THE PRICING OF DERIVATIVES ON ASSETS WITH QUADRATIC VOLATILITY

CHRISTIAN ZÜHLSDORFF

Abstract. The basic model of financial economics is the Samuelson model of geometric Brownian motion because of the celebrated Black-Scholes formula for pricing the call option. The asset volatility is a linear function of the asset values and the model guarantees positive asset prices. We show that the pricing PDE can be solved if the volatility function is a quadratic polynomial and give explicit formulas for the call option: a generalization of the Black-Scholes formula for an asset whose volatiliy is affine, a formula for the Bachelier model with constant volatility and a new formula in the case of quadratic volatility. The implied Black-Scholes volatilities of the Bachelier and the affine model are frowns, the quadratic specifications also imply smiles.

JEL Classification G13

Keywords option pricing, quadratic volatility

In their seminal article Black and Scholes (1973) derive a formula for the value of a call option if the underlying asset follows geometric or economic Brownian motion, a model introduced by Samuelson (1964). Half a century before Louis Bachelier (1900) already tried to evaluate derivatives if changes in asset prices are normally distributed. This can be translated to a model where volatility is constant with respect to the asset value.

In this paper we show that the pricing partial differential equation (PDE) can be solved for general quadratic volatility functions, i.e., functions that are the product of a time dependent function and a quadratic polynomial. To exclude negative asset prices an absorbing boundary in zero is possible. In section 1 we derive the pricing PDE and the general solution in the case of quadratic volatility. In section 2 we give formulas for the value of a call option for the three possible specifications: constant, affine and quadratic. In section 3 we plot the Black-Scholes implied volatilities of the specifications which exhibit "smiles" and "frowns". Section 4 concludes.

1. The pricing partial differential equation

We model an arbitrage-free frictionless financial market where traders can costlessly store money and trade in an asset whose price X we take to be an element of the space $\mathcal{M}_{c,loc}$ of continuous local martingales with respect to $(\Omega, (\mathcal{F}_t), P^*)$ which is a stochastic base whose filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed to satisfy the usual conditions. As our objective is pricing, P^* is already the martingale measure. Denote the domain of the price process X by

$$\mathcal{D} = [l, r] \qquad \Leftrightarrow\infty \leq l < r \leq \infty$$

it is the smallest interval s.t.

$$P^*\{X_t \notin \mathcal{D}\} = 0 \quad \forall t.$$

Further we assume that $0 \in \mathcal{D}$ and that the boundaries l and r are absorbing if they are attainable. The usual choice is $l = 0$ and $r = \infty$ to model an asset, if the model has a positiv probability of attaining the left boundary in zero this can be interpreted as bankruptcy. But also other choices of the boundaries are reasonable, e.g. $l = 0$ and $r = 1$ for a zero-bond.

Assumption 1.1. *The quadratic variation $\langle X\rangle$ of X is P^*-a.s. absolutely continuous and*

$$\frac{d\langle X\rangle_t}{dt} = \frac{1}{2}\sigma^2(t, X_t)$$

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Financial support by the Deutsche Forschungsgemeinschaft is gratefully acknowledged.
mailto: christian@addi.finasto.uni-bonn.de .

for a deterministic function $\sigma(t,x)$ which is jointly measurable in (t,x) and strictly postive on the domain of X. By (Karatzas and Shreve 1991, Rem. 3.4.3) it is equivalent to assume that there is a Brownian motion W with respect to $(\Omega,(\mathcal{F}_t),P^*)$ s.t.

$$dX_t = \sigma(t,X_t)\ dW_t.$$

A european option on the asset is given by its payoff function g and its maturity T. Its value is the expectation under the martingale measure P^*:

$$V(t,x) = E^*\left[g(X_T) \mid X_t = x\right]$$

The Feynman-Kac theorem (e.g. Karatzas and Shreve 1991) tells us that under suitable smoothness conditions on g the value function V satisfies the PDE

$$V_t + \frac{1}{2}\sigma^2 V_{xx} = 0 \tag{1}$$

with terminal value

$$V(T,x) = g(x). \tag{2}$$

The boundary conditions in l and r are

$$V(t,l) = g(l) \qquad V(t,r) = g(r) \qquad \forall t \tag{3}$$

as we assumed absorbing barriers. If e.g. $r = \infty$ the condition is interpreted as

$$\lim_{x\to\infty} \frac{V(t,x)}{g(x)} = 1.$$

A *(self-financing) portfolio* is given by its initial value π_0 and a $\mathcal{F}$-previsible X-integrable process ϕ (the *portfolio strategy*), its *value process*

$$\pi_t = \pi_0 + \int_0^t \phi_s\ dX_s$$

fulfills the self-financing condition

$$d\pi = \phi\ dX.$$

If we apply Itô's formula to the value function

$$dV(t,X) = (V_t + \frac{1}{2}\sigma^2 V_{xx})\ dt + V_x\ dX = V_x\ dX$$

we see that the Δ-hedge $\phi_t = V_x(t,X_t)$ gives the hedge portfolio for the claim $g(X_T)$.

Assumption 1.2. *The function σ can be split up $\sigma(t,x) = \sigma(t)\ p(x)$ for a strictly positive bounded function σ and a quadratic polynomial $p(x) = a + bx + cx^2$.*

It is a common myth of stochastic calculus that one needs a linear growth condition on the dispersion function to ensure existence of a (strong) solution of a stochastic differential equation (SDE), a recent example is Dumas, Fleming and Whaley (1998). But this specification is locally Lipschitz which implies strong uniqueness and existence of a non-exploding weak solution, both results together in turn imply the existence of a strong solution (Karatzas and Shreve 1991, 5.2.5, 5.5.4, 5.3.23).

We give a constructive example:

$$dX = X^2\ dW \qquad X_0 = x$$

By Itô's formula the invers process $R = 1/X$ fulfills

$$dR = X\ dt \Leftrightarrow dW = \frac{1}{R}\ dt \Leftrightarrow dW \qquad R_0 = \frac{1}{x}.$$

So R is a Bessel process of dimension 3 which especially implies

$$\lim_{t\to\infty} X = 0 \qquad P^* \Leftrightarrow \text{a.s.}$$

as R reaches ∞ a.s. (see Karatzas and Shreve 1991, 3.3.24). Contrary to intuition the process does not explode but converges to zero for almost all paths.

If we model with quadratic volatility under the pricing measure there is no problem with existence of such a process. The right question to ask is which kind of dynamics under the historic measure allow a risk-neutral dynamics like this, i.e. what forms of drift are consistent with no-arbitrage.

We return to the question of pricing.

Proposition 1.3. *Under the assumptions **1.1–1.2** the value of a contingent claim $g(X_T)$ is given by a value function $V(t, X_t)$ of the following form:*

$$V(t,x) = \gamma(\tau^2(t))\ \xi(Z(x))\ h(\tau^2(t), Z(x)) \tag{4}$$

The time and space changes τ^2 and Z are defined by

$$\tau^2(t) = \int_t^T \sigma^2(u)du, \qquad Z(x) = \int^x \frac{1}{p(y)}dy. \tag{5}$$

The function h satisfies the heat equation

$$h_{\tau^2} = \frac{1}{2}h_{zz} \tag{6}$$

in the interval $\mathcal{R} = Z(\mathcal{D}) = [L, R]$ for $L = Z(l)$ and $R = Z(r)$ with the initial condition

$$h(0,z) = \frac{g(Z^{-1}(z))}{\xi(z)}$$

and the boundary condition

$$h(\tau^2, B) = \frac{1}{\gamma(\tau^2)}\frac{g(b)}{\xi(B)} = \frac{h(0,B)}{\gamma(\tau^2)} \qquad B \in \{L, R\}. \tag{7}$$

The correction functions γ and ξ are given by

$$\gamma(\tau^2) = \exp(-\varsigma\tau^2/8) \quad \varsigma = b^2 - 4ac \qquad \xi(z) = \sqrt{p(Z^{-1}(z))}.$$

The **Proof** that (4) is the solution of the PDE (1–2–3) is in appendix A.

2. Formulas for the call option

In this section, we give closed form solutions for the call option

$$g(x) = (x - k)^+ \qquad k \geq 0$$

for the three possible specifications of the volatility function: constant, affine and quadratic.

The density of the one-dimensional normal distribution is $\varphi(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}$ and the corresponding distribution function $\Phi(x) = \int_{-\infty}^x \varphi(y)\ dy$. All proofs are in appendix B.

Constant Volatility: $\quad p = 1$

The first model of asset prices and as well the first description of Brownian motion, the thesis of Louis Bachelier (1900) (english translation Bachelier 1964), is also the first attempt to evaluate an option given the dynamics of the asset. The Bachelier model is generally assumed to imply that the price process can get negative, i.e. $\mathcal{D} = (-\infty, \infty)$. Take

$$dX = \sigma\ dW \qquad \text{for a} \quad \sigma > 0$$

as specification under the pricing measure. The pricing formula is

$$\text{Bac}(k,x,\tau) = \tau\,[d\Phi(d) + \varphi(d)] \qquad d = \frac{x-k}{\tau}.$$

If we model the asset as positive, i.e. with absorbtion in zero, the pricing formula is

$$\text{Bac}^a(k,x,\tau) = \text{Bac}(k,x,\tau) - \text{Bac}(k,-x,\tau).$$

Affine Volatility: $\quad p(x) = x - l \qquad l \leq 0$

The famous model by Samuelson (1964) with a linear volatility function corresponds to the choice $l = 0$, the call price is given by the Black-Scholes formula:

$$\text{BS}(k,x,\tau) = x\Phi(d + \tau/2) - k\Phi(d - \tau/2) \qquad d = \frac{\log x - \log k}{\tau}$$

It is easy to verify that the general pricing formula is

$$\text{Affi}(k,x,\tau,l) = \text{BS}(k-l, x-l, \tau).$$

If we choose to model with absorbtion in zero we have to subtract a correction term given by

$$\text{AffiC}(k,x,\tau,l) = l\left(\frac{k-l}{l}\frac{x-l}{l}\Phi(d-\tau/2) - \Phi(d+\tau/2)\right)$$

with

$$d = \frac{1}{\tau} \log \left(\frac{l}{x \Leftrightarrow l} \frac{l}{k \Leftrightarrow l} \right).$$

Quadratic Volatility with two roots: $p(x) = (x \Leftrightarrow l)(r \Leftrightarrow x) \qquad d = r \Leftrightarrow l > 0$

This model was used by Rady and Sandmann (1994) and Miltersen, Sandmann and Sondermann (1997) with $l = 0$ and $r = 1$ for LIBOR rates. The pricing formula is

$$\text{Q2}(k, x, \tau, l, r) = \frac{1}{d} \left[(x \Leftrightarrow l)(r \Leftrightarrow k)\Phi(e + d\tau/2) \Leftrightarrow (k \Leftrightarrow l)(r \Leftrightarrow x)\Phi(e \Leftrightarrow d\tau/2) \right]$$

with

$$e = \frac{1}{d\tau} \log \left(\frac{x \Leftrightarrow l}{r \Leftrightarrow x} \frac{r \Leftrightarrow k}{k \Leftrightarrow l} \right).$$

Quadratic Volatility with no roots:

$$p(x) = 1 + \left(\frac{x \Leftrightarrow m}{d} \right)^2 \qquad d > 0$$

The parameters m and d give the location and slope of the quadratic volatility: the parabola has its minimum in m and doubles its value in $m \pm d$. The space transformation is

$$Z(x) = d \left(\frac{\pi}{2} + \arctan \frac{x \Leftrightarrow m}{d} \right)$$

and the solution of the pricing PDE is

$$\text{Q0}(k, x, \tau, m, d) = \frac{1}{\sin(z/d)} \frac{2}{d\pi} \sum_{n>0} c_n \exp\left((1 \Leftrightarrow n^2)\tau^2/2d^2 \right) \sin_n z \qquad \sin_n z = \sin(nz/d)$$

for coefficients given by equation (12) in appendix B.

3. Implied Volatilities

In this section we give examples of the behaviour of (Black-Scholes) implied volatilities for the different specifications.

Take $X_0 = 100$, a maturity $T = 1$, and choose specifications of the volatility functions such that the at-the-money volatilities equal a Black-Scholes volatiliy of 20%. Most important, we will see that the corrections for an absorbing barrier in 0 are so small that they are only of academic interest. We first look at three affine volatilities:

Bac is a Bachelier model with constant volatility $\sigma = 20$, the value of the correction in 0 is $1.1 \cdot 10^{-6}$.

Sam a Samuelson model with $\sigma = .2$ whose implied volatility is constant.

Affine is an affine model with $l = \Leftrightarrow 100$ and $\sigma = .1$, the correction in 0 is $1.15 \cdot 10^{-11}$.

Figure 1 plots the different specifications and figure 2 gives their implied volatilities. It is obvious that the implied volas of the affine model vary between the Bachelier ($l = \Leftrightarrow \infty$) and the Black-Scholes ($l = 0$) implied volatility.

Q0-0 is a quadratic model with no root, $m = 0$, $d = 100$ and $\sigma = .11$, the correction is $9.9 \cdot 10^{-13}$.

Q0-1 is a quadratic model with no root, $m = d = 100$ and $\sigma = .2$, the correction is $2.8 \cdot 10^{-4}$.

Q2 is a quadratic model with two roots in $l = 0$ and $r = 200$, $\sigma = .002$.

Figure 3 plots the different specifications and figure 4 gives their implied volatilities. To make the implied volatility smile we have to use an input function that is above the linear one. Together with the affine implied volatilities a wide range of smiles and frowns are possible.

4. Conclusion

If we want to price real world options we have to incorporate interest rates. Let r denote the (deterministic) short rate and define the bank account:

$$db = rb\, dt \qquad b_0 = 1$$

Under the risk-neutral measure the dynamics of the discounted asset $X^b = X/b$ has to be a martingale and the price of the call with strike k is

$$V(t, X_t) = E^* \left[\frac{(X_T \Leftrightarrow k)^+}{b_T} \mid \mathcal{F}_t \right] = E\left[(X_T^b \Leftrightarrow k/b_T)^+ \mid \mathcal{F}_t \right].$$

The pricing formulae from section 2 hold with k replaced by $k^b = k/b_T$ if we assume that the discounted asset price $X^b = X/b$ satisfies assumptions **1.1**–**1.2**.

For the Dumas et al. (1998) model with quadratic volatility for the forward price of the asset the formulas could immediately be used.

Generally, we have to assume the quadratic volatility for the price of the asset discounted by some numeraire N, whereas for the linear model this is implied for the discounted asset $X^N = X/N$ for any numeraire process:

$$dX = \cdots dt + \sigma X \; dW \implies dX^N = \cdots dt + \sigma X^N \; dW$$

The usual trick to make a numeraire change to evaluate an option that is homogeneous of degree one in the asset price does only work if the volatility of the asset stays the same under the new numeraire, i.e., only in the case of linear volatility. This is what makes the Samuelson model of geometric Brownian motion the most convenient one for pricing.

Appendix A. Solving the pricing PDE

The proof we give is a generalization of the one given by (Rady and Sandmann 1994) for a model with two roots in $\{0, 1\}$ for LIBOR rates (see also Rady 1997). In this section we omit function arguments and subscripts denote partial differentials. Note that for the time τ and space Z functions defined by (5) it holds:

$$\tau_t^2 = \Leftrightarrow \sigma^2 \qquad Z_x = \frac{1}{p} \qquad Z_{xx} = \Leftrightarrow \frac{p_x}{p^2} \qquad Z_z^{-1} = p(Z^{-1})$$

Suppose that the solution of the pricing PDE (1) is of the form

$$V(t, x) = \gamma(\tau^2(t)) \; \xi(Z(x)) \; h(\tau^2(t), Z(x))$$

where h is a solution to the heat equation

$$h_{\tau^2} = \frac{1}{2} h_{zz}.$$

Then

$$V_{\tau^2} = \xi(\gamma_{\tau^2} h + \gamma h_{\tau^2}) \qquad V_{xx} = \frac{\gamma}{p^2}\left[\xi_{zz} h + 2\xi_z h_z + \xi h_{zz} \Leftrightarrow p_x(\xi_z h + \xi h_z) \right]$$

and the PDE becomes

$$\begin{aligned} 0 &= V_{\tau^2} \Leftrightarrow \frac{1}{2} p^2 V_{xx} \\ &= \gamma\xi\underbrace{(h_{\tau^2} \Leftrightarrow \frac{1}{2} h_{zz})}_{=0} \Leftrightarrow \gamma h_z \underbrace{(\xi_z \Leftrightarrow \frac{1}{2} p_x \xi)}_{E_1} \Leftrightarrow h\underbrace{(\frac{1}{2}\gamma(\xi_{zz} \Leftrightarrow p_x \xi_z) \Leftrightarrow \gamma_{\tau^2}\xi)}_{E_2}. \end{aligned}$$

To solve the PDE it must hold $E_1 = E_2 = 0$. For E_1 this is equivalent to

$$\xi_z = \frac{1}{2} p_x \xi \tag{8}$$

which implies ($x = Z^{-1}(z)$)

$$\xi(z) = \exp\left[\frac{1}{2} \int^z p_x(Z^{-1}(y)) dy \right] = \sqrt{p(Z^{-1}(z))}$$

as

$$\int^z p_x(Z^{-1}(y)) dy = \int^{Z^{-1}(z)} \frac{p_x(v)}{p(v)} \; dv = \log[p(Z^{-1}(z))].$$

Now (8) implies

$$\xi_{zz} = \frac{1}{4} \xi (2 p p_{xx} + p_x^2)$$

so $E_2 = 0$ is equivalent to

$$\gamma_{\tau^2} = \frac{1}{4}\gamma(pp_{xx} - p_x^2/2) = -\varsigma\gamma/8.$$

It follows from the theory of ordinary differential equations that the term in braces is constant iff p is a quadratic polynomial 1.2 and we can solve for

$$\gamma(\tau^2) = \exp(-\varsigma\tau^2/8).$$

The initial condition is

$$g(x) = V(0,x) = \xi(Z(x))\; h(0, Z(x)) \quad \Leftrightarrow \quad h(0,z) = \frac{g(Z^{-1}(z))}{\xi(z)}. \tag{9}$$

Appendix B. Solving the PDE for the call option

Notice the following specifics which will ease the computation of the formulas: The boundary condition for h in L will always be of Dirichlet type as

$$h(\tau^2, L) = (Z^{-1}(L) - k)^+ = 0.$$

Suppose that we have solved the pricing PDE (1–2–3) on the domain (l, r) for an $l < 0$. If we want to find a solution on the domain $(0, r)$ the only characteristic of the PDE that changes is the boundary condition which is for the call to be 0 in 0. The obvious solution to this is to take

$$h_0(\tau^2, z) = h(\tau^2, z) - h(\tau^2, 2z_0 - z) \qquad z_0 = Z(0) \tag{10}$$

as the solution to the heat equation on $\mathcal{R} = (z_0, R)$.

The transition density of a one-dimensional Wiener process is

$$p_\tau(x,y) = P\{W_{\tau^2} \in dy \mid W_0 = x\} = \frac{1}{\tau}\varphi\left(\frac{x-y}{\tau}\right).$$

Constant Volatility: $p = 1$

It holds $\varsigma = 0$, $\gamma \equiv 1$,

$$Z(x) = x, \qquad Z^{-1}(z) = z, \qquad \text{and} \quad \xi(Z^{-1}(z)) = 1.$$

The domain for this specification is the whole real line $\mathcal{D} = (-\infty, \infty)$. There is no boundary condition as $\mathcal{R} = (-\infty, \infty)$. The fundamental solution for the heat equation in $\mathbb{R}$ is p_τ, so with the substitution $y = x - \tau\widetilde{y}$

$$\begin{aligned}
\operatorname{Bac}(k,x,\tau) = h(\tau^2, x) &= \int_{-\infty}^{\infty} h(0,y)p_\tau(x,y)\; dy \\
&= \int_{-\infty}^{\infty} (y-k)^+ p_\tau(x,y)\; dy \\
&= \int_{-\infty}^{d} (x - \tau y - k)\varphi(y)\; dy \\
&= (x-k)\Phi(d) + \tau\varphi(d) \qquad \text{for} \quad d = \frac{x-k}{\tau}.
\end{aligned}$$

By (10) the price for the case of an absorbing boundary in 0 is easily obtained as $z_0 = 0$:

$$\operatorname{Bac}^a(k,x,\tau) = \operatorname{Bac}(k,x,\tau) - \operatorname{Bac}(k,-x,\tau)$$

Affine Volatility: $p(x) = x - l \qquad l \leq 0$

We have

$$\varsigma = 1 \qquad \gamma(\tau^2) = \exp(-\tau^2/8) \qquad Z(x) = \log(x-l) \qquad Z^{-1}(z) = l + e^z \qquad \xi(z) = e^{z/2}$$

The initial value is

$$h(0,z) = (l + e^z - k)^+ e^{-z/2}.$$

For the domain $\mathcal{D} = (l, \infty)$, $\mathcal{R} = \mathbb{R}$, we get

$$\begin{aligned} h(\tau^2, z) &= \int_{-\infty}^{\infty} h(0,y) p_\tau(z,y)\ dy \\ &= \int_{-\infty}^{d} (l + e^{z-\tau y} \Leftrightarrow k) e^{-\frac{z-\tau y}{2}} \varphi(y)\ dy \\ &= e^{\tau^2/8} \left(e^{z/2} \Phi(d + \tau/2) \Leftrightarrow (k \Leftrightarrow l) e^{-z/2} \Phi(d \Leftrightarrow \tau/2) \right). \end{aligned}$$

with

$$d = \frac{z_x \Leftrightarrow z_k}{\tau} = \frac{\log(x \Leftrightarrow l) \Leftrightarrow \log(k \Leftrightarrow l)}{\tau}.$$

Combining the factors we get

$$\begin{aligned} \mathrm{Affi}(k, x, \tau, l) &= \gamma(\tau^2)\ e^{z_x/2}\ h(\tau^2, z_x) \\ &= (x \Leftrightarrow l)\Phi(d + \tau/2) \Leftrightarrow (k \Leftrightarrow l)\Phi(d \Leftrightarrow \tau/2) \\ &= \mathrm{Affi}(k \Leftrightarrow l, x \Leftrightarrow l, \tau, 0) \\ &= \mathrm{BS}(x \Leftrightarrow l, k \Leftrightarrow l, \tau). \end{aligned}$$

As $z_0 = \log(\Leftrightarrow l)$ the correction term is

$$\begin{aligned} \mathrm{AffiC}(k, x, \tau, l) &= \gamma(\tau^2)\ e^{z_x/2}\ h(\tau^2, 2z_0 \Leftrightarrow z_x) \\ &= l \left(\frac{k \Leftrightarrow l}{l} \frac{x \Leftrightarrow l}{l} \Phi(c \Leftrightarrow \tau/2) \Leftrightarrow \Phi(c + \tau/2) \right) \end{aligned}$$

for

$$c = \frac{2z_0 \Leftrightarrow z_x \Leftrightarrow z_k}{\tau} = \frac{1}{\tau} \log \left(\frac{l}{x \Leftrightarrow l} \frac{l}{k \Leftrightarrow l} \right).$$

Quadratic Volatility with two roots: $p(x) = (x \Leftrightarrow l)(r \Leftrightarrow x) \qquad d = r \Leftrightarrow l > 0$

It is

$$\varsigma = d^2 \qquad \gamma(\tau^2) = \exp(\Leftrightarrow d^2 \tau^2 / 8)$$

$$Z(x) = \frac{1}{d} \log \left(\frac{x \Leftrightarrow l}{r \Leftrightarrow x} \right) \qquad Z^{-1}(z) = l + \frac{d}{1 + e^{-dz}} \qquad \xi(z) = d \frac{e^{dz/2}}{e^{dz} + 1}$$

The initial value is

$$h(0, z) = \left(\frac{r \Leftrightarrow k}{d} e^{dz/2} \Leftrightarrow \frac{k \Leftrightarrow l}{d} e^{-dz/2} \right)^+$$

so for $\mathcal{D} = \mathcal{R} = \mathbb{R}$:

$$\begin{aligned} \gamma(\tau^2) h(\tau^2, z) &= \gamma(\tau^2) \int_{z_k}^{\infty} \left(\frac{r \Leftrightarrow k}{d} e^{dy/2} \Leftrightarrow \frac{k \Leftrightarrow l}{d} e^{-dy/2} \right) p_\tau(z, y)\ dy \\ &= \gamma(\tau^2) \int_{-\infty}^{\frac{z-z_k}{\tau}} \left(\frac{r \Leftrightarrow k}{d} e^{d(z-\tau y)/2} \Leftrightarrow \frac{k \Leftrightarrow l}{d} e^{-d(z-\tau y)/2} \right) \phi(y)\ dy \\ &= \frac{r \Leftrightarrow k}{d} e^{dz/2} \Phi \left(\frac{z \Leftrightarrow z_k}{\tau} + \frac{\tau}{2} \right) \Leftrightarrow \frac{k \Leftrightarrow l}{d} e^{-dz/2} \Phi \left(\frac{z \Leftrightarrow z_k}{\tau} \Leftrightarrow \frac{\tau}{2} \right) \end{aligned}$$

Combining the factors we get the formula Q2.

Quadratic with no root:

$$p(x) = 1 + \left(\frac{x \Leftrightarrow m}{d} \right)^2 \qquad d > 0$$

This implies

$$\varsigma = \Leftrightarrow 4/d^2 \qquad \gamma(\tau^2) = \exp(\tau^2 / 2d^2)$$

$$Z(x) = d \left(\frac{\pi}{2} + \arctan \frac{x \Leftrightarrow m}{d} \right) \qquad Z^{-1}(z) = m + d \tan \left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2} \right)$$

$$\xi(z) = \sqrt{1 + \tan^2\left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2}\right)} = \frac{1}{\cos\left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2}\right)} = \frac{1}{\sin z/d}$$

The initial value is

$$\begin{aligned} h(0,z) &= \cos\left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2}\right)\left[m + d\tan\left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2}\right) \Leftrightarrow k\right]^+ \\ &= \left[(m \Leftrightarrow k)\cos\left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2}\right) + d\sin\left(\frac{z}{d} \Leftrightarrow \frac{\pi}{2}\right)\right]^+ \\ &= \left[(m \Leftrightarrow k)\sin z/d \Leftrightarrow d\cos z/d\right]^+ \end{aligned}$$

The general solution of the heat equation (Carslaw and Jaeger 1959, 3.3) on the finite domain $\mathcal{R} = (0, d\pi)$ with Dirichlet boundary condition is given by

$$h(\tau^2, z) = \frac{2}{d\pi}\sum_{n>0} a_n \exp\left(\Leftrightarrow n^2\tau^2/2d^2\right)\sin_n z \qquad \sin_n z = \sin(nz/d) \tag{11}$$

with coefficients

$$a_n = \int_0^{d\pi} h(0,z)\sin_n z\ dz.$$

The general integrals are

$$S_n = \int \sin_1 \sin_n\ d\lambda = \begin{cases} \frac{1}{2}\left(z \Leftrightarrow \frac{d}{2}\sin_2 z\right) & n = 1 \\ \frac{d}{2}\left(\frac{\sin_{n+1}}{n+1} \Leftrightarrow \frac{\sin_{n-1}}{n-1}\right) & n > 1 \end{cases}$$

$$C_n = \int \cos_1 \sin_n\ d\lambda = \begin{cases} \Leftrightarrow \frac{d}{2}\cos_1^2 & n = 1 \\ \frac{d}{2}\left(\frac{\cos_{n+1}}{n+1} + \frac{\cos_{n-1}}{n-1}\right) & n > 1 \end{cases}$$

so our coefficients are

$$a_n = \left((m \Leftrightarrow k)S_n \Leftrightarrow dC_n\right)\Big|_{Z(k)}^{d\pi}.$$

The boundary condition given by the general formula (7) in $R = d\pi$ is

$$h(\tau^2, d\pi) = de^{-\tau^2/2d^2}.$$

The solution for a time dependent boundary is by (Carslaw and Jaeger 1959, 3.5) a sum (11) with coefficients

$$b_n = n(\Leftrightarrow 1)^{n+1}\int_0^t e^{(n^2-1)\lambda/2d^2}d\lambda = \begin{cases} t & n = 1 \\ n(\Leftrightarrow 1)^{n+1}d^2\frac{e^{(n^2-1)t/2d^2}-1}{n^2-1} & n > 1 \end{cases}$$

For our special problem we get the coefficients

$$c_n = a_n + b_n. \tag{12}$$

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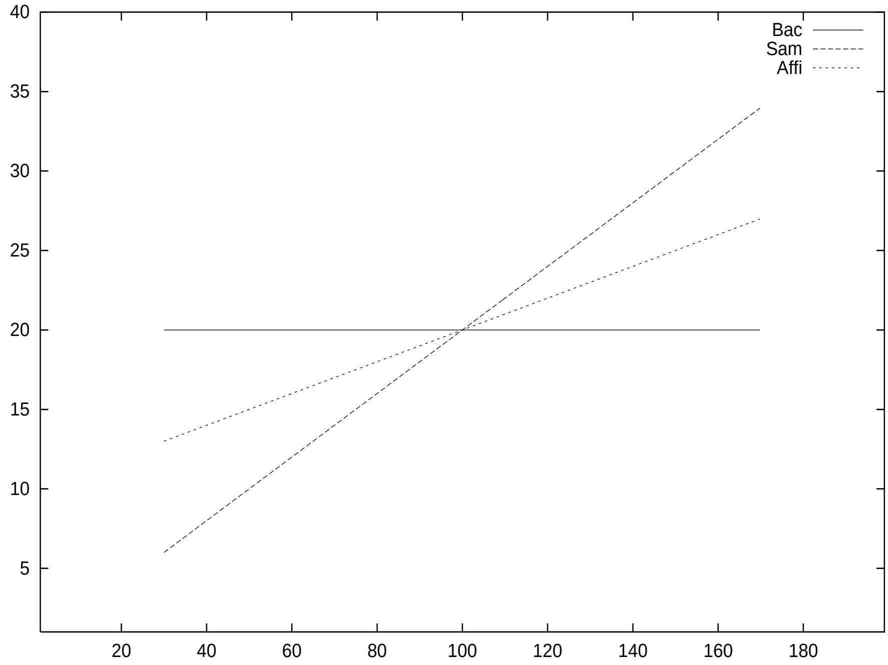

Figure 1. Affine Volatility Functions

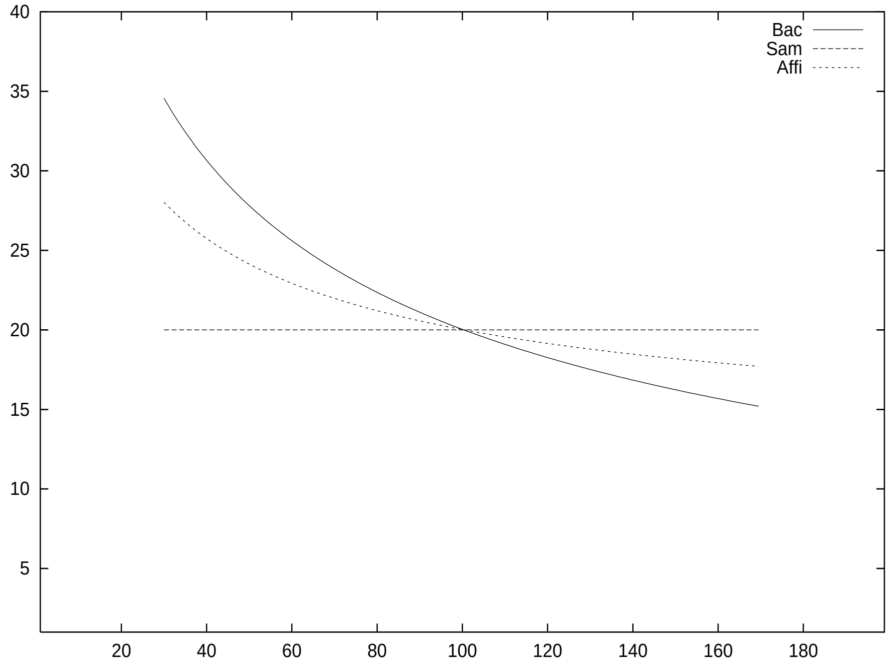

Figure 2. Affine Implied Volatilities

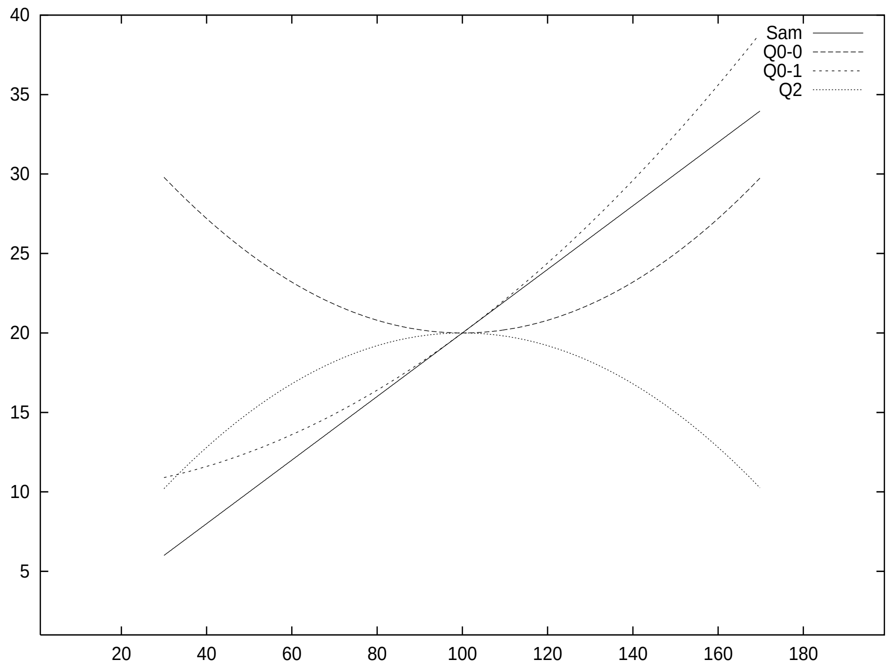

Figure 3. Quadratic Volatility Functions

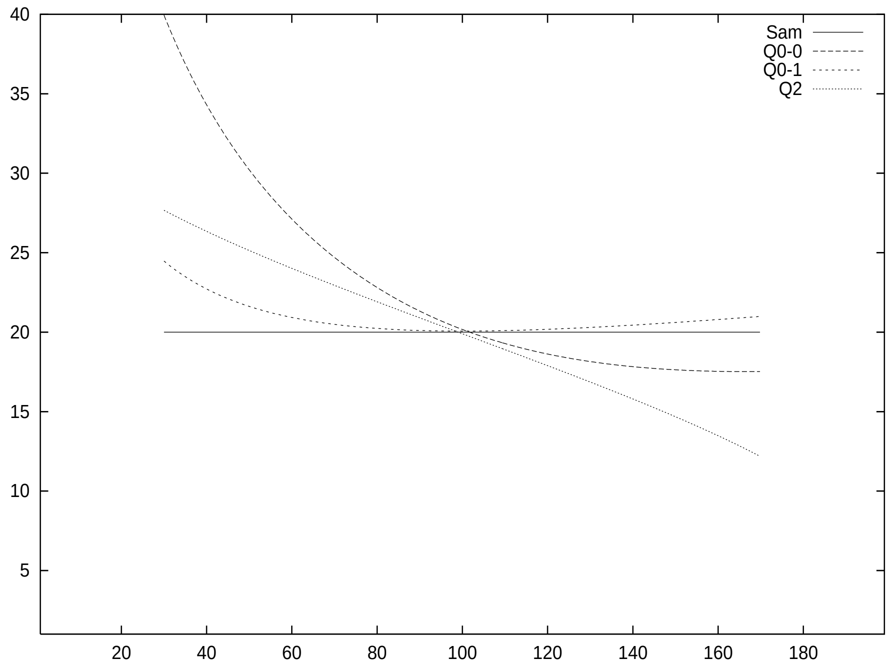

Figure 4. Quadratic Implied Volatilities