sum_{j=1}^m \zeta_t^{(j)}\vartheta_t^{(j)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)} d\left(W_t^{(j)} - \int_0^t \bar{\lambda}_u^{(j)}du\right) \\
&\quad + \sum_{k=1}^n \zeta_t^{(m+k)} d\left(N_t^{(k)} - \int_0^t \left(1+\bar{\kappa}_u^{(k)}\right)\nu_u^{(k)}du\right) \\
&= \sum_{j=1}^m \zeta_t^{(j)}\vartheta_t^{(j)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}dW_t^{(j)} - \sum_{j=1}^m \zeta_t^{(j)}\bar{\lambda}_t^{(j)}dt \\
&\quad + \sum_{k=1}^n \zeta_t^{(m+k)} d\left(N_t^{(k)} - \int_0^t \nu_u^{(k)}du\right) - \sum_{k=1}^n \zeta_t^{(m+k)}\bar{\kappa}_t^{(k)}\nu_t^{(k)}dt \\
&= \sum_{j=1}^m \zeta_t^{(j)}\vartheta_t^{(j)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}dW_t^{(j)} + \sum_{k=1}^n \zeta_t^{(m+k)}d\bar{N}_t^{(k)} \\
&\quad - \sum_{j=1}^m \zeta_t^{(j)}\bar{\lambda}_t^{(j)}dt - \sum_{k=1}^n \zeta_t^{(m+k)}\bar{\kappa}_t^{(k)}\nu_t^{(k)}dt, \quad \text{for } t \in [0,T]. \qquad \text{(A.31)}
\end{aligned}$$

Equation (5.15) equals the following equations:

$$-\bar{\lambda}_t^{(j)} = \sum_{i=1}^d \sigma_t^{(i,j)}\bar{\theta}_t^{(i)} + \bar{\vartheta}_t^{(j)}, \quad \text{for } t \in [0,T] \text{ and } j = 1,...,m \qquad \text{(A.32)}$$

and

$$-\bar{\kappa}_t^{(k)} = \sum_{i=1}^d \rho_t^{(i,k)}\bar{\theta}_t^{(i)} + \bar{\vartheta}_t^{(m+k)}, \quad \text{for } t \in [0,T] \text{ and } k = 1,...,n, \qquad \text{(A.33)}$$

Equation (5.16):

$$\begin{aligned}
dR_t^M &= \sum_{j=1}^m \zeta_t^{(j)}\vartheta_t^{(j)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}dW_t^{(j)} + \sum_{k=1}^n \zeta_t^{(m+k)}d\bar{N}_t^{(k)} \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}\left(\sum_{i=1}^d \sigma_t^{(i,j)}\bar{\theta}_t^{(i)} + \bar{\vartheta}_t^{(j)}\right)dt \\
&\quad + \sum_{k=1}^n \zeta_t^{(m+k)}\left(\sum_{i=1}^d \rho_t^{(i,k)}\bar{\theta}_t^{(i)} + \bar{\vartheta}_t^{(m+k)}\right)\nu_t^{(k)}dt \\
&= \sum_{j=1}^m \zeta_t^{(j)}\vartheta_t^{(j)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}dW_t^{(j)} + \sum_{k=1}^n \zeta_t^{(m+k)}d\bar{N}_t^{(k)} \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}\sum_{i=1}^d \sigma_t^{(i,j)}\bar{\theta}_t^{(i)}dt + \sum_{j=1}^m \zeta_t^{(j)}\bar{\vartheta}_t^{(j)}dt \\
&\quad + \sum_{k=1}^n \zeta_t^{(m+k)}\sum_{i=1}^d \rho_t^{(i,k)}\bar{\theta}_t^{(i)}\nu_t^{(k)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\bar{\vartheta}_t^{(m+k)}\nu_t^{(k)}dt \\
&= \sum_{j=1}^m \zeta_t^{(j)}\vartheta_t^{(j)}dt + \sum_{k=1}^n \zeta_t^{(m+k)}\vartheta_t^{(m+k)}\nu_t^{(k)}dt \\
&\quad + \sum_{j=1}^m \zeta_t^{(j)}dW_t^{(j)} + \sum_{k=1}^n \zeta_t^{(m+k)}d\bar{N}_t^{(k)}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{d} \bar{\theta}_t^{(i)} \underbrace{\left(\sum_{j=1}^{m} \sigma_t^{(i,j)} \zeta_t^{(j)} + \sum_{k=1}^{n} \rho_t^{(i,k)} \zeta_t^{(m+k)} \nu_t^{(k)} \right)}_{=0} dt \\
&+ \sum_{j=1}^{m} \zeta_t^{(j)} \bar{\vartheta}_t^{(j)} dt + \sum_{k=1}^{n} \zeta_t^{(m+k)} \bar{\vartheta}_t^{(m+k)} \nu_t^{(k)} dt \\
= &\sum_{j=1}^{m} \zeta_t^{(j)} \vartheta_t^{(j)} dt + \sum_{k=1}^{n} \zeta_t^{(m+k)} \vartheta_t^{(m+k)} \nu_t^{(k)} dt \\
&+ \sum_{j=1}^{m} \zeta_t^{(j)} dW_t^{(j)} + \sum_{k=1}^{n} \zeta_t^{(m+k)} d\bar{N}_t^{(k)} \\
&+ \sum_{j=1}^{m} \zeta_t^{(j)} \bar{\vartheta}_t^{(j)} dt + \sum_{k=1}^{n} \zeta_t^{(m+k)} \bar{\vartheta}_t^{(m+k)} \nu_t^{(k)} dt, \quad \text{for } t \in [0,T]. \qquad \text{(A.34)}
\end{aligned}
$$

Equation (5.17):

$$
\begin{aligned}
& d\left\langle M^{(i)}, R^M \right\rangle_t \\
= & d\left\langle \int_0^{\cdot} \sum_{j=1}^{m} S_{u-}^{(i)} \sigma_u^{(i,j)} d\widehat{W}_u^{(j)} + \int_0^{\cdot} \sum_{k=1}^{n} S_{u-}^{(i)} \rho_u^{(i,k)} d\widehat{\bar{N}}_u^{(k)}, \right. \\
& \int_0^{\cdot} \sum_{j=1}^{m} \zeta_u^{(j)} \vartheta_u^{(j)} dt + \int_0^{\cdot} \sum_{k=1}^{n} \zeta_u^{(m+k)} \vartheta_u^{(m+k)} \nu_u^{(k)} dt \\
& \left. + \int_0^{\cdot} \sum_{j=1}^{m} \zeta_u^{(j)} d\widehat{W}_u^{(j)} + \int_0^{\cdot} \sum_{k=1}^{n} \zeta_u^{(m+k)} d\widehat{\bar{N}}_u^{(k)} \right\rangle_t \\
= & d\left\langle \int_0^{\cdot} \sum_{j=1}^{m} S_{u-}^{(i)} \sigma_u^{(i,j)} d\widehat{W}_u^{(j)}, \int_0^{\cdot} \sum_{j=1}^{m} \zeta_u^{(j)} d\widehat{W}_u^{(j)} \right\rangle \\
& + d\left\langle \int_0^{\cdot} \sum_{k=1}^{n} S_{u-}^{(i)} \rho_u^{(i,k)} d\widehat{\bar{N}}_u^{(k)}, \int_0^{\cdot} \sum_{k=1}^{n} \zeta_u^{(m+k)} d\widehat{\bar{N}}_u^{(k)} \right\rangle \\
= & d\left\langle \int_0^{\cdot} \sum_{j=1}^{m} S_{u-}^{(i)} \sigma_u^{(i,j)} d\left(W_u^{(j)} - \int_0^u \widehat{\lambda}_s^{(j)} ds \right). \right. \\
& \left. \int_0^{\cdot} \sum_{j=1}^{m} \zeta_u^{(j)} d\left(W_u^{(j)} - \int_0^u \widehat{\lambda}_s^{(j)} ds \right) \right\rangle \\
& + d\left\langle \int_0^{\cdot} \sum_{k=1}^{n} S_{u-}^{(i)} \rho_u^{(i,k)} d\left(N_u^{(k)} - \int_0^u \left(1 + \widehat{\kappa}_s^{(k)}\right) \nu_s^{(k)} ds \right), \right. \\
& \left. \int_0^{\cdot} \sum_{k=1}^{n} \zeta_u^{(m+k)} d\left(N_u^{(k)} - \int_0^u \left(1 + \widehat{\kappa}_s^{(k)}\right) \nu_s^{(k)} ds \right) \right\rangle \\
= & \sum_{j=1}^{m} S_{t-}^{(i)} \sigma_t^{(i,j)} \zeta_t^{(j)} dt \\
& + \sum_{k=1}^{n} S_{t-}^{(i)} \rho_t^{(i,k)} \zeta_t^{(m+k)} \left(1 + \widehat{\kappa}_t^{(k)}\right) \nu_t^{(k)} dt \\
= & S_{t-}^{(i)} \underbrace{\left(\sum_{j=1}^{m} \sigma_t^{(i,j)} \zeta_t^{(j)} dt + \sum_{k=1}^{n} \rho_t^{(i,k)} \zeta_t^{(m+k)} \nu_t^{(k)} dt \right)}_{=0} \\
& + S_{t-}^{(i)} \sum_{k=1}^{n} \rho_t^{(i,k)} \zeta_t^{(m+k)} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} dt \\
= & S_{t-}^{(i)} \sum_{k=1}^{n} \rho_t^{(i,k)} \zeta_t^{(m+k)} \widehat{\kappa}_t^{(k)} \nu_t^{(k)} dt, \quad \text{for } t \in [0,T]. \qquad \text{(A.35)}
\end{aligned}
$$

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