Discussion Paper No. B–446

Valuation of Barrier Options in a Black–Scholes Setup with Jump Risk

Dietmar P.J. Leisen*

This version: January 12, 1999.

Abstract

This paper discusses the pitfalls in the pricing of barrier options using approximations of the underlying continuous processes via discrete lattice models. These problems are studied first in a Black–Scholes model. Improvements result from a trinomial model and a further modified model where price changes occur at the jump times of a Poisson process. After the numerical difficulties have been resolved in the Black–Scholes model, unpredictable discontinuous price movements are incorporated.

Keywords

binomial model, option valuation, lattice–approach, barrier option

JEL Classification

C63, G12, G13

* Stanford University, Hoover Institution, Stanford, CA 94305, U.S.A., and University of Bonn, Department of Statistics, Adenauerallee 24–42, 53113 Bonn, Germany, email: leisen@hoover.stanford.edu

We are grateful to Dieter Sondermann and Holger Wiesenberg for comments and assistance. Financial support from the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at Rheinische Friedrich-Wilhelms-Universität Bonn is gratefully acknowledged.

1 Introduction

Over time, barrier options have become increasingly popular to reduce the cost of plain vanilla options while incorporating individual views of market participants concerning the asset evolution in an easy way: the payoff of barrier options depends on whether the asset price path will (not) cross a prespecified boundary. Of course, similar to standard options, they are also used to insure against price drops below the barrier. Standard barrier options do then either pay off a call or a put at the maturity date. For all these eight standard barrier options with single barrier, closed form solutions are available in the Black–Scholes setup (see Rubinstein and Reiner (1991) and Carr (1995)). For double barrier options, however, analytical approximations are known only in some specific cases (see Kunitomo and Ikeda (1992)). Numerical procedures have to be applied to come up with prices, especially in those generalizations of the Black–Scholes setup, in which jumps are possible.

In this paper we are interested in the "efficient" pricing of barrier options using approximations of the underlying process. It is well known that binomial models suffer from numerical deficiencies with barrier options: increasing the refinement, prices converge erratically in a saw–tooth manner to the continuous time price and even high refinements do not ensure adequate accuracy. Many authors addressed this problem and suggested adjustments. Ritchken (1995), and Cheuk and Vorst (1996) constructed trees where the nodes lie on the barrier. Figlewski and Gao (1998) refine the tree further at the barrier. Our improvements are related to the finite difference approach (see Boyle and Tian (1998)): we start from a discretization of the asset space, instead of discretizing time first, as previous approaches did. With barrier options this is a straightforward way to have nodes on critical levels by construction. We recover binomial models with the specific refinement of Boyle and Lau (1994). To incorporate our approach in full generality we make use of trinomial models.

Since discretizing the asset space breaks up the time discretization, we allow for random trading, similar to Leisen (1998b) for American put options, and Rogers and Stapleton (1998) for barrier options. However, the latter did not recognize the numerical deficiencies resulting from the strike at maturity, which will be prevented in our trinomial model. We also differ from their approach by assuming that trading dates are the jump times of a Poisson process, which results in simple valuation formulas. Our parameters are set in accordance with a convergence theorem of the processes which also ensures convergence of prices. The model smoothes the convergence structure even better. The order of convergence increases from 1/2 to 1 and removes the wavy patterns. Using extrapolation we are able to obtain even quadratic order.

After the numerical deficiencies have been removed in the Black–Scholes setup, we turn to the framework of Merton (1976), adding a compound Poisson process to the Black–Scholes model. This models the empirical observation of sudden strong price changes. An extension of binomial models to this setup was proposed by Amin (1993), which unfortunately inherits from there its poor convergence properties, especially with barrier options. Our model with random jump times driven by a Poisson process allows us in a simple way and intuitive way to incorporate jumps. It inherits the extraordinary convergence properties from the trinomial model in its randomized version. Again, a theorem ensures convergence to the continuous time solution.

The remainder of the paper is organized as follows. In section 2 we present the jump–diffusion setup as well as the barrier option contract. Section 3 explains the pitfalls in discretizing according to CRR. In section 4 these difficulties are circumvented using a suitable trinomial model. This model is then randomized. Section 5 incorporates "strong" jumps. Section 6 discusses barrier option valuation and the price accuracy in our framework. Section 7 concludes the paper.

2 The Setup

On a probability space $(\Omega, \mathcal{F}, P)$ we study an economy (B, S), consisting of the Bond B and the stock S. The interest rate r is assumed to be constant over time ($B_t = \exp\{rt\}$) and the stock price evolves under the objective measure P according to

$$
\begin{aligned}
& S_t = S_0 \cdot G_t \cdot J_t \ , && (1)\\
\text{where} \quad & G_t = \exp\{\mu t + \sigma W_t\} \ , && (2)\\
\text{and} \quad & J_t = \prod_{i=1}^{N_t} V_i \ . && (3)
\end{aligned}
$$

$(N_t)_t$ is a Poisson process with constant parameter λ, $(V_i)_i$ a sequence of non–negative iid random variables, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$.

The process $(G_t)_t$ is the continuous process known as geometric Brownian motion and has stationary, Gaussian returns. It was suggested by Samuelson (1965) to model the evolution of a stock. Since Black and Scholes (1973) it is used as one of the standard financial models. S evolves according to G until the next jump time τ of the Poisson process at which N changes from, say, i to $i+1$. We then observe a per–cent change $V_i \Leftrightarrow 1$, i.e., the stock changes value from $S_{\tau-}$ before the jump to $S_{\tau-} \cdot V_i$.

So, the two parts can be interpreted as follows: $(G_t)_t$ models the "typical" evolution of the stock under the "normal" arrival of information, whereas $(J_t)_t$ models jumps in the stock prices, due to some rare strong information shock. Since the Poisson process is "memoryless," the expected time until the next shock occurs is equal to $1/\lambda$, independent of current time. Merton (1976) studied this model under the assumptions that the $(V_i)_i$ are lognormally distributed random variables and λ and $(N_t)_t$ are specific for the firm under consideration.

The Black–Scholes setup is a so–called *complete market*, i.e., any contingent claim on the stock can be hedged (see Duffie (1992)). The no–arbitrage principle is sufficient to price any claim. In the language of Harrison and Kreps (1979) this means that there is a probability measure Q, equivalent to P, and under which $(G_t/B_t)_t$ is a martingale. Such a measure gives rise to a linear pricing operator; it is completely specified for the (B, G) market by $\mu^G = r \Leftrightarrow \sigma^2/2$.

In our setup, however, we are in an incomplete market; the unforeseeable jump can not hedged. Under the assumption of no–arbitrage there is a multiplicity of equivalent martingale measure (EMM) under which agents do evaluate this risk. The mathematical finance literature suggests the use specific EMM, derived from hedging criteria (see Föllmer and Sondermann (1986) and Föllmer and Schweizer (1991)). When our setup is used to model "market crashes," i.e., $(J_t)_t$ represents jumps in the aggregate or market portfolio, the procedure taken in the mathematical finance literature seems appropriate. If the asset $(J_t)_t$ would be traded in the market, the three assets (B, S, J) would constitute a complete market. As it is, however, not traded, under the EMM Q chosen by the market for valuation, only $(S_t)_t$ has to be a martingale, but $(J_t)_t$ not; the description of $(J_t)_t$ (under P) and the choice of Q together with the Girsanov–Theorem are equivalent. Our view on this is that of specifying under an appropriate measure the process $(J_t)_t$ by

$$J_t = J_0 e^{(\nu - \lambda E[V_i - 1])t} \prod_{i=1}^{N_t} V_i \ , \tag{4}$$

for some $\nu \in \mathbb{R}$, which will be explained below. Using the description of the stochastic exponential we derive $E[J_t/J_0] = e^{\nu t}$ (see Protter (1990) or Jacod and Shiryaev (1987)). Using the Girsanov–Theorem as in the Black–Scholes setup gives then the EMM for the (B, S, J) market as a change in the Wiener measure for which $\mu^S = r \Leftrightarrow \sigma^2/2 \Leftrightarrow \nu$. For a detailed discussion, we refer to Wiesenberg (1998).

Then $\tilde{\nu} = \ln E[J_t/J_0]/t \Leftrightarrow r = \nu \Leftrightarrow r$ can be interpreted as the excess return on the risky process $(J_t)_t$ over the riskless rate. With exogenously fixed parameters λ and (V_i), ν specifies the risk–premium. As this is a "free" parameter, we can treat the choice problem of an EMM as the specification problem of an appropriate risk–premium in the market. Hamilton (1995), Bates (1996), and Trautmann and Beinert (1995) discussed the implementation, i.e., how to infer the properties of the jump component $(J_t)_t$ in the stock process $(S_t)_t$.

Similar to Merton (1976) we are interested in the case where jumps are firm specific, and therefore uncorrelated with the market as a whole. In the sense of the CAPM this is non systematic risk, has a β of zero and therefore the premium is zero. Bates (1996) gives statistical evidence that the risk premium ν is non–zero, i.e., the jump–risk is correlated to the market as a whole, and "crashes" need to be taken into account. However we use the Merton (1976) model as a starting point for our presentation; our approach is general and can easily be generalized to a non–zero ν. Section 6 studies numerical simulations in the case of a firm allowing for ruin ($V_i \equiv 0$).

Barrier options are a type of exotic options where the payoff depends on the crossing of predetermined levels, which may be either discretely or continuously monitored. In the event of a crossing, depending on the specification, a lump–sum (called "rebate") might be payed, another asset might be activated (called "knock–in") or deactivated ("knock–out"). Typical examples are those where the secondary asset is a plain vanilla option, e.g., a down (up) and in (out) barrier option pays this off, if some barrier $H < S_0$ ($H > S_0$) is (not) crossed and nothing otherwise.

We are interested here in continuously monitored constant barrier options, where the final payoff is that of a plain vanilla European option. We explicitly allow for multiple barrier options. The barrier option payoff depends on a "choice variable" as follows: The set Σ contains all those paths ω, where the terminal payoff will be activated. Then the terminal payoff is $1_{\omega \in \Sigma} f(S_T)$, where f is the payoff function at maturity, and its price is $E\left[e^{-rT} 1_{\omega \in \Sigma} f(S_T)\right]$.

3 Binomial Pitfalls

This section studies the numerical deficiencies with barrier option pricing in the Black–Scholes setup, i.e., the model $S_t = S_0 \cdot G_t$, using binomial models. Starting with CRR many authors presented so called binomial models for the asset evolution. We recall here briefly the CRR model to present the main difficulties. The specification of a refinement n of the time axis $[0, T]$ yields a discretization set $\mathcal{T}^n = \{0 = t_{n,0} < t_{n,1} \ldots < t_{n,n} = T\}$ of equidistant trading dates, i.e., $t_{n,i+1} \Leftrightarrow t_{n,i} = \Delta t_n = \frac{T}{n}$. The logarithmic asset price evolves in some fixed grid, with grid points at distance Δx_n. More specifically it is supposed that from one date to the next the asset can jump only to the next adjacent node, i.e., we model the (per–period) return by

$$\overline{R}_{n,i} \sim \begin{cases} +\Delta x_n & ; q_n \\ \Leftrightarrow\Delta x_n & ; 1 \Leftrightarrow q_n \end{cases} , \tag{5}$$

and the stock process by

$$\overline{G}_t^{(n)} := \prod_{i=1}^{N_t^{(n)}} \overline{R}_{n,i} \ , \tag{6}$$

$$\text{with} \qquad N_t^{(n)} := \left\lfloor \frac{t}{\Delta t_n} \right\rfloor \ . \tag{7}$$

Let us denote by $X_t = \ln G_t$ ($\overline{X}_t^{(n)} = \ln \overline{G}_t^{(n)}$) the (discrete) logarithmic process. To match the continuous per–period variance, we set

$$\Delta x_n = \sigma \sqrt{\Delta t_n} \ . \tag{8}$$

For pricing, only the evolution of the processes under the risk–neutral probability measures $Q^{(n)}$ matters. It corresponds here to the risk–neutral probability $Q^{(n)}$ of the continuous process, and is represented by the probability q_n for an up–move, such that $E[\overline{R}_{n,1}] = \exp\{r\Delta t_n\}$.

Theorem 1 *If* $E\left[\ln \overline{R}_{n,1}\right] = \left(r \Leftrightarrow \frac{\sigma^2}{2}\right)\Delta t_n$, *then* $E\left[\overline{R}_{n,1}\right] = \exp\{r\Delta t_n\} + \mathcal{O}(\Delta t_n)$. *On the other hand, if the martingale measure condition* $E[\overline{R}_{n,1}] = \exp\{r\Delta t_n\}$ *holds, then* $E\left[\ln \overline{R}_{n,1}\right] = \left(r \Leftrightarrow \frac{\sigma^2}{2}\right)\Delta t_n + \mathcal{O}(\Delta t_n^{3/2})$.

In both cases we have $\overline{S}^{(n)} \stackrel{d}{\Longrightarrow} S$.

PROOF. The condition $\exp\{r\Delta t_n\} = E[\overline{R}_{n,1}]$ is equivalent to

$$\begin{aligned} & 1 + r\Delta t_n + \mathcal{O}(\Delta t_n^2) \\ &= \exp\{r\Delta t_n\} = q_n \exp \Delta x_n + (1 \Leftrightarrow q_n) \exp \Leftrightarrow\Delta x_n \\ &= q_n \left(1 + \Delta x_n + \Delta x_n^2/2\right) + (1 \Leftrightarrow q_n)\left(1 \Leftrightarrow \Delta x_n + \Delta x_n^2/2\right) + \mathcal{O}(\Delta t_n^{3/2}) \\ &= 1 + \Delta x_n(2q_n \Leftrightarrow 1) + \Delta x_n^2/2 + \mathcal{O}(\Delta t_n^{3/2}) \end{aligned}$$

which is equivalent to $\left(r \Leftrightarrow \frac{\sigma^2}{2}\right)\Delta t_n = (2q_n \Leftrightarrow 1)\sigma\sqrt{\Delta t_n} + \mathcal{O}(\Delta t_n^{3/2}) = E[\ln \overline{R}_{n,1}] + \mathcal{O}(\Delta t_n^{3/2})$. This and the converse — which follows similarly — prove the first two assertions. Then Donsker's theorem proves $\overline{X}^{(n)} \stackrel{d}{\Longrightarrow} X$ and the proof concludes, observing that the exponential function is continuous.

According to theorem 1 there are two alternative ways of specifying q_n. First, looking at the processes G and $\overline{G}^{(n)}$ we can match their risk–neutral drift $E[\overline{R}_{n,i}] = \exp\{r\Delta t_n\}$. Second, we can take $E\left[\ln \overline{R}_{n,1}\right] = \left(r \Leftrightarrow \frac{\sigma^2}{2}\right)\Delta t_n$, to match the drift of the logarithmic processes X and $\overline{X}^{(n)}$. In the limit the same processes and thus the same prices will result. We will use the second one to specify the risk–neutral probability, and require $E\left[\overline{R}_{n,1}\right] = \mu\Delta t_n$, where $\mu = r \Leftrightarrow \frac{\sigma^2}{2}$. Easy calculations reveal that

$$q_n = \frac{1}{2} + \frac{\mu\Delta t_n \Leftrightarrow \kappa_n}{2\Delta x_m} \tag{9}$$

$$= \frac{1}{2} + \frac{r \Leftrightarrow \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t_n}\ . \tag{10}$$

Figure 1 presents a pricing example for a down–and–out call with strike

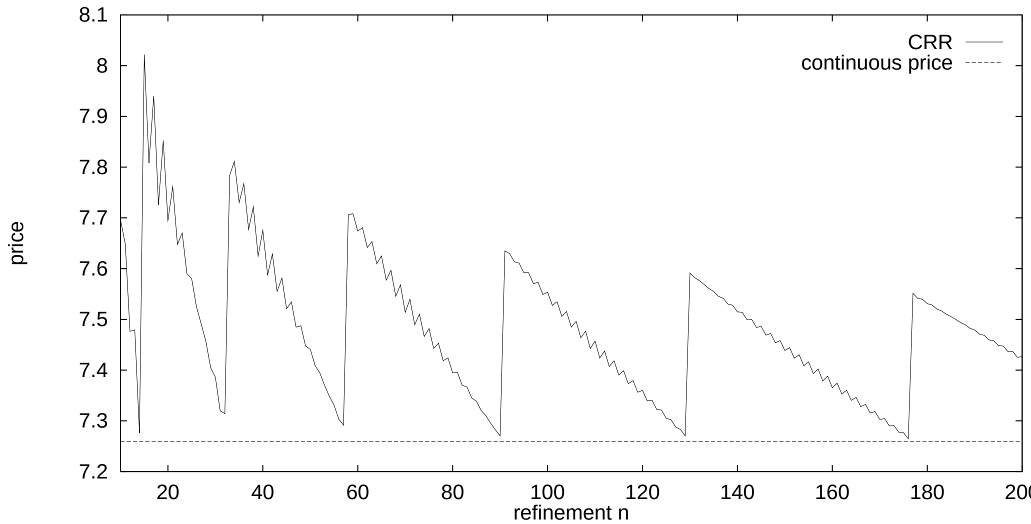

Fig. 1. price picture for the barrier option, depicting the convergence structure

$X = 110$ and a barrier at $H = 90$ written on a stock when today's stock price S_0 is 100, the volatility is $\sigma = 0.2$ and the interest rate is $r = 0.1$. We observe a unregular convergence to the continuous time solution. The structure exhibits the typical "saw–tooth" pattern well known in the literature. We observe also the "odd–even ripple." Figure 2 depicts the error on a log–log scale. Since the function $1/\sqrt{n}$ is an appropriate upper bounding line, the order of convergence is 1/2. Compared to the pricing of plain vanilla options, where the order is 1 we lose 1/2. We see that we have quite high errors in the worst case. Furthermore we deduce that we need at least a refinement of $1/\sqrt{n} = 0.01 \Leftrightarrow n = 10000$ to ensure "penny–accuracy."

The convergence is slower than in a standard European call and put option case, since in binomial models the whole probability mass is concentrated in

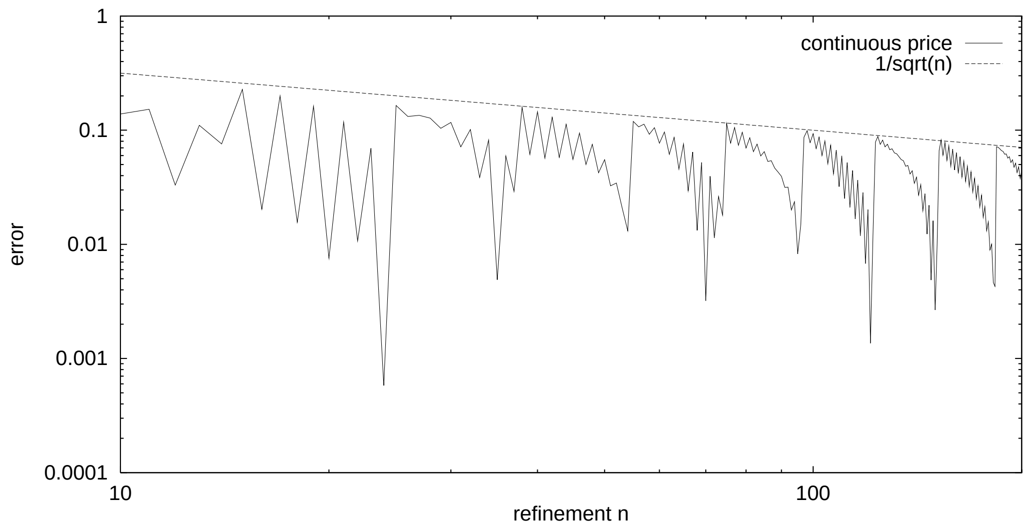

Fig. 2. error picture for the barrier option

the tree–nodes. The difference in probability between two adjacent nodes is known to be of the order $\mathcal{O}(1/\sqrt{n})$ (see Feller (1966)). Now, if, resulting from an increase in the refinement by one, a node "jumps" over the barrier layer, taking the corresponding probability mass with, we will observe a similar numerical effect. To prevent this for barrier options, Ritchken (1995) pointed out that it has to be ensured that there lies a node exactly on the barrier for any refinement. We characterize all (for double or even multi barrier options) barrier lines as a "critical line." Similarly Boyle and Lau (1994) argued to take depending on $m \in \mathbb{N}$ only the refinements

$$\left\lfloor \frac{m^2 \sigma^2 T}{\left(\ln \frac{S_0}{H}\right)^2} \right\rfloor . \tag{11}$$

These refinements are exactly the one's before an entire layer jumps over the barrier, again. Pricing errors are reduced to a size comparable to those of call options.

4 How to discretize properly

This section discusses the construction of a trinomial model in a first approach to remove the pitfalls of the previous section. Our approach to this problem is as follows: Discretizing time, and *then* studying the resulting X–grid, runs into problems. We are discretizing the wrong thing; the X–axis by a refinement m needs to be discretized first, yielding Δx_m, and then the refinement n_m of the t–axis should be set appropriately, rewriting equation (8) as

$$\Delta t_n = \left(\frac{\Delta x_m}{\sigma}\right)^2 \quad . \tag{12}$$

$$\Longleftrightarrow \qquad n_m = \left\lfloor T\left(\frac{\sigma}{\Delta x_m}\right)^2 \right\rfloor \quad . \tag{13}$$

For a down–and–out call option to place grid points on the barrier H requires $\Delta x_m = \frac{\ln S_0/H}{m}$. Interestingly, this way we recover formula (11) derived by Boyle and Lau (1994) as the resulting refinement n_m in equation (13). In the symmetrical case where $|S_0 \Leftrightarrow X| = |S_0 \Leftrightarrow H|$ this improves the accuracy and reduces the oscillations to a size comparable to European call options. To encounter this problem in the general case we need to ensure that nodes also lie on the strike.

We will now present a first approach to resolve this difficulty in full generality. Later then, we explain it on a concrete example. We call *critical layer* all barrier lines (possibly many), the strike and the current asset price. Let us denote by $\mathcal{L}$ the ordered set $l_1 < \ldots < l_L$ of critical layers, and by $\mathcal{L}' = \mathcal{L} \setminus \{l_1, l_L\}$ the inner points. First we define variables for $0 < i < L \Leftrightarrow 1$, $\Delta x_{m,i} = \frac{l_{i+1} - l_i}{m}$, and then $\Delta x_{m,L} = \Delta x_{m,L-1}$ and $\Delta x_{m,0} = \Delta x_{m,1}$, and for $x \in [l_{i-1}, l_i[$: $\Delta x^u_m(x) = \Delta_{m,i-1}$ and for $x \in]l_{i-1}, l_i]$: $\Delta x^d_m(x) = \Delta_{m,i}$. This defines a discrete time–homogeneous grid for the stock values over time. Moreover we define the minimum $\Delta x_m = \min_i \Delta x_{m,i}$ of all. We model the return, depending on some $x \in \mathbb{R}$, by a trinomial random variable $\overline{R}_{m,i}$ of the form:

$$\overline{R}_{m,i}(x) \sim \begin{cases} \Delta x^u_m(x) & ; p_m(x) \\ 0 & ; 1 \Leftrightarrow p_m(x) \Leftrightarrow q_m(x) \\ \Leftrightarrow \Delta x^d_m(x) & ; q_m(x) \end{cases} \quad . \tag{14}$$

For $\Delta t_n, n_m$ defined by equation (12),(13), we call the processes

$$\overline{X}^{(m)}_t = \sum_{i=1}^{N^{(m)}_t} \overline{R}_{m,i}\left(\overline{X}^{(n)}_{t-}\right) \tag{15}$$

$$\overline{G}^{(m)}_t = \exp \overline{X}^{(n)}_t \tag{16}$$

$$N^{(m)}_t = \left\lfloor \frac{t}{\Delta t_m} \right\rfloor \tag{17}$$

the *Trinomial Adjusted (TA)* model.

The choice of $\Delta x^u_m(x), \Delta x^d_m(x)$ will give us sufficient degrees of freedom to place nodes on critical "lines," like the barrier and the strike price. Here, we deal with a trinomial model in order to get the variance right despite the complication that discretizations are not constant and change at critical lines. The process seems to be path–dependent; however this is only conditionally on the actual state. It remains recombining and therefore computationally simple, as it can be handled as any standard trinomial model for calculations. The dependence on x is easy to resolve. We will drop it in the sequel to present the main ideas.

Let us explain our approach in detail for a down–and–out call option in figure 3 where a strike price is at $X = 120$ and a barrier is at $H = 90$. We see that

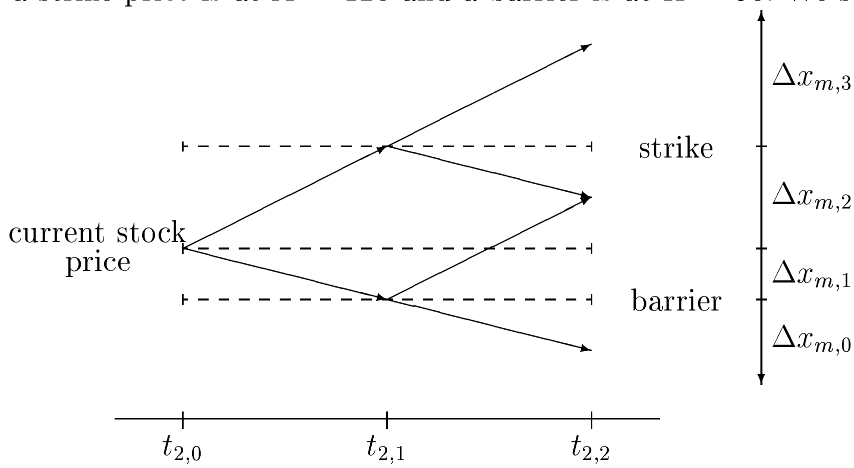

Fig. 3. Example dynamics

we can distinguish $L = 3$ critical layers and four different ranges: one below the barrier, one between the barrier and the current asset price, one between the strike and the current asset price and one above the strike. In each of the two inner ranges we need a different Δx_m. To place nodes on critical layers, we take between the barrier and the current asset price $\Delta x_{m,1} = |\ln H/S_0|/m$ and between the strike and the current asset price $\Delta x_{m,2} = |\ln K/S_0|/m$. This defines our grid for the asset evolution. As there is no clear choice for $\Delta x_{m,3}$ ($\Delta x_{m,0}$), our approach takes $\Delta x_{m,2}$ ($\Delta x_{m,1}$) for simplicity. Please note that although we focus here on a down–and–out call for ease of exposition, our approach is general and can easily be generalized, e.g. to double barrier options.

A consistency requirement is $\overline{G}^{(m)} \overset{d}{\Longrightarrow} G$. Since the exponential function is continuous, this is equivalent to $\overline{X}^{(m)} \overset{d}{\Longrightarrow} X$. So, setting $\mu = r \Leftrightarrow \frac{\sigma^2}{2}$, according to Donsker's theorem the following two conditions are sufficient:

$$\frac{E\left[\overline{R}_{m,i}\right]}{\Delta t_n} \overset{n}{\Leftrightarrow\!\!\to} \mu \ , \tag{18}$$

$$\text{and} \quad \frac{\mathrm{Var}\left[\overline{R}_{m,i}\right]}{\Delta t_n} \overset{n}{\Leftrightarrow\!\!\to} \sigma^2 \ . \tag{19}$$

We require them to hold with equality:

$$
\begin{aligned}
\Delta x_m^u p_m \Leftrightarrow \Delta x_m^d q_m &= \mu \Delta t_n \ , && (20)\\
(\Delta x_m^u)^2 p_m + (\Delta x_m^d)^2 q_m &= \sigma^2 \Delta t_n \ , && (21)
\end{aligned}
$$

which can be resolved easily

$$
p_m = \frac{\left(\mu \Delta x_m^d + \sigma^2\right) \Delta t_n}{\Delta x_m^u (\Delta x_m^u + \Delta x_m^d)} \ , \tag{22}
$$

$$
\text{and} \qquad q_m = \frac{(\Leftrightarrow \mu \Delta x_m^u + \sigma^2) \Delta t_n}{\Delta x_m^d (\Delta x_m^u + \Delta x_m^d)} \ . \tag{23}
$$

Theorem 2 *For sufficiently high refinements, all probabilities* $p_m, q_m, 1 \Leftrightarrow p_m \Leftrightarrow q_m$ *are positive and* $\overline{X}^{(m)} \stackrel{d}{\Longrightarrow} X$ *and* $\overline{G}^{(m)} \stackrel{d}{\Longrightarrow} G$.

PROOF. Positivity of the probabilities follows from $\Delta t_n = (\Delta x_n / \sigma)^2$, and equations (22), (23) since

$$
\begin{aligned}
& 0 < \frac{\Delta x_m^2}{\Delta x_m^u (\Delta x_m^u + \Delta x_m^d)} \cdot \left(\frac{\Delta x_m}{\sigma}\right)^2 \ < \ 1 \ , \\
\text{and} \qquad & \frac{\mu \Delta x_m^d}{\Delta x_m^u (\Delta x_m^u + \Delta x_m^d)} \cdot \left(\frac{\Delta x_m}{\sigma}\right)^2 \stackrel{m}{\Leftrightarrow\!\!\to} 0 \ .
\end{aligned}
$$

The observation that the barrier option with a put as payoff–function is continuous and bounded, the above theorem and put–call parity ensure convergence of approximate values calculated to their continuous time counterpart. Please note, that this price consistency holds whenever we know that processes are consistent in the limit.

Table 1 gives a pricing example for a barrier option using CRR, TA, and the modified Richardson extrapolation rule (see Leisen (1998a))

$$
\pi_m^e = (n_{m+1} \pi_{m+1} \Leftrightarrow n_m \pi_m) / (n_{m+1} \Leftrightarrow n_m) \ , \tag{24}
$$

iterating $m = 1, \ldots, 8$. The RT model will be introduced in the following section. Figure 4 contains the error picture for the same parameter constellation as in table 1. Here we take only specific refinements; we see that the convergence structure is much smoother than in figure 2. However it is not sufficiently smooth to apply extrapolation, which do therefore not depict. For TA, in comparison to CRR, errors are drastically reduced and extrapolation gives quickly “penny–accuracy.” We do also observe that the convergence structure is fairly smooth.

m	n_m	π_{CRR}	π_{TA}	π^e_{TA}	π_{RT}	π^e_{RT}
1	8	8.44693	7.89878	7.93419	6.60928	8.14613
2	34	8.17859	7.92586	7.92367	7.78452	7.97775
3	78	8.04017	7.92463	7.98542	7.89352	7.97775
4	140	7.97567	7.95155	7.98644	7.93082	7.97837
5	220	8.03844	7.96423	7.97371	7.94811	7.97904
6	316	8.02729	7.96711	7.97778	7.95751	7.97889
7	430	7.97612	7.96994	7.98570	7.96318	7.97884
8	562	8.0086	7.97364	7.97786	7.96686	7.97883

Table 1
Pricing example with S=100, X=110, H=85, T=1, r=0.1, $\sigma = 0.2$. The continuous time price is 7.97888.

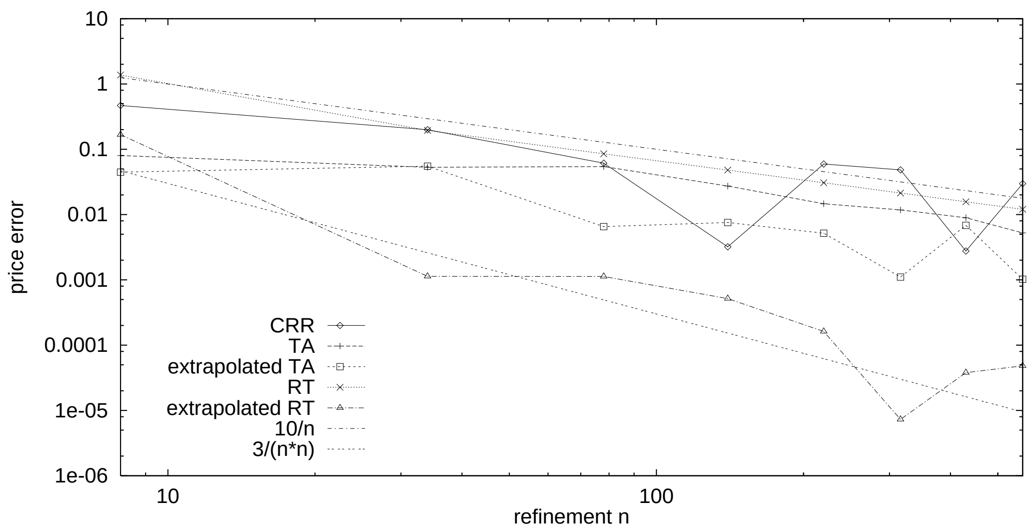

Fig. 4. error picture for the barrier option

5 Randomizing and Jump–diffusions

We will now randomize the previous model. This will be an easy and straightforward way to incorporate the additional jumps which characterize the jump–diffusion model in difference to the Black–Scholes setup. It turns out, in accordance to Leisen (1998b), that such a randomized model yields even better convergence results.

We start with a sequence of Poisson processes $N^{(m)} = (N^{(m)}_t)_{t\geq 0}$ where $N^{(m)}_t$ has parameter $(\lambda_m)_m$. $N^{(m)}_t$ is described by interarrival times $\tau_{m,i}$, independent exponentially distributed random variables with parameter $1/\lambda_m$ and $N^{(m)}_t = \max\{n \,|\sum_{i=1}^{n} \tau_{m,i} \leq t\}$. In the previous section we approximated the process

X between two trading dates $t_{n,i}, t_{n,i+1} \in \mathcal{T}^n$ by iid random variables $\overline{R}_{m,i}$. Similarly here we will now approximate it by random variables $\overline{R}_{m,i}$ between two interarrival times $\tau_{m,i}, \tau_{m,i+1}$, which of the same form. All $\Delta x_{m,l}$ ($l = 1, \ldots, L$) are defined as in the last section described to place nodes on critical "lines," like the barrier and the strike price. Then study the discrete processes $\overline{X}_t^{(m)}$ and $\overline{G}_t^{(m)}$, defined similar to (15)–(17) by

$$
\begin{aligned}
\overline{X}_t^{(m)} &= \sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i} \ , \\
\overline{G}_t^{(m)} &= \exp \overline{X}_t^{(m)} \ .
\end{aligned}
$$

and we require, since $1/\lambda_m = E[\tau_{m,i+1} \Leftrightarrow \tau_{m,i}]$,

$$
E\left[\overline{R}_{m,i}\right] \lambda_m \overset{n}{\Leftrightarrow\!\!\rightarrow} r \Leftrightarrow \frac{\sigma^2}{2} \ , \tag{25}
$$

$$
\text{and} \qquad \operatorname{Var}\left[\overline{R}_{m,i}\right] \lambda_m \overset{n}{\Leftrightarrow\!\!\rightarrow} \sigma^2 \ . \tag{26}
$$

Theorem 3 *Under conditions (25) and (26) we have* $\overline{X}^{(m)} \overset{d}{\Longrightarrow} X$ *and* $\overline{G}^{(m)} \overset{d}{\Longrightarrow} G$.

PROOF. Let us define the two sequences (depending on m) of processes $\left(M_t^{(m)}\right)_t$ and $\left(A_t^{(m)}\right)_t$ by

$$
\begin{aligned}
M_t^{(m)} &= \sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i} \Leftrightarrow \left(r \Leftrightarrow \frac{\sigma^2}{2}\right) t \ , \\
\text{and} \qquad A_t^{(m)} &= \sigma^2 t \ .
\end{aligned}
$$

Then for each m, the processes $\left(M_t^{(m)}\right)_t$ and $\left(\left(M_t^{(m)}\right)^2 \Leftrightarrow A_t^{(m)}\right)_t$ are martingales. As the jump sizes are of order Δx_m and vanish in the limit, we deduce from the Martingale Central Limit Theorem as stated in Ethier and Kurtz (1986) that $M \overset{d}{\Longrightarrow} \sigma W$. This is also sufficient for $\overline{G}^{(m)} \overset{d}{\Longrightarrow} G$, since the exponential function is continuous.

We require equations (25) and (26) to be fulfilled with equality, i.e.,

$$
\begin{aligned}
(\Delta x_m^u)^2 p_m + (\Delta x_m^d)^2 q_m &= \frac{\sigma^2}{\lambda_m} \ , \\
\Delta x_m^u p_m \Leftrightarrow \Delta x_m^d q_m &= \frac{r \Leftrightarrow \frac{\sigma^2}{2}}{\lambda_m} \ ,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
p_m &= \frac{\sigma^2 + \left(r \Leftrightarrow \frac{\sigma^2}{2}\right) \Delta x_m^u}{\lambda_m((\Delta x_m^u)^2 + \Delta x_m^u \Delta x_m^d)} \ , \\
\text{and} \qquad q_m &= \frac{\sigma^2 \Leftrightarrow \left(r \Leftrightarrow \frac{\sigma^2}{2}\right) \Delta x_m^d}{\lambda_m((\Delta x_m^d)^2 + \Delta x_m^u \Delta x_m^d)} \ .
\end{aligned}
$$

Similarly to Leisen (1998b) we take

$$
\lambda_m = \left(\frac{\sigma}{\Delta x_m}\right)^2 \ .
$$

This choice is justified from the assumptions and since $E[\tau_{m,1}] = 1/\lambda_m$ which corresponds to Δt_n in equation (13). Then $p_m, q_m \in [0,1], p_m + q_m \leq 1$ which makes them feasible transition probabilities and the Poisson process $N^{(m)}$ is stationary, which will make valuations in the next section especially easy to perform. We call this model the *Randomized Trinomial (RT)* model. The only difference to the process studied in the last section is the driving process. Whereas there it was $\lfloor t/\Delta t_n \rfloor$, here it is the Poisson process $N_t^{(m)}$ with parameter λ_m.

This model can also be used easily to construct an approximation in the jump–diffusion setup. Startin with a sequence $(N^{(m)}, \overline{R}_m)_m$ of the above type, where here $\mu = r \Leftrightarrow \frac{\sigma^2}{2} \Leftrightarrow \lambda E[V_i \Leftrightarrow 1]$, we define the process

$$
\overline{N}^{(m)} = N + N^{(m)} \ , \tag{27}
$$

which is a Poisson process with intensity $\bar{\lambda}_m = \lambda + \lambda_m$, the sequence of random variables

$$
\overline{R}'_{m,i} \sim \begin{cases} V_i & ; \frac{\lambda}{\lambda+\lambda_m} \\ \overline{R}_{m,i} & ; \frac{\lambda_m}{\lambda+\lambda_m} \end{cases} \ , \tag{28}
$$

and the processes

$$
\overline{Y}_t^{(m)} = \sum_{i=1}^{\overline{N}_t^{(m)}} \overline{R}'_{m,i} \ , \tag{29}
$$

$$
\text{and} \qquad \overline{S}_t^{(m)} = \exp \overline{Y}_t^{(m)} \ . \tag{30}
$$

The counterpart to theorem 3 is:

Theorem 4 *For the sequence of discrete models $(\overline{Y}^{(m)})_m$ defined by equations (27)–(30) above we have $\overline{Y}^{(m)} \stackrel{d}{\Longrightarrow} Y$ and $\overline{S}^{(m)} \stackrel{d}{\Longrightarrow} S$, where S is the process defined in equation (1) and $Y = \ln S$.*

PROOF. Denote by h the function $h : x \mapsto x + \sum_{i=1}^{N} U_i$ on $\mathcal{D}$. Since

$$\sum_{i=1}^{N^{(m)}} \overline{R}^{(m)} \stackrel{d}{\Longrightarrow} \left(r \Leftrightarrow \frac{\sigma^2}{2}\right) t + \sigma W_t \ ,$$

and the latter is continuous, we conclude using VI.1.23 and VI.3.8 (ii) of Jacod and Shiryaev (1987):

$$\overline{Y}^{(m)} = h\left(\sum_{i=1}^{N^{(m)}} \overline{R}'_{m,i}\right) \stackrel{d}{\Longrightarrow} h\left(\left(r \Leftrightarrow \frac{\sigma^2}{2}\right) t + \sigma W_t\right) = Y \ .$$

6 Implementing barrier option valuation in the randomized models

Since the structure in the randomized models differs only in the specification of the return variable, we will treat pricing in those models for a general $(\overline{R}''_{m,i})$ which is then either $(\overline{R}_{m,i})$ or $(\overline{R}'_{m,i})$. The value V_m of a barrier option is given by

$$\begin{aligned} V_m &= e^{-rT} E\left[1_\Sigma f(\overline{S}_T)\right] \\ &= e^{-rT} E\left[E[1_\Sigma f(\overline{S}_T) | \overline{N}_T^{(m)}]\right] \\ &= e^{-rT} \sum_{n=0}^{\infty} E\left[1_\Sigma f(\overline{S}_T) | \overline{N}_T^{(m)} = n\right] \cdot P\left[\overline{N}_T^{(m)} = n\right] \ . \end{aligned}$$

This splits up the valuation task by conditioning first on $\overline{N}_T^{(m)}$ and then averaging over all possible values. This is feasible, due to the independence of $\overline{N}^{(m)}$ and the random variables in the sequence $\overline{R}''_m$. Let us now denote by Σ_n the "choice variable" corresponding to Σ in the n–step tree with return modeled by $\overline{R}''_{m,i}$, and define

$$\begin{aligned} \Phi_n^{(m)} &= E\left[1_\Sigma f(\overline{S}_T) | \overline{N}_T^{(m)} = n\right] \\ &= E\left[1_{\Sigma_n} f(\overline{S}_T) | \overline{N}_T^{(m)} = n\right] \end{aligned}$$

Therefore

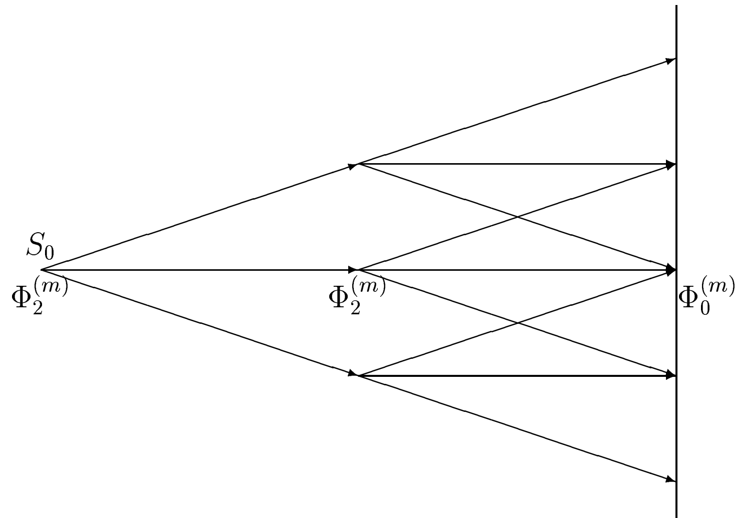

Fig. 5. A trinomial grid

$$V_m = e^{-(r+\bar{\lambda}_m)T} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}_m T)^n}{n!} \Phi_n^{(m)} \tag{31}$$

We cut off the infinite sum in equation (31) at an appropriate γ_m such that $\lim_{m\to\infty} P[N_T^{(m)} \in \{0,\ldots,\gamma_m\}] = 1$. Due to the Central Limit Theorem for renewals,

$$\frac{N_T^{(m)} \Leftrightarrow \bar{\lambda}_m T}{\sqrt{\bar{\lambda}_m T}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,1) \ ,$$

setting $\gamma_m = 2\lfloor \bar{\lambda}_m T \rfloor$ is *one* appropriate choice. In the sequel we adopt as our cut–off

$$V_m \approx e^{-(r+\bar{\lambda}_m)T} \sum_{n=0}^{2\lfloor \bar{\lambda}_m T \rfloor} \frac{(\bar{\lambda}_m T)^n}{n!} \cdot \Phi_n^{(m)} \ . \tag{32}$$

The value $\Phi_n^{(m)}$ can be interpreted as the value calculated by backward–induction in an n–step tree grid with return $(\overline{R}''_{m,i})_i$ exactly as in the CRR model, if we *do not* perform discounting (see figure 5 for $\overline{R}''_{m,i} = \overline{R}_{m,i}$). Please note that for any $\tilde{n}$ calculating $\Phi_{\tilde{n}}^{(m)}$ gives us $\Phi_{n'}^{(m)}$ for any $n' = 0,\ldots,\tilde{n}$ as intermediate calculations. Thus we can calculate prices as intermediate calculations in an $2\lfloor \bar{\lambda}_m T \rfloor$ step tree and computing prices in our model is comparable to a trinomial model (and therefore to the CRR model) in terms of the computational burden. In order to compare both approaches properly depending on its complexity, we index calculations in the RT by $2\lfloor \bar{\lambda}_m T \rfloor$.

As explained on figure 3 we adopt the discretizations $\Delta x_{m,0} = \Delta x_{m,1} = |\ln H/S_0|/m$ and $\Delta x_{m,2} = \Delta x_{m,3} = |\ln K/S_0|/m$ to place *all* nodes on critical

m	n_m	π_{RT}	π^e_{RT}	ϵ_{RT}	ϵ^e_{RT}
1	8	11.0891	13.6483	2.2053	0.3539
2	34	13.0461	13.2948	0.2483	0.0004
3	78	13.1864	13.2930	0.1080	0.0015
4	140	13.2336	13.2937	0.0608	0.0008
5	220	13.2555	13.2945	0.0390	0.0001
6	316	13.2673	13.2943	0.0271	0.0001
7	430	13.2745	13.2942	0.0200	0.0002
8	562	13.2791	13.2942	0.0153	0.0002

Table 2
Ruin pricing example with S=100, X=110, H=85, T=1, r=0.1, $\sigma = 0.2$, $\lambda = 0.1$

"lines." Table 1 presents prices and errors. We see that RT yields even better price approximations than TA. By extrapolation we get extremely accurate price approximations. This becomes even more apparent in the error picture 4. We see that prices converge with order one and extrapolated prices seem to converge even with order two. Moreover we observe a gain in accuracy by extrapolation in comparison to the extrapolated TA model.

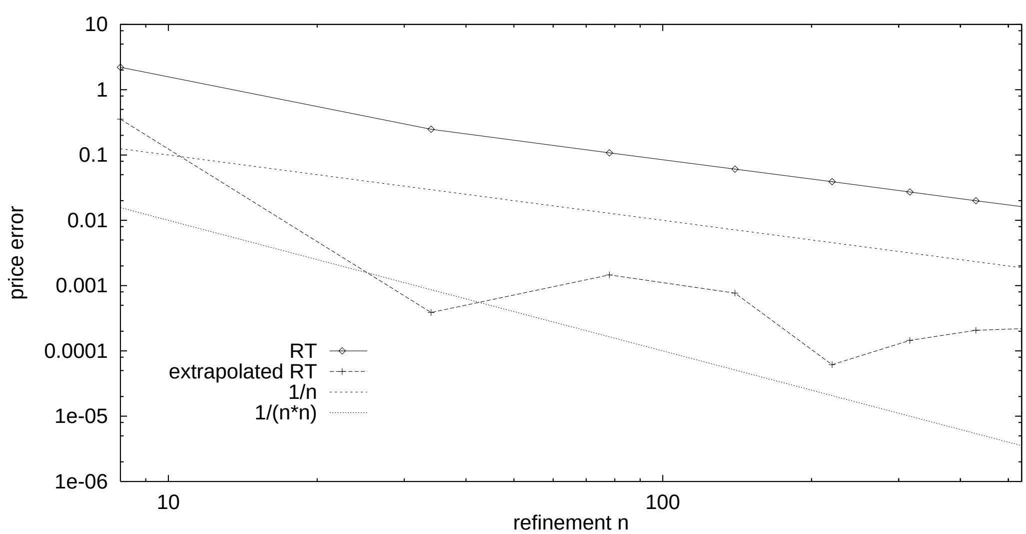

Fig. 6. error picture for the barrier option

We now discuss the accuracy in a ruin setup ($V_i \equiv 0$) with $\lambda = 0.1$. Table 2 and figure 6 present values calculated for this case. We also depict errors ϵ and extrapolated prices (errors) π^e (ϵ^e), calculated according to equation (24). We calculated 13.2944 as the price in the CRR model with a refinement of 100000. This is an estimation of the continuous time price. In figure 2 we saw that $1/\sqrt{n}$ was an appropriate upper bound for the error. If we assume a similar error bound here, then the error of our estimate is less than 0.0031. Here we see very impressively the slow convergence of the CRR model. Immediately (with $m = 2$) we fall below this level. We do not perform a graphical analysis of the order, since we can not calculate sufficiently accurate values; iterating the RT higher than $m = 2$ to compare the accuracy is even doubtful. It is astonishing that the remarkable convergence properties carry over from the Black–Scholes setup to the jump–diffusion case.

7 Conclusion

This paper constructs a randomized trinomial model. This is a natural way to approximate jump–diffusions. The parameters are set such as to get consistency with the continuous–time processes. We discuss an easy approximation to jump–diffusions and how efficient numerical approximations for barrier options result in the Black–Scholes and the jump–diffusion setup.

References

Amin, Kaushik I., Jump Diffusion Option Valuation in Discrete Time, *The Journal of Finance*, 1993, *43*, 1833–1863.

Bates, David S., Dollar Jump Fears, 1984–1992: Distributional Abnormalities Implicit in Currency Futures Options, *Journal of International Money and Finance*, 1996, *15* (1), 65–93.

Black, Fischer and Myron Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 1973, *81*, 637–659.

Boyle, Phelim P. and Sok Hoon Lau, Bumping up against the Barrier with the Binomial Method, *The Journal of Derivatives*, 1994, pp. 6–15.

—— and Yisong Tian, An explicit finite difference approach to the pricing of barrier options, *Applied Mathematical Finance*, 1998, pp. 17–43.

Carr, Peter, Two extensions to barrier option valuation, *Applied Mathematical Finance*, 1995, *2*, 173–209.

Cheuk, Terry H.F. and Ton C.F. Vorst, Complex Barrier Options, *The Journal of Derivatives*, 1996, *4*, 8–22.

Duffie, Darrell, *Dynamic Asset Pricing Theory*, Princeton University Press, 1992.

Ethier, Stuart N. and Thomas Kurtz, *Markov Processes: Characterization and Convergence*, John Wiley & Sons, 1986.

Feller, William, *An Introduction to Probability Theory and Its Applications*, Vol. 2, John Wiley & Sons, 1966.

Figlewski, Stephen and Bin Gao, The Adaptive Mesh Model: A New Approach to Efficient Option Pricing, Discussion Paper, New York University Stern School of Business 1998.

Föllmer, Hans and Dieter Sondermann, Hedging of Non–Redundant Contingent Claims, in Werner Hildenbrand and Andreu Mas-Colell, editors, *Contributions to Mathematical Economics*, North–Holland, 1986, chapter 12, pp. 205–223.

—— and Martin Schweizer, Hedging of Contingent Claims under Uncomplete Information, in Mark H.A. Davis and Robert J. Elliott, editors, *Stochastics Monographs*, Vol. 5, Gordon and Breach, 1991.

Hamilton, James D., *Time Series Analysis*, Princeton University Press, 1995.

Harrison, J. Michael and David M. Kreps, Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory*, 1979, *20*, 381–408.

Jacod, Jean and Albert N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer–Verlag, 1987.

Kunitomo, N. and M. Ikeda, Pricing options with curved boundaries, *Journal of Mathematical Economics*, 1992, *8*, 275–298.

Leisen, Dietmar P.J., Pricing the American Put Option: A detailed convergence analysis for Binomial Models, *Journal of Economic Dynamics and Control*, 1998, *22*, 1419–1444.

——, The Random–Time Binomial Model, forthcoming: Journal of Economic Dynamics and Control 1998.

Merton, Robert C., Option Pricing when Underlying Stock Returns are Discontinous, *Journal of Financial Economics*, 1976, *3*, 125–144.

Protter, Philipp, *Stochastic Integration and Differential Equations: A new approach*, Springer–Verlag, 1990.

Ritchken, Peter, On Pricing Barrier Options, *The Journal of Derivatives*, 1995, *3*, 19–28.

Rogers, Chris and Emily Stapleton, Fast Accurate Binomial Pricing, *Finance and Stochastics*, 1998, *2*, 3–17.

Rubinstein, Mark and E. Reiner, Breaking down the barriers, *RISK*, 1991, *4 (8)*, 28–35.

Samuelson, Paul A., Rational Theory of Warrant Pricing, *Industrial Management Review*, 1965, *6*, 13–31.

Trautmann, Siegfried and Michaela Beinert, Stock Price Jumps and Their Impact on Option Valuation, Discussion Paper, University of Mainz 1995.

Wiesenberg, Holger, Modeling Market Risk in a Jump–Diffusion Setting: A generalized Hofmann–Platen–Schweizer Model, Discussion Paper B–424, University Bonn 1998.