ASIAN EXCHANGE RATE OPTIONS UNDER STOCHASTIC INTEREST RATES: PRICING AS A SUM OF DELAYED PAYMENT OPTIONS

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Abstract. The aim of the paper is to develop pricing formulas for European type Asian options written on the exchange rate in a two currency economy. The exchange rate as well as the foreign and domestic zero coupon bond prices are assumed to follow geometric Brownian motions. As a special case of a discrete Asian option we analyse the delayed payment currency option and develop closed form pricing and hedging formulas.

The main emphasis is devoted to the discretely sampled Asian option. It is shown how the value of this option can be approximated as the sum of Black-Scholes options. The formula is obtained under the application of results developed by Rogers and Shi (1995a) and Jamshidian (1991). In addition bounds for the pricing error are determined.

1. Introduction

No Asian option is traded as a standardized contract in any organized exchange. However, they are extremely popular in the OTC market among institutional investors. Milevsky and Posner (1997) mention that the estimated outstanding volume is in the range from five to ten billion USD.

Several reasons for introducing Asian options are presented in the literature. A corporation expecting to have payments in foreign currency claims can reduce its average foreign currency exposure by using Asian options. Because the average, which is the underlying asset in these contracts, tends to be less volatile than the exchange rate itself, the Asian option is (normally) priced more cheaply than the standard option. The hedging costs for the firm is therefore reduced.

Another reason for introducing Asian options was to avoid speculators in arranging price manipulation of the underlying asset close to the maturity date.

The value of an Asian option depends at any point in time on the spot exchange rate and on the history of the spot exchange rate: the Asian option is path–dependent. This increases the complexity of both

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TABLE 1. Valuation methods for Asian Options

	Method	References
*	Monte Carlo Simulation	Kemna and Vorst (1990)
*	Approximation via geometric average	Vorst (1992)
	Laplace transformation	Geman and Yor (1993)
*	Edgeworth expansion	Turnbull and Wakeman (1991) Levy (1992) Jacques (1996)
	Binomial trees	Hull and White (1993)
	Fast Fourier Transformation	Carverhill and Clewlow (1990)
*	Numerical solution of the fundamental second order pde	Kemna and Vorst (1990)
*	Numerical solution of the variable reduced second order p.d.e.	Rogers and Shi (1995a) Alziary, Décamps and Koehl (1997) He and Takahashi (1996)
*		Rogers and Shi (1995a)

pricing and hedging. In addition the probability distribution of the arithmetic average is unknown if we, as is usually the case, assume that the exchange rate and the relevant bond prices follow standard lognormal processes. Numerical techniques must be relied on in order to determine the prices of Asian options in general, and this brings us in a difficult position to generate the hedging strategy. The hedging strategy is determined through the price sensitivity of the Asian option to changes in the average. However, as the average itself is not a traded asset we need to develop how the average can be duplicated through a self financing strategy in traded assets. The main problem is clearly that we do not have a closed form solution giving us the price of the Asian option, we have at best a good approximation. No guarantee exists, however, that the sensitivity of the approximation is just similar or close to the sensitivity of the true price.

The numerical techniques applied for valuing Asian options are numerous. A majority of the methods can be extended to the situation we consider with a stochastic developing term structure in both the domestic and the foreign country. With the symbol * Table 1 indicates those methods which in their conception are applicable to the valuation problem under stochastic interest rates[1]. Concerning the hedging of Asian options only a few papers exist. In Alziary, Décamps and Koehl (1997) hedging strategies are analysed in the situation where the term structure of interest rates develops in a deterministic manner, and this can obviously not be the relevant case for exchange rate options. Alziary et al use a variable reduction approach introduced by Rogers and Shi (1995a) to price the Asian option and they continue showing that the option's delta can be derived from this pde. In their paper they furthermore derive

[1] See e.g. Nielsen and Sandmann (1996) and Schmidli (1997).

useful expressions for the difference between the European and the Asian option. Turnbull and Wakeman (1991) give a rough estimate of the delta based on an Edgeworth series expansion. El Karoui and Jeanblanc-Picquè (1993) show that the price of an Asian option on a stock is equal to a European option on a fictitious asset which has a random volatility. They propose a super hedging strategy build an Black and Scholes (1973) formula with volatility equal to the upper bound on the stochastic volatility. Jacques (1996) uses approximate price formulas based on the lognormal approximation and based on an inverse Gaussian approximation. In both situations he derives a formula for the hedging portfolio and shows through numerical examples that these formulae are efficient in the sense that the replicating strategy is close to the intrinsic value of the Asian option at maturity. Jacques finds, comparing the two approximations, that they are equally efficient.

From the mathematical point of view the main difficulty of pricing and hedging an Asian option is to determine the distribution of the arithmetic average. Some of the mentioned approximation techniques can be interpreted as a change to a more convenient distribution. On one hand this seems to be a reasonable approach for two reasons. First, the continuously growing literature in finance based on the lognormality assumption is in no way a verification of this distribution for the changes of a financial asset. Second, empirical evidence for this distribution is certainly doubtful, but a uniformly better distribution cannot be identified. On the other hand the objective of financial modeling does amount to something more than the calculation of numbers. The objective is to clarify dependencies. To precisely measure the size of these dependencies, we have to study examples, i.e. to specify distributions.

A theoretical model should be understood as a reference model. In financial markets the relevance of a reference model for practical purposes is whether or not it is accepted and serves as a guideline to clarify relationships and the impact of decisions. The Black and Scholes (1973) model has become the most widely accepted model for the analysis of derivative assets. Although theorists and practitioners are aware of its weaknesses, this model is used as the reference model in finance. With a view to the mathematical problems in the situation of Asian options the tentative to leave the Black and Scholes model may be strong, but the consistency with results already understood and recognized seems more important to us.

The problem of pricing and hedging an Asian option proves to be much more difficult when the bond markets are described by a model allowing for stochastic term structure developments. It is the extension to such a situation which will be the aim of this paper.

Following the description in Section 2 of the financial market model to be used in this paper, we analyse in Section 3 the Asian put–call parity, the duplication of the arithmetic average, and the delayed payment exchange rate option. In the classical Black and Scholes (1973) setting where the underlying asset is a non-dividend paying stock, the put–call parity, e.g. Stoll (1968), is given by

$$C_e(t) - P_e(t) \quad = \quad S(t) - e^{-r(T-t)}\, K$$

where C_e and P_e denote the price of an European call and put option respectively. The relationship is based on a pure arbitrage argument. At time zero the call option is bought and the put option is sold. The price of this portfolio is equal to the spot price of the underlying stock reduced by the discounted value of the exercise price, K. Without performing any trade during the lifetime of the options this portfolio duplicates $S(T) - K$ at the maturity date of the options. The relationship is based on a buy and hold strategy and it does not depend on the specific model chosen for the development of the stock price. The reason being that the underlying asset itself is a marketed and non-dividend paying asset.

Denoting by $A(T)$ the arithmetic average of the exchange rate at the Asian option's maturity date, T, the Asian put–call parity at time T takes the form:

$$C_a(T) - P_a(T) = A(T) - K.$$

where C_a and P_a denote the price of an Asian call and put option respectively. This contrast to the previous situation $A(T)$ is not a traded asset. It means that we cannot expect that the trading strategy to duplicate the average will be as simple as for the standard European options.

In Section 4 we discuss the pricing of Asian options. The method suggested by Vorst is analysed and adapted to our setting and we generalize the Rogers and Shi approach to the situation of an Asian exchange rate option with stochastic interest rates, i.e. we have to take into consideration the time dependent, multi–dimensional volatility structure. Applying a result from Jamshidian (1991) we derive an analytical closed form solution for the approximation. Finally, we conclude in Section 5.

2. The financial market model

The model of the financial market we consider is a two country model. The exchange rate between the domestic and the foreign country is assumed to be stochastic. In addition we consider a stochastic behaviour of the interest rate market in both countries, and apply the bond price oriented approach by Geman, El Karoui and Rochet (1995). The model is based on the international financial market model derived by Amin and Jarrow (1991), and we restrict ourselves to the situation with deterministic volatility functions. As discussed in the introduction, existing results presume deterministic interest rates. In the case of the Asian exchange rate option this assumption seems questionable to us. In order to compare these results with those implied by a stochastic behaviour of the interest rate markets we are staying within the class of lognormal processes.

The continuous time model is defined on the finite time interval $[0, T]$. Let $(\Omega, \mathcal{F}, P_d^*)$ be a filtered probability space where the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by an n-dimensional ($n \geq 3$) Brownian motion $\{W_d^*(t)\}_{t\in[0,T]}$ under P_d^*. Following Geman, El Karoui and Rochet (1995), the domestic interest rate market is characterized by the price process of the family of the domestic zero coupon bonds, i.e.

Assumption 2.1:

For any maturity $t' \in [0, T]$ the price process $\{D^d(t, t')\}_{t\in[0,t']}$ of the domestic default free zero coupon *bond with national value 1 and maturity t' satisfies the stochastic differential equation*

$$\begin{aligned} dD^d(t,t') &= r^d(t)D^d(t,t')dt + D^d(t,t')\sigma^d(t,t')\cdot dW_d^*(t) \quad \forall\, t\in[0,t'] \\ D^d(t',t') &= 1 \quad P_d^* \quad \textit{a.s. in the domestic currency,} \end{aligned} \tag{2.1}$$

where $\{r^d(t)\}_{t\in[0,T]}$ denotes the domestic continuously compounded spot rate process[2] *and the volatility structure satisfies the following requirements:*

i) *for any $t'\in[0,T]$ the volatility function*

$$\sigma^d(\cdot,t'):[0,t']\to\mathbb{R}^n$$

is continuous and square integrable on $[0,t']$ with $\sigma(t',t')=0$,

ii) $\left|\left|\frac{\partial\sigma^d(t,t')}{\partial t'}\right|\right|$ *is bounded on $\{(t,t')|0\le t\le t', t'\in[0,T]\}$,*

iii) *there exists a real number $H>0$ such that*

$$\left|\left|\frac{\partial\sigma^d(t,t'+\delta)}{\partial t'} - \frac{\partial\sigma^d(t,t')}{\partial t'}\right|\right| \le H\cdot\delta \quad \forall\, t\le t' \quad \forall\,\delta>0.$$

With Assumption 2.1 the domestic interest rate market is arbitrage free and P_d^* is the unique (domestic) martingale measure (see Geman, El Karoui and Rochet (1995)). The solution for the domestic zero coupon bond is given by

$$D^d(t,t') = D^d(0,t)\exp\left\{\int_0^t r^d(u)du - \frac{1}{2}\int_0^t ||\sigma^d(u,t')||^2du + \int_0^t \sigma^d(u,t')\cdot dW_d^*(u)\right\} \quad . \tag{2.2}$$

As in Frey and Sommer (1996), the foreign interest rate market and the exchange rate are modeled under the domestic martingale measure P_d^*, i.e. with reference to the contribution by Amin and Jarrow (1991), we assume

Assumption 2.2:

a) *For all $t'\in[0,T]$ the foreign zero coupon bond market is determined by*

$$\begin{aligned} dD^f(t,t') &= \left[r^f(t) - \sigma^f(t,t')\cdot\sigma^x(t)\right]D^f(t,t')dt + D^f(t,t')\sigma^f(t,t')\cdot dW_d^*(t) \\ D^f(t',t') &= 1 \qquad P_d^* \quad \textit{a.s. in the foreign currency,} \end{aligned} \tag{2.3}$$

b) *The exchange rate process $\{X(t)\}_{t\in[0,T]}$ in units of the domestic currency per one unit of the foreign currency satisfies*

$$dX(t) = \left[r^d(t) - r^f(t)\right]X(t)dt + X(t)\sigma^x(t)\cdot dW_d^*(t) \quad , \tag{2.4}$$

where $\{r^f(t)\}_{t\in[0,T]}$ is the process of the foreign spot rate and the volatility functions are assumed to satisfy the same requirements as in Assumption 2.1.

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[2] By $x\cdot y$ for $x,y\in\mathbb{R}^n$ we denote the standard scalar product, i.e. $x\cdot y=\sum_{i=1}^n x_iy_i$ and $||x|| := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ denotes the Euclidian norm.

Given the spot rate processes in both countries the bank account is defined by

$$(2.5)\qquad \beta^d_{t,T} := \exp\left\{\int_t^T r^d(u)du\right\} \quad \text{for the domestic country, resp. by}$$

$$\beta^f_{t,T} := \exp\left\{\int_t^T r^f(u)du\right\} \quad \text{for the foreign country.}$$

Obviously, the expected discounted value of the foreign zero coupon bond is not a martingale under P^*_d, since P^*_d is the domestic martingale measure. But for the process $\left\{Z^d(t,t') := X(t)D^f(t,t')\right\}_{t\in[0,t']}$, i.e. the value of the foreign zero coupon bond denoted in the domestic currency, this is true, since

$$(2.6)\qquad \begin{aligned} dZ^d(t,t') &= r^d(t)Z^d(t,t')dt + Z(t,t')\left[\sigma^x(t) + \sigma^f(t,t')\right]\cdot dW^*_d(t) \\ Z^d(t',t') &= X(t')D^f(t',t') \qquad P^*_d \quad a.s. \text{ in the domestic currency.} \end{aligned}$$

3. The Asian Put–Call Parity

The payoff of an European type Asian option is determined by the average value of the underlying asset, i.e. the exchange rate and the strike price. Let T denote the maturity date of the contract, then the continuous time average exchange rate in the domestic currency is defined as

$$(3.1)\qquad A_c(T) := \frac{1}{T}\int_0^T X(u)D^f(u,u)du,$$

whereas for given averaging times $\{t_1 < \ldots < t_N = T\}$ the discrete average is determined by

$$(3.2)\qquad A_d(T) := \frac{1}{N}\sum_{i=1}^N X(t_i)D^f(t_i,t_i) \quad .$$

For so-called fixed strike Asian options the strike price K (in the domestic currency) is a constant, in the case of a floating strike Asian option, the strike price is equal to the exchange rate at maturity. The payoff of an European type Asian option at maturity is therefore determined by

$$(3.3)\qquad [A_l(T) - K_T]^+ := \max\{A_l(T) - K_T, 0\}$$

$$(3.4)\qquad [K_T - A_l(T)]^+ := \max\{K_T - A_l(T), 0\} \quad ,$$

where $l \in \{c,d\}$ indicates continuous time or discrete time averaging and $K_T \in \{K, X(T)D^f(T,T)\}$ a fixed or floating strike specification. For a fixed strike, $K_T = K$, the payoff (3.3) corresponds to a fixed strike Asian call, (3.4) to a fixed strike Asian put. In the situation $K_T = X(T)D^f(T,T)$ the payoff of a floating strike Asian call is determined by (3.4), whereas (3.3) defines a floating strike Asian put. With $C_a(t)$ as abbreviation for the value of an Asian call at time t and $P_a(t)$ as the value of an Asian put the Asian put-call parity is given by

$$(3.5)\qquad \begin{aligned} C_a(t) - P_a(t) &= \Pi_t(A_l(T)) - KD^d(t,T) &&\text{for the fixed strike, and} \\ C_a(t) - P_a(t) &= X(t)D^f(t,T) - \Pi_t(A_l(T)) &&\text{for the floating strike} \end{aligned}$$

specification, where $\Pi_t(A_l(T))$ denotes the value at time t of the continuous or discrete time average. The Asian put-call parity indicates that the difference between the value of an Asian call and an otherwise identical Asian put at time t is determined by the difference of the value at time t of the average and the present value of the strike, i.e. $KD^d(t,T)$ in the case of a fixed strike and $X(t)D^f(t,T)$ in the case of a floating strike option. The value of the average is determined by the expected discounted value under the domestic martingale measure P_d^*. In order to calculate this expected discounted value, we apply the change of measure technique to the domestic forward risk adjusted measure. By Grisanov's Theorem the domestic T-forward risk adjusted measure P_d^T is defined as an equivalent measure with respect to P_d^* with the Radon-Nikodym derivative

$$\left.\frac{dP_d^T}{dP_d^*}\right|_t = \frac{(\beta_{t,T}^d)^{-1}D^d(T,T)}{E_{P_d^*}[(\beta_{t;T}^d)^{-1}D^d(T,T)|\mathcal{F}_t]} = \exp\left\{-\frac{1}{2}\int_t^T ||\sigma^d(u,T)||^2du + \int_t^T \sigma^d(u,T)\cdot dW_d^*(u)\right\}$$

where $dW_d^T(t) = dW_d^*(t) - \sigma^d(t,T)dt$ defines a vector Brownian motion under P_d^T. With the domestic zero coupon bond $D^d(t,T)$ resp. the foreign bond $D^f(t,T)$ as a numeraire, the stochastic differential equations (2.1), (2.3) and (2.4) can be rewritten as

$$\begin{aligned}
d\left(\frac{D^d(t,t')}{D^d(t,T)}\right) &= \frac{D^d(t,t')}{D^d(t,T)}\eta^d(t,t',T)\cdot dW_d^T(t)\\
d\left(\frac{D^f(t,t')}{D^f(t,T)}\right) &= \frac{D^f(t,t')}{D^f(t,T)}\eta^f(t,t',T)\cdot[dW_d^T(t) - \eta^x(t,T)dt]\\
d\left(\frac{X(t)D^f(t,T)}{D^d(t,T)}\right) &= \frac{X(t)D^f(t,T)}{D^d(t,T)}\eta^x(t,T)\cdot dW_d^T(t)
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\text{where}\quad \eta^d(t,t',T) &:= \sigma^d(t,t') - \sigma^d(t,T)\\
\eta^f(t,t',T) &:= \sigma^f(t,t') - \sigma^f(t,T)\\
\eta^x(t,T) &:= \sigma^x(t) + \sigma^f(t,T) - \sigma^d(t,T) \quad .
\end{aligned}$$

The solution for the domestic and the foreign zero coupon bonds under the domestic T-forward risk adjusted measure imply for $t' = t$ that

$$\frac{D^d(t,T)}{D^d(0,T)} = \frac{D^d(t,t)}{D^d(0,t)}\exp\left\{\frac{1}{2}\int_0^t ||\eta^d(u,t,T)||^2du - \int_0^t \eta^d(u,t,T)\cdot dW_d^T(u)\right\} \tag{3.7}$$

$$\frac{D^f(t,T)}{D^f(0,T)} = \frac{D^f(t,t)}{D^f(0,t)}\exp\left\{\int_0^t \eta^f(u,t,T)\cdot[\eta^x(u,T) + \frac{1}{2}\eta^f(u,t,T)]dt - \int_0^t \eta^f(u,t,T)\cdot dW_d^T(u)\right\}. \tag{3.8}$$

Applying (3.8) to the solution for the exchange rate process we can conclude that[3]

$$\begin{aligned}(3.9)\qquad X(t) &= X(0)\frac{D^f(0,T)}{D^f(t,T)}\frac{D^d(t,T)}{D^d(0,T)}\exp\left\{-\frac{1}{2}\int_0^t ||\eta^x(u,T)||^2 du + \int_0^t \eta^x(u,T)\cdot dW_d^T(u)\right\} \\ &= X(0)\frac{D^f(0,t)D^d(t,t)}{D^d(0,t)D^f(t,t)} \\ &\quad \cdot\exp\left\{-\frac{1}{2}\int_0^t \eta^x(u,t)\cdot(\eta^x(u,t)+2\eta^d(u,t,T))du + \int_0^t \eta^x(u,t)\cdot dW_d^T(u)\right\}\end{aligned}$$

Setting $n(t) = \max\{i|t_i < t\}\forall t \geq t_1$ and $n(t) = 0$ for $t < t_1$ the value of the discrete average is determined by the expected discounted discrete average under the domestic martingale measure, i.e.

$$\begin{aligned}(3.10)\qquad \Pi_t(A_d(T)) &= E_{P_d^*}[(\beta_{t,T}^d)^{-1}A_d(T)|\mathcal{F}_t] \\ &= E_{P_d^T}[D^d(t,T)A_d(T)|\mathcal{F}_t] \\ &= \frac{D^d(t,T)}{N}\left[\sum_{i=1}^{n(t)}\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)} + \sum_{i=n(t)+1}^{N} E_{P_d^T}\left[\left.\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right|\mathcal{F}_t\right]\right] \\ &= \frac{D^d(t,T)}{N}\left[\sum_{i=1}^{n(t)}\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right. \\ &\quad \left. + X(t)\sum_{i=n(t)+1}^{N}\frac{D^f(t,t_i)}{D^d(t,t_i)}\exp\left\{-\int_t^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du\right\}\right]\end{aligned}$$

where the first term of (3.10) just covers the known average at time t determined by the exchange rates prior to time t. For $t = 0$ equation (3.10) simplifies to

$$(3.11)\qquad \Pi_0(A_d(T)) = \frac{X(0)D^d(0,T)}{N}\sum_{i=1}^{N}\frac{D^f(0,t_i)}{D^d(0,t_i)}\exp\left\{-\int_0^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du\right\},$$

[3] If both interest rate markets are deterministic, i.e. $\eta^d(t,t',T) = 0 = \eta^f(t,t',T)$ the exchange rate process (3.9) is determined by

$$X(t) = X(0)\frac{D^f(0,t)D^d(t,t)}{D^d(0,t)D^f(t,t)}\exp\left\{-\frac{1}{2}\int_0^t ||\sigma^x(u)||^2\, du + \int_0^t \sigma^x(u)\cdot dW_d^T\right\}$$

which coincides with the Black-Scholes model applied to exchange rate options by Garman and Kohlhagen (1983) which implies for $\frac{T}{N} =: \Delta t \rightarrow 0$ that the value of the continuous time average is equal to[4]

$$\text{(3.12)} \quad \lim_{\Delta t \rightarrow 0} \Pi_0(A_d(T)) = \frac{X(0)D^d(0,T)}{T} \int_0^T \left(\frac{D^f(0,v)}{D^d(0,v)} \exp\left\{ - \int_0^v \eta^x(u,v) \cdot \eta^d(u,v,T) du \right\} \right) dv$$

$$= D^d(0,T) E_{P_d^T} \left[\frac{1}{T} \int_0^T \frac{X(v) D^f(v,v)}{D^d(v,v)} dv \right] = \Pi_0(A_c(T)) \quad .$$

3.1. **Duplication of the Arithmetic Average.** The duplication of the discrete arithmetic average in the case of deterministic interest rate markets is straightforward. According to (3.11) we buy

$$\text{(3.13)} \qquad \frac{D^d(0,T)}{N} \sum_{i=1}^{N} \frac{D^f(0,t_i)}{D^d(0,t_i)}$$

units of the foreign currency and invest these units in the foreign bond market. More precisely we take a long position of $\frac{1}{N} \frac{D^d(0,T)}{D^d(0,t_i)}$ units in the zero coupon bond with maturity date $t_i \in t_1, ..., t_N = T$. At times, $t_i \in t_1, ..., t_N$, the payoff of the long position in the zero coupon bonds with maturity t_i is converted by the prevailing exchange rate to take a long position in the domestic zero coupon bonds with maturity date T. Under deterministic interest rates the following relationship between initial forward and future spot prices holds:

$$\text{(3.14)} \qquad \frac{D^d(0,T)}{D^d(0,t_i)} = D^d(t_i,T).$$

With an increase in the long position of the domestic zero coupon bond at each time t_i by $\frac{1}{N}X(t_i)$ units, the final payoff of this self-financing strategy is just equal to $\frac{1}{N}\sum_{i=1}^{N} X(t_i)$, the payoff defined by the arithmetic average. The average can with deterministic interest rates be duplicated by a trading strategy which is close to a buy and hold strategy. Trades are only performed at the sampling dates.

We will now turn our attention to find the portfolio strategy to duplicate the arithmetic average where the term structure of interest rates develops in a stochastic manner. To ease the understanding, and also to highlight the importance of having stochastic interest rates in the two countries, we will consider only a single term in the average, assuming that the 'average' consists only of the spot exchange rate at time t_i and that the payment takes place at time $T > t_i$. Such a contract will be referred to as a delayed payment exchange rate contract.

The payment at time T, the maturity of the contract, is $X(t_i)D^f(t_i,t_i)$ which is known at time $t_i < T$. The value of this payment at time t_i is $X(t_i)D^f(t_i,t_i)D^d(t_i,T)$. This is not the value of a domestic traded asset, whereas its two components, $X(t_i)D^f(t_i,t_i)$ and $D^d(t_i,T)$, are traded assets. The value at

[4] As in Vorst (1992) the expected continuous time average in the case of deterministic interest rates and a flat yield curve is given by

$$\lim_{\Delta t \rightarrow 0} \Pi_0(A_d(T)) = X(0)\frac{e^{-r^d T}}{T} \int_0^T e^{-(r^f - r^d)v} dv$$

$$= \frac{X(0)}{T} \frac{1}{r^d - r^f} \left[e^{-r^f T} - e^{-r^d T} \right] \underset{T \rightarrow \infty}{\longrightarrow} 0$$

time $t < t_i$ of the contract, denoted by $V(t,t_i)$, can be obtained from the above derived expression for the conditional expected value of the discounted arithmetic average.

$$\begin{aligned} V(t,t_i) &:= E_{P_d^*}\left[\left(\beta_{t,T}^d\right)^{-1} X(t_i) D^f(t_i,t_i)|\mathcal{F}_t\right] \\ &= E_{P_d^T}\left[D^d(t,T)\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}|\mathcal{F}_t\right] \\ &= D^d(t,T)X(t)\frac{D^f(t,t_i)}{D^d(t,t_i)}\exp\left\{-\int_t^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du\right\} \quad . \end{aligned} \tag{3.15}$$

The procedure to obtain the portfolio of traded assets is the same as in the former case, the only difference is that the holdings of the different claims will now explicitly depend on the volatility structure of the model through the term $\exp\left\{-\int_t^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du\right\}$. Observe that $\eta^d(u,t_i,T)$ equals 0 if the domestic interest rate is deterministic, which will give us back the pure deterministic model. If, on the other hand, the domestic market is stochastic but the foreign market is deterministic, we will still have the influence from the volatility structure on the holdings and no buy and hold strategy exists. This means that the delayed payment structure has importance for the hedging strategy. Call and put options on exchange rates are very popular instruments in the market and the above discussion shows that the investor has to take into consideration the settlement risk, as the settlement often takes place with a delay from the option's maturity by 2 banking days.

The delayed exchange rate value calculated in (3.15) can be duplicated through the following strategy:

$$\begin{aligned} \phi^1(t,t_i) &:= \frac{\partial V(t,t_i)}{\partial X(t)D^f(t,t_i)} = \frac{V(t,t_i)}{X(t)D^f(t,t_i)} \\ &\quad \text{number of foreign zero coupon bonds with maturity } t_i, \\ \phi^2(t,t_i) &:= \frac{\partial V(t,t_i)}{\partial D^d(t,t_i)} = -\frac{V(t,t_i)}{D^d(t,t_i)} \\ &\quad \text{number of domestic zero coupon bonds with maturity } t_i, \\ \phi^3(t) &:= \frac{\partial V(t,t_i)}{\partial D^d(t,T)} = \frac{V(t,t_i)}{D^d(t,T)} \\ &\quad \text{number of domestic zero coupon bonds with maturity } T. \end{aligned} \tag{3.16}$$

Generalizing this result in an obvious manner the terminal value of the discrete arithmetic average exchange rate $A_d(T)$ can be duplicated by the following self-financing portfolio strategy at time $t \le T$

a) Foreign zero coupon bond market:
   For each $t_i > t$ hold a long position of $\frac{\phi^1(t,t_i)}{N}$ foreign zero coupon bonds with maturity t_i.
b) Domestic zero bond market
   i) For each $t_i > t$ take a short position of $\frac{\phi^2(t,t_i)}{N}$ domestic zero coupon bonds with maturity t_i.
   ii) For each $t_i > t$ hold a long position of $\frac{\phi^3(t)}{N}$ domestic zero coupon bonds with maturity T.
   iii) For each $t_i \le t$ hold a long position of $\frac{X(t_i)}{N}$ domestic zero coupon bonds with maturity T.

We observe that with stochastic interest rates the duplicating trading strategy implies continuous trading on the foreign and domestic market. Furthermore, the difference between the value of Asian call and put options is not completely determined by the market prices of the foreign and domestic zero coupon bonds and the exchange rate since it depends on the volatility structure. With respect to this, the arithmetic average and the future price under stochastic interest rates share the same property.

3.2. **Delayed Payment Exchange Rate Options.** A further application of the derived duplication strategy is the pricing and hedging of a delayed payment exchange rate option. The payoff of this option at maturity T is determined by $\left[X(\tau)D^f(\tau,\tau)-K\right]^+$ resp. $[K-X(\tau)D^f(\tau,\tau)]^+$ where $\tau<T$. The structure is similar to an usual European exchange rate call or put option. The difference is the delayed payment. By the change of measure technique, the arbitrage price C_d of the delayed payment exchange rate call is equal to

$$\begin{aligned} C_d(X(t),K,t,\tau,T) &= E_{P_d^*}\left[\left(\beta_{t,T}^d\right)^{-1}\left[X(\tau)D^f(\tau,\tau)-K\right]^+\Big|\mathcal{F}_t\right] \\ &= E_{P_d^*}\left[\left(\beta_{t,T}^d\right)^{-1}\left[\frac{V(\tau,\tau)}{D^d(\tau,T)}-K\right]^+\Bigg|\mathcal{F}_t\right] \\ &= D^d(t,T)E_{P_d^T}\left[\left[\frac{V(\tau,\tau)}{D^d(\tau,T)}-K\right]^+\Bigg|\mathcal{F}_t\right] \quad . \end{aligned} \tag{3.17}$$

The ratio process is a lognormal martingale under the domestic T-forward risk adjusted measure, i.e.

$$d\left(\frac{V(t,\tau)}{D^d(t,T)}\right)=\frac{V(t,\tau)}{D^d(t,T)}\eta^x(t,\tau)\cdot dW_d^T(t) \tag{3.18}$$

which implies that the solution for the delayed exchange rate call option is determined by

$$\begin{aligned} C_d(X(t),K,t,\tau,T) &= V(t,\tau)N(d_1)-KD^d(t,T)N(d_2) \\ d_{1,2} &= \frac{\ln\left(\frac{V(t,\tau)}{KD^d(t,T)}\right)\pm\frac{1}{2}\int_t^\tau||\eta^x(u,\tau)||^2du}{\left(\int_t^\tau||\eta^x(u,\tau)||^2du\right)^{\frac{1}{2}}} \\ &= \frac{\ln\left(\frac{X(t)D^f(t,\tau)}{KD^d(t,\tau)}\right)-\int_t^\tau\eta^x(u,\tau)\cdot\eta^d(u,\tau,T)du\pm\frac{1}{2}\int_t^\tau||\eta^x(u,\tau)||^2du}{\left(\int_t^\tau||\eta^x(u,\tau)||^2du\right)^{\frac{1}{2}}} \quad . \end{aligned} \tag{3.19}$$

Since $V(t,\tau)$ is not a traded asset the duplicating portfolio cannot be determined in terms of this value process. By applying our former results the duplicating strategy in terms of traded assets is given by

$$\begin{aligned} &C_d(X(t),K,t,\tau,T) \\ &= \left[X(t)D^f(t,\tau)\phi^1(t,\tau)+D^d(t,\tau)\phi^2(t,\tau)+D^d(t,T)\phi^3(t,\tau)\right]\cdot N(d_1)-KD^d(t,T)N(d_2) \\ &= X(t)D^f(t,\tau)\cdot\phi^1(t,\tau)N(d_1)+D^d(t,\tau)\cdot\phi^2(t,\tau)N(d_1)+D^d(t,T)\cdot\left[\phi^3(t,\tau)N(d_1)-KN(d_2)\right] \end{aligned}$$

where, obviously, the partial derivatives

$$\begin{aligned}\frac{\partial C_d(X(t),K,t,\tau,T)}{\partial X(t)D^f(t,\tau)} &= \phi^1(t,\tau)N(d_1)\\ \frac{\partial C_d(X(t),K,t,\tau,T)}{\partial D^d(t,\tau)} &= \phi^2(t,\tau)N(d_1)\\ \frac{\partial C_d(X(t),K,t,\tau,T)}{\partial D^d(t,T)} &= \phi^3(t,\tau)N(d_1)-KN(d_2)\end{aligned}$$

are the corresponding hedge ratios. For the standard European type exchange rate option, i.e. $T=\tau$ the pricing formula (3.19) coincides with the usual option pricing formula

$$\begin{aligned}C_e(X(t),K,t,\tau) &= X(t)D^f(t,\tau)N(d_1)-D^d(t,\tau)KN(d_2)\\ &= C_d(X(t),K,t,\tau,\tau),\end{aligned}$$

since (3.16) implies $\phi^1(t,\tau)=1, \phi^2(t,\tau)=-\phi^3(t)$ and $\eta^d(t,\tau,T)=0$ for $T=\tau$.

4. Pricing and Hedging of Asian Options: The Approximation Method

The calculation of the value of an Asian option and the hedge ratio requires the application of numerical methods and the specification of the volatility functions. With the Assumptions 2.1 and 2.2, the volatility functions of the exchange rate process and the domestic and foreign zero coupon bond markets are restricted to be non-stochastic. As a consequence, the discretely or continuously sampled arithmetic average of the exchange rate represents a weighed average of multi-dimensional lognormally distributed variables. A variety of numerical approaches for the pricing of Asian options can be considered in the case of one-dimensional lognormally distributed variables with constant volatility. Obviously these methods must be generalized in severals ways, since our situation implies time-dependent volatility functions, a complicated correlation structure and a multi-dimensional distribution.

As already discussed in a similar but less complex situation by Nielsen and Sandmann (1996) the Fast Fourier transformation technique applied in a paper by Carverhill and Clewlow (1990) to the pricing of Asian options can not be generalized to the situation of stochastic interest rates. The Edgeworth expansion which was applied by Turnbull and Wakeman (1991) involves complicated calculations of the first four moments of the arithmetic average. In the case of stochastic interest rates these calculations must be done in a recursive and hence by a slow algorithm similar to the one proposed by Nielsen and Sandmann. The Laplace transformation approach introduced by Geman and Yor (1993) is associated with the numerical inversion of a non-trivial Laplace transformation. Although this approach is of high mathematical elegance, the generalization of this method to the multi-dimensional case does not lead to a profound economic interpretation. Finally, it seems to us that a multinomial approximation based on the binomial approach of Hull and White (1993) is not appropriate in our case and would raise complex convergency questions. Therefore, we will not consider these methods.

The numerical analysis will involve two methods: a generalization of the Vorst (1992) approximation, and an adoption of the Rogers and Shi (1995a) approach.

The Vorst approximation is based on a payoff argument, i.e. a higher and lower payoff in terms of a geometric average option. Since the latter option possesses a closed form solution in the case of non stochastic volatility functions the approximation involves no further numerical problems and endows us with hedging strategies for closely related payoffs.

Rogers and Shi propose an approximation in the case of non stochastic interest rates. In addition, they are able to derive upper and lower bounds which are tight compared to the bounds derived by Vorst. Since the method is very efficient from the numerical point of view in the one-dimensional case, it seems to be interesting to generalize the method to our situation and to ask for some economic interpretation.

For simplicity of the exposition we restrict ourselves to the fixed strike Asian option. The floating strike case is covered if we apply a further change of measure, i.e. by choosing $X(T)D^f(T,T)$ as a numeraire. Note, that by

$$\begin{aligned} \left.\frac{dP_d^T}{dP_X}\right|_t &= \frac{X(T)D^f(T,T)}{E_{P_d^T}[X(T)D^f(T,T)|\mathcal{F}_t]} \\ &= \exp\left\{-\frac{1}{2}\int_t^T ||\eta^x(u,T)||^2\,du + \int_t^T \eta^x(u,T)\cdot dW_d^T(u)\right\} \end{aligned} \tag{4.1}$$

the new probability measure is defined. Furthermore, $dW_X(t) = dW_d^T - \eta^x(t,T)dt$ defines a vector Brownian motion under P_X and the arbitrage price of a floating strike Asian option can be expressed by

$$E_{P_d^*}\left[\left(\beta_{t,T}^d\right)^{-1}[X(T)-A_l(T)]^+|\mathcal{F}_t\right] = X(0)D(0,T)E_{P_X}\left[[1-\tilde{A}_l(T)]^+|\mathcal{F}_t\right] \tag{4.2}$$

with $\tilde{A}_d(T) = \frac{1}{N}\sum_{i=1}^{N}\frac{X(t_i)}{X(T)}$ resp. $\tilde{A}_c(T) = \frac{1}{T}\int_0^T \frac{X(u)}{X(T)}du$ and

$$\begin{aligned} \frac{X(t)}{X(T)} &= \frac{D^f(0,t)}{D^f(0,T)}\frac{D^d(0,T)}{D^d(0,t)}\exp\left\{\int_0^T \eta^x(u,T)\cdot\eta^d(u,t,T)du\right\} \\ &\exp\left\{-\frac{1}{2}\int_0^T ||\eta^x(u,t)1_{u\le t}-\eta^x(u,T)||^2\,du\int_0^T (\eta^x(u,t)1_{u\le t}-\eta^x(u,T))\cdot dW_X(u)\right\} \quad . \end{aligned} \tag{4.3}$$

Thus the floating strike case can be solved as the fixed strike case by changing from the domestic T-forward risk adjusted measure to the measure P_X and adjusting the volatility functions.

4.1. **Generalization of the Vorst (1992) approximation.** As already mentioned, the Vorst approximation is based on the dominance of the arithmetic over the geometric average, i.e.

$$\begin{aligned} A_d(T) &\ge G_d(T) := \left(\prod_{i=1}^{N} X(t_i)D^f(t_i,t_i)\right)^{\frac{1}{N}} \quad , \\ A_c(T) &\ge G_c(T) := \exp\left\{\frac{1}{T}\int_0^T \ln\left(X(u)D^f(u,u)\right)du\right\} \quad . \end{aligned} \tag{4.4}$$

If the geometric average is lognormally distributed the arbitrage price of an European type geometric average call or put option has a closed form solution (for $l \in \{c, d\}$)

$$
\begin{aligned}
(4.5)\qquad C_g(t,K,T) &= D^d(t,T)\left[E_{P_d^T}[G_l(T)|\mathcal{F}_t]N(d_1) - K\,N(d_2)\right] \\
(4.6)\qquad P_g(t,K,T) &= D^d(t,T)\left[K\,N(-d_2) - E_{P_d^T}[G_l(T)|\mathcal{F}_t]N(-d_1)\right] \\
E_{P_d^T}[G_l(T)|\mathcal{F}_t] &= \exp\left\{m_{G_l}(t) + \frac{1}{2}\sigma^2_{G_l}(t)\right\} \\
m_{G_l}(t) &= E_{P_d^T}[\ln G_l(T)|\mathcal{F}_t] \\
\sigma^2_{G_l}(t) &= V_{P_d^T}[\ln G_l(T)|\mathcal{F}_t] \\
d_1 &= \frac{m_{G_l}(t) - \ln(K) + \sigma^2_{G_l}(t)}{\sigma_{G_l}(t)} \quad , \quad d_2 = d_1 - \sigma_{G_l}(t).
\end{aligned}
$$

The Vorst approximation for an Asian option and the bounds are given by ($l \in \{d, c\}$)

$$
\begin{aligned}
(4.7)\qquad C_g(t,K,T) &\leq C_a(t,K,T) \approx D^d(t,T)\left[E_{P_d^T}[G_l(T)|\mathcal{F}_t]N(d_1') - K\,N(d_2')\right] \\
&\leq C_g(t,K,T) + D^d(t,T)E_{P_d^T}[A_l(T) - G_l(T)|\mathcal{F}_t] \\
(4.8)\qquad P_g(t,K,T) &\geq P_a(t,K,T) \approx D^d(t,T)\left[K\,N(-d_2') - E_{P_d^T}[G_l(T)|\mathcal{F}_t]N(-d_1')\right] \\
&\geq P_g(t,K,T) - D^d(t,T)E_{P_d^T}[A_l(T) - G_l(T)|\mathcal{F}_t]
\end{aligned}
$$

with

$$
d'_{1/2} = \frac{m_{G_l}(t) - \ln(K') \pm \sigma^2_{G_l}(t)}{\sigma_{G_l}(t)} \quad , \quad K' = K - E_{P_d^T}[A_l(T) - G_l(T)|\mathcal{F}_t].
$$

Since the expected value of the discretely and continuously sampled arithmetic average under the domestic T-forward risk adjusted measure is given by (3.11) and (3.12) respectively, we only need to determine the mean and variance of the logarithmic geometric average

Proposition 4.1:

Suppose the domestic and the foreign interest rate markets and the exchange rate satisfy the Assumptions

2.1 and 2.2. The expected value of the geometric exchange rate average is determined by

$$
\begin{aligned}
E_{P_d^T}[G_l(T)|\mathcal{F}_t] &= \exp\left\{ m_{G_l}(t) + \frac{1}{2}\sigma^2_{G_l}(t) \right\} \\
m_{G_d}(t) &:= \frac{1}{N}\left[\sum_{i=1}^{n(t)} \ln\left(\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)} \right) \right. \\
&\quad \left. + \sum_{i=n(t)+1}^{N} \left(\ln\left(X(t)\frac{D^f(t,t_i)}{D^d(t,t_i)} \right) - \frac{1}{2}\int_t^{t_i} \eta^x(u,t_i)\cdot\left(\eta^x(u,t_i) + 2\eta^d(u,t_i,T)\right) du \right) \right] \\
\sigma^2_{G_d}(t) &:= \frac{1}{N^2}\sum_{i=n(t)}^{N-1} \left(\int_{\max\{t_i,t\}}^{t_{i+1}} \left\| \sum_{j=i+1}^{N} \eta^x(u,t_j) \right\|^2 du \right) \\
m_{G_c}(t) &:= \frac{1}{T}\left[\int_{t_0}^{t} \ln\left(\frac{X(v)D^f(v,v)}{D^d(v,v)} \right) dv + \int_t^T \ln\left(\frac{X(t)D^f(t,v)}{D^d(t,v)} \right) dv \right. \\
&\quad \left. -\frac{1}{2}\int_t^T \left(\int_t^v \eta^x(u,v)\cdot\left[\eta^x(u,v) + 2\eta^d(u,v,T)\right] du \right) dv \right] \\
\sigma^2_{G_c}(t) &:= \frac{1}{T^2}\left[\int_t^T \left\| \int_u^T \eta^x(u,v)dv \right\|^2 du \right]
\end{aligned}
$$

Remark:

If $t = t_0$, i.e. the sampling period is just starting, the complexity of the above expressions are reduced, since $n(t_0) = 0$ and those parts which take care of the past exchange rate evolution are dropped.

Proof. The price process of the exchange rate under the domestic T-forward risk adjusted measure is given by (3.9). For the discrete geometric average this implies

$$
\begin{aligned}
V_{P_d^T}[\ln G_d(T)\big|\mathcal{F}_t] &= V_{P_d^T}\left[\frac{1}{N}\sum_{i=n(t)+1}^{N} \int_t^{t_i} \eta^x(u,t_i)\cdot dW_d^T(u) \right] \\
&= \frac{1}{N^2}\, V_{P_d^T}\left[\sum_{i=n(t)}^{N-1} \int_{\max\{t,t_i\}}^{t_{i+1}} \left(\sum_{j=i+1}^{N} \eta^x(u,t_j) \right)\cdot dW_d^T(u) \right] \\
&= \frac{1}{N^2}\sum_{i=n(t)}^{N-1} \left(\int_{\max\{t,t_i\}}^{t_{i+1}} \left\| \sum_{j=i+1}^{N} \eta^x(u,t_j) \right\|^2 du \right)
\end{aligned}
$$

$$\begin{aligned}
E_{P_d^T}[\ln G_d(T)|\mathcal{F}_t] &= E_{P_d^T}\left[\frac{1}{N}\sum_{i=1}^{N}\ln\left(\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right)\Bigg|\mathcal{F}_t\right] \\
&= \frac{1}{N}\left[\sum_{i=1}^{n(t)}\ln\left(\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right)+\sum_{i=n(t)+1}^{N}E_{P_d^T}\left[\ln\left(\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right)\Bigg|\mathcal{F}_t\right]\right] \\
&= \frac{1}{N}\left[\left(\sum_{i=1}^{n(t)}\ln\left(\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right)\right)\right. \\
&\quad \left.+\sum_{i=n(t)+1}^{N}\left(\ln\left(X(t)\frac{D^f(t,t_i)}{D^d(t,t_i)}\right)-\frac{1}{2}\int_t^{t_i}\eta^x(u,t_i)\cdot\left(\eta^x(u,t_i)+2\eta^d(u,t_i,T)\right)du\right)\right].
\end{aligned}$$

The last equality uses the independency of the n-dimensional Brownian motion. Similarly, the first and second moments in the continuous time case are determined by

$$\begin{aligned}
E_{P_d^T}\left[\ln G_c(T)\Big|\mathcal{F}_t\right] &= E_{P_d^T}\left[\frac{1}{T}\int_0^T\ln\left(\frac{X(v)D^f(v,v)}{D^d(v,v)}\right)dv\Bigg|\mathcal{F}_t\right] \\
&= \frac{1}{T}\left[\int_0^t\ln\left(\frac{X(v)D^f(v,v)}{D^d(v,v)}\right)dv+\int_t^T\ln\left(\frac{X(t)D^f(t,v)}{D^d(t,v)}\right)dv\right. \\
&\quad \left.-\frac{1}{2}\int_t^T\left[\int_t^v\eta^x(u,v)\cdot\left(\eta^x(u,v)+2\eta^d(u,v,T)\right)du\right]dv\right] \\
V_{P_d^T}\left[\ln G_c(T)\Big|\mathcal{F}_t\right] &= V_{P_d^T}\left[\frac{1}{T}\int_t^T\ln\left(X(v)D^f(v,v)\right)dv\Bigg|\mathcal{F}_t\right] \\
&= \frac{1}{T^2}\cdot V_{P_d^T}\left[\int_t^T\left(\int_t^v\eta^x(u,v)\cdot dW_d^T(u)\right)dv\Bigg|\mathcal{F}_t\right] \\
&= \frac{1}{T^2}\cdot V_{P_d^T}\left[\int_t^T\left(\int_u^T\eta^x(u,v)dv\right)\cdot dW_d^T(u)\Bigg|\mathcal{F}_t\right]=\frac{1}{T^2}\cdot\left[\int_t^T\left|\left|\int_u^T\eta^x(u,v)dv\right|\right|^2du\right]
\end{aligned}$$

where again the independency condition of the n-dimensional Brownian motion is applied to justify the linearity of the variance operator. □

The maximum approximation error $\bar{\varepsilon}_{Vorst}$ for an European type Asian call or put option implied by the Vorst (1992) approach is given by the discounted difference between the expected arithmetic and geometric average under the domestic T-forward risk adjusted measure. With the solution for the expected arithmetic average (3.10) and Proposition 4.1 the maximum approximation error $\bar{\varepsilon}_{Vorst}$ can be computed in terms of the volatility structure and the initial market prices.

4.2. **The Conditional Expectation Approach.** In their paper on the value of an Asian option Rogers and Shi (1995a) derive an approximation by use of a conditional expectation. The computation of the approximation can be done very fast numerically. In addition, the approximation error turns out to be quite small for an appropriate choice of the conditioning variable. This very nice approximation was introduced by Rogers and Shi for the Black and Scholes (1973) framework.

The objective of this section is to generalize the Rogers and Shi approach to the situation of an Asian exchange rate option with stochastic interest rates, i.e. we have to take care of the time dependent, multi-dimensional volatility structure. The generalization proves to be straightforward.

Whether the approximation is useful or not depends on the size of the approximation error. The latter depends on the choice of the conditioning random variable. The choice of this random variable is of course related to the specific contract under consideration. Like Rogers and Shi (1995b) we can not determine the conditioning variable which minimizes the approximation error. Instead, we follow their argument and motivate a specific choice.

Let Z be a random variable, $A_l(T)$ the discretely or continuously sampled arithmetic average and K the fixed strike of the Asian option. The forward value of an Asian option is equal to the expected value of the terminal payoff under the domestic T-forward risk adjusted measure P_d^T. For simplicity of the expressions, denote by $E^t[.]$ the conditional expection with respect to the σ-algebra $\mathcal{F}_t$. As in Rogers and Shi (1995a) this expectation satisfies the following relation

$$E^t_{P_d^T}\left[[A_l(T)-K]^+\right] = E^t_{P_d^T}\left[E^t_{P_d^T}\left[[A_l(T)-K]^+|Z\right]\right] \geq E^t_{P_d^T}\left[\left[E^t_{P_d^T}[A_l(T)-K|Z]\right]^+\right]. \tag{4.9}$$

The difference between the unconditional expectation and the lower bound, i.e. the forward value of the approximation error can be estimated by ε_{RS} :

$$\begin{aligned} 0 &\leq E^t_{P_d^T}\left[E^t_{P_d^T}\left[[A_l(T)-K]^+|Z\right] - \left[E^t_{P_d^T}[A_l(T)-K|Z]\right]^+\right] \\ &= \frac{1}{2}E^t_{P_d^T}\left[E^t_{P_d^T}\left[|A_l(T)-K| \ |Z\right] - \left|E^t_{P_d^T}[A_l(T)-K|Z]\right|\right] \\ &\leq \frac{1}{2}E^t_{P_d^T}\left[E^t_{P_d^T}\left[\left(\left|A_l(T)-K-E^t_{P_d^T}\left[A_l(T)-K|Z\right]\right|\right) \left|Z\right]\right]\right. \\ &\leq \frac{1}{2}E^t_{P_d^T}\left[\left(V_{P_d^T}[A_l(T)|Z]\right)^{\frac{1}{2}}\right] \leq \frac{1}{2}\left(E^t_{P_d^T}\left[V_{P_d^T}[A_l(T)|Z]\right]\right)^{\frac{1}{2}} =: \varepsilon_{RS} \end{aligned} \tag{4.10}$$

The main difference to the original Rogers and Shi approach is that we have to consider a n-dimensional random variable Z instead of a one-dimensional[5]. More precisely we restict Z by

Assumption 4.1:

a) *Z is a normally distributed n-dimensional random variable under the domestic forward risk adjusted measure P_d^T.*

b) *$Z = (Z_1, \cdots, Z_n)^T$ satisfies the following normalizing and independency conditions*

   i) $E_{P_d^T}[Z_j] = 0, \quad V_{P_d^T}[Z_j] = 1 \qquad \forall j = 1, \cdots, n$

   ii) $cov_{P_d^T}(Z_j, Z_k) = 0 \qquad \forall j \neq k \quad \forall j,k = 1, \cdots, n$

   iii) $cov_{P_d^T}(Z_j, W^T_{d,k}) = 0 \qquad \forall j \neq k \quad \forall j,k = 1, \cdots, n.$

[5]The same argument can be applied to the floating strike Asian option. Instead of the measure P_d^T the problem should be considered under the measure P_X (see (4.1)).

The lower bound for the arbitrage price of an Asian option is now determined by

Proposition 4.2:

Let Z be a random variable satisfying Assumption 4.1 and suppose that the exchange rate and the domestic and foreign zero coupon bond market satisfy the Assumption 2.1 and 2.2 resp. Denote by $K(t)$ the difference between the strike K and the known average at the t, i.e. $K(t) := K - \frac{T}{t}A_c(t)$ or $K(t) := K - \frac{N}{n(t)}A_d(t)$ resp.. The lower bound for an Asian exchange rate option with $K(t) > 0$ is given by

$$
\begin{aligned}
(4.11) \qquad & D^d(t,T)E^t_{P^T_d}[[A_d(T) - K]^+] \\
& \geq D^d(t,T)E^t_{P^T_d}\left[\left[\left(\frac{1}{N}\sum_{i=n(t)+1}^{N} F(t,t_i)\exp\left\{m_t(t_i)\cdot Z + \frac{1}{2}\nu_t^2(t_i,t_i)\right\}\right) - K(t)\right]^+\right]
\end{aligned}
$$

in the case of a discrete Asian option, and

$$
\begin{aligned}
(4.12) \qquad & D^d(t,T)E^t_{P^T_d}[[A_c(T) - K]^+] \\
& \geq D^d(t,T)E^t_{P^T_d}\left[\left[\frac{1}{T}\int_t^T F(t,u)\exp\left\{m_t(u)\cdot Z + \frac{1}{2}\nu_t^2(u,u)\right\}du - K(t)\right]^+\right]
\end{aligned}
$$

in the case of a continuous Asian option, where

$$
\begin{aligned}
F(t,\tau) &:= X(t)\frac{D^f(t,\tau)}{D^d(t,\tau)}\exp\left\{-\frac{1}{2}\int_t^\tau \eta^x(u,\tau)\cdot\left[\eta^x(u,\tau) + 2\eta^d(u,\tau,T)\right]du\right\} \quad \in \mathbb{R} \\
\nu_t^2(\tau,s) &:= \int_t^{\min\{s,\tau\}} \eta^x(u,\tau)\cdot\eta^x(u,s)du - m_t(\tau)\cdot m_t(s) \quad \in \mathbb{R} \\
m_t(\tau) &:= (m_{t,1}(\tau),\cdots,m_{t,n}(\tau))^T \quad \in \mathbb{R}^n \\
m_{t,k(t)} &:= E^t_{P^T_d}\left[Z_k\int_t^\tau \eta^x_k(u,\tau)\cdot dW^T_{d,k}(u)\right]
\end{aligned}
$$

Proof. For simplicity of the proof we consider the situation $t = 0$ and omit the subscript t. The assumptions on Z imply for the conditional expectation

$$
\begin{aligned}
E_{P^T_d}\left[\int_0^t \eta^x(u,t)\cdot dW^T_d(u)\middle| Z\right] &= \sum_{k=1}^{N} E_{P^T_d}\left[\int_0^t \eta^x_k(u,t)dW^T_{d,k}(u)\middle| Z_k\right] \\
&= \sum_{k=1}^{N} E_{P^T_d}\left[Z_k\int_0^t \eta^x_k(u,t)dW^T_{d,k}(u)\right]Z_k = m(t)\cdot Z
\end{aligned}
$$

and for the conditional covariance

$$\begin{aligned}
\nu^2(s,t) &:= \operatorname{cov}_{P_d^T}\left(\int_0^t \eta^x(u,t)\cdot dW_d^T(u), \int_0^s \eta^x(u,s)\cdot dW_d^T(u) \,\middle|\, Z \right) \\
&= \sum_{k=1}^N \operatorname{cov}_{P_d^T}\left(\int_0^t \eta_k^x(u,t) dW_{d,k}^T(u), \int_0^s \eta_k^x(u,s) dW_{d,k}^T(u) \,\middle|\, Z \right) \\
&= \sum_{k=1}^N \operatorname{cov}_{P_d^T}\left(\int_0^t \eta_k^x(u,t) dW_{d,k}^T(u), \int_0^s \eta_k^x(u,s) dW_{d,k}^T(u) \,\middle|\, Z_k \right) \\
&= \sum_{k=1}^N \left[\int_0^{\min\{s,t\}} \eta_k^x(u,t)\eta_k^x(u,s) du \right. \\
&\qquad \left. - E_{P_d^T}\left[Z_k \int_0^t \eta_k^x(u,t) dW_{d,k}^T(u) \right] E_{P_d^T}\left[Z_k \int_0^s \eta_k^x(u,s) dW_{d,k}^T(u) \right] \right] \\
&= \int_0^{\min\{s,t\}} \left(\sum_{k=1}^N \eta_k^x(u,t)\eta_k^x(u,s) \right) du - \sum_{k=1}^N m_k(t) m_k(s) \\
&= \int_0^{\min\{s,t\}} \eta^x(u,t)\cdot\eta^x(u,s) du - m(t)\cdot m(s)
\end{aligned}$$

The conditional expected value of the exchange rate is determined by

$$\begin{aligned}
E_{P_d^T}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)} \,\middle|\, Z \right] &= E_{P_d^T}\left[F(0,t) \exp\left\{ \int_0^t \eta^x(u,t)\cdot dW_d^T(u) \right\} \,\middle|\, Z \right] \\
&= F(0,t)\exp\left\{ m(t)\cdot Z + \frac{1}{2}\nu^2(t,t) \right\}
\end{aligned}$$

which obviously implies

$$E_{P_d^T}\left[E_{P_d^T}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)} \,\middle|\, Z \right] \right] = E_{P_d^T}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)} \right]$$

and with (4.9) the lower bounds are determined. □

So far the lower bound is depending on the specific choice of the conditioning random variable Z. Following Rogers and Shi (1995a) we fix Z such that the conditional variance of the terminal payoff is small. In order to calculate the conditional variance at time t note that

$$\begin{aligned}
&\operatorname{cov}_{P_d^T}^t \left(\exp\left\{ \int_t^\tau \eta^x(u,\tau)\cdot dW_d^T(u) \right\}, \exp\left\{ \int_t^\tau \eta^x(u,s)\cdot dW_d^T(u) \right\} \,\middle|\, Z \right) \\
= \; &\exp\left\{ (m_t(\tau)+m_t(s))\cdot Z + \frac{1}{2}(\nu_t^2(\tau,\tau)+\nu^2(s,s)) \right\} \left(\exp\{\nu_t^2(s,\tau)\} - 1 \right)
\end{aligned}$$

The conditional variance for the discrete Asian option at time t is equal to

$$\begin{aligned} & V^t_{P^T_d}[A_d(T)|Z] \\ = & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N} F(t,t_i)F(t,t_j) \\ & \exp\left\{(m_t(t_i)+m_t(t_j))\cdot Z + \frac{1}{2}(\nu_t^2(t_i,t_i)+\nu_t^2(t_j,t_j))\right\}\left(\exp\{\nu_t^2(t_i,t_j)\}-1\right) \end{aligned}$$

and for the continuous Asian option

$$\begin{aligned} & V^t_{P^T_d}[A_c(T)|Z] \\ = & \frac{1}{T^2}\int_t^T\left(\int_t^T F(t,u)F(t,s)\exp\left\{(m_t(u)+m_t(s))\cdot Z + \frac{1}{2}(\nu_t^2(u,u)+\nu_t^2(s,s))\right\}\left(\exp\{\nu_t^2(u,s)\}-1\right)ds\right)du \end{aligned}$$

Using the approximation $\exp\{x\}\approx 1+x$ and the fact $\nu_t^2(s,t)=\nu_t^2(t,s)$ the conditional variance can be approximated by

$$\begin{aligned} & V^t P^T_d[A_d(T)|Z] \\ \approx & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N}\left(X(t)^2\frac{D^f(t,t_i)}{D^d(t,t_i)}\frac{D^f(t,t_j)}{D^d(t,t_j)}\nu_t^2(t_i,t_j)\right. \\ & \left[1-\frac{1}{2}\int_t^{t_i}\eta^x(u,t_i)\cdot(\eta^x(u,t_i)+2\eta^d(u,t_i,T))du + m_t(t_i)\cdot Z + \frac{1}{2}\nu_t^2(t_i,t_i)\right. \\ & \left.\left.-\frac{1}{2}\int_t^{t_j}\eta^x(u,t_j)\cdot(\eta^x(u,t_j)+2\eta^d(u,t_j,T))du + m_t(t_j)\cdot Z + \frac{1}{2}\nu_t^2(t_j,t_j)\right]\right) \\ = & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\left(X(t)^2\frac{D^f(t,t_i)}{D^d(t,t_i)}\left[1-\int_t^{t_i}\left(\eta^x(u,t_i)\cdot(\eta^x(u,t_i)+2\eta^d(u,t_i,T))\right)du + 2m_t(t_i)\cdot Z + \nu_t^2(t_i,t_i)\right]\right. \\ & \left.\cdot\left(\sum_{j=n(t)+1}^{N}\nu_t^2(t_i,t_j)\frac{D^f(t,t_j)}{D^d(t,t_j)}\right)\right) \end{aligned}$$

Since by definition

$$\sum_{j=N(t)+1}^{N}\nu_t^2(t_i,t_j)\frac{D^f(t,t_j)}{D^d(t,t_j)} = \mathrm{cov}^t_{P^T_d}\left(\int_t^{t_i}\eta^x(u,t_i)\cdot W^T_d(u),\sum_{j=n(t)+1}^{N}\frac{D^f(t,t_j)}{D^d(t,t_j)}\int_t^{t_j}\eta^x(u,t_j)\cdot dW^T_d(u)\middle| Z\right)$$

the conditional covariance at time t is approximated by zero if we set the random variable $Z = (Z_1, \cdots, Z_n)^T$ equal to

$$
\begin{aligned}
(4.13) \qquad Z_k &= \frac{1}{\alpha_k}\left(\sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta_k^x(u,t_j)dW_{d,k}^T(u)\right) \qquad \forall k = 1,\cdots,n \\
\alpha_{t,k}^2 &= E_{P_d^T}^t\left[\left(\sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta_k^x(u,t_j)dW_{d,k}^T(u)\right)^2\right] \\
&= \sum_{j=n(t)+1}^{N-1}\left(\int_{t_j}^{t_{j+1}}\left(\sum_{i=j+1}^{N}\frac{D^f(t,t_i)}{D^f(t,t_i)}\eta_k^x(u,t_i)\right)^2 du\right) \\
\Longrightarrow \quad m_{t,k}(t_j) &= E_{P_d^T}^t\left[Z_k \int_t^{t_j} \eta_k^x(u,t_j)dW_{d,k}^T(u)\right] \\
&= \frac{1}{\alpha_k}\sum_{i=n(t)+1}^{N}\left[\int_t^{\min\{t_j,t_i\}}\left(\frac{D^f(t,t_i)}{D^d(t,t_i)}\eta_k^x(u,t_i)\eta_k^x(u,t_j)\right)du\right]
\end{aligned}
$$

Alternatively we could use the approximation

$$
\begin{aligned}
& V_{P_d^T}^t[A_d(T)|Z] \\
\approx\ & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N} X^2(t)\nu_t^2(t_i,t_j) \\
& \left[1+\ln F(t,t_i)+m_t(t_i)\cdot Z+\frac{1}{2}\nu_t^2(t_i,t_i)+\ln F(t,t_j)+m_t(t_j)\cdot Z+\frac{1}{2}\nu_t^2(t_j,t_j)\right] \\
=\ & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\left(X^2(t)\left[1+2\ln F(t,t_i)+2m_t(t_i)\cdot Z+\nu_t^2(t_i,t_i)\right]\sum_{j=n(t)+1}^{N}\nu_t^2(t_i,t_j)\right)
\end{aligned}
$$

This suggests a specification of the random variable Z equal to

$$
\begin{aligned}
(4.14) \qquad Z_k &= \frac{1}{\alpha_k}\left(\sum_{j=n(t)+1}^{N}\int_t^{t_j}\eta_k^x(u,t_j)\cdot dW_{d,k}^T(u)\right) \\
\alpha_{t,k}^2 &= E_{P_d^T}^t\left[\left(\sum_{j=n(t)+1}^{N}\int_t^{t_j}\eta_k^x(u,t_j)\cdot dW_{d,k}^T(u)\right)^2\right] = \sum_{j=n(t)+1}^{N-1}\left(\int_{t_j}^{t_{j+1}}\left(\sum_{i=j+1}^{N}\eta_k^x(u,t_i)\right)^2 du\right) \\
\Longrightarrow \quad m_{t,k}(t_j) &= \frac{1}{d_k}\sum_{i=n(t)+1}^{N}\left[\int_t^{\min\{t_j,t_i\}}\left(\eta_k^x(u,t_i)\cdot\eta_k^x(u,t_j)\right)du\right]
\end{aligned}
$$

In case of a continuously sampled Asian option, the same argument implies that the conditional covariance is approximated by zero if we choose Z equal to

$$
\begin{aligned}
(4.15) \qquad Z_k &= \frac{1}{\alpha_k}\int_t^T \left(\int_t^s \eta_k^x(u,s)dW_{d,k}^T(u)\right)\frac{D^f(t,s)}{D^d(t,s)}ds \\
\alpha_{t,k}^2 &= V_{P_d^T}^t[Z_k] = \int_t^T \left(\int_u^T \eta_k^x(u,s)\frac{D^f(t,s)}{D^d(t,s)}ds\right)^2 du \\
\Longrightarrow \quad m_{t,k}(\tau) &= \frac{1}{\alpha_k}\int_t^\tau \left(\int_u^T \eta_k^x(u,s)\frac{D^f(t,s)}{D^d(t,s)}ds\right)\cdot \eta_k^x(u,\tau)du \quad .
\end{aligned}
$$

Using the alternative approximation the Z vector is found to be

$$
\begin{aligned}
(4.16) \qquad Z_k &= \frac{1}{\alpha_k}\int_t^T\int_t^s \eta_k^x(u,s)dW_{d,k}^T(u)ds \\
\alpha_{t,k}^2 &= V_{P_d^T}^t[Z_k] = \int_t^T\left(\int_u^T \eta_k^x(u,s)ds\right)^2 du \\
\Longrightarrow \quad m_{t,k}(\tau) &= \frac{1}{\alpha_k}\int_t^\tau\left(\int_u^T \eta_k^x(u,s)ds\right)\cdot \eta_k^x(u,\tau)du \quad .
\end{aligned}
$$

Numerical integration then has to be performed to find the approximate price represented by the lower bound.

4.3. **The Conditional Expectation Approach. Closed Form Solution.** In this section the n–dimensional random variable, Z, will be replaced by a 1–dimensional random variable and it is shown that a closed form solution for the lower bound in this situation can be derived. This closed form solution can be interpreted as a portfolio of European type options, where each option has a structure similar to the delayed exchange rate option discussed in section 3.2. Of course the results can be applied to the Black-Scholes model, i.e. to the case of deterministic interest rates and constant volatility for the underlying asset.

Proposition 4.3:

Let Z be a one-dimensional standard normally distributed random variable and suppose that the exchange rate and the domestic and foreign zero coupon bond markets satisfy the Assumption 2.1 and 2.2 resp. Denote by $K(t)$ the difference between the strike K and the known average at the t, i.e. $K(t) := K - \frac{T}{t}A_c(t)$ or $K(t) := K - \frac{N}{n(t)}A_d(t)$ resp.. The lower bound for an Asian exchange rate option with $K(t) > 0$ is given by

$$
\begin{aligned}
(4.17) \qquad & D^d(t,T)E_{P_d^T}^t[[A_d(T)-K]^+] \\
& \geq D^d(t,T)E_{P_d^T}^t\left[\left[\left(\frac{1}{N}\sum_{i=n(t)+1}^N F(t,t_i)\exp\left\{m_t(t_i)Z+\frac{1}{2}\nu_t^2(t_i,t_i)\right\}\right)-K(t)\right]^+\right]
\end{aligned}
$$

in the case of a discrete Asian option, and

$$D^d(t,T)E^t_{P^T_d}[[A_c(T)-K]^+] \tag{4.18}$$
$$\geq D^d(t,T)E^t_{P^T_d}\left[\left[\frac{1}{T}\int_t^T F(t,u)\exp\left\{m_t(u)Z+\frac{1}{2}\nu_t^2(u,u)\right\}du-K(t)\right]^+\right]$$

in the case of a continuous Asian option, where

$$\begin{aligned}
F(t,\tau) &:= X(t)\frac{D^f(t,\tau)}{D^d(t,\tau)}\exp\left\{-\frac{1}{2}\int_t^\tau \eta^x(u,\tau)\cdot\left[\eta^x(u,\tau)+2\eta^d(u,\tau,T)\right]du\right\} \quad \in \mathbb{R}\\
\nu_t^2(\tau,s) &:= \int_t^{\min\{s,\tau\}} \eta^x(u,\tau)\cdot\eta^x(u,s)du - m_t(\tau)m_t(s) \quad \in \mathbb{R}\\
m_t(\tau) &:= E^t_{P^T_d}\left[Z\int_t^\tau \eta^x(u,\tau)\cdot dW^T_d(u)\right]
\end{aligned}$$

Proof. For simplicity of the proof we consider the situation $t=0$ and omit the subscript t. The assumptions on Z imply for the conditional expectation

$$E_{P^T_d}\left[\int_0^t \eta^x(u,t)\cdot dW^T_d(u)\middle| Z\right] = E_{P^T_d}\left[Z\int_0^t \eta^x(u,t)\cdot dW^T_d(u)\right]Z = m(t)Z$$

and for the conditional covariance

$$\begin{aligned}
\nu^2(s,t) &:= \operatorname{cov}_{P^T_d}\left(\int_0^t \eta^x(u,t)\cdot dW^T_d(u), \int_0^s \eta^x(u,s)\cdot dW^T_d(u)\middle| Z\right)\\
&= \int_0^{\min\{s,t\}} \eta^x(u,t)\cdot\eta^x(u,s)du - m(t)m(s)
\end{aligned}$$

The conditional expected value of the exchange rate is determined by

$$\begin{aligned}
E_{P^T_d}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)}\middle| Z\right] &= E_{P^T_d}\left[F(0,t)\exp\left\{\int_0^t \eta^x(u,t)\cdot dW^T_d(u)\right\}\middle| Z\right]\\
&= F(0,t)\exp\left\{m(t)Z+\frac{1}{2}\nu^2(t,t)\right\}
\end{aligned}$$

which obviously implies

$$E_{P^T_d}\left[E_{P^T_d}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)}\middle| Z\right]\right] = E_{P^T_d}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)}\right]$$

and with (4.9) the lower bounds are determined. □

The formal difference between the exact equation for the value of an Asian option and the lower bound is that we are not integrating with respect to a n-dimensional Brownian motion. Applying an argument by Jamshidian (1991), a closed form solution for the lower bound of a fixed strike discretely sampled Asian option can now be derived.

Theorem 4.1:

Under the Assumptions of Proposition 4.3 and the assumption that $sign(m(t_i)) = sign(m(t_j)) \forall i, j$, the lower bound of a fixed strike discretely sampled Asian call option at time t with $K(t) > 0$ is equal to

$$D^d(t,T)\frac{1}{N}\sum_{i=n(t)+1}^{N}\left(E^t_{P^T_d}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]N\left(-z^* + m_t(t_i)\right) - K_i(t)N(-z^*)\right) \tag{4.19}$$

where $N(\cdot)$ denotes the standard normal distribution and z^* is the unique solution at time t of the problem

$$\frac{1}{N}\sum_{i=n(t)+1}^{N} F(t,t_i)\exp\left\{m_t(t_i)z + \frac{1}{2}\nu_t^2(t_i,t_i)\right\} = K(t)$$

and

$$K_i(t) := F(t,t_i)\exp\left\{m_t(t_i)z^* + \frac{1}{2}\nu_t^2(t_i,t_i)\right\} \quad .$$

Proof. To simplify the proof it is sufficient to consider the situation at time zero. The lower bound of the fixed strike discretely sampled Asian call option is equal to

$$D^d(0,T)E_{P^T_d}\left[\left[\frac{1}{N}\sum_{i=1}^{N}F(0,t_i)\exp\left\{m(t_i)Z + \frac{1}{2}\nu^2(t_i,t_i)\right\} - K\right]^+\right]$$

Due to the *sign* condition on the $m(t_i)$ there exists a unique z^* such that the arithmetic average is equal to K. Formally we can now apply Jamshidian's (1991) argument, i.e.

$$\begin{aligned}
& E_{P^T_d}\left[\left[\frac{1}{N}\sum_{i=1}^{N}F(0,t_i)\exp\left\{m(t_i)Z + \frac{1}{2}\nu^2(t_i,t_i)\right\} - K\right]^+\right] \\
= & \int_{z^*}^{+\infty}\frac{1}{\sqrt{2\pi}}\left[\frac{1}{N}\sum_{i=1}^{N}F(0,t_i)\exp\left\{m(t_i)x + \frac{1}{2}\nu^2(t_i,t_i)\right\} - K\right]\exp\left\{-\frac{x^2}{2}\right\}dx \\
= & \frac{1}{N}\left[\sum_{i=1}^{N}\left(F(0,t_i)\exp\left\{\frac{1}{2}\nu^2(t_i,t_i) + \frac{1}{2}m(t_i)^2\right\}\int_{z^*}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-m(t_i))^2}{2}\right\}dx\right.\right. \\
& \left.\left. -K_i\int_{z^*}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx\right)\right] \\
= & \frac{1}{N}\left[\sum_{i=1}^{N}\left(F(0,t_i)\exp\left\{\frac{1}{2}\nu^2(t_i,t_i) + \frac{1}{2}m(t_i)^2\right\}N(-z^* + m(t_i)) - K_iN(-z^*)\right)\right]
\end{aligned}$$

where by definition

$$\begin{aligned}
& F(0,t_i)\exp\left\{\frac{1}{2}\nu^2(t_i,t_i) + \frac{1}{2}m(t_i)^2\right\} \\
= & \frac{X(t_i)D^f(0,t_i)}{D^d(0,t_i)}\exp\left\{-\int_0^{t_i}\eta^x(u,t_i)\eta^d(u,t_i,T)du\right\} \\
= & E_{P^T_d}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]
\end{aligned}$$

□

The condition on $sign(m(t_i))$ in Theorem 4.1 is difficult to analyse in general. Assuming a Vasicek term structure model for the domestic as well as for the foreign bond market it turns out that the condition is fulfilled for some correlation structures. On the other hand it is also possible to find situations where the condition is not fulfilled.

The lower bound for a fixed strike discrete Asian put option is equal to

$$\frac{D^d(t,T)}{N}\left(\sum_{i=n(t)+1}^{N} K_i(t)N(z^*) - E^t_{P^T_d}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]N(z^* - m_t(t_i))\right) \tag{4.20}$$

In the case of continuous sampling the lower bounds can be expressed by

$$\frac{D^d(t,T)}{T}\left[\int_t^T E^t_{P^T_d}\left[\frac{X(u)D(u,u)}{D^d(u,u)}\right]N(-z^* + m_t(u))du - K(t)N(-z^*)\right] \tag{4.21}$$

for the fixed strike call and by

$$\frac{D^d(t,T)}{T}\left[K(t)N(z^*) - \int_t^T E^t_{P^T_d}\left[\frac{X(u)D^f(u,u)}{D^d(u,u)}\right]N(z^* - m_t(u))du\right] \tag{4.22}$$

for the fixed strike put, where z^* is the solution of

$$\frac{1}{T}\int_t^T F(t,u)\exp\left\{m_t(u)z + \frac{1}{2}\nu_t^2(u,u)\right\}du = K - \frac{1}{T}\int_0^t X(u)D^f(u,u)du =: K(t).$$

In both situations the value of z^* can be computed with standard algorithms. Furthermore, applying the change of measure to P_X, similar closed form solutions can be derived to approximate the floating strike Asian option.

So far the closed form solution of the lower bound depends on the specific choice of the conditioning random variable Z. As Rogers and Shi (1995a), we fix Z such that the conditional variance of the terminal payoff is small. In order to calculate the conditional variance at time t note that

$$\begin{aligned}
& \operatorname{cov}^t_{P^T_d}\left(\exp\left\{\int_t^\tau \eta^x(u,\tau)\cdot dW^T_d(u)\right\}, \exp\left\{\int_t^s \eta^x(u,s)\cdot dW^T_d(u)\right\}\bigg| Z\right) \\
= \ & \exp\left\{(m_t(\tau) + m_t(s))Z + \frac{1}{2}(\nu_t^2(\tau,\tau) + \nu^2(s,s))\right\}\left(\exp\{\nu_t^2(s,\tau)\} - 1\right)
\end{aligned}$$

The conditional variance for the discrete Asian option at time t is equal to

$$\begin{aligned}
& V^t_{P^T_d}[A_d(T)|Z] \\
= \ & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N} F(t,t_i)F(t,t_j) \\
& \exp\left\{(m_t(t_i) + m_t(t_j))Z + \frac{1}{2}(\nu_t^2(t_i,t_i) + \nu_t^2(t_j,t_j))\right\}\left(\exp\{\nu_t^2(t_i,t_j)\} - 1\right)
\end{aligned}$$

and for the continuous Asian option

$$
\begin{aligned}
& V^t_{P_d^T}[A_c(T)|Z] \\
= \; & \frac{1}{T^2}\int_t^T \left(\int_t^T F(t,u)F(t,s)\exp\left\{ (m_t(u)+m_t(s))Z + \frac{1}{2}(\nu_t^2(u,u)+\nu_t^2(s,s)) \right\} \left(\exp\{\nu_t^2(u,s)\} - 1 \right) ds \right) du
\end{aligned}
$$

Using the approximation $\exp\{x\} \approx 1 + x$ and the fact $\nu_t^2(s,t) = \nu_t^2(t,s)$ the conditional variance can be approximated by

$$
\begin{aligned}
& V^t P_d^T[A_d(T)|Z] \\
\approx \; & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N} \Bigg(X(t)^2 \frac{D^f(t,t_i)}{D^d(t,t_i)}\frac{D^f(t,t_j)}{D^d(t,t_j)}\nu_t^2(t_i,t_j) \\
& \qquad \Bigg[1 - \frac{1}{2}\int_t^{t_i} \eta^x(u,t_i)\cdot(\eta^x(u,t_i) + 2\eta^d(u,t_i,T))du + m_t(t_i)Z + \frac{1}{2}\nu_t^2(t_i,t_i) \\
& \qquad -\frac{1}{2}\int_t^{t_j} \eta^x(u,t_j)\cdot(\eta^x(u,t_j) + 2\eta^d(u,t_j,T))du + m_t(t_j)Z + \frac{1}{2}\nu_t^2(t_j,t_j) \Bigg] \Bigg) \\
= \; & \frac{1}{N^2}\sum_{i=n(t)+1}^{N} \Bigg(X(t)^2 \frac{D^f(t,t_i)}{D^d(t,t_i)} \left[1 - \int_t^{t_i} \left(\eta^x(u,t_i)\cdot(\eta^x(u,t_i) + 2\eta^d(u,t_i,T)) \right) du + 2m_t(t_i)Z + \nu_t^2(t_i,t_i) \right] \\
& \qquad \cdot \left(\sum_{j=n(t)+1}^{N} \nu_t^2(t_i,t_j)\frac{D^f(t,t_j)}{D^d(t,t_j)} \right) \Bigg)
\end{aligned}
$$

Since by definition

$$
\sum_{j=N(t)+1}^{N} \nu_t^2(t_i,t_j)\frac{D^f(t,t_j)}{D^d(t,t_j)} \;=\; \mathrm{cov}^t_{P_d^T}\left(\int_t^{t_i} \eta^x(u,t_i)\cdot dW_d^T(u), \sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta^x(u,t_j)\cdot dW_d^T(u) \middle| Z \right),
$$

the conditional covariance at time t is approximated by zero if we set the random variable Z equal to

$$
\begin{aligned}
Z &= \frac{1}{\alpha_t}\left(\sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta^x(u,t_j)\cdot dW_d^T(u)\right) \\
\alpha_t^2 &= V_{P_d^T}^t\left[\sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta^x(u,t_j)\cdot dW_d^T(u)\right] \\
&= \sum_{k=1}^{n}\left(V_{P_d^T}^t\left[\sum_{j=n(t)}^{N-1} \int_{\max\{t,t_j\}}^{t_{j+1}} \left(\sum_{i=j+1}^{N} \frac{D^f(t,t_i)}{D^f(t,t_i)}\eta_k^x(u,t_i)\right)\cdot dW_{d,k}^T(u)\right]\right) \\
&= \sum_{k=1}^{n}\left(\sum_{j=n(t)}^{N-1}\left(\int_{\max\{t,t_j\}}^{t_{j+1}} \left(\sum_{i=j+1}^{N} \frac{D^f(t,t_i)}{D^f(t,t_i)}\eta_k^x(u,t_i)\right)^2 du\right)\right) \\
\Longrightarrow \quad m_t(t_i) &= E_{P_d^T}^t\left[Z\int_t^{t_i} \eta_k^x(u,t_i)\cdot dW_d^T(u)\right] \\
&= \frac{1}{\alpha_t}\sum_{j=n(t)+1}^{N}\left[\frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{\min\{t_j,t_i\}} \eta^x(u,t_i)\cdot\eta^x(u,t_j)du\right]
\end{aligned}
\tag{4.23}
$$

Alternatively we could use the approximation

$$
\begin{aligned}
V_{P_d^T}^t[A_d(T)|Z] &\approx \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N} X^2(t)\nu_t^2(t_i,t_j) \\
&\quad \left[1+\ln F(t,t_i)+m_t(t_i)Z+\frac{1}{2}\nu_t^2(t_i,t_i)+\ln F(t,t_j)+m_t(t_j)Z+\frac{1}{2}\nu_t^2(t_j,t_j)\right] \\
&= \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\left(X^2(t)\left[1+2\ln F(t,t_i)+2m_t(t_i)Z+\nu_t^2(t_i,t_i)\right]\sum_{j=n(t)+1}^{N}\nu_t^2(t_i,t_j)\right)
\end{aligned}
$$

This suggests a specification of the random variable Z equal to

$$
\begin{aligned}
Z &= \frac{1}{\alpha_t}\left(\sum_{j=n(t)+1}^{N} \int_t^{t_j} \eta^x(u,t_j)\cdot dW_d^T(u)\right) \\
\alpha_t^2 &= \sum_{k=1}^{n}\left(\sum_{j=n(t)}^{N-1}\left(\int_{\max\{t,t_j\}}^{t_{j+1}} \left(\sum_{i=j+1}^{N} \eta_k^x(u,t_i)\right)^2 du\right)\right) \\
\Longrightarrow \quad m_t(t_i) &= \frac{1}{\alpha_t}\sum_{j=n(t)+1}^{N}\left[\int_t^{\min\{t_j,t_i\}} \eta^x(u,t_i)\cdot\eta^x(u,t_j)du\right]
\end{aligned}
\tag{4.24}
$$

In case of a continuously sampled Asian option, the same argument implies that the conditional covariance is approximated by zero if we choose Z equal to

$$
\begin{aligned}
(4.25) \qquad Z &= \frac{1}{\alpha_t} \int_t^T \left(\int_t^s \eta^x(u,s) \cdot dW_d^T(u) \right) \frac{D^f(t,s)}{D^d(t,s)} ds \\
\alpha_t^2 &= \sum_{k=1}^n \left[\int_t^T \left(\int_u^T \eta_k^x(u,s) \frac{D^f(t,s)}{D^d(t,s)} ds \right)^2 du \right] \\
\Longrightarrow \quad m_t(\tau) &= \frac{1}{\alpha_t} \int_t^T \frac{D^f(t,s)}{D^d(t,s)} \left(\int_t^{\min\{\tau,s\}} \eta^x(u,s) \cdot \eta^x(u,\tau) du \right) ds \quad .
\end{aligned}
$$

Using the alternative approximation Z is found to be

$$
\begin{aligned}
(4.26) \qquad Z &= \frac{1}{\alpha_t} \int_t^T \left(\int_t^s \eta^x(u,s) \cdot dW_d^T(u) \right) ds \\
\alpha_t^2 &= \sum_{k=1}^n \left[\int_t^T \left(\int_u^T \eta_k^x(u,s) ds \right)^2 du \right] \\
\Longrightarrow \quad m_t(\tau) &= \frac{1}{\alpha_t} \int_t^T \left(\int_t^{\min\{\tau,s\}} \eta^x(u,s) \cdot \eta^x(u,\tau) du \right) ds \quad .
\end{aligned}
$$

The approximation error to the price of an exchange rate Asian option is now determined by equation (4.10). For a discrete Asian option the approximation error is estimated by:

$$
\begin{aligned}
& D^d(t,T)\, \varepsilon_{RS} \\
= \; & \frac{D^d(t,T)X(t)}{N} \Bigg[\sum_{i=n(t)+1}^{N} \sum_{j=n(t)+1}^{N} \Bigg(\frac{D^f(t,t_i)}{D^d(t,t_i)} \frac{D^f(t,t_j)}{D^d(t,t_j)} \exp\{m_t(t_i) m_t(t_j)\} \\
& \exp\left\{ -\frac{1}{2} \int_t^{t_i} \eta^x(u,t_i) \cdot \eta^d(u,t_i,T) du - \frac{1}{2} \int_t^{t_j} \eta^x(u,t_j) \cdot \eta^d(u,t_j,T) du \right\} \left(\exp\left\{ \nu_t^2(t_i,t_j) \right\} - 1 \right) \Bigg) \Bigg]^{\frac{1}{2}},
\end{aligned}
$$

and in the case of a continuous Asian option by:

$$
\begin{aligned}
& D^d(t,T)\, \varepsilon_{RS} \\
= \; & \frac{D^d(t,T)X(t)}{T} \Bigg[\int_t^T \Bigg(\int_t^T \frac{D^f(t,t_i)}{D^d(t,t_i)} \frac{D^f(t,t_j)}{D^d(t,t_j)} \exp\{m_t(t_i) m_t(t_j)\} \\
& \exp\left\{ -\frac{1}{2} \int_t^u \eta^x(v,u) \cdot \eta^d(v,u,T) dv - \frac{1}{2} \int_t^s \eta^x(v,s) \cdot \eta^d(v,s,T) dv \right\} \left(\exp\left\{ \nu_t^2(u,s) \right\} - 1 \right) du \Bigg) ds \Bigg]^{\frac{1}{2}} .
\end{aligned}
$$

5. Conclusion

Foreign exchange rate Asian options have been analysed with the primary aim to find a good approximation for their pricing. A model for the foreign exchange should describe not only the exchange rate itself but also the term structure of interest rates in the two countries. The total correlation structure in this two-country economy will be important for the pricing purpose. The correlation structure influences the Asian put-call parity relationship, and it turns out that the pricing strategy applied to establish the put-call parity has to be continuously updated. This is in contrast to the classical put-call parity relationship in the Black and Scholes setting.

An important feature of the Asian option is the delayed payment structure. It is argued that this feature is also important for ordinary options and its influence on pricing is analysed.

The Asian option is in this paper priced through two different approximation methods, the Vorst (1992) approximation and, as the paper's main contribution to the literature, through the Rogers and Shi (1995a) approach combined with an exact pricing of their approximated lower bound. This exact pricing takes the form of a sum of delayed payment options. The pricing error for the Rogers and Shi approximation is developed.

Furthermore, we conjecture to apply the methodology to other financial contracts with a similar mathematical structure. Examples are basket options and more generally n-color rainbow type options. As in our case the methodology may imply an approximation to the value of these options by a portfolio of simpler financial contracts.

It is ongoing research to do numerical analysis of the theoretical results presented in this paper and to extend the results to cover the important topic of hedging long term Asian options in an international setting.

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