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The Time Optimal Transition of Eastern Germany's Productivity

by

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1 Introduction

Among the countries of former socialist regimes East Germany clearly plays a special role. Some important problems arising in the reform process from a centrally planned to a market economy have been readily solved with the German unification in 1990. East Germany owns political and social stability, the institutional infrastructure necessary to support market principles and has a stable monetary system. Nevertheless, big effort is still required to solve the economic problems. The German government must cope with the economic transition, which means in first place to rise productivity and employment in East Germany with the goal of reaching West German levels.

In the beginning of the German unification in 1990, political forces and economic leaders believed that this adjustment would be possible within a few years. The chancellor of the Federal Republic of Germany, Helmut Kohl, promised that the Neue Bundesländer would have become flourishing landscapes until the next elections.[3] This belief was shared by the major part of the German population. A population survey shows that in 1990 expectations about the period of the transition process were very optimistic. On average, more than 80 percent of the German population expected that equal productivity in both parts of Germany would have been reached until the year 2000.

But soon the belief in a fast economic transition faded away. Table 1 shows the development of the gross domestic product for the first six years after the unification of Germany. In 1996, the ratio of the gross domestic product per capita was just 43.26 per cent. Although the ratio raised heavily in the first years after German unification, today it is clear that it will take a long time to get the ratio close to one. The data for both parts of Germany reproduced in table 1 indicate that the real world does not perform as expected. Since 1993, the steps to close the gap have been very small. Therefore it is clear that it can be closed only in the long run.

To support the adjustment process of Eastern Germany, the government soon decided to levy an extra tax. This so called solidarity contribution was imposed in 1991 and 1992 for the first time at a rate of 7.5 per cent on the income tax yield. For the following two years, the solidarity contribution was

[3] Helmut Kohl: Fernseh- und Hörfunkansprache vom 2. Oktober 1990, reprinted in: Presse- und Informationsamt der Bundesregierung [7, S. 660–662, 662].

Table 1: Real Gross National Product in Eastern and Western Germany in prices of 1991

Year	Gross National Produkt[a]		Employed Persons[b]		Gross National Produkt per Employed[c]		
	East	West	East	West	East	West	Ratio[d]
1991	206.0	2 647.6	7 321	29 189	28 138.2	90 705.4	31.02
1992	222.1	2 694.3	6 387	29 455	34 773.8	91 471.7	38.02
1993	238.1	2 644.5	6 208	29 005	38 353.7	91 173.9	42.07
1994	258.3	2 706.8	6 303	28 654	40 980.5	94 465.0	43.98
1995	274.5	2 748.3	6 416	28 461	42 783.7	96 563.7	44.31
1996	285.4	2 779.2	6 603	27 818	43 222.8	99 906.5	43.26

Source: Statistical Yearbook of Germany and own calculations
[a] In Billion DM.
[b] On average per year, in 1000.
[c] In DM.
[d] In per cent.

stopped without any obvious economic reason. Finally, it was again levied in 1995. Since that time the solidarity tax rate remained constant at 7.5 per cent of the income tax yield up to the year 1998, when it was reduced to 5.5 per cent.

These ups and downs of the solidarity tax rate lead to the question if there is a pattern over time which is optimal in some sense. The purpose of this paper is the development of an optimal solidarity tax policy. The optimal policy is understood as that policy which moves GNP of both parts of Germany to the steady–state growth path in minimal time. This means that the optimal policy minimizes the time required to achieve equal standards of living in both parts. The framework for this attempt is a neoclassical growth model, and the solution will be derived by means of optimal control theory.

In section two we start with the presentation of the growth model, develop a simple tax policy and introduce the allocation parameter which controls the allocation of the tax revenue to the two countries. The allocation parameter was introduced by Rahman [8], [9] and Intrilligator [5] who solved the problem of maximizing production in a two-country–model with fixed production coefficients. In the next section the resulting control problem is solved by exhibiting a policy satisfying the sufficient optimality conditions. The fourth and last section shows some policy implications and summerizes the results.

2 The model

Although the following analysis is valid in general for any two countries, we are primarily concerned with the German reunification. Keeping German reunification in mind, we choose the standard neoclassical growth model as a basis. This is, because Ackermann [1] showed that the *iron law of convergence* formulated by Barro [2] and Barro/Sala–i–Martin [3], [4] is not valid for Germany. The estimation of the convergence coefficient β lead to a rate of convergence of 3.66 per cent for Western Germany. Hence, it is almost twice as high as the rate of convergence predicted by the iron law of convergence, where β was estimated to be around 0.02.

Prior to the introduction of the solidarity tax, each of the two countries developes according to the standard neoclassical growth model as introduced by Robert Solow [11] and Trevor Swan [12]. Therefore capital accumulation per capita follows the basic differential equation

$$\dot{k}_i(t) = sf[k_i(t)] - (\delta + n)k_i(t), \qquad i = 1, 2, \tag{1}$$

where the parameters have the usual meaning: s denotes the constant saving rate, δ the depreciation rate and n the population growth rate. The function $f(\cdot)$ stands for the aggregate neoclassical production function (in part., $f'(0+) = \infty$, $f' > 0$ and $f'' < 0$) and the subscript $i = 1, 2$ identifies the country.

Throughout the paper we will use the subscript 1 for the initially poorer country and the subscript 2 for the initially wealthier one. Thus, in the context of German reunification, the index 1 stands for Eastern Germany and the index 2 for Western Germany. Setting the starting time of the problem to zero, it follows that $k_1(0) < k_2(0)$. Motivated by the evidence found by Ackermann [1, p. 27–28] we set $k_2(0) = \overline{k}$, where $\overline{k}$ stands for the steady–state capital intensity, i.e. the capital intensity for which $\dot{\overline{k}} = 0$ or, equivalently, $sf(\overline{k}) = (\delta + n)\overline{k}$. It should, however, be noted that the analysis below works for any $k_2(0) \leq \overline{k}$.

When the government levies the solidarity contribution, which is modelled as a linear income tax, total tax revenue is given by

$$T(t) = \tau_t L_1(t) f[k_1(t)] + \tau_t L_2(t) f[k_2(t)] \tag{2}$$

where τ_t denotes the tax rate to be optimized and $L_i(t)$, $i = 1, 2$, the population in country i. We assume that the government sets an upper bound for the tax rate, denoted $\overline{\tau}$. Higher tax rates cannot be enforced politically. The control region for the tax rate is therefore given by the interval $[0, \overline{\tau}]$. The tax revenue is then allocated to the two countries as additional investment. The fraction of this investment allocated to country 1 is controlled by the allocation parameter β_t, which has to be optimized in conjunction with τ_t. Hence the fraction allocated to country 2 is given by $(1 - \beta_t)$. Define ρ to be the population ratio between the two countries: $\rho = L_1(t)/L_2(t)$ which is constant over time because the population growth rates are equal in both countries. The additional investment per capita allocated to country 1 is then given by

$$\beta_t \frac{T(t)}{L_1(t)} = \beta_t \tau_t \Big(f[k_1(t)] + \frac{1}{\rho} f[k_2(t)] \Big), \tag{3a}$$

and the part allocated to country 2 by

$$(1 - \beta_t) \frac{T(t)}{L_2(t)} = (1 - \beta_t) \tau_t \rho \Big(f[k_1(t)] + \frac{1}{\rho} f[k_2(t)] \Big). \tag{3b}$$

With these additional investments capital accumulation now developes according to

$$\dot{k}_1(t) = \beta_t t_1(t) + s(1 - \tau_t) f[k_1(t)] - \alpha k_1(t) \tag{4a}$$

and

$$\dot{k}_2(t) = (1 - \beta_t) t_2(t) + s(1 - \tau_t) f[k_2(t)] - \alpha k_2(t). \tag{4b}$$

In (4a) and (4b) we used $\alpha = (\delta + n)$ as a shorthand notation. Furthermore, we set $t_i(t) = T(t)/L_i(t)$, $i = 1, 2$.

The assumption of a constant saving rate seems rather implausible, since each individuum will adjust its consumption and hence savings in dependency of its income. Nevertheless it must be introduced in order to find a solution for the control problem at all. In our opinion, this can be justified as an approximation of real behavior for small tax rates.

To complete the model, we have to model the behavior of the government. As already mentioned, the goal of the government is to minimize the period

$[0, T]$ required to reach equal productivity or, equivalently, to achieve the same standard of living in both countries. The government has to choose trajectories of the tax rate τ_t and of the allocation parameter β_t to reach that goal. In addition, the initially wealthier country should not be punished by the transition process. It is therefore demanded that the capital intensity reaches the steady–state capital intensity $\overline{k}$ at the end of the transition process. This goal can be written as

$$k_1(T) = k_2(T) = \overline{k}, \tag{5}$$

where the required period T has to be determined endogenously as a part of the solution of the control problem.

3 Formulation of the problem

As a result of these considerations we arrive at the following control problem:

$$\begin{aligned}
&\min_{\substack{0\le\tau_t\le\overline{\tau}\\ 0\le\beta_t\le 1}} T\\
&\text{s.t.}\;\; \dot{k}_1(t) = \beta_t\tau_t\Big[f[k_1(t)] + \frac{1}{\rho}f[k_2(t)]\Big]\\
&\qquad\qquad + s(1-\tau_t)f[k_1(t)] - \alpha k_1(t),\\
&\qquad\;\; \dot{k}_2(t) = (1-\beta_t)\rho\tau_t\Big[f[k_1(t)] + \frac{1}{\rho}f[k_2(t)]\Big]\\
&\qquad\qquad + s(1-\tau_t)f[k_2(t)] - \alpha k_2(t),\\
&\qquad\;\; k_1(0) = k_1^0,\;\; k_2(0) = k_2^0,\\
&\qquad\;\; k_1(T) = k_2(T) = \overline{k},
\end{aligned}$$

with $k_1^0 < k_2^0 \le \overline{k}$ and the equilibrium capital intensity determined by

$$sf(\overline{k}) = \alpha\overline{k}. \tag{6}$$

The Hamiltonian corresponding to this problem is given by

$$\begin{aligned}
&H(k_1, k_2, p_0, p_1, p_2, \beta, \tau) =\\
&\quad -p_0 + p_1\Big[\beta\tau[f(k_1) + \frac{1}{\rho}f(k_2)] + s(1-\tau)f(k_1) - \alpha k_1\Big]\\
&\quad + p_2\Big[(1-\beta)\tau\rho[f(k_1) + \frac{1}{\rho}f(k_2)] + s(1-\tau)f(k_2) - \alpha k_2\Big]
\end{aligned} \tag{7}$$

or, equivalently,

$$\begin{aligned} &H(k_1,k_2,p_0,p_1,p_2,\beta,\tau) = \\ &\quad -p_0 + p_1[sf(k_1)-\alpha k_1] + p_2[sf(k_2)-\alpha k_2] \\ &\quad +(p_1-\rho p_2)\beta\tau[f(k_1)+\frac{1}{\rho}f(k_2)] \\ &\quad -\tau[p_1 sf(k_1)+p_2 sf(k_2)-p_2\rho f(k_1)-p_2 f(k_2)]. \end{aligned} \tag{8}$$

The responses $k_1^*(t)$ and $k_2^*(t)$ describe the total investment in this two–country–model. The adjoint variables $p_1(t), p_2(t)$ have therefore a straight economic interpretation. They can be understood as the shadow prices for an investment in either country. The higher the shadow price for an investment the more should be invested. Therefore the investment in the country with the higher shadow price is more favourable, as long as the two adjoint variables are not equal. This than leads to the conclusion that the total amount of additional investment should be allocated to the country with the higher shadow price.

By Pontryagin's maximum principle, a necessary condition for a feasible control (β_t^*, τ_t^*), $0 \le t \le T^*$, with response $(k_1^*(t), k_2^*(t))$ to be optimal is then the existence of adjoint variables $(p_0, p_1(t), p_2(t))$ s. t. the following is true:

$$\dot{p}_1 = -p_1[[\tau_t^*(\beta_t^*-s)+s]f'(k_1^*)-\alpha] - p_2(1-\beta_t^*)\tau_t^*\rho f'(k_1^*) \tag{9a}$$

$$\dot{p}_2 = -p_2[[\tau_t^*(1-\beta_t^*-s)+s]f'(k_2^*)-\alpha] - p_1\beta_t^*\tau_t^*\frac{1}{\rho}f'(k_2^*) \tag{9b}$$

$$(p_0, p_1(t), p_2(t)) \neq (0,0,0) \quad \forall t \in [0, T^*] \tag{10}$$

$$\begin{aligned} H^*(t) :=& H[k_1^*(t), k_2^*(t), p_0, p_1(t), p_2(t), \beta_t^*, \tau_t^*] \\ =& M[t, k_1^*(t), k_2^*(t)] \quad \forall\, t \in [0, T^*] \end{aligned} \tag{11}$$

where

$$M[t,k_1,k_2] = \max_{\substack{0\le\tau\le\overline{\tau} \\ 0\le\beta\le 1}} H[k_1,k_2,p_0,p_1(t),p_2(t),\beta,\tau]$$

and

$$H^*(t) = 0 \quad \text{on } [0, T^*]. \tag{12}$$

Somewhat more precisely, (9a, b) is to be understood in the sense that the costates $p_1(t)$, $p_2(t)$ are absolutely continuous and (9a, b) hold almost everywhere on $[0, T^*]$. Actually, if we admit only piecewise continuous controls, the derivatives $\dot{p}_1(t)$, $\dot{p}_2(t)$ will exist at all points of $[0, T^*]$, except the (finitely many) jump points of the controls. The same applies to the state equations.

It follows from (11) that—if $\tau_t^* \neq 0$—the function $p_1 - \rho p_2$ ist a switching function for β_t^*:

$$\beta_t^* = \begin{cases} 1 & \text{if } p_1(t) > \rho p_2(t), \\ \textit{singular} & \text{if } p_1(t) = \rho p_2(t), \\ 0 & \text{if } p_1(t) < \rho p_2(t). \end{cases} \tag{13}$$

The singular value of β_t^* can easily be determined.

Lemma 1

If β_t^* is singular on some open interval I and $\tau_t^* \neq 0$ on I, then

$$\beta_t^* = \frac{\rho}{1+\rho}.$$

Proof 1

Differentiating the singularity condition $p_1 - \rho p_2 = 0$ and inserting the right hand sides of (9a,b), we obtain after some straight forward manipulations the condition

$$p_1[f'(k_2^*) - f'(k_1^*)][s + \tau_t^*(1-s)] = 0 \quad \text{on } I.$$

Since the expression in the second square brackets is always positive, this can only be the case if

$$p_1 = 0 \quad \text{or} \quad f'(k_1^*) = f'(k_2^*).$$

The first alternative is ecxcluded by (12) together with (10). Hence, from the second alternative,

$$k_1^* = k_2^* \quad \text{and} \quad \dot{k}_1^* = \dot{k}_2^* \quad \text{on } I.$$

Inserting the right hand sides of the system equations for the derivatives, this means

$$\beta_t^* \tau_t^* (1 + \frac{1}{\rho}) + s(1-\tau_t^*) = (1-\beta_t^*)\tau_t^*(1+\rho) + s(1-\tau_t^*).$$

If $\tau_t^* \neq 0$ on I, this implies that

$$\beta_t^* = \frac{1+\rho}{2+\rho+1/\rho} = \frac{\rho(1+\rho)}{1+2\rho+\rho^2} = \frac{\rho}{1+\rho}.$$

◆

The structure of τ_t^* is less easy to grasp. Actually, it seems rather forbidden to try to describe the class of all extremal solutions, hoping that there will be only one, which would solve the problem, given some existence result. We shall therefore pursue the reverse way and exhibit one particular feasible extremal solution, i.e. a control (β_t, τ_t) s.t.

1. (β_t, τ_t) steers the corresponding response $(k_1(t), k_2(t))$ from (k_1^0, k_2^0) to $(\overline{k}, \overline{k})$ in some time T;
2. there exist adjoint variables $(p_0, p_1(t), p_2(t))$, s.t. (9a)–(12) (without the asterisks) are satisfied.

This, in conjunction with a suitable sufficiency condition, will show that the candidate control (β_t, τ_t) is indeed optimal.

Our candidate is the control

$$\beta_t = \begin{cases} 1 & \text{for } 0 \le t < \tilde{t} \\ \frac{\rho}{1+\rho} & \text{for } \tilde{t} \le t \le T, \end{cases} \tag{14}$$

$$\tau_t = \overline{\tau} \text{ for } 0 \le t \le T. \tag{15}$$

Here $\tilde{t}$ ist the first time at which the responses k_1 and k_2 coincide:

$$\tilde{t} = \inf\{t \ge 0 : k_1(t) = k_2(t)\},$$

and T is the first time at which *both* k_1 and k_2 reach $\overline{k}$. The choice (14) of β_t is motivated through the fact that the first country possesses the lower capital intensity up to the point $\tilde{t}$. Hence, it has a higher marginal rate of productivity throughout the interval $[0, \tilde{t})$. Allocating the tax revenue of both countries to country one therefore clearly contributes stronger to the growth process of capital than allocating just a fraction of total savings to country one.

The candidate for the tax rate should be choosen as high as possible, hence at the maximum rate that can be enforced politically, because the higher the tax rate is the higher is the amount of additional savings, provided the savings rate remains constant. As already mentioned, for small tax rates the assumption of a constant savings rate seems to be plausible, although the individuals in reality will optimize a intertemporal utility function subject to disposable income. The changes in the fraction of disposable income are small if the tax rate is small. Therefore this effect seems neglectible.

In order to show that (14), (15) defines an extremal solution, let us first consider the corresponding responses. During the first regime, i.e. for $0 \le t < \tilde{t}$, the system equations become

$$\begin{aligned} \dot{k}_1 &= \overline{\tau}[f(k_1) + \frac{1}{\rho}f(k_2)] + s(1-\overline{\tau})f(k_1) - \alpha k_1 \\ &= \overline{\tau}(1-s)f(k_1) + \frac{\overline{\tau}}{\rho}f(k_2) + sf(k_1) - \alpha k_1 \\ &> 0 \end{aligned} \tag{16a}$$

certainly as long as $k_1 < \overline{k}$, and

$$\dot{k}_2 = s(1-\overline{\tau})f(k_2) - \alpha k_2, \tag{16b}$$

hence

$$\dot{k}_2 \gtrless 0 \quad \text{for} \quad k_2 \lessgtr \hat{k},$$

with $\hat{k}$ defined by

$$s(1-\overline{\tau})f(\hat{k}) = \alpha\hat{k}. \tag{17}$$

Note that $\hat{k} < \overline{k}$. If k_2 starts at $k_2^0 < \hat{k}$ ($k_2^0 > \hat{k}$), then it will rise (fall) under regime 1 in a strictly monotone way to $\hat{k}$, but will never reach it under regime 1. This is because if the solution to (16b) should hit $\hat{k}$ at some finite point $\hat{t}$: $k_2(\hat{t}) = \hat{k}$, this would provide a nontrivial solution (in some neighborhood of $\hat{t}$, corresponding to the initial condition $k_2(\hat{t}) = \hat{k}$ which is different from the equilibrium solution $k_2 \equiv \hat{k}$ (note that $\hat{k}$ has been defined as the equilibrium point of the right hand side of (16b)). Uniqueness of solutions of such differential equations (the function on the right hand side of 16b) ist Lipschitz in a neighborhood of $\hat{k}$) prevents this from happening. If k_2 starts at $k_2^0 = \hat{k}$, then it will stay there as long as regime 1 lasts. In any case, for $k_2^0 > 0$, there is some lower bound $m > 0$ for the values of k_2 and hence $f(k_2)$ during regime 1. Therefore, from (16a),

$$\begin{aligned} \dot{k}_1 &\ge \overline{\tau}(1-s)f(k_1) + \frac{\overline{\tau}}{\rho}m + sf(k_1) - \alpha k_1 \\ &\ge \frac{\overline{\tau}}{\rho}m > 0 \end{aligned}$$

as long as $k_1 < \overline{k}$. This means that, under regime 1, k_1 will rise to any level $< \overline{k}$ in finite time and hence meet $k_2(t)$ at some finite time $\tilde{t}$. In case $k_2^0 = \hat{k}$, $k_2 \equiv \hat{k}$ and $k_1(\tilde{t}) = k_2(\tilde{t}) = \hat{k}$. Otherwise, $\tilde{k} = k_i(\tilde{t}) < \hat{k}$ or $> \hat{k}$. In any case, $k_i(\tilde{t}) < \overline{k}$.

At time $\tilde{t}$, the regime switches to $\beta_t = \rho/(1+\rho)$. The system equations then become

$$\dot{k}_1 = \overline{\tau}(1-s)f(k_1) + sf(k_1) - \alpha k_1, \tag{18a}$$
$$\dot{k}_2 = \overline{\tau}(1-s)f(k_2) + sf(k_2) - \alpha k_2, \tag{18b}$$

with initial condition $k_1(\tilde{t}) = k_2(\tilde{t}) = \tilde{k}$. Hence the joint path of k_1 and k_2 for $t \geq \tilde{t}$ is given by

$$\dot{k} = \overline{\tau}(1-s)f(k) + sf(k) - \alpha k\,, \quad k(\tilde{t}) = \tilde{k}.$$

As long as $k < \overline{k}$, $\dot{k} > 0$, and $\overline{k}$ is reached at some finite time T since

$$\dot{k} \geq \overline{\tau}(1-s)f(k) \geq \overline{\tau}(1-s)f(\tilde{k}) > 0$$

for $k \leq \overline{k}$. This shows that the proposed control (14), (15) is indeed feasible, i.e. condition 1 is satisfied. For later use, note that

$$k_1(t) < \overline{k}\,, \quad k_2(t) < \overline{k} \text{ for all } 0 \leq t < T. \tag{19}$$

Let us now turn to the construction of the adjoint variables (p_0, p_1, p_2). Choosing $p_0 = 1$, (10) certainly will be satisfied . To obtain p_1, p_2 solve the adjoint equations (9a, b) backward in time, starting at $t = T$, under regime 2:

$$\dot{p}_1 = -p_1\left\{\left[\overline{\tau}\left(\frac{\rho}{1+\rho} - s\right) + s\right]f'(k) - \alpha\right\} - p_2\frac{\rho}{1+\rho}\overline{\tau}f'(k) \tag{20a}$$
$$\dot{p}_2 = -p_2\left\{\left[\overline{\tau}\left(\frac{\rho}{1+\rho} - s\right) + s\right]f'(k) - \alpha\right\} - p_1\frac{1}{1+\rho}\overline{\tau}f'(k). \tag{20b}$$

It is then easily calculated that

$$\dot{p}_1 - \rho\dot{p}_2 = -\left[s(1-\overline{\tau})f'(k) - \alpha\right](p_1 - \rho p_2). \tag{21}$$

In order to ensure that the singular control $\beta_t = \rho/(1+\rho)$ is indeed part of an extremal solution, $p_1 - \rho p_2 = 0$ must be fulfilled on $[\tilde{t}, T]$, corresponding to a choice of end conditions $p_1(T)$, $p_2(T)$ s.t.

$$p_1(T) = \rho p_2(T). \tag{22}$$

The Hamiltonian on $(\tilde{t}, T]$ is given by

$$\begin{aligned} H(t) &= -1 + p_1(t)\frac{1+\rho}{\rho}\left[[1+\overline{\tau}(1-s)]f[k(t)] - \alpha k(t)\right] \\ &= -1 + p_1(t)\frac{1+\rho}{\rho}r[k(t)], \end{aligned} \tag{23}$$

with $r(k) = [1 + \overline{\tau}(1-s)]f(k) - \alpha k$. In order to meet the requirement (12) at $t = T$, the end condition $p_1(T)$ must be chosen so that

$$H(T) = -1 + p_1(t)\frac{1+\rho}{\rho} r(\overline{k}) = 0. \tag{24}$$

Since $r(\overline{k}) > 0$, $p_1(T) > 0$ and, by (22), $p_2(T) > 0$. Note that the pair $(p_1(T), p_2(T))$ of end conditions is uniquely determined by (22) and (23). The costate p_1 then evolves according to the homogeneous linear differential equation

$$\dot{p}_1 = -p_1\Big[[s + \overline{\tau}(1-s)]f'(k) - \alpha\Big].$$

Together with the end condition $p_1(T) > 0$ this implies that $p_1 = \rho p_2 > 0$, in particular

$$\tilde{p} = p_1(\tilde{t}) = \rho p_2(\tilde{t}) > 0. \tag{25}$$

Finally, during regime 2,

$$\begin{aligned} M(t, k_1, k_2) = &-1 + p_1(t)[f(k_1) - \alpha k_1 + \frac{1}{\rho}(f(k_2) - \alpha k_2)] \\ &+ \max_{0 \le \tau \le \overline{\tau}} \tau p_1(t)(1-s)[f(k_1) + \frac{1}{\rho} f(k_2)] \\ = &-1 + p_1(t)[f(k_1) - \alpha k_1 + \frac{1}{\rho}(f(k_2) - \alpha k_2)] \\ &+ \overline{\tau} p_1(t)(1-s)[f(k_1) + \frac{1}{\rho} f(k_2)], \end{aligned} \tag{26}$$

so that, comparing with (23), (11) is indeed satisfied along $k_1(t) = k_2(t) = k(t)$.

Coming now to regime 1, the adjoint equations are given by

$$\dot{p}_1 = -p_1[[\overline{\tau}(1-s) + s]f'(k_1) - \alpha] \tag{27a}$$
$$\dot{p}_2 = -p_2[s(1-\overline{\tau})f'(k_2) - \alpha] - p_1 \frac{1}{\rho} \overline{\tau} f'(k_2). \tag{27b}$$

They have too be solved on $[0, \tilde{t}]$ with the end conditions (25). As a consequence, the solution to the homogeneous linear differntial equation (27a) staisfies $p_1 > 0$ on $[0, \tilde{t}]$. Next, observe that

$$\begin{aligned} \overbrace{p_1 - \rho p_2}^{\cdot} = &\ \alpha(p_1 - \rho p_2) + s(1-\overline{\tau})[\rho p_2 f'(k_2) - p_1 f'(k_1)] \\ &+ \overline{\tau} p_1 [f'(k_2) - f'(k_1)] \\ = &\ (p_1 - \rho p_2)[\alpha - s(1-\overline{\tau})f'(k_2)] + \nu(t) \end{aligned} \tag{28}$$

with

$$v(t) = \big[s(1-\overline{\tau}) + \overline{\tau}\big] p_1 \big[f'(k_2) - f'(k_1)\big].$$

Since $p_1 > 0$ and $f'(k_2) < f'(k_1)$ during regime 1, $v(t) < 0$. By the variation of parameters formula it follows that

$$(p_1 - \rho p_2)(t) = -e^{A(t)} \int_t^{\tilde{t}} e^{-A(s)} v(s) \mathrm{d}s > 0 \quad \text{for } 0 \le t < \tilde{t},$$

with

$$A(t) = -\int_t^{\tilde{t}} \big(\alpha - s(1-\overline{\tau}) f'[k_2(s)]\big) \mathrm{d}s.$$

This shows that $p_1 > \rho p_2$ on $[0, \tilde{t})$. A similar argument shows that $p_2 > 0$ on $[0, \tilde{t})$. The maximized Hamiltonian is given by

$$\begin{aligned} M(t, k_1, k_2) = &- 1 + p_1(t)[sf(k_1) - \alpha k_1] + p_2(t)[sf(k_2) - \alpha k_2] \\ &+ \max_{\substack{0 \le \tau_t \le \overline{\tau} \\ 0 \le \beta_t \le 1}} \Big[[p_1(t) - \rho p_2(t)] \beta \tau [f(k_1) + \frac{1}{\rho} f(k_2)] \\ &- \tau \big[p_1(t) s f(k_1) + p_2(t) s f(k_2) - p_2(t) \rho f(k_1) - p_2(t) f(k_2)\big] \Big] \\ = &- 1 + p_1(t)[sf(k_1) - \alpha k_1] + p_2(t)[sf(k_2) - \alpha k_2] \\ &+ \max_{0 \le \tau_t \le \overline{\tau}} \tau \Big[(1-s) p_1(t) f(k_1) + (1-s) p_2(t) f(k_2) \\ &+ \frac{1}{\rho} [p_1(t) - \rho p_2(t)] f(k_2) \Big] \\ = &- 1 + p_1(t)[sf(k_1) - \alpha k_1] + p_2(t)[sf(k_2) - \alpha k_2] \\ &+ \overline{\tau} \Big[(1-s) p_1(t) f(k_1) + (1-s) p_2(t) f(k_2) \\ &+ \frac{1}{\rho} [p_1(t) - \rho p_2(t)] f(k_2) \Big], \end{aligned} \tag{29}$$

showing that, along $k_1(t)$, $k_2(t)$,

$$H(t) = M\big(t, k_1(t), k_2(t)\big),$$

so that (11) is satisfied. (12) follows in the usual way from from (24) and the fact that the system equations are autonomuous.

Gathering the results, we have shown that our candidate control (14), (15) gives indeed rise to a feasible extremal solution. Moreover, it is obvious that the right hand sides of the system equations are strictly concave functions in (k_1, k_2) (for any admissible choice of β_t, τ_t). One may therefore evoke any sufficiency theorems for time optimal control problems (cf., e.g., Seierstad/Sydsæter [10, S. 146–147]) to show that (14), (15) is the unique optimal control. Indeed, in our case, the argument is so simple that we present it here. To this end, let us return to the $*$–notation for our candidate control and responses corresponding to it, and let, for brevity, $g(k_1, k_2, \beta, \tau) = (g_1, g_2)'$ denote the functions defining the system dynamics (i. e. the functions on the right hand side of the system equations). Write $k^*(t) = \big(k_1^*(t), k_2^*(t)\big)'$ and $k(t) = \big(k_1(t), k_2(t)\big)'$ for our candidate response and the response to any other admissible control (β_t, τ_t) resp., and (committing some abuse of notation) $g^* = g^*(t) = g\big(k_1^*(t), k_2^*(t), \beta_t^*, \overline{\tau}\big)$ and $g = g(t) = g\big(k_1(t), k_2(t), \beta_t, \tau_t\big)$ as well as $p(t) = (p_1(t), p_2(t))'$ for the vector of adjoint variables belonging to our candidate. Then, if there is an admissible control (β_t, τ_t), $0 \le t \le T$, s.t. $T < T^*$, i. e.

$$k_1(T) = k_2(T) = \overline{k},$$

we may conclude that

$$\begin{aligned} 0 &= \int_0^T [\dot{p} + p g_k^*](k - k^*)\mathrm{dt} \\ &= p(T)\big[k(T) - k^*(T)\big] - \int_0^T p\big[g - g^* - g_k^*(k - k^*)\big]\mathrm{dt} \\ &> p(T)\big[k(T) - k^*(T)\big] = p_1(T)\big[\overline{k} - k_1^*(T)\big] + p_2(T)\big[\overline{k} - k_2^*(T)\big] \\ &> 0, \end{aligned}$$

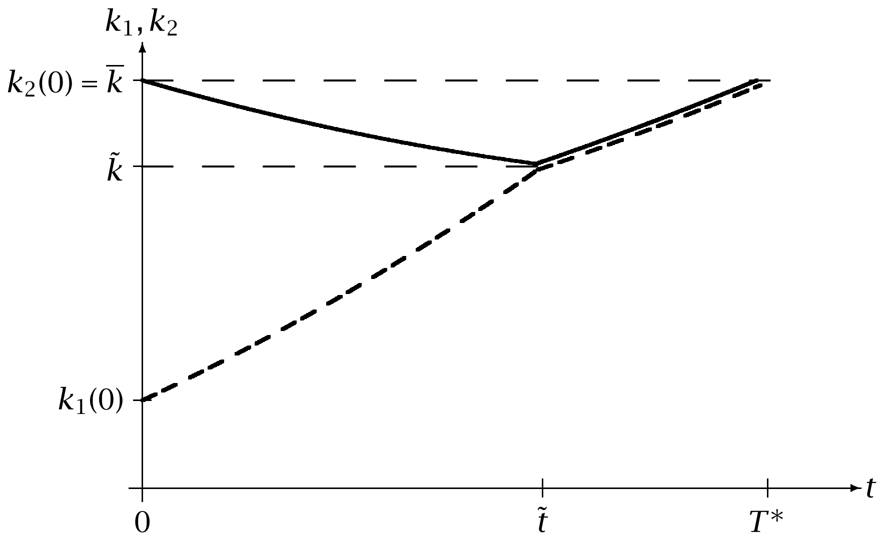
leading to a contradiction. The reasons for the above relations are as follows:

- The first equality by the adjoint equations (9a, b).
- The second equality by partial integration noting that $k(0) = k^*(0)$.
- The first inequality by (strict) concavity of g_1, g_2 (in k) and $p_1 > 0$, $p_2 > 0$.
- The last inequality by $p_1(T) > 0$, $p_2(T) > 0$ together with (19) (for $t = T$; note that the T appearing in (19) is now T^*).

4 Summary, Extensions and Policy Implications

It was shown that the time optimal solidarity tax rate should remain constant throughout the whole transition period $[0, \tilde{t}]$ and the following period of catching up to the steady–state growth path $(\tilde{t}, T^*]$. In addition, the solidarity tax rate should be choosen at the upper bound $\overline{\tau}$. Therefore, the observed ups and downs of the solidarity tax rate cannot be optimal. Nevertheless they may be explained through the political business cycle. The elections to the German Bundestag in 1994 and 1998 could be the reason for the lowering of the solidarity tax rate.

Figure 1: The development of the capital intensity in the two countries for $k_2(0) = \overline{k}$ and $k_1(0) < k_2(0)$



Based on a numerical solution for Cobb–Douglas production functions.
The solid line illustrates the development for country 2 and the dashes line for country 1. At the point $\tilde{t}$ the capital intensities coincide for the first time. At the point T^* the goal of the adjustment process is reached.

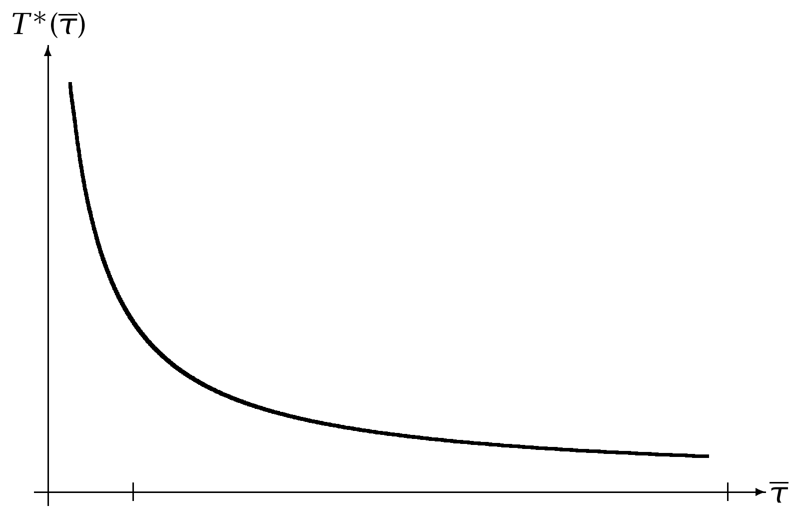
The development of the capital intensities under optimal control is illustrated in figure 1. For this calculations a Cobb–Douglas production function[9] was choosen to represent the production possibilities of both countries. The solution of the differential equations (4a, b) is only possible by numerical integration[10], because equation (4a) cannot be solved algebraicly over the interval $[0, \tilde{t})$. Therefore, the point $\tilde{t}$ and in the following the point T^* can only be determined numerically, too. The initially wealthier country experiences losses in capital per capita and, even more important from the government point of view, losses in disposable income during the whole transition period.

Calculating the transition period for different values of the optimal tax rate $\overline{\tau}$ shows that the transition period decreases when the tax rate increases, but with dimishing rates. This relationship is illustrated in figure 2 for reasonable

[9] Since we are interested in explicit solutions of the development of the capital intensity we need an explicit production function. The Cobb–Douglas production function was proofed to be valid in the long run and follows from a fairly general aggregation of individual production functions. See Krelle [6, pp. 71–72].

[10] A fifth-order Runge–Kutta algorithm was choosen because of its simple implementation and the sufficient accuracy.

Figure 2: The minimal adjustment time T^* for Eastern Germany as a function of the income tax rate $\overline{\tau}$



Based on a numerical solution for Cobb–Douglas production functions.

parameter values. While the optimal trajectory of the solidarity tax rate is now solved, the government faces the problem to determine the upper bound of the solidarity tax rate. If the transition period would be the only objective, the upper bound should be choosen as high as possible. But with respect to other objectives lying outside the model, e. g. disposable income, a trade–off between these opposite goals may exist.

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