# Optimal Risk/Dividend Distribution Control Models. Applications to Insurance

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**Abstract**

The current paper presents a short survey of stochastic models of risk control and dividend optimization techniques for a financial corporation. While being close to consumption/investment models of Mathematical Finance, dividend optimization models possess special features which do not allow them to be treated as a particular case of consumption/investment models.

In a typical model of this sort, in the absence of control, the reserve (surplus) process, which represents the liquid assets of the company, is governed by a Brownian motion with constant drift and diffusion coefficient. This is a limiting case of the classical Cramer-Lundberg model in which the reserve is a compound Poisson process, amended by a linear term, representing a constant influx of the insurance premiums. Risk control action corresponds to reinsuring part of the claims the cedent is required to pay simultaneously diverting part of the premiums to a reinsurance company. This translates into controlling the drift and the diffusion coefficient of the approximating process. The dividend distribution policy consists of choosing the times and the amounts of dividends to be paid put to shareholders. Mathematically, the cumulative dividend process is described by an increasing functional which may or may not be continuous with respect to time.

The objective in the models presented here is maximization of the dividend pay-outs. We will discuss models with different types of conditions imposed upon a company and different types of reinsurances available, such as proportional, noncheap, proportional in a presence of a constant debt liability, excess-of-loss. We will show that in most cases the optimal dividend distribution scheme is of a barrier type, while the risk control policy depends substantially on the nature of reinsurance available.

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# 1 Introduction

There are a lot of problems in insurance and finance which can be set up as optimization problems, in which the decision maker has an option to dynamically control certain variables, simultaneously affecting the state of the controlled process as well as the the objective function, whose value he wants to maximize (minimize). In actuarial science one of this type of problems is to find the optimal rate of dividend pay-outs for an insurance company. This problem was discussed in the literature for quite a while, e.g., [15], [7], [8], [12], [25],[26]. In his speech to the Royal Statistical Society of London in 1967, K. Borch pointed out the value of the control theory for actuarial science:

*The theory of control processes seems to be "tailor-made" for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed, if actuaries and engineers had realized that they were studying the same problems and joined forces over 50 years ago. A little reflection should teach us that a "highly specialized" problem may, when given the proper mathematical formulation, be identical to a series of other, seemingly unrelated problems.*

In the last 25 years there have been quite a few attempts in which the insurance surplus was treated as a diffusion process (e.g., see [30], [38], [22]), in contrast to the more established Cramer-Lundberg model in which the surplus process is described by a compound Poisson process with drift (see the original paper by Lundberg [43]). Diffusion process modeling of the surplus gave rise to a whole new development within the "optimization" area of the actuarial science. It allowed to use the techniques of optimal diffusion control to those actuarial problems in which the surplus could be treated as a linear diffusion. Recently the interplay between finance, insurance and control attracted a lot of attention. The dividend optimization/risk control models which can describe the behavior of an insurance corporation, started being thoroughly developed. This paper is devoted to some recent advances in the diffusion control models in the actuarial science.

Another closely related area, in which diffusion control models recently gained prominence, is Mathematical Finance. One of the first attempts to describe the stock price fluctuation via a Brownian motion can be traced back to Bachelier [4]. The limitations of the arithmetic Brownian motion, used by Bachelier hindered further development of this model. The classical paper of Black and Scholes [5], introduced logarithmic Brownian motion as the model of the stock price process and set the foundation for the option pricing theory. During the same period of time Merton published his seminal paper on the optimal consumption/investment strategy for a small investor [45]. In this model an individual faces a problem of dynamically trading his portfolio, using the proceeds to finance his consumption. The stocks available for trading have prices modeled by diffusion processes. The ultimate objective is to maximize the total discounted expected utility of consumption. In spite (or maybe because) of several technical flaws and gaps, this paper attracted a lot of attention and made a strong impact on the future development of the field of consumption/investment models. In fact, this paper perpetuated numerous corrections and amelioration of Merton's original version (e.g., [41], [40], [14], [56], [57] [21] to name a few), which in turn stimulated development of an entirely new area within the classical finance, based primarily on the tools and techniques of the control theory. A comprehensive list of literature on diffusion consumption/investment models can be found in [46] and [55].

Since for a long period of time ruin probabilities were of a major interest in mathematical insurance, the first diffusion optimization models in insurance dealt with minimization of the ruin probabilities –or equivalently– maximization of survival probabilities (see [1], [7], [10], [18], [22], [23], [25], [27], [28], [29], [38], [52], [53], [58]. A more detailed reference can be found in two monographs by Buhlmann and Gerber [12] and [26]). Beginning from the middle of 90's we see a series of papers, which use diffusion control in dividend optimization or similar models ( see [39], [10], [11], [51], [3], [36], [37], [61], [60]). The dividend optimization/risk control models in many instances can be viewed as consumption/investment models with linear utility function and with risky assets governed by an arithmetic Brownian motion, rather than by a logarithmic one. In a certain sense a dividend optimization insurance problem is a problem of a small investor living in a "Bachelier world". In this world the nature of the assets' growth is linear rather than exponential, and every investor has a linear utility function. It should be also mentioned that the dividend optimization models possess some additional features not always present in the classical consumption/investment schemes, such as singularity of the dividend distribution process (see [39], [51], [3], [36]), in most cases inevitability of bankruptcy, etc. Moreover, in the "Black-Scholes-Merton's world", linear utility functions make the optimization problem trivial as a recent work by Radner and Shepp [51] shows. That might be the reason for the most of diffusion optimization models in insurance to be developed "from scratch" even though some of those dealt with portfolio management and insurance (e.g., [11]) whose frameworks are very close to those the of classical mathematical finance.

First dividend optimization problems were formulated for the Cramer-Lundberg, compound Poisson, models (see [24], [12]). This setting has more a intuitive appeal. In fact, to understand how one arrives at a diffusion control model of an insurance company, it is better to start with a more "tangible" Cramer-Lundberg model of the reserve (surplus) and its diffusion limit. Assume that claims arrive at a Poisson rate $\lambda$ and the size of $i$-th claim is $U_i$, where $\{U_i\}$ are iid with mean $m$ and variance $s^2$. If $r_t$ represents the reserve of the company at time $t$ then

$$r_t = r_0 + pt - \sum_{i=1}^{A(t)} U_i, \tag{1.1}$$

where $p$ is the amount of premium per unit time received by the insurance company and $r_0$ is the initial reserve. If one makes change of time and normalizes the state space: $r_t \mapsto r_{nt}/\sqrt{n}$, then the limiting process $R_t$ satisfies

$$dR_t = \mu dt + \sigma dW_t \tag{1.2}$$

with $W_t$ being a standard Brownian motion and

$$\mu = p - \lambda m, \quad \sigma^2 = \lambda(m^2 + s^2). \tag{1.3}$$

It should be mentioned that the diffusion approximation (1.2) is suitable for big portfolios, that is, for the case in which an individual claim is negligible compared to the size of the total reserve. Motivations for and relevant references on this and more complicated examples of diffusion approximations in risk theory can be found in Iglehart [38], Grandell [27], [28], [29], Emanuel *et al.* [18], Harrison [30], Asmussen [1], Schmidli [52], [53] and Møller [48].

In the next sections we will describe different control functions based on diffusion approximation (1.2). In most cases (1.2) represents the dynamics of the reserve process when no control actions are taken. Depending on the type of reinsurance (risk control) modes and constraints on the dividend pay-out rates, we will get different types of diffusion control models: regular, singular, mixed with various drift/diffusion control functions. In Section 2 we present models, in which the only available control is related to the dividend pay-outs. We develop the optimality equation and explain how one come about its solution. This is done in sufficient details to give understanding of the general methods and techniques employed. Section 3 deals with the model in which only the level of reinsurance is controlled, while the dividend pay-out scheme is fixed. The type of reinsurance considered is *proportional reinsurance* which requires the reinsurer to cover a fraction of each claim equal to the fraction of total premiums he receives from the cedent. Here we also present a sufficiently detailed analysis. In subsequent sections we will only outline the major steps since many technical procedures are similar to those described in Section 2 and Section 3. Next we deal with the model which combines the features of the two previous ones. We allow for both dividend and reinsurance control. We treat the case of dividend pay-out rate being bounded as well as the case of unrestricted rate.

In Section 5 we consider a model in which the company faces additional debt liabilities, which must be amortized at a constant rate. Next section is devoted to the model of the so-called *noncheap reinsurance*. This is the case when for insuring $\alpha$ fraction of each claim, the reinsurer need to be paid $\chi\alpha, \chi > 1$ of all premiums. We show that after a change of parameters this model becomes equivalent to the model with debt liabilities. In section 7 we consider the case of excess-of-loss reinsurance. The cedent pays each claim up to $b$, called the retention level, while the reinsurer pays everything in in excess of $b$. It turns out that the analysis in this case depends on whether or not the claim size distribution has a bounded support.

In Section 8 we outline some open problems. A generalized Ito's formula and a simplified version of the one dimensional Skorohod problem are presented in Appendix.

The results of Section 2 were developed in [39] and [3]. The proportional reinsurance model with no dividend control was considered in [36]. The model of Section 3 was considered in [37]. The method of solution used there was first developed by Radner and Shepp in [51]. Debt liability scheme can be found in [61]. The case of minimization of ruin probability in the presence of liabilities was also considered in [11]. Reduction of noncheap reinsurance to a debt liability problem is first done in this paper. A similar case of cheap reinsurance with no dividend control was studied in [35]. The excess-of-loss model can be found in [2].

## 2 Dividend Control Model

2.1. **Setting of the problem.** As always, a rigorous mathematical setting of a stochastic control problem starts from a triple $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and a standard Brownian motion $W_t$ adapted to $\mathcal{F}_t$.

In a pure dividend control model the dynamics of the reserve process is the same as in (1.2) with and extra term $-dL_t$ added to the right hand side of the equation. Thus

$$dR_t = \mu dt + \sigma dW_t - dL_t, \tag{2.1}$$

$$R_{0-} = x \tag{2.2}$$

The control functional $L_t$ represents the cumulative amount of dividends paid-out up to time $t$. The major requirement on $L_t$ is that it is nonnegative and adapted to the filtration $\mathcal{F}_t$. The latter is the mathematical expression of the fact that in making the policy decision one can use only the past history but not the future information. In addition there is a technical requirement that $L_t$ is right continuous with left limits (*cadlag*). Any functional satisfying these conditions is called an *admissible control* or a *policy.*

Once a dividend distribution policy is chosen, the bankruptcy time $\tau$ is defined as

$$\tau = \inf\{t \geq 0 : R_t \leq 0\}. \tag{2.3}$$

With each control functional $L$ we associate its performance index

$$J_x(L) = E\int_0^\tau e^{-ct} dL_t, \tag{2.4}$$

where the integral in the right hand side of (2.4) is the Lebesgue-Stiltjes integral. The objective is to find

$$V(x) = \sup_L J_x(L). \tag{2.5}$$

and the functional $L_t^*$ such that

$$V(x) = J_x(L^*). \tag{2.6}$$

The function $V$ is called *the value function* or *the optimal return function* and the functional $L^*$ is called the *optimal control* or the *optimal policy.*

There are two cases to be considered in this setting. The first case is when the rate of dividend payments is bounded by a constant $M$. Thus (2.1) can be rewritten as

$$dR_t = (\mu - l(t))dt + \sigma dW_t,$$

where $l(t)$ is an $\mathcal{F}_t$-adapted process chosen by a controller, subject to

$$0 \leq l(t) \leq M.$$

In the second case there are no restrictions on the pay-out rate, in which case the functional $L_t$ is a general increasing right continuous functional. For convenience purposes, in the sequel we will write $J_x(l(\cdot))$ instead of $J_x(L)$, whenever $L'(t) = l(t)$ is a priori bounded.

2.2. **Bounded rate of dividends.** In this case the solution can be found via the classical stochastic control theory (see [19]). Prior to dealing with the analytical part of the problem one needs to establish

**Lemma 2.1** *The function $V$ is a nonnegative concave function.*

The formal proof is rather simple and we omit it. There is a natural economic interpretation of this fact. Shares in two insurance companies with identical market parameters will not give better return on investment than one company with a combined capital.

Our next step is to find the function $V$. To understand the equation this function must satisfy, consider for each $y > 0$ a control $l_y(t)$ such that

$$EJ_y(l_y(t)) \;\geq\; V(y) - \epsilon.$$

Fix $x > 0$ and let

$$l^\epsilon(t) \;=\begin{cases} l, & 0 \leq t \leq \delta, \\ l_{R_\delta,}(t-\delta) & t > \delta. \end{cases}$$

The meaning of $l^\epsilon(t)$ is the following: we use dividend rate $l$ on the interval $[0, \delta]$ and then switch to the $\epsilon$–optimal process $l_{R_\delta}(\cdot)$ corresponding to reserve level $R_\delta$ at the time $\delta$. Then we can write

$$\begin{aligned} V(x) &\geq E_x\left\{J_x(l^\epsilon(\cdot))\right\} \\ &\geq l\delta P_x(\tau > \delta) + E_x\left[\int_\delta^\tau l_{R(\delta)}(t-\delta)e^{-ct}dt\,;\,\tau > \delta\right] \\ &= l\delta P_x(\tau > \delta) + E_x\left[1_{\{\tau>\delta\}}E\left[\int_\delta^\tau l_{R(\delta)}(t-\delta)e^{-ct}dt \mid \mathcal{F}_\delta\right]\right] \\ &\geq l\delta P_x(\tau > \delta) + E_x\left[E_{R(\delta)}\left[e^{-c\delta}\int_0^\infty l_{R(\delta)}(s)e^{-cs}ds\,;\,\tau > \delta\right]\right] \\ &\geq l\delta P_x(\tau > \delta) + (1 - c\delta)\left[E_x V(R(\delta)) - \epsilon\right] + o(\delta). \end{aligned} \tag{2.7}$$

Inequality (2.7) should be valid for all $l \in [0, M]$. It can be shown (see Fleming & Rishel [19]) that this inequality is tight for at least one $l \in [0, M]$. Dividing by $\delta$ and letting $\delta \to 0$, we arrive at the so–called Hamilton–Jacobi–Bellman (HJB) equation for the optimal return function:

$$\max_{0\leq l\leq M}\left[\frac{1}{2}\sigma^2 V''(x) + (\mu - l)V'(x) - cV(x) + l\right] = 0. \tag{2.8}$$

Since the reserve $x = 0$ corresponds to the bankruptcy state, the value function must vanish at this point

$$V(0) = 0. \tag{2.9}$$

To find a solution to (2.8)-(2.9), one should recall that $V$ is concave and that the expression in the left hand side of the equation (2.9) is a linear function or $l$. Thus for each $x$ the maximizer of the left hand side of (2.8) is either $l = 0$ if $V'(x) > 1$ of $l = M$, if $V'(x) \leq 1$. In view of concavity of $V$, the set $\{x : V'(x) > 1\}$ (if nonempty) is an interval $[0, u)$ and by virtue of (2.8) for all $x \in [0, u)$, the function $V$ satisfies

$$\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - cV(x) = 0, \tag{2.10}$$

while for $x \geq u$

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - cV(x) + M = 0. \tag{2.11}$$

Denote

$$\theta_{\pm}(y) = \frac{-y \pm \sqrt{y^2 + 2c\sigma^2}}{\sigma^2} \tag{2.12}$$

and put $\theta_1 = \theta_+(\mu), \theta_2 = \theta_-(\mu)$ and $\hat{\theta} = \theta_-(\mu - M)$. The solution to (2.10), (2.9) is

$$C(e^{\theta_1 x} - e^{\theta_2 x}),$$

while the solution to (2.11) is given by

$$\frac{M}{c} + C_1 e^{\hat{\theta} x} + C_2 e^{\theta - (\mu - M)x}.$$

In order for the above function to be concave one must have $C_2 = 0$ and $C_1 < 0$. Thus to find the solution we must determine three unknown constants: $C, C_1$ and $u$. To this end we use the so-called " principle of smooth fit". The function $V$ must be continuous with a continuous first derivative. In addition the derivative of $V$ at $u$ must be equal to 1 (simple argument shows that this would imply continuity of the second derivative at $u$ as well). This gives us three equations.

$$C(e^{\theta_1 u} - e^{\theta_2 u}) = \frac{M}{c} + C_1 e^{\hat{\theta} u}, \tag{2.13}$$

$$C(\theta_1 e^{\theta_1 u} - \theta_2 e^{\theta_2 u}) = 1, \tag{2.14}$$

$$C_1 \hat{\theta} e^{\hat{\theta} u} = 1. \tag{2.15}$$

**Theorem 2.1** *If*

$$\alpha = \frac{M}{c} - \frac{1}{\hat{\theta}} > 0 \tag{2.16}$$

*then the solution of (2.13)-(2.15) exists and unique with*

$$u = \frac{1}{\theta_1 + \theta_2} \log \frac{1 + \alpha\theta_2}{1 - \alpha\theta_1} > 0,$$

$$C = ((e^{\theta_1 u} - e^{\theta_2 u})^{-1} > 0,$$

$$C_1 = \frac{1}{\hat{\theta}} e^{-\hat{\theta} u} < 0.$$

*The solution of (2.8) is given by*

$$V(x) = \begin{cases} C(e^{\theta_1 x} - e^{\theta_2 x}), & 0 \le x < u, \\ \frac{M}{c} + C_1 e^{\hat{\theta} x}, & u \le x < \infty \end{cases}$$

*If* $\alpha \le 0$ *then the solution to (2.8) is given by*

$$V(x) = \frac{M}{c}(1 - e^{\hat{\theta} x}).$$

In order to construct the optimal control one should start with identifying the optimal feedback control function $\mathcal{L}^*(x)$. The latter is defined as the argmax of the left hand side of the equation (2.8). The structure of our solution shows that in the case of $\alpha > 0$

$$\mathcal{L}^*(x) = M 1_{x > u},$$

while if $\alpha \le 0$, then

$$\mathcal{L}^*(x) \equiv M.$$

After the the optimal feedback $\mathcal{L}^*(x)$ is determined, one can find the optimal reserve process $R^*(t)$ as a solution to the following stochastic differential equation

$$dR_t^* = (\mu - \mathcal{L}^*(R_t^*))dt + \sigma dW_t, \tag{2.17}$$

$$R_0^* = x.$$

When the solution to (2.17) is obtained, the optimal control functional $l^*$ is determined via

$$l^*(t) = \mathcal{L}^*(R_t^*). \tag{2.18}$$

To see that (2.17), in fact, yields the optimal process one needs to prove the following verification lemma.

**Lemma 2.2** *If $V$ is a concave solution of (2.8), (2.9), then for any control $l(\cdot)$*

$$V(x) \geq J_x(l(\cdot)).$$

*If*

$$\mathcal{L}^*(x) = \arg\max_{l \leq M} \left[\frac{1}{2}\sigma^2 V''(x) + (\mu - l)V'(x) - cV(x) + l\right],$$

*then for $l^*(t)$ given by (2.17), (2.18)*

$$J_x(l^*(\cdot)) = V(x).$$

*Proof* Let $l(t)$ be any control and $R_t$ be the corresponding reserve process. Then by virtue of Ito's formula (see[17], Ch. 12)

$$E[V(e^{-cT\wedge\tau}R_{T\wedge\tau})] - V(x) = -E\int_0^{T\wedge\tau} e^{-ct}\left(\frac{1}{2}\sigma^2 V''(R_t) + (\mu - l(t))V'(R_t) - cV(R_t)\right)dt. \quad (2.19)$$

In view of (2.8) the integrand in the right hand side of (2.19) is greater of equal to $-e^{-ct}l(t)$. Thus

$$E[e^{-cT\wedge\tau}V(R_{T\wedge\tau})] - V(x) \leq -E\int_0^{T\wedge\tau} e^{-ct}l(t)dt. \quad (2.20)$$

Since $V$ has at most a linear growth and $R_t \leq \mu t + \sigma W_t$, we have $V(R_t) \leq K(1 + \mu t + \sigma W_t)$. Also $e^{-cT\wedge\tau}V(R_{T\wedge\tau}) = e^{-cT}1_{\tau>T}V(R_T) \leq e^{-cT}V(R_{T\wedge\tau})$. Therefore

$$\lim_{t\to\infty} E[e^{-cT\wedge\tau}V(R_{T\wedge\tau})] = 0.$$

Thus one can pass to a limit in (2.20) to obtain

$$V(x) \geq E\int_0^{\infty} e^{-ct}l(t)dt = J_x(l(\cdot)). \quad (2.21)$$

This proves the first part of the verification lemma. To prove the second part one needs to notice that the inequalities in (2.20) and (2.21) become equalities, when $l(t)$ is replaced by $l^*(t) = \mathcal{L}^*(R^*(t))$.

In any control problem, in order to complete its solution, one always needs to find the optimal process and to go through a verification lemma like the one above. Nevertheless, in essence the problem is solved, once the solution to HJB equation is found and the optimal feedback is determined. Existence of the optimal feedback control shows that the optimal policy depends only on the current level of reserve. The optimal control functional is obtained via a solution to a stochastic differential equation.

The dividend optimization problem considered in this section yields a rather simple control mode. When (2.16) holds the optimal policy consists of not paying dividends until the reserve level reaches $u$ and paying the maximal possible rate whenever the reserve level is above $u$. If (2.16) is not true then the optimal policy is always to pay the maximal rate.

2.3. **Unrestricted dividend rates.** Suppose we consider the same control problem as in the previous subsection but without any bound on the dividend rates. Deriving HJB equation (2.8) and trying to solve it formally, we will see that as a function of $l$, the left hand side of (2.8) is linear. Therefore, its maximizer is either 0 or $\infty$. To overcome this difficulty, we have to change the control functional. Namely, $L_t$ represents the cumulative amount of dividends paid out up to time $t$. We will consider only admissible controls, that is those functionals $L_t$ which are nonnegative nondecreasing $\mathcal{F}_t$-adapted *cadlag* processes. Once an admissible control $L$ is chosen, the dynamics of the reserve process is given by (2.1), (2.2). To understand the equation the optimal return function $V$, given by (2.5), (2.6) satisfies in this case, define $L^y_\cdot$ by

$$EJ_y(L^y_\cdot) \;\geq\; V(y) - \epsilon.$$

and put

$$L_t(\epsilon) \;=\; \begin{cases} 0, & t < \delta, \\ L^{R_\delta}_{t-\delta}, & t < \delta \end{cases}$$

(assuming $L_t \equiv L_\tau$, for $t > \tau$). The policy $L_\cdot(\epsilon)$ prescribes to pay no dividends before time $\delta$; during this period of time the reserve evolves as a Brownian motion with a constant drift and a constant diffusion coefficient, and gets to the level $R_\delta$. Then we switch to the policy $L^{R_\delta}_{\cdot-\delta}$, which yields, no less than $V(R_\delta) - \epsilon$. Since such policy is suboptimal

$$\begin{aligned} V(x) \;&\geq\; E\int_0^\tau e^{-ct} dL_t(\epsilon) \equiv E\int_0^\infty e^{-ct} dL_t(\epsilon) \\ &=\; E\int_\delta^\infty e^{-ct} dL^{R_\delta}_{t-\delta} = E\left[e^{-c\delta}\left[E\int_0^\infty e^{-cu} dL^{R_\delta}_u | \mathcal{F}_\delta\right]\right] \\ &\geq\; e^{-c\delta} E[V(R_\delta) - \epsilon] \end{aligned}$$

In view of arbitrariness of $\epsilon$, we arrive at

$$V(x) - e^{-c\delta} EV(R_\delta) \geq 0.$$

Assuming twice continuous differentiability of $V$,

$$e^{-c\delta} EV(R_\delta) \;=\; (1 - c\delta)\left(V(x) + \delta\left(\frac{\sigma^2}{2} V''(x) + \mu V'(x)\right)\right) \;+\; o(\delta).$$

Dividing by $\delta$ and letting $\delta \to 0$, we obtain

$$\frac{\sigma^2}{2}V''(x) + \mu V'(x) - cV(x) \ \le \ 0. \tag{2.22}$$

This is one of the so–called "variational inequalities" that $V$ should satisfy. To obtain another inequality, fix $x$ and $\delta > 0$ and let $L^y_\cdot$ be defined as above. Consider $L_t(\epsilon) = \delta + L_t^{x-\delta}$. The policy $L_\cdot(\epsilon)$ in this case consists of paying instantaneously $\delta$ as dividend (thus reducing reserve to $x - \delta$) and then following the policy $L_\cdot^{x-\delta}$. Since $L_\cdot(\epsilon)$ is suboptimal

$$\begin{aligned} V(x) \ &\ge \ E\int_0^\tau e^{-ct}dL_t(\epsilon) = \delta + E\int_0^\tau e^{-ct}dL_t^{x-\delta} \\ &\ge \ \delta + V(x-\delta) - \epsilon. \end{aligned}$$

Arbitrariness of $\epsilon$ results in $V(x) - V(x-\delta) \ge \delta$ and consequently

$$V'(x) \ge 1. \tag{2.23}$$

A more refined argument shows that one of the inequalities (2.22)-(2.23) must be tight and the function $V$ must satisfy

$$\max\left\{\frac{\sigma^2}{2}V''(x) + \mu V'(x) - cV(x), 1 - V'(x)\right\} = 0 \tag{2.24}$$

at least in viscosity sense (see [20]). However, in our case the value function turns out to be twice continuously differentiable and must satisfy (2.24) in the classical sense. In addition the function $V$ is subject to (2.9) for the same reasons as before.

To solve the control problem our first step would be finding a twice continuously differentiable solution to (2.24), (2.9). To understand, how one can derive such a solution, let us assume that the optimal return function $V$ is concave. (After $V$ is found we will see that this assumption holds.) Then $V'$ is decreasing and there exists a point $u$ such that $V'(u) = 1$, while $V'(x) < 1$ for all $x < u$. Thus (2.22) is tight for all $x < u$. On the other hand due to (2.24) and concavity, the inequality (2.23) must be tight for all $x \ge u$. The solution of the equality (2.22) subject to (2.9) is given by

$$V(x) = C(e^{\theta_1 x} - e^{\theta_2 x}),$$

with $C$ being a free constant and $\theta_1$ and $\theta_2$ the same as in the previous subsection. The solution to the equality $V'(x) = 1$ is given by a linear function $x + C_1$. To find $C$, $C_1$ and $u$, we again use "the principle of smooth fit". Equalizing the values, derivatives and the second derivatives from the right and from the left at the point $u$, we get

$$C(e^{\theta_1 u} - e^{\theta_2 u}) = C_1 + u,$$

$$C({\theta_1}^{\theta_1 u} - \theta_2 e^{\theta_2 u}) = 1,$$

$$C({\theta_1}^2 e^{\theta_1 u} - {\theta_2}^2 e^{\theta_2 u}) = 0.$$

A sequential solution of those three equations with respect to $u$, $C$ and $C_1$ yields

$$u = \frac{2}{\theta_1} \log \left| \frac{\theta_2}{\theta_1} \right| = \frac{\sigma^2}{\sqrt{\mu^2 + 2\sigma^2 c}} \log \frac{\sqrt{\mu^2 + 2\sigma^2 c} + \mu}{\sqrt{\mu^2 + 2\sigma^2 c} - \mu}, \tag{2.25}$$

$$C = \frac{1}{\theta_1 e^{\theta_1 u} - \theta_2 e^{\theta_2 u}} \tag{2.26}$$

and

$$C_1 = C(e^{\theta_1 u} - e^{\theta_2 u}) - u. \tag{2.27}$$

As soon as those free parameters are found, we can write an explicit expression for the solution to (2.24).

**Theorem 2.2** *Let $u, C$ and $C_1$ be given by (2.25)-(2.27).Let*

$$V(x) = \begin{cases} C(e^{\theta_1 x} - e^{\theta_2 x}), & x \le u \\ C_1 + x - u, & x \ge u \end{cases} \tag{2.28}$$

*Then $V$ is a concave twice continuously differentiable solution to (2.24), (2.9).*

Our next step is to find the optimal policy, whose performance index coincides with the value function. In this case, however, we do not have any more the advantage of determining the optimal feedback function from the solution to HJB equation. In fact, to find the optimal control one must find a solution $(R_t^*, L_t^*)$ to the Skorohod problem in $(-\infty, u]$ (see Appendix, Section 9.2). To determine the optimality of $(R_t^*, L_t^*)$, we need to prove the following verification lemma.

**Lemma 2.3** *Let $V$ be given by (2.28). Then for any control functional $L$*

$$J_x(L) \le V(x).$$

*If $(R_t^*, L_t^*)$ is a solution to the Skorohod problem in $(-\infty, u]$ then*

$$J_x(L^*) = V(x).$$

*Proof* Let $L_t$ be any control functional and $R_t$ be the corresponding reserve process and $\tau$ be the bankruptcy time. Then (9.1) implies

$$\begin{aligned} V(x) &= E(e^{-cT\wedge\tau}V(R_{T\wedge\tau})) \\ &- E\int_0^{T\wedge\tau} e^{-ct}\left(\frac{1}{2}\sigma^2 V''(R_t)+\mu V'(R_t)-cV(R_t)\right)dt \\ &+ E\int_0^{T\wedge\tau} e^{-ct}V'(R_t)dL_t^c - E\sum_{0\le t\le T\wedge\tau} e^{-ct}(V(R_t)-V(R_{t-})) \end{aligned} \tag{2.29}$$

By virtue of (2.22) the integrand in the first integral in the right hand side of (2.29) is nonpositive. In view of (2.23) the integrand in the second integral in the right hand side is not smaller than $e^{-ct}$. By the same token

$$V(R_t)-V(R_{t-}) = V(R_{t-}-(L_t-L_{t-}))-V(R_{t-}) \ge -(L_t-L_{t-})$$

Thus (2.29) yields

$$\begin{aligned} V(x) &\ge E(e^{-cT\wedge\tau}V(R_{T\wedge\tau})) + E\int_0^{T\wedge\tau} e^{-ct}dL_t^c + E\sum_{0\le t\le T\wedge\tau} e^{-ct}(L_t-L_{t-}) \\ &= E(e^{-cT\wedge\tau}V(R_{T\wedge\tau})) + E\int_0^{T\wedge\tau} e^{-ct}dL_t. \end{aligned}$$

By letting $T\to\infty$, we get first assertion of the verification lemma. To get the second statement, assume for simplicity that $x\le u$ and apply (2.29) to the pair $(R_t^*, L_t^*)$.

$$\begin{aligned} V(x) = E(e^{-cT\wedge\tau}V(R^*_{T\wedge\tau})) - E\int_0^{T\wedge\tau} e^{-ct}\left(\frac{1}{2}\sigma^2 V''(R_t^*)+\mu V'(R_t^*)-cV(R_t^*)\right)dt \\ +E\int_0^{T\wedge\tau} e^{-ct}V'(R_t^*)dL_t^*. \end{aligned} \tag{2.30}$$

Since (2.22) is tight for all $x\le u$ and $R_t^*\le u$ for all $t>0$, the first integral in the right hand side of (2.30) vanishes. Since $dL_t^* = 1_{R_t^*=u}dL_t^*$, the quantity $V'(R_t^*)$ in the second integral can be replaced by $V'(u)\equiv 1$. Combining these facts together, we get

$$V(x) = E(e^{-cT\wedge\tau}V(R^*_{T\wedge\tau})) + E\int_0^{T\wedge\tau} e^{-ct}dL_t^*. \tag{2.31}$$

Letting $T\to\infty$, and using the fact that $1_{\tau<\infty}R^*_\tau = 0$ as well as a linear growth of $V$ at infinity, we deduce that the first term in the r.h.s. of (2.31) tends to 0, whereas the second converges to $J_x(L^*)$. This proves the second part of the verification theorem.

The economic interpretation of this optimal policy is somewhat different from the one we have in the case of bounded dividend rate. The level $u$ is the reserve level which under the optimal policy should be never exceeded. Whenever, the reserve becomes larger than $u$, everything in excess, should be paid out as dividends. We will call such a policy a *barrier* policy with the barrier $u$.

## 3 Risk control model

This model deals with the case, when the control of the risk is done via the so-called *proportional reinsurance*. Proportional reinsurance means that it is possible for the cedent to divert $(1-a)$ fraction of all premiums to the reinsurance company with the obligation from the latter to pay $(1-a)$ fraction of each claim.To understand the nature of the control functional and the dynamics of the controlled process, start again from the Cramer-Lundberg model. Suppose the reinsurance rate $1-a$ is fixed. Then

$$r_t = r_0 + apt - \sum_{i=1}^{A(t)} aU_i,$$

Taking the usual diffusion approximation $r_t \mapsto r_{nt}/\sqrt{n}$, we get as a limit a Brownian motion with drift $a\mu$ and diffusion coefficient $a\sigma$, where $\mu$ and $\sigma$ are given by (1.3). If the reinsurance is dynamically chosen, then we model the reserve process as

$$dR_t = a(t)\mu dt + a(t)\sigma dW_t,$$

$$R_0 = x,$$

where the control functional $a(t)$ represents the fraction of claims the cedent insures itself (while $1-a(t)$ is the reinsured fraction) at time $t$. As usual $a(t)$ should be adapted to the filtration $\mathcal{F}_t$ and in addition for each $t \geq 0$

$$0 \leq a(t) \leq 1.$$

In this model the dividend distributions are not controlled, rather we assume that the reserve is kept in a bank and the interest it gains is continuously paid out as dividends. Thus the performance index associated with each control functional $a(t)$ is defined as

$$J_x(a(\cdot)) = E\int_0^\tau e^{-ct} R_t dt,$$

where $\tau$ is the bankruptcy time given by (2.3) and the optimal return function is given by

$$V(x) = \sup_{a(\cdot)} J_x(a(\cdot)).$$

This is a regular stochastic control problem. The function $V$ satisfies a standard Hamilton-Jacobi-Bellman equation

$$\max_{a\in[0,1]}\left[\frac{\sigma^2a^2}{2}V''(x)+\mu aV'(x)-cV(x)+x\right]=0. \tag{3.1}$$

Simple probabilistic arguments show that $V$ is concave and satisfies (2.9). Thus our main analytical problem is to find a concave solution to (3.1) subject to (2.9).

Let $A^*(x)$ be the maximizer of the l.h.s. of (3.1). If $0<A^*(x)<1$ then

$$A^*(x)=-\frac{\mu V'(x)}{\sigma^2V''(x)}. \tag{3.2}$$

The function $A^*(x)$ is the optimal feedback control function. It shows the amount of risk (measured in terms of nonreinsured fraction of each claim) to be taken if the current reserve is $x$. Assume that there exists a point $u_1$ such that for all $x\in(0,u_1)$ the expression in the r.h.s. of (3.2) is strictly positive and does not exceed 1. Then substituting $A^*(x)$ from (3.2) into (3.1), we get

$$-\frac{\mu^2\left[V'(x)\right]^2}{2\sigma^2V''(x)}-cV(x)+x=0, \text{for all } 0<x<u_1. \tag{3.3}$$

Concavity of $V$ implies existence of a function $X:R\to[0,\infty)$, such that $-\log(V'(X(z)))=z$. It is easily seen, that

$$V'(X(z))=e^{-z} \text{ and } V''(X(z))=\frac{-e^{-z}}{X'(z)}. \tag{3.4}$$

Substituting $x=X(z)$ into (3.3) and using (3.4), we get an equation for $X$.

$$\frac{\mu^2}{2\sigma^2}X'(z)e^{-z}-cV(X(z))+X(z)=0.$$

Differentiating both parts of this equation with respect to $z$, we obtain

$$X''(z)-(1+c\beta-\beta e^z)X'(z)=0, \tag{3.5}$$

where

$$\beta=2\sigma^2/\mu^2.$$

The solution to (3.5) can be expressed as

$$X(z)=k_1\frac{\Gamma(c\beta+1)}{\beta^{c\beta+1}}\mathcal{G}(e^z)+k_2=k_1\mathcal{G}(e^z)+k_2,$$

where $\mathcal{G}$ is the cumulative distribution function of a Gamma distribution with parameters $(c\beta+1,1/\beta)$. From the definition of $X$ we have $-\log(V'(x))=\log(\mathcal{G}^{-1}(\frac{x-k_2}{k_1}))$, or

$$V'(x)=\frac{1}{\mathcal{G}^{-1}(\frac{x-k_2}{k_1})}. \tag{3.6}$$

A simple but tedious analysis shows that in the above formula $k_2 = 0$. On the other hand, according to our assumptions $A^*(x) = 1$ for all $x \geq u_1$. Therefore on $[u_1, \infty)$ the function $V$ satisfies

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - cV(x) + x = 0.$$

Any concave solution of this equation is given by

$$V(x) = x/c + \mu/c^2 + Ke^{\hat{\theta}x} \tag{3.7}$$

where $\hat{\theta} = \theta_-(\mu - M)$ (see (2.12) ) and $K < 0$ is a free constant. Thus taking into account (3.6), (2.9) and (3.7), we can suggest the solution to (3.1) in the form

$$V(x) = \begin{cases} \displaystyle\int_0^x \frac{1}{\mathcal{G}^{-1}(z/k_1)} dz, & 0 \leq x < u_1, \\ x/c + \mu/c^2 + Ke^{\hat{\theta}x}, & x > u_1. \end{cases} \tag{3.8}$$

There are three unknown constants in this solution: $k_1, K$ and $u_1$. To determine them we use the principle of smooth fit. Making the value the first and second derivative continuous at the point $u_1$, we get three equations whose solution yield

$$(k_1, u_1) = \left( \frac{\sigma^2}{\mu\alpha g(\alpha)}, \frac{\sigma^2 \mathcal{G}(\alpha)}{\mu\alpha g(\alpha)} \right). \tag{3.9}$$

$$K = e^{-\hat{\theta}u_1}\left(\int_0^{u_1} 1/\mathcal{G}^{-1}(z/k_1)\, dz - u_1/c - \mu/c^2\right), \tag{3.10}$$

where

$$\alpha = c\left(1 + \frac{\mu}{\sigma^2 \hat{\theta}}\right). \tag{3.11}$$

These computations enable one to prove the following theorem.

**Theorem 3.1** *Let $V$ be given by (3.8), with $k_1$, $u_1$ and $K$ given by (3.9), (3.10). Then $V$ is a concave solution of the Hamilton-Jacobi-Bellman equation (3.1) subject to (2.9).*

**Corollary 3.1** *The optimal feedback control function is given by*

$$A^*(x) = \begin{cases} \dfrac{\mathcal{G}^{-1}(\frac{x}{k_1})g\left(\mathcal{G}^{-1}(\frac{x}{k_1})\right)}{\alpha g(\alpha)}, & x < u_1, \\ 1, & x > u_1, \end{cases}$$

*where $g$ is the density of $\mathcal{G}$ and $\alpha$ is given by (3.11)*

Determining the optimal return function $V$ and the optimal feedback control $A^*(x)$ in essence completes the solution of the problem. The proof of verification theorem in this case is similar to the one in Section 2. The only technical difference lies in the fact that $V$ might have infinite derivatives at 0, which would require localization technique while applying Theorem 9.1.

Economic interpretation of the obtained results is the following. If the reserve is above $u_1$, then it is optimal to take the maximal risk, using no reinsurance. If the reserve level is below $u_1$, then the optimal fraction which must be reinsured at each time $t$ is $1 - A^*(x)$, where $x$ is the current reserve.

# 4 Risk and dividend control model

Suppose, that a financial corporation has an option to chose the amount and time of the dividend distribution in addition to choosing the business policy (reinsurance fraction in the case of an insurance company). We describe control $\pi$ by a two-dimensional stochastic process $\{a_\pi(t), L_t^\pi\}$, where $0 \le a_\pi(t) \le 1$ corresponds to the risk exposure at time $t$ ( while $1-a_\pi(t)$ being the reinsurance fraction) and $L_t^\pi \ge 0$ is a non-decreasing process whose value corresponds to the cumulative amount of the dividends distributed up to time $t$. The dynamics of the reserve process under this policy is

$$dR_t = \mu a_\pi(t)dt + \sigma a_\pi(t)dW_t - dL_t^\pi, \tag{4.1}$$

$$R_{0-} = x. \tag{4.2}$$

The performance index of each policy is defined by the expected present value of the cumulative dividend distributions

$$J_x(\pi) = E\int_0^\tau e^{-ct}dL_t^\pi, \tag{4.3}$$

and the optimal return function is defined as

$$V(x) = \sup_\pi J_x(\pi). \tag{4.4}$$

The policy $\pi^*$ is optimal if

$$V(x) = J_x(\pi^*). \tag{4.5}$$

As in Section 2 we will consider the cases, of bounded and of unrestricted dividend rates respectively.

4.1. **Bounded dividend rate.** Suppose that the dividend rate is bounded by a constant $M < \infty$. That is

$$L_t^\pi = \int_0^t l_\pi(s)ds, \qquad 0 \le l_\pi(s) \le M,$$

and (4.1) and (4.3) can be rewritten as

$$dR_t = (\mu a_\pi(t) - l_\pi(t))dt + \sigma a_\pi(t)dW_t,$$

$$J_x(\pi) = E\int_0^{\tau_\pi} e^{-ct} l_\pi(t)dt.$$

The function $V$ given by (4.4) satisfies a standard HJB equation (see [19])

$$\max_{a\in[0,1],l\in[0,M]}\left[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - l)V'(x) - cV(x) + l\right] = 0, \tag{4.6}$$

with the boundary condition (2.9).

To find a solution to (4.6), assume that $V$ is concave (this assumption will be verified a posteriori). Let $u_1 = \inf\{u : V'(u) = 1\}$. Then due to concavity of $V$ the maximizer of the left hand side of (4.6) is

$$\mathcal{L}^*(x) = \begin{cases} 0 & x < u_1, \\ M & x \ge u_1. \end{cases} \tag{4.7}$$

Therefore for all $x < u_1$

$$\max_{a\in[0,1]}\left[\frac{1}{2}\sigma^2 a^2 V''(x) + \mu a V'(x) - cV(x)\right] = 0. \tag{4.8}$$

Let $A^*(x)$ be the maximizer of the left hand side of (4.8). Let $O \subseteq [0, u_1)$ be such that $0 < A^*(x) < 1$ for all $x \in O$. Then

$$A^*(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)},\ x \in O. \tag{4.9}$$

Substitution of (4.9) into (4.8) results in

$$-\frac{\mu^2\left[V'(x)\right]^2}{2\sigma^2 V''(x)} - cV(x) = 0 \text{ for all } x \in O. \tag{4.10}$$

The solution of (4.10), (2.9) can be found "from scratch": $V(x) = C_1 x^\gamma$, where

$$\gamma = \frac{c}{\frac{\mu^2}{2\sigma^2} + c}. \tag{4.11}$$

Then

$$A^*(x) = -\frac{\mu x}{\sigma^2(\gamma - 1)}.$$

Putting

$$u = \frac{\sigma^2}{\mu}(1-\gamma), \tag{4.12}$$

we find that $O = (0, u)$. On the other hand $A^*(x) = 1$ for $x > u$. Inserting the latter into (4.8), we obtain the following solution for $u < x < u_1$

$$V(x) = C_2 e^{\theta_2 x} + C_3 e^{\theta_1 x}, \tag{4.13}$$

where $\theta_1 = \theta_+(\mu)$ and $\theta_2 = \theta_-(\mu)$, given by (2.12).

If $x > u_1$, then $\mathcal{L}^*(x) = M$ and $V$ must satisfy

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - cV(x) + M = 0. \tag{4.14}$$

Any concave solution of (4.14) is given by

$$V(x) = \frac{M}{c} + C_4 e^{\hat{\theta} x},$$

where $\hat{\theta} = \theta_-(\mu - M)$. Thus we conjecture the solution to (4.6) as

$$V(x) = \begin{cases} C_1 x^{\gamma}, & x < u, \\ C_2 e^{\theta_2 x} + C_3 e^{\theta_1 x}, & u < x < u_1, \\ \frac{M}{c} + C_4 e^{\hat{\theta} x}, & x > u_1, \end{cases} \tag{4.15}$$

with $u$ is given by (4.12) and $C_1, C_2, C_3, C_4$ and $u_1$ being unknown constants. For the function $V$ to be twice continuously differentiable we must have the value the first and second derivatives to be continuous at $u$ and $u_1$. In view of validity of (4.10) in $O$, (4.13) in $[u, u_1)$ and (4.14) in $[u_1, \infty)$, it is sufficient to check continuity of any two of those quantities. Recalling that under our assumption $V'(u_1) = 1$, we get five equations:

$$V(u-) = V(u+), V'(u-) = V'(u+), V(u_1-) = V(u_1+), V'(u_1-) = 1, V'(u_1+) = 1.$$

Solution of these equations yields

$$u_1 = u + \frac{1}{\theta_1 - \theta_2} \log\left(\frac{\hat{\theta} - \theta_2}{\theta_1 - \hat{\theta}}\right), \tag{4.16}$$

$$C_1 = -2\mu\theta_1\theta_2\sigma^2 (e^{\theta_2(u_1-u)} + e^{\theta_1(u_1-u)}), \tag{4.17}$$

$$C_2 = \frac{e^{-\theta_2 u}}{\theta_2 (e^{(u_1-u)\theta_2} + e^{(u_1-u)\theta_1})}, \tag{4.18}$$

$$C_3 = \frac{e^{-\theta_1 u}}{\theta_1(e^{\theta_2(u_1-u)} + e^{\theta_1(u_1-u)})}, \tag{4.19}$$

$$C_4 = \frac{1}{\hat{\theta}} e^{-\hat{\theta} u_1}. \tag{4.20}$$

To tie up loose ends, we need to verify that $u_1$ given by (4.16) is not smaller than $u$, otherwise the assumptions used in obtaining (4.15) fail.

**Lemma 4.1** *$u_1 > u$ if and only if*

$$M > \frac{\mu}{2} + \frac{c\sigma^2}{\mu}. \tag{4.21}$$

This enables us to formulate the following

**Theorem 4.1** *Assume (4.21) holds. Then $V$ given by (4.15) with $u, C_1, C_2, C_3, C_4$ and $u_1$ defined via (4.12) and (4.16)-(4.20) is a concave twice continuously differentiable solution of (4.6). (2.9).*

**Corollary 4.1** *The optimal feedback control $A^*$ is equal to*

$$A^*(x) = \begin{cases} x/u, & x < u, \\ 1, & x \geq u. \end{cases}$$

The economic interpretation of this solution is the following. If the maximal dividend rate is high enough (i.e., (4.21) holds), then the optimal policy is to take the risk linearly proportional to the current amount of reserve, until the reserve reaches the level $u$. Above this level it is optimal to take the maximal risk. The dividends start being paid when the reserve level exceeds $u_1, u_1 > u$. They are always paid at the maximal rate.

It is important to note that under the optimal policy, the reserve process behaves like a logarithmic Brownian motion on $[0, u]$, as an a arithmetic Brownian motion on $[u, u_1]$ and as arithmetic Brownian motion with different parameters on $[u_1, \infty)$. In particular, under the optimal proportional reinsurance policy, the bankruptcy time is infinite.

4.2. **Small bound on dividend rate**. The calculations in the previous subsection do not show us the nature of the optimal return function, when (4.21) is not true. Obviously, we cannot assume that there exist $u < u_1$, such that both (4.7) and (4.9) (with $O = [0, u]$) are valid. In fact, it is possible to show that the assumption of an existence of $u$, such that $A^*(x) = 1$ for $x \geq u$ also leads to a contradiction. Thus we conjecture that there exists $u_1$, such that (4.7) is satisfied and in addition

$$A^*(x) = a^* \equiv const < 1 \tag{4.22}$$

for all $x > u_1$. Since $\mathcal{L}^*(x) = 0$ for $x < u_1$, we see that $V$ satisfies (4.10) on $[0, u_1]$, therefore $V(x) = C_1 x^\gamma$, where $\gamma$ is given by (4.11). In view of (4.22) and (4.8)) we have two equations that $V$ must satisfy

$$\frac{1}{2}\sigma^2(a^*)^2 V''(x) + (\mu a^* - M)V'(x) - cV(x) + M = 0,$$

$$-\frac{\mu V'(x)}{\sigma^2 V''(x)} = a^*.$$

The only pair $(a^*, V(x))$, which satisfies both equations is

$$a^* = \frac{M}{\frac{\mu}{2} + \frac{c\sigma^2}{\mu}}, \tag{4.23}$$

and

$$V(x) = \frac{M}{c} + C_2 e^{-\frac{\mu}{a^*\sigma^2}x},$$

where $C_2$ is a free constant. Note that due to $M \le \frac{\mu}{2} + \frac{c\sigma^2}{\mu}$, the right hand side of (4.23) does not exceed 1. To find $C_1, C_2$ and $u_1$, we should use $V(u_1-) = V(u_1+), V'(u_1-) = V'(u_1+) = 1$. Those 3 equations result in

$$u_1 = \frac{M\gamma(1-\gamma)}{c},\ C_1 = u_1^{1-\gamma}/\gamma,\ C_2 = -\gamma\frac{M}{c}e^{\frac{cu_1}{M\gamma}}. \tag{4.24}$$

The following theorem proves that thus constructed function $V$ is the solution to the HJB equation.

**Theorem 4.2** *Suppose that (4.21) does not hold. Let $\gamma$ and $u_1$ be given by (4.11) and (4.24) respectively. Then*

$$V(x) = \begin{cases} \frac{u_1}{\gamma}\left(\frac{x}{u_1}\right)^\gamma & x \le u_1, \\ \frac{M}{c}(1 - \gamma e^{-\frac{c}{M\gamma}(x-u_1)}) & x > u_1 \end{cases}$$

*is a concave twice continuously differentiable solution of (4.6), (2.9).*

From our construction, we see that $A^*(x) = a^*(\frac{x}{u_1} \wedge 1)$. The economic interpretation of the obtained results is the following. The optimal risk exposure is proportional to the current level of reserve, and reaches its maximum $a^*$ at $u_1$. When the reserve exceeds $u_1$, the maximal rate of dividends is paid out, while the risk exposure remains $a^*$.

In essence, this results show that if maximal allowable dividend rate is small, then there is no need to take the maximal risk, no matter, how big is your reserve. Higher risk, results in potentially higher reserve levels in the future. However, the presence of discounting and limitation on the pay-out rate preclude from cashing in on those high reserve. From (4.21) it is also seen that the higher the discount rate the more likely one would not need to take the maximal risk at any time. As in the previous case, the bankruptcy time is infinite under the optimal policy.

4.3. **Unrestricted dividend rate.** Since the dynamics of reserve is given by (4.1), (4.2), with $L_t$ being an arbitrary increasing *cadlag* adapted process, we have a so-called mixed regular-singular stochastic control problem. If one uses heuristic arguments similar to those employed in Sections 2 and 3, then the HJB equation for the optimal return function $V$ can be derived in the following form

$$\max\left(\max_{a\in[0,1]}\left[\frac{1}{2}\sigma^2 a^2 V''(x)+\mu a V'(x)-cV(x)\right], 1-V'(x)\right)=0. \tag{4.25}$$

Assuming concavity, we derive an existence of $u_1>0$, such that

$$V'(x)=1,\ x\geq u_1$$

and for all $x<u_1$

$$\max_{a\in[0,1]}\left(\frac{1}{2}\sigma^2 a^2 V''(x)+\mu a V'(x)-cV(x)\right)=0.$$

Assuming existence of $u<u_1$, such that argmax of the left hand side of (4.25) is between 0 and 1 for all $x<u$ and is equal to 1 for all $u\leq x\leq u_1$, we can repeat the procedure in the previous subsections and arrive at the conjecture for the optimal return function $V$:

$$V(x)=\begin{cases} C_1 x^{\gamma}, & x<u,\\ C_2 e^{\theta_2 x}+C_3 e^{\theta_1 x}, & u<x<u_1,\\ x+C_4, & x>u_1.\end{cases} \tag{4.26}$$

The following theorem concludes the solution of the analytical part of the problem.

**Theorem 4.3** *Let $V$ be defined by (4.26), where $u$ is given by (4.12), $C_1, C_2, C_3$ are given by (4.17)-(4.19),*

$$u_1=u+\frac{1}{\theta_1-\theta_2}\log\left(\frac{-\theta_2}{\theta_1}\right),$$

*and*

$$C_4=\frac{\mu}{c}-u_1.$$

*Then $V$ is a concave twice continuously differentiable solution of (4.25), (2.9).*

Recalling the expression for $A^*(x)$, we get

$$A^*(x)=\frac{x}{u}\wedge 1.$$

Notice that due to $|\theta_2| > \theta_1$, the value for $u_1$ is always greater than $u$. The optimal reserve process $R_t^*$ would be a reflected at $u_1$ diffusion with drift $\mu A^*(x)$ and a diffusion coefficient $\sigma A^*(x)$. The optimal control functional $L_t^*$ is the one, which maintains the reflection of $R_t^*$ from the boundary $u_1$, that is, the couple $(R_t^*, L_t^*)$ forms a solution to the Skorohod problem in $(\infty, u_1]$. To prove this fact, one needs to go through a verification lemma, similar to the one in Section 2.

It is important to notice that under the optimal control the reserve process $R_\cdot^*$ will be a logarithmic Brownian motion on $[0, u]$ and an arithmetic Brownian motion on $[u, u_1]$ reflected at the upper boundary. In particular, the bankruptcy time is infinite.

Economically it means that we should use the policy, with risk proportional to the current reserve level, until the reserve reaches $u$ and and we must assume the maximal risk when the reserve is above this level. If the reserve reaches $u_1 > u$, then the dividends should be paid. The optimal policy is the barrier policy with the barrier $u_1$.

# 5 Dividend optimization with debt liabilities

5.1. **Formulation of the optimization problem**. Suppose, that in the situation described in the previous section there is a constant debt payment such as "coupon" bonds or amortization of a loan. Those payments must be made no matter what and cannot be controlled. If we start with the Cramer-Lundberg model, then the dynamics of the reserve process becomes

$$r_t = r_0 + pt - \delta t - \sum_{i=1}^{A(t)} U_i, \tag{5.1}$$

where $\delta$ is a constant rate of liability payments. If there had been no risk control (i.e., no reinsurance), then we would be able to replace $p$ by $p - \delta$ in (5.1) and the problem with debt liability would have been equivalent to the problem of Section 2 with $\mu$ replaced by $\mu - \delta$. If we use risk control and reinsure $1 - a$ fraction of each claim, then $p$ and $U_i$ in the right hand side of (5.1) are replaced by $ap$ and $aU_i$ respectively. In the diffusion limit the dynamics becomes

$$dR_t = (a_\pi(t)\mu - \delta)dt + a_\pi(t)\sigma dW_t - dL_t^\pi,$$

where $\pi = (a_\pi(t), L_t^\pi)$ is a policy, whose first component $0 \leq a_\pi(t) \leq 1$ is the fraction of claims insured by the cedent at time $t$ (that is, $1 - a(t)$ is the reinsurance fraction) and the second component $L_t$ is the cumulative amount of dividends paid-out up to $t$. Those two processes must satisfy the same admissibility requirements as described in Sections 2 and 3.

For each policy $\pi$ we define its performance index by (4.3) and the optimal return function $V$ is defined via (4.4). The Hamilton-Jacobi-Bellman equation for $V$ is close to (4.25), in fact, it differs from (4.25) only by an extra term $-\delta$ in front of $V'$, namely

$$\max\left(\max_{a\in[0,1]}\left[\frac{1}{2}\sigma^2a^2V''(x)+(\mu a-\delta)V'(x)-cV(x)\right],1-V'(x)\right)=0. \tag{5.2}$$

Now our main problem is to find a concave solution to (5.2) subject to (2.9). Despite ostensible similarity with the model of the previous section, the analytical part of the liability problem would require a rather different approach. The next theorem shows that for very large $\delta$ the problem becomes trivial.

**Theorem 5.1** *If $\mu \le \delta$ then $V(x)=x$.*

A consequence of this theorem is that $L_t^* \equiv x$ for all $t\ge 0$ and $\tau=0$ under the optimal policy. This agrees with the intuition: if the expected per unit time profit is less than the unit rate of liability payments, then potentially we cannot earn more than the amount of reserve that we have at the very beginning. The optimal policy in this case is to declare bankruptcy, distributing all the reserve as dividends. In sequel, we will consider only the case of $\mu>\delta$.

If we proceed like we did in the previous section and conjecture that there always exists $u>0$ such that $A^*(x)<1$ for all $x\in(0,u)$, then we might not be able to find a a concave solution to (5.2) if $\delta$ is close to $\mu$. Therefore we start our preliminary analysis from an extreme assumption that $A^*(x)=1$ for all $x\ge 0$. In view of concavity there exists a point $u_1$, such that $V'(x)>1$ for all $x\le u_1$ and $V'(x)=1$ for all $x\ge u_1$. As a result (5.2) reduces to

$$\frac{1}{2}\sigma^2V''(x)+(\mu-\delta)V'(x)-cV(x)=0 \tag{5.3}$$

on $[0,u_1]$. The solution of this equation subject to (2.9) is $V(x)=C(e^{\zeta_+x}-e^{\zeta_-x})$, where $\zeta_+=\theta_+(\mu-\delta)$ and $\zeta_-=\theta_-(\mu-\delta)$ (see (2.12)). Substituting this expression back into (5.2), we see that the assumption, that the maximizer $A^*(x)$ of the right and side is equal to 1, is valid iff for all $x$.

$$\frac{-\mu(\zeta_+e^{\zeta_+x}-\zeta_-e^{\zeta_-x})}{\sigma^2({\zeta_+}^2e^{\zeta_+x}-{\zeta_-}^2e^{\zeta_-x})}\ge 1. \tag{5.4}$$

Computing the left hand side of (5.4) for $x=0$, we obtain

$$-\frac{\mu}{\sigma^2}\frac{V'(0)}{V''(0)}=-\frac{\mu}{\sigma^2}\frac{1}{\zeta_++\zeta_-}=\frac{\mu}{2(\mu-\delta)}.$$

The above expression is greater or equal to 1 iff

$$\mu \leq 2\delta. \tag{5.5}$$

Accordingly, we will consider two different cases: the first when (5.5) is valid, the second when it is not.

5.2.**The case of** $\mu \leq 2\delta$.

**Theorem 5.2** *If* $\delta < \mu \leq 2\delta$*, then*

$$V(x) =) = \begin{cases} C(e^{\zeta_+ x} - e^{\zeta_- x}), & x \leq u_1, \\ C(e^{\zeta_+ u_1} - e^{\zeta_- u_1}) + x - u_1, & x > u_1, \end{cases} \tag{5.6}$$

*where*

$$u_1 = 2\frac{1}{\zeta_+ - \zeta_-}\log|\zeta_-/\zeta_+|,$$

$$C = (\zeta_+ e^{\zeta_+ u_1} - \zeta_- e^{\zeta_- u_1})^{-1}.$$

*is a concave twice continuously differentiable solution of (5.2), (2.9).*

With a hind side, we can claim that in this case the problem is equivalent to dividend control problem of Section 2 if we replace the drift $\mu$ in (2.1) by $\mu - \delta$. The construction of the optimal policy is identical to a similar procedure in Section 2. The optimal dividend pay-out scheme is to pay everything in excess of the $u_1$, whenever the reserve reaches this level. Under the optimal policy, the probability of ruin is 1.

From the economic point of view we see, that if the expected per unit time profit is not large compared to the liabilities, then it is optimal to take the maximal risk, gambling on the increase of the reserve sufficiently enough to avoid immediate bankruptcy.

5.3. **The case of** $\mu > 2\delta$. In this case, as the expression (5.5) shows, there must be a region $O$ such that $0 < A^*(x) < 1$, when $x \in O$. Differentiating (5.2) with respect to $a$, we see, that $A^*(x)$ is given by (3.2). Assuming convexity, set $u_1 = \inf\{x : V'(x) = 1\}$ and suppose $O = (0, u)$, with $u < u_1$. Then, substituting (3.2) into (5.2), we get

$$-\alpha\frac{V'^2(x)}{V''(x)} - \delta V'(x) - cV(x) = 0, \tag{5.7}$$

where $\alpha = \frac{\mu^2}{2\sigma^2}$. We will seek a concave solution to (5.7), (2.9). Let $X(z)$ be an inverse function to $V'(x)$, that is, $V'(X(z)) = z$. Differentiating, we get $V''(X(z)) = 1/X'(z)$ and substituting this into (5.7) yields

$$-\alpha z^2 X'(z) - \delta z - cV(X(z)) = 0. \tag{5.8}$$

Differentiating this equation with respect to $z$ results in

$$-\alpha z^2 X''(z) - (2\alpha + c)zX'(z) - \delta = 0.$$

Solution to this linear ODE can be obtained by "brute force":

$$X(z) = Cz^{-1-\frac{c}{\alpha}} + C_1 - \frac{\delta}{\alpha + c}\log z, \tag{5.9}$$

where $C$ and $C_1$ are free constants. The function $X(z)$ given by (5.9) is, however, not a solution to the equation (5.8) but to the equation obtained by the differentiation of (5.8). Because of that one extra free constant appears in the solution. To get rid of this extra degree of freedom we should use the condition (2.9). That is, substituting $X(z) = 0$ into (5.8), and using (2.9), we get two equations

$$Cz^{-1-\frac{c}{\alpha}} + C_1 - \frac{\delta}{\alpha + c}\log z = 0,$$

$$-\alpha z^2[C(-1 - \frac{c}{\alpha})z^{-2-\frac{c}{\alpha}} - \frac{\delta}{\alpha + c}z^{-1}] - \delta z = 0.$$

Excluding $z$ from these equations, we get

$$C_1 = -\frac{\delta c}{(\alpha + c)^2} + \frac{\delta\alpha}{(\alpha + c)^2}\log C + \frac{\delta\alpha}{(\alpha + c)^2}\log\frac{(\alpha + c)^2}{\delta c}. \tag{5.10}$$

Denote by $X_C(z)$ the function given by (5.9) with $C_1$ expressed via $C$ as in (5.10). For any $C > 0$ the function $X_C$ is monotone on $[0, \infty)$, decreasing from $+\infty$ to $-\infty$. As a result

$$V(x) = \int_0^x X_C^{-1}(y)dy, \quad x \leq u.$$

Our next step would be to determine the constant $C$ and the point $u$. This will be done in several stages. Denote

$$z_1 = V'(u).$$

Since by the definition of the region $O$, we must have $A^*(u) = 1$, we can substitute our solution into (3.2) and arrive at

$$-\frac{\mu}{\sigma^2}z_1 X_C'(z_1) = 1.$$

Solving this equation for $C$, we get its expression through $z_1$

$$C = z_1^{1+\frac{c}{\alpha}}\frac{\mu(c + \alpha(1 - \frac{2\delta}{\mu}))}{2(\alpha + c)^2}. \tag{5.11}$$

Note that since $\mu > 2\delta$ the above expression is positive if $z_1$ is so.

Next we have to write down a solution on $[u, u_1]$. On this interval $V$ satisfies a linear differential equation (5.3), whose general solution can be written in the form

$$V(x) = \frac{C_2}{\zeta_+}\exp(\zeta_+(x-u)) + \frac{C_3}{\zeta_-}\exp(\zeta_-(x-u)),$$

with two free constants $C_2$ and $C_3$. For $x \geq u_1$,

$$V(x) = x - u_1 + C_4.$$

To determine $C, C_3, C_4, u$ and $u_1$, recall that by definition

$$u = X_C(z_1), \tag{5.12}$$

where $C$ is given by (5.11). In addition applying the principle of smooth fit at $u$ and $u_1$ for the first and the second derivatives:

$$V'(u-) = V'(u+), V''(u-) = V''(u+), V'(u_1-) = V'(u_1+), V'(u_1-) = V''(u_1+),$$

we get 5 equations, which determine $C_2, C_3, u\ u_1,\ z_1$, namely

$$C_2 = \frac{1}{(\frac{1+\mu/(\sigma^2\zeta_+)}{1+\mu/(\sigma^2\zeta_-)})^{\zeta_+/(\zeta_+-\zeta_-)} + (\frac{-\zeta_- -(\mu/\sigma^2)}{\zeta_+ +(\mu/\sigma^2)})(\frac{1+\mu/(\sigma^2\zeta_+)}{1+\mu/(\sigma^2\zeta_-)})^{\zeta_-/(\zeta_+-\zeta_-)}}, \tag{5.13}$$

$$C_3 = C_2\frac{\zeta_+ + (\mu/\sigma^2)}{-\zeta_- - (\mu/\sigma^2)}, \tag{5.14}$$

$$z_1 = C_2\frac{\zeta_+ - \zeta_-}{-\zeta_- - (\mu/\sigma^2)}, \tag{5.15}$$

$C$ determined via $z_1$ by (5.11), $u$ determined via (5.12) and

$$u_1 = u + \frac{1}{\zeta_+ - \zeta_-}\log(\frac{1+\mu/(\sigma^2\zeta_+)}{1+\mu/(\sigma^2\zeta_-)}), \tag{5.16}$$

To determine $C_4$, we must equate $V(u_1+)$ and $V(u_1-)$, which yields

$$C_4 = \frac{C_2}{\zeta_+}\exp(\zeta_+(u_1-u)) + \frac{C_2}{\zeta_-}\exp(\zeta_-(u_1-u)). \tag{5.17}$$

Computing the unknown constants, enables us to formulate the following

**Theorem 5.3** *Let $C, C_1, C_2, C_3, C_4, u$ and $u_1$ are determined via (5.11), (5.10), (5.12), (5.13), (5.14), (5.15), (5.16) and (5.17). Then*

$$V(x) = \begin{cases} \int_0^x X_C^{-1}(z)dz, & x < u, \\ \frac{C_2}{\zeta_+}\exp(\zeta_+(x-u)) + \frac{C_3}{\zeta_-}\exp(\zeta_-(x-u)), & u \le x < u_1, \\ C_4 + x - u_1, & x \ge u_1 \end{cases} \tag{5.18}$$

*is a concave twice continuously differentiable solution of (5.2), (2.9).*

**Corollary 5.1** *The optimal feedback risk control function is given by*

$$\begin{aligned} A^*(x) &= -\frac{\mu}{\sigma^2}(X_C^{-1}(x))X_C'(X_C^{-1}(x)) \qquad (5.19) \\ &\equiv -\frac{\mu}{\sigma^2}\frac{1}{\zeta_+}\frac{1+(C_3/C_2)\exp((\zeta_- - \zeta_+)(x-u))}{1+(C_3\zeta_-)/(C_2\zeta_+)\exp((\zeta_- - \zeta_+)(x-u))}, \quad x < u \end{aligned}$$

*and $A^*(x) = 1$ for $x \ge u$.*

It can be shown that $A^*(x)$ given by (5.19) is an increasing function with $A^*(0) > 0$. This makes it markedly different from the case of zero liability. In particular, the bankruptcy time under the optimal policy is finite with probability 1. The optimal dividend distribution scheme is similar to that of other cases: it is of the barrier type with the barrier $u_1$.

# 6 Noncheap reinsurance

In the previous sections, we assumed that the safety loading of the the reinsurer is the same as that of the cedent. In many cases the reinsurer requires higher relative safety loading (see a more detailed discussion on safety loading in the next section), which results in the cedent paying larger fraction of premiums, than the fraction which is reinsured. We call this *noncheap reinsurance*. Suppose that the cedent decides to reinsure $1-a$ fraction of all claims. Then he is required to divert $\chi(1-a)$ fraction of premiums to reinsurer where $\chi > 1$. The money which is paid to the reinsurer in excess of $(1-a)$ fraction of premiums is called *transaction cost*.

Let $\pi = (a_\pi(t), L_t^\pi)$ be an admissible control, defined as in Section 4. Then under this policy the dynamics of the reserve is

$$dR_t = (\mu - (1-a_\pi(t))\chi\mu)dt + a_\pi(t)\sigma dW_t - dL_t^\pi, \tag{6.1}$$

Our aim is to maximize the expected total dividend pay-outs until the time of bankruptcy, that is, to find $V$ given by (4.3), (4.4) and find the optimal policy $\pi^*$, such that

$$J_x(\pi^*) = V(x).$$

If we rewrite the equation (6.1) in the form

$$dR_t = (a_\pi(t)\chi\mu - (\chi - 1)\mu)dt + a_\pi(t)\sigma dW_t - dL_t^\pi, \tag{6.2}$$

then we see that the transaction cost problem is mathematically equivalent to the problem with debt liabilities, if we put

$$\mu' = \chi\mu, \delta = (\chi - 1)\mu. \tag{6.3}$$

Using this isomorphism, we can formulate the following theorem.

**Theorem 6.1** *Let the dynamics of the reserve process be described by (6.2). If $\chi \geq 2$ then the optimal return function (4.4) is given by (5.6) with $\mu$ and $\delta$ in the expression for all constants replaced by $\mu'$ and $\delta$ of (6.3).*

*If $\chi < 2$, then the optimal return function $V$ is given by (5.18), with the same substitution for $\mu$ and $\delta$ in the formulae for all free constants involved.*

As a consequence, we see, that when $\chi \geq 2$, it is optimal always to take the maximal risk, that is, not to use reinsurance at all. If $\chi < 2$, then the optimal feedback risk control function $A^*$ is monotone, with $A^*(0) > 0$. In other words, even when the reserve approaches zero, it is not optimal to reinsure 100% of all claims. The dividend distribution policy is of the same barrier type as before. In both cases, under the optimal policy, the probability of ruin is 1.

# 7 Excess-of-loss reinsurance

7.1. **Specifics of the excess-of-loss reinsurance**. Another type of reinsurance employed by many companies consists of reinsuring not a fixed fraction of each claim, but the amount of the claim in excess of a given level $b$, called the *retention level*. If the claim size is $U$, then the cedent pays $U \wedge b = U^{(b)}$, while the reinsurer picks up the rest of the claim. To derive the dynamics of the reserve process, let us start from the Cramer-Lundberg model, written in a slightly different form. Let us assume that in (1.1) the premium $p$ is calculate via the *expected value principle*, that is,

$$p = (1 + \eta)\lambda EU^{(b)},$$

where $\eta > 0$ is a *relative safety loading*. Suppose that the retention level $b$ is fixed and the reinsurer uses the same safety loading as the cedent. Then, the reserve level $r_t$ at the time $t$ is

$$r_t = r_0 + p^{b,\eta}t - \sum_{i=1}^{A(t)} U_i^{(b)},$$

where

$$p^{b,\eta} = (1+\eta)\lambda EU^{(b)}.$$

Denote $\mu(b) = EU^{(b)}$ and $\sigma(b) = E((U^{(b)})^2)$ (thus $\mu(\infty) = \mu$ and $\sigma^2(\infty) = EU^2$). Then, when $\eta \to 0$, the process $\{\eta r_{t/\eta^2}\}_{t\geq 0}$ converges in distribution to a Brownian motion with drift $\lambda\mu(b)$ and diffusion coefficient $\lambda\sigma(b)$. Without loss of generality, we can put $\lambda = 1$. Considering the retention level to be dynamically controlled, we arrive at the following diffusion control model.

A strategy $\pi = (b_\pi(t), L_t^\pi)$ is a pair of $\mathcal{F}_t$ adapted processes, where $b_\pi(t) \geq 0$ and $L_t^\pi$ is an increasing process subject to the same conditions as the ones in Section 2. Under the policy $\pi$, the dynamics of the reserve process satisfies

$$R_t = x + \int_0^t \mu(b_\pi(s))ds + \int_0^t \sigma(b_\pi(s))\, W_s - L_t^\pi,$$

where $x$ is the initial reserve level. The objective is to maximize the expected present value of the cumulative dividend distributions, that is to find $V$ defined by (4.3), (4.4) and the policy $\pi^* = (b^*(t), L_t^*)$ subject to (4.5).

To make the problem more tractable, we first make a change of control variables. Instead of the retention level $b$ we will use $m = \mu(b)$ as an independent control. The following lemma shows consistency of this representation.

**Lemma 7.1** *Let $F$ be the distribution function of the claim size $U$ and*

$$\infty \geq N = \inf\{x : F(x) = 1\} \equiv \sup\{x : \bar{F}(x) > 0\}, \tag{7.1}$$

*The function $\mu(b)$ is a continuous increasing function of $b$ on $[0, N]$ and there exists an inverse function $b(\mu)$, for which*

$$b(\mu(y)) = y. \tag{7.2}$$

*The function $\phi(m) \equiv \sigma^2(b^{-1}(m))$ is a strictly increasing convex function of $m$ on the interval $[0, \mu]$ with*

$$\phi(0) = 0, \phi(\mu) = \sigma^2(\infty), \frac{d\sigma^2(0)}{dm} = 0, \frac{d\sigma^2(\mu)}{dm} = 2N.$$

7.2 **Unrestricted dividend rate.** This is a mixed regular-singular control problem. The HJB equation for the optimal return function $V$ is

$$\max\left(\max_{m\in[0,\mu)}\left[\frac{\phi(m)}{2}V''(x) + mV'(x) - cV(x)\right], 1 - V'(x)\right) = 0. \tag{7.3}$$

It turns out that the nature of the solution to (7.3) (2.9) depends on whether or not the distribution the claim size $U$ has a bounded support, that is whether or not $N$ given by (7.1) is finite.

7.3. **The case of the claim size distribution with unbounded support**. Suppose $V$ is concave (will be shown a posteriori) and $u_1 = \inf\{x : V'(x) = 1\}$. Then for $x \le u_1$

$$\max_{m\in[0,\mu)} \left[\frac{\phi(m)}{2}V''(x) + mV'(x) - cV(x)\right] = 0. \tag{7.4}$$

Let $\mathcal{M}^*(x) \equiv m(x)$ be the maximizer of the left hand side of (7.4). Then differentiating the left hand side of (7.4) with respect to $m$,

$$-\frac{V'(x)}{V''(x)} = \frac{1}{2}\frac{d}{dm}\phi(m(x)) = b(m(x)), \tag{7.5}$$

where $b(y)$ is defined by (7.2). Substituting (7.5) into (7.4), we arrive at

$$\frac{-\phi(m)}{2b(m)}V'(x) + mV'(x) - cV(x) = 0, \tag{7.6}$$

with $m = m(x)$. Differentiating (7.6) with respect to $x$, and using (7.5) once more, leads to

$$\left[\frac{\phi(m)}{2b^2(m)}b'(m)m' - c\right]V'(x) + \left[-\frac{\phi(m)}{2b(m)} + m\right]V''(x) = 0$$

and eventually to

$$\left[\frac{\phi(m)}{2b^2(m)}b'(m)m' - c + \frac{\phi(m)}{2b^2(m)} - \frac{m}{b(m)}\right]V'(x) = 0.$$

Since $V'(x) \ge 1$, we can divide by $V'(x)$ and obtain an equation for $m$

$$m' = \frac{2cb^2(m) - \phi(m) + 2mb(m)}{\phi(m)b'(m)}.$$

Put

$$G(m) = \int_0^m \frac{\phi(y)b'(y)}{2cb^2(y) - \phi(y) + 2yb(y)}dy.$$

Then

$$m(x) = G^{-1}(x + C)$$

for some unknown constant $C$. Making a conjecture $m(0) = 0$, we get $C = 0$. Equation (7.5) shows

$$\frac{1}{b(m(x))} = (\log(V'(x)))'$$

Therefore, in view of $V'(u_1) = 1$ ($u_1$ is still needed to be determined), we can write

$$V(x) = \int_0^x e^{\int_z^{u_1} \frac{1}{b(m(y))}dy}dz.$$

Since $V'(x) = 1$ for $x \geq u_1$, we conjecture

$$V(x) = \begin{cases} \int_0^x e^{\int_z^{u_1} \frac{1}{b(m(y))} dy} dz, & x < u_1, \\ x - u_1 + C_1, & x > u_1. \end{cases} \tag{7.7}$$

By construction the derivative of $V$ at point $u_1$ is equal to 1. Continuity of the second derivative at $u_1$ implies

$$0 = V''(u_1-) = \frac{1}{b(m(u_1-))}.$$

Therefore $b(m(u_1-)) = \infty$, or, in view of $N = \infty$,

$$m(u_1) = G^{-1}(u_1) = \mu \tag{7.8}$$

(see Lemma 7.1). Thus

$$u_1 = G(\mu). \tag{7.9}$$

After $u_1$ has been found, we can determine $C_1$ by continuity of $V$ at $u_1$.

$$C_1 = \int_0^{u_1} e^{\int_z^{u_1} \frac{1}{b(m(y))} dy} dz. \tag{7.10}$$

**Theorem 7.1** *Let $N = \infty$ and $u_1$ and $C_1$ be given by (7.9), (7.10). Then $V(x)$ defined by (7.7) is a twice continuously differentiable concave solution of (7.3), (2.9).*

From the construction of the solution it follows that the optimal retention level $\mathcal{M}^*(x)$ as a function of the current reserve $x$ is equal to $b(G^{-1}(x))$. It is an increasing function, which means that with higher reserves we need less reinsurance. In contrast to the previous Section, the optimal policy always require to have some reinsurance; there is no time interval when it is optimal to take the full risk. The optimal dividend distribution policy is of a barrier type with the maximal reserve level (barrier) equal to $u_1$.

7.4. **The case of the claim size distributions with bounded support.** If we follow the same route as in the case of unbounded support then we can come as far as (7.8), but at this point there could be no further progress, in view of the fact that $b(m(u_1-)) < \infty$. Therefore we must proceed with the assumption that there exists $u < u_1$ such that (7.6) holds on $[0, u]$, while on $[u, u_1]$ the function $V$ satisfies

$$\frac{\sigma^2}{2} V''(x) + \mu V'(x) - cV(x) = 0. \tag{7.11}$$

Proceeding as above, we can solve (7.6) for $x \le u$:

$$V(x) = C \int_0^x e^{\int_z^u \frac{1}{b(m(y))} dy} dz, \tag{7.12}$$

where $C$ is a free constant. Arguments identical to those of the previous subsection show that

$$u = G(\mu).$$

Solution of (7.11) can be written as

$$V(x) = C_1 e^{\theta_1(x-u_1)} + C_2 e^{\theta_2(x-u_1)}, \tag{7.13}$$

where $\theta_1 = \theta_+(\mu), \theta_2 = \theta_-(\mu)$ calculated via (2.12). For $x > u_1$, the derivative of $V$ is 1 and

$$V(x) = x - u_1 + C_3. \tag{7.14}$$

Due to the "smooth fit" at $u$ and $u_1$,

$$V'(u-) = V'(u+), V''(u-) = V''(u+), V(u_1-) = V(u_1+), \tag{7.15}$$
$$V'(u_1-) = V'(u_1+), V''(u_1-) = V''(u_1+).$$

Note that we do not need to write $V(u-) = V(u+)$, in (7.15) since this will follow automatically from the continuity of the first and the second derivative. In addition substituting (7.12)-(7.14) into five equations of (7.15), we can solve for $C, C_1, C_2, C_3$ and $u_1$.

$$C_1 = \frac{-\theta_2}{\theta_1(\theta_1 - \theta_2)}, C_2 = \frac{\theta_1}{\theta_1(\theta_1 - \theta_2)}, u_1 = u + \frac{1}{\theta_1 - \theta_2} \log\left(\frac{\theta_2(1 + N\theta_1)}{\theta_1(1 + N\theta_2)}\right), \tag{7.16}$$

$$C = C_1 \theta_1 e^{\theta_1(u_1 - u)} + C_2 \theta_2 e^{\theta_2(u_1 - u)}, C_3 = \frac{\mu}{c}. \tag{7.17}$$

Those calculations enables us to conclude with the following theorem.

**Theorem 7.2** *Suppose $N < \infty$. Let*

$$V(x) = \begin{cases} C \int_0^x e^{\int_z^u \frac{1}{a(u(y))} dy} dz, & x < u, \\ C_1 e^{\theta_1(x-u_1)} + C_2 e^{\theta_2(x-u_1)}, & u < x < u_1, \\ \frac{\mu}{c} + x - u_1, & x > u_1, \end{cases}$$

*where $u$, $C, C_1, C_2, C_3$ and $u_1$ are given by (7.13), (7.16) and (7.17). Then $V$ is a twice continuously differentiable concave solution to (7.3), (2.9).*

The optimal feedback risk control policy $\mathcal{M}^*(x)$ as a function of the current reserve $x$ is equal to $b(G^{-1}(x))$ for $0 \le x \le u$ and $b(x) = N$ if $x > u$. That is, if the reserve level exceeds $u$ then there should be no reinsurance. The optimal dividend distribution policy is of a barrier type with the level $u_1$ being the barrier.

7.5. **Bounded dividend rate. Unbounded support for the claim size distribution..** In the case of the bounded dividend rate

$$L_\pi(t) = \int_0^t l_\pi(s)ds, \ 0 \le l_\pi(s) \le M,$$

and the HJB equation becomes

$$\max_{m\in[0,\mu],l\in[0,M]} \left[\frac{\phi(m)}{2}V''(x) + (m-l)V'(x) - cV(x) + l\right] = 0. \tag{7.18}$$

If we assume that there are $u < u_1$ such that $u_1 = \sup\{x : V'(x > 1\}$ and there should be no reinsurance whenever the reserve level is above $u$, then we will arrive at a contradiction. Similarly, we will arrive at a contradiction if we assume that the level $u_1$ satisfies $G^{-1}(u_1) = \mu$ as in section 7.3. Thus our assumption is that for all $x > u_1$ the optimal retention level remains constant. In this case, we can proceed as in Sections 7.3, 7.4 and obtain a solution for $V$ in the form

$$V(x) = \begin{cases} \int_0^x e^{\int_z^{u_1} \frac{1}{b(m(y))}dy} dz, & x < u_1, \\ \frac{M}{c} + \frac{1}{\tilde{\theta}(m_1)} e^{\tilde{\theta}(m_1)(x-u_1)}, & x \ge u_1, \end{cases}$$

where

$$\tilde{\theta}(m) = \frac{-(m-M) - \sqrt{(m-M)^2 + 2c\phi(m)}}{\phi(m)} < 0$$

and $u_1$ and $m_1$ are unknown constants to be determined. Equalizing the value of the second derivative from the left and from the right at $u_1$ results in the equation

$$\frac{1}{b(m_1)} = -\tilde{\theta}(m_1). \tag{7.19}$$

The above equation has a unique solution because both sides of (7.19) are continuous functions, the left hand side is strictly decreasing on $[0,\mu]$ from $+\infty$ to 0 (recall that $b(m)$ is strictly increasing from 0 to $\infty$), while the right hand side is strictly increasing on $[0,\mu]$. Arguments similar to those of Section 7.31 show

$$u_1 = G(m_1). \tag{7.20}$$

**Theorem 7.3** *Let $m_1$ be the root of (7.19) and $u_1$ be given by (7.20). Then the function $V$ of (7.19) is a concave twice continuously differentiable solution of (7.18), (2.9).*

The optimal feedback risk control policy is having the retention level $b(G^{-1}(x))$, if the current reserve level is $x$ and $x < u_1$. Whenever $x \geq u_1$, the retention level is constant and equal to $b(m_1)$. The dividend distribution policy is not to pay any dividends until the reserve level reaches $u_1$ and to pay the maximal rate, whenever the reserve exceeds $u_1$.

7.6. **Bounded dividend rate. Bounded support for the claim size distributions.**. Here we have the same HJB equation (7.18). Intuitively, it is clear that if $N$ is large enough, we should have the same type of the solution as in Theorem 7.3. For relatively small $N$, the same intuition suggests that we should have an interval, when no reinsurance is necessary. In fact, this is true and below we present the results without specifying details of the procedure. By and large the proof goes along the same lines as in Sections 7.1-.7.5.

**Theorem 7.4** *Suppose*

$$M > cN - \frac{\sigma^2}{2} + \mu, \tag{7.21}$$

*Let $\hat{\theta} = \theta_-(\mu - M)$ and*

$$D = \frac{(\theta_1 + \frac{1}{N})(\theta_2 - \hat{\theta})}{(\theta_2 + \frac{1}{N})(\theta_1 - \hat{\theta})} > 1,$$

$$u = G(\mu),$$

$$u_1 = u + \frac{1}{\theta_1 - \theta_2} \log D,$$

$$C_1 = \frac{D^{\frac{-\theta_2}{\theta_1 - \theta_2}}}{\theta_2} \left( \frac{1}{\theta_1 - \theta_2} \right) < 0,$$

$$C_2 = -\frac{D^{\frac{-\theta_1}{\theta_1 - \theta_2}}(\theta_2 - \hat{\theta})}{\theta_1(\theta_1 - \theta_2)(\theta_1 - \hat{\theta})} > 0,$$

$$C = C_1\theta_2 + C_2\theta_1.$$

*Then*

$$V(x) = \begin{cases} C \int_0^x e^{\int_z^u \frac{1}{b(m(y))} dy} dz, & x < u, \\ C_1 e^{\theta_2(x-u)} + C_2 e^{\theta_1(x-u)}, & u < x < u_1, \\ \frac{M}{c} + \frac{1}{\hat{\theta}} e^{\hat{\theta}(x-u_1)}, & x > u_1 \end{cases}$$

*is a concave twice continuously differentiable solution of (7.18), (2.9).*

Theorems 7.4 shows that if $M$ and $N$ are subject to (7.21), then the optimal retention level is equal to $b(G^{-1}(x))$, when the current reserve is equal to $x < u$. There should be no reinsurance if $u \le x \le u_1$. The dividends should be paid at the maximal rate whenever the reserve exceeds $u_1$.

**Theorem 7.5** *Suppose (7.21) is not true. Let $m_1$ be the root of (7.19) and $u_1$ be given by (7.20). Then*

$$V(x) = \begin{cases} \int_0^x e^{\int_z^{u_1} \frac{1}{b(m(y))} dy} dz, & x < u_1, \\ \frac{M}{c} + \frac{1}{\tilde{\theta}(m_1)} e^{\tilde{\theta}(m_1)(x-u_1)}, & x \ge u_1, \end{cases}$$

*is a concave twice continuously differentiable solution of (7.18), (2.9).*

From this theorem it follows that if (7.21) fails then the optimal retention level as a function of the current reserve $x$ is equal to $b(G^{-1}(x))$, when $x < u_1$ and it is equal to $b(m_1)$ for all $x > u_1$. The dividends should be paid at the maximal rate when the reserve is above $u_1$.

# 8 Open problems

In the models presented here, the dividend distribution is frictionless, it does not require any set-up costs. Suppose each time the dividends are distributed there is a non recoverable cost $K$ associated with it. In this case, there is no possibility of a continuous distribution, rather the pay-outs must be done is lump sums. The control functional $L_t$ is represented in the form

$$L_t = \sum_{\eta_k \le t} \xi_k,$$

where $0 \le \eta_1 < \eta_2, \ldots, < \eta_k, < \ldots$ is a sequence of stopping times with respect to $\mathcal{F}_t$ and $\xi_k$ are $\mathcal{F}_{\eta_k}$ measurable random variables. The sequence $\eta_k$ is the sequence of times when the dividends are paid out, while $\xi_k$ is the $k$th amount paid out. The performance index associated in this case with a policy $\pi$ is

$$J_x(\pi) = E[\int_0^\tau e^{-ct} dL_t - \sum_{\eta_k < \tau} K e^{-c\eta_k}].$$

This model should be solved via *impulse control* techniques. The HJB equation for this problem would be quasiviriational inequality which involves both second order differential operator and a difference operator. A partial solution to this problem is done in [39], where the case of a pure dividend control was considered.

Another type of unsolved problems relates to existence of a "dissolution value" of a company at the time of bankruptcy. This will change the boundary condition (2.9) to

$$V(x) = P,$$

where $P$ is the terminal value. For the case of proportional reinsurance with unrestricted dividend rate this problem was solved in [60]. Other cases are still open.

Very closely related to the above is the problem with nonterminal bankruptcy. This should be the model in which upon reaching the bankruptcy state, the company does not go out of business, rather it stays in this state a random amount of time (akin to reorganization under Chapter 11 of US Bankruptcy Code) and then resumes "business as usual". In the consumption/investment models in finance a diffusion limit description of such behavior would be via Brownian motion with delayed reflection at 0 (see [57]). It appears that a similar approach might work in the dividend optimization models as well.

A special interest is a problem of the excess-of-loss reinsurance in which the reinsurer's safety loading increases with the retention level the cedent chooses. As was shown in [54], absence of arbitrage in the reinsurance market might require an increase in safety loading when the retention level increases.

Another interesting and important extension would be to allow the reserve to accrue interest. That would add a term $rR_t$ to the drift in the dynamics of $R_t$. For example, in the case of proportional reinsurance the dynamics becomes

$$dR_t = (\mu a_\pi(t) + rR_t)dt + \sigma a_\pi(t)dW_t - dL_t^\pi.$$

The major analytical difficulty in solving the HJB arise from the fact that it will contain an additional term $rx$ in front of the first derivative.

A more sophisticated version of the same problem would be the one in which it is allowed to invest a part or all of the reserve in the stock market. In this case we will have additional control variables which determine the fraction of the reserve invested in risky or risk free asset. This is in addition to reinsurance control parameters. Some work related tot he case, when reserve is invested in stock market (but with no control of those investments) is done in [49], [50]. For the objectives of minimizing the ruin probability and maximization of the growth rate, this problem was solved in [10], [11].

The most difficult extension of the investment possibility problem would be to the original Cramer-Lundberg model, that would require modeling of the reserve by the sum of two controlled processes: one being a compound Poisson, the other a diffusion process. The resulting Hamilton-Jacobi-Bellman equation becomes an integro-differential equation, whose closed form solution would be hard to obtain if at all. In this regard, the computational methods should become of a major interest.

# 9 Appendix

9.1. **Generalized Ito's formula.**

**Theorem 9.1** *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space with filtration $\mathcal{F}_t$ and a standard Brownian motion $W_t$, adapted to $\mathcal{F}_t$. Let $L_t$ be a right continuous with left limits (cadlag) increasing $\mathcal{F}_t$-adapted process. Suppose $X_t$ satisfies the following stochastic differential equation*

$$X_t = x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s - L_t.$$

*Denote by $L_t^c = L_t - \sum_{s \le t}(L_s - L_{s-})$ the continuous part of $L$. Then for any twice continuously differentiable function $V(x)$ with bounded first and second derivatives and for any stopping time $\tau$ with respect to $\mathcal{F}_t$,*

$$\begin{aligned} e^{-c\tau}V(X_\tau) - V(x) \quad &= \int_0^\tau e^{-ct}\left(\frac{1}{2}\sigma^2(X_t)V''(X_t) + \mu(X_t)V'(X_t) - cV(X_t)\right)dt \\ &+ \int_0^\tau e^{-ct}\sigma(X_t)V'(X_t)dW_t - \int_0^\tau e^{-ct}V'(X_t)dL_t^c - \sum_{0\le t\le\tau} e^{-ct}(V'(X_t) - V(X_{t-})). \end{aligned}$$

*In particular*

$$\begin{aligned} V(x) \quad &= \quad E(e^{-c\tau}V(X_\tau)) \qquad (9.1) \\ &- \quad E\int_0^\tau e^{-ct}\left(\frac{1}{2}\sigma^2(X_t)V''(X_t) + \mu(X_t)V'(X_t) - cV(X_t)\right)dt \\ &+ \quad E\int_0^\tau e^{-ct}V'(X_t)dL_t^c + E\sum_{0\le t\le\tau} e^{-ct}(V(X_t) - V(X_{t-})) \end{aligned}$$

The proof of this theorem can be found in [47] p. 301 or [31] Ch.4.

9.2. **Skorohod problem on a real line**. Let $\mu(x)$ and $\sigma(x)$ be Lipschitz continuous functions. Let $u \in R$ be fixed. A solution to the Skorohod problem in $(-\infty, u]$ with initial position $x$ is a pair $(X_t, L_t)$ of cadlag $\mathcal{F}_t$-adapted processes, such that $L_t$ is nonnegative and increasing and

$$X(t) = x + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW_s - L_t, \qquad (9.2)$$

$$X_t \leq u, \quad \text{for all } t \geq 0, \tag{9.3}$$

$$\int_0^\infty 1_{X(s)<u} dL_s = 0. \tag{9.4}$$

The resulting process $X_t$ is a continuous for all $t > 0$ diffusion process on $(-\infty, u]$ with drift $\mu(\cdot)$ and diffusion coefficient $\sigma(\cdot)$ reflected at the upper boundary $u$. Note, that if $x > u$, then $L_0 = x - u$ and $X_0 = u$. By convention, $X_{0-} = x$ and $L_{0-} = 0$ and we assume that both $X$ and $L$ have a discontinuity at 0 if $x > u$.

Existence and uniqueness of a solution to the Skorohod problem in a much more general setting is proved in [42]. In the case of $\sigma(x)$ and $\mu(x)$ being constants, the solution to the Skorohod problem (9.2)-(9.4) can be written in a closed form (see [31] Ch.2)

$$L_t = \max\{\max_{s \leq t}(x + \mu s + \sigma W_s - u), 0\}, \tag{9.5}$$

$$X_t = x + \mu t + \sigma W_t - L_t. \tag{9.6}$$

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