Multi-Fractal Processes as Models for Financial Returns:

A First Assessment

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Abstract: Multi-fractal processes have been proposed as a new formalism for modeling the time series of returns in finance. The major attraction of these processes is their capability of generating various degrees of long-memory in different powers of returns - a feature that has been found to characterize virtually all financial prices. Furthermore, elementary variants of multi-fractal models are very parsimonious formalizations as they are essentially one-parameter families of stochastic processes. The aim of this paper is to provide a first assessment of the goodness-of-fit of this new class of models by applying them to four long time series from different financial markets (one exchange rate, two stock market indices and the price of gold). Our results are very encouraging in that the estimated models provide an astonishingly good fit to the unconditional distribution of the data and do even outperform estimates from a GARCH(1,1) specification. However, we also remark that a trade-off exists between goodness-of-fit for the unconditional distribution and the capability of the estimated processes to match the autocorrelation patterns of various moments.

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1. Introduction

While so-called uni-fractal or self-similar processes (like fractional Brownian motion) have been known for quite some time in the stock market and exchange rate literature, more general multi-fractal processes have only been considered as candidate generating mechanisms for financial prices very recently. After some earlier attempts at recovering traces of multi-fractal behavior (Vassilicos, Demos and Tata, 1993, Ghasghaie, S. *et al.*, 1996) this topic has been taken up in a couple of recent papers. Among these contributions, Schmitt, Schertzer and Lovejoy (1999) and Vandewalle and Ausloos (1998a, b) also concentrate on statistical analyses which aims at demonstrating the multi-fractal nature of various financial records, while Mandelbrot, Fisher and Calvet (1997) proceed one step further by proposing a compound stochastic process as a generating mechanism of stock returns and exchange rate changes in which a multi-fractal cascade plays the role of a time transformation. The message of these papers is quite unequivocal in indicating that the data under consideration consistently exhibit features that have been found to characterize multi-fractal processes in other environments (e.g. statistical analyses of turbulence[1]). However, the methods employed by these authors differ quite fundamentally from the usual techniques used to estimate and evaluate time series models in economics. Although a comparison of simulated multi-fractal processes with empirical data (Fisher, Calvet and Mandelbrot, 1997; Mandelbrot, 1999) suggests that they are, in fact, able to reproduce to a large extent the empirical characteristics of financial returns, no assessment of goodness-of-fit is tried in these papers. A comparison of the performance of multi-fractals with, for example, GARCH processes as a candidate alternative, is hampered by the fact, that 'time series' of multi-fractal processes are generated by algorithms that are of a combinatorial nature (see below for an example) rather than iterative mechanisms. Nevertheless, in order to get some impression of the explanatory power of this new model, some efforts towards an assessment of its goodness of fit beyond visual arguments seems indispensable.

The purpose of this paper is to go one (modest) step towards such an assessment of the empirical performance of multi-fractal cascade models. To this end, we estimate the parameters of simple multi-fractal processes for the time series of daily variations of various financial data from different sources: two stock market indices (the German DAX and the New York Stock Exchange Composite Index), an exchange rate (Deutsche Mark/U.S$), and the daily price of gold from the London Precious Metal Exchange.[2] Although we consider rather long time series (extending over several decades in all cases), we find that even the most simple *one-parameter* models of the multi-fractal type can provide a perplexingly good fit to the unconditional distribution of the data. In formal terms, even with estimation of only one parameter, we would often not reject the hypothesis that the empirical data and synthetic data from the pertinent cascade model share the same *unconditional* distribution when performing tests of the Kolmogorov-Smirnov type. Comparing the performance of the new model with that of its main competitor from the econometrics literature, the GARCH model, we find that the multi-fractal cascades outperform a GARCH(1,1) specification in terms of the Kolmogorov-Smirnov statistic.

However, the main attraction of multi-fractal processes is their capability of matching elementary properties of the *conditional* distribution, i.e. long-term dependence in various powers. When comparing the theoretical autocorrelation structure of our estimated cascade models with descriptive

[1] The similarities in the time series characteristics of financial data and data from turbulent flows has stimulated a discussion about similarities in the generating mechanisms among physicists, cf. Vassilicos, 1995; Gashghaie et al., 1996, and Mantegna and Stanley, 1996.

[2] Details on the time horizon and length of each series can be found in Table 1.

empirical statistics (Hurst exponents and fractional differencing parameter), we find that there is a certain trade-off between achieving goodness-of-fit for the unconditional distribution and the task of matching the autocorrelation patterns of various moments. In particular, the best fits of the empirical density come along with an unrealistically *high* degree of long-term dependence in squared and absolute returns.

2. The Multi-Fractal Model [3]

The multi-fractal model put forward in Mandelbrot, Calvet and Fisher (1997) postulates that returns { r(t) } follow a compound process:

(1) $r(t) = B_H[\theta(t)]$.

In this notation, $B_H[\ ]$ is a fractional Brownian motion with index H, and $\theta(t)$ is the distribution function of a multi-fractal measure which plays the role of a time-deformation. Both component processes are assumed to be independent of each other. With a time-homogeneous Brownian process B_H, the multi-fractal measure, $\theta(t)$, is responsible for changes in the scale of the fluctuations which may generate heteroscedasticity of the overall dynamics. In contrast to the GARCH model and its descendants, the above cascade model is scale-free and, therefore, one and the same specification can be applied to data of different sampling frequencies. This feature is highlighted by Fisher, Calvet, and Mandelbrot in their analysis of both high-frequency and daily returns of the Deutschmark/U.S.$ exchange rate.

The physics literature knows several models of multiplicative cascades which could be used for concretizing the time-transformation $\theta(t)$. Mandelbrot, Calvet and Fisher focus on the so-called Binomial and Log-normal cascades, while Schmitt, Schertzer and Lovejoy (1999) estimate the parameters of the Log-Levy model for a number of foreign exchange rates. To get a basic idea of this approach, it is useful to first have a look at one of the simplest cases, the Binomial model.

In their original form, multi-fractal cascades are operations performed on probability measures. The ‘cascade’ starts with assigning uniform probability to the interval [0,1]. In the first step, this interval is split up into two subintervals of equal length, which receive a fraction p_1 and $1 - p_1$, respectively, of the total probability mass. In the next step, each subinterval is again split up into two subintervals, which again receive fractions p_1 and $1 - p_1$ of the probability mass of their ‘mother’ intervals. In principle, this procedure is, then, repeated *ad infinitum*. The successive results of this splitting process at steps 2, 6, and 12 are shown in the first three panels of *Fig. 1*. The lower panel shows an example of a compound process where the same cascade is used as a time transformation and the index of the Brownian motion is H = 0.5.

Insert Fig. 1 about here

It is easy to envisage more or less complicated variants of this general procedure: first, the probabilities could be assigned in a systematic fashion (e.g. always assigning probability p_1 to the left

[3] Tel (1988), Falconer (1990) and Evertz and Mandelbrot (1992) are recommendable introductory sources to multi-fractal processes.

hand descendant and 1 - p_1 to the right-hand descendant of a mother interval). Alternatively, this assignment could be made randomly, which is the case in the illustration in *Fig. 1*. Going beyond the Binomial model, one could think of more than two subintervals to be generated in each step (which leads to multinomial cascades) or of generating random numbers for p_1 in each step instead of using the same constant value in each iteration. The Log-normal and Log-Levy models mentioned above are examples of the latter type of multi-fractal measures.

In order to present the distinguishing features of multi-fractal measures, we first note that the scaling of the measure, denoted by μ, within some small box of size ε, obeys a law of the form:

(2) $\mu([0,\varepsilon] \sim \varepsilon^{\alpha}$ for $\varepsilon \to 0$.

The exponent α in eq. (2) is known as the *Hölder exponent* and may be obtained as:

(3) $$\alpha = \lim_{\varepsilon \to 0} \frac{\log \mu([0,\varepsilon])}{\log \varepsilon}.$$

While *uni*-fractal measures are characterized by one single Hölder exponent[4], for *multi*-fractal measures (like the ones resulting from the above cascade models), the Hölder exponent is not unique. The behavior of such measures can, thus, only be described by the *distribution* of their Hölder exponents. This distribution is denoted as the *multi-fractal spectrum* $f(\alpha)$.

Denoting by $N_\varepsilon(\alpha_i)$ the subset of intervals of length ε with a certain Hölder exponent α_i, it can be shown that for ε -> 0, this number behaves like:

(4) $$N_\varepsilon(\alpha_i) \sim \varepsilon^{-f(\alpha_i)}.$$

$f(\alpha_i)$ can, then, also be interpreted as the fractal dimension of the subset of boxes carrying Hölder exponent α_i. Eq. (4) suggests that the spectrum $f(\alpha)$ may be estimated by a box-counting algorithm for $N_\varepsilon(\alpha_i)$. In fact, this line of attack is pursued in several applications in the natural sciences as well as in an early application of the multi-fractal framework to economic data (Vassilicos, Demos and Tata, 1993).

Another approach for determining $f(\alpha)$ is known as the *method of moments*. Here, one considers the quantities:

(5) $$S(\varepsilon, q) = \sum_{j=1}^{N(\varepsilon)} \mu_j^q,$$

which give the sums over all boxes of the *q*-th power of the mass contained in each box, i.e. the q-th moment of the measure. The sum in eq. (5) can be written as the sum of the product of boxes $N_\varepsilon(\alpha_i)$ times the *q*-th power of their pertinent mass ε^{α_i}:

[4] The concept of the Hölder exponent is, therefore, a generalization of that of the Hurst exponent which is used for the single exponents of uni-fractal processes.

$$(6)\quad S(\varepsilon,q) = \sum_i N_\varepsilon(\alpha_i)\cdot(\varepsilon^{\alpha_i})^q = \sum_i \varepsilon^{\alpha_i\cdot q - f(\alpha_i)}.$$

For different powers *q*, different elements give the dominating contribution to the sum on the right-hand side. Hence, replacing measures by distributions, it turns out that the dominating contribution to different moments comes from different fractal subsets of the support. Considering a multi-fractal distribution as a mathematical model of financial returns, this finding can be related to the well-known differences in the behavior of various powers of returns.

In order to get a handle on the behavior of the so-called *partition functions* $S(\varepsilon,q)$, we can replace the whole sum by its dominant contribution leading to (cf. Tel, 1988, or Evertz and Mandelbrot, 1992):

$$(7)\quad S(\varepsilon,q) \sim \varepsilon^{\tau(q)} \qquad \text{with: } \tau(q) = \arg\min_\alpha [q\alpha - f(\alpha)].$$

After computing an estimate of the empirical analogue of $\tau(q)$, i.e. the empirical *scaling function* of moments, $f(\alpha)$ can be obtained from (7) by way of a *Legendre transformation*:

$$(8)\quad f(\alpha) = \arg\min_q [q\alpha - \tau(q)].$$

Redefining $\tau(q) = q\cdot H_q - 1$, we can highlight the key difference between uni-fractal and multi-fractal processes: for the former H_q is a constant and, hence, $\tau(q)$ is linear in *q*. Multi-fractal processes, on the contrary are characterized by continuously changing H_q and hence, a nonlinear development of $\tau(q)$. It is this feature which makes these later formalisms an attractive model of financial returns. In fact, variability of H over various powers has been found to be a pervasive feature of financial data. The first systematic inquiry into the behavior of various measures of long-term dependence with varying powers q has been contributed by Ding, Engle and Granger (1993) and their findings have been confirmed in a number of other studies recently (Lux, 1996; Mills, 1997). The consensus now is that this feature appears in virtually all financial prices (Anderson and Bollerslev, 1997; Lobato and Savin, 1998). It is noteworthy that, although the above authors did not refer to multi-fractality in their papers, they did already point to empirical regularities of the type depicted in eq. (7) that are consistent with the multi-fractal model. Their basic message is, therefore, very similar to that of the recent contributions from physicists (Schmitt, Schertzer and Lovejoy, 1999; Vandewalle and Ausloos, 1998a, b) who, however, concentrated on deriving the spectrum of Hölder exponents (eq. (8)). As the $f(\alpha)$ spectrum and the scaling of moments are related by the Legendre transformation, the finding of a non-trivial spectrum confirms what has been found to be a salient feature of the data in the economics literature. The virtue of this alternative approach is, of course, to go beyond a description of stylized facts and to propose a new explanatory model that accounts for these facts.

3. Estimation of the f(a) spectrum

In our application, we simplify from the outset the multi-fractal model, eq. (1), put forward by Mandelbrot *et al*. by assuming that H = 0.5. This means we restrict the price process assuming that (in transformed time) the logs of prices follow a (Wiener) Brownian motion instead of fractal

Brownian motion with arbitrary H. The reason is that evidence in favor of $H \neq 0.5$ is weak in that statistical tests can usually not reject the null hypothesis $H = 0.5$ (cf. Lo, 1991; Goetzman, 1991; Mills, 1993),[5] while absolute and squared returns have values of H significantly exceeding 0.5. Furthermore, dependence in absolute returns is stronger than in squared returns (Ding *et al.*, 1993). Hence, the picture from the literature as well as from the analysis of our time series is that long-term dependence (which shows up in an estimate $H > 0.5$) is confined to various powers of returns, but is almost absent in the raw data. In order to model long-term dependence in the powers, we do not need to assume a fractional Brownian motion of returns. This feature of the data can be accounted for by the introduction of the multi-fractal time-transformation alone.

In estimating the multi-fractal spectrum of our returns time series, we note that under our assumption of Brownian motion of prices changes in transformed time the spectrum of the compound process $r(t) = B_H[\theta(t)]$ is related to the spectrum of the multi-fractal time-transformation $\theta(t)$ in the following way (cf. Mandelbrot, Calvet and Fisher):

$$(9) \quad f_r(\alpha) = f_\theta(\alpha H) = f_\theta(\alpha/2).$$

Now we turn to the empirical estimation of the parameters of multi-fractal models. Using the method of moments as described above, we consider the behavior of powers of returns:[6]

$$(10) \quad S(\Delta t, q) = \sum_{t=1}^{\mathrm{int}[T/\Delta t]} \{|p(t+\Delta t) - p(t)|^q\}$$

Eq. (10) has to be interpreted as a time-series analogue of eq. (5): p() is the logarithm of the price, and various powers of returns replace the powers of the measure μ. Furthermore, instead of dividing the support into finer and finer intervals, we consider different levels of time aggregation (thus, Δt replaces the box length ε).

As the first step, we compute the empirical *partition functions* $S(\Delta t, q)$ and use them to estimate the scaling function $\tau(q)$ from regressions in log co-ordinates. The upper panel of Fig. 2 shows a selection of partition functions for some low (left-hand side) and higher moments (right-hand side) for the German stock market index DAX.[7] As can be observed, the empirical behavior is very close to the presumed linear shape for moments of small order, while the fluctuations around the regression line become more pronounced for higher powers. This is, however, to be expected as the influence of chance fluctuations is magnified with higher powers q.

The resulting scaling function for moments in the range [-10, 20] is exhibited in the lower left panel of *Fig. 2*, For comparison, the broken line shows the behavior expected with Wiener Brownian motion, i.e. scaling according to $q/2 - 1$. There is a clear deviation from pure Brownian motion. The qualitative picture is the same found by Mandelbrot *et al.* as well as Schmitt, Schertzer and Lovejoy. Finally, the last step consists in computing the multi-fractal $f(\alpha)$ spectrum. The lower right-hand panel of *Fig. 2* is a visualization of the Legendre transformation. The spectrum is obtained by drawing lines of slope q and intercept $-\tau(q)$ for various q. If the underlying data indeed exhibits

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[5] It is also well-known that the R/S and other estimation methods are positively biased around $H = 0.5$ which may explain some (seemingly significant) findings of H in excess of one half in the earlier literature (cf. North and Halliwell, 1994).

[6] Int[.] denotes the integer part of the argument in brackets.

[7] Here and in the following the plots from the other three time series are almost identical.

multi-fractal properties, these lines would turn out to constitute the envelop of the distribution $f(\alpha)$. As can be seen, a convex envelope emerges from our scaling functions. Again, this outcome is shared by all other studies available hitherto, which may suggest that such a shape of the spectrum is a robust feature of financial data. As emphasized by Mandelbrot *et al.*, some very simple cascade model give rise to similar $f(\alpha)$ spectra and may therefore be considered as candidate models for financial data.

Insert Fig. 2 about here

4. Spectra of Binomial and Log-normal Models

The binomial cascade has already been used as an illustration above. In its simplest form, it consists in splitting a bounded support repeatedly into two subintervals of equal length and assigning them the fractions p_1 and $1 - p_1$ of the mass of the mother interval. It has been shown that the resulting spectrum of Hölder exponents has a closed-form solution which reads:

$$(11)\quad f(\alpha) = -\frac{\alpha_{max}-\alpha}{\alpha_{max}-\alpha_{min}}\log_2\left(\frac{\alpha_{max}-\alpha}{\alpha_{max}-\alpha_{min}}\right)-\frac{\alpha-\alpha_{min}}{\alpha_{max}-\alpha_{min}}\log_2\left(\frac{\alpha-\alpha_{min}}{\alpha_{max}-\alpha_{min}}\right).$$

The spectrum is, thus, restricted to an interval $[\alpha_{min}, \alpha_{max}]$ with the minimum and maximum attainable exponent being defined by the binomial structure: assuming $p_1 \geq 0.5$, we have $\alpha_{min} = -\log_2(p_1)$ and $\alpha_{max} = -\log_2(1-p_1)$.

The second model, the Log-normal cascade, assumes that the multipliers in each step are random draws from a Log-normal distribution instead of being fixed quantities as in the Binomial model. More concretely, denoting by M the multipliers (the analogue of p_1 and $1 - p_1$ above), $\log_2(M)$ is assumed to follow a Normal distribution with mean λ and variance σ^2. While in the binomial case, the total mass is preserved in each iteration, this is obviously not the case when using two random numbers to split up the mass of each mother interval. However, one can introduce restrictions in order to preserve the mass *on average*, which would prevent a collapse of the measure to zero or explosion to infinity (and would, therefore, prevent non-stationarity in a time series context). The necessary requirement is $E[M] = 0.5$, which can be guaranteed by appropriate choice of σ^2 (or, *vice versa*, appropriate choice of λ if σ^2 is given).[8] The Log-normal cascade with this restriction, therefore, boils down to a one-parameter model as well which is fully defined by the parameter λ. Its fractal spectrum is given by:

$$(12)\quad f(\alpha) = 1 - \frac{(\alpha-\lambda)^2}{4(\lambda-1)}.$$

Note that from a given spectrum like (11) or (12) the Hölder exponent of the *q*th moment is easily obtained by applying the inverse Legendre transformation. Visually, this corresponds to identifying the resulting exponent by a local slope equal to *q* of the spectrum. From the hump-shaped

[8] Note that without such restriction $E[M] = \exp(-\lambda \ln(2) + 0.5\, \sigma^2 (\ln(2))^2)$

appearance of both theoretical distributions, it is obvious that the relevant Hölder exponent at q = 2 will be smaller than the one at q = 1, which is accordance with the usual finding of higher persistence in absolute returns than in squared values.

For the application of these models to our data set, we have to keep in mind, that the cascade models are used for the volatility or time deformation $\theta(t)$ and that the returns themselves result from the compound process $B_{.5}[\theta(t)]$. When fitting the empirical spectrum in Fig. 2, we, therefore, have to take into account the shift in the spectrum as detailed in eq. (9).

5. Estimation of Cascade Models and Comparison with GARCH

Unfortunately, there is hardly any statistical theory available for determination of the parameters of a cascade model. We therefore have to resort to *ad hoc* methods for estimating p_1 or λ. We pursue two approaches here: first, we compute the best fit of (11) and (12) for the empirical spectrum using a least square criterion which is the common approach pursued in physical applications. To this end, we restrict our attention to the positively sloped, left-hand part of the spectrum. The reason is, that the right-hand arm is computed from partition functions with negative powers and is, therefore, strongly affected by chance fluctuations due to the Brownian process. In fact, performing experiments with synthetic data from multi-fractal processes with both Binomial and Log-normal time transformation, we found, that the location of the downward sloping part was strongly biased and, even with a symmetrical theoretical spectrum, often showed the same skewness as our empirical spectra. As a consequence, a fit based on the left-hand arm alone seems preferable.[9] Results from this procedure are exhibited in *Table 1*.

In order to have some indication of goodness-of-fit of these estimates beyond estimation of the $f(\alpha)$ spectrum, we applied the Kolmogorov-Smirnov test to Monte Carlo simulations of the cascade models with estimated parameters p_1 or λ. As an alternative for the traditional estimation method based on the distribution of Hölder exponents, we also performed a grid search for the variant of the multi-fractal cascades that achieves the best fit to the unconditional distribution of the empirical data. Since we are dealing with a one-parameter family of stochastic processes in both the Log-normal ad Binomial cases, such an approach can be implemented at reasonable computational costs. The criterion used here was again the Kolmogorov-Smirnov statistic.

Let us shortly recall the details of the Kolmogorov-Smirnov tests: given two random samples of size n, $X_1, X_2, ..., X_n$, and m, $Y_1, Y_2, ..., Y_m$, the Kolmogorov-Smirnov statistic (denoted by K in the following) is defined as the supremum of the absolute vertical distances between the empirical distribution functions, $S_1(x)$ and $S_2(x)$:

(13) $$K = \sup_x \left|S_1(x) - S_2(x)\right| .$$

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[9] It may be added that fits with both arms gave inferior results throughout and sometimes even led to violations of the restrictions of the underlying model. Note also that a bias towards skewness on the right implies also that our empirical $f(\alpha)$ shape does not necessarily speak against a more symmetric shape as would be implied by the Binomial and Log-normal models.

If K exceeds the 1 - α quantile of its theoretical distribution, the null hypothesis H_0: $S_1(x) = S_2(x)$ can be rejected in favor of the alternative $S_1(x) \neq S_2(x)$. This test for identical distribution functions of two random samples is denoted the Smirnov test. If instead of comparing two empirical samples, one wishes to test whether the empirical distribution function, say $S_1(x)$, follows some hypothesized distribution function $F^*(x)$, one would use the closely related statistic of the Kolmogorov test:

$$\text{(13a)} \quad K' = \sup_x \left| S_1(x) - F^*(x) \right|.$$

Quantiles of both test statistics can be found in Conover (1980). Our application here lies somewhere in between the designs of the Smirnov and Kolmogorov test: we are able to estimate the parameters of a stochastic model, but lack an analytical solution to the resulting unconditional distribution. Our solution consists in performing Monte Carlo simulations of the multi-fractal cascade models with estimated parameters p_1 and λ and comparing the unconditional distributions from the simulated time series with that of the underlying financial data. In order to facilitate comparisons, we used simulated time series of the same sample size as the empirical data. In this case (i.e. m = n) the 95% and 99% quantiles of the Kolmogorov test are given by $1.36\sqrt{n}$ and $1.63\sqrt{n}$, while the pertinent quantiles of the Smirnov test are $1.92\sqrt{n}$ and $2.30\sqrt{n}$.

Because of the 'non-standard' application of the Kolmogorov-Smirnov type test here, we are, however, careful in avoiding an interpretation in terms of 'rejection' and 'acceptance', but prefer to simply interpret the resulting K's as measures of similarity of the distributions under consideration. *Table 1* exhibits the means and standard deviations of the K statistic obtained for both the $f(\alpha)$ estimates and those obtained by grid-search for all four financial time series. Means and standard deviations are computed from 2,000 Monte Carlo samples in each case. As the time series differ in size, the K statistics have been multiplied by $\sqrt{n}$ to facilitate comparability.

As can also be observed in *Table 1*, results turned out to be quite different for both estimation methods. First, the parameter estimates from the least-square fit of the spectrum did not appear to generate unconditional distributions with a particularly good fit. With $K\sqrt{n}$ hovering between 4.34 for the Log-normal model in the case of the U.S./DM exchange rate[10] and a highest value of 31.56 for the Binomial model with the NYSE composite index, the performance did not appear to be particularly encouraging.

The picture changes, however, dramatically, when we turn to the results from our more direct grid search of the cascade parameters. In all cases, results turned out to be very clear in that the behavior of the Kolmogorov-Smirnov statistic showed a smooth variation with a unique global minimum whose parameters are exhibited in *Table 1*. Judged by the K statistics, a tremendous improvement could be obtained in all cases with the resulting unconditional distributions showing much greater similarity to the empirical data than before. In most cases, the mean values over 2,000 Monte Carlo replications are within the 95% quantile of the Smirnov or even the Kolmogorov test.

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[10] Interestingly, the Log-normal model has also been estimated for the U.S.$/DM exchange rate in Calvet, Fisher and Mandelbrot (1997). From the information given in their paper, one can infer an estimate of λ = 1.11 which differs somewhat from our result (λ = 1.03). Due to a somewhat different sample horizon and their allowance of $H \neq 0.5$, results are, however, not directly comparable.

It is also interesting to note that in both variants of the cascade models the K minimizing values of the parameters p_1 and λ are smaller than those obtained by the fit of $f(\alpha)$. As a consequence, volatility bursts in the simulations are more moderate with the parameters from the grid search.

It is also worthwhile noticing that under our grid search approach both cascade models perform more or less equally well and that the results for all four financial time series are very similar in terms of parameter estimates and goodness of fit. This is a signature of the statistical similarity of data from different financial markets and suggests also that the multi-fractal model may be able to capture some basic properties of these data. Note also that the fit is the more remarkable as no attempts have been made at all to account for skewness in the data.

To introduce a benchmark for the performance we also estimated GARCH(1,1) models and again explored goodness-of-fit by means of Monte Carlo simulations using the Kolmogorov-Smirnov criterion. Because of the higher computational costs involved with a grid search over a higher-dimensional parameter space, we estimated parameters by a standard maximum likelihood procedure.[11] Results are also shown in *Table 1.* Consistent with experience from the literature, parameter estimates are also remarkably constant across markets. Results from Monte Carlo simulations show, however, that the fit of the unconditional distribution is considerably worse than that of the multi-fractal models.

Insert Table 1 about here

Having explored the goodness-of-fit of the *unconditional* distribution function that can be achieved by multi-fractal processes we turn back to the characteristics of the *conditional* distribution. As we already said, the main motivation for our interest in the cascade models is their potential ability to generate varying degrees of long-term dependence in moments which is what we observe in empirical records. The $f(\alpha)$ spectrum, in principle, encapsulates all available information on this long-range autocorrelation structure. Parameter estimation based on the spectrum, therefore, should lead to selection of models that largely share the same pattern of long-memory as the underlying data.

But how good is this fit of the conditional distribution? In order to get some clue to the performance of the Binomial and Log-normal cascades in this respect, we compared the theoretical Hölder exponents at powers q = 1 and q = 2 with empirical estimates.[12] Theoretical numbers are easily obtained by applying the inverse Legendre transformation to eq. (11) and (12) and solving for q = 1 and 2. In the lower right panel of *Fig. 1*, the resulting exponents $\alpha(q = 1)$ and $\alpha(q = 2)$ are identified as points with local slope equal to 1 or 2, respectively. As for empirical estimation, we used both the time-honored rescaled range method (R/S) and the Geweke/Porter-Hudak (GPH) method for estimating the parameter of fractional differentiation d by a regression in the frequency domain (Geweke and Porter-Hudak, 1983). The R/S technique gives a point estimate of the Hölder exponent itself, while the estimate d from the second approach is related to the Hölder exponent by: $\alpha = d +$

[11] One may argue that this different way of estimation, which does not use the minimization of the Kolmogorov-Smirnov distance directly, could introduce a bias against the GARCH model. However, although not identical, ML estimation and minimization of K are closely related criteria. Furthermore, we performed a local grid search around the estimated parameters and could not find improvements to the K statistic by parameter variation.

[12] There is no need to give results for GARCH processes here as they are characterized by exponential decay of the autocorrelation function and are, therefore, unable to match the pertinent empirical findings.

0.5. In order to facilitate comparison, we add 0.5 to the estimates of the fractional differencing parameter in *Table 2*. While it is known that the R/S method is quite robust (Brock and de Lima, 1995), there is still no asymptotic distribution theory available for this method. We, therefore, confine ourselves to the point estimates, while for the (GPH) method 95% confidence intervals from the asymptotic distribution of the estimates can be computed and are also given in the table.

Our empirical findings confirm the usual picture: we always find $\alpha > 0.5$ for both absolute and squared returns. Furthermore, absolute returns have a larger exponent than squared returns in all cases except for the somewhat unusual estimates for the gold price with the GPH method. Comparing the theoretical Hölder exponents from the fits of the spectrum, we see that the correspondence between the empirical estimates and those implied by the pertinent multi-fractal model is best for the German DAX, while the theoretical exponents appear to underestimate the degree of long-term dependence in the NYCI and the Gold price and overstates long-range correlations in the US$-DM exchange rate. It is also remarkable, that both the Binomial and Log-normal model always lead to very similar results.

Turning to the performance of the second set of estimates form the grid search, we see that they uniformly overstate the degree of long-range dependence. The perplexingly good fit to the unconditional distribution, therefore, seems to come at the cost of a poor performance with respect to the characteristics of the autocorrelation structure. Thus, it seems that we face a certain trade-off between both criteria.

Insert Table 2 about here

5. Conclusion

The purpose of this paper was to contribute to an evaluation of the recent proposal of multi-fractal processes as a model for financial returns. From their very construction, these processes are able to account for the pervasive finding of long-memory effects in volatility. They also allow to capture a broader spectrum of dependence structures than models of the uni-fractal type in that different degrees of auto-correlation in various powers of returns can be explained *within* these models. However, up to now, evaluation of these models has been restricted to demonstrating their visual similarity with empirical records. As other models (e.g. GARCH) are also very similar to empirical records upon inspection, an assessment should ideally rely on less subjective criteria.

In the absence of any standard methods of inference we tried a grid search for the parameter values of the Log-normal and Binomial cascade models using the Kolmogorov-Smirnov distance as a selection criterion. As it turned out, these most elementary multi-fractal models achieved an impressive fit for various empirical data, although these are essentially *one-parameter* families of stochastic processes and no adjustments for skewness have been made. However, we also noticed a certain trade-off in that the better fit of the unconditional distribution comes at the cost of deviations of the implied autocorrelation structure from the empirical findings. Similarly as with the GARCH approach, various refinements could be developed to overcome this deficiencies of the most elementary multi-fractal models. Obvious avenues for improvements are the use of more complicated cascade processes (e.g. multinomial models) and the choice of alternative distributions for the price increment other than the Normal.

Nevertheless, taking into account the very simple structure of the one-parameter families of stochastic processes used in this paper, the results reported above appear quite remarkable. In our view, they confirm that this new model is worthwhile further considerations by economists. A major obstacle to its widespread use is the combinatorial character of existing multi-fractal models and the lack of standard time-series techniques for estimation and statistical inference. As concerns future research, what is most urgently needed is a reformulation of the multi-fractal model in terms of iterative processes[13] for which one could develop more standard tools of estimation and statistical inference. Appropriate refinements may lead to rush of new results and new models for the volatility of financial prices.

References:

Anderson, T.G. and T. Bollerslev (1997) Heterogeneous Information Arrivals and Return Volatility Dynamics: Uncovering the Long Run in High Frequency Data, *Journal of Finance*, **52**, 975 - 1005.

Conover, W.J., 1980, *Practical Nonparametric Statistics*, Second Edition, New York: John Wiley.

Brock, W. A. and P.J.F. de Lima (1995) Nonlinear Time Series, Complexity Theory, and Finance, in: G. Maddala and C. Rao, eds., *Handbook of Statistics*, Vol. 14. North-Holland, Amsterdam.

Ding, Z., C.W.J. Granger and R.F. Engle (1993) A Long Memory Property of Stock Market Returns and a New Model, *Journal of Empirical Finance*, **1**, 83 - 106.

Evertz, C.J.G. and B. Mandelbrot (1992) Multifractal Measures, in: Peitgen, H.-O., H. Jürgens and D. Saupe, eds., *Chaos and Fractals: New Frontiers of Science*. Berlin: Springer.

Falconer, K. (1990) *Fractal Geometry: Mathematical Foundations and Applications*. New York: John Wiley.

Fisher, A., L. Calvet and B. Mandelbrot (1997) *Multifractality of Deutschemark/US Dollar Exchange Rates*. mimeo: Cowles Foundation for Research in Economics.

Geweke, J. and S. Porter-Hudak, 1983, The Estimation and Application of Long-Memory Time Series Models, *Journal of Time Series Analysis* **4**, 221 - 238.

Ghasghaie, S. *et al*. (1996) Turbulent Cascades in Foreign Exchange Markets, *Nature* **381**, 767 - 770.

Goetzman, W.N. (1993) Patterns in Three Centuries of Stock Market Prices, *Journal of Business,* **66**, 249 - 270.

Lo, A.W. (1991) Long-term Memory in Stock Market Prices, *Econometrica*, **59**, 1279 - 1313.

Lobato. I.N. and N.E. Savin (1998) Real and Spurious Long-Memory Properties of Stock Market Data, *Journal of Business and Economics Statistics*, **16**, 261 - 283.

Lux, T. (1996) Long-term stochastic dependence in financial prices: evidence from the German stock market, *Applied Economics Letters*, **3**, 701 - 6.

Mandelbrot, B. (1999) A Multifractal Walk down Wall Street, *Scientific American*, February, 50 - 53.

[13] A first step into this direction can be found in Marsan, Schertzer and Lovejoy (1996).

Mandelbrot, B., A. Fisher and L. Calvet (1997) *A Multifractal Model of Asset Returns*. mimeo: Cowles Foundation for Research in Economics.

Mantegna, R. N. and H. E. Stanley (1996) Turbulence and Financial Markets, *Nature* **383**, 587 - 588.

Marsan, D., D. Schertzer and S. Lovejoy (1996) Causal Space-Time Multifractal Processes: Predictability and Forecasting of Rain Fields, *Journal of Geophysical Research* **101**, 26,333 - 26,236.

Mills, T.C. (1997) Stylized Facts of the Temporal and Distributional Properties of Daily FT-SE Returns, *Applied Financial Economics*, **7**, 599 - 604.

North, C.P. and D.I. Halliwell (1994) Bias in Estimating Fractal Dimension with the Rescaled-Range (R/S) Technique, *Mathematical Geology*, **26**, 531 - 55.

Schmitt, F., D. Schertzer and S. Lovejoy (1999), Multifractal Analysis of Foreign Exchange Data, *Applied Stochastic Models and Data Analysis*, in press.

Tel, T. (1988) Fractals, Multifractals, and Thermodynamics, *Zeitschrift für Naturforschung*, **43**, 1154 - 1174.

Vandewalle, N. and M. Ausloos (1998a) Sparseness and Roughness of Foreign Exchange Rates, *International Journal of Modern Physics C*, **9**, 711 - 719.

Vandewalle, N. and M. Ausloos (1998b) Multi-affine Analysis of Typical Currency Exchange Rates, *European Physical Journal B*, **4**, 257 - 261.

Vassilicos, J.C. (1995) Turbulence and Intermittency, *Nature*, **374**, 408 - 409.

Vassilicos, J.C., A. Demos and F. Tata (1993) No Evidence of Chaos But Some Evidence of Multifractals in the Foreign Exchange and the Stock Market, in: Crilly, A.J., R.A. Earnshaw and H. Jones, eds., *Applications of Fractals and Chaos*. Berlin: Springer.

Table 1: Estimates of the Parameters of Binomial and Log-normal Cascades

Model	**Method**	**Data**	**Parameter estimates**	**Kolmogorov distance . $n^{0.5}$**
Binomial (Parameter: p_1)	LS-Fit of $f(\alpha)$	DAX	$p_1 = 0.6991$	26.23 (1.29)†
		NYCI	$p_1 = 0.7735$	31.56 (1.00)†
		US$-DM	$p_1 = 0.6133$	6.70 (0.71)†
		Gold	$p_1 = 0.7495$	20.98 (1.29)†
Log-normal (Parameter: λ)	LS-Fit of $f(\alpha)$	DAX	$\lambda = 1.0920$	21.11 (2.58)
		NYCI	$\lambda = 1.1776$	25.53 (2.83)
		US$-DM	$\lambda = 1.0295$	4.34 (1.33)
		Gold	$\lambda = 1.1473$	15.85 (2.65)
Binomial (Parameter: p_1)	Minimization of Kolmog. distance	DAX	$p_1 = 0.5605$	1.71 (0.28)
		NYCI	$p_1 = 0.5721$	1.24 (0.36)
		US$-DM	$p_1 = 0.5662$	1.20 (0.31)
		Gold	$p_1 = 0.5974$	1.38 (0.22)
Log-normal (Parameter: λ)	Minimization of Kolmog. distance	DAX	$\lambda = 1.0116$	1.88 (0.50)
		NYCI	$\lambda = 1.0158$	1.46 (0.46)
		US$-DM	$\lambda = 1.0128$	1.42 (0.47)
		Gold	$\lambda = 1.0138$	4.23 (0.72)

continued...

GARCH (1,1)	Maximum Likelihood	DAX	$\alpha_0 = 4.14 \cdot 10^{-6}$ (5.07)§ $\alpha_1 = 0.1520$ (7.61) $\beta_1 = 0.8186$ (49.86)	2.17 (0.69)
		NYCI	$\alpha_0 = 1.04 \cdot 10^{-6}$ (4.44)§ $\alpha_1 = 0.0787$ (3.81) $\beta_1 = 0.9093$ (48.18)	3.81 (0.73)
		US$-DM	$\alpha_0 = 8.50 \cdot 10^{-7}$ (3.74)§ $\alpha_1 = 0.1049$ (8.26) $\beta_1 = 0.8836$ (80.45)	2.25 (0.94)
		Gold	$\alpha_0 = 1.14 \cdot 10^{-6}$ (2.46)§ $\alpha_1 = 0.1023$ (6.78) $\beta_1 = 0.8853$ (54.25)	5.51 (1.00)

† The displayed values of Kolmogorov distances are the means over 2,000 Monte Carlo replications, standard deviation are given in parenthesis.
§ t-values of the GARCH parameters are given in parenthesis.

Note: The 95% (99%) points of the *Kolmogorov* tests (i.e. comparison of empirical distribution with hypothesized distribution function are $1.36/\sqrt{n}$ and $1.63/\sqrt{n}$, respectively. The 95% and 99% points for the *Smirnov* test (i.e. comparison of independent samples) are $1.92/\sqrt{n}$ and $2.30/\sqrt{n}$, respectively (cf. Conover, 1980, c. 6).

The time intervals and number of observations are:
DAX: 10/59 - 12/98 (n = 9818),
NYCI: 01/66 - 12/98 (n = 8308),
US$-DM: 01/74 - 12/98 (n = 6140),
Gold price: 01/78 - 12/98 (n = 5140).

Table 2: Empirical and Theoretical Hurst/Hölder Exponents

Model	Method	Data	a at q = 2	a at q = 1
Binomial	LS-Fit of f(α)	DAX	0.71	0.88
		NYCI	0.51	0.77
		US$-DM	0.89	0.96
		Gold	0.57	0.81
Log-normal	LS-Fit of f(α)	DAX	0.72	0.91
		NYCI	0.47	0.82
		US$-DM	0.91	0.97
		Gold	0.56	0.85
Binomial	Min. of Kolmog. distance	DAX	0.97	0.98
		NYCI	0.96	0.98
		US$-DM	0.96	0.99
		Gold	0.92	0.97
Log-normal	Min. of Kolmog. distance	DAX	0.97	0.99
		NYCI	0.95	0.98
		US$-DM	0.96	0.99
		Gold	0.96	0.99
Empirical Estimates	Hurst Exponent from R/S	DAX	0.77	0.85
		NYCI	0.80	0.88
		US$-DM	0.75	0.86
		Gold	0.85	0.93
Empirical Estimates	GPH:† $d + 0.5$	DAX	0.69 (0.55, 0.83)	0.85 (0.71, 0.99)
		NYCI	0.65 (0.51, 0.79)	0.91 (0.76, 1.05)
		US$-DM	0.74 (0.58, 0.90)	0.79 (0.63, 0.95)
		Gold	1.12 (0.95, 1.28)	1.07 (0.90, 1.23)

† Asymptotic 95% confidence intervals of the Geweke/Porter-Hudak estimates of the parameter of fractional differencing are given in parentheses. For information about the data, see *Table 1*.

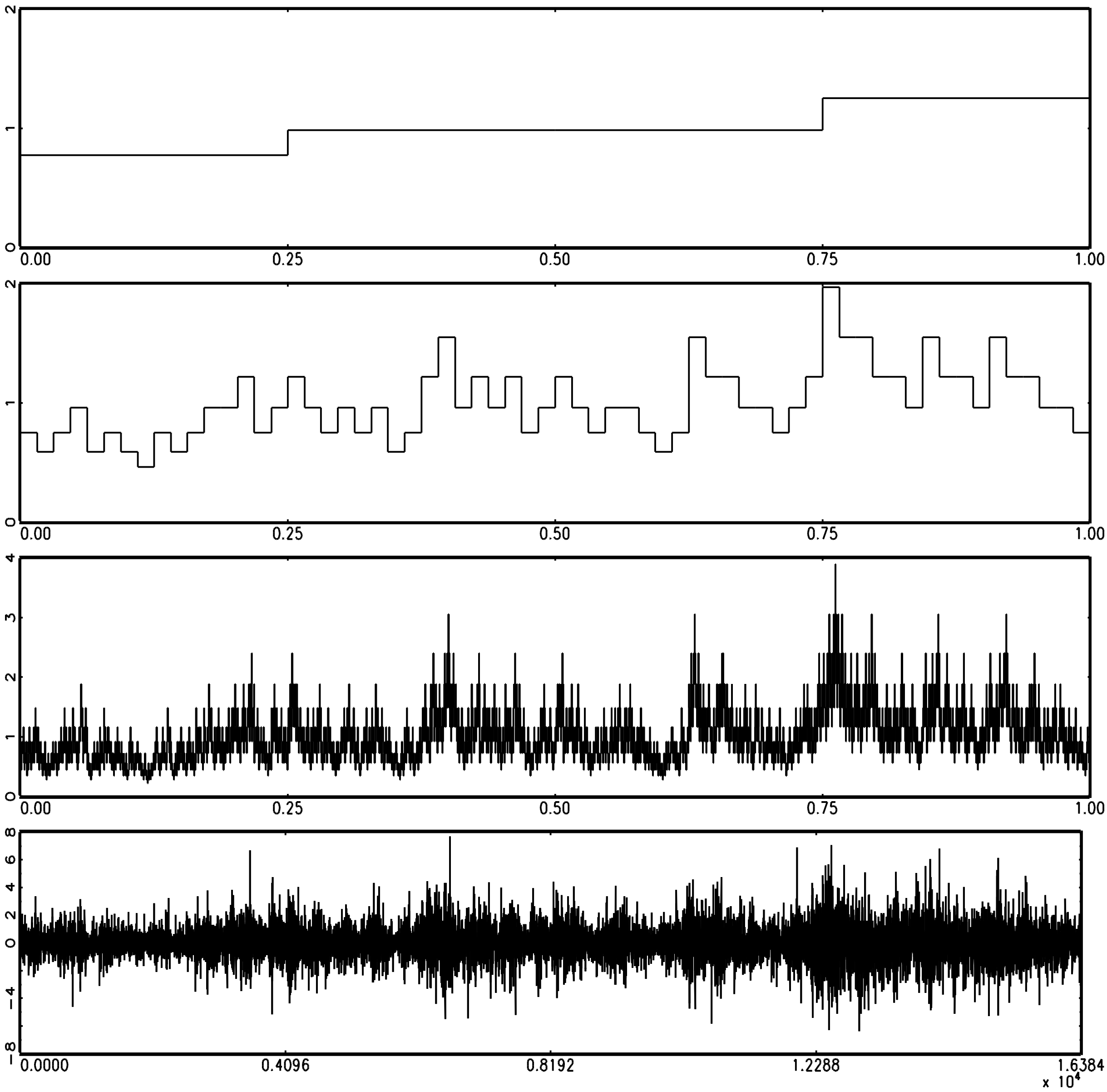

Fig. 1: Development of a Binomial Cascade and its Use as a Time Deformation. The upper panels of the figure show (from top to bottom) the development of a binomial measure after 2, 6, and 12 iterations of the cascade. In the lower panel, a compound process is illustreted in which the same cascade is used as a time transformation device. Superimposed is a Wiener Brownian motion (H = 0.5). The parameter p_1 used in this example was 0.5605 which gave the best fit of the multi-fractal model with binomial time transformation for German stock returns under minimization of the Kolmogorov-Smirnov distance. Note that the integrals under the first three curves are equal to 1 in each iteration.

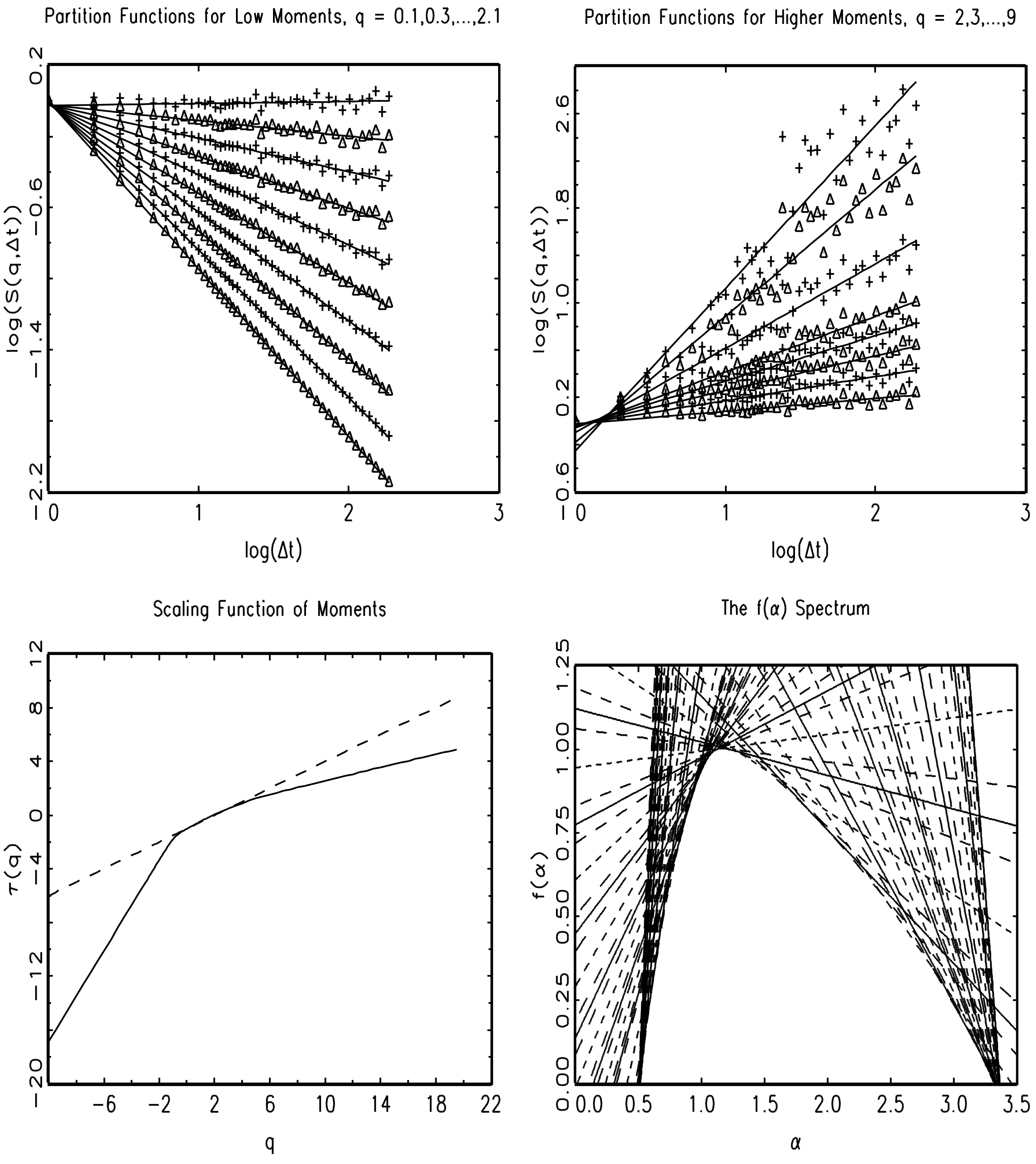

Fig. 2: Scaling and Multi-Fractal Spectrum of DAX Returns. The *upper panel* shows the partition functions obtained for a variety of (positive) moments ranging from q = 0.1 to q = 9. While we observe an almost perfectly linear relationship for the lower moments, there is more randomness in the scaling of higher moments. The *bottom panel* shows that the deviation form the expected behavior $\tau(q) = q/2 - 1$ under Brownian motion (*left*), and the $f(\alpha)$ spectrum of Hölder exponents obtained from the Legendre transformation (*right*).