Abschlussbericht des Sonderforschungsbereichs 303

"Information and Coordination of Economic Activities"

Acknowledgement

We gratefully acknowledge financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, throughout 1985 – 1999. We also acknowledge the permission to reprint the following articles:

Bös, D., and C. Lülfesmann (1996): The Hold-up Problem in Government Contracting. Scandinavian Journal of Economics 98, pp. 53-74 (Blackwell Publishers).

Christopeit, N., and M. Musiela (1994): Existence of Arbitrage - Free Measures. Stochastic Analysis and Applications 12 (1), pp. 41-63 (Marcel Dekker, Inc., N.Y.).

Corneo, G., and O. Jeanne (1997): Conspicuous Consumption, Snobbism and Conformism. Journal of Public Economics 66, pp. 55-71 (Elsevier Science).

Föllmer, H., and D. Sondermann (1986): Hedging of Non-Redundant Contingent Claims. In: Hildenbrand, W., and A. Mas-Colell (eds.), Contributions to Mathematical Economics, in Honor of Gérard Debreu, North-Holland, Amsterdam, pp. 205-223 (Elsevier Science).

Härdle, W., W. Hildenbrand, and M. Jerison (1991): Empirical Evidence on the Law of Demand. Econometrica 59, pp. 1525-1549 (The Econometric Society).

Hildenbrand, W., and A. Kneip (1998): Demand Aggregation under Structural Stability. Journal of Mathematical Economics 31, pp. 81-110 (Elsevier Science).

Miltersen, K.R., K. Sandmann, and D. Sondermann (1997): Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates. The Journal of Finance 52 (1), pp. 409-430 (Blackwell Publishers).

Nagel, R. (1995): Unraveling in Guessing Games: An Experimental Study. American Economic Review 85, pp. 1313-1326 (American Economic Association).

Neumann, M.J.M., and J. v. Hagen (1994): Real Exchange Rates within and between Currency Areas: How far Away is EMU? Review of Economics and Statistics 76, pp. 236-244 (MIT Press Journals).

Nöldeke, G., and K.M. Schmidt (1995): Option Contracts and Renegotiation: A Solution to the Hold-Up Problem. RAND Journal of Economics 26, pp. 163-179 (The RAND Journal of Economics).

Schlag, K. (1998): Why Imitate, and if so, How? A Bounded Approach to Multi-Armed Bandits. Journal of Economic Theory 78(1), pp. 130-156 (Academic Press, Inc.).

Schweizer, U. (1989): Litigation and Settlement under Two-sided Incomplete Information. Review of Economic Studies 56, pp. 163-178 (Review of Economic Studies Ltd.).

Selten, R., M. Mitzkewitz, and G.R. Uhlich (1997): Duopoly Strategies Programmed by Experienced Players. Econometrica 65, pp. 517-555 (The Econometric Society).

Content

Chapter 1: Introduction

Werner Hildenbrand

1.

The Sonderforschungsbereich 303 "Information and the Coordination of Economic Activities" ended after 15 years of financial support on December 31st, 1999. During these 15 years economics and, in particular, quantitative economics and economic theory has changed noticeably. Some economists speak of a paradigm change, others of a crisis of neoclassical economic theory. Obvious symptoms for this change are, on the one side, the great popularity and the rapid dissemination of *game theory* and on the other side the fast increase in acceptance of *experimental economics* by the profession.

It is not exaggerated to speak of a triumphal march of game theory. There was the extraordinarily fast development in game theory, in particular in non-cooperative game theory, which is apparent from the enormous increase in publications and the founding of new scientific journals specializing in game theory. Perhaps more important, one could observe a general penetration of game theoretic thinking in almost all fields of economics. Indeed, with few exceptions, presentations at international conferences on economic theory are based on game theoretic approaches.

This boom in game theory - which naturally had its own inner dynamics - was favoured, as it seems to me, by a distinct fall of interest in general equilibrium models, in particular in Walrasian general equilibrium theory. Many economic theorists of great distinction, in particular among the mathematical economists who made outstanding contributions to Walrasian general equilibrium theory, changed their interest in favour of game theory. The reason for the loss of interest in Walrasian general equilibrium theory is of course well-known. Indeed, the results by Sonnenschein, Mantel and Debreu in the early seventies showed that the Walrasian model does not have the necessary structure which is required for using these models to analyse concrete economic problems. In short, the problem with Walrasian equilibrium models is not the possible lack of existence of equilibria but the multiplicity of equilibria. This multiplicity has the consequence that, in general, a comparative static analysis with definite predictions is not possible.

The above mentioned developments in economics – the rise of game theory and experimental economics - which could be observed world-wide and in which the Sonderforschungsbereich 303 played an important role, naturally left deep traces in the development of the Sonderforschungsbereich 303 during the last 15 years. From a methodological point of view this development is quite remarkable. It is known that game theory assumes a high degree of rationality of the participating players, while experimental economics, in any case explorative experimental economics as it is pursued in Bonn, seems to reject the basic hypothesis of game theory, i.e., rationality in the sense of maximising behaviour.

Despite the fact that the general interest of the profession and the methodological approaches changed clearly in the past 15 years it seems to me that the definition of economics by Keynes is still valid:

"Economics is a science of thinking in terms of models joined to the art of choosing models which are relevant to the contemporary world."

There is full agreement in the profession and among the members of the Sonderforschungsbereich 303 about the meaning of "thinking in terms of models". There is, however, a disagreement – what else could be expected – about the "art of choosing models which are relevant to the contemporary world". Thinking in terms of models means to obey the strict rules of the axiomatic deductive method. This requires a formal or mathematical formulation of the model with an explicit statement of all assumptions. During the process of deduction, that is to say, the process of proving certain propositions, no further implicit assumption can be made, even if the temptation to do so is strong. This axiomatic deductive method requires, as is well-known, a great discipline of thinking. I am safe in claiming that all research papers of the Sonderforschungsbereich 303 are exemplary in this respect.

Every modelling of a real economic phenomenon requires a certain degree of abstraction. That is to say, certain aspects of the real phenomenon in question are consciously neglected in order to isolate its relevant aspects. A model necessarily is a simplification, often even a gross simplification. About this point there is a general consensus since it would not be meaningful to simultaneously model the great variety of all aspects of the real phenomenon. Obviously, I have in mind here the art of simplification. Economic modelling is artificial and an art since there is no generally valid procedure which describes how to obtain a satisfactory economic model. Furthermore, in analysing a model one often has to add simplifying hypotheses since otherwise the model does not allow a qualitative or explicit analysis. These additional hypotheses often are difficult to justify, sometimes they are even in open conflict with empirical evidence or with knowledge from experimental economics. These additional hypotheses exclusively serve one purpose: they allow to prove certain propositions. In this sense they are ad hoc. The final result of such a modelling process is surely a mathematical model, however, often it is just a caricature of the original real economic phenomenon.

There is disagreement in the profession and also among the members of the Sonderforschungsbereichs 303 about the epistemological value of such ad hoc specified models. What do we learn about the real economic phenomenon if we astutely analyse the properties of such models? This is of course an old dispute in the methodology of economics which has been answered in quite different ways. Ariel Rubinstein writes about this point in the Journal of Economic Literature1:

"It would be no exaggeration to state that economic theory is in a methodological mess. The main problem is the vague connection between economic theory and reality. Economic theories are meant to be about the real world. But economic models do not fit or even approximate any reasonable picture of that world. Worse still, economic theory uses assumptions that are easily refuted. Thus, the relevance of the conclusions seems to depend on whether the theory miraculously produces accurate predictions, which it usually does not."

Naturally, economics was always in a methodological mess, and I expect that it will remain so in the future. In a certain sense this is what makes economics interesting. The above mentioned change in the fields of research has revived this methodological dispute. Economic theory today consists of a great variety of quite special models where often slide modifications of the model lead to radically different conclusions. It is not clear what we learn by astutely analysing all these "little" models. Economics needs a Newton or Einstein, perhaps a new Walras or Keynes would suffice.

[1] Rubinstein, A. (1999): Book Review on: Simon, Herbert A., An Empirically Based Microeconomics. *Journal of Economic Literature 37(4)*, p. 1711.

2.

I would like to make a few comments on the importance of the Sonderforschungsbereich 303 for the Department of Economics at the University of Bonn. The Sonderforschungsbereich 303 was for the Department of Economics much more than a mere generous financial supplement to the relatively modest basic provision that the University of Bonn offers to their professors. Naturally, without this generous financial support by the Deutsche Forschungsgemeinschaft many of the scientific activities would not have been possible. With certainty, however, a financial support with the same volume given separately to the various projects of the Sonderforschungsbereich 303 would not have had the same effect. Indeed, the setting-up of a Sonderforschungsbereich promotes cooperation, even more, it enforces the cooperation of the professors involved, who, as is sometimes said, tend by nature to eccentricity ("Eigenbrödelei"). In addition, a Sonderforschungsbereich gives the necessary incentives and furthers the motivation for creating a stimulating scientific environment which is necessary for research on a high level. It seems to me that a solid foundation for a successful Sonderforschungsbereich rests on two pillars:

1. A research oriented department, that is to say, a department whose professors give high priority to research and teaching.

2. Encouragement of the new generation of academics, since it is this new generation, the graduate students and post-doc fellows ("Habilitanden") which fill a Sonderforschungsbereich with life.

The encouragement of a new generation of academics of a high scientific level can only be achieved in an institutionalised graduate school. The traditional German way of writing a Ph.D. thesis which essentially consists of a bilateral relationship between the graduate student and the "Doktorvater" does not lead, with few exceptions, to the desired maximal achievements.

The two pillars support each other. Indeed, a department can only attract research oriented professors if there are highly qualified graduate students and conversely, a Graduate School of high scientific level is only possible with research oriented professors. At the time of application of the Sonderforschungsbereich 303 in 1984 one of the pillars was already established. Due to a successful hiring policy during the period of support of the Sonderforschungsbereich 21 under the chairmanship of Wilhelm Krelle the department had already a critical number of research oriented professors in order to dare a new application. The prospect of a new Sonderforschungsbereich on the other hand facilitated some new appointments such as, to give one example, the appointment of Reinhard Selten who moved from Bielefeld to Bonn in 1984. In all appointments which were made by the department the scientific qualification and productivity with regard to the needs and orientation of the Sonderforschungsbereich 303 had absolute priority. This hiring policy was only possible since all members of the department, not only the members of the Sonderforschungsbereich 303, fully supported all activities of the SFB.

At the time of application of the Sonderforschungsbereich 303 the second pillar consisted of the "European Doctoral Program in Quantitative Economics". The European Doctoral Program was founded in 1977 by the London School of Economics, the Université Catholique de Louvain and the Department of Economics in Bonn. Later, the European Doctoral Program was extended by the École des Hautes Etudes en Sciences Sociales, Paris, and the University Pompeu Fabra, Barcelona. The special feature of the European Doctoral Program is that every graduate student has to spend a full year in one of the partner universities. The graduate student is free to choose his partner university as well as the university where he wants to submit his thesis. During the time of support of the Sonderforschungsbereich 303 the European Doctoral Program was extended by a Postgraduate Research Group (Graduiertenkolleg) "Interaktive ökonomische Entscheidungen" and finally, in 1998, we achieved our goal, the founding of the Bonn Graduate School of Economics.

It is a well-known, even though regrettable fact that the applicants for the Bonn Graduate School of Economics from German Universities are older than those from other European Universities. The consequence of this fact was that the staff-members of the Sonderforschungsbereich 303 have been older as compared to those in other European research centres. To change this situation the Department of Economics as the first department in Germany in 1993 introduced the Credit Point System. Among many other advantages this system led to a definite shortening of the time for graduation. Yet, the main advantage was that visiting professors of the Sonderforschungsbereich 303 could take part in teaching on an advanced level. Thus, also the students of the Department of Economics took advantage of the visiting professors of the Sonderforschungsbereich 303. The driving force for all reforms in the Department of Economics has been, without doubt, the Sonderforschungsbereich 303.

One often can hear the assertion that the whole should be more than the sum of its parts. Applied to a Sonderforschungsbereich this is only the case if all individual projects cooperate with each other, and if one succeeds to create a stimulating intellectual atmosphere in which a creative scientific dispute is possible. It was never our goal to form a completely homogeneous group. We purposely wanted a certain variety of methodological approaches and we wanted to keep the spirit of dissent alive.

The numerous Sonderforschungsbereich 303 seminars and the "jours fixes" certainly created - to a large extend - the stimulating intellectual atmosphere of the Sonderforschungsbereich 303. In the SFB research seminars scientists from all over the world presented their latest results and thus stimulated the research of the SFB members. On the other hand the visiting scholars reported on our research in their home universities which led to the world-wide recognition of the Sonderforschungsbereich 303.

3.

The success of the Sonderforschungsbereich 303 can naturally be evaluated by different points of view. With certainty I can claim that the Sonderforschungsbereich 303 had been recognised in Germany and internationally as an active research centre of great distinction. The profession wanted to be informed what happened in the Sonderforschungsbereich 303 in Bonn. Indeed, the "Sonderforschungsbereich-303-News" which had been published weekly (all together 524 issues) have been read world-wide. For many scholars, in particular those from the United States, a visit at the Sonderforschungsbereich 303 in Bonn had absolute priority while visiting Europe. The Sonderforschungsbereich 303 did not only influence the development in economics in Bonn and other German universities, but its influence went much further, it left its traces world-wide. This is an unquestionable fact. Yet, it is much more difficult to answer the question about the quality of the research produced by the Sonderforschungsbereich 303. If one accepts the hypothesis that the quality of the research correlates positively with the quantity of the publications then the above question has a simple answer. The members of the Sonderforschungsbereich 303 published more than 1.000 research papers, a great part of which was published in scientific journals of high international recognition.

How can one decide whether these publications are of high quality? Is quality a purely subjective opinion of the referee? Surely not, otherwise the process of refereeing of the Deutsche Forschungsgemeinschaft would be totally dependent on the choice of the referees, thus more or less arbitrary. In order to put up for discussion the quality of research of the Sonderforschungsbereich 303 13 publications were selected and reprinted in this report. This is a delicate experiment: to choose 13 publications out of 681 publications in scientific journals. In a first step every Project (Teilprojekt) was assigned a quota based on its total publications. Then every "Teilprojekt" decided among its members which paper to propose as candidates for publication. This led to 23 proposed papers. To make the final choice all members of the Sonderforschungsbereich 303 spent three days in a closed meeting at "Schloß Friedewald". All papers have been presented and discussed in detail. Finally, 13 papers had been selected. In the election all members of the Sonderforschungsbereich 303 had equal vote. The selected papers are reprinted in Chapter 3 and thus allows readers to form their own opinion about the quality of research of the Sonderforschungsbereich 303.

Chapter 2: Projects and Results

1 Experimental Economics

Reinhard Selten

1.1 Main research topic

The project was mainly concerned with experimental research on behaviour in interactive decision situations. However, also some basic questions in individual decision making were explored. The research had a strong exploratory component with the aim of developing new descriptive theories of boundedly rational decision procedures. Moreover, accompanying theoretical investigations complemented the experimental work.

Important topics of experimental research on interactive decision making were behaviour in coalition games, in two-person bargaining under incomplete information, in auctions and markets, in reciprocity games, and in normal form games.

1.2 Methodological approach

Experiments are often designed with the purpose to either confirm or refute special theoretical hypotheses. However, testing prespecified hypotheses is not the only possible purpose of experiments. Much of the experimental research at the project had a strong exploratory component: One tries to create a theoretically interesting economic situation in the laboratory and observes how subjects behave. It is the purpose of exploratory experimental work to enhance the formation of new descriptive theories.

The exploratory character of many experiments is a special feature of the research at the Bonn laboratory. There was always much emphasis on the development of new descriptive theories, much more so than in the United States where at least in the first years experimentation was predominantly oriented towards traditional theory. Over the years the methodological differences between experimental work in Germany and elsewhere decreased. Our behavioural outlook gained more acceptance and exploratory methods also spread to some extend.

Another difference of our work compared to that of most other experimental economists concerns the methods of evaluation. More than others we insists on independence requirements by experimental design.

1.2.1 Experimental procedures

The project was engaged in different methods of experimentation: Sessions in the computerised laboratory, paper and pencil experiments, “mensa” experiments, video experiments, strategy studies.

The Bonn Laboratory of Experimental Economics has 18 terminals in cubicles permitting anonymous interaction among subjects. This is the environment of our computerised laboratory sessions. Our paper and pencil experiments also were done with especially recruited subjects. We did not perform classroom experiments with participants of courses mainly because of the reduced control connected with this procedure which is very popular at other places.

Another procedure used by us is the “mensa” experiment. Subjects are recruited in the student cafeteria and make their decisions on the spot. Video experiments involve teams who jointly have to make a decision in a video taped group discussion. In a video experiment two or more teams may interact by formal bids and offers.

In a strategy study a group of subjects is engaged in a more long-term interaction, e.g. in the framework of a student seminar. The participants first gain experience within planing the game and then write strategy programs, directly in computer code or as flow charts. The strategies are matched against each other in a tournament. On the basis of feedback on the tournament result the participants then have the opportunity to revise their strategies. Several rounds of strategy revisions and tournaments eventually lead to a final tournament. The final strategies are the main focus for evaluation. Some strategy studies were organised as international competitions with young economists as participants. The strategies and tournament results were communicated by mail.

In all our experiments except some strategy studies participants were motivated by success dependant money payments. In strategy studies in the framework of students seminars participants cannot be adequately paid for intensive work during the whole semester. Instead of that they were motivated by grades mainly based on success in the final tournament.

1.2.2 Methods of evaluation

In view of our emphasis on independence requirements satisfied by design or analysis is mainly based on non-parametric tests. We avoid econometric techniques which require additional independence assumptions. Since there is only one economic history, macro-econometric models must rely on build-in independence assumptions. Experiments, however, can be repeated and independence should be generated by repetition.

An important problem in experimental research is the comparison of the predictive success of different theories. It is not always clear which methods should be used for this purpose. In order to clarify this question for different theory types, axiomatic characterisations have been supplied for the difference measure of predictive success for area theories (Selten 1991) and for the quadratic scoring rule (Selten 1998).

A problem of exploratory experimental research is the description of typical behaviour. For this purpose a measure of typicity has been introduced (Selten, Mitzkewitz, and Uhlich 1997, reprinted in this volume). Later a deeper understanding and a further elaboration of this measure to larger context has been supplied by Kuon (1993).

The evaluation of video experiments is very labour intensive. First one must prepare typewritten word protocols because video tapes cannot be directly evaluated. The evaluation of the word protocols than permits conclusions about the underlying reasoning processes. This information hardly can be obtained in an easier way. Video experiments also have been explored elsewhere, e.g. by Professor Tack and his group in Saarbrücken, but undoubtedly more emphasis is put on them in the Bonn Laboratory than in other centres in experimental economics.

1.2.3 Experimental software

The programming of computerised laboratory experiments used to be a very tedious task. Therefore, in our laboratory the toolbox RatImage (Abbink and Sadrieh 1995)[2] was developed for this purpose. This system reduced a work of months to a work of weeks. RatImage was also thoroughly documented. This enabled other laboratories, e.g. Amsterdam, Barcelona, Berlin, Jerusalem, Valencia, and Strasbourg to make use of the system. Afterwards similar systems were developed elsewhere, e.g. z-Tree by Fischbacher (1998)[3] in Zurich. However, RatImage still offers some advantages and continues to be used.

1.3 Selected results

In the first years of the work of the project, it was an important aim to extend the theory of equal division payoff bounds (Selten 1983)[4] for zero-normalised three-person characteristic function games to the more general case of three-person games with non-zero payoffs for one-person coalition. This work has led to the descriptive theory of proportional-division-payoff bounds (Uhlich 1990). The experiments were run in our computerised laboratory.

Normative game-theory has produced many theories for characteristic function games. These theories define solution concepts by abstract requirements and then find the structure of the predicted results by mathematical deduction. The structure of the behavioural theories elaborated in the project is very different. They directly describe the procedure in which the solution is found. In this sense one can speak of procedural theories.

1.3.1 Two-person bargaining

An experimental investigation of alternating bid bargaining in two-person characteristic function games with non zero conflict payoffs led to the descriptive theory of the negotiation agreement area (Kuon and Uhlich 1993). A deeper understanding of motivation and aspiration adaptation in such games was provided by a series of video experiments (see 2.1) by Hennig-Schmidt (1999).

A monograph of Kuon (1994) presents an extensive study of two-person bargaining under incomplete information with anonymous formal alternating bids. This study involves a theoretical analysis, experiments with spontaneous play and an application of the strategy method.

Other important results in this area are the paper by Mitzkewitz and Nagel (1993) on ultimatum games with incomplete receiver information and a monograph on damage claims bargaining under incomplete information (Ryll 1996).

1.3.2 Auctions and markets

An experimental study on sealed bid auctions presented a new approach to the explanation of bidding behaviour by *learning direction theory* (Selten and Buchta 1999).

---

[2] Abbink, K., A. Sadrieh (1995): RatImage – Research Assistance Toolbox for Computer-Aided Human Behavior Experiments, University Bonn, SFB Discussion Paper B-325.

[3] Fischbacher, U. (1998): Z-Tree: A Toolbox for Readymade Economic Experiments. Working Paper University of Zurich.

[4] Selten, R. (1983): Equal Division Payoff Bounds for Three-Person Characteristic Function Experiments. In: R. Tietz (ed.), *Aspiration Levels in Bargaining and Economic Decision Making*, Lecture Notes in Economics and Mathematical Systems, 213, Springer-Verlag Berlin, Heidelberg, New York, Tokyo, 1983, pp. 255-275.

A monograph by Sadrieh (1998) presented a breakthrough in the theory of the double auction and reported experimental results.

An application of the strategy method (see 2.1.) to a twenty times repeated, numerically specified asymmetric Cournot duopoly with linear costs and demand has led to a descriptive duopoly theory in which players try to achieve co-operative goals by "*measure for measure*" policies (Selten, Mitzkewitz, and Uhlich 1997). This theory is very different from traditional and game-theoretical approaches to oligopoly which always are based on optimisation on the basis of quantitative expectations. Contrary to this, the new theory neither involves quantitative expectations nor optimisation. It is not result oriented but procedural.

Basic questions arising in connection with financial markets were explored in an experimental individual behaviour study on option valuation (Abbink and Kuon 1996[5], Abbink and Kuon 2000[6]). A related market study reports interactive option pricing experiments (Kuon 1999)[7]. This work has led to a new behavioural approach to option pricing.

As an accompanying theoretical investigation auctions with external effects have been explored (Moldovanu, Jehiel, and Stacchetti 1996)[8].

1.3.3 Reciprocity games

Reciprocity is the behavioural tendency to respond as indicated by the phrase "I do unto you, as you do unto me". Reciprocity games are simple extensive games terminating after relatively few moves providing opportunity for reciprocal behaviour. A number of such games was explored in the laboratory: the ultimatum game with incomplete receiver information (Mitzkewitz and Nagel 1993), the uncertain reward ultimatum game (Abbink, Bolton, Sadrieh, and Tang 1996)[9], the covered response ultimatum game (Abbink, Sadrieh, and Zamir 1999)[10], the moonlighting game (Abbink, Irlenbusch, and Renner, forthcoming), an investigation of non-enforceable contracts (Irlenbusch and Schade, 1999), an investment game in a video study (Jacobsen and Sadrieh, 1996)[11] and a real effort setup (Fahr and Irlenbusch forthcoming).

A motivation similar to but different from reciprocity is explored in an experiment on a solidarity game (Selten and Ockenfels 1998).

The research on reciprocity games contributed to the lively international discussion on the nature of other-directed motivation and helped to clarify several important problems in this area.

---

[5] Abbink, K., B. Kuon (1996): An experimental investigation of the option pricing approach. SFB Discussion Paper B-376, University of Bonn.

[6] Abbink, K., B. Kuon (2000): Der Fluch der Erfahrung: Professionelle Trader versus Studenten in einem Optionsbewertungsexperiment. To appear in: Innovative Kapitalanlagekonzepte, E. Hehn (ed.), Gabler Verlag, Wiesbaden, 2000.

[7] Kuon, B. (1999): Information Aggregation, Speculation, and Arbitrage in an Option Market Experiment, mimeo.

[8] Jehiel, P., B. Moldovanu, and E. Stacchetti (1996): How (not) to Sell Nuclear Weapons. *American Economic Review 86*(4), 814-829.

[9] Abbink, K., G. E Bolton, A. Sadrieh, F.-F. Tang (1996): Adaptive Learning versus Punishment in Ultimatum Bargaining. SFB Discussion Paper B-381, University of Bonn.

[10] Abbink, K., A. Sadrieh, S. Zamir (1999): The Covered Response Ultimatum Game. SFB Discussion Paper B-416, University of Bonn.

[11] Jacobsen, E., A. Sadrieh, (1996): Experimental Proof for the Motivational Importance of Reciprocity. SFB Discussion Paper B-386, University of Bonn.

1.3.4 Normal form games

A special normal form game is the guessing game (Nagel 1995, reprinted in this volume) which led to a theory of levels of reasoning.

Learning in the context of normal form games played by randomly matched populations was experimentally explored in the dissertation by Fang-Fang Tang (1996)[12]. Experimentation with the normal-form version of the centipede game resulted into a successful application of learning direction theory (Nagel and Tang 1998)[13].

1.3.5 Individual behaviour

A series of experiments explored lottery choice and lottery evaluation behaviour in tasks with direct money payoffs compared to other tasks with payoffs in terms of probabilities of winning the high price in a binary lottery. It was shown that contrary to theoretical expectations, the deviations from expected value maximisation are greater in the binary lottery case. The reasons for this are discussed (Selten, Sadrieh, and Abbink 1999).

1.4 Open problems

1.4.1 Coalition games

The development of reasonable descriptive theories for three person games in characteristic function form has been brought to a closure even if it is not inconceivable that the discussion in this area will be open up again. However, a generalisation to more than three players is still an open problem. In view of the great size of the parameter space involved it is unclear how this problem should be approached.

1.4.2 Two-person bargaining

What has been said to coalition games can also be applied here. Especially, as far as two-person bargaining with incomplete information is concerned, it is an open question how the results of Bettina Rockenbach (Kuon 1994) can be transferred to more general forms of incomplete information.

1.4.3 Auctions and markets

The work of Abdolkarim Sadrieh (1998) on the alternating bid double auction succeeds in answering an important theoretical question. However, the theory concerns only markets in which each participant either buys or sells one unit. Sadrieh's experiment concerns the multiple unit case to which his theory is not directly applicable. This raises a theoretical question and also creates a need for further experimental research.

The theory of Selten, Mitzkewitz, Uhlich (1997) seems to be transferable to general two person supergames, but not necessarily to dynamic games. The strategy study of Claudia Keser (1992) shows how co-operative goals are formed in these games but it is not clear whether the approach can be generalised to larger classes of dynamic games.

---

[12] Tang, F.-F. (1996): Anticipatory Learning in Two-Person Games: An Experimental Study, Ph.D. – Thesis, University Bonn

[13] Nagel, R., F.-F. Tang (1998): An Experimental Study on the Centipede Game in Normal Form - An Investigation on Learning, *Journal of Mathematical Psychology*, forthcoming.

1.4.4 Reciprocity games

The work of our project in this area has shown how different motivations combine and interact in reciprocity games. Thus, it has been shown that in ultimatum games the genuine desire for punishment and punishment for the purpose of educating the player on the other side are both present. However, such contributions to the discussion of motivational problems are not more than steps towards the development of a procedural descriptive theory of motivational reciprocity games.

1.4.5 Normal form games

The theory proposed by Nagel (1995) for her guessing game was confirmed by other studies. The comparison of a great number of different learning theories in randomly matched repetitions of three by three games supports payoff sum re-enforcement learning (Roth and Erev 1995)[14]. However, in the mean time new theories have been proposed which need to be examined with the same methodology.

A strategy study on randomly generated three by three normal forms games has been conducted by Selten, Abbink, Buchta, and Sadrieh (1999)[15], but the evaluation is not yet completely finished. As far as games with pure strategy equilibria are concerned, the study yields a clear answer. The final strategies select pure equilibrium strategies and even largely succeed in co-ordinating at the maximum joint equilibrium pair. However, a less clear picture emerges for games without pure equilibria. Behaviour in such games is still an open problem.

1.4.6 Individual behaviour

Our results on lottery choice behaviour show that there is a need for a radically new start of theorising in this area. Our work has provided hints about the direction in which one has to go but we are still very far from an adequate descriptive theory. Perhaps lottery choice, which is very important for normative decision theory, is not the right point of departure for behavioural decision theory.

1.4.7 Learning direction theory

One of the most important results of our work was the emergence of learning direction theory. This theory applies to situations in which a decision parameter has to be fixed in a number of consecutive repetitions of the same problem and in which, after each choice, feedback is supplied permitting qualitative conclusions about what could have been better. Behaviour then has a tendency to change in the direction suggested by these conclusions if it is changed at all.

The origin of learning direction theory (Selten and Stoecker, 1986)[16] predates the work of the projects, but only in the course of this work it became visible that the idea has applicability far beyond the original context. There are now more than a dozen studies which confirm learning direction theory. Moreover, it only gradually became clear to us what are the distinctive features of learning direction theory compared to familiar re-enforcement theories (Selten and Buchta, 1999).

---

[14] Roth, A. E. and I. Erev (1995): Learning in Extensive Form Games: Experimental Data and Simple Dynamic Models in Intermediate Term. *Games and Economic Behavior 8*, 164-212.

[15] Selten, R., K. Abbink, J. Buchta, A. Sadrieh: How to play 3 x 3 games, in preparation.

[16] Selten, R., R. Stoecker, (1986): End Behavior in Sequences of Finite Prisoner's Dilemma Supergames, *Journal of Economic Behavior and Organization* 7, 47-70.

Learning direction theory is a qualitative theory. It is not yet completely clear how it should be complemented by quantitative assumptions. Unfinished work by Selten, Abbink, and Cox[17] about a winner's curse situation promises some progress in this direction.

1.5 Selected Publications

Abbink, K., B. Irlenbusch, and E. Renner: The Moonlighting Game, An Experiment on Reciprocity and Retribution. *Journal of Economic Behavior and Organization*, forthcoming.

Artale, A. (1997): Litigation and Settlement in a Game with Incomplete Information. Springer Lecture Notes in Economics and Mathematical Systems, 440, New York, Berlin, Tokyo.

Brennscheidt, G. (1993): Predictive Behavior: An Experimental Study, Springer Lecture Notes in Economics and Mathematical Systems, 403, New York, Berlin, Tokyo.

Chen, Y., and F.F. Tang (1998): Learning and Incentive Compatible Mechanisms for Public Goods Provision: An Experimental Study. *The Journal of Political Economy*, 106, 633-662.

Dahlhäuser, H. (1996): Prominenz der Preise in einem Warenhauskatalog. *zfbf, Schmalenbachs Zeitschrift für betriebswirtschaftliche Forschung*, pp. 711-737.

Fahr, R., and B. Irlenbusch: Fairness as a Constraint on Trust in Reciprocity – Earned Property Rights in a Reciprocal Exchange Experiment, *Economics Letters*, forthcoming.

Friedman, D. (1996): Equilibrium in Evolutionary Games: Some Experimental Results; *Economic Journal 106*, pp. 1-25.

Friedman, D., and T. Cason (1997): Price Formation in Single Call Markets; *Econometrica 65*, pp. 311-345.

Gardner, R., A. Herr and J. Walker (1997): Appropriation Externalities in the Commons: Repetition, Time Dependence, and Group Size; *Games and Economic Behavior 19*, pp. 77-96.

Hennig-Schmidt, H. (1999): *Bargaining in a Video Experiment - Determinants of Boundedly Rational Behavior*; Lecture Notes in Economics and Mathematical Systems, Vol. 467, Springer-Verlag, Berlin-Heidelberg-New York.

Irlenbusch, B., and L. Schade (1999): Zur Wirksamkeit nicht bindender Verträge - Eine experimentelle Untersuchung. *zfbf, Schmalenbachs Zeitschrift für betriebswirtschaftliche Forschung*, pp. 730-752.

Jehiel, P., Moldovanu, B. and E. Stacchetti (1996): How (not) to Sell Nuclear Weapons; *American Economic Review 86*, pp. 814-829.

Keser, C. (1992): Experimental Duopoly Markets with Demand Inertia: Game-Playing Experiments and the Strategy Method; *Lecture Notes in Economics and Mathematical Systems, Vol. 391*, Springer-Verlag, Berlin.

Kuon, B. (1993): Measuring the Typicalness of Behavior; *Mathematical Social Sciences 26*, pp. 35-49.

Kuon, B., and G. Uhlich (1993): The Negotiation Agreement Area: An Experimental Analysis of Two-Person Characteristic Function Games; *Group Decision and Negotiation 2*, pp. 323-345.

Kuon, B. (1994): Two-Persons Bargaining Experiments with Incomplete Information. Springer Lecture Notes in Economics and Mathematical Systems, 412, New York, Berlin, Tokyo.

Mitzkewitz, M., and R. Nagel (1993): Experimental Results on Ultimatum Games with Incomplete Information; *International Journal of Game Theory 22*, pp. 171-198.

[17] Selten, R., K. Abbink, and R. Cox: Learning Direction Theory and the Winner' s Curse, in preparation.

Moldovanu, B. (1992): Coalition-Proof Nash Equilibria and the Core in Three-Player Games; *Games and Economic Behavior 4*, pp. 565-581.

Moldovanu, B., and E. Winter (1993): Order Independent Equilibria; *Games and Economic Behavior*, 9, 21-35.

Nagel, R. (1995): Unraveling in Guessing Games: An Experimental Study; *American Economic Review 85*, pp. 1313-1326.

Nagel, R., and N. J. Vriend (1999): An experimental study of adaptive behavior in a oligopolistic market game. *Evolutionary Economics* 9, 27-65.

Nagel, R., and N. J. Vriend (1999): Inexperienced and Experienced Players in an Oligopolistic Market Game with Minimal Information. *Industrial and Corporate Change* 8 (1), 41-65.

Ryll, W. (1996): Litigation and Settlement in a Game with Incomplete Information. Springer Lecture Notes in Economics and Mathematical Systems, 440, New York, Berlin, Tokyo.

Sadrieh, A. (1998): *The Alternating Double Auction Market – A Game Theoretic and Experimental Investigation*; Lecture Notes in Economics and Mathematical Systems, Vol. 466, Springer-Verlag, Berlin - Heidelberg-New York.

Selten, R. (1991): Evolution, Learning, and Economic Behavior; *Games and Economic Behavior 3*, pp. 3-24.

Selten, R. (1991): Properties of a Measure of Predictive Success; *Mathematical Social Sciences 21*, pp. 153-167.

Selten, R. (1992): A Demand Commitment Model of Coalition Bargaining; in: R. Selten (ed.), *Rational Interaction*, Springer-Verlag, Berlin, pp. 245-282.

Selten, R. (1998): Aspiration Adaptation Theory. *Journal of Mathematical Psychology 42*, 191-214.

Selten, R. (1998): Axiomatic Charaterization of the Quadratic Scoring Rule. *Experimental Economics 1*, 43-62.

Selten, R. (1998): Features of Experimentally Observed Bounded Rationality. *European Economic Review 42*, 413-436.

Selten, R. (1999): *Game Theory and Economic Behaviour – Selected Essays*. Edward Elgar Publishing Inc.

Selten, R., M. Mitzkewitz and G.R. Uhlich (1997): Duopoly Strategies Programmed by Experienced Players; *Econometrica 65*, pp. 517-555.

Selten, R., and A. Ockenfels (1998): An Experimental Solidarity Game. *Journal of Economic Behavior and Organization 34*, 517-539.

Selten, R., and J. Buchta (1999): Experimental Sealed Bid First Price Auctions With Directly Observed Bid Function. In: Games and Human Behavior – Essays in Honor of A. Rapoport, D. Budescu, I. Erev, R. Zwick (ed.), Lawrence Erlbaum Associates, Inc., Mahwa, NJ, USA.

Selten, R., A. Sadrieh, and K. Abbink (1999): Money Does Not Induce Risk Neutral Behavior, but Binary Lotteries do Even Worse. *Theory and Decision 46*, 211-249.

Sippel, R. (1997): An Experiment on the Pure Theory on Consumer's Behaviour. *Economic Journal 107*, 1431-1444.

Uhlich, G.R. (1990): Descriptive Theories of Bargaining: An Experimental Analysis of Two- and Three-Person Characteristic Function Bargaining; *Lecture Notes in Economics and Mathematical Systems, Vol. 341*, Springer-Verlag, Berlin, Heidelberg, New York.

van Damme, E., R. Selten and E. Winter (1990): Alternating Bid Bargaining with a Smallest Money Unit; *Games and Economic Behavior 2*, pp. 188-201.

2 Contract Theory

Urs Schweizer

2.1 Main research topic

The main research topics of the project were concerned with the functioning of economic systems for which the information is asymmetrically distributed among the involved parties. At the early stage of the project, the focus was on banking regulation and competition among private banks and, more generally, on competition under asymmetric information. In 1987, Urs Schweizer took over to direct the project from Martin Hellwig who had left the University of Bonn. Henceforth, instead of anonymous market transactions, the project was shifting attention mainly to contract theory and the principal-agent paradigm. Contracts were understood in a broad sense, including general mechanisms under both voluntary and enforced participation. Important research topics have included the theory of the firm, transactions within firms versus market transactions and the interplay between contractual arrangements and market structure.

2.2 Methodological approach

The traditional property rights approach keeps focussing on the notoriously difficult concept of transaction costs. Contract theory, in contrast, does not rely on this concept. Rather, the starting point is the assumed distribution of information among the involved parties. To emphasize this approach, the distribution of information was referred to as the contract-specific environment. Hence, to fully capture a transaction problem, it is not enough to take all choice variables into account. Details of the contract-specific environment have to be specified as well.

The contract-theoretic approach explores the impact of a given contract-specific environment on the strategic interaction among individual agents. To this end, a decision theory is needed which is able to cope with uncertainty and private information. Since the theory of boundedly rational behavior has not yet reached a level of maturity such that it could be applied to the general subject of contract theory, we took resort to the more fully developed theory of fully rational behavior. Heavy use has been made of the solution concepts of non-cooperative game theory. While rational choice theory allows for an unified treatment, it may well fail to take account of certain aspects of real life contracting.

Rational choice theory, if taken literally, leads to the presumption that parties sign contracts only which are verifiable in front of courts and which are encompassing in the sense that the obligations of all parties are fully specified for all contingencies. It should be pointed out that an encompassing contract need not be optimal. Its crucial property rather stems from the fact that an encompassing contract gives rise to a game in normal or extensive form. Therefore, rational decisions under any encompassing contract can be captured as a Nash equilibrium of the game as induced by the contract. Since this solution concept is available for situations including asymmetrically informed parties, it well serves our general purpose.

As in microeconomic theory, parties are assumed to have preferences over the set of allocations or outcomes but not over the procedure by which they were reached. For this reason, contracts can only indirectly be evaluated, namely according to the outcome they are expected to lead to under rational play. While game theory deals with the solution of a given game, the scope of contract theory is broader. In fact, given any transaction problem, the main question concerns the optimal contract for that situation. In other words, since encompassing contracts induce games, contract theory deals with the design of games. As a consequence, contract theory and mechanism design are closely related subjects. In order to find the optimal contract for a given contract-specific environment, heavy use of the revelation principle has to be made. This principle turns out to be a powerful tool. Unfortunately, it also tends to obscure the limits between different institutional arrangements.

As a remedy, at later stages, we shifted attention to the theory of incomplete contracts as pioneered by Grossman and Hart. The model of relationship-specific investments and the closely related hold-up problem serve as the main paradigm of the approach. Due to the long-term nature of relationship-specific investments, renegotiations by mutual agreement of contractual obligations becomes a crucial issue. In spite of several attempts, unfortunately, it has turned out to be impossible to justify the incompleteness of ex-ante contracts as a stringent consequence of anticipated renegotiations. Nevertheless, the issue of renegotiations has become an important subject of research on its own grounds.

2.3 Selected results

After having described the methodological approach in general terms, we now want to summarize some results of the project. They have been selected from a great variety of papers, the selection criterion being that they forcefully illustrate the general approach of contract theory. From the early period of the project where market transactions were studied, we want to emphasize two papers by Bester (1988,1989). The goal was to explain price dispersion as it occurs in reality but as it remains difficult to explain in theory. By introducing a bargaining procedure explicitly modeled as a non-cooperative game, Bester has successfully replaced the purely anonymous view of the market process In this way, he was able to establish, among other results, the existence of equilibrium in a Hotelling-type model of one-dimensional space. The famous principle of minimal differentiation was not confirmed by Bester's approach.

Gale and Hellwig (1985) and Güth and Hellwig (1986) were early contributions strictly following the contract-theoretic approach as outlined above. In each paper, the optimal arrangement for a given contract-specific environment was found. In the first paper, the solution can nicely be interpreted as a standard credit contract closely resembling arrangements known from real life. The solution as found in the second paper is framed as a direct mechanism and, as such, does not readily lend itself to an equally nice interpretation. It allows, however, to establish an impossibility result in the sense that no mechanism can exist which is ex-post efficient, balanced and which satisfies all participation constraints. In other words, the ex-post efficient solution cannot be the result of voluntary contracting. The paper extends earlier findings of Myerson and Satterthwaite in a substantial way.

Emons (1988) considers an interesting contract specific environment. His analysis provides a deeper understanding of the role of warranties. The aim is not to find the optimal contract but rather to investigate a given institutional arrangement. The contribution nicely fits into the general approach of the project, illustrating the role of asymmetrically distributed information as a potential source of frictions. Emons and Sobel (1991) study different liability rules, their performance being evaluated according to the criterion of efficiency. As is the case with encompassing contracts in general, rules of liability also give rise to games in normal form such that rational behavior can be captured by the solution concepts of non-cooperative game theory.

Schweizer (1988) provides a non-cooperative approach to the Coase Theorem. In a simple setting of external effects, it is shown that including steps of the bargaining process may lead to an efficient outcome even if decisions are governed by non-cooperative behavior. The findings are in contrast with those of the market failure literature. Moreover, the paper also introduces a setting of one-sided asymmetric information such that the efficient outcome can no longer result from voluntary contracting. In this sense, the model illustrates again that asymmetrically distributed information may be the source of frictions. Schweizer (1989) reinforces this view in a model of pre-trial negotiations with two-sided asymmetric information. While, for suitable parameter configurations, costly litigation could be avoided under pooling equilibria, refinement criteria for Nash equilibrium as proposed for signaling games in general would rule out such efficient outcomes. Schweizer (1990) attempts to view political rules in the sense of Buchanan from a contract-theoretic perspective. Conceptually, the paper leaves many questions unanswered. The model, however, has served as a framework for an experimental investigation as well as a theoretical extension due to Wessels (1993).

Jost (1996) and Kessler (1998) are interesting contributions to traditional principal-agent theory. Starting form a given contract-specific environment, they both follow a strictly contract-theoretic approach. Both papers uncover unexpected phenomena which are based on asymmetrically distributed information. In spite of their seemingly simple structure, the results require a sophisticated game-theoretic analysis.

As far as the theory of incomplete contracts is concerned, several results of the project deserve to be emphasized. Nöldeke and Schmidt (1995) deal with the hold-up problem in a setting where parties anticipate renegotiations to take place within a non-cooperative bargaining procedure as pioneered by Hart and Moore. The paper leads to two important insights. First, option contracts may well be suited to sustain an efficient outcome. Second, inefficiency results based on anticipated renegotiations may not nearly be as robust as Hart and Moore have suggested. The paper of Nöldeke and Schmidt is frequently referred to in the literature.

Marin and Schnitzer (1995) explain potential advantages of barter trade versus monetary exchange in a contract-theoretic setting. Their analysis was stimulated by looking at a sample of real foreign trade contracts. The paper has played a decisive role when Monika Schnitzer was awarded the “Preis der Nordrhein-Westfälischen Akademie in Düsseldorf”. Schmidt and Schnitzer (1995) deal with privatization from the perspective of incomplete contract theory. Their analysis has received much attention. Schnitzer (1995) provides important insights into the theory of the firm by looking at the constitution of a firm from a contract-theoretic perspective. A recent contribution to the theory of incomplete contracts is due to Rosenkranz and Schmitz (1999) who have investigated information sharing within strategic alliances.

To conclude, the book on contract theory by Schweizer (1999) should be mentioned. The book provides a systematic exposition of both traditional principal-agent theory and the more recent theory of incomplete contracts. A whole chapter is devoted to anticipated renegotiations within settings of relationship-specific investments. The book summarizes the conceptual aspects of contract theory as they have guided the research of the project. Without the intensive collaboration within the project and with other projects of the “Sonderforschungsbereich 303”, this book could not have been written.

2.4 Open problems

The project has made substantial progress in terms of both conceptual issues and applications of the economic theory of contracts. Important contributions were obtained in the field of principal-agent theory and the theory of incomplete contracts. The impact of anticipated renegotiations on relationship-specific investments was investigated in detail. Yet, we were unable to provide a satisfactory justification for the restricted use of ex-ante contractual arrangements which is at the heart of the incomplete-contract approach. In other words, the theory of incomplete contracts still lacks a proper underpinning. As a conjecture, it could well turn out that many institutional details as observed in reality cannot be understood from the view of fully rational behavior. In fact, Oliver Williamson keeps advancing the view that boundedly rational behavior is a necessary ingredient for institutional analysis. Unfortunately, he too seems far away from having the corresponding theory at his disposal. In any case, one might have to look for help beyond the tool kit of microeconomic analysis and applied game theory to gain further insights into details of institutional arrangements.

2.5 Selected Publications

Artale, Angelo und Hans Peter Grüner (1999) : A Model of Stability and Persistance in a Democracy, erscheint in *Games and Economic Behavior*.

Bester, H. (1985): Screening versus Rationing in Credit Markets with Imperfect Information, *American Economic Review 75*, 850-855.

Bester, H. (1988): Bargaining, Search Costs and Equilibrium Price Distributions; *Review of Economic Studies 55*, pp. 201-214.

Bester, H. (1989): Non-Cooperative Bargaining and Spatial Competition; *Econometrica 57*, pp. 97-113.

Breuer, W. (1997): Hedging von Wechselkursrisiken bei internationalen Ausschreibungen; *Zeitschrift für Betriebswirtschaft 67, Supplementary Issue 2*, pp. 81-103.

Breuer, W. (1997): Kreditgenossenschaften, Managementsteuerung und der Markt für Unternehmenskontrolle; *Kredit und Kapital 30*, pp. 219-249.

Breuer, W., M. Gürtler, and J. Schuhmacher (1999): Die Bewertung betrieblicher Realoptionen. In: *Betriebswirtschaftliche Forschung und Praxis 51*, 213-232.

Emons, W. (1988): Warranties, Moral Hazard, and the Lemons Problem; *Journal of Economic Theory 46*, 16-33.

Emons, W. (1990): Efficient Liability Rules for an Economy with Non-identical Individuals; *Journal of Public Economics 42*, 89-104.

Emons, W., and J. Sobel (1991): On the Effectiveness of Liability Rules when Agents Are Not Identical; *Review of Economic Studies 58*, pp. 375-390.

Funk, P. (1995): Bertrand and Walras Equilibria in Large Economies; *Journal of Economic Theory 67*, pp. 436-466.

Funk, P. (1996): Auctions with Interdependent Valuations; *International Journal of Game Theory 25*, pp. 51-64.

Grüner, H.-P. (1997): A Comparison of Three Institutions for Monetary Policy when Central Bankers Have Private Objectives; *Public Choice 62*, pp. 172-193.

Haller, H. (1985): The Principal-Agent Problem with a Satisficing Agent, *Journal of Economic Behavior and Organization 6*, 354-379.

Haller, H. (1991): Corporate Production and Shareholder Cooperation under Uncertainty; *International Economic Review 32*, pp. 823-842.

Hellwig, M. (1987): Some Recent Developments in the Theory of Competition in Markets with Adverse Selection, *European Economic Review 31*, pp. 319-325.

Hellwig, M. (1988): A Note on the Specification of Inter-Firm Communication in Insurance Markets with Adverse Selection; *Journal of Economic Theory 46*, 154-163.

Hellwig, M., and B. Allen (1986): Bertrand-Edgeworth Oligopoly in Large Markets, *Review of Economic Studies 53*, 175-204.

Hellwig, M., and D. Gale (1985): Incentive-Compatible Debt Contracts: The One-Period Problem, *Review of Economic Studies 52*, 647-663.

Hellwig, M., and W. Güth (1986): The Private Supply of a Public Good, *Zeitschrift für Nationalökonomie/Journal of Economics*, Supplementum 5, 121-159.

Hellwig, M., and W. Leininger (1987): On the Existence of Subgame-Perfect Equilibrium in Infinite-Auction Games of Perfect Information, *Journal of Economic Theory 43*, pp. 55-75.

Hellwig, M., W. Leininger, Ph. Reny and A. Robson (1990): Subgame Perfect Equilibrium in Continuous Games of Perfect Information: An Elementary Approach to Existence and Approximation by Discrete Games; *Journal of Economic Theory 52*, 406-422.

Jost, P.-J. (1996): On the Role of Commitment in a Principal-Agent Relationship with an Informed Principal; *Journal of Economic Theory 68*, pp. 510-530.

Kamecke, U. (1989): Non-Cooperative Matching Games; *International Journal of Game Theory 18*, pp. 423-431.

Kamecke, U. (1992): On the Uniqueness of the Solution to a Large Linear Assignment Problem; *Journal of Mathematical Economics 21*, pp. 509-521.

Kamecke, U. (1993): The Role of Competition for an X-inefficiently Organized Firm; *International Journal of Industrial Organization 11*, pp. 391-405.

Kessler, A. (1997): The Value of Ignorance; *Rand Journal of Economics*, 29, 339-354.

Kessler, A.S. (1999): On Monitoring and Collusion in Hierarchies, *Journal of Economic Theory*, forthcoming.

Leininger, W. (1986): The Existence of Perfect Equilibria in a Model of Growth with Altruism between Generations, *Review of Economic Studies 53*, 349-367.

Leininger, W. (1990): Patent Competition, Rent Dissipation, and the Persistence of Monopoly: The Role of Research Budgets; *Journal of Economic Theory 53*, 146-172.

Leininger, W., P.B. Linhart and R. Radner (1989): Equilibria of the Sealed-Bid Mechanism for Bargaining with Incomplete Information; *Journal of Economic Theory 48*, pp. 63-106.

Marin, D., and M. Schnitzer (1995): Tying Trade Flows: A Theory of Countertrade with Evidence; *American Economic Review 85*, pp. 1047-1064.

Nöldeke, G. (1997): On Testing for Financial Market Equilibrium under Asymmetric Information; *Journal of Political Economy 105*, pp. 1107-1113.

Nöldeke, G., and E. van Damme (1990): Signalling in a Dynamic Labour Market; *Review of Economic Studies 57*, 1-23.

Nöldeke, G., and K.M. Schmidt (1995): Option Contracts and Renegotiation: A Solution to the Hold-Up Problem; *RAND Journal of Economics 26*, pp. 163-179.

Rosenkranz, S., and P. Schmitz (1999): Know-How Disclosure and Incomplete Contracts. *Economics Letters* 63, pp. 181-185.

Schmidt, K. (1993): Commitment through Incomplete Information in a Simple Repeated Bargaining Model; *Journal of Economic Theory 60 (1)*, pp. 114-139.

Schmidt, K. (1993): Reputation and Equilibrium Characterization in Repeated Games with Conflicting Interests; *Econometrica 61 (2)*, pp. 325-351.

Schmidt, K. (1996): The Costs and Benefits of Privatization - An Incomplete Contracts Approach; *Journal of Law, Economics and Organization 12*, pp. 1-24.

Schmidt, K., and M. Schnitzer (1993): Privatization and Management Incentives in the Transition Period in Eastern Europe; *Journal of Comparative Economics 17*, pp. 264-287.

Schmidt, K., M. Cripps and J. Thomas (1996): Reputation in Perturbed Repeated Games; *Journal of Economic Theory 69*, pp. 387- 410.

Schmidt, K.M., and M. Schnitzer (1995): The Interaction of Explicit and Implicit Contracts; *Economics Letters 48*, pp. 193-199.

Schmitz, P. (1998): Randomization in Coalition Contracts. *Public Choice 94*, 341-353.

Schnitzer, M. (1994): Dynamic Duopoly and Best Price Clauses; *RAND Journal of Economics, Vol. 25*, pp. 186-196.

Schnitzer, M. (1995): Breach of Trust in Takeovers and the Optimal Corporate Charter; *Journal of Industrial Economics 43*, pp. 229-259.

Schweizer, U. (1988): Externalities and the Coase Theorem: Hypothesis or Result?, *JITE (Zeitschrift für die gesamte Staatswissenschaft) 144 (2)*, pp. 245-266.

Schweizer, U. (1989): Litigation and Settlement under Two-sided Incomplete Information; *Review of Economic Studies 56*, pp. 163-178.

Schweizer, U. (1990): Calculus of Consent: A Gametheoretic Perspective; *Journal of Institutional and Theoretical Economics (Zeitschrift für die gesamte Staatswissenschaft) 146*, No. l, pp. 28-54.

Schweizer, U. (1996): Endogenous Fertility and the Henry George Theorem; *Journal of Public Economics 61*, pp. 209-228.

Schweizer, U. (1999): *Vertragstheorie*; Mohr Siebeck, Tübingen.

van Damme, E. (1989): Renegotiation-Proof Equilibria in Repeated Prisoner's Dilemma; *Journal of Economic Theory 47*, pp. 206-217.

van Damme, E. (1989): Stable Equilibria and Forward Induction; *Journal of Economic Theory 48*, pp. 476- 496.

van Damme, E., and W. Güth (1986): A Comparison of Pricing Rules for Auctions and Fair Division Games, *Social Choice and Welfare 3*, 177-198.

Wessels, J.H. (1993): Redistribution from a Constitutional Perspective; *Constitutional Political Economy 4 (3)*, pp. 425-448.

Winter, E. (1989): A Value for Cooperative Games with Levels Structure of Cooperation; *International Journal of Game Theory 18*, pp. 227-240.

Winter, E., and G. Owen (1992): The Multilinear Extension and the Coalition Value; *Games and Economic Behavior 4*, pp. 582-587.

Winter, E., E. van Damme and R. Selten (1990): Alternating Bid Bargaining with Smallest Money Unit; *Games and Economic Behavior 2*, pp. 188-201.

3 Contract Theory and Public Economics

Dieter Bös

3.1 Main research topic

Government activities are often enacted by means of contracts. For this reason, the application of recent developments in modern contract theory to public economics is a promising and important field of research. Principal agent theories, in particular the literature on adverse selection and moral hazard, address the question how to design efficient contractual arrangements between economic parties when they have asymmetric information on relevant variables. Complete contracts, which explicitly take provision for every possible state of the world, play a major role in public economics when it comes to the relation between a regulator on the one hand, and owner and management of a firm on the other: since the regulator is less well informed than the management, he has to design a proper mechanism to elicit the relevant piece of information from the agent. This theory of regulation has been applied to public and to private firms, in particular to privatized firms. It has even been the basis of various sophisticated theories of privatization, where a public firm is described by a two-stage principal-agent approach (minister and manager), whereas the privatized firm is characterized by a three-stage principal-agent relationship (regulator, private shareholders and manager): in the case of privatization the private shareholders cause an information barrier between regulator and manager, which leads to decisive changes in the incentives of the management. Incomplete contract theory as developed by Grossman-Hart-Moore is of high explanatory value when one can reasonably assume that fully state-contingent contracts are not feasible. For example, in public procurement the relationship-specific investments of a government buyer and a private seller may not be verifiable before a court though observed by both parties. Since the contract cannot be conditioned on effort levels, the players are in subsequent renegotiations not sufficienty rewarded for their investments, and underinvestment potentially arises. Incomplete-contracting theory offers contractual remedies to overcome this hold-up by means of (often) simple arrangements; renegotiation of initial contracts turns out to be a decisive instrument for the achievement of an efficient outcome. This feature is well in line with real-life behavior, but in strong contrast to the usual theory of public economics which considers renegotiations as a sign of mismanagement. Incomplete contracts also play an important role in the theory of fiscal federalism where the relationships among governments of different or identical hierarchy levels are considered. For example, intergovernmental cost-sharing and subsidy schemes play an important role in the governance of political unions. The optimal design of these highly incomplete arrangements is of vital interest for the efficient operation of federal structures.

3.2 Methodological approach

Our studies on asymmetric-information problems (adverse selection and moral hazard) in public finance apply standard methods from principal agent theory. When writing a contract, parties to a relationship (e.g., government and potential private owner in a privatization context, representatives from central and local governments) have to take into account certain informational restrictions. For example, government officials may be unable to observe the effort level that is expended by a public manager, or a firm's production costs may be private information in a regulation context. The task of the model builder is to analyze which contract leads to the best results. Unfortunately, however, complete contracting theory suffers from the problem that institutional issues (as the notions of „ownership“, „political decision rights“ etc.) often cannot usefully be addressed. Under an optimal complete contract, all agents in an economy are connected by a network of contracts that prescribes behavior in any verifiable contingency. For this reason, the recent Grossman-Hart-Moore theory of property rights and incomplete contracts provides a useful tool to overcome these limitations. Under the (empirically very reasonable) postulate that contracts between economic parties cannot be fully state contingent, it can be shown that observed characteristics of relationships between economic agents (certain contract forms, the occurence of renegotiation) as well as a variety of real-life institutional features can be rationalized.

In contrast to standard contract theory that mostly analyzes relationships between private utility-maximizing parties, public economic theory must take stronger normative presumptions. Because the government is an involved player in the relationships we analyze, it is interesting to compare different possible objective functions with regard to their economic implications. In some cases, it is useful to view government officials as pursuing idiosyncratic goals. Often, however, we also suppose that officials behave completely benevolent, i.e., in the interest of their constituency. This modelling choice is particularly appropriate to answer the question whether even a government with the best possible objectives cannot do better (and sometimes even worse) than a private principal.

3.3 Selected results

In the following only contributions are mentioned which are directly concerned with the topic of contract theory and public economics. Other important contributions of our research project (on the theory of taxation and social insurance) are listed in subsection 6.

3.3.1 Principal-agent models on privatization

Bös and Peters (1988) develop a theory of the privatization of public enterprises where the relevant decisions are modelled as a three-stage process comprising a technological management, the board of the firm and a privatization body. It is shown that privatization implies a reduction in the bureaucratization of the firm, but never leads to the optimal extent of control and of the quality of supply. Bös and Peters (1991) present a simple principal-agent model comparing a public and a privatized firm which differ in the objective of the board of the firm (welfare versus profit). They show the particular inefficiencies in the management’s behavior in the public firm. In his book on „Privatization“ Bös (1991) devotes several chapters to principal-agent models of public and privatized firms. Taking account for the fact that incentive schemes in reality often take a linear form, he investigates various types of linear incentive incomes which are conditioned on profit or on the percentage reduction of unit costs. He also discusses managerial incentives in the case of partial privatization. Bös and Peters (1995) analyze inefficiencies that result from team production and from the management’s choice of the size of teams: public firms are more likely to choose a technology where cost inefficiency and shirking in teams reinforce each other. Bös and Harms (1997) present a model on mass privatization. They argue that this form of privatization is in the interest of the incumbent managers of the firm because a greater dispersion of shares reduces his control by the firm's new owners. However, since mass privatization makes the management quasi residual claimant of the profits of the firm, mass privatization leads to an efficient allocation. Finally, Kessler and Lülfesmann (1999)[18] compare the productive efficiency of public and privatized firms. In a

[18] A.S. Kessler and Ch. Lülfesmann (1999): Monitoring and Internal Efficiency: A Comparison of Public and Private Ownership, Discussion Paper No. A-608, Sonderforschungsbereich 303, University of Bonn.

standard adverse-selection model with managerial effort, they first show that equilibrium effort and therefore productive efficiency is higher in a public firm. However, when either principal has access to costly monitoring, a private owner always monitors more frequently than a government. In many situations, the internal efficiency of the firm is then higher under private governance.

3.3.2 Principal-agent models and regulatory policy

In the third edition of his North-Holland Advanced Textbook on „Pricing and Price Regulation“ Bös (1994) introduced a special part on the new economics of price regulation presenting pricing policies which result if a regulator alternatively maximizes welfare or optimizes political or bureaucratic aims in a contract-specific environment with asymmetric information: the manager of the firm is better informed about costs or demand than the regulator. Soft budget constraints are modelled in a similar environment with a lobbyist taking the role of the agent of the regulator who wants the parliament to spend more money for the regulated firm.

3.3.3 Incomplete contracts and public procurement

Bös and Lülfesmann (1996) consider a two-period public procurement model where government and private supplier can expend cost-reducing or value-enhancing effort at the first stage, and actually produce and trade at the second stage. This paper shows that a first best result can be attained if the shadow costs of public funds are negligibly low or if the government procurement agency ex-ante can commit not to distort the supplier's ex-post profits. The contract must be written in such a way that regardless of the underlying supports of costs and benefit distributions renegotiation inevitably occurs in some states of nature. This renegotiation always increases the ex-ante fixed trade price. In a follow-up paper Bös and Lülfesmann (1996)[19] confine attention to one-sided specific investments of the private supplier, but suppose that the quality of the delivered good is unverifiable ex ante. Now, public procurement is in general unable to implement a first-best outcome, and underinvestments prevail. In contrast, a profit-maximizing seller (private procurement) always attains the efficiency goal. This result shows that welfare-maximizing behavior can lead to a suboptimal outcome in a multi-stage game.

According to a recent judgement of the European Commission in public procurement award and actual contract will have to be separated. This separation has both positive and negative consequences. Bös and Kolmar (2000)[20] deal with the positive consequence that an inefficient award can be corrected in the time interval between award and contracting. The authors show that efficiency can be increased by post-award, pre-contract negotiations between the award-winning seller and one of the ‚losing‘ sellers even if one allows for pre-award negotiations. The efficiency gains can be higher if the award is given to an inferior seller instead of to the seller with the highest reputation for quality (!). Second, the authors show that under certain conditions post-award, pre-contract rent-seeking activities also increase efficiency. This is always the case if the procurement agency is corrupt, but may also occur in the case of lobbying. Bös (1999c)[21] is devoted to a particular negative consequence, namely a double inefficiency which is caused by the separation between award and contract: (i) Since the award

[19] Bös, D. and Lülfesmann, Ch. (1996): Holdups, Quality Choice and the Achilles´ Heel in Government Contracting, Discussion Paper No. A-481, Sonderforschungsbereich 303, University of Bonn.

[20] Bös, D. and Kolmar, M. (2000): Self-Correcting Mechanisms in Public Procurement: Why Award and Contract Should be Separated. Mimeo, University of Bonn.

[21] Bös, D. (1999c): Inefficient R&D in Public Procurement: Negative Consequences of a Separation Between Award and Actual Contract, CESifo Working Paper 208, München.

does not definitely determine the contractor, not only the award-winning seller has an incentive to relationship-specific investments, but also other potential sellers who see a chance to gain the contract in a law suit. However, all relationship-specific investments of sellers who do not get the contract are pure waste. (ii) Since it is not a priori clear whether the agency will sign the contract with the award-winning seller or with some competitor, the sellers do not get the correct incentives for the efficient extent of relationship-specific investments.

3.3.4 Incomplete contracts in privatization and regulation

Lülfesmann (1998)[22] analyzes the optimal privatization decision of a rational and welfare-maximizing government. He shows that privatization generates a credible commitment not to accept high costs, and privatization leads to efficiency gains when the firm's future survival does not critically depend on the invention of a superior production technology. Bös (1999a) applies the incomplete-contract approach to a hold-up problem in the case of privatization in a transition economy: if an enterprise is restructured prior to privatization, underinvestment in restructuring is very likely. Efficient restructuring can be guaranteed if this restructuring is performed by either the buyer or by a cash-revenue maximizing privatization agency. Both-sided efficient restructuring can never be achieved. Bös (1999b) deals with price regulation of a monopolistic distribution grid which sells a license to some retailer. The regulator aims at attaining efficient sale of the license and efficient relationship-specific investments of the agents. The first best can be achieved by a sequential regulatory mechanism which gives the seller an option to grant the license but allows the buyer to make counteroffers. This sequential mechanism runs counter to the usual price-cap idea since possible upward but never downward renegotiation of the regulated prices is the vehicle to attain the first best.

3.3.5 Incomplete contracts in health economics

Bös and DeFraja (1998)[23] study the effects of non-contractability of investment on the choices made by a health authority and the hospital with which it contracts for the provision of a specific service. The authors deal with a situation where the parties do not sign a contract before making their investment choices. For this reason inefficiency emerges: compared with any Pareto efficient outcome, the quality of the service chosen by the hospital is too low, and the health authority relies too much on outside providers.

3.3.6 Property rights theory and fiscal federalism

Lülfesmann (1999)[24] applies the property rights theory to the question of whether political decisions should be decentralized. In a model where regions can at a first stage expend effort that increases the expected valuation of policy projects which can subsequently be implemented, centralized and decentralized governance induce too little effort when each region bears the costs of its policies. Conversely, when linear matching grants are fixed at a constitutional prestage, subsidiarity implements efficient decisions even if policies give rise to externalities on other regions. In contrast, under centralized governance cost sharing has no positive effect when the implementation of policies requires the unanimous consent of all regions, and in general fails to reach the efficiency frontier under majority rule.

[22] Lülfesmann, Ch. (1997): When Should We Privatize? An Incomplete-Contracts Approach, Discussion Paper No. A-547, Sonderforschungsbereich 303, University of Bonn.
[23] Bös, D. and DeFraja, G. (1998): Contracts for Health Services: Quality versus Excess Capacity, Discussion Paper No. A-578, Sonderforschungsbereich 303, University of Bonn.
[24] Lülfesmann, Ch. (1999): Central Governance or Subsidiarity: A Property-Rights Appproach to Federalism. Mimeo, University of Bonn.

3.4 Open problems

3.4.1 Principal-agent models with multi-dimensional private information

The usual literature from Mirrlees‘ income-tax model to Laffont-Tirole‘s procurement and regulation models only deals with one private-information parameter. In fact, if there is more than one continuous private-information parameter, there is (yet) no fully satisfactory solution approach. In the recent literature (for instance Armstrong or Rochet and Choné) either strong assumptions on consumers' utilities and on the distribution of the various private-information parameters are imposed or the optimal solution of a multidimensional screening problem is characterized by adapting the notion of sweeping operator used in potential theory; however, this characterization result is not constructive: it does not tell how to find the optimal solution.

3.4.2 Privatization in transition economies

There is yet no fully satisfactory theory of privatization in transition economies. Such theories should start from the practice of renegotiations of privatization contracts to develop a more advanced theory of renegotiation, getting rid of the many ad-hoc assumptions which characterize the present theorizing on the subject.

3.4.3 Incomplete contracts and fiscal federalism

The theory of incomplete contracts has been developed as a part of the theory of industrial organization and, therefore, at the present state almost exclusively deals with contracts between private parties. Contracts between various public partners, such as in fiscal federalism, have only rarely been investigated and offer an important field for future research.

3.5 Selected Publications

Bös, D. (1985): Public Sector Pricing, *Handbook of Public Economics*, A. Auerbach, M. Feldstein (eds.), *Vol. I,* North Holland, Amsterdam - New York - Oxford, 129-211.

Bös, D. (1986): Public Enterprise Economics: Theory and Application*; Advanced Textbooks in Economics, Vol. 23*, North Holland, Amsterdam, New York, Oxford. 2. Auflage 1999.

Bös, D. (1987): Privatization of Public Enterprises; *European Economic Review 31*, pp. 352-360.

Bös, D. (1988): Recent Theories on Public Enterprise Economics; *European Economic Review 32*, pp. 409-414.

Bös, D. (1991): *Privatization: A Theoretical Treatment*; Oxford University Press, Oxford.

Bös, D. (1993): Privatization in Europe: A Comparison of Approaches; *Oxford Review of Economic Policy 9 (1)*, pp. 95-111. Nachdruck in: E.E. Bailey und J. Rothenberg Pack (Hsg.): Privatisation in the European Union.Theory and Policy Perspectives. Lond-New York 1998, S. 49-69.

Bös, D. (1994): Pricing and Price Regulation: An Economic Theory for Public Enterprises and Public Utilities; *Advanced Textbooks in Economics, Vol. 34,* Elsevier/North Holland, Amsterdam, New York, Oxford.

Bös, D. (1997): Privatization in Eastern Germany: The Never-Ending Story of the Treuhand. In: H. Giersch (ed.): *Privatization at the End of the Century*, Springer, Berlin-Heidelberg, 175-197.

Bös, D. (1999a): Privatization and Restructuring: An Incomplete - Contract Approach; *JITE (Journal of Institutional and Theoretical Economics) 155*, pp. 362-382.
Bös, D. (1999b): Incomplete Contracting and Price Regulation; *Journal of Public Economics 73,* pp. 353-371.
Bös, D. (2000): Earmarked Taxation: Welfare versus Political Support; forthcoming in: *Journal of Public Economics.*
Bös, D., and S. Cnossen (eds.) (1992): *Fiscal Implications of an Aging Population*; Springer-Verlag, Berlin-Heidelberg-New York-Tokyo.
Bös, D. and P. Harms (1997): Mass Privatization, Management Control and Efficiency. *Journal of Public Economics 64,* 343-357.
Bös, D., and C. Lülfesmann (1996): The Hold-up Problem in Government Contracting; *Scandinavian Journal of Economics 98*, pp. 53-74; Nachdruck in: *Journal of Financial Management and Analysis 9* (1996), pp. 21-34.
Bös, D., and L. Nett (1990): Privatization, Price Regulation, and Market Entry - An Asymmetric Multistage Duopoly Model; *Journal of Economics / Zeitschrift für Nationalökonomie, Vol. 51*, pp. 221-257.
Bös, D., and L. Nett (1991): Employee Share Ownership and Privatization: A Comment; *Economic Journal 101*, pp. 966-969.
Bös, D., and W. Peters (1988): Privatization, Internal Control and Internal Regulation; *Journal of Public Economics 36*, pp. 231-258.
Bös, D., and W. Peters (1991): A Principal-Agent Approach on Manager Effort and Control in Privatized and Public Firms; in: Ott, A., and K. Hartley (eds.), *Privatization and Economic Efficiency*, Aldershot, UK, pp. 26-52. Nachdrucke in: V. Wright und L. Perotti Hsg.): Privatization and Public Policy, Aldershot, forthcoming; P. Cook und C. Kirkpatrick (Hsg.): Privatisation in Developing Countries, Aldershot, forthcoming.
Bös, D., and W. Peters (1995): Double Inefficiency in Optimally Organized Firms; *Journal of Public Economics 56*, pp. 355-375.
Bös, D. and Ch. Seidl (eds.) (1986): Welfare Economics of the Second Best, *Journal of Economics/ Zeitschrift für Nationalökonomie*, Supplementum 5.
Bös, D. and G. Tillmann (1985): An 'Envy Tax': Theoretical Principles and Application to the German Surcharge on the Rich, *Public Finance / Finances Publiques 40*, 35-63.
Bös, D., and G. Tillmann (1989): Equitability and Income Taxation; in: D. Bös and B. Felderer (eds.), *The Political Economy of Progressive Taxation*, Berlin-Heidelberg-New York-Tokyo, pp. 75-99.
Bös, D., and R.K. von Weizsäcker (1989): Economic Consequences of an Aging Population; *European Economic Review 33*, pp. 345-354.
Bös, D., and H.G. Zimmermann (1987): Maximizing Votes under Imperfect Information, *European Journal of Political Economy 3*, pp. 523-553.
Corneo, G. (1995): Social Custom, Management Opposition, and Trade Union Membership; *European Economic Review 39*, pp. 275-292.
Corneo, G. and Grüner, H.-P. (2000): Social Limits to Redistribution*; American Economic Review*, forthcoming.
Corneo, G. and O. Jeanne (1997): Snobs, Bandwagons, and the Origin of Social Customs in Consumer Behavior; *Journal of Economic Behavior and Organization 32*, pp. 333-348.
Corneo, G. and O. Jeanne (1997): Conspicuous Consumption, Snobbism and Conformism; *Journal of Public Economics 66*, pp. 55-71.
Corneo, G. and O. Jeanne (1998): Social Organization, Status, and Savings Behavior; *Journal of Public Economics 70,* pp. 37-51.
Ebert, U. (1987): Size and Distribution of Incomes as Determinants of Social Welfare, *Journal of Economic Theory 41*, pp. 23-33.
Ebert, U. (1988): A Family of Aggregative Compromise Inequality Measures; *International Economic Review 29*, pp. 363-376.
Ebert, U. (1988): Rawls and Bentham Reconciled; *Theory and Decision 24*, pp. 215-223.

Ebert, U. (1988): Measurement of Inequality: An Attempt at Unification and Generalization; *Social Choice and Welfare 5,* pp. 147-169.

Ebert, U. (1992): A Reexamination of the Optimal Nonlinear Income Tax, *Journal of Public Economics 49,* pp. 47-73.

Gaube, T. (2000): When Do Distortionary Taxes Reduce the Optimal Supply of Public Goods?; *Journal of Public Economics,* forthcoming.

Janeba, E. (1995): Corporate Income Tax Competition, Double Taxation Treaties, and Foreign Direct Investment; *Journal of Public Economics 56,* pp. 311-325.

Janeba, E. (1996): Foreign Direct Investment under Oligopoly: Profit Shifting or Profit Capturing?, *Journal of Public Economics 60,* pp. 423-445.

Janeba, E., and W. Peters (1999): Tax Evasion, Tax Competition and the Gains from Non-Discrimination: The Case of Interest Taxation in Europe; forthcoming in *Economic Journal.*

Keuschnigg, Ch. (1991): The Transition to a Cash Flow Income Tax; *Swiss Journal of Economics and Statistics 127*, pp. 113-140.

Kolmar, M. (2000): Optimal Intergenerational Redistribution in a Two-Country Model with Endogenous Fertility; *Public Choice,* forthcoming.

Lang, G. (1990): Intergenerational Contracts and Their Decomposition: An Extension; *Journal of Economics / Zeitschrift für Nationalökonomie, Vol. 52*, pp. 177-189.´

Lang, G. (1992): Dynamic Efficiency and Capital Accumulation; *European Journal of Political Economy 8,* pp. 153-174.

Nett, L. (1993): Negative Effects of Competition in a Medical-Service Market; *Economics Letters 40*, pp. 481-485.

Nett, L. (1994): Why Private Firms are More Innovative than Public Firms; *European Journal of Political Economy 10*, pp. 639-653.

Nett, L. (1995): Tax Authority to the European Parliament?; *Public Choice 82*, pp. 341-357.

Peters, W. (1988): Cost Inefficiency and Second Best Pricing; *European Journal of Political Economy 4*, pp. 29-44.

Peters, W. (1988): A Pension Insurance System in an Overlapping Generations Model; *JITE (Journal of Institutional and Theoretical Economics) 144*, pp. 813-830.

Peters, W. (1991): Public Pensions in Transition - An Optimal Policy Path; *Journal of Population Economics 4*, pp. 155-175.

Peters, W. (1995): Public Pensions, Family Allowances and Endogenous Demographic Change; *Journal of Population Economics 8*, pp. 161-183.

Tillmann, G. (1985): Existence and Stability of Rational Expectation-Equilibria in a Simple Overlapping Generation Model, *Journal of Economic Theory 36*, 333-351.

Tillmann, G. (1989): Equity, Incentives, and Taxation; *Lecture Notes in Economics and Mathematical Systems, Vol. 329*, Springer-Verlag Berlin-Heidelberg-New York-Tokyo.

von Weizsäcker, R.K. (1989): Demographic Change and Income Distribution; *European Economic Review 33,* pp. 377-388.

von Weizsäcker, R.K. (1993): *A Theory of Earnings Distribution*; Cambridge, UK.

von Weizsäcker, R.K. (1995): Public Pension Reform Demographics and Inequality; *Journal of Population Economics 8*, pp. 205-221.

von Weizsäcker, R.K. (1996): Educational Choice, Lifetime Earnings Inequality, and Conflicts of Public Policy, *Journal of Income Distribution 6*, pp. 67-89.

von Weizsäcker, R.K. (1996): Distributive Implications of an Aging Society, *European Economic Review 40,* pp. 729-746.

4 Systems of Local Interaction

Avner Shaked

4.1 Main research results

Research in this project centered on the following topics:

1. Evolution of cooperation in structured populations
2. Creation of social networks
3. Analysis of imitation as a learning method
4. Evolution of play in extensive games

Like many economists, biologists and sociologists, we were also intrigued by the existence of cooperation among individuals. Why should an individual help another when helping is costly? Game theory explains cooperation by reciprocation in a repeated interaction: Individuals can punish and reward other individuals' past behavior. This, as is well known, can lead to cooperation for fear of punishment. Biologists explain that cooperation has begun in the family, kin selection is advantageous to the gene. As a result an individual will help his kin to a degree depending on how closely they are related.

We provide a different explanation for cooperation which does not require the individuals to be related (as in the biological explanation) nor do we require that they possess a sophisticated memory (as assumed by the theory of repeated games). We show how cooperation may emerge in a population endowed with a local interaction structure, in which individuals are boundedly rational and in which they react to changes in their environment by imitating their more successful neighbors.

Our starting point is that individuals are either not fully rational or they are not fully informed about their environment. Thus, they are unable to use pure analytic methods to deduce which action is the best for them under the current circumstances. Instead, we assume that individuals can observe the actions and the payoffs of other individuals. In the absence of a theory about what they should do, they simply imitate the action taken by the more successful of the individuals they observe. Imitation is a very natural way to learn, infants learn by imitating the behavioral patterns of their parents, and imitation remains an important source of learning throughout human (and animals') adult life.

Our main assumption is that the population is not fully mixed (panmictic), an individual meets only a subgroup of the population: his neighbors. Thus, each individual operates in a different environment, and two individuals who use the same strategy may earn a different payoff. A group of cooperators who support each other will earn a higher payoff than a group of non cooperators. If individuals change their behavior by imitating their more successful neighbors, then cooperation may spread in the population.

4.2 Methodological approach

We, mostly, used pure analysis in our research. Most studies of structured populations (populations with a local interaction structure) use simulations, we have succeeded to prove our results analytically, albeit for simple models. To test our results in more elaborate models we used simulations.

We have also run experiments testing the robustness of the backward induction solution concept (we do not report on it ihere since the paper is still in refereeing process).

4.3 Selected results

4.3.1 Evolution of cooperation in structured populations

In a series of papers we investigated the evolution of structured populations under various imitationn patterns. Imitation can be deterministic (imitating the strategy whose average payoff is the highest) or stochastic (choosing whom to imitate is stochastic, the probability of choosing someone increases with their payoff). We have shown that under these conditions cooperation emerges as an evolutionarily stable mode of behavior and that it resembles kin selection. Each individual will cooperate with his neighbors to a degree which is related to the size of the neighborhoods, the size of the neighborhoods plays a role similar to the degree of kinship in kin-selection.

4.3.2 Creation of social networks

In the models described above, the local interaction structure (the social network) was taken as given. A natural question arises in this context: How is the network created? Towards the end of the project we approached this problem. The environment of an individual may be formed by a search process, he may look for suitable partners. We investigated how the search policy of the individuals leads to a formation of different networks. We have taken an example of a labor market in which each firm seeks a worker and each worker a firm. The workers may be skilled or less skilled and in addition they are either green or red. Their color bears no relation to their skill level. There is clearly an equilibrium in which firms, when looking for a worker, ignore his color and look for any worker. However, there is an equilibrium in which firms look only for a red worker, although they will bargain with a green worker who has found them and will give him his market value. The social network created by this equilibrium is discriminatory, there will be less green workers in the industry and their wages will be lower.

4.3.3 Analysis of imitation as a learning method

We have mentioned before that imitation is a very natural method of learning, however, imitation can take many forms. In a series of papers, Karl Schlag discovers imitation rules that are optimal in certain environments.

Schlag begins with a population facing a multi-armed bandit. A new individual replaces one who just died in a population, he learns the action and payoff of the deceased and observes the action and payoff of another individual. Schlag shows that the following imitation rule is optimal for the individual: He should not switch to the other's strategy if it earned a lower payoff than his (inherited) strategy. If the other's strategy has a higher payoff then he should switch to it with probability proportional to the difference in payoffs.

Schlag continues his analysis by showing that if all individuals adopt this imitation rule then the population will evolve according to the replicator dynamics. The replicator dynamics is derived from Darwinian fitness considerations and a great deal is known about it. It is therefore useful to know that a natural learning rule leads to the same dynamics. In addition, Schlag shows that the optimal rule leads the population to a desirable outcome, all individuals will learn to use the best arm of the bandit.

Monotone dynamics are particularly liked in the literature of evolutionary economics. In a monotone dynamic a strategy which currently earns a high payoff will become more popular in the population compared to a strategy with a current lower payoff . Schlag shows (in a model with two distinct populations) that the only learning rules leading to a monotonic dynamic are imitation rules. No other learning rules including the ones in which individuals use extensive information about their environment, lead to monotonic dynamics.

4.3.4 Evolution of play in extensive games

In all the works mentioned above it was assumed that an individual may observe the strategy of another and can therefore imitate it. This is not necessarily the case when the interaction between individuals takes the form of an extensive game. There, an individual may observe only that part of the strategy that was executed, he may not be able to have access to the other's global plan, of which only a part was carried out.

Nöldeke & Samuelson analyze an extensive game in which players have strategies and conjectures about others, their learning rule is the best response rule and they update their strategies and beliefs according to what they actually observed. Nöldeke & Samuelson analyze under what condition the population will converge to one of two types of equilibria the celebrated backward induction solution and the forward induction solution.

Schlag continues with his investigation of imitation rules, and considers a 'parallel' bandit: A random bandit chosen by a lottery. Each individual has a global plan telling him which arm he should choose in each bandit. Schlag tests his imitation rules in this environment and shows that some will lead the population to playing the correct arms in all bandits.

4.4 Open problems

Through our research we have developed useful intuition concerning evolution of behavior in populations with a local interaction structure. Although our analytic methods and results are not directly applicable to more elaborate networks, they do point out a direction in which the population will evolve. Using simulations we have tested these predictions in a number of models and confirmed it.

An interesting open question in this field is how to model the formation of networks, we have modestly begun working on it in a model of a labor market, but there are many possible generalizations, each suitable to other applications of the theory to economic and social network.

4.6 Selected Publications

Eshel, I., D. Samuelson and A. Shaked (1997): Altruists, Egoists, and Hooligans in a Local Interaction Model; *American Economic Review* 88, pp. 157-179

Eshel, I., D. Herreiner, L. Samuelson, E. Sansone, and A. Shaked (2000): Cooperation, Mimesis and Local Interaction; *Sociological Methods and Research*, forthcoming

Eshel, I., E. Sansone, and A. Shaked (1999): The Emergence of Kinship Behavior in Structured Populations of Unrelated Individuals; *International Journal of Game Theory*, 1999, 28 (4), 447-463.

Herreiner, D. (1999): Local Interaction as a Model of Social Interaction? The Case of Cooperation; T. Brenner (ed.); Computational Techniques for Modelling Learning in Economics; in: *Advances in Computational Economic Series*, Boston; Kluwer Academic Publishers, pp. 221-239

Herreiner, D., and A. Sela (1999): Fictitious Play in Coordination Games; *International Journal of Game Theory*, 28 (2) pp. 189-197

Mailath, G.J., L. Samuelson, and A. Shaked (1997): Correlated Equilibria and Local Interactions; *Economic Theory* 3 (9) pp. 551-556

Mailath, G.J., L. Samuelson, and A. Shaked (1999): Endogenous Inequality in Integrated Labor Markets with two-sided Search; *American Economic Review*; forthcoming

Monderer, D., and A. Sela (1996): A 2x2 Games without the Fictitious Play Property; *Games and Economic Behavior* 14, pp. 144-148

Monderer, D., D. Samet, and A. Sela (1997): Belief Affirming in Learning Processes; *Journal of Economic Theory* 2, pp. 38-452

Nöldeke, G., and L. Samuelson (1993): An Evolutionary Analysis of Backward and Forward Induction; *Games and Economic Behavior* 5, pp. 425-454

Nöldeke, G. and L. Samuelson (1997): A Dynamic Model of Equilibrium Selection in Signaling Markets; *Journal of Economic Theory* 73 (2) pp. 118-156

O'Neill, B. (1999): Honor Symbols and War; University of Michigan Press; forthcoming

Pollock, G. (1999): Evolutionary Games and Computer Simulation in Studies of Social Structure; *Sociological Methods and Research*; forthcoming

Pollock, G. (1999): Evolutionary Games and Social Structure; *Rationality and Society*; forthcoming

Pollock, G., and K. Schlag (1999): Social Roles as an Effective Learning Mechanism; *Rationality and Society*; forthcoming

Schlag, K. (1998): Why imitate, and if so, How? A Boundedly Rational Approach to Multi-armed Bandits; *Journal of Economic Theory* 78 (1) pp. 130-156

Schlag, K. (1998): On the Dynamic (In)Stability of Backward Induction; *Journal of Economic Theory* 83, pp. 260-285

Schlag, K. (1998): Which one Should I Imitate? *Journal of Mathematical Economics* 31 (4) pp. 493-522

Schlag, K., and A. Sela (1998): You Play (an Action) Only Once; *Economic Letters* 59 (3) pp. 299-303

5 Econometric Analysis of Time Variable and Feedback Systems

Peter Schönfeld

5.1 Main research topics

The project mainly investigated (1) stochastic dynamic systems under incomplete information, (2) matrix theory and the generalized linear regression model, and (3) information and transformation processes in macro-economic systems.

5.2 Methodological approach

5.2.1 Stochastic dynamic systems under incomplete information

In stochastic dynamic systems contributions to systems with unknown parameters and to systems with endogenous formation of expectations and prediction feedback have been made.

Methodologically, the martingale convergence theory and its generalizations to semi-martingales and almost-supermartingales served as a unifying instrument to tackle quite different problems of stochastic dynamic systems. This approach was applied, e.g., to prove the strong consistency of the least-squares estimator in the general framework of time-continuous semi-martingale models and of Kruskal´s monotone regression least-squares estimator in non-parametric regression models. Mainly, however, it was used to investigate the convergence properties of systems with endogenous expectations and prediction feedback.

Systems with endogenous formation of expectations and prediction feedback are complex due to their self-referential character. In self-referential models the agents learn about the relations among the economic variables but these relationships by themselves are affected by the expectations. The relations observed by the agents change permanently as long as the agents are learning and changing their behaviour in the light of what they learnt. In the period of learning the observed relations coincide neither with the equilibrum relationships nor with the expectations. In the realistic set-up of bounded rationality agents are always in error. Nonetheless learning procedures can converge to rational expectations equilibria asymptotically (in the long run). The usual approaches lead to a dynamic stochastic system with time variable stochastic parameters that are determined by learning schemes of the type of adaptive algorithms (e.g. least-squares learning, stochastic gradient algorithm, self-tuning regulator learning). Rational expectations equilibria are defined by the fixed points of the feedback functions that characterise the system. The main subject of investigation was the analysis of the (almost sure) convergence of the parameter process to such fixed points. Two cases of different complexity can be distinguished. In the “static“ case lagged endogenous variables are absent from the structural equations of the system. The stochastic system becomes dynamic, in an intricate non-linear and non-stationary way indeed, only through the endogenous formation of expectations and its impact on the endogenous process via the prediction feedback. Still more complicated is the “dynamic“ case where the presence of lagged endogenous variables introduces a second source of stochastic dynamics into the system. While the “static“ case can be analysed by the martingale approach it turns out that a direct algebraic approach is much more natural. This is the algebraic stochastic approximation approach by Walk (1985)[17],

[17] Walk, H. (1985): Almost Sure Convergence of Stochastic Approximation Processes; *Statistics and Decisions*,Supplement Issue No. 2, pp. 137-142.

Walk/Zsidó (1989)[18]. It exploits only Banach space properties and can be generalized to be applicable to the present framework. This method led to a definitive general theory of the “static“ case. In the “dynamic“ case it appears that only a genuinely stochastic approach will do. Essentially, the reason is that the Borel-Cantelli “0-1“ world does not apply any longer. There can be convergence with positive probability and divergence with positive probability at the same time. Our analysis of the "dynamic" case rests on convergence results for almost-supermartingales, Robbins/Siegmund (1971)[19]. In contrast, the well-known contribution by Marcet/Sargent (1989)[20] adopts Ljung´s ordinary differential equation approach. Ljung´s approach applies to the “static“ case as well as to the “dynamic“ case. But it relies on a "projection device" for appropriate correction of the learning procedure and thus requires insight into the model structure that barely can be expected from agents .

5.2.2 Matrix theory and the generalized linear regression model

In matrix theory and in the generalized linear regression model considerable progress has been made. The various concepts of generalized matrix inversion, matrix monotonicity and matrix order relations [see, e.g., Werner (1986), (1991), Jain/Mitra/Werner (1996)] proved instrumental in developing the linear model into a highly diversified central subject of recent statistical research. The careful classification of the various types of generalized inverses and the precise characterization of their rôle in oblique and orthogonal projections onto subspaces along other subspaces has led to a powerful algebraic-geometric method to analyse the general linear model.

5.2.3 Information and transformation processes in macro-economic systems

In macro-economic modelling a basic objective of research was to improve the performance of econometric forecasting models by including latent variables like intentions, moods, expectations, waves of optimism or pessimism, etc. Usually, observable indicators are available for these latent variables. The so-called PLS method by Herman Wold, the LISREL method by Karl Jöreskog and Kalman filter methods have been analysed theoretically and applied in practice.

As a consequence of the actual political and economic situation after the breakdown of the communist system and the reunification of Germany the process of the transition from a planned economy to a market economy became an important object of study. The process of transition was analysed by use of special models on the base of neoclassical growth theory and control theory.

5.3 Selected results

5.3.1 Stochastic dynamic systems under incomplete information

In Christopeit (1986), Christopeit/Tosstorff (1987) least-squares estimation in linear and monotone regression models was generalized to a framework that includes the class of auto-

---

[18] Walk, H., and L. Zsidó (1989): Convergence of the Robbins-Monro Method for Linear Problems in a Banach Space, *Journal of Mathematical Analysis and Applications 139*, pp.152-177.

[19] Robbins, H., and D. Siegmund (1971): A Convergence Theorem for Non Negative Almost Supermartingales and Some Applications; in: F.S. Rustagi (ed.), *Optimizing Methods in Statistics*, Academic Press, New York, pp. 233-257.

[20] Marcet, A., and T. J. Sargent (1989): Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models, *Journal of Economic Theory 48*, pp. 337-368.

regressive models and thus can serve to analyse control and filter theoretic problems in these models. In particular, the framework in Christopeit (1986) comprises time continuous semi-martingale models and thus provides a base for estimation in many financial market models as analysed by project group B3. The monotone regression paper arose in part from problems of nonparametric estimation of Engel curves (project group A3). Here, however, kernel estimation has proved more fruitful and the monotone regression problem has not attracted much recent attention. Contributions have also been made to the adaptive control of stochastic dynamic systems, e.g. Christopeit (1994b), using methods of stochastic asymptotics and stochastic analysis. Methodologically similar but thematically quite apart is a contribution to the estimation of extreme value distributions [Christopeit (1994a)] which, e.g., are pertinent to assessing earthquake risks.

In self-referential systems, in the "static" case of absence of lagged endogenous variables from the system, a complete and definitive solution for least-squares learning was obtained for the framework of stationary-ergodic exogenous variables and errors and linear feedback functions, Kottmann (1989), (1990, Diss.), Mohr (1990, Diss.). In this framework, quite generally, variables may be multivariate and expectations may be about endogenous variables from different periods. The manifolds of rational expectations equilibria and of the limit points of least-squares learning as well as the conditions of their coincidence have been explored fully for various scenarios within this framework. The analysis is based on Walk´s algebraic stochastic approximation approach that had to be generalized to comply with the present framework. In the "dynamic" case where lagged endogenous variables are among the explanatory variables of the system important special cases of the convergence problem could be solved. The most comprehensive results refer to the "purely autoregressive" case where no uncontrollable (exogenous) variables enter the system. An interesting contribution is to the super-rational learning approach. In this approach, agents know the structure of the model but are uncertain about parameters. They can use their expectations as control variables in order to achieve minimal prediction error. This leads to a unique particular rational expectations equilibrium called the optimal expectations equilibrium. It was shown that in situations of multiple rational expectations equilibria the modified self-tuning regulator learning converges to the optimal expectations equilibrium, Kottmann (1990). In the "dynamic" case within the framework of linear feedback functions the strong consistency of the parameter process for a rational expectations equilibrium could be shown under a persistent excitation condition on the exogenous inputs and some stability conditions. As yet general results are only available for the stochastic gradient learning algorithm and, partly, depend on endogenously determined regularity conditions [Zenner (1996)]. Nevertheless these results are among the most general available and, e.g., lend themselves to analyse, for the first time in full rigour, a realistic version of the well-known Cyert/DeGroot (1974)[21] model. Although there is ample evidence from special cases and Monte Carlo studies that the least-squares algorithm will grossly outperform the stochastic gradient algorithm as a convergent learning procedure the problem of convergence is still unsettled for least-squares.

5.3.2 Matrix theory and the generalized linear regression model

It is known since long that in the linear model of statistics all the information apart from the sample information can be gained by inversion of a partitioned matrix (IPM) - the so called fundamental matrix of the linear model. C. R. Rao extended this result to singular models using generalized inverses. In Werner (1987) remarkable simplifications in the numerics of calculating the IPM generalized inverses are given. Here the concept of weak bicomplementarity proved particularly helpful. Major contributions have also been made to the algebra of the

[21] Cyert, R. M., and M. H. DeGroot (1974): Rational Expectations and Bayesian Analysis, *Journal of Political Economy 82*, pp. 521-536.

inequality constrained generalized least squares (ICGLS) problem. Inequality constraints on parameters are frequently among the a priori specifications of the regression model and should be used in the estimation of the parameters. In Werner/Yapar (1996a) a closed-form representation of the ICGLS selections was achieved under very general assumptions (nearly without rank conditions). These results vastly generalize the results by Firoozi (1990)[22]and Werner (1990b). As it is shown how ICGLS selections are related to the unconstrained GLS selections and to the equality constrained GLS selections further research on this topic is to be expected. Methodologically, some of these results were used in Werner/Yapar (1995) to generalize results by Aigner/Balestra (1988)[23] and Nurhonen/Puntanen (1992)[24].

Various contributions have been made to the theory of best linear unbiased estimators (BLUE) in the general linear model. In Schönfeld/Werner (1987) the class of BLUE representations was extended to a general class of nontraditional estimators. In Werner/Yapar (1996b) a new BLUE decomposition and properties of the dispersion matrix of the BLUE were given. In Puntanen/Styan/Werner (1998) two new proofs of the BLUE property were presented. One of them is based on a projector theoretic approach which was already adopted in Werner/Yapar (1996a) and could prove useful in unifying the theory of testing "non-testable" hypotheses.

5.3.3 Information and transformation processes in macro-economic systems

In the analysis of information processes by modelling latent variables such as intentions, moods, expectations, waves of optimism and pessimism, as a major achievement, the practice of Kalman filtering has been improved sensibly. The time dependent parameter estimates gained from Kalman filtering improved the forecasting abilities of the macro-econometric models considerably. LISREL proved superior to the PLS method. Also errors-in-variables methods were applied to latent variable models. Like the PLS method they did not prove very encouraging due to their sensitivity to misspecification. For details see Kirchen (1988), Krelle (1989a). As an application of latent variables, technical progress being the main determinant of long-term cycles of economic growth was considered in detail. The basic assumption was that the rate of technical progress may be explained by the state of technical knowledge in the society and the transfer of this knowledge into the economic usage. These, in turn, depend on the intensity of innovations and the flexibility of the organizational structure. Indicators for these latent variables were used to estimate parameters by the method of MIMIC and DY-MIMIC, see Badke (1990), Krelle (1987).

Concerning the transition of a planned economy to a market economy, it was found that in almost all cases it is optimal to subsidize firms for a while in the process of transition if otherwise they go bankrupt, Krelle (1994). According to neoclassical growth theory, under normal conditions, the backward country would reach the advanced economy only asymptotically. Encouragingly, in the control theoretic approach the backward country could reach the advanced country in finite time, see Ackermann (1998).

5.3.4 Results reflecting cooperation within the Sonderforschungsbereich

[22]Firoozi, F. K. (1990): A transformation of the inequality-constrained linear model; *Linear Algebra Appl. 133*, pp. 153-163.

[23]Aigner, D. J., and P. Balestra (1988): Optimal experimental design for error components models; *Econometrica 56*, pp. 955-971.

[24]Nurhonen, M., and S. Puntanen (1992): A property of partitioned generalized regression; *Commun. Statist.-Theory Meth. 21(6)*, pp. 1579-1583.

There have been quite a few contributions from project group B1 that relate to problems explored in other project groups. Thus, Brennscheidt (1993) studied predictive learning behaviour from the point of view of experimental economics (project group B4). Various contributions are pertinent to the subjects of project group B3. Thus Cron (1997, Diss.) studied the asymptotic properties of kernel estimators in general autoregressive conditionally heteroskedastic models under assumptions appropriate for models of financial markets. Christopeit/Musiela (1994) gave conditions for the existence and nonexistence of arbitrage-free measures in models of markets for contingent claims. Their criteria can serve to eliminate models that are flawed by the nonexistence of an arbitrage-free measure and hence do not allow for a meaningful valuation of the contingent claims. This paper was chosen for reprint as it reflects the strength of the SFB in advanced stochastics.

5.4 Open problems

5.4.1 Stochastic dynamic systems under incomplete information

In the framework of self-referential systems, in the "dynamic" case, a major open question still is the convergence of the least-squares learning algorithm. The Ljung approach circumvents the main technical difficulty by introducing the ad hoc "projection device". This may work well in engineering applications where the practitioner may have a feeling when a system runs out of control and what the appropriate direction of correction should be. Economic agents, in general, will not have this kind of insight into the mechanism of a self-referential system. A solution of the least-squares problem would also be a major contribution to the theory of the self-tuning regulator in stochastic control. Markov theory may be powerful enough to solve this problem. The main obstacle is the utterly unhandy form of the state space representation of self-referential systems under the least-squares recursion.

5.4.2 Matrix theory and the generalized linear regression model

Concerning the linear regression model the stochastic properties of the inequality constrained generalized least squares (ICGLS) estimator remain unsettled. The closed-form representation of the ICGLS selections that has been achieved may be helpful in this context but is still too complex to allow for an easy solution of this long-standing open question in regression theory.

5.5 Selected Publications

Ackermann, M. (1998): Die optimale Angleichung der neuen Bundesländer an die Lebensverhältnisse in Westdeutschland; in: *Dynamische Wirtschaftstheorie Bd. 18*, Peter Lang - Europäischer Verlag der Wissenschaften, Frankfurt/M.

Badke, M. (1990): Eine Theorie des technischen Fortschritts - Empirische und theoretische Untersuchung des gesamtwirtschaftlichen Produktivitätsfortschritts in der Bundesrepublik Deutschland, Verlag Dr. Kovac, Hamburg.

Brennscheidt, G. (1993): Predictive Behavior: An Experimental Study; *Lecture Notes in Economics and Mathematical Systems 403*, Springer-Verlag, Berlin, Heidelberg et al.

Christopeit, N. (1986): Quasi-least-squares Estimation in Semimartingale Regression Models, *Stochastics 16*, 255-278.

Christopeit, N. (1994a): Estimating Parameters of an Extreme Value Distribution by the Method of Moments; *Journal of Statistical Planning and Inference 41*, pp. 173-186.

Christopeit, N. (1994b): Some Comments on a Single Nonlinear Filter with Application to Adaptive Control; *European Journal of Operations Research 73*, pp. 252-264.

Christopeit, N., and G. Tosstorff (1987): Strong Consistency of Least-Squares Estimators in the Monotone Regression Model with Stochastic Regressors, *The Annals of Statistics 15*, 568-586

Christopeit, N., and M. Musiela (1994): On the Existence and Characterization of Arbitrage-Free Measures in Contingent Claim Valuation; *Stochastic Analysis & Applic. 12*, pp. 41-63.

Jain, S.K., S.K. Mitra and H.J. Werner (1996): Extensions of G-based Matrix Partial Orders; *SIAM J. Matrix Anal. Appl. 17*, pp. 834-850.

Kirchen, A. (1988): Schätzung zeitveränderlicher Strukturparameter in ökonometrischen Prognosemodellen; in: *Mathematical Systems in Economics*, Athenäum Verlag, Frankfurt/M.

Kottmann, T. (1989): OLS-Estimation and Rationality in Linear Models with Forecast Feedback; in: N. Christopeit, K. Helmes, M. Kohlmann (eds.), *Proceedings of the 4th Bad Honnef Conference, Springer Lecture Notes in Control and Information Sciences 126,* Springer-Verlag, Berlin.

Krelle, W. (1987): Long-Term Fluctuation of Technical Progress and Growth; in: *Zeitschrift f. die gesamte Staatswissenschaft JITE 143,3*, pp. 379-401.

Krelle, W. (1989a): Growth, Decay and Structural Change; in: W. Krelle (ed.), *The Future of the World Economy*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, pp. 3-37.

Krelle, W. (1989b): Lange Wellen der wirtschaftlichen Entwicklung. Tatsachen und Erklärungen; *Walter-Adolf-Jöhr-Vorlesung 1988,* Forschungsgemeinschaft für Nationalökonomie (ed.), St. Gallen.

Krelle, W. (1990): Latente Variable in ökonometrischen Prognosesystemen; in: G. Nakhaeizadeh and K.-H. Vollmer (eds.), *Neuere Entwicklungen in der angewandten Ökonometrie*, Physica-Verlag, Heidelberg, pp. 1-19.

Krelle, W. (1991): Probleme des Übergangs von einer Planwirtschaft zu einer Marktwirtschaft; in: U. Jens (ed.), *Der Umbau - Von der Kommandowirtschaft zur Ökosozialen Marktwirtschaft*, Nomos, Baden-Baden, pp. 15-34.

Krelle, W. (1991): The Relation Between Economic Growth and Structural Change; in: P. Hackl, and A.W. Westlund (eds.), *Economic Structural Change*, Springer-Verlag, Berlin, Heidelberg, New York, pp. 257-290.

Krelle, W. (1993): A Problem of the Transition from a Planned to a Market Economy: Should Firms Be Subsidized in Order to Avoid Their Bankruptcy?; in: E.W. Diewert, K. Spremann, F. Stehling (eds.), *Mathematical Modelling in Economics - Essays in Honor of Wolfgang Eichhorn*, Springer-Verlag, Berlin, Heidelberg et al., pp. 632-649.

Krelle, W. (1994): On the Subsidization of Firms in the Process of Transition from a Planned to a Market Economy; *OR-Spektrum 16*, pp. 135-144.

Krelle, W. (1996): Entwicklung und Aufrechterhaltung moralischer Standards; in: Ulrich Immenga, Wernhard Möschel, Dieter Reuter (eds.), *Festschrift für Ernst-Joachim Mestmäcker*, Baden-Baden, Nomos Verlagsgesellschaft, 1996, pp. 227-241.

Krelle, W., and H. Sarrazin (1989): Growth of West German Economy: Forecast by the Bonn Modell 11; in: W. Krelle (ed.), *The Future of the World Economy*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, pp. 399-405.

Krelle, W., and M. Ackermann (1994): Sollen Unternehmen während des Übergangs von einer Plan- zu einer Marktwirtschaft subventioniert werden, wenn sie andernfalls in Konkurs gehen würden?; in: Horst Albach (ed.), *Globale Soziale Marktwirtschaft: Ziele - Wege - Akteure*, Gabler (ZfB-Edition), Wiesbaden, pp. 125-139.

Puntanen, S., G.P.H. Styan and H.J. Werner (1998): Two Matrix-based Proofs that the Linear Estimator Gy is the Best Linear Unbiased Estimator. *Journal of Statistical Planning and Inference*, forthcoming.

Schönfeld, P., and H.J. Werner (1987): A Note on C. R. Rao's Wider Definition BLUE in the General Gauss-Markov Model, *Sankhya, Vol. 49 Series B,* pp. 1-8.

Werner, H.J. (1985): Weak Complementarity and Missing Observations, *Methods of Operations Research 50*, 421-439.

Werner, H.J. (1986): Generalized Inversion and Weak Bi-Complementarity, *Linear and Multilinear Algebra 19*, 357-372.

Werner, H.J. (1987): C. R. Rao's IPM Method: A Geometric Approach, in: M.L. Puri et al. (eds.), *New Perspectives in Theoretical and Applied Statistics*, John Wiley & Sons, New York, pp. 367-382.

Werner, H.J. (1990a): A Closed Form Formula for the Intersection of Two Complex Matrices under the Star Order; *Linear Algebra Appl. 140*, pp. 13-30.

Werner, H.J. (1990b): On Inequality Constrained Generalized Least-Squares Estimation; *Linear Algebra Appl. 127*, pp. 379-392.

Werner, H.J. (1991): Some Further Results on Matrix Monotonicity; *Linear Algebra Appl. 150*, pp. 371-392.

Werner, H.J., and C. Yapar (1995): More on Partitioned Possibly Restricted Linear Regression; in: Tiit, E.-M. et al. (eds.), *New Trends in Probability and Statistics, Vol. 3, Multivariate Statistics, Proceedings of the 5th Tartu Conference*, VSP, Zeist, and TEV, Vilnius, pp. 57-66.

Werner, H.J., and C. Yapar (1996a): On Inequality Constrained Generalized Least Squares Selections in the General Possibly Singular Gauss/Markov Model: A Projector Theoretical Approach; *Linear Algebra Appl. 237/238*, pp. 359-393.

Werner, H.J., and C. Yapar (1996b): A BLUE Decomposition in the General Linear Regression Model; *Linear Algebra Appl. 237/238*, pp. 395-404.

Zenner, M. (1996): Learning to Become Rational. The Case of Self-Referential Autoregressive and Non-Stationary Models; in: *Lecture Notes in Economics and Mathematical Systems No. 439*, Springer Verlag, Berlin, Heidelberg, New York.

6 Stochastics of Financial Markets

Dieter Sondermann

6.1 Main Research Topics

6.1.1 Incomplete Financial Markets

This project was a central research topic and has been pursued during the whole research period. It has laid the foundations to a broad new research area with numerous applications in many fields of financial economics, like option pricing, stock price modelling, term structure models, credit risks and risk management.

In the early eighties there existed a well developed theory on option pricing based on self-financing hedging strategies which perfectly replicate the option. By arbitrage arguments the initial investment into the replication portfolio gives the exact price of the option. But this means that the option is redundant, since it can be replicated by the already existing financial instruments. The state of the art was well described by Hakansson: "But if this is the case, the option adds nothing new to the market and no social welfare can arise – the option is perfectly redundant .... So we find ourselves in the awkward position of being able to derive unambiguous values only for redundant assets and unable to value options which do have social value" (Hakansson (1979), Journal of Financial and Quantitative Analysis , p. 723). To overcome this problem was the motivation for the research project on incomplete markets.

6.1.2 Term Structure Models

The second major research topic was centered around the modelling of the term structure of interest rates and the pricing and hedging of interest rate derivatives. Since the mid 80ies the pricing and hedging of interest rate derivative securities has become a key subject in the financial market literature. The essential contribution made by the project B3 is the "log-normal" or "market model" of an interest rate market. For several reasons the article by Miltersen, Sandmann and Sondermann (1997) opened a new approach to term structure models. The so-called market model enjoys now increasing popularity not only by academics but also by financial practioners.

6.2 Methodological approach

6.2.1 Incomplete Financial Markets

In the paper by Föllmer-Sondermann (1986), reprinted in this volume, basic concepts for the study of incomplete financial markets were introduced. The starting point was to shift the emphasis from the value process of the hedge portfolio to the hedge cost process, enlarge the class of admissible portfolio strategies by introducing the concept of 'mean-self-financing', and introduce a global risk-minimizing criterion. For the case where the asset process is considered under a given martingale measure a complete solution for the optimal hedging strategy is obtained via the Kunita-Watanabe decomposition for martingales.

Important contributions to the further development were made by Martin Schweizer, who extended the Föllmer-Sondermann approach from the martingale to the semi-martingale case. This case has turned out to be substantially more subtle and not all questions have been answered yet. A first issue in the semi-martingale case is the precise formulation of the optimality criterion. After showing by a counterexample that the approach of Föllmer/Sondermann no longer works in the semi-martingale case, the early work by Martin Schweizer (1990, 1991) started with local risk-minimization where the objective is to minimize conditionally expected squared cost increments in an asymptotic sense, i. e., over arbitrarily small time intervals. Solving this problem has led to the introduction of the so-called *minimal martingale measure* and a new decomposition for semi-martingales that has become known as the *Föllmer-Schweizer decomposition* (the terminology is due to C. Stricker).

One criticism against local risk-minimization has been that the criterion is rather technical due to its infinitesimal nature. In a second thrust of research, M. Schweizer (1992, 1995) has therefore introduced and developed the concept of *mean-variance hedging* for semi-martingales where the goal is to minimize expected squared total hedging costs over all self-financing trading strategies. The naturally associated valuation is determined by the so-called *variance-optimal martingale measure* and the question of existence of mean-variance optimal strategies has led to new problems and results in stochastic analysis itself. Key techniques used here include BMO theory and weighted norm inequalities; a more detailed overview is given in the financial introduction to Delbaen/Monat/Schachermayer/Schweizer/Stricker (Finance and Stochastics, 1997).

6.2.2 Term Structure Models

The starting point was the development of a simple binomial short rate model (Sandmann-Sondermann (1993)). Instead of the usual modelling of continuously compounded rates this model was formulated in terms of effective rates with discrete compounding. Such rates are applied in practice, but till then were largely ignored by financial economists. It turned out that this shift to effective or to nominal rates (like LIBOR, FIBOR, EURIBOR) was a crucial step for the results obtained later on. The next step was to derive the continuous time limit of the binomial model and to show that contrary to other lognormal models the model had a stable lognormal limit distribution, i. e., rates and rollovers did not explode with shrinking time intervals.

In addition, Sandmann and Sondermann (1997) studied the problems of negative interest rates and the reasons for the instability of traditional lognormal interest rate models. A comparison of the stochastic PDE’s, which determine the interest rate dynamics showed that the problems with the lognormal structure indeed arise from modelling the wrong rate, namely the continuously compounded rate. If nominal or effective rates are lognormally distributed instead, interest rates stay positive, expected rollover returns are finite, i. e., the model is stable, and also Eurodollar Futures can be priced. Two further important steps towards the continuous time market model were the comparative study of zero coupon bond price models by Rady and Sandmann (1994) and the continuous time interest rate model by Sandmann, Sondermann and Miltersen (1995). In the framework of effective rates Sandmann et al. were able to show for the first time that Black’s caplet formula is consistent with an arbitrage free model of the interest market. Circulated since 1994 this paper laid the foundations of the “lognormal” or “market model” of the term structure of interest rates.

6.3 Selected Results

6.3.1 Incomplete Financial Markets

In Föllmer-Sondermann (1986) a complete solution for the hedging problem in incomplete markets is obtained for the martingale case. The Kunita-Watanabe decomposition for martingales allows the separation of the hedgeable risk from the intrinsic risk. There exists a unique globally risk minimizing hedging strategy for non-redundant contingent claims. The concepts developed in this paper have formed the basis of numerous research papers on incomplete markets with applications to many different fields in finance, in particular option pricing and hedging for general price processes, stochastic volatilities, credit risks and risk management.

The concepts of the *minimal martingale measure* and the *variance optimal martingale measure*, first introduced by M. Schweizer, have become standard tools for pricing and hedging in incomplete markets, known as the Föllmer-Schweizer approach. This approach has been developed in a series of papers with increasing generality and is now being widely used by many other authors for diverse applications. One advantage of this approach is that it can be implemented reasonably easily because the ingredients required for the construction of the optimal strategy can be red off almost immediately form the semi-martingale decomposition of Föllmer-Schweizer. The minimal martingale measure has also turned up later in different contexts in other areas of mathematical finance and is often a natural first candidate for derivative pricing in an incomplete setting. Subsequent extensions have replaced the quadratic L^2- by an L^p-criterion and very recent work shows an asymptotic connection to maximizing expected exponential utility. The overall picture that has emerged is one of a natural duality between the choice of a hedging criterion and the selection of a particular martingale measure for option valuation. One major contribution in project B3 has been to recognize this link and to work it out explicitly in a number of cases. More detailed information can be found in recent surveys by Pham and Schweizer which also contain a total of about 75 references.

6.3.2 Term Structure Models

The main result is the development of the now so-called *Market Model* by Miltersen-Sandmann-Sondermann (1997), reprinted in this volume. Rather than following the traditional approach based on continuously compounded interest rates, now observable rates like forward LIBOR or swap rates are the primary processes. This shift to nominal rates leads to interest rate models which better reflect market practice. The substantial consequence of this new approach is that the widely used pricing formula for caplets, the Black formula, can be rationalized in and is consistent with an arbitrage free model of the interest rate market. Instead of calibrating the parameters of traditional models by means of a root search procedure to the market, the “market model” can be fitted directly to basic market segments like the cap or swaption market by specifying (implicite) observable lognormal market volatilities.

The “market models” are by now the subject of active research and implementation by practioners. Important contributions and further developments, including the pricing and hedging of swaptions, were made by Brace-Gatarek-Musiela (Math. Fin., 1997), Jamshidian (Finance Stoch., 1997) and Musiela-Rutkowski (Finance Stoch., 1997).

6.3.3 The Bonn Financial Data Bank

A further result is the development of the Bonn Financial Data Bank. This data bank is devoted to the interest rate market and its derivatives. It comprises daily (and partly intra-day high frequency) time series on money market interest rates, LIFFE Futures and Options, and swap yields for the major currencies. It is accessible via the internet. Time series can be directly downloaded via a userfriendly graphical surface. Forming a solid base for empirical studies on term structures and interest rate derivatives, the Bonn Data Bank constitutes a supplement of the Deutsche Finanzmark Datenbanken in Mannheim (bonds), Karlsruhe (stocks and stock options) and Aachen (balance sheet data).

6.4 Open problems

6.4.1 Incomplete Financial Markets

Research is still going on vigorously in many directions. One important open problem is the explicit description of the variance-optimal martingale measure in nontrivial situations, because at present examples like stochastic volatility models with nonzero correlation or most models with discontinuous asset prices still lead to unsolved questions. New applications in risk management and insurance mathematics have recently appeared and led to a demand for an even more general treatment. Other open questions concern the behaviour of optimal strategies under convergence along a sequence of models or the influence of a change of filtration, i. e., problems of restricted information or situations with insider information.

6.4.2 Term Structure Models

An open problem with the lognormal market model is the consistent choice of market rates as primary inputs. Not all rates can be lognormal at the same time. This is the so-called interpolation problem. The market model has been extended by Jamshidian (Finance Stoch. 1997) from LIBOR rates to the swap market. But a model combining both markets in a consistent way is still an open problem. Further open problems are the application of the market model to the pricing and hedging of exotic interest rate option, in particular of path dependent options.

6.5 Selected Publications

Evstigneev, I., and K. Schürger (1994): A Limit Theorem for Random Matrices with a Multiparameter and its Application to a Stochastic Model of a Large Economy; *Stochastic Processes Appl. 52*, pp. 65-74.

Föllmer, H. and D. Sondermann, (1986): Hedging of Non-Redundant Contingent Claims, *Contributions to Mathematical Economics, in Honor of Gerard Debreu*; W. Hildenbrand and A. Mas-Colell (eds.), North-Holland, Amsterdam, 205-223.

Föllmer, H., and M. Schweizer (1991): Hedging of Contingent Claims under Incomplete Information; in: M.H.A. Davis, and R.J. Elliott (eds.), *Applied Stochastic Analysis, Stochastic Monographs, Vol. 5*; Gordon and Breach, London, New York, pp. 389-414.

Föllmer, H., and M. Schweizer (1993): A Microeconomic Approach to Diffusion Models for Stock Prices; *Mathematical Finance 3 (1)*, pp. 1-23.

Föllmer, H (1994): Stock Price Fluctuations as a Diffusion in a Random Environment; *Phil. Trans. R. Soc. Lond. A 347*, pp. 471- 483.

Frey, R., and D. Sommer (1996): A Systematic Approach to Pricing and Hedging International Derivatives with Interest Rate Risk: Analysis of International Derivatives under Stochastic Interest Rates; *Applied Mathematical Finance 3*, pp. 295-318.

Frey, R., and A. Stremme (1997): Market Volatility and Feedback Effects from Dynamic Hedging; *Mathematical Finance 7*, pp. 351-374.

Frey, R. (1998): Perfect Option Hedging for a large Trader, *Finance and Stochastics 2*, pp. 115-142

Leisen, D. (1998): Pricing the American Put Option: A Detailed Convergence Analysis for Binomial Models. *Journal of Economic Dynamics and Control 22 (8-9)*, 1419-1444.

Leisen, D.P.J., and M. Reimer (1996): Binomial Models for Option Valuation - Examining and Improving Convergence; *Applied Mathematical Finance 3*, pp. 319-346.

Miltersen, K.R., K. Sandmann and D. Sondermann (1997): Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates; *The Journal of Finance 52*, pp. 409-430.

Müller, S., and M. Klein (1991): ECU-Interest Rates and ECU Basket Adjustments: *Ökonomische Prognose-, Entscheidungs- und Gleichgewichtsmodelle,* W. Krelle (ed.), VCH Verlagsgesellschaft Weinheim, 107-122

Nielsen, J.A., and K. Sandmann (1995): Equity-linked life insurance: A model with stochastic interest rates; *Insurance Mathematics & Economics 16*, pp. 225-253.

Nielsen, J.A., and K. Sandmann (1996): Uniqueness of the Fair Premium for Equity-Linked-Life-Contracts; *The Geneva Papers on Risk and Insurance Theory 21*, pp. 65-102.

Rady, S., and K. Sandmann (1994): The Direct Approach to Debt Option Pricing; *The Review of Futures Markets 13/2*, pp. 461-514.

Sandmann, K., and D. Sondermann (1990): Zur Bewertung von Caps und Floors; *Zeitschrift für Betriebswirtschaftslehre 11*, pp. 1205-1238.

Sandmann, K. (1993): The Pricing of Options with an Uncertain Interest Rate: A Discrete Time Approach; *Mathematical Finance 3 (2)*, pp. 201-216.

Sandmann, K., and D. Sondermann (1993): A Term Structure Model and the Pricing of Interest Rate Derivatives; *in: Review of Future Markets,* Vol. 12/2, pp. 391-430

Sandmann, K., D. Sondermann and K. Miltersen (1995): Closed Form Term Structure Derivatives in a Heath-Jarrow-Morton Model with Log-normal Annually Compounded Interest Rates; *Research Symposium Proceedings CBOT*, pp. 145-164.

Sandmann, K., and D. Sondermann (1997): A Note on the Stability of Lognormal Interest Rate Models and the Pricing of Eurodollar Futures; *Mathematical Finance 7*, pp. 119-125.

Schlögl, E., and D. Sommer (1998): Factor Models and the Shape of the Term Structure. *The Journal of Financial Engineering 7*, 79-88.

Schönbucher, P.H. (1998): Term Structure Modelling of Defaultable Bonds. *Review of Derivatives Research 2 (2/3)*, 161-192.

Schürger, K. (1991): Almost Subadditive Extensions of Kingman's Ergodic Theorem; *Ann. Probab. 19*, pp. 1575-1586.

Schürger, K. (1996): On the Existence of Equivalent Tau-Measures in Finite Discrete Time; *Stoch. Proc. Appl. 61*, pp. 109-128.

Schweizer, M. (1990): Risk-Minimality and Orthogonality of Martingales; *Stochastics and Stochastics Reports 30*, pp. 123-131.

Schweizer, M. (1991): Option Hedging for Semimartingales; *Stochastic Processes and Their Applications 37*, pp. 339-363.

Schweizer, M. (1992): Mean-Variance Hedging for General Claims; *Annals of Applied Probability 2*, pp. 171-179.

Schweizer, M. (1995): Variance, Optimal Hedging in Discrete Time; *Mathematics of Operations Research 20/1*, pp. 1-32.

Sommer, D. (1997): Pricing and Hedging of Contingent Claims in Term Structure Models with Exogenous Issuing of New Bonds; *European Financial Management 3*, pp. 269-292.

Sondermann, D. (1987): Currency Option: Hedging and Social Value, *Eur. Econ. Review 31*, pp. 246-256

Sondermann, D. (1991): Reinsurance in Arbitrage-Free Markets; *Insurance: Mathematics and Economics 10*, pp. 191-202

Werner, J. (1987): Arbitrage and the Existence of Competitive Equilibrium, *Econometrica 55/6*, pp. 1403-1418

Wiesmeth, H. (1990): The State Preference Approach to General Equilibrium in Corporate Finance; *European Economic Review 34*, 1247-1264.

7 Macroeconomic Institutions and Structures

Manfred M.J.M. Neumann

7.1 Main research topic

The guiding notion of our main research topic was developing a theory of monetary policy making that can explain the actual behaviour of central bankers under different institutional settings as regards their dependence on the government.

Our work was motivated by the observation that since the seminal contribution by Barro-Gordon the game-theoretic literature on the inflation bias of time consistent monetary policy has been dominated by the assumption that the authorities maximise a social welfare function. The underlying assumption of a representative agent who controls the central bank and the government is unsatisfactory as it precludes to study the policy implications of agent heterogeneity. Moreover, in most of the literature institutional differences as regards the dependence of central bankers on the government are not explicitly modelled but are equated with differences in the central bankers' preference for inflation. The implications derived by this literature are that an independent central bank reduces but does not eliminate the inflation bias and produces less business cycle stabilisation than a government dependent central bank. However, both implications are rejected by cross-country evidence.

7.2 Methodological approach

To develop models of monetary policy making that are capable of explaining the main facts, requires some background knowledge about the actual policy behaviour and, consequently, the empirical investigation of relevant aspects. In this type of work we have applied the standard methods, from statistical data analysis to refined econometric methods; see, for example, Klein and Neumann (1990) on seigniorage; v. Hagen (1999) and Neumann (1997) on monetary targeting; Weber (1999) on credibility; Neumann and Weidmann (1998) on interest rate policy; Wesche (1997) on money demand.

As regards the model building, the natural approach is to identify the major deficiencies of previous analytical work and to try to eliminate them by exploring the implications of a richer structure. Two main strands of ideas have been followed: (i) to model the interdependencies between the incumbent government and the central bank under different institutional characteristics, and (ii) to replace the representative agent by a median voter whose preferences differ from social welfare.

7.2 Selected results

In this sub-section we only report on three innovative contributions to the theory of central bank independence. Other important studies on policy and related issues are listed in sub-section 6.

In a first attempt at modelling the interdependencies between the elected government and the central bank in a more satisfactory fashion v. Hagen, Fratianni, and Waller (1997) introduce the notion of personal independence, proposed by Neumann (1991), by modelling the central banker's expected remuneration and his reappointment probability. Since for a given degree of independence the reappointment probability is affected by the incumbent's chance of re-election, the stabilisation performance of the central bank can be positively or negatively biased. As a result, the model permits to understand why cross-country studies find that the variance of output is uncorrelated with the degree of central bank independence.

In search for a more descriptive theory, Herrendorf and Neumann (1998, 2000) eliminate the standard assumption that the objective of monetary policy is social welfare. Modelling the labour market, they replace the representative agent by insiders and outsiders or, as an alternative, by senior and junior workers. The utility functions of these individuals only differ with respect to desired per capita employment. Given that in the insider-outsider set-up the median voter is an insider and can do away with institutions, the independent central bank avoids the inflation bias in the interest of the insider. The government dependent central bank, in contrast, is forced to collect seigniorage and to promote the incumbent's reelection prospects. The degree of output stabilisation reflects the insider's interest but is unaffected by the central bank's status.

A richer set of implications as regards the behaviour of central bank derives when agent heterogeneity is combined with a model of democratic government that provides each type of agent with a probability of winning elections or becoming member of the central bank board. Herrendorf and Neumann (1998) introduce the citizen-candidate model with two different campaign issues, hence four types of candidates. They then analyse staggered partisan appointments to the central bank board as well as strategic delegation. They are able to prove the new result that an equilibrium condition exists under which junior as well as senior governments delegate monetary policy to an anti-inflation central bank board even though the labour market is distorted.

7.4 Open problems

hile we can explain under which conditions all agents endorse the coexistence of the labour market distortion and of central bank independence, there are open ends. For example, the analysis does not rule out that the labour market distortion will be kept but the central bank status be changed after each electoral swing in politics. To get rid of this unattractive implication, it might be necessary to model majority votes in parliament and to introduce costs of changing institutions.

7.5 Selected Publications

Herrendorf, B., and M.J.M. Neumann (1998): The Political Economy of Inflation and Central Bank Independence; *Centre for Economic Policy Research, London No. 1787;* to appear.

Herrendorf, B., and M.J.M. Neumann (2000): A Non-normative Theory of Inflation and Central Bank Independence; *Review of World Economics 136(2)*.

Klein, M., and M.J.M. Neumann (1990): Seigniorage: What Is It and Who Gets It?; *Review of World Economics 126*, 205-221.

Klein, M., and S. Müller (1992): ECU Interest Rates and ECU Basket Adjustments:An Arbitrage Pricing Approach; *Journal of Banking and Finance 16*, 137-153.

Lux, T. (1998): The Socio-Economic Dynamics of Speculative Markets: Interacting Agents, Chaos, and the Fat Tails of Return Distributions; *Journal of Economic Behavior and Organisation 33*, 143-165.

Lux, T., and M. Marchesi (1998): Scaling and Criticality in a Stochastic Multi-Agent Model of a Financial Market; *Nature 397*, 498-500.

Neumann, M.J.M. (1990): Implementing Monetary Policy in Germany, in: Board of Governors of the Federal Reserve System (ed.); *Financial Sectors in Open Economics: Empirical Analysis and Policy Issues*, Washington, D.C., 499-528.

Neumann, M.J.M. (1990): Messung von Mindestreserveeffekten; *Review of World Economics 126*, 761-764.

Neumann, M.J.M. (1991): Precommitment by Central Bank Independence; *Open Economies Review 2,* 95-112.

Neumann, M.J.M. (1992): Seigniorage in the U.S.; *Review, Federal Reserve Bank of St. Louis 74,* 3-28.

Neumann, M.J.M. (1996): A Comparative Study of Seigniorage: Japan and Germany; *Monetary and Economic Studies 14*, 143-193.

Neumann, M.J.M. (1997): Monetary Targeting in Germany, in: I. Kuroda (ed.); *Towards More Effective Monetary Policy*, MacMillan, London, 176-198.

Neumann, M.J.M. (1999): Monetary Stability: Threat and Proven Response; in: Deutsche Bundesbank (ed.); *Central Bank and the Currency in Germany since 1948*, Oxford: Oxford University Press, 269-306.

Neumann, M.J.M., and J. v. Hagen (1990): Relative Price Risk in an Open Economy with Fixed and Flexible Exchange Rates; *Open Economies Review 1*, 269-289.

Neumann, M.J.M., and J. v. Hagen (1991): Conditional Relative Price Variance and Its Determinants: Open Economy Evidence from Germany; *International Economic Review 32*, 195-208.

Neumann, M.J.M., and J. Weidmann (1998): The Information Content of German Discount Rate Changes; *European Economic Review 42,* 1667-1682.

v. Hagen, J. (1997): Monetary Policy and Institutions in the EMU; *Swedish Economic Policy Review 4,* 52-127.

v. Hagen, J. (1999): Monetary Targeting in Germany; *Journal of Monetary Economics 43*, 681-701.

v. Hagen, J., T. Bayoumi and B. Eichengreen (1997): European Monetary Unification: Implications of Policy for Research; *Open Economies Review,* 71-91.

v. Hagen, J., and I. Fender (1998): Central Banking in a More Perfect Financial System; *Open Economies Review 9*, 493-531.

v. Hagen, J., M. Fratianni and C. Waller (1997): Central Banking as a Political Principal Agent Problem; *Economic Inquiry*, 378-394.

v. Hagen, J., and G. Hammond (1998): Regional Insurance against Asymmetric Shocks – An Empirical Study for the EC; *The Manchester School 66*, 331-353.

v. Hagen, J., and M.J.M. Neumann (1994): Real Exchange Rates within and between Currency Areas: How far Away is EMU?; *Review of Economics and Statistics 76*, 236-244.

v. Hagen, J., and M.J.M. Neumann (1996): A Framework for Monetary Policy under EMU, in: Deutsche Bundesbank (ed.); *Monetary Policy Strategies in Europe*, München, 141-165.

Weber, A.A. (1995): Exchange Rates and the Effectiveness of Central Bank Intervention: New Evidence from the G-3 and the EMS; in: E. Girardin (ed.), *European Currency Crisis and After*, Manchester University Press, Manchester, England, 202-235.

Weber, A.A. (1998): Ursachen Spekulativer Attacken: Eine empirische Analyse; in: E. Baltensperger (ed.); *Spekulation, Preisbildung und Volatilität auf Finanz- und Devisenmärkten,* Berlin: Duncker & Humblot, 55-93.

Weber, A.A. (1999): *Monetary Policy and Credibility: Theories and Facts*. Kluwer Academic Publishers, Dodrecht, Netherlands, forthcoming.

Wesche, K. (1997): The Demand for Divisia Money in a Core Monetary Union; *Federal Reserve Bank of St. Louis Review 79 (5)*, 51-60.

Wesche, K. (1997): The Stability of European Money Demand: An Investigation of M3H; Open Economies Review 8 (4), 371-391

Wesche, K. (1998): *Die Geldnachfrage in Europa: Aggregationsprobleme und Empirie*. Wirtschaftswissenschaftliche Beiträge, Bd. 154, Physica-Verlag, Heidelberg.

8 Aggregation

Werner Hildenbrand

8.1 Main research topic

The main research topic can be described as the search for and a justification of a sufficiently rich structure of standard equilibrium models; a sufficiently rich structure should imply that equilibria are well-determined.

The motivation for this research topic is based on the well-known fact that interactive decision models with "rational" economic agents – where "rational" simply means that individual behaviour is modelled as a result of a maximization problem under suitable constraints - , in general, do not possess sufficient structure in order to go beyond an existence proof of equilibria. This unsatisfactory feature of the various equilibrium models discussed in the literature thus is not the possible non-existence, yet the multiplicity of equilibria. This situation is unsatisfactory since an equilibrium theory with many possible equilibria is incomplete and is not very useful for economic analysis (e.g. a comparative statics analysis).

This general lack of structure was worked out very sharply before the SFB 303 began in the case of Walrasian equilibrium models. The 'disaggregation results' due to Sonnenschein, Mantel, Debreu and others essentially say that the hypothesis of rational economic agents alone does not imply any useful structure of the excess demand function of a Walrasian economy. The research project 'Aggregation' – where aggregation is understood over a large and suitably defined heterogeneous population – aims at opposing the negative view which might be drawn from the 'disaggregation results'.

The lack of structure of Walrasian equilibrium models is mainly due to the demand side of the economy. Therefore, emphasis is put on modelling the consumption sector of an economy. The goal is, starting from microeconomic assumptions, to derive an aggregate demand function which has a sufficiently rich structure such as the 'Law of Demand' or alternative properties.

8.2 Methodological approach

The common practise in the literature to escape the dilemma of the lack of structure of equilibrium models is to add specific assumptions to the maximization problems which describe the behaviour of the economic agents. This typically amounts to postulating quite special utility – or production functions. These ad hoc assumptions are difficult to justify given the empirical evidence and the knowledge from experimental economics. A model based on such ad hoc specifications then often degenerates to a mere illustrative example. The robustness of conclusions that are derived in this way is difficult to judge.

An alternative methodological approach to obtain the required structure is based on the structure-creating effects that are the results of the aggregation process over a heterogeneous population. Instead of restricting the behavioural relation on the micro level (for example by assuming a specific utility function) one considers now restrictions on the distribution across the population of agents' characteristics (for example by assuming a uniform or log-normal distribution of income). To model the distribution of agents' characteristics amounts to assuming explicitly that agents differ in their characteristics. That is to say, the population is viewed to be heterogeneous. A specific assumption on the distribution of agents' characteristics then means a specific form of heterogeneity. The early contribution by Houthakker (1956)[25] and a paper by Hildenbrand (1983)[26] show (in quite different contexts) that assumptions on the distribution of agents' characteristics can indeed lead to strong properties of the aggregate relation which are not shared by the micro relation.

The method of aggregation with respect to a distribution of agents' characteristics requires, of course, a justification of the distributional assumption. In the case where agents' characteristics are observable one can rely for such a justification on cross-section data. An essential part of the research in the project 'Aggregation' was to analyse cross-section data. For this non-parametric statistical methods had to be developed. The methodological approach used in this project can be described as a dialogue between data analysis and modelling of economic hypothesis.

8.3 Selected results

In the following only contributions are mentioned which are directly concerned with the topic of aggregation. Other important contributions to equilibrium analysis of interactive decision models are listed in sub-section 6.

Weak axiom of revealed preferences
The demand theory based on the weak axiom of revealed preferences and its relation to the Slutsky decomposition was analysed in Hildenbrand and Jerison (1989) and John (1995). The relation between the axiom of revealed preferences and rationality was analysed in Jerison and Jerison (1993) and (1996). For a standard Walrasian economy it was shown in Hildenbrand (1989) and Grodal and Hildenbrand (1989) that the weak axiom of revealed preferences for market demand is a very strong assumption. It was shown that the axiom of revealed preferences is not satisfied generically. A laboratory experiment on the weak axiom was carried out by Sippel (1997).

Disaggregation
The well-known disaggregation theorem of Debreu has been considerably strengthened by Kirman and Koch (1986). They show that the excess demand in an exchange economy with identical preferences and collinear endowments has no identifiable structure which goes beyond the trivial ones: homogeneity and Walras law. In Hens and Bottazzi (1996) and Hens and Gottardi (1998) the disaggregation results are analysed for exchange economies under uncertainty and incomplete markets.

Law of Demand
The Law of Demand (monotonicity) for the mean demand function of a heterogeneous population of households (with a price-independent income distribution) is based on two hypotheses which are of quite different nature. First, the weak axiom of revealed preferences on the household level (a weak form of individual rationality) and second, the hypothesis of increasing dispersion or spread of households' demand. This last hypothesis is an assumption of the entire population of households, thus it is a distributional assumption which cannot be derived from acceptable assumptions on individual behaviour. However, the hypothesis of increasing disperion or spread can be tested (falsified) by a statistical analysis of cross-section data of households' income and expenditures on various consumptions categories. Such an empirical test has been carried out for the U.K. Family Expenditure Survey and the French Enquête

[25] Houthakker, H. S. (1955) "The Pareto distribution and the Cobb-Douglas-production function in activity analysis", *Review of Economic Studies 23*, 27-31.
[26] Hildenbrand, W. (1983) "On the 'Law of Demand'", *Econometrica 5*, 997-1019.

Budget de Famille. It turned out that for no year the hypothesis was rejected. This research project is presented in Hildenbrand (1994). Relevant contributions to the foundation of the Law of Demand produced in the SFB are Hildenbrand and Hildenbrand (1986), Härdle and Stoker (1989), Härdle, Hildenbrand and Jerison (1981), Grodal and Hildenbrand (1992), Hildenbrand and Kneip (1993), and Jerison (1999).

Consumption function

In analysing the Law of Demand the distribution of agents' characteristics is fixed and the impact of hypothetical price changes is analysed. If one wants to model the evolution over time of consumption expenditures one has to model the evolution over time of the distribution of agents' characteristics and analyse the impact on consumption of these changes in the distribution of agents' characteristics. In Hildenbrand and Kneip (1999) hypostheses are formulated which describe the evolution over time of the distribution of households' characteristics under the hypothesis of structural stability. The formulation of these hypothesis is motivated by a statistical analysis of cross-section data. It is then shown that the aggregation process leads to a simple expression for the change over time of aggregate consumption.

8.4 Open problems

The Law of Demand is derived for a consumption sector with a price independent income distribution. If households' income depends on the price system as in a Walrasian model then the situation is much more complicated. Only very partial results are known (for example in an exchange economy with co-linear endowments). What is missing is a satisfactory theory of the process which determines the income distribution.

An important open problem is an aggregation analysis of dynamic micro relations. Such a theory is needed for a well-founded theory of the consumption function as used in macroeconomics. If the dynamic micro relation is derived from an intertemporal (utility maximization) decision problem under uncertainty, many explanatory variables are unobservable, such as life-cycle income and expectation functions. Modelling the distribution of these explanatory variables is very delicate. The model by Hildenbrand and Kneip (1999) can at best be considered as a first step. Most likely, in order to formulate reasonable hypotheses one has to stratify the population by observable household attributes. Then it is necessary to model the evolution over time of the distributions of observable household´s attributes. This requires a detailed statistical analysis of cross-section data of household's attributes. Such an analysis is in principle possible, yet it is very time consuming.

8.5 Selected Publications

Bettzüge, M.O. (1998): An Extension of a Theorem by Mitjushin and Polterovich to Incomplete Markets. *Journal of Mathematical Economics 30*, 285-300.

Bewley, T. (1999): Why Wages Don’t Fall During a Recession. Harvard University Press, Cambridge.

Bewley, T. (1998): Why not Cut Pay? *European Economic Review 42*, 459-490.

Dierker, E., and J. Lenninghaus (1986): Surplus Maximization and Pareto Optimality; *Contributions to Mathematical Economics, In Honor of Gérard Debreu*, W. Hildenbrand and A. Mas-Colell (eds.), North-Holland, Amsterdam, 143-166.

Dierker, E., and W. Neuefeind (1988): Quantity Guided Price Settings; *Journal of Mathematical Economics 17*, pp. 249-259.

Dierker, E., R. Guesnerie and W. Neuefeind (1985): General Equilibrium when some Firms Follow Special Pricing Rules; *Econometrica 53*, 1369-1393.

Dierker, H., and B. Grodal (1986): Non-existence of Cournot-Walras Equilibrium in a General Equilibrium Model with Two Oligopolists; *Contributions to Mathematical Economics, In Honor of Gérard Debreu*, W. Hildenbrand and A. Mas-Colellm (eds.), North-Holland, Amsterdam, 167-185.

Evstigneev, I., and K. Schürger (1994): A Limit Theorem for Random Matrices with a Multiparameter; *Stochastic Processes and their Appl. 52*, pp. 65-74.

Evstigneev, I., and M. Taksar (1995): Stochastic Equilibria on Graphs II; *Journal of Mathematical Economics 24*, pp. 383-406.

Evstigneev, I., and M.I. Taksar (1994): Stochastic Equilibria on Graphs, I; *Journal of Math. Economics 24*, pp. 401-433.

Evstigneev, I., W. Hildenbrand and M. Jerison (1997): Metonymy and Cross Section Demand; *Journal of Mathematical Economics 28*, pp. 397-414.

Grodal, B., and W. Hildenbrand (1992): Cross-Section Engel Curves, Expenditure Distributions and the 'Law of Demand'; in: L. Phlips, and L.D. Taylor (eds.), *Aggregation, Consumption and Trade, Essays in Honor of H.S. Houthakker*, Kluwer Academic Publishers, Dordrecht, Boston, London, pp. 37-53.

Guesnerie, R., and M. Jerison (1991): Taxation as a Social Choice Problem; *Journal of Public Economics 44*, pp. 37-63.

Härdle, W. (1990): Applied Nonparametric Regression; *Econometric Society Monograph Series 19*, Cambridge University Press, Cambridge, London, New York.

Härdle, W., and J. Hart (1993): A Bootstrap Test for Positive Definiteness of Income Effects Matrices; *J. of Econometric Theory 8*, pp. 276-290.

Härdle, W., and M. Jerison (1991): Cross Section Engel Curves over Time; *Recherches Economiques de Louvain 57*, pp. 391-431.

Härdle, W., and A. Kneip (1999): Testing a regression model when we have smooth alternatives in mind; *Scandinavian Journal of Statistics, 26*, 221-238

Härdle, W., and E. Mammen (1993): Comparing Nonparametric versus Parametric Regression Fits; *Annals of Statistics 21*, pp. 1926-1947.

Härdle, W., and J.S. Marron (1991): Bootstrap Simultaneous Error Bars for Nonparametric Regression; *Annals of Statistics 19*, pp. 778-796.

Härdle, W., and T. Stoker (1989): Investigating Smooth Multiple Regression by the Method of Average Derivatives; *Journal of the American Statistical Association 84*, pp. 986-995.

Härdle, W., and A.B. Tsybakov (1993): How Sensitive Are Average Derivatives?; *Journal of Econometrics 58*, pp. 31-48.

Härdle, W., J. Hart, J.S. Marron and A.B. Tsybakov (1992): Bandwidth Choice for Average Derivative Estimation; *Journal of the American Statistical Association 87*, pp. 218-226.

Härdle, W., W. Hildenbrand and M. Jerison (1991): Empirical Evidence on the Law of Demand; *Econometrica 59*, pp. 1525-1549.

Härdle, W., J.S. Marron and M. Wand (1989): Bandwidth Choice for Density Derivatives; *Journal of the Royal Statistical Society, Series B 52*, pp. 223-232.

Hens, T. (1992): A Note on Savage´s Theorem with a Finite Number of States; *Journal of Risk and Uncertainty 5*, pp. 63-71.

Hens, Th. (1997): Stability of Tatonnement Processes of Short Period Equilibria with Rational Expectations; *Journal of Mathematical Economics 28*, pp. 41-67.

Hens, Th. (1998): Do Sunspots Matter when Spot Market Equilibria Are Unique? *Econometrica*, forthcoming.

Hens, Th. (1998): Incomplete Markets. In: Kirman, A. (ed.), *Elements of General Equilibrium Theory*, Blackwell Publishers, Oxford, 139-210.

Hens, T., and J.-M. Bottazi (1996): On Excess Demand Functions with Incomplete Markets; *Journal of Economic Theory 68*, pp. 49-63.

Hens, Th., and P. Gottardi (1998): On the Disaggregation of Excess Demand Functions when Markets Are Incomplete: The Case of Nominal Assets. *Economic Theory*, forthcoming.

Hens, T., and A. Loeffler (1995): A Note on Gross Substitution in Financial Markets; *Economics Letters 49*, pp. 39-43.
Hens, Th., J.-M. Bottazzi and A. Löffler (1998): Market Demand Functions in CAPM. *Journal of Economic Theory 79*, 192-206.
Hildenbrand, K., and W. Hildenbrand (1986): On the Mean Income Effect: A Data Analysis of the U.K. Family Expenditure Survey; *Contributions to Mathematical Economics, In Honor of Gérard Debreu*, W. Hildenbrand and A. Mas-Colell (eds.), North Holland, Amsterdam, 247-268.
Hildenbrand, W. (1989): Facts and Ideas in Microeconomic Theory; *European Economic Review 33*, pp. 251-276.
Hildenbrand, W. (1989): The Weak Axiom of Revealed Preference for Market Demand is Strong; *Econometrica 57*, pp. 979-985.
Hildenbrand, W. (1994): *Market Demand: Theory and Empirical Evidence*, Princeton University Press.
Hildenbrand, W. (1998): An Introduction to Demand Aggregation. *Journal of Mathematical Economics (Special Issue on Aggregation)31*, pp. 1-14.
Hildenbrand, W. (1998): How Relevant Are Specifications of Behavioral Relations on the Micro-Level for Modelling the Time Path of Population Aggregates? *European Economic Review 42*, 437-458.
Hildenbrand, W. (1998): Zur Relevanz mikroökonomischer Verhaltenshypothesen für die Modellierung der zeitlichen Entwicklung von Aggregaten. Schriften des Vereins für Socialpolitik, Bd. 261. *Zeitschrift für Wirtschafts- und Sozialwissenschaften*, Duncker & Humblot, Berlin, 195-218.
Hildenbrand, W., and B. Grodal (1989): The Weak Axiom of Revealed Preference in a Productive Economy, *The Review of Economic Studies 56*, pp. 635-639.
Hildenbrand, W., and M. Jerison (1989): The Demand Theory of the Weak Axioms of Revealed Preference; *Economics Letters 29*, pp. 209-213.
Hildenbrand, W., and A. Kirman (1988): *Equilibrium Analysis*; North-Holland, Amsterdam.
Hildenbrand, W., and A. Kneip (1993): Family Expenditure Data, Heteroscedasticity and the 'Law of Demand'; *Ricerche Economiche 47*, pp. 137-165.
Hildenbrand, W. and Kneip, A. (1999): Demand Aggregation under Structural Stability; *Journal of Mathematical Economics (Special Issue on Aggregation) 31*, 81-110
Hildenbrand, W., and H. Sonnenschein (eds.) (1991): *Handbook of Mathematical Economics, Vol. IV*; North-Holland, Amsterdam, New York, Oxford, Toronto.
Hildenbrand, W., A. Kneip, and K. Utikal (1998): Une analyse non parametrique des distributions du revenu et des caracteristiques des menages; *Revue de Statistique Appliquée*, 47, 39-56
Jerison, D., and M. Jerison (1993): Approximately Rational Consumer Demand; *Economic Theory 3*, pp. 217-241.
Jerison, M. (1993): Russel on Gorman's Engel Curves: A Correction; *Economics Letters 43*, pp. 171-175.
Jerison, M. (1998): Dispersed Excess Demands, the Weak Axiom and Uniqueness of Equilibrium. *Journal of Mathematical Economics*, forthcoming.
Jerison, M., and D. Jerison (1996): A Discrete Characterization of Slutsky Symmetry; *Economic Theory 8*, pp. 229-237.
John, R. (1995): The Weak Axiom of Revealed Preference and Homogeneity of Demand Functions; *Economics Letters 47*, pp. 11-16.
John, R. (1997): A Simple Cycle Preserving Extension of a Demand Function; *Journal of Economic Theory 72*, pp. 442-445.
John, R., and H.E. Ryder (1985): On the Second Optimality Theorem of Welfare Economics; *Journal of Economic Theory 36*, 176-185.

Kirman, A.P., and K.J. Koch (1986): Market Excess Demand in Exchange Economies with Identical Preferences and Collinear Endowments; *Review of Economic Studies 53*, 457-463.

Kneip, A. (1994): Nonparametric Estimation of Common Regressors for Similar Curve Data; *Annals of Statistics 22*, pp. 1386-1428.

Kneip, A. (1997): Behavioral Heterogeneity and Structural Properties of Aggregate Demand; *Journal of Mathematical Economics,* to appear.

Kneip, A. (1998): Behavioral Heterogeneity and Structural Properties of Aggregate Demand. *Journal of Mathematical Economics*, forthcoming.

Kneip, A. (1999): Behavioral heterogeneity and structural properties of aggregate demand; *Journal of Mathematical Economics, 31*, 49-79

Laitenberger, M. (1996): Existence of Financial Market Equilibria with Transaction Costs; *Ricerche Economiche 50*, pp. 69-77.

Nehring, K. and C. Puppe (1999): On the Multi-Preference Approach to Evaluating Opportunities, *Social Choice and Welfare*, 41-63.

Puppe, C. (1998): Individual Freedom and Social Choice, in: *Freedom in Economics: New Perspectives in Normative Analysis*, edited by J.F.Laslier, M.Fleurbaey, N.Gravel und A.Trannoy, London: Routledge, 49-68.

Puppe, C. (1998): Individual Freedom and Social Choice. In: Laslier, J.F., M. Fleurbaey, N. Gravel and A. Trannoy (eds.), *Freedom in Economics: New Perspectives in Normative Analysis*, Routledge, London, 49-68.

Puppe, C., and R. Kerschbamer (1998): Voluntary Contributions when the Public Good Is Not Necessarily Normal. *Journal of Economics 68*, 175-192.

Sippel, R. (1997): An Experiment on the Pure Theory of Consumer’s Behaviour; *The Economic Journal 107*, pp. 1431-1444.

Trockel, W. (1987): Market Demand by Non-Convex Preferences; *Rendiconti del Seminario Matematico e Fisico di Milano*, *LVII*, pp. 311-320.

RAND Journal of Economics
Vol. 26, No. 2, Summer 1995
pp. 163–179

Option contracts and renegotiation: a solution to the hold-up problem

Georg Nöldeke*

and

Klaus M. Schmidt*

In this article, we analyze the canonical hold-up model of Hart and Moore under the assumption that the courts can verify delivery of the good by the seller. It is shown that no further renegotiation design is necessary to achieve the first best: simple option contracts, which give the seller the right to take the delivery decision and specify payments depending on whether delivery takes place, allow implementation of efficient investment decisions and efficient trade.

1. Introduction

■ In a seminal article, Hart and Moore (1988) considered a buyer-seller relationship with observable but unverifiable investment decisions. They argued that contractual incompleteness, due to nonverifiability of the relevant state of the world, combined with the parties' inability to prevent *ex post* renegotiation will lead to underinvestment in such a classical hold-up problem. This result has attracted considerable attention because it seems to provide a theoretical foundation for the rapidly growing literature on incomplete contracts, which tries to explain economic institutions, such as the allocation of ownership rights or the financial structure of the firm, as second-best solutions to incentive problems in a world in which comprehensive contracts cannot be written.

In this article, we argue that the underinvestment problem in the Hart–Moore model can be overcome if the parties can write simple option contracts. An option contract gives the seller the right (but not the obligation) to deliver a fixed quantity of the good and makes the buyer's contractual payment contingent on the seller's delivery decision. Note that an option contract is feasible only if it is possible to enforce payments conditional on the seller's delivery decision, that is, the court must be able to observe whether the seller

---

* University of Bonn.

An earlier version of this article was circulated under the title "Unverifiable Information, Incomplete Contracts, and Renegotiation." We would like to thank Dieter Balkenborg, Frank Bickenbach, Oliver Hart, Bengt Holmström, Kai-Uwe Kühn, Albert Ma, Bentley MacLeod, Benny Moldovanu, John Moore, Monika Schnitzer, Urs Schweizer, two anonymous referees, and in particular Mike Riordan for helpful comments and discussions. This research was initiated during the International Summer School of the Center for the Study of the New Institutional Economics in Wallerfangen, Germany, August 1990. We are grateful to Professor Rudolf Richter for providing this stimulating atmosphere. Financial support by Deutsche Forschungsgemeinschaft, SFB 303 at the University of Bonn, is gratefully acknowledged.

delivered the good to the buyer. This possibility is explicitly ruled out by Hart and Moore who assume that, if trade fails, a court cannot distinguish whether the seller refused to supply or whether the buyer refused to take delivery. It is this assumption, and only this assumption, from the original Hart–Moore model that we need to abandon in order to achieve the first best.

To put our contribution into perspective, it is useful to relate our results to Aghion, Dewatripont, and Rey (1994). These authors have shown that the underinvestment problem can be solved if renegotiation design is possible, in the sense that the contractual environment allows (i) allocation of all bargaining power in the renegotiation game to one of the contracting parties and (ii) specification of an appropriate default point that obtains if renegotiation breaks down. The logic behind this result is that the party who has all the bargaining power in the renegotiation game becomes residual claimant on total surplus (minus a constant) and thus has the right incentives to invest. Incentives for the other party are then provided through the effect his investment has on the value of the default point. In a second step, the authors present a model of the renegotiation process—quite different from the one assumed by Hart and Moore—that achieves the required renegotiation design through the use of specific performance clauses and penalties for delay in the original contract.[1]

In contrast to Aghion, Dewatripont, and Rey, Hart and Moore took the renegotiation process as exogenously given. We shall show that, given their renegotiation process, every option contract results in the allocation of all bargaining power to the buyer in the renegotiation game. Hence, property (i) obtains naturally and the buyer has the right incentives to invest. Although all option contracts result in the same allocation of bargaining power, different option contracts will induce different default points for the renegotiation game. Because the seller has the right to decide whether to deliver, the (implicit) default point for renegotiation is given by whatever delivery decision the seller prefers to make under the terms of the initial contract. We shall show that adjusting the option price (i.e., the additional payment required from the buyer if the seller exercises his option to deliver) provides us with enough flexibility to achieve property (ii) and thus provide the seller with the correct investment incentives. Because we use the same renegotiation game as Hart and Moore, our approach highlights that it is Hart and Moore's assumption that the court cannot observe whether the seller delivered the good that is crucial to the underinvestment result (and not the exogenously given renegotiation game).[2]

In order to further contrast our analysis with Aghion, Dewatripont, and Rey, consider the case in which the good to be traded is indivisible and at most one unit can be traded (which is the only case considered by Hart and Moore and also the focus of most of our article). In this case, Aghion, Dewatripont, and Rey rely on explicit randomization to achieve the first best, whereas no such randomization is necessary with an option contract. Aghion, Dewatripont, and Rey proceed by designing a contract that gives all the bargaining power in their renegotiation game to one party, say, the buyer. The problem then becomes providing the seller with the correct incentives. If no trade is specified as a default point, the seller has no incentive to invest. If trade of one unit is specified as a default

[1] Chung (1991) also shows that the first best can be achieved if (i) and (ii) are satisfied. However, whereas Aghion, Dewatripont, and Rey offer an explicit contract, which generates a renegotiation game satisfying (i) and (ii), Chung just assumes that these conditions are satisfied.

[2] Remarks to this effect can already be found in Aghion, Dewatripont, and Rey and Hermalin and Katz (1993). Hermalin and Katz do not elaborate the point. Aghion, Dewatripont, and Rey observe that it is possible to allocate all bargaining power to one party in the original Hart–Moore model by choosing the price differential in the original contract appropriately, so that the only element of renegotiation design lacking from their model seems to be the ability to assign a default point different from no trade. This argument is incomplete in that it ignores the fact that, once different default points are introduced in the Hart–Moore model, it may no longer be the case that the price differential influences the distribution of bargaining power as in Hart and Moore. Indeed, the price differential in an option contract has no effect on the distribution of bargaining power. Instead it serves to shift the default point.

point, overinvestment will be induced if the probability that trade is efficient is less than one. To avoid this underinvestment (overinvestment) problem, Aghion, Dewatripont, and Rey propose that the initial contract should specify "trade with probability q" as a default point, where q is chosen to provide just the right investment incentives for the seller. Of course, this contract requires that the probability of trade would be enforced by a court if renegotiation fails.

Suppose now that the renegotiation process is the same as in Hart and Moore and consider an option contract that gives the seller the right to supply the good at price p_1 or not to supply and receive p_0. In this case, it is always the seller who has to be convinced (through renegotiation) to take the efficient action. The renegotiation game used by Hart and Moore has the property that the buyer can bribe the seller to do the right thing by making him just indifferent between trade and no trade. Thus, the buyer becomes residual claimant on the margin and, as in Aghion, Dewatripont, and Rey, is induced to invest efficiently. What about the seller? There are two possible default points of renegotiation that determine his utility: If the difference between p_1 and p_0 is higher than his production cost, the seller will enforce trade. If, however, the difference between p_1 and p_0 is smaller than his production cost, he will choose not to trade. Because the seller's production costs are a random variable, by varying $p_1 - p_0$, we can vary the probability of the two default points and give the seller, in expectation, just the right incentives to invest. The main difficulty in showing this result is that, in contrast to Aghion, Dewatripont, and Rey, there is a feedback effect from investment decisions to the (expected) default point induced by an option contract: The seller's investment affects the distribution of his production costs and thus the probability that "trade at p_1" arises as the default point of contract renegotiation.

Although in most of our article we deal with the case in which at most one unit of an indivisible good can be traded, our main results can be generalized to the case in which there are different levels of quantity and/or quality from which to choose. In this case, an option contract specifies one particular specification of the good and a price to be paid if the seller chooses to deliver exactly this specification. If any other specification is delivered (and the contract has not been renegotiated), the buyer is not required to pay more than the base payment, which he would have to pay even if the seller delivered nothing. We provide a simple condition under which the first best can be implemented if such an option contract is enforced by the courts. This condition is automatically satisfied if trade is a zero-one decision. Our result is in stark contrast to the incomplete contracts literature (e.g., Grossman and Hart (1986)) which argues that contracts are incomplete because of the difficulty to specify in advance the good to be traded contingent on a complex state of the world. Our result can be interpreted as showing that a contingent contract is often not necessary but that the first best can be achieved if it is possible to contract on at least one specification.

There is a large and growing recent literature dealing with contractual remedies to the hold-up problem. Rogerson (1992) shows that sequential mechanisms from the implementation literature can be used to achieve the first best under a variety of informational assumptions. However, these mechanisms are typically not renegotiation-proof. MacLeod and Malcomson (1993) and Edlin and Reichelstein (1993) consider a hold-up problem with a different renegotiation game. They focus on the case in which only one party has to make a relationship-specific investment and show that simple contracts can achieve the first best in this case. For some special cases, their results carry over if both parties have to invest.[3]

[3] MacLeod and Malcomson (1993) show that the first best can be achieved if (i) investments are not relationship specific but there is a switching cost or (ii) if investments are specific and there is an observable variable that is correlated with the investment levels and that can be contracted upon. Edlin and Reichelstein (1993) can implement the first best if the effect of investments and the effect of the state of the world enter the production costs of the seller (and the valuation of the buyer) in an additively separable manner.

More closely related to our article is the contribution by Hermalin and Katz (1993). These authors consider an environment in which the buyer's valuation and the seller's cost are stochastically independent. They show that a fill-in-the-price contract can achieve the first best in the absence of renegotiation. We show that these fill-in-the-price contracts can easily be embedded in the extensive form of our model in which they correspond to a menu of option contracts from which one party is allowed to choose after costs and benefits have been realized. The particular contract suggested by Hermalin and Katz will indeed not be renegotiated in equilibrium. Whereas our simple contract specifies only two prices to achieve efficient investments and relies on renegotiation to achieve efficient trade, writing a more elaborate fill-in-the-price contract thus avoids renegotiation while still achieving the first best under Hermalin and Katz's independence assumption.

We organize the remainder of the article as follows. In Section 2, we briefly summarize the model of Hart and Moore and show how the outcome of their renegotiation game is affected if we allow for option contracts. In Section 3, we show that an option contract can achieve the first best. In Section 4, we show that there are interesting cases in which renegotiation never occurs in equilibrium and discuss fill-in-the-price contracts. In Section 5, we extend our main results to the case in which there are different levels of quantity and/or quality of the good from which to choose. In Section 6, we conclude and discuss some further extensions.

2. Description of the model

■ Consider a buyer and a seller both of whom are risk neutral. At some initial date 0, they can write a contract specifying the terms of trade of one unit of an indivisible good which they may want to exchange at some future date 2. After date 0 but before date 1, the buyer and the seller make relationship-specific investments $\beta \in [0, \bar{\beta}]$ and $\sigma \in [0, \bar{\sigma}]$, respectively. These investments are sunk. The buyer's valuation $v(\omega^B, \beta)$ and the seller's production costs $c(\omega^s, \sigma)$ are determined by their relationship-specific investments and the realization of the state of the world, $\omega = (\omega^B, \omega^S)$, which is realized at date 1.[4] Let ω be distributed on $\Omega = [0, 1]^2$ according to the continuous joint density function $f(\omega)$. The marginal densities are denoted by $f^B(\omega^B) = \int_0^1 f(\omega^B, \omega^S)d\omega^S$ and $f^S(\omega^S) = \int_0^1 f(\omega^B, \omega^S)d\omega^B$.

Let $h^B(\beta)$ and $h^S(\sigma)$ denote the strictly increasing and continuous cost functions for the investments. Furthermore, assume that $v(\omega^B, \beta)$ and $c(\omega^S, \sigma)$ are continuous in both arguments and strictly positive. Finally, suppose that production costs are nonincreasing in σ for all ω^S.

Let $q \in \{0, 1\}$ be the level of trade and p the (possibly negative) net payment of the buyer to the seller. Then the utilities of the buyer and the seller after date 2 are given by

$$u^B = q \cdot v(\omega^B, \beta) - p - h^B(\beta) \tag{1}$$

$$u^S = p - q \cdot c(\omega^S, \sigma) - h^S(\sigma). \tag{2}$$

The problem of the parties at date 0 is to design a contract that implements efficient investment and trade decisions, i.e., that maximizes expected total surplus

$$W(\beta, \sigma) = \int_0^1 \int_0^1 [v(\omega^B, \beta) - c(\omega^S, \sigma)]^+ f(\omega)d\omega^S d\omega^B - h^B(\beta) - h^S(\sigma), \tag{3}$$

[4] Note that the specification of $v(\cdot, \beta)$ and $c(\cdot, \sigma)$ assumes that there are no direct externalities of the investments. However, there is of course an indirect externality because the investments affect the probability of trade. It is this indirect externality that is the focus of Williamson (1985) and Grossman and Hart (1986). See also Section 6.

where we shall frequently use the notation $[\cdot]^+ = \max\{0, \cdot\}$ throughout the remainder of this article. Given our continuity assumptions, $W(\beta, \sigma)$ is continuous in β and σ. The boundedness assumption on β and σ thus implies that the set of maximizers of $W(\cdot, \cdot)$ is always nonempty. Denote by (β^*, σ^*) a pair of first-best investment levels that maximizes (3). For convenience, we assume that (β^*, σ^*) is unique. Also, let $Q^*(\omega, \beta, \sigma) = \text{argmax}_q\{q \cdot [v(\omega^B, \beta) - c(\omega^S, \sigma)]\}$ denote the set of *ex post* efficient levels of trade.

The first best could easily be achieved if it were possible to contract upon the level of investment. However, we assume that, although investments β and σ as well as the state of the world ω (and so v and c) are perfectly observable by both agents, they cannot be verified to any third party, e.g., the courts. Thus, the contract cannot enforce outcomes contingent on these variables.

Trade takes place ($q = 1$) if and only if the seller delivers the good at date 2 and the buyer accepts delivery. Hart and Moore assume that the courts can only observe whether $q = 0$ or 1, but if $q = 0$, they cannot distinguish whether the seller or the buyer was unwilling to trade. In contrast, we assume that the courts can observe whether the seller delivered the good, $d = 1$, or not, $d = 0$. Thus, in our model, it is possible to write an initial contract signed at date 0 that specifies two different prices (p_1, p_0) depending on whether $d = 1$ or $d = 0$. The initial contract could, in principle, also be conditional on verifiable messages exchanged between the parties. Because we want to show that a simple option contract implements the first best already, we do not need to consider these more complicated mechanisms.

After date 1, the initial contract can be renegotiated. To simplify the proof of the following result, we assume that there is only one point in time between dates 1 and 2 at which the parties can send signed contract offers $(\tilde{p}^i_0, \tilde{p}^i_1)$, $i = S, B$, to each other.[5] After trade decisions have been made at date 2, the parties can decide simultaneously whether to present any renegotiation offers they have received to the court. The court can observe delivery and will enfore payments as specified in the initial contract unless

(a) exactly one party has produced a contract signed by the other party that specifies different terms of trade, or

(b) both parties produced identical contracts signed by the other party and specifying different terms of trade,

in which cases the payments of the new contract(s) are enforced.

As Hart and Moore, we are interested in the case in which renegotiation is costless. However, if sending renegotiation offers is costless, the renegotiation subgame that occurs after date 1 may have multiple subgame-perfect equilibrium outcomes. To obtain Proposition 1, which summarizes the outcome of the renegotiation game after an option contract (p_0, p_1) as defined above has been signed, we thus focus on the subgame-perfect equilibrium strategies in which an agent makes a renegotiation offer only if doing so strictly increases his expected payoff. This result is the counterpart to Proposition 1 of Hart and Moore.

Proposition 1. Let (p_0, p_1) be the initial option contract signed at date 0. Given investment levels $\beta \in [0, \bar{\beta}]$ and $\sigma \in [0, \bar{\sigma}]$, the traded quantity satisfies $q \in Q^*(\omega, \beta, \sigma)$ and the payment of the buyer to the seller is given by

[5] Hart and Moore allow for a finite number of renegotiation dates and arbitrarily complex contract offers. However, they do not properly specify strategy spaces. In order to have a well-defined game and to keep the exposition self-contained, we consider a simplified renegotiation process. Alternatively, the same arguments used by Hart and Moore to support their Proposition 1 could be used to obtain our Proposition 1. See Appendix A of their article.

(i) if $p_1 - p_0 \leq c(\omega^S, \sigma)$, then $p = p_0 + q \cdot c(\omega^S, \sigma)$
(ii) if $p_1 - p_0 > c(\omega^S, \sigma)$, then $p = p_1 - c(\omega^S, \sigma) + q \cdot c(\omega^S, \sigma)$.

Proof. See Appendix A.

Although the formal proof is relegated to Appendix A, the basic intuition for this result is easy to understand and will be explained in the remainder of this section.[6]

Given the initial contract with prices p_0 and p_1, the seller is willing to trade if and only if $p_1 - p_0 > c$. If the seller chooses $d = 0$, then $q = 0$ follows automatically. If he chooses $d = 1$, then it is a dominant strategy for the buyer to accept delivery (because $v > 0$ and the payment of the buyer is independent of whether he accepts delivery), so $q = 1$. Suppose the privately optimal decision of the seller is also socially optimal. In these cases, there is no scope for renegotiation: Efficient trade decisions will already result from the original contract, and each player can guarantee himself the corresponding payoff by not making a renegotiation offer and withholding any offer he might have received.

However, if the seller's privately optimal delivery decision is not efficient, there is scope for renegotiation. Note that renegotiation can only succeed if the buyer offers a new contract. To see this, suppose the buyer made no offer at the renegotiation stage. Then, no matter what new contract has been sent by the seller, the buyer can always induce the courts to enforce the old contract (p_0, p_1) by withholding any renegotiation offer he received. Therefore, the seller will not make the efficient trading decision until he has a new contract in hand, offered and signed by the buyer, which guarantees him at least what he could get from sticking to the old contract and taking the inefficient action. Thus, the buyer must give in and adjust prices such that they are more favorable for the seller if he makes the efficient delivery decision. On the other hand, the buyer need not give in too much. He makes the renegotiation offer, so he can suggest new prices that make the seller just indifferent whether to reverse his delivery decision. Hence, the buyer has all the bargaining power in the renegotiation game. In case (i), if $v > c$, the buyer thus needs to raise the delivery price to $p_0 + c$ to induce the seller to produce and deliver. Note that, because $v > c$, it is profitable for the buyer to do so instead of forgoing delivery under the original contract. By the same argument, the buyer needs to raise the no-delivery payment to $p_1 - c$ in case (ii), provided that trade is inefficient ($v < c$), in order to induce the seller to forgo production; because $v < c$, he will choose to do so.

Let us finally compare Proposition 1 with the corresponding Proposition 1 in Hart and Moore. They assume that the courts cannot observe delivery (d) but only whether trade took place (q). Thus, a Hart–Moore contract also consists of two prices p_0 and p_1, but now p_i is the payment if $q = i$, $i \in \{1, 2\}$. The analysis is very similar except for the following two cases:

(a) If $v < c < p_1 - p_0$, a Hart–Moore contract is not renegotiated and yields $q = 0$ and $p = p_0$. Renegotiation is not necessary because the buyer, who does not want to trade, can prevent inefficient trade unilaterally, guaranteeing himself $u^B = -p_0 \geq c - p_1$.

(b) If $p_1 - p_0 > v > c$, a Hart–Moore contract is renegotiated, whereas an option contract is not. Trade is efficient, but without renegotiation of the Hart–Moore contract, the buyer would veto trade. Thus, in equilibrium, the seller has to offer to lower the trade payment to $p = v + p_0$, and payoffs are $u^S = v + p_0 - c$ and $u^B = -p_0$.

3. Efficient option contracts

■ What investment incentives are given by an option contract? Using Proposition 1, we can derive the expected utilities of the parties as a function of their investment choices

[6] Throughout the following heuristic discussion, we shall ignore cases in which either $p_1 - p_0 = c(\omega^S, \sigma)$ or $v(\omega^B, \beta) = c(\omega^S, \sigma)$.

and the initial contract. To do so, it will be convenient to choose a slightly different parameterization of an option contract, namely, to identify an option contract with a pair (p_0, k), where p_0 is a base payment, which has to be made anyway, and $k = p_1 - p_0$ is the option price. Denote the expected utility of agent i by $U^i(\sigma, \beta, p_0, k)$. We then have

Corollary 1. The expected utilities of the agents are given by

$$U^B(\sigma, \beta, p_0, k) = -h^B(\beta) - p_0 + \int_0^1 \int_0^1 [v(\omega^B, \beta) - c(\omega^S, \sigma)]^+ f(\omega) d\omega^B d\omega^S$$
$$- \int_0^1 [k - c(\omega^S, \sigma)]^+ f^S(\omega^S) d\omega^S \tag{4}$$

and

$$U^S(\sigma, \beta, p_0, k) = -h^S(\sigma) + p_0 + \int_0^1 [k - c(\omega^S, \sigma)]^+ f^S(\omega^S) d\omega^S. \tag{5}$$

Proof. Using Proposition 1, the date 2 payoffs of the buyer and the seller are given by

$$u^B = -h^B(\beta) + \begin{cases} -p_0 & \text{in (i) if } v(\omega^B, \beta) < c(\omega^S, \sigma) \\ v - p_0 - c & \text{in (i) if } v(\omega^B, \beta) \geq c(\omega^S, \sigma) \\ c - p_0 - k & \text{in (ii) if } v(\omega^B, \beta) < c(\omega^S, \sigma) \\ v - p_0 - k & \text{in (ii) if } v(\omega^B, \beta) \geq c(\omega^S, \sigma) \end{cases} \tag{6}$$

and

$$u^S = -h^S(\sigma) + \begin{cases} p_0 & \text{in (i)} \\ p_0 + k - c & \text{in (ii)} \end{cases}. \tag{7}$$

Integrating over ω^B, ω^S yields (4) and (5). *Q.E.D.*

Note that, for any option contract, the buyer's payoff is simply total surplus minus an expression that does not depend on his investment decision. This fact can be understood by noting that, whenever renegotiation occurs, the renegotiated price is determined by the seller's cost and thus independent of the buyer's investment decision. Consequently, the buyer receives the full marginal return on his investment if and only if trade is efficient. Hence, given the investment choice of the seller, the buyer will always invest efficiently. The problem is thus reduced to find an option contract that induces the seller to choose the efficient investment level σ^*.

Given an option contract (p_0, k), the seller chooses σ to solve

$$\max_{\sigma} U^S(p_0, k, \sigma) = -h^S(\sigma) + p_0 + \int_0^1 [k - c(\omega^S, \sigma)]^+ f^S(\omega^S) d\omega^S, \tag{8}$$

where we dropped β as an argument of his utility function because his payoff is independent of the buyer's investment decision. Note that the set of maximizers of (8) is nonempty for every given option contract and depends only on the option price k. We let $\Sigma(k)$ denote the set of maximizers of (8) for a given k.

Lemma 1 shows that there always exist option prices $\underline{k}$ and $\bar{k}$, $\underline{k} < \bar{k}$, such that the seller is induced to underinvest (overinvest) relative to the efficient level σ^*. The intuition for this result can be seen from Corollary 1: If a sufficiently small option price has been chosen, then $[k - c(\omega^S, \sigma)]^+ = 0$ with probability 1 and the seller's payoff at date 2 (net of investment costs) is simply p_0 and thus independent of his investment level. Consequently, the seller will not invest. On the other hand, if the option price is chosen sufficiently high, then $[k - c(\omega^S, \sigma)]^+ = k - c(\omega^S, \sigma)$ with probability 1. Now the seller's payoff at date 2 is $p_1 - c$. Thus, an investment in cost reduction will pay off with probability 1. Because the probability that trade is efficient is less than or equal to 1, such an option price will provide the seller with an incentive to overinvest.

Lemma 1. Let $\bar{k} = \max_{\omega^S,\sigma} c(\omega^S, \sigma)$. Then

$$\sigma \in \Sigma(0) \Rightarrow \sigma \leq \sigma^*$$
$$\sigma \in \Sigma(\bar{k}) \Rightarrow \sigma \geq \sigma^*.$$

Proof. See Appendix A.

The possibility to induce overinvestment (underinvestment) by varying the option price suggests that it should be possible to find an option price k^* in $[\underline{k}, \bar{k}]$, which gives the seller the desired incentive to choose his first-best investment level. Whenever the seller's maximization problem for a given option price is sufficiently well behaved, this is indeed the case. In particular, we can now state our main result.

Proposition 2. Suppose the seller's maximization problem (8) has a unique solution $\sigma(k)$ for all $k \in [0, \bar{k}]$. Then there exists an option contract (p_0, k), which implements efficient investment and trade decisions. Furthermore, any division of the *ex ante* surplus can be achieved by choosing p_0 appropriately.

Proof. By Berge's maximum theorem, the function $\sigma(k)$ is continuous, and by the intermediate value theorem and Lemma 1, it follows that there exists $k^* \in [0, \bar{k}]$ such that $\sigma(k^*) = \sigma^*$. Given any option contract (p_0, p_1) with $p_1 = p_0 + k^*$, the seller will thus choose the efficient investment level σ^*. Anticipating this, the unique best response of the buyer is to choose β^*. Hence, every such contract implements first-best investment decisions, and renegotiation yields efficient trade by Proposition 1. Finally, note that we are free to choose the base payment p_0. Thus, it follows immediately from the expressions in Corollary 1 that any division of the *ex ante* surplus can be achieved. *Q.E.D.*

The uniqueness assumption in Proposition 2 is essentially a continuity requirement, ensuring that, by varying the option price, it is possible to fine-tune the investment incentives of the seller. Obviously, this assumption is satisfied if the seller's payoff function given by (8) is strictly quasiconcave in σ. Note, however, that the standard assumptions that $h^S(\sigma)$ and $c(\cdot, \sigma)$ are convex in σ are not sufficient to guarantee this property. The problem is that a variation in σ not only affects $h^S(\cdot)$ and $c(\cdot)$, but also the set of states of the world in which $k - c(\omega^S, \sigma) \geq 0$ and thus the probability that the seller's investment pays off. If for some (k, σ) this effect is too strong, it may generate convexities in the seller's utility function. Appendix B discusses explicit conditions on the underlying cost functions $h^S(\cdot)$ and $c(\cdot, \cdot)$ that ensure that the seller's problem is strictly concave in σ for all possible option prices.

How are our findings different from those of Hart and Moore? Given a Hart–Moore contract, at least one party will block trade whenever trade is inefficient. Furthermore, if $c < p_1 - p_0 < v$, trade is efficient and will take place because both parties are willing to trade. In these cases, private and social marginal returns of investments coincide. In all other cases, however, at least one party has an incentive to underinvest: If $v > c > p_1 - p_0$, the buyer's incentives are fine but the seller's marginal return is 0, so he will underinvest. If $p_1 - p_0 > v > c$, an opposite result is obtained. The seller has the right incentives, but the buyer's investment does not pay off. Hart and Moore's underinvestment result stems from the fact that, in general, it is impossible to choose $p_1 - p_0$ such that the probabilities of these two cases vanish at the same time. In contrast, given our option contracts, the seller is induced to overinvest if $v < c < p_1 - p_0$ and to underinvest if $v > c > p_1 - p_0$. By choosing k appropriately, it is possible to balance the probabilities of these cases such that, on average, the seller has just the right incentives to invest.

4. Efficiency without renegotiation

■ Our argument for the efficiency of simple option contracts relies on the assumption that the parties can use costless renegotiation to avoid *ex post* inefficient delivery decisions by the seller. Although this is in the tradition of the contributions by Hart and Moore (1988) and Aghion, Dewatripont, and Rey (1994), the question remains whether there are circumstances in which the terms of the original contract can be designed to induce both efficient investment decisions and *ex post* efficient delivery decisions without renegotiation. Addressing this issue will also allow us to explain how our article relates to the recent work by Hermalin and Katz (1993), which does not use renegotiation to achieve the first best.

Suppose first that, given the optimal investment levels, trade is efficient with probability 1. In this case, the first best can be achieved without renegotiation by choosing a sufficiently high option price.

Proposition 3. Suppose that, given the efficient investment choices (β^*, σ^*), trade is efficient with probability 1. Then any option contract with $k = \bar{k}$ implements efficient investment decisions and the initial contract is renegotiated with probability 0.

Proof. See Appendix A.

The result in Proposition 3 is stronger than the corresponding "no renegotiation" result in Hart and Moore, which not only requires that trade be efficient with probability 1 but also that one find a constant k such that, with probability 1, $v(\omega^B, \beta^*) \geq k \geq c(\omega^S, \sigma^*)$. The reason why this additional condition appears in their result but not in ours is simple. To ensure that an option contract is not renegotiated, it suffices to ensure that the seller always prefers to deliver under the original contract. This can be done by choosing k sufficiently large. With a Hart–Moore contract, on the other hand, the buyer has the power to veto trade under the original contract. Thus, to avoid renegotiation, it is also necessary to ensure that the buyer wants to trade under the original contract, which requires $k \leq v(\omega^B, \beta^*)$ for all ω^B.

A different approach to avoid renegotiation (which applies more generally) is to specify a more complicated initial contract that requires one party to send a verifiable message to the other party after the uncertainty has been resolved. This is the approach suggested by Hermalin and Katz (1993). Under the additional assumption that the seller's cost and the buyer's valuation are stochastically independent, these authors show that fill-in-the-price contracts can implement the first best. Their idea is easily embedded in our extensive form: Suppose that the initial contract specifies that, after date 1, the buyer has to announce an option price $k \in \mathbb{R}$. This announcement can be verified by the court. The initial contract also specifies a real-valued function $p_0(k)$ with the interpretation that $(p_0(k), k)$ is the option contract in force if the buyer announces the option price k.

Suppose that renegotiation of the option contract selected by the buyer is not feasible. Under this condition, Proposition 1 in Hermalin and Katz shows that, if the initial contract specifies the menu of option contracts given by

$$p_0(k) = \int_0^1 [v(\omega^B, \beta^*) - k]^+ f^B(\omega^B) d\omega^B + t, \tag{9}$$

where t is an arbitrary constant, then the buyer will select the option price $k = c(\omega^S, \sigma)$ and the seller will take the efficient delivery decision in equilibrium. Furthermore, this contract induces efficient investment decisions. To see this, note that, because $k = c(\omega^S, \sigma)$, the seller's expected payoff is given by

$$U^S(\sigma) = \int_0^1 p_0(c(\omega^S, \sigma)) f^S(\omega^S) d\omega^S + t$$

$$= \int_0^1 \left[\int_0^1 [v(\omega^B, \beta^*) - c(\omega^S, \sigma)]^+ f^B(\omega^B) d\omega^B \right] f^S(\omega^S) d\omega^S + t. \tag{10}$$

If ω^B and ω^S are stochastically independent, this expression equals the expected social surplus (given β^*) as a function of σ, so the seller has just the right incentives to invest.[7] On the other hand, by setting the option price k equal to the seller's cost, the buyer extracts all the surplus from the seller (minus the constant $p_0(k)$, which is independent of the buyer's investment). Hence, given that the seller chooses σ^*, the buyer is residual claimant of social surplus on the margin and will also invest efficiently.

Let us now allow for renegotiation of the option contract selected by the buyer. That is, suppose that, after the buyer has made his selection from the menu of option contracts, the parties are free to renegotiate the resulting contract, as in Section 2. Clearly, this will not affect Hermalin and Katz's argument if it is the case that the buyer will still find it optimal to select the option price $k = c(\omega^S, \sigma)$ because then $(p_0(k), k)$ induces efficient trade and there is nothing to be renegotiated. As the following result shows, this is indeed the case.

Proposition 4. Suppose the initial contract specifies the menu of option contracts given by (9). Then for all ω there is an equilibrium in which the buyer selects the option price $k = c(\omega^S, \sigma)$ and the resulting contract is not renegotiated.

Proof. See Appendix A.

This result shows that an initial contract as specified in (9) implements an efficient allocation without renegotiation if the buyer's valuation and the seller's cost are stochastically independent. The tradeoff, however, is that the parties have to specify a more complicated menu of contracts initially.

5. Variable quantities and/or qualities

■ In this section, we discuss briefly a simple extension of our main result to the case where $q \in Q$ and Q is some (finite or infinite) subset of an Euclidean space of possible quantities and/or qualities of the good to be produced and consumed. We shall derive a simple condition, which is necessary for an option contract to implement the first best. This condition is automatically satisfied if $q \in \{0, 1\}$. Assuming (as in Proposition 2) that the seller's maximization problem has a unique solution, this condition is also sufficient to guarantee implementation of an efficient allocation.

The case discussed in this section is a strict generalization of the case considered in Sections 2 and 3. In the general case, an option contract specifies some level of quantity and/or quality q_1 and two prices, p_0 and p_1. We assume that the court can distinguish whether q_1 was delivered. The contract says that, if the seller delivers q_1, the buyer is required to pay the price p_1. If any other $q \neq q_1$ is delivered and if the contract has not been renegotiated, then the buyer can keep q and must pay only p_0. In the renegotiation game, each agent may propose a new contract $(\tilde{p}_0, \tilde{p}, \tilde{q})$ specifying some $\tilde{q}$, a price $\tilde{p}$ if

[7] If ω^B and ω^S are not independent, there is no obvious way to design a fill-in-the-price contract along the lines of Hermalin and Katz such that both parties have the right incentives to invest.

$\tilde{q}$ is delivered, and a no-trade payment $\tilde{p}_0$.[8] The utilities of the buyer and the seller are given by

$$u^B = v(q, \omega^B, \beta) - p - h^B(\beta) \tag{11}$$

$$u^S = p - c(q, \omega^S, \sigma) - h^S(\sigma). \tag{12}$$

Trade of q takes place if and only if the seller delivers q at date 2 and the buyer accepts delivery. Let $q_0 \in Q$ denote the event of no trade with $v(q_0, \cdot) = c(q_0, \cdot) = 0$. For all $q \neq q_0$, the valuation of the buyer and the production cost of the seller are strictly positive. As before, we also assume that production costs are nonincreasing in σ and continuous. The first-best investment levels maximize

$$W(\beta, \sigma) = \int_0^1 \int_0^1 \left[\max_{q \in Q} \{v(q, \omega^B, \beta) - c(q, \omega^S, \sigma)\} \right] f(\omega) d\omega^B d\omega^S - h^B(\beta) - h^S(\sigma). \tag{13}$$

Assume that there exists a unique pair (β^*, σ^*) maximizing this expression, and let $Q^*(\omega, \beta, \sigma) = \text{argmax}_{q \in Q}\{v(q, \omega^B, \beta) - c(q, \omega^S, \sigma)\}$ be nonempty. The following proposition summarizes the outcome of the renegotiation game and is the counterpart of Proposition 1:

Proposition 5. Let (p_0, p_1, q_1) be the initial option contract signed at date 0. Given investment levels $\beta \in [0, \bar{\beta}]$ and $\sigma \in [0, \bar{\sigma}]$, the specifications of trade satisfies $q \in Q^*(\omega, \beta, \sigma)$ and the payment from the buyer to the seller is given by

(i) if $p_1 - p_0 \leq c(q_1, \omega^S, \sigma)$, then $p = p_0 + c(q, \omega^S, \sigma)$

(ii) if $p_1 - p_0 > c(q_1, \omega^S, \sigma)$, then $p = p_1 + c(q, \omega^S, \sigma) - c(q_1, \omega^S, \sigma)$.

The formal proof is omitted because it is a simple generalization of the proof of Proposition 1. To give some intuition for it, consider two cases in turn.

(i) If $p_1 - p_0 < c(q_1, \omega, \sigma)$, then in the absence of renegotiation, the seller will refuse to trade ($q = q_0$). Clearly, delivering q_1 is not profitable. Furthermore, it is never profitable for the seller to deliver any other $\tilde{q} \neq q_1$ because production costs are positive, whereas the payment p_0 is independent of whether he delivers $\tilde{q}$ or refuses to trade. If $q_0 \in Q^*(\omega, \beta, \sigma)$, i.e., no trade is efficient, then there is no scope for renegotiation and the outcome is given by (q_0, p_0). So suppose that $q_0 \notin Q^*$. The seller is only willing to deliver $\tilde{q} \neq q_0$ if he gets a renegotiation offer signed by the buyer saying that $\tilde{q}$ will be traded for payment $\tilde{p}$, where $\tilde{p}$ has to be large enough to give the seller at least the utility of his default point (q_0, p_0). Hence, the buyer's renegotiation offer will satisfy

$$\tilde{p} - c(\tilde{q}, \omega^S, \sigma) = p_0 \tag{14}$$

and the seller will accept this contract, deliver $\tilde{q}$, and enforce $\tilde{p}$. Because the buyer

[8] Although an option contract unambiguously specifies what payments should be enforced by the courts, it could be argued that such a contract is unlikely to be enforceable in practice. In particular, if the seller chooses a specification q that departs only slightly from the q_1 agreed upon in the contract, the price drops from p_1 to p_0 even if the utility loss incurred by the buyer is small. However, when the courts feel that damage payments depart too much from actual (or expected) damages, they may dismiss them as inadequate or punitive and refuse to enforce them. See Edlin and Reichelstein (1993) and the literature cited there. This problem is common to most theoretical analyses of contracts. Note, however, that the general logic of our arguments applies even if the courts enforce p_1 for any q delivered by the seller, which is in a neighborhood of q_1, as long as this implicit option contract induces overinvestment. We are grateful to Mike Riordan for this observation.

can extract all the surplus from the seller, the optimal renegotiation offer satisfies $\tilde{q} \in Q^*(\omega, \beta, \sigma)$.

(ii) If $p_1 - p_0 > c(q_1, \omega, \sigma)$, the default point of the seller is to deliver q_1. If q_1 is the efficient level of trade, there is nothing to renegotiate, so suppose that $q_1 \notin Q^*(\omega, \beta, \sigma)$. Again, the buyer has to make an offer that induces the seller not to deliver q_1. Thus, in equilibrium, the buyer will offer $(\tilde{q}, \tilde{p})$ such that

$$\tilde{p} - c(\tilde{q}, \omega^S, \sigma) = p_1 - c(q_1, \omega^S, \sigma) \tag{15}$$

and $\tilde{q} \in Q^*(\omega, \beta, \sigma)$. Again, the seller will accept this offer, deliver $\tilde{q}$, and enforce $\tilde{p}$.

Anticipating this renegotiation outcome, the expected utilities of the agents are given by

$$U^B(\sigma, \beta, p_0, k, q_1) = \int_0^1 \int_0^1 \max_{q \in Q} [v(q, \omega^B, \beta) - c(q, \omega^S, \sigma)] f(\omega) d\omega^B d\omega^S$$
$$- h^B(\beta) - p_0 - \int_0^1 [k - c(q_1, \omega^S, \sigma)]^+ f^S(\omega^S) d\omega^S \tag{16}$$

$$U^S(\sigma, \beta, p_0, k, q_1) = -h^S(\sigma) + p_0 + \int_0^1 [k - c(q_1, \omega^S, \sigma)]^+ f^S(\omega^S) d\omega^S. \tag{17}$$

For any option contract, the buyer's expected payoff coincides with social welfare minus a term that is independent of his investment decision. Thus, the buyer will always invest efficiently. The investment incentives of the seller depend on the choice of q_1 and k, and we let $\Sigma(q_1, k)$ denote the set of maximizers of (17) given these parameters. Also, let $\bar{k}(q_1) = \max_{\omega^S, \sigma} c(q_1, \omega^S, \sigma)$.

The crucial consideration in determining whether it is possible to achieve the first best with an option contract is whether it is possible to induce the seller to invest at least the efficient amount by specifying a sufficiently high option price. In Lemma 2, we give a necessary and sufficient condition for this to be the case. The condition requires that there exists a q_1 such that the seller is induced to overinvest if he receives the returns for his investment with probability 1. Note that, if $Q = \{0, 1\}$, this condition is always satisfied, as was shown by Lemma 1.

Lemma 2. There exists an option contract that induces the seller to invest at least the efficient amount, i.e., $\exists(q_1, k)$: $\sigma^* \leq \max\Sigma(q_1, k)$, if and only if there exists $q_1 \in Q$ such that

$$\exists \sigma \geq \sigma^*: \sigma \in \underset{\sigma'}{\operatorname{argmax}} \int_0^1 - c(q_1, \omega^S, \sigma') f^S(\omega^S) d\omega^S - h^S(\sigma'). \tag{18}$$

In particular, the first best cannot be implemented if (18) fails for all q_1.

Proof. See Appendix A.

The economic interpretation of (18) will be discussed after Proposition 6, which is the counterpart to Proposition 2.[9]

[9] As in Proposition 2, the following result imposes a uniqueness requirement on the solution to the seller's maximization problem. Note that this assumption is only required for one particular choice of q_1. Specifically, it suffices to find a specification q_1 such that (18) holds and such that $c(q_1, \cdot, \cdot)$ satisfies the conditions discussed in Appendix B.

Proposition 6. Suppose there exists a q_1 satisfying (18). Furthermore, assume that, given this q_1, there exists a unique $\sigma(k)$ maximizing the seller's payoff function (17) for all $k \in [0, \bar{k}(q_1)]$. Then there exists an option contract (p_0, p_1, q_1) that implements efficient investment and trade decisions. Furthermore, any division of the *ex ante* surplus can be achieved by choosing p_0 appropriately.

Proof. Let q_1 be as specified in the statement of the proposition. Then $\sigma(0) = 0$ (cf. the proof of Lemma 1) and $\sigma(\bar{k}(q_1)) \geq \sigma^*$ (cf. Lemma 2). The result then follows from the uniqueness assumption on $\sigma(k)$ as in the proof of Proposition 2. *Q.E.D.*

In the literature on incomplete contracts, it has often been claimed that q is noncontractible *ex ante* because it is too difficult to specify the good in advance, in particular because the optimal specification may depend on a complex realization of the state of the world. The above proposition shows that optimal investment incentives can be given with a simple option contract in which only one specification of the good (q_1) has to be described.[10] If this good were traded with probability 1, the seller would be induced to overinvest. By choosing the option price k appropriately, the incentives to overinvest (if q_1 is the default point) and to underinvest (if q_0 is the default point) can be balanced such that the seller will invest efficiently, whereas renegotiation ensures efficient trade.

However, a necessary condition for an option contract to implement the first best is that (18) holds. If q is interpreted as the quantity to be produced, this condition is innocuous. It is natural to assume that the marginal benefit of investment is nondecreasing with the quantity of trade in all states of the world. Thus, it would be sufficient to pick q_1 as the largest q that is traded with positive probability in the first best in order to induce the seller to overinvest. If q represents different levels of quality, which can be ordered along the real line such that higher levels of quality make a higher level of investment more desirable in all states of the world, (18) is also unproblematic. However, if q stands for different specifications of the good and, if for any specification, the productivity of the investment depends on the realization of the state of the world, then there are natural examples in which (18) is violated. As an illustration, suppose that, for any given $q_1 \in Q$, the investment pays off only for some states of the world but is unproductive in others. Thus, if any fixed q_1 is traded with probability 1, only some modest investment is optimal. On the other hand, the *ex post* efficient q^* depends on the state of the world. Thus, given that $q^*(\omega)$ will be traded, the investment may now pay off in all states of the world. Hence, the socially optimal investment level σ^* may be higher than the optimal investment level given any fixed $q_1 \in Q$.

A related problem that may prevent the implementation of the first best with a simple option contract is the possibility that investments may be multidimensional. For example, suppose that the seller has a choice between investing in a general purpose technology or in one of several specific purpose technologies that are only useful if a given specification is produced. Because there is uncertainty about the optimal specification *ex ante,* it may be efficient to invest in the general purpose technology, whereas an option contract will provide the seller with a strong incentive to invest in a technology that is tailored to the specification in the option contract. Although the analysis of the hold-up problem with multidimensional investment decisions is beyond the scope of this article, it clearly is an important topic for future research.

[10] Note that the incomplete contracts literature typically assumes that it is possible to contractually describe a single specification *ex post,* so it should also be possible to do this *ex ante*. However, it is essential that this specification be described unambiguously, which may be more difficult if the good does not yet exist (e.g., if some research is necessary to develop it).

6. Conclusions

■ We have shown that, in the canonical hold-up model introduced by Hart and Moore (1988), the first best can be achieved if the courts can verify delivery of the good by the seller. This can be done using a very simple contract that does not rely on renegotiation design or complicated revelation mechanisms. The crucial feature of the contracts we used is that one of the parties can decide unilaterally whether trade takes place. This is why we called them option contracts.

Throughout the article, we restricted attention to the case in which both parties are risk neutral and in which there are only indirect externalities of the investment decisions.[11] This is the case considered in most of the hold-up and incomplete contracts literature. We question whether an option contract can implement the first best in more complex environments in which these assumptions are relaxed. There is no hope that an option contract can allocate risk efficiently. This would require that the default point of renegotiation vary continuously with the realization of the state of the world, which is impossible to achieve with the very simple instrument considered in this article.[12] Option contracts are more successful in dealing with one-sided direct externalities. Suppose that the investment of the seller affects the valuation of the buyer directly, e.g., because it has an impact on the quality of the good. In this case, an option contract can induce both parties to invest efficiently, provided that it is still possible to induce the seller to overinvest. In this case, we can choose the option price such that, in expectation, the seller has just the right incentives to invest. Given that he chooses the first-best level σ^*, the unique best response of the buyer is to also invest efficiently. On the other hand, if there is a two-sided direct externality, option contracts fail to implement the first best. Although the seller can be given the right incentives as just described, the buyer will not take into account the impact of his investment on the costs of the seller.[13] An interesting question for future research is whether there are other contracts that give optimal investment incentives in this case and, in particular, whether the first best can be achieved using simple real-world contracts, which do not rely on complex revelation mechanisms.

Appendix A

■ Proofs of Propositions 1, 3, and 4 and proofs of Lemmas 1 and 2 follow.

Proof of Proposition 1. We first show that the seller does not make a renegotiation offer in equilibrium. Given our assumption that an agent makes a renegotiation offer only if it strictly increases his payoff, it suffices to show that the seller has nothing to lose from dropping any renegotiation offer he might make. To see this, consider a subgame starting after renegotiation offers $(\bar{p}_0^i, \bar{p}_1^i)$, $i \in \{B, S\}$, have been made[14] and let $p_d^* = \max\{p_d, \bar{p}_d^B\}$. We argue that the seller's continuation payoff cannot exceed $\max_d\{p_d^* - dc\}$. Suppose the seller chooses $d = 0$. The buyer can ensure that his payment does not exceed p_0^* by withholding any contract the seller may have sent. Hence, the seller's expected continuation payoff (after he has chosen $d = 0$), which is uniquely determined because the contract submission game is zero sum, must be smaller than p_0^*. If the seller chooses $d = 1$, an equivalent argument implies that the seller's expected continuation payoff cannot exceed $p_1^* - c$. We should note that the seller can ensure the payoff $\max_d\{p_d^* - dc\}$ by not making a renegotiation offer, choosing the appropriate delivery decision, and submitting the contract that specifies the price p_d^* to the court. Because the buyer has only the initial contract to submit, the payment p_d^* will then be enforced by the court.

Consider now a subgame that results after the buyer (or no agent) has made a renegotiation offer. In a subgame-perfect equilibrium, the seller must choose d to maximize $\max\{\bar{p}_d^B, p_d\} - d \cdot c$ and then submit the

[11] The externalities are indirect because $\beta(\sigma)$ does not affect the cost of the seller (valuation of the buyer) directly but affects his utility only indirectly through the probability of trade.

[12] Aghion, Dewatripont, and Rey (1994) show that this can be achieved using a complex revelation mechanism.

[13] This can be seen from the buyer's utility function (4) given in Corollary 1. If β affects c, the last term of this expression is no longer a constant.

[14] To simplify notation, we let $(\bar{p}_0^i, \bar{p}_1^i) = (p_0, p_1)$ if player i did not make a renegotiation offer.

most profitable contract to the court. If the buyer does not make a renegotiation offer, the seller will thus choose not to deliver if strict inequality holds in case (i) and will choose to deliver in case (ii). For each of these cases, two subcases have to be distinguished:

(ia) $p_1 - p_0 < c(\omega^S, \sigma)$ and $v(\omega^B, \beta) \le c(\omega^S, \sigma)$. In this case, $q = 0$ is an efficient outcome, and the seller does not want to trade given the initial prices. It is easy to see that there does not exist a renegotiation offer the buyer could make that would increase his payoff. Hence, the buyer does not make a renegotiation offer. In any subgame-perfect equilibrium, the seller will thus not deliver, implying $q = 0$ and a transfer payment $p = p_0$.

(ib) $p_1 - p_0 < c(\omega^S, \sigma)$ and $v(\omega^B, \beta) > c(\omega^S, \sigma)$. In this case, trade would be efficient, but if the buyer does not make a renegotiation offer, the seller is not going to trade because not delivering gives him $p_0 > p_1 - c$. Because the seller's delivery decision depends only on the difference between the trade and the no-trade price, the buyer will not offer to raise the no-trade price in a subgame-perfect equilibrium. The buyer could send a renegotiation offer to the seller, raising the trade payment to $\tilde{p}_1^B = p_0 + c + \epsilon$ and leaving the no-trade price unchanged. For all $\epsilon > 0$, the seller will respond by delivering and enforcing the payment $\tilde{p}_1^B$. If the seller also responds by delivering for $\epsilon = 0$, then it is optimal for the buyer to offer a new contract with $\tilde{p}_1^B = p_0 + c(\omega^S, \sigma)$. Indeed, no best response would exist for the buyer if the seller were not to deliver given this offer. Thus, it follows that, in a subgame-perfect equilibrium, the buyer offers a new contract with $\tilde{p}_1^B = p_0 + c(\omega^S, \sigma)$ and the seller chooses to deliver and then enforces the transfer $\tilde{p}_1^B$. Note that, because $v > 0$, every best response of the buyer specifies that he accepts delivery. Hence, $q = 1$.

(iia) $p_1 - p_0 > c(\omega^S, \sigma)$ and $v(\omega^B, \beta) < c(\omega^S, \sigma)$. No trade would be efficient, but if the buyer does not make a renegotiation offer, the seller will choose to deliver. As in case (ib), the buyer can strictly increase his utility by making a renegotiation offer, in this case, one that raises the no-trade payment to $\tilde{p}_0^B = p_1 - c + \epsilon$. For all $\epsilon > 0$, the seller will respond to such an offer by not delivering, implying $q = 0$, and enforcing the price $\tilde{p}_0^B$. Hence, in equilibrium, the buyer will offer $\tilde{p}_0^B = p_1 - c$ and the seller will not deliver and enforce $\tilde{p}_0^B$.

(iib) $p_1 - p_0 > c(\omega^S, \sigma)$ and $v(\omega^B, \beta) \ge c(\omega^S, \sigma)$. In this case, the seller wants to deliver given the old prices and trade is efficient. As in case (ia), there is no renegotiation offer the buyer could make that would improve his utility. Thus, the seller will deliver and the transfer is p_1.

Finally, we need to consider the case $p_1 - p_0 = c(\omega^S, \sigma)$. Then the seller is indifferent whether to deliver under the terms of the initial contract. If $v = c$, any decision by the seller results in efficient trade and the buyer is indifferent between receiving the good and paying p_1 and not receiving it and paying p_0, so that he has nothing to gain from making a renegotiation offer. If $v \neq c$, a best response for the buyer only exists if the seller takes the efficient delivery decision, either without receiving a renegotiation offer or after receiving a renegotiation offer that specifies exactly the same prices as the initial contract. *Q.E.D.*

Proof of Lemma 1. Let $k = 0$. Because production costs are nonnegative, the seller's problem is then to maximize

$$U^S(p_0, 0, \sigma) = p_0 - h^S(\sigma). \tag{A1}$$

Because h^S is increasing in σ, the only solution to this problem is $\sigma = 0$. Hence, because $\sigma^* \ge 0$, the first claim follows.

Let $k = \bar{k}$. Given (p_0, k), the seller's problem is to maximize

$$U^S(p_0, k, \sigma) = \int_0^1 (k - c(\omega^S, \sigma))f^S(\omega^S)d\omega^S + p_0 - h^S(\sigma). \tag{A2}$$

This problem has the same set of maximizers as the problem

$$\max_\sigma \int_0^1 \int_0^1 (v(\omega^B, \beta^*) - c(\omega^S, \sigma))f(\omega)d\omega^S d\omega^B - h^S(\sigma) - h^B(\beta^*), \tag{A3}$$

which, in turn, is the same as

$$\max_\sigma W(\beta^*, \sigma) - \int_0^1 \int_0^1 [c(\omega^S, \sigma) - v(\omega^B, \beta^*)]^+ f(\omega)d\omega^S d\omega^B. \tag{A4}$$

Because $c(\omega^S, \sigma)$ is nonincreasing in σ for all ω^S, the integral added to $W(\beta^*, \sigma)$ is nondecreasing in σ. Hence, this problem cannot have a maximizer $\sigma < \sigma^*$. *Q.E.D.*

Proof of Proposition 3. From Lemma 1 $\sigma \in \Sigma(\bar{k}) \Rightarrow \sigma \geq \sigma^*$. Consider (A4) in the proof of Lemma 1. Under the stated condition

$$\int_0^1 \int_0^1 [c(\omega^S, \sigma) - v(\omega^B, \beta^*)]^+ f(\omega) d\omega^S d\omega^B = 0 \tag{A5}$$

for all $\sigma \geq \sigma^*$. Thus, for all $\sigma \geq \sigma^*$ the seller's maximization problem is equivalent to maximizing $W(\beta^*, \sigma)$ plus a constant. Consequently, σ^* is the unique optimal investment choice for the seller. Hence, the buyer will also choose $\beta = \beta^*$. *Q.E.D.*

Proof of Proposition 4. It follows from Proposition 1 that, if the buyer selects an option contract that makes it optimal for the seller to take the efficient delivery decision, then this contract will not be renegotiated and the seller will indeed trade efficiently. Hence, for all such choices, the buyer's continuation payoff is as specified in Hermalin and Katz (1993). We must still show that the buyer cannot strictly gain by selecting a contract that would be renegotiated instead of choosing $k = c(\omega^S, \sigma)$. There are two cases to consider.

First, suppose $v(\omega^B, \beta) > c(\omega^S, \sigma)$, that is, trade is efficient. It follows from Proposition 1 that renegotiation will occur if and only if the buyer selects $k < c(\omega^S, \sigma)$. The resulting option contract $(p_0(k), k)$ will be renegotiated to the contract $(p_0(k), c(\omega^S, \sigma))$ and the seller will deliver, so the resulting payment by the buyer is given by $p_0(k) + c(\omega^S, \sigma)$. If the buyer chooses $k = c(\omega^S, \sigma)$ instead, the resulting option contract will not be renegotiated, the seller will deliver, and the buyer's payment is given by $p_0(c(\omega^S, \sigma)) + c(\omega^S, \sigma)$. Because $k < c(\omega^S, \sigma)$ implies $p_0(k) \geq p_0(c(\omega^S, \sigma))$, it follows that choosing $k < c(\omega^S, \sigma)$ cannot increase the buyer's payoff.

Second, suppose $v(\omega^B, \beta) < c(\omega^S, \sigma)$, that is, trade is inefficient. It follows from Proposition 1 that renegotiation will occur if and only if the buyer selects an option price $k > c(\omega^S, \sigma)$. The resulting option contract $(p_0(k), k)$ will be renegotiated to the contract $(p_0(k) + k - c(\omega^S, \sigma), c(\omega^S, \sigma))$, the seller will not deliver, and the buyer's payment is given by $p_0(k) + k - c(\omega^S, \sigma)$. If the buyer chooses $k = c(\omega^S, \sigma)$ instead, the resulting option contract will not be renegotiated, the seller will not deliver, and the buyer's payment is given by $p_0(c(\omega^S, \sigma))$. Because $k > c(\omega^S, \sigma)$ implies $p_0(c(\omega^S, \sigma)) - p_0(k) \leq k - c(\omega^S, \sigma)$, it follows that the buyer cannot increase his payoff by choosing $k > c(\omega^S, \sigma)$. *Q.E.D.*

Proof of Lemma 2. Note that, for all $q_1 \in Q$ and $k \geq \bar{k}(q_1)$,

$$\Sigma(q_1, k) = \underset{\sigma}{\operatorname{argmax}} - \int_0^1 c(q_1, \omega^S, \sigma) f^S(\omega^S) d\omega^S - h^S(\sigma). \tag{A6}$$

Suppose q_1 satisfies (18). Then (A6) implies $\max\Sigma(q_1, \bar{k}(q_1)) \geq \sigma^*$.

Suppose q_1 does not satisfy (18). We show that $\forall k$: $\max\Sigma(q_1, k) < \sigma^*$. For $k \geq \bar{k}(q_1)$, this follows from (A6). Consider $k < \bar{k}$. Then

$$U^S(\sigma, \beta, p_0, \bar{k}(q_1), q_1) - U^S(\sigma, \beta, p_0, k, q_1) = \int_0^1 [\bar{k}(q_1) - \max\{k, c(q_1, \omega^S, \sigma)\}]^+ f^S(\omega^S) d\omega^S.$$

Because production costs are nonincreasing in σ, this expression is nondecreasing in σ. It follows that $\forall k < \bar{k}(q_1)$: $\sigma \in \Sigma(q_1, k) \Rightarrow \sigma \leq \max\Sigma(q_1, \bar{k}(q_1)) < \sigma^*$. Hence, if there is no q_1 satisfying (18), then $\forall(q_1, k)$: $\max \Sigma(q_1, k) < \sigma^*$. *Q.E.D.*

Appendix B

■ Conditions on the cost functions $h^S(\cdot)$ and $c(\cdot, \cdot)$ follow.

Assume that c and h^S are twice continuously differentiable with $\partial c/\partial\sigma < 0$, $\partial^2 c/\partial\sigma^2 \geq 0$, $\partial c/\partial\omega^S > 0$, and $\partial h^S/\partial\sigma > 0$. For every option price k and investment level σ, there exists a unique $\omega^S(k, \sigma) \in [0, 1]$ such that

$$\omega^S < \omega^S(k, \sigma) \Rightarrow c(\omega^S, \sigma) < k$$
$$\omega^S > \omega^S(k, \sigma) \Rightarrow c(\omega^S, \sigma) > k.$$

The seller's utility function can thus be written as

$$U^S(p_0, k, \sigma) = \int_0^{\omega^S(k,\sigma)} [k - c(\omega^S, \sigma)] f^S(\omega^S) d\omega^S + p_0 - h^S(\sigma). \tag{B1}$$

Taking the derivative with respect to σ, we obtain

$$\frac{\partial U^S(p_0, k, \sigma)}{\partial \sigma} = -\int_0^{\omega^S(k,\sigma)} \frac{\partial c(\omega^S, \sigma)}{\partial \sigma} f^S(\omega^S) d\omega^S - \frac{dh^S(\sigma)}{d\sigma}. \tag{B2}$$

For a given k, $\omega^S(k, \sigma)$ is almost everywhere differentiable in σ with

$$\frac{\partial \omega^S(k, \sigma)}{\partial \sigma} = \begin{cases} 0 & \text{if} \quad k < c(1, \sigma) \\ -\dfrac{\dfrac{\partial c(\omega^S(k, \sigma), \sigma)}{\partial \sigma}}{\dfrac{\partial c(\omega^S(k, \sigma), \sigma)}{\partial \omega^S}} & \text{if} \quad c(0, \sigma) < k < c(1, \sigma). \\ 0 & \text{if} \quad k > c(1, \sigma) \end{cases} \tag{B3}$$

$\partial U^S/\partial \sigma$ is also almost everywhere differentiable in σ with

$$\frac{\partial^2 U^S}{\partial \sigma^2} = -\int_0^{\omega^S(k,\sigma)} \frac{\partial^2 c(\omega^S, \sigma)}{\partial \sigma^2} f^S(\omega^S) d\omega^S - \frac{d^2 h^S(\sigma)}{d\sigma^2} - \frac{\partial \omega^S(k, \sigma)}{\partial \sigma} \cdot \frac{\partial c(\omega^S(k, \sigma), \sigma)}{\partial \sigma} \cdot f^S(\omega^S(k, \sigma)). \tag{B4}$$

Whereas the first two terms in this expression are negative if both h^S and c are convex in σ, the third term will be strictly positive whenever $c(0, \sigma) < k < c(1, \sigma)$. Hence, assuming that either h^S or c is strictly convex will not suffice to imply that the seller's maximization problem is strictly concave in σ for all k. However, it is easy to state a condition that ensures that h^S is sufficiently convex to make the seller's problem strictly concave. In particular, let

$$\xi(\sigma) = \max_{\omega^S} \frac{\left(\dfrac{\partial c(\omega^S, \sigma)}{\partial \sigma}\right)^2}{\dfrac{\partial c(\omega^S, \sigma)}{\partial \omega^S}} f(\omega^S) \tag{B5}$$

and suppose that

$$\frac{d^2 h^S(\sigma)}{d\sigma^2} > \xi(\sigma). \tag{B6}$$

Then the first derivative $\partial U^S/\partial \sigma$ is strictly decreasing in σ for all k.

References

AGHION, P., DEWATRIPONT, M., AND REY, P. "Renegotiation Design with Unverifiable Information." *Econometrica,* Vol. 62 (1994), pp. 257–282.

CHUNG, T.-Y. "Incomplete Contracts, Specific Investments, and Risk Sharing." *Review of Economic Studies,* Vol. 58 (1991), pp. 1031–1042.

EDLIN A. AND REICHELSTEIN, S. "Holdups, Standard Breach Remedies, and Optimal Investment." Discussion Paper. University of California at Berkeley, 1993.

GROSSMAN, S.J. AND HART, O.D. "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration." *Journal of Political Economy,* Vol. 94 (1986), pp. 691–719.

HART, O.D. AND MOORE, J. "Incomplete Contracts and Renegotiation." *Econometrica,* Vol. 56 (1988), pp. 755–785.

HERMALIN, B.E. AND KATZ M.L. "Judicial Modification of Contracts between Sophisticated Parties: A More Complete View of Incomplete Contracts and Their Breach." *Journal of Law, Economics and Organization,* Vol. 9 (1993), pp. 230–255.

MACLEOD, W.B. AND MALCOMSON, J.M. "Investments, Holdup, and the Form of Market Contracts." *American Economic Review,* Vol. 83 (1993), pp. 811–837.

ROGERSON, W.P. "Contractual Solutions to the Hold-Up Problem." *Review of Economic Studies,* Vol. 59 (1992), pp. 777–793.

WILLIAMSON, O.E. *The Economic Institutions of Capitalism.* New York: Free Press, 1985.

Review of Economic Studies (1989) 56, 163-178
0034-6527/89/00130163$02.00

Litigation and Settlement under Two-Sided Incomplete Information

URS SCHWEIZER
University of Bonn

First version received August 1986; *final version accepted October* 1988 (*Eds.*)

This paper deals with a simple game of litigation and settlement with incomplete information. Parties are assumed to have the choice between settling their dispute out of court or resorting to costly litigation. The set of sequential equilibria is characterized and conditions are given under which an efficient equilibrium does exist. Efficient equilibria, however, will be ruled out by various tests of refinement. A comparative statics analysis is carried out with respect to the quality of private information which parties are assumed to receive before any moves have to be made.

INTRODUCTION

It seems to be a basic tenet of the economic analysis of law that voluntary exchange and transaction enhance efficiency. In particular, if few parties are involved and if it causes no difficulties to discover who it is that one wishes to deal with then transaction costs may be expected to be low and, hence, parties should be able to reach some mutually beneficial agreement. Such seems to be the case if some dispute arises between a potential plaintiff and a defendant. Here, parties always have the opportunity to voluntarily settle their dispute out of court instead of resorting to costly litigation. The fact that cases go to litigation appears to violate the principle according to which parties will voluntarily transact whenever some mutually beneficial transaction exists. Various attempts have been made to resolve the puzzle. Most of them (see Posner (1977, p. 435)) argue along the following lines. Suppose plaintiff P claims that defendant D owes P damages amounting to W dollars. The plaintiff's case if litigated will be found meritorious with probability π, whereas it will be judged to lack merit with complementary probability. According to the so-called British rule, all litigation costs of the (fixed) amount C have to be borne by the losing party. The defendant's expected loss L and the plaintiff's expected gain G from litigation amount to $L=\pi(W+C)$ and $G=L-C$. Since the loss exceeds the gain, there must exist prices S in the range $G<S<L$. The defendant would prefer to pay such a price if, by doing so, litigation were avoided whereas the plaintiff, by accepting such settlement terms, would also fare better. Therefore, voluntary transactions are predicted to lead to the case being settled at terms in the range $G<S<L$. In order to explain the fraction of cases that actually go to trial, parties are usually assumed to hold differing views about the outcome of trial. If the probability that P wins at trial as perceived by P sufficiently exceeds the probability of the same event but as perceived by D then P's expected gain could well be higher than D's expected loss. In this case, settlement terms would not exist which both parties were willing to accept. Landes (1971),

Posner (1977), Gould (1973), Shavell (1982) and Adams (1981) among others have proposed such deviating views on the part of parties in order to resolve the puzzle of why cases are ever litigated.

Obviously, decisions based on unqualified judgement and faulty views can lead to litigation and, in fact, may actually do so quite frequently in real life. But, given that the rationality of man is stressed elsewhere, the approach does not appear to be in accord with the usual reasoning in the economic analysis of law. To preserve the postulate of rationality, deviating assessments should rather be attributed to parties holding private information on the merit of the case. Rational parties, however, will also recognize that the opponent may possess private information which is relevant for the case. As negotiations proceed, they attempt to infer as much information as possible from observing their opponent's strategic behaviour. To describe such a situation properly, the setting has to be one of a game of incomplete information where strategic interaction, including transmission of private information, is taken into account. In such a framework, however, simply comparing gains and losses from litigation will not be sufficient to predict whether the case ends in court or whether parties agree to settle.

Games of litigation and settlement have been studied before by, among others, Samuelson (1982), Salant and Rest (1982), P'Ng (1983), Bebchuk (1984), Salant (1984) and Reinganum and Wilde (1986). What most of these papers have in common is that the game is formulated in extensive form and essentially consists of a sequence of two steps. After the suit has been filed, one party, in the first step, proposes settlement terms which, in the second step, the other party either accepts or rejects. If the last-moving party accepts, the case is settled out of court at the proposed terms. Otherwise, it will be litigated. Except for Samuelson (1982), who discusses a numerical example in a slightly different framework, the above articles deal with a setting of one-sided incomplete information. They might be distinguished according to whether the informed player moves first (P'Ng, Salant and Rest, Salant) or last (Bebchuk). They might further be classified according to whether settlement terms are given exogenously (Samuelson, Salant and Rest, P'Ng) or whether they are determined endogenously (Bebchuk, Salant). If the informed player moves last, private information will not be transmitted in any direct way. Here the equilibrium turns out to be unique. If, however, the informed player is assumed to move first, strategic transmission of private information has to be taken into account. The framework then corresponds to a signalling game and, as such, will typically allow for a large set of equilibria.

The present paper deals with the issue of litigation versus settlement from the view of a framework of two-sided asymmetric information which contains the information setup of previous articles as limiting cases. It is organized as follows. In Section 1, details of the game are introduced. In Section 2, the game's set of sequential equilibria will be characterized. First, pooling equilibria are shown to exist under which, of course, no transmission of information takes place. If the game admits efficient equilibria at all, i.e. equilibria where litigation is avoided with certainty, then they have to be of the pooling type. Put differently, if the second-moving party is able to learn part of his opponent's private information then the case will end in court with positive probability. Second, equilibria are shown to exist where the first-moving party's private information is fully transmitted but where that party uses pure strategies only. Such equilibria are of the separating type. They have the case litigated with positive probability. Moreover, among the separating equilibria, there exists one which is most efficient in the sense that it leads to minimum ex-ante probability of litigation. The corresponding equilibrium outcome turns out to survive the various tests of refinement. Third, semi-pooling equilibria will also exist. Here, some but not all private information is transmitted and, hence, the case will end up in court with positive probability as well.

In Section 3, direct mechanisms are considered which are ex-post efficient, individually rational and Bayesian incentive compatible. Such mechanisms are shown to exist if, and only if, the defendant's expected loss of litigation if he holds favourable private information exceeds the plaintiff's expected gain if he, too, has received a favourable signal. If parameter values are such that this condition is not met then no efficient mechanism will exist. It then follows from the revelation principle that, for such values, no bargaining procedure, no matter how sophisticated, will ever have an efficient equilibrium. As a consequence, the performance of bargaining under incomplete information should not be measured against efficiency but rather against the outcome under the optimum direct mechanism.

In Section 4, the cutting power of various tests of refinement of sequential equilibrium will be explored in the framework of our simple game of litigation and settlement. The intuitive criterion proposed by Cho and Kreps (1987) turns out to have very little cutting power. In terms of equilibrium outcomes, not a single one will be ruled out. The stronger criterion of divinity as proposed by Banks and Sobel (1987) has more cutting power. In particular, some of the pooling equilibria will fail to pass the test. If, however, efficient equilibria exist then one of them, as well as some of the separating ones, will survive the test of divinity. In particular, the test does not allow to discriminate between efficient and inefficient equilibria. The test of universal divinity, however, singles out a unique equilibrium outcome. It is the most efficient one among the separating equilibria where the first-moving party uses pure strategies only. This outcome or, to be precise, its adjoining one in a discretized version of the game, will also be the one which is stable in the sense of Kohlberg and Mertens (1986). It follows that, if an efficient equilibrium happens to exist, it will be ruled out by these tests, at least, by these which have substantial cutting power.

In Section 5, finally, the unique outcome singled out by the tests of refinement will be the subject of a comparative statics analysis with respect to the quality of private information. Here, quality refers to a party's ability to predict the outcome of litigation. For the unique equilibrium, the probability of litigation turns out to increase as one of the parties receives more accurate information. The corresponding loss of efficiency is due to the fact that, under such a separating equilibrium, the weak type of the first-moving party must be given no incentives to mimic its strong counterpart. In terms of expected payoffs, it always turns out to be worthwhile for the second-moving player to receive more accurate private information. As for the first-moving player, in general no such prediction can be made. He might well suffer from receiving more accurate information. In a setting of incomplete information, there need not be a first-mover's advantage!

1. A GAME OF LITIGATION AND SETTLEMENT

Details of the game are as follows. In a preliminary move, nature provides parties with information on the merits of the case. The defendant D's information is denoted by i and consists of either $i = b$ (bad news for him) or $i = g$ (good news for him). The plaintiff P receives signal j which, again, refers either to $j = b$ (bad for P) or $j = g$ (good for P). Private information is assumed to be independent. Information state (i, j) occurs with probability $p_i q_j$. In state (i, j), the plaintiff's case, if litigated, will be found meritorious with probability π_{ij}. It is assumed that a party's bad or good news is bad or good for him irrespective of which signal the opponent has received. As for D, bad news $(i = b)$

means a high probability of P's winning at trial, whereas, for P, bad news ($j=b$) consists of a low such probability and, vice versa, with respect to good news $i=g$ and $j=g$, respectively. In other words, the following set of inequalities is assumed to hold throughout the paper:

$$\pi_{gb}<\pi_{bb}<\pi_{bg} \quad \text{and} \quad \pi_{gb}<\pi_{gg}<\pi_{bg}.$$

In state (i,j), the defendant's expected loss from litigation amounts to $L_{ij}=\pi_{ij}(W+C)$, the plaintiff's gain to $G_{ij}=L_{ij}-C$. From the interim view, i.e. right after nature's move, parties have obtained their private information but they do not know the full state of the world. Let D_i denote the defendant having obtained information i. Then D_i's expected loss from litigation amounts to $L_i=q_bL_{ib}+q_gL_{ig}$. Similarly, for plaintiff P_j holding information j, the expected gain from litigation amounts to $G_j=p_bG_{bj}+p_gG_{gj}$.

After this preliminary stage, it is the defendant's turn to propose settlement terms. His strategy is denoted by

$$s_i(S)=\text{prob}\,\{D_i \text{ offers terms } S\}.$$

At the next (and final) stage, the plaintiff either accepts the proposed terms or, else, the case will be litigated. His strategy is denoted by

$$a_j(S)=\text{prob}\,\{P_j \text{ accepts terms } S\}.$$

In a sequential equilibrium (see Kreps and Wilson (1982)), the second-moving party P is assumed to form beliefs about the opponent's private information. Let $\mu_i(S)$ denote the probability that D has obtained information i as perceived by P, given that D has offered terms S. For out-of-equilibrium messages, beliefs are in no way restricted. If, however, terms S are offered with positive probability, beliefs will be updated according to Bayes' rule:

$$[p_bs_b(S)+p_gs_g(S)]\mu_i(S)=p_is_i(S).$$

For a given set of beliefs, plaintiff P_j's updated expected gain from litigation amounts to

$$\gamma_j(S)=\mu_b(S)G_{bj}+\mu_g(S)G_{gj}.$$

His move is sequentially rational if he accepts terms exceeding the updated expected gain and if he rejects terms below this gain. Irrespective of beliefs, the plaintiff's expected gain will always be lower if he has obtained bad news rather than good news, i.e. $\gamma_b(S)<\gamma_g(S)$. Therefore, whenever P_g accepts terms S, P_b will do the same. It follows that both plaintiffs' strategies can be recovered from the ex-ante expected move $a(S)=q_ba_b(S)+q_ga_g(S)$. If $a(S)\leqq q_b$, then $a_g(S)=0$, whereas if $a(S)>q_b$ then $a_b(S)=1$. In the following, the plaintiff's strategy will always be denoted by his expected move $a(S)$.

To complete the set of conditions characterizing sequential equilibrium, let $U_i(S,a)$ denote defendant D_i's expected payoff if he proposes settlement terms S which are accepted with probability a. When $a\leqq q_b$ the reference loss is L_i but if anyone accepts it must be P_b. This happens with probability a. When it does, D saves L_{ib} but loses S. Therefore, $U_i(S,a)=-L_i+a(L_{ib}-S)$ if $a\leqq q_b$. When $a>q_b$, settlement terms S are accepted with probability a, but if anyone rejects it must be P_g. This happens with probability $1-a$. When it does, D_i's expected loss amounts to L_{ig}. Therefore, $U_i(S,a)=-aS-(1-a)L_{ig}$ if $a>q_b$.

Given the plaintiff's strategy $a(S)$, sequential rationality requires that defendant D_i offers only such settlement terms as yield the highest payoff. Therefore, if defendant D_i's expected payoff from the game amounts to u_i then $u_i \geqq U_i[S, a(S)]$ for all S, with equality prevailing if terms S are ever to be offered in equilibrium, i.e. if $s_i(S) > 0$. As a final piece of notation, "indifference curves" $A_i = A_i(S, u_i)$ to be defined in the range $S < L_{ig}$ are introduced by their implicit definition $U_i[S, A_i(S, u_i)] = u_i$. These are indifference curves in the sense that if the plaintiff were to respond according to $a(S) = A_i(S, u_i)$ then defendant D_i would be indifferent between any pair of settlement terms. Put differently, given the plaintiff's strategy $a(S)$, sequential rationality for D_i requires that

$$a(S) \leqq A_i(S, u_i) \quad \text{for all } S, \tag{1}$$

with equality prevailing if terms S are ever offered, i.e. if $s_i(S) > 0$. The equilibrium analysis will be carried out in terms of these indifference curves. Useful properties are listed in the following Lemma.

Lemma 1. *In equilibrium, defendant D_i never proposes settlement terms exceeding L_i. Moreover, associated with any sequential equilibrium, there exists a unique intersection (S^*, a^*) of indifference curves, i.e. $a^* = A_b(S^*, u_b) = A_g(S^*, u_g)$. At such an intersection, the slope of A_g exceeds that of A_b. Moreover, $A_i(S, u_i)$ tends to infinity as S approaches L_{ig}.*

Proof. See appendix. ||

2. THE SET OF SEQUENTIAL EQUILIBRIA

Among all beliefs the plaintiff may have, the most optimistic ones are those where P firmly believes that D has obtained bad news $i = b$. Under such optimistic beliefs, the plaintiff accepts terms S with probability $a^b(S)$ where $a^b(S) = 0$ for $S < G_{bb}$, $a^b(S) = q_b$ for $G_{bb} < S < G_{bg}$ and $a^b(S) = 1$ for $G_{bg} < S$. Similarly, the plaintiff's most pessimistic beliefs are those where he firmly believes that D has received good news $i = g$. Under such pessimistic beliefs, P accepts terms S with probability $a^g(S)$ where $a^g(S) = 0$ for $S < G_{gb}$, $a^g(S) = q_b$ for $G_{gb} < S < G_{gg}$ and $a^g(S) = 1$ for $G_{gg} < S$. For plaintiff's strategy $a(S)$ to be sequentially rational, it necessarily must hold that $a^b(S) \leqq a(S) \leqq a^g(S)$. In particular, it follows that the plaintiff accepts settlement terms $S > G_{bg}$ with certainty, terms $S > G_{bb}$ with probability of at least q_b whereas he rejects terms $S < G_{gb}$ with certainty. Therefore, in sequential equilibrium, defendant D_i's expected payoff u_i must satisfy

$$u_i \geqq u_i^0 = \text{Max}\,\{U_i(G_{bg}, 1),\ U_i(G_{bb}, q_b),\ U_i(G_{gb}, 0)\}. \tag{2}$$

In Figure 1, a parameter configuration is depicted where $u_b^0 = U_b(G_{bg}, 1) = -G_{bg}$ and where the corresponding indifference curve $A_b(S, u_b^0)$ cuts the line $a = q_b$ (point 4 in Figure 1) to the right of G_{gb} such that

$$0 < a^0 = A_b(G_{gb}, u_b^0) = \frac{L_b - G_{bg}}{L_{bb} - G_{gb}} < q_b. \tag{3}$$

Other configurations would be where point 4 is to the left of G_{gb} or where the lowest indifference curve above $a^b(S)$ is given by $u_b^0 = U_b(G_{bb}, q_b)$. To fix ideas, however, we shall focus on configurations such as the one depicted in Figure 1. To see that this configuration can actually occur, take the following numerical values of parameters: $W = 11$, $C = 1$, $q_b = 5/12$, $q_g = 7/12$, $\pi_{gb} = 1/6$, $\pi_{bb} = 1/4$ and $\pi_{bg} = 1/3$. Then $L_b = 43/12$, $G_{bg} = 3$, $L_{bb} = 3$, $G_{gb} = 1$ and, hence $a^0 = 7/24$. Since $q_b = 5/12 > 7/24 = a^0$, condition (3) is met for this numerical example.

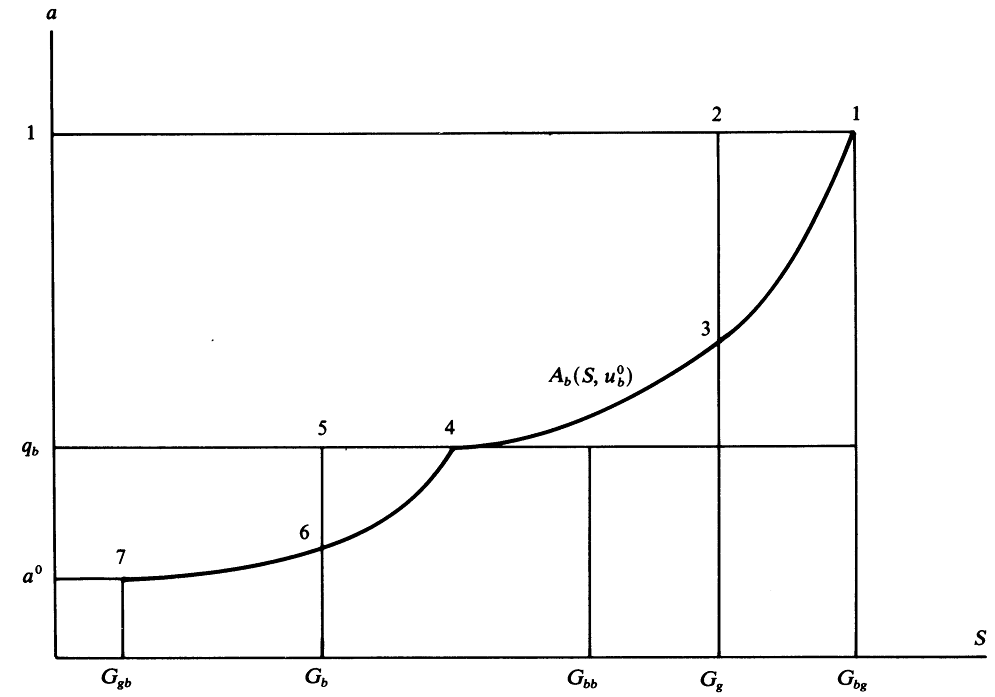


FIGURE 1
The set of sequential equilibria

2.1. *Equilibria without transmission of information*

Suppose the defendant, irrespective of his private information, always chooses the same strategy, i.e. $s_b(S) = s_g(S)$ for all terms S. In this case, the plaintiff is unable to learn anything about the defendant's private information. Moreover, it follows from equilibrium condition (1) that the defendant offers settlement terms $S = S^*$ associated with the intersection of indifference curves (see Lemma 1) with certainty, i.e.

$$s_b(S^*) = s_g(S^*) = 1. \tag{4}$$

At such terms, the plaintiff's "updated" beliefs are of course equal to the corresponding *ex-ante* probabilities, i.e. $\mu_b(S^*) = p_b$ and $\mu_g(S^*) = p_g$. He accepts such terms S^* with probability $a^0(S^*)$ where $a^0(S)$ denotes the plaintiff's sequentially rational response under such beliefs. Formally speaking, a^0 is to be defined as $a^0(S) = 0$ for $S < G_b$, $a^0(S) = q_b$ for $G_b < S < G_g$ and $a^0(S) = 1$ for $G_g < S$. If the plaintiff accepts terms $S = S^*$ with probability $a^0(S^*)$, defendant D_i's expected payoff ($i = b$ and $i = g$) amounts to u_i where

$$A_i(S^*, u_i) = a^0(S^*). \tag{5}$$

Conditions (2), (4) and (5) fully characterize the set of all pooling equilibria. For the parameter configuration depicted in Figure 1, such equilibria correspond to intersections of indifference curves along 12, 23, 45, 56 in Figure 1. The set of pooling equilibria contains as a subset the set of all (ex-post) efficient equilibria, i.e. of equilibria which avoid litigation with certainty. In Figure 1, intersections of indifference curves associated with efficient equilibria occur along 12. Whether such equilibria exist depends on whether the defendant's expected payoffs associated with an intersection along 12 satisfy equilibrium condition (2). Among these, the most likely to exist is as follows. The defendant proposes settlement terms, irrespective of his private information, amounting to the plaintiff P_g's expected gain of litigation, i.e. $s_b(G_g) = s_g(G_g) = 1$. The plaintiff, irrespective of his information, is willing to accept such terms $S^* = G_g$. For this to be an equilibrium, the defendant should have no incentive to deviate. In particular, defendant D_g should prefer paying the price $S^* = G_g$ to enforcing litigation, i.e.

$$G_g \leqq L_g. \tag{6}$$

Moreover, defendant D_g should also prefer paying $S^* = G_g$ to offering terms $S = G_{bb}$ which plaintiff P_b at least would accept, i.e.

$$G_g \leqq q_b G_{bb} + q_g L_{gg}. \tag{7}$$

Therefore, conditions (6) and (7) must hold in order to ensure the existence of an efficient sequential equilibrium. If one of them fails to hold then the above game admits inefficient equilibria only. The issue of efficiency will be further explored in Section 3.

2.2. *Equilibria with complete transmission of information*

In this subsection, equilibria are considered which fully reveal the defendant's private information and where the defendant uses pure strategies only. Suppose D_g offers settlement terms $S = S_g$ whereas D_b offers terms $S = S_b$, i.e.

$$s_g(S_g) = 1 \quad \text{and} \quad s_b(S_b) = 1. \tag{8}$$

If the plaintiff faces terms $S = S_g$ he knows for sure that the defendant has obtained information $i = g$. Therefore, his response is sequentially rational if he accepts with probability $a^g(S_g)$. Similarly, if the plaintiff is offered settlement terms $S = S_b$ he accepts with probability $a^b(S_b)$. In this case, the expected payoffs u_b and u_g of defendant D_b and D_g are to be calculated as

$$A_g(S_g, u_g) = a^g(S_g) \quad \text{and} \quad A_b(S_b, u_b) = a^b(S_b). \tag{9}$$

In equilibrium, defendants must have an incentive to reveal their private information truthfully which means that inequalities

$$A_g(S_g, u_g) \leqq A_b(S_g, u_b) \quad \text{and} \quad A_b(S^b, u_b) \leqq A_g(S^b, u_g)$$

have to be met. It then follows from Lemma 1 that $S_g < S^* < S_b$ where, as usual, S^* denotes those terms at which indifference curves intersect, i.e.

$$A_b(S^*, u_b) = A_g(S^*, u_g). \tag{10}$$

The set of separating equilibria where the defendant never uses mixed strategies is fully characterized by equations (8)–(10). In the configuration of Figure 1, these equilibria are as follows. Defendant D_b proposes settlement terms $S_b = G_{bg}$ which the plaintiff accepts whereas D_g proposes terms $S_g = G_{gb}$ which plaintiff P_b accepts with probability $a(G_{gb})/q_b \leqq a^0/q_b$ (cf. (3)) and which are rejected by plaintiff P_g. Corresponding intersections of indifference curves are on the segment 13467 of Figure 1. Observe that, for such equilibria, the case will always be litigated with positive probability. The equilibrium with intersection at point 1 of Figure 1 which is characterized by

$$A_g(G_{gb}, u_g) = a^0 = A_b(G_{gb}, u_b^0)$$

has the lowest ex-ante probability of litigation among all such separating equilibria. The corresponding equilibrium outcome will turn out to survive all tests of refinement (see Section 4). Its comparative statics properties will be explored in Section 5.

2.3. *Other equilibria*

In spite of the fact that we have already found a vast number of equilibria, the set is still not complete. There exist, for instance, equilibria where defendant D_g uses a mixed strategy, proposing a mixture of different terms $S_g < S^*$ with probabilities $s_g(S_g)$ and $s_g(S^*)$, respectively, whereas defendant D_b always offers terms S^* with certainty, i.e. $s_b(S^*) = 1$. Such equilibria are of the semi-pooling type. In Figure 1, the set of corresponding intersections of indifference curves is to be found within areas 123 and 456. A typical semi-pooling equilibrium would be as follows. Suppose D_b always offers terms S^* where $G_g < S^* < G_{bg}$ which are accepted with probability $a^* < 1$, i.e. (S^*, a^*) is within area 123 of Figure 1. Response a^* is sequentially rational if defendant D_g proposes terms S^* with such a probability $s_g(S^*)$ that P_g becomes indifferent between accepting these terms and going to trial, i.e. the updated expected gain $\gamma_g(S^*)$ must be equal to S^*. This is possible because of the range from which S^* is taken. With complementary probability $s_g(G_{gb}) = 1 - s_g(S^*)$, defendant D_g proposes terms $S = G_{gb}$. This would be consistent with equilibrium. Other semi-pooling equilibria look similar. For lack of space, we abstain from rigorously characterizing the full set of semi-pooling equilibria.

3. A DISCOURSE ON EFFICIENT MECHANISMS

In Section 2.1, it has been shown that if one of conditions (6) and (7) is violated then our game of litigation and settlement never admits an efficient equilibrium. The question then arises whether such inefficiency is due to the simple take-it-or-leave-it bargaining procedure or whether some more fundamental impediment to efficiency is involved. To explore this issue, suppose there were some more sophisticated bargaining model which allows for an efficient equilibrium. Then, according to the revelation principle, there must exist a direct mechanism which is Bayesian incentive-compatible and produces the same efficient (no litigation!) outcome. Such a direct mechanism asks players to reveal their private information and, given that state (i, j) is revealed, requires the defendant to pay a certain amount $T(i, j)$ to the plaintiff. If information is truthfully revealed, D_i's and P_j's interim expected payoffs amount to

$$u^D(i) = -T(i, b)q_b - T(i, g)q_g$$

and

$$u^P(j) = T(b, j)p_b + T(g, j)p_g.$$

Since the mechanism is incentive compatible, it should never hurt to tell the truth which, in this simple context, just means that

$$u^D(b) = u^D(g) \quad \text{and} \quad u^P(b) = u^P(g). \tag{11}$$

Moreover, since the mechanism results from applying the revelation principle to a bargaining procedure where each party has the option to go to trial directly the payoffs must be at least as high as if this option were taken, i.e.

$$u^D(i) \geqq -L_i \quad \text{and} \quad u^P(j) \geqq G_j \tag{12}$$

for all i and j. This last condition corresponds to the usual requirement of individual rationality.

Since the mechanism is purely redistributive it follows that

$$p_b u^D(b) + p_g u^D(g) + q_b u^P(b) + q_g u^P(g) = 0$$

and, by (11), that $u^D(g) = -u^P(g)$. But then the condition (12) of individual rationality implies that

$$-L_b < -L_g \leqq u^D(g) = -u^P(g) \leqq -G_g < -G_b.$$

In other words, such an efficient mechanism exists only if $G_g \leqq L_g$, which is one of the two conditions (see (6)) needed to ensure that the simple take-it-or-leave-it bargaining procedure also admits an efficient equilibrium. Suppose that condition (6) is met but (7) fails to hold. In this case, our game of litigation and settlement does not have an efficient equilibrium. There exists, however, a direct and efficient mechanism which outperforms all equilibria of the game. If, however, $G_g > L_g$ then no bargaining procedure, no matter how sophisticated, will ever admit an efficient equilibrium. This result, of course, is in the same spirit as the findings of Myerson and Satterthwaite (1983).

4. REFINEMENT

In Section 2.1, the exact condition was stated under which the game has an efficient equilibrium. But even if the condition happens to be met, many other equilibria will exist which fail to be efficient. Kohlberg and Mertens (1986), Banks and Sobel (1987) and Cho and Kreps (1987) have proposed concepts of refinement to reduce the number of equilibria. In the following, the cutting power of various tests will be explored in the framework of our game of litigation and settlement. Most of the criteria turn out either to have little cutting power or, else, to reject efficient equilibria.

4.1. *The intuitive criterion*

In a first step, all message response pairs (S, a) for which a is never a best response, irrespective of P's beliefs are deleted. The set of remaining responses is denoted by $R(S)$. It consists of all responses which are between the sequentially rational moves for the most pessimistic and the most optimistic beliefs (see Section 2), i.e. $R(S) = [a^b(S), a^g(S)]$. The intuitive criterion restricts out-of-equilibrium beliefs as follows. Let u_b and u_g denote defendant D_b and D_g's expected (interim) payoffs for a given equilibrium. Consider any out-of-equilibrium message S and suppose that $u_b > (<) \max U_b(S, a)$ but $u_g < (>) \max U_g(S, a)$ where maxima are taken with respect to all responses a of $R(S)$. Then defendant D_b (D_g) could not expect to fare better by proposing terms S as compared to the equilibrium, no matter what the plaintiff's reaction would be. Defendant D_g (D_b), however, would fare better for at least some responses. In this case, the intuitive criterion has the plaintiff believing that such a message must have been proposed by defendant D_g (D_b, respectively). It is not difficult to see that the intuitive criterion rules out some sequential equilibrium profiles. At the level of equilibrium outcomes, however, the criterion has no cutting power. Take, in particular, any efficient pooling equilibrium if one exists (cf. Section 2.1) and let S^* denote the terms proposed by the defendant, irrespective of his private information. These terms are accepted with probability $a^0(S^*) = 1$. For messages $S > S^*$ and any reaction of the plaintiff, the defendant would do worse by proposing terms S as compared to the equilibrium payoff, no matter what information D has received. For messages $S < S^*$, however, D_b as well as D_g might possibly wish to defect from equilibrium. In other words, the premises of the test are never met and, hence, the efficient equilibrium cannot be ruled out.

Moreover, some tedious but straightforward analysis leads to the conclusion that any other equilibrium outcome of our game of litigation and settlement will survive the intuitive criterion as well. For lack of space, no details of such an analysis will be given. In any case, the intuitive criterion does not allow to predict whether inefficient equilibria or, if they exist, efficient ones are more plausible.

4.2. *Divinity*

The criteria of divinity and universal divinity restrict out-of-equilibrium beliefs in a more stringent way than the intuitive criterion does. Take any out-of-equilibrium message S and suppose that, for all responses for which defendant D_b (D_g) wishes to defect, defendant D_g (D_b) also wishes to defect and that there exist reactions for which D_g (D_b) but not D_b (D_g) wishes to defect. Then the test of divinity has the plaintiff believing that D_g (D_b) is more likely than D_b (D_g, respectively) to have proposed such terms S. For universal divinity, defendant D_g (D_b, respectively) is believed to having offered terms S with certainty.

The criterion of divinity has strictly more cutting power than the intuitive test. Consider, e.g. an efficient pooling equilibrium where the defendant proposes settlement terms S^* strictly exceeding plaintiff P_g's expected gain from litigation, i.e. $G_g < S^*$. Let us apply the test to some out-of-equilibrium message S in the range $G_g < S < S^*$. As for divinity, the plaintiff believes defendant D_g to be more likely than D_b to have proposed such terms S. It follows that P_g's updated expected gain $\gamma_g(S)$ from litigation will never exceed the corresponding gain as expected ex-ante, i.e. $\gamma_g(S) \leqq G_g$. Hence, the plaintiff will accept terms in the above range with certainty. Such a reaction, however, would not be consistent with equilibrium. Therefore, the only efficient equilibrium to survive the test of divinity is the one where the above range is empty, i.e. where the defendant proposes settlement terms $S^* = G_g$. As for universal divinity, even such an equilibrium would be ruled out. Take any out-of-equilibrium message S in the range $G_{gg} < S < G_g$. Universal divinity has the plaintiff believing that the defendant must have received his good news. Therefore, P_g's updated expected gain would be $\gamma_g(S) = G_{gg}$. The plaintiff would accept such terms with certainty, a reaction inconsistent with equilibrium. Hence, the test of universal divinity rules out all efficient equilibria. Moreover, it can be shown to have substantial cutting power beyond the set of efficient equilibria.

Other (inefficient) pooling equilibria are ruled out for similar reasons. As far as separating equilibria are concerned, only the one with the lowest probability of litigation will survive. In Figure 1, this equilibrium has indifference curves intersecting at point 7 (see Section 2.2). Take any other separating equilibrium with intersection S^* along 67 but strictly to the right of point 7. This means that terms S exist in the range $G_{gb} < S < S^* \leqq G_b$. Under universal divinity, such terms S are believed to come from defendant D_g and, hence, will be accepted by plaintiff P_b because his updated expected gain from litigation amounts to $\gamma_b(S) = G_{gb} < S$. Such a reaction would not be consistent with equilibrium. Separating equilibria with intersection of indifference curves along 1 3 4 6 can be ruled out similarly, and the same holds true for semi-pooling equilibria (see Section 2.3). To summarize, the only equilibrium outcome surviving the test of universal divinity is characterized by indifference curves intersecting at point 7 of Figure 1. This outcome corresponds to the most efficient (though still inefficient) equilibrium among all separating equilibria. Its properties will be further explored in Section 5. We point out that Reinganum and Wilde (1986) have also explored the cutting power of the universal divinity criterion in their game of litigation and settlement which is one of one-sided incomplete information. They arrive, of course, at a very similar conclusion, namely that it must be the unique separating equilibrium which survives the criterion.

4.3. *Stability*

Kohlberg and Mertens (1986) have proposed the notion of stability as a criterion to reduce the set of equilibria. Their concept which is of use beyond the class of signalling games has substantial cutting power in our case as well. Cho and Kreps (1987) have established that the criteria reported in Sections 4.1 and 4.2 will never rule out any stable equilibrium outcome. Since we have argued in the previous subsection that only one outcome survives the criterion of universal divinity, that outcome must also be the stable one.

This statement is not entirely correct. Stability refers to finite games only. To apply the notion to our game of litigation and settlement, the following discretized version must be considered. The defendant has to select settlement terms from a given finite sequence of values. The sequence may include all critical values such as $S = G_{bg}$ and $S = G_{gb}$ as well as sufficiently many intermediate values. But it has to remain finite. In this case, however, the reasoning of Section 4.2 leading to a unique universally divine outcome must be slightly modified. Let G_{gb^+} denote the smallest settlement terms of the given finite sequence which are strictly higher than $S = G_{gb}$. If indifference curves intersect at $S^* = G_{gb^+}$, the outcome cannot be ruled out because there are no terms of the sequence in the range between G_{gb} and S^*. This outcome which, in an obvious sense, might be called the adjoining one, would pass the test of universal divinity as well. It follows that two candidates are left which could possibly qualify as stable outcomes. In fact, it can be shown that only the equilibrium outcome adjoining the most efficient separating one but not that equilibrium itself will be stable. Obviously, however, this adjoining outcome approaches the original one as the sequence of prespecified values is refined. Under the stable outcome, the case will be litigated with positive probability, no matter whether an efficient equilibrium exists or not.

5. THE VALUE OF PRIVATE INFORMATION

The test of universal divinity as well as the criterion based on stability allow us to predict some essentially unique equilibrium outcome. In the configuration of Figure 1, it is the separating equilibrium (its adjoining one, respectively) where defendant D_b proposes settlement terms $S_b = G_{bg}$ which the plaintiff, irrespective of his private information, accepts with certainty and where defendant D_g proposes terms $S_g = G_{gb}$ which the plaintiff accepts with probability a^0 (see Section 2.2). This probability is to be determined as follows. Plaintiff P_g holding good information rejects terms $S_g = G_{gb}$ because he expects a higher return from going to trial. Plaintiff P_b is indifferent between accepting terms $S_g = G_{gb}$ and going to trial. To sustain the separating equilibrium, however, defendant D_b having received his bad signal should be given no incentive to mimic his strong counterpart D_g. Let a_b^0 denote D_b's probability of accepting terms $S_g = G_{gb}$. In equilibrium, D_b's expected payoff amounts to paying settlement terms $S_b = G_{bg}$. If D_b were to mimic D_g his expected payments would be

$$q_b[a_b^0 G_{gb} + (1 - a_b^0) L_{bb}] + q_g L_{bg} = L_b + a^0(G_{gb} - L_{bb})$$

and should not be lower than $S_b = G_{bg}$. At the most efficient equilibrium among the separating ones, payments will be equal, i.e.

$$L_b - G_{bg} = a^0(L_{bb} - G_{gb}). \qquad (13)$$

Moreover, from the ex-ante view, the defendant's and the plaintiff's expected payoffs amount to

$$u^D = -p_b G_{bg} - p_g L_g + p_g a^0 C \tag{14}$$

and

$$u^P = p_b G_{bg} + p_g (L_g - C), \tag{15}$$

respectively. This equilibrium outcome has been identified as the only one to survive the various tests of refinement.

In the following, a comparative statics analysis with respect to the quality of private information will be carried out for the above equilibrium. Here, quality is meant to refer to how accurately the outcome of litigation can be predicted. The quality of information would be perfect if a party, based on its private signal, knew who would win if the case ended in court. At less than perfect information, an exogenous shift of probabilities in this direction is said to improve that party's information quality.

To begin with, consider the case where the information quality of the plaintiff is improved. If he receives his bad signal, then the probability of his winning at trial decreases, i.e. $d\pi_{ib} < 0$, irrespective of the defendant's information. If, however, the plaintiff obtains his good signal this probability increases, i.e. $d\pi_{ig} > 0$. For simplicity, it is assumed that the amount of change in probabilities is independent of the other party's information, i.e.

$$d\pi_{ib} = d\pi_b < 0 \quad \text{and} \quad d\pi_{ig} = d\pi_g > 0$$

for $i = b, g$. In order to make the parties' expected payoffs commensurable from the ex-ante view, it is assumed that the ex-ante probability of the plaintiff's winning at trial remains constant, i.e. $q_b d\pi_b + q_g d\pi_g = 0$. It is then easy to calculate the impact of such an exogenous shift on the equilibrium outcome. Since $dL_b = 0$ and $dL_{bb} = dG_{gb}$ it follows from (13) that $da^0 < 0$. In other words, as the accuracy of the plaintiff's signal improves, acceptance of the proposed settlement terms becomes less likely and, hence, the ex-ante probability of litigation increases. This loss of efficiency results from the fact that, given improved quality of information, the plaintiff's threat must be increased to keep defendant D_b holding bad information from mimicking his strong counterpart D_g. Let us now discuss the impact on expected payoffs. Basic intuition seems to suggest that the plaintiff's payoff increases as he receives more accurate information whereas the defendant suffers from facing a more accurately informed opponent. Indeed, differentiating equations (14) and (15) easily leads to the conclusion that such intuition is correct, i.e. $du^P > 0$ and $du^D < 0$.

The above result, however, turns out to depend crucially on the assumption that it is the second-moving player whose information quality has been improved. To see this point consider, now, an exogenous shift in the probability of the plaintiff's winning at trial such that

$$d\pi_{bj} = d\pi_b > 0 \quad \text{and} \quad d\pi_{gj} = d\pi_g < 0$$

holds. As the defendant receives his bad signal, his expected loss at trial increases whereas this loss decreases if he obtains his good signal. The amount of change is, again, assumed to be independent of the opponent's private information. Such a shift corresponds to an improvement of the defendant's information quality. Moreover, it is assumed that, from the ex-ante view, the shift is neutral in the sense that the ex-ante probability remains constant, i.e. $p_b d\pi_b + p_g d\pi_g = 0$. Since $dL_b = dG_{bg}$ and since $d(L_{bb} - G_{gb}) > 0$ it follows from (13) that $da^0 < 0$. Therefore, in this case too, improved information quality requires a higher probability of litigation in order to prevent the weak defendant from mimicking his strong counterpart. Differentiating (15) leads to the conclusion that the plaintiff's expected payoff remains unaffected, i.e. $du^P = 0$, as he faces a more accurately informed opponent. To explain such findings, one might attempt to argue that, due to the disadvantage of the second move, the plaintiff's expected payoff anyway remains at the lower bound of individual rationality and, hence, remains constant as the first-moving party's information quality improves. But, in a setting of incomplete information, the ordering of moves does not easily translate into a corresponding level of advantage. This fact becomes obvious if (14) is differentiated in order to calculate the impact on the defendant's expected payoff. Since $d(p_b G_{gb} + p_g L_g) = 0$, it follows that $du^D < 0$. Surprisingly, for the defendant as the first-moving player, more accurate information reduces his expected payoff! At first glance, this result might appear counter-intuitive. But one should keep in mind that the above formulation presupposes, of course, that the second moving party is fully aware of the fact that his opponent receives more accurate information. In any case, it seems dangerous to argue in terms of an advantage of the first move.

As a matter of fact, the unique equilibrium outcome of our game might be compared to the corresponding outcome as the ordering of moves is reversed. The reverse game will not be discussed in detail. It turns out, however, that examples can be constructed such that a party's ex-ante expected payoff is lower in the game where it moves first as compared to the reversed game. Only from the interim view where a party has obtained its good information, is it always worthwhile to move first. Under bad news, this need not be the case.

CONCLUDING REMARKS

This paper has dealt with a game of litigation and settlement under two-sided incomplete information. The bargaining process is of the simple take-it-or-leave-it type. While, for appropriate parameter values, the game allows for ex-post efficient equilibria, such equilibria have been shown not to survive the various tests of refinement proposed in the literature. For other values, our game has no efficient equilibrium at all. The range of parameter values has also been identified for which a direct, ex-post efficient, individually rational and Bayesian incentive-compatible mechanism does exist. This range is slightly larger than the one under which our game has an efficient equilibrium. For parameter values outside of this range, ex-post efficiency is not to be reached by any bargaining game, no matter how sophisticated, nor by any direct mechanism. In this case, efficiency no longer appears to be the proper point of reference. The outcome of any particular game should rather be compared with the direct mechanism which is optimum in the sense that the case will be litigated with minimum ex-ante probability. An interesting topic for further research would be to investigate whether the outcome of the optimum direct mechanism could also be sustained, not only as a sequential equilibrium but even as a stable one of some suitably defined model of bargaining. If that were true then voluntary exchange could be said to lead, if not to an efficient outcome, to the optimum outcome which can conceivably be reached under incomplete information. The basic tenet of economic analysis of law would have to be modified accordingly. In particular, only if the optimum direct mechanism strictly outperforms the stable equilibria of any bargaining model can it justly be claimed that the essential message of the Coase Theorem does not hold in a setting of incomplete information. The stable outcome of our simple game is typically outperformed by the optimum direct mechanism. But some more sophisticated model of bargaining might well turn out to do better in this respect!

As a final remark, it should be pointed out that the present paper casts the tradeoff between litigation and settlement entirely in terms of the Coase Theorem. Therefore, the measure of efficiency is a very restricted one because, in reality, litigation does not merely exist to redistribute wealth but also to provide incentives to potential sources of harm to others. P'ng (1987) provides a framework with which to analyse the incentives for suit, settlement, and trial, as well as incentives for care in activities that may lead to accidental injury. Here, of course, further aspects must be taken into account as far as the true measure of efficiency is concerned. The present paper neglects such additional difficulties and, instead, focuses on the bargaining part of the problem.

APPENDIX

Proof of Lemma 1. For defendant D_i (where $i=b, g$), an indifference curve $A_i(S, u_i)$ has been defined for settlement terms in the range $S<L_{ig}$. Notice that indifference curves are monotonically increasing in their range of definition. First, we show that

$$u_g \geqq -L_g \quad \text{and} \quad u_b > -L_b \tag{A1}$$

must hold in equilibrium. Defendant D_i always can ensure that he obtains at least as much as if the case were litigated, i.e. $u_i \geqq -L_i$. As for defendant D_b, he has the option to propose settlement terms G_{bb} which would be accepted with probability q_b at least, i.e. $u_b \geqq U_b(G_{bb}, q_b) = -L_b + q_b(L_{bb} - G_{bb}) > -L_b$. Therefore, (A1) is established.

Second, we show that defendant D_i never proposes settlement terms exceeding L_i, i.e.

$$S \leqq L_i \quad \text{if } s_i(S) > 0. \tag{A2}$$

To establish (A2), suppose terms S are proposed in equilibrium, i.e. $s_b(S) + s_g(s) > 0$. Three cases must be distinguished according to whether (i): $a(S)=0$ or (ii): $a_g(S)=0$ but $a_b(S)>0$ or (iii): $a_g(S)>0$ and $a_b(S)=1$. If (i) then $U_i[S, a(S)] = -L_i$ and, by (A1), $s_b(S)=0$. Therefore, it must be D_g who has proposed these terms which means that plaintiff P_b's expected gain amounts to G_{gb}. Since, in case (i), $a(S)=0$ it follows that $S \leqq G_{gb} < L_{gb} < L_g$. Hence, (A2) is settled in sub-case (i). If (ii) then $U_i[S, a(S)] = -L_i + a(S)(L_{ib} - S)$. In this case, it follows from (A1) that $S \leqq L_{ib} < L_i$ whenever defendant D_i proposes terms S. Hence, (A2) is settled in sub-case (ii) as well. As for sub-case (iii), finally, suppose defendant D_i proposes terms S with positive probability, i.e. $s_i(S)>0$. Then, by (A1), $U_i[S, a(S)] = -a(S)S - [1 - a(S)]L_{ig} \geqq -L_i$. Since $-L_{ig} < -L_i$ it follows that $-S > -L_i$. Therefore, (A2) is established for all sub-cases.

Third, we show that, for values occurring in equilibrium, indifference curves have at least one point of intersection. To this end, let S_i $(i=b, g)$ denote terms which defendant D_i proposes with positive probability. It follows from equilibrium constraint (1) that

$$a(S_b) = A_b(S_b, u_b) \leqq A_g(S_b, u_g)$$

and

$$a(S_g) = A_g(S_g, u_g) \leqq A_b(S_g, u_b).$$

Therefore, by continuity, terms S must exist such that $A_g(S, u_g) = A_b(S, u_b)$ as was to be shown.

Fourth, we must show that the point of intersection is unique. Let (S^*, a^*) denote any such point, i.e.

$$a^* = A_b(S^*, u_b) = A_g(S^*, u_g).$$

If $a^* \leqq q_b$, derivatives of indifference curves at S^* can be calculated as

$$(L_{bb} - S^*)A_b' = a^* = (L_{gb} - S^*)A_g'.$$

Since $L_{gb} < L_{bb}$ it follows that $A_b' < A_g'$ at $S = S^*$. If, however, $q_b > a^*$ derivatives are to be calculated from

$$(L_{bg} - S^*)A_b' = a^* = (L_{gg} - S^*)A_g'.$$

Since $L_{gg} < L_{bg}$ it follows that, in this case, too, $A_b' < A_g'$ at $S = S^*$. If, however, the slope of A_b is smaller than the one of A_g at any point of intersection, there cannot be more than one such point. Lemma 1 is established. ||

Acknowledgement. The author wishes to thank Martin Hellwig and Eric van Damme for many helpful discussions and suggestions as well as two referees for helpful comments. Remaining errors and shortcomings are entirely my own responsibility. Financial support by the Deutsche Forschungsgemeinschaft through Sonderforschungsbereich 303 is gratefully acknowledged.

REFERENCES

ADAMS, M. (1981), *Ökonomische Analyse des Zivilprozesses* (Königstein/Ts.: Athenäum).

BANKS, J. S. and SOBEL, J. (1987), "Equilibrium Selection in Signaling Games", *Econometrica*, **55**, 647-661.

BEBCHUK, L. A. (1984), "Litigation and Settlement under Imperfect Information", *Rand Journal of Economics*, **15**, 404-415.

CHO, I.-K. and KREPS, D. M. (1987), "Signaling Games and Stable Equilibria", *Quarterly Journal of Economics*, **102**, 179-222.

GOULD, J. (1973), "The Economics of Legal Conflict", *Journal of Legal Studies*, **2**, 279-300.

KOHLBERG, E. and MERTENS, J.-F. (1986), "On the Strategic Stability of Equilibria", *Econometrica*, **54**, 1003-1037.

KREPS, D. and WILSON, R. (1982), "Sequential Equilibria", *Econometrica*, **50**, 863-894.

LANDES, W. (1971), "An Economic Analysis of the Court", *Journal of Law and Economics*, **14**, 61-107.

MYERSON, R. B. and SATTERTHWAITE, M. A. (1983), "Efficient mechanisms for bilateral trading", *Journal of Economic Theory*, **29**, 265-281.

P'NG, I. P. L. (1983), "Strategic Behavior in Suit, Settlement, and Trial", *Bell Journal of Economics*, **14**, 539-550.

P'NG, I. P. L. (1987), "Litigation, Liability, and Incentives for Care", *Journal of Public Economics*, **34**, 61-85.

POSNER, R. A. (1971) *Economic Analysis of Law* (Boston: Little, Brown).

REINGANUM, J. F. and WILDE, L. L. (1986), "Settlement, Litigation, and the Allocation of Litigation Costs". *Rand Journal of Economics*, **17**, 557-566.

SALANT, S. W. (1984), "Litigation of Settlement Demands Questioned by Bayesian Defendants" (Social Science Working Paper 516, California Institute of Technology).

SALANT, S. W. and REST, G. (1982), "Litigation of Questioned Settlement Claims: A Bayesian Nash-Equilibrium Approach" (Rand Corporation Discussion Paper p-6809).

SAMUELSON, W. (1982), "Negotiation Versus Litigation" (School of Management Working Paper 49/82, Boston University).

SHAVELL, S. (1982), "Suit, Settlement, and Trial: A Theoretical Analysis under Alternative Methods for the Allocation of Legal Costs", *Journal of Legal Studies*, **11**, 55-81.

Scand. J. of Economics 98(1), 53–74, 1996

The Hold-up Problem in Government Contracting*

Dieter Bös and Christoph Lülfesmann
University of Bonn, D-53113 Bonn, Germany

Abstract

A two-period procurement model is considered in an incomplete-contract framework. In contrast to Hart and Moore (1988), the welfare-maximizing government, as the buyer, is able to accomplish *ex-ante* optimal contracts which guarantee first-best specific investments of both buyer and seller. These contracts are precisely characterized. Regardless of the underlying supports of cost and benefit distributions, renegotiation inevitably occurs in some states of nature. This renegotiation always increases the *ex-ante* fixed trade price. Hence, the empirical observation of soft budget constraints in government contracting can be rationalized. Furthermore, in accordance with common beliefs, the seller's rents accrue only at the production stage.

I. Introduction

It is a common belief that various inefficiencies are inherent when governments buy goods or services. One potential source of this belief is the increase in prices which is notoriously observed in large procurement programs. These post-contractual price adjustments are often claimed to result from commitment failures which are seen as a specific feature of economic activities of the public hand. In this paper we challenge these common beliefs. Instead, we show in a theoretical model that there is a rationale in the upward renegotiation of contracted prices in public procurement. Clearly, when standard goods are bought by governments, there is no justification for a deviation from initial contract terms. In many cases, however, the government does not simply buy standardized goods which are part of the usual supply of the private firm with whom the

* An earlier version of this paper was presented at the EEA Congress in Maastricht, September 1994, the *Scandinavian Journal of Economics'* Conference in Aarhus, September 1994, the Annual Congress of the *Verein für Socialpolitik* in Jena, September 1994 and at seminars in Aberdeen, Bonn, Boston, Kingston, London, Montreal, Philadelphia, Quebec, Saarbrücken, Toronto, York and Washington, DC. We are grateful to seminar participants and to Nico Hansen, Anke Kessler, Georg Nöldeke, Klaus Schmidt, Steinar Vagstad and two anonymous referees for helpful comments. Financial support from *Deutsche Forchungsgemeinschaft, Sonderforschungsbereich 303* at the University of Bonn is gratefully acknowledged.

government makes a contract, but rather specific goods whose technology is (at least partly) unknown at the date the project is started. These sophisticated projects are the focus of our analysis. Procurement in these cases can be characterized as a two-step process consisting of innovation and production. The goods can only be supplied if, prior to production, the private contractor engages in specific innovative activities, for instance in the development of a new hospital technology or the special design of a particular building for child care or for elderly handicapped people. Hence, innovation is the first part of the contractual relationship between government and a private firm; production and trade are the second part. The innovative effort of the private supplier is relationship specific, at least to a great extent: special technological innovations which are useful in constructing a particle accelerator of a government research institution are practically worthless if the project is not completed.

While preparing a procurement project, the government also has to perform specific investment expenditures for complementary goods which are essential in ensuring the success of the project. For example, one can think of investment in infrastructure when a new hospital or a new university campus is to be built, the government employment of scientific specialists if the above-mentioned particle accelerator is projected, etc.

Any sort of government procurement has much to learn from experiences in military procurement. This is an area where relationship-specific innovations are eminently important. Many technological developments which are useful in constructing defense equipment can only be used when purveying for the public hand. Government's development of a particular radar system to increase the value of a fighter aircraft project has a negligibly low market value if the project does not pass the blue-print stage. The recent practice of the U.S. Department of Defense has switched from cost-plus to fix-price contracts for innovations; see Kovacic (1991). The same policy can be recommended for any form of procurement, as shown by the first-best result of this paper.

Typically, the relationship between the government and the firm during the initial innovation phase is governed by an incomplete contract. The reasons for *ex-ante* contract incompleteness can easily be isolated: since both the amount of the parties' specific investments is nonverifiable and contracts cannot be made contingent on costs or gross welfare, there is room only for very rough contracts to be written at the *ex-ante* stage. In the case of private procurement, according to Williamson (1985), a hold-up problem arises in such a setting. Since the division of the net surplus from trade cannot be fixed *ex-ante*, the parties cannot be prevented from renegotiating the initial contract terms when the net value of the project has finally become clear. Since, however, the specific investments are sunk at this date, they do not influence the outcome of the renegotiations. Accordingly, since the investments cannot be protected by an *ex-ante* contract, the respective investor anticipates his exploitation and underinvests in relationship-specific assets.

In their seminal paper, Hart and Moore (1988) presented a formal analysis of the hold-up problem in a model where one unit of a homogeneous good may be traded between a private seller and a private buyer who both engage in relationship-specific investments prior to production.[1] Assuming that contracts are incomplete, they concluded that a first-best outcome cannot be achieved because the specific investments will not be chosen optimally. As the subsequent literature has shown the crucial point driving their inefficiency result is the assumption that only "at-will" contracts can be written at the beginning of the relationship. This means that, in case of legal disputes between the parties, the court is unable to decide which party is responsible for an eventual breach of the initial contract. Of course, the court observes whether the project has been cancelled, but it cannot assign the responsibility for that event to any one party. Accordingly, the inclusion of breach penalties into the initial contract is infeasible; the completion of the project after the initial innovation phase is a voluntary decision of both agents.

By deviating from this decisive assumption, other authors arrived at a first-best result. Chung (1991) and Aghion, Dewatripont and Rey (1994) showed that for variable quantities a first-best result can be attained if "specific-performance" contracts are available, that is if the trade of a positive quantity can be enforced by the court in the case of disagreement between the parties. Assigning an adequately chosen default option to one player and making the other player a residual claimant in renegotiations,[2] both players are given the right incentives to invest efficiently. Nöldeke and Schmidt (1995) further strengthened the Chung/Aghion-Dewatripont-Rey result by considering the original indivisible-good setting. They allow for "option contracts" under which one party unilaterally can insist on trade. Thereby, a first-best result is achieved. If renegotiations occur in their model, a renegotiation game of the Hart–Moore style is employed where (endogenously) all bargaining power rests with the buyer.

In contrast to Hart and Moore, the above-mentioned papers share the assumption that a court can verify who is guilty for not trading in the case of an *ex-post* cancellation of the project. Implicitly, this approach expresses

[1] Tirole (1986) analyzed a procurement model of a similar spirit; however, in the main part of his paper he assumes asymmetric information between the actors, and contracts which are even more incomplete than those in Hart and Moore. He always arrives at an over- or underinvestment result.

[2] This is an assumption in Chung. It results from the renegotiation design which is modeled in Aghion, Dewatripont and Rey.

the view that the exact nature of the good at stake is known and verifiable at the beginning of the relationship since otherwise the seller would be free to deliver some different (and cheap) good to the buyer, who would be unable to reject the delivery. The Hart–Moore voluntary trade assumption, however, fits into a setting where the precise design of the project is not quite clear at the starting date. In our paper, we stick to the Hart-Moore assumption of at-will contracting as this modelling is most natural in our context.

Since this paper is on public procurement, in contrast to the other papers on the hold-up problem, we deal with a buyer who is a government agency and is interested in maximizing welfare. The seller is a private contractor who maximizes profit. In the main part of the paper, we assume an environment which is characterized by negligible shadow costs of public funds. Under this assumption, we show that by means of an appropriately chosen *ex-ante* contract, the first best can be achieved. Moreover, our result carries over to the case of significant shadow costs if the government *ex ante* can commit not to distort the suplier's *ex-post* profits. This requirement means that the shadow costs of the seller's *ex-post* realized profits do not influence the government's investment behavior and its consent to trade. This commitment device is in line with the results of Rogerson's (1989) empirical study of defense companies which compete for the production stage of procurement projects. The author claims that the overall profit of the supplier should be reduced to zero, but this does not necessarily require zero profit at each step of the procurement process: "In fact, the major theoretical point [...] is that there is a very good reason to structure the regulatory process so that negative economic profit is earned in the innovative phase and positive profit is earned in the production phase."

If such a commitment is not feasible, efficient investments in general cannot be attained: whatever initial contract has been written, the government has an incentive to reduce the *ex-post* rents of the seller by lowering the probability of final trade, that is by reducing its own specific investments. Accordingly, if significant shadow costs and a commitment failure occur jointly, both-sided efficient investments cannot be supported in the subgame-perfect equilibrium. This result implies that in contrast to common beliefs, optimal government behavior in procurement should be characterized by soft renegotiation behavior.

Under commitment or vanishing shadow costs, the optimal *ex-ante* contract induces a fundamental dichotomy: if it is *ex-ante* clear that the subsequent benefits of the project will always exceed its costs, the optimal contract has to set a trade price which will never induce renegotiation; if, however, the buyer and the seller know that with certain probability the benefit of the project may fall below its costs, then it is *never* optimal to fully exclude the possibility of renegotiation. If renegotiation actually occurs in such a case, it leads to an *ex-post* trade price in excess of the *ex-ante* price. Moreover, we prove that upward renegotiation is not only a casual feature under the optimal contract, but that there is a positive probability of renegotiation under the optimal contract in any possible setting.

Our upward-renegotiation result is opposite to the outcome of Hart and Moore for the case of one-sided investments of the seller:[3] in their setting, *ex-ante* prices must be chosen in such a way that the first-best outcome requires a downward renegotiation of the *ex-ante* contracted trade price in every state of the world.[4] The difference is due to the fact that the seller can only be made a residual claimant for his cost savings if he receives the social value of the relationship in every state of the world. In private contracting under the Hart–Moore renegotiation game, which assigns the bargaining power to the party which agrees to efficient trade under the initial prices, this can only be ensured if the *ex-ante* trade price is chosen so high that the buyer always refuses delivery unless there is downward renegotiation. In contrast, since in our framework the government always agrees to trade if this is efficient, downward renegotiation is no matter of concern.

Summarizing, the negative evaluation of soft budget constraints must be challenged: in our setting soft budget constraints, that is the abandonment of *ex-ante* fixed prices in combination with soft renegotiation behavior in the case of significant shadow costs, appear as a necessary prerequisite for obtaining the first-best outcome.[5]

The paper is organized as follows. In Section II we present the model, in particular the sequence of events. In Section III we deal with an example which helps to lay out explicitly a simplified version of the game. In Section IV, the general version of our model is presented and solved; moreover, we briefly deal with the implementation of the optimal contract from the government's viewpoint. A brief conclusion follows.

[3] Note that our model in some sense corresponds to the one-sided investment case of Hart and Moore (1988), Proposition 3, case (2), since the government will always invest efficiently, given its welfare-maximizing behavior.

[4] Nöldeke and Schmidt arrive at renegotiation cases in which either the trade price or the non-trade price is increased.

[5] This is in sharp contrast to the informal literature on the subject. In the defense procurement context, Kovacic (1991), for example, judges renegotiation as a major weakness of fix-price contracts.

II. The Model

The Stages of the Game

The two agents of our model are a procurement agency and a private contractor. The private contractor is to be chosen by means of some bidding process or is perhaps directly chosen by the procurement agency which knows that he is the only potential supplier. Both agency and contractor are risk neutral. At all stages of the game both agents have symmetric information: each agent observes the levels of both relationship-specific investments as soon as they are made; both agents simultaneously learn nature's move determining the value and costs of the project. In order to lay the game structure open, we illustrate the sequence of events in Figure 1.

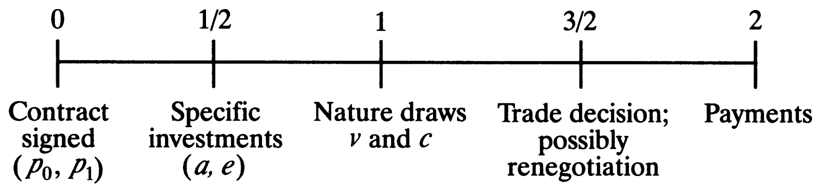


Fig. 1. Game structure.

At *date 0* the agents write a contract which governs their complete future relationship concerning the trade of one unit of an indivisible good (in the following called the "project"). This contract is incomplete. Although we consider a multi-stage game with observed actions, a third party — the court — can only observe whether there has been trade and whether the corresponding payments have been made. This assumption is fully in line with the usual motivation for incomplete contracts. An outsider like the court can hardly verify the benefits and costs of the project: the benefits are rather subjective in nature and — even if complete accounting data are available — the firm cannot be prevented from shifting costs between different activities. Hence, if the project is not completed *ex post*, the court cannot assign responsibility for the breach of the contract to any one party. Accordingly, the *ex-ante* contract can only be conditioned on the *ex-post* verifiable events "trade" or "no trade" and on the *ex-post* verifiable payment of the government. Let $q = 1$ or 0 be the quantity to be traded. For these two cases prices p_1 and p_0, respectively, are fixed in the contract:

$$q = 1 \Leftrightarrow p = p_1; \qquad q = 0 \Leftrightarrow p = p_0. \tag{1}$$

In the no-trade case the private contractor will receive some price p_0 which can be interpreted as a cancellation fee or a reward for his relationship-specific effort.[6] In the trade case, he will receive p_1 which pays for both the costs of the specific investment and the costs of the subsequent completion of the project at stake. Alternatively, one can think of p_0 as the reward for innovation and $(p_1 - p_0)$ as the reward for production.

After signing the contract, the procurement agency and the private contractor engage in relationship-specific investments, say at *date 1/2*. We denote the government's investments by a and the investments of the firm by e. The investment levels are commonly observed by the two parties, but are not verifiable before a court. The associated expenditures are assumed to be convex in the arguments and written as $\mu(a)$ and $\psi(e)$.

At *date 1*, the state of the world $\omega(e,a)\in\Omega$ is drawn by nature and both agents come to know the realized values of benefits and costs. We denote v as the procurement agency's valuation of benefits and c as the private contractor's project completion costs. Both values refer to one unit of the relevant good, the project. They will accrue at date 2 if and only if trade takes place and the firm completes the project. We assume that nature draws (stochastically independent) realized values of benefits and costs from given sets of possible values, v_i, $i = 1,\dots,I$, and c_j, $j = 1,\dots,J$. According to nature's random draw, the values occur with conditional probabilities $\pi_i(a)$ and $\rho_j(e)$, where the probabilities depend on the agents' investments. Higher a increases the probability of a higher value of the project, higher e increases the probability of lower costs.

Between *dates 1* and *2* the final decision on the completion of the project has to be taken, say at *date 3/2*. At this date, renegotiations on the contractual terms are possible between the parties. We assume that only the procurement agency has the right to open renegotiations and hence the agency has all the bargaining power. This treatment of the government as a Stackelberg leader makes it possible to forgo the explicit modeling of the renegotiation process since this is not the focus of our study. Note, however, that this explicit assumption on the distribution of bargaining power is only made for convenience. It results endogenously if we employ a renegotiation game of the Hart and Moore (1988) type, as long as messages cannot be verified.[7] Moreover, note that none of the results of this paper is sensitive to the distribution of *ex-post* bargaining power.

---

[6] As we shall see, however, in equilibrium p_0 can take a negative value. This can occur in particular if the relative value of the seller's specific investments is low as compared with the completion costs.

[7] In an alternative setting, the full bargaining power of the government also arises in a changing-offers sequential-bargaining game. If the supplier at date 1 has to pay a certain amount of money (a "hostage") to the government, which is repaid without interest when the final trade decision has been taken, the buyer's full bargaining power is attained by an appropriate choice of this hostage; see Aghion *et al.* (1994).

At *date 2*, finally, the physical completion of the project takes place (if agreed upon) and the corresponding payments are provided.

Setup and First-best Benchmark

The procurement agency is a welfare maximizer. We assume that it has a lexicographic preference ordering with respect to allocative efficiency and payments. This modeling is equivalent to assuming an objective function reflecting costs of raising public funds λ, where $\lambda \to 0$; see Laffont and Tirole (1993). As argued above, introducing significant shadow costs does not influence our results qualitatively if the government agency can commit to ignore the shadow costs of the firm's profits after the initial contract has been written. Since a welfare-maximizing government would always prefer such a commitment, in some sense our approach reflects a long-term benevolence assumption of government behavior. While it could save on expenditures in certain states of nature by distorting the seller's *ex-post* rents, such a behavior would be short-sighted from a welfare point of view since investment incentives are undermined. For an analysis of significant shadow costs under commitment and non-commitment, see the Appendix.

When the final trade decision has to be taken, the agency will only care about allocative efficiency. If trade is efficient but the supplier credibly refuses trade under the initial terms of contract, however, the agency uses its bargaining power in order to ensure the lowest possible trade price. By a slight abuse of notation,[8] the agency's objective function is

$$\mathscr{W} = \begin{cases} \mathscr{E}_{\omega(e,a)}\{v_i - c_j | q = 1\} - \mu(a) - \psi(e) & \text{at dates 0, 1/2,} \\ q(v_i - c_j) & \text{at date 3/2,} \end{cases} \tag{2}$$

where the expectation operator refers to states of the world, conditional on the parties' specific investments. Considering the expectation operator implies that the application of the trade rule (i.e., the subgame-perfect continuation of the game) is internalized.

The private contractor maximizes expected profit

$$\Pi = \begin{cases} \mathscr{E}_{\omega(e,a)}\{p^{\mathrm{T}} - p_0 - c_j | q = 1\} + p_0 - \psi(e) & \text{at dates 0, 1/2,} \\ q(p^{\mathrm{T}} - p_0 - c_j) + p_0 & \text{at date 3/2,} \end{cases} \tag{3}$$

[8] For a precise formulation of the government agency's utility function, one must define $G = G(x,y)$, where x denotes the level of allocative efficiency and y the payments to the firm. In this formulation, lexicographic preferences over these two arguments can be expressed as follows: $G > \tilde{G} \Leftrightarrow$ (a) $x = \tilde{x}$ and $y < \tilde{y}$ or (b) $x > \tilde{x}$.

where p^T is the realized trade price, that is either the *ex-ante* contracted price p_1 or the modified price resulting from *ex-post* renegotiations. Note that the supplier's participation constraint requires Π to be nonnegative at date 0.

For later reference, we derive the first-best benchmark which requires two notions of efficiency. First, *ex-post efficiency* refers to the trade rule of the model, that is to the decisions made at date 3/2. It requires trade to take place if this increases welfare, that is

$$q^* = 1 \Leftrightarrow v_i \geq c_j; \qquad q^* = 0 \Leftrightarrow v_i < c_j. \tag{4}$$

Ex-ante efficiency refers to the optimal choice of the specific investments a and e, that is, to decisions at date 1/2:

$$(a^*, e^*) \in \operatorname{argmax}_{a,e} \mathscr{W} \quad = \quad \mathscr{E}_{\omega(e,a)}\{v_i - c_j \,|\, q^* = 1\} - \mu(a) - \psi(e). \tag{5}$$

a^* and e^* are used as a benchmark to be compared with the actual choice of investments resulting from the two agents' investment game at stage 1/2. We assume that there is a unique solution of the benchmark model.[9] This solution can be described by two first-order conditions which are necessary and sufficient for an interior solution $a^*, e^* > 0$.

In the subgame-perfect equilibrium of the game between agency and seller, a *first-best result* is attained if at date 0 the production reward $(p_1 - p_0)$ can be chosen so as to induce both *ex-ante* and *ex-post* efficiency in the framework of our model. Note that by arbitrary choices of the absolute values of p_0 (or p_1, alternatively) any distribution of *ex-ante* utilities of the parties can be achieved; in particular, the *ex-ante* profits of the firm can be reduced to zero.

III. A Simple Example

In order to provide a more intuitive flavor of the general results to be stated in Section IV, we start with a simple example. In this example we assume that there are only two possible realizations of benefits and of costs,

$$v \in \{\bar{v}, \underline{v}\}; \qquad c \in \{\bar{c}, \underline{c}\}, \tag{6}$$

with $\bar{v} > \bar{c} > \underline{v} > \underline{c}$. Nature decides at date 1 which realizations occur; the corresponding probabilities are denoted as $\pi(a) = \Pr\{v = \bar{v} \,|\, a\}$ and $\rho(e) = \Pr\{c = \underline{c} \,|\, e\}$ where $\pi'(a)$, $\rho'(e) > 0$, $\pi''(a)$, $\rho''(e) \leq 0$. Higher investments increase the probability of low costs and of high benefits, respec-

[9] Technically, the existence of an interior solution is ensured since expected welfare as defined in (7) is concave in both of its arguments and the Inada conditions are assumed to be fulfilled. The maximum is unique if one assumes $|\mathscr{W}_{ii}| > |\mathscr{W}_{ij}|$, $i, j \in a, e$.

tively. We are looking for the subgame-perfect equilibrium of the game under consideration.

Ex-post Efficiency

Solving the model by backward induction, we begin with the question of *ex-post* efficiency.[10] *Ex-post* efficiency requires the completion of the project if and only if the project's *ex-post* net value is positive, that is $v-c \geq 0$. Since at date 3/2 we consider a (constrained) bargaining game under symmetric information between the parties, achieving *ex-post* efficiency in the game is no serious matter of concern. Due to its welfare objective, the government regardless of contracted prices always agrees to trade if this is efficient. The firm, on the other hand, is willing to trade under the initial prices iff $p_1 - c \geq p_0$, that is, when its net payoff under trade exceeds its default option. If trade is efficient but this inequality does not hold, it is rational for the procurement agency to offer a new increased trade price $p^{\mathrm{T}} = p_0 + c$ which makes the firm just indifferent between trade and no trade.[11] In the subgame-perfect equilibrium, the firm accepts this offer and the project is completed. As we see, *ex-post* efficiency is attained, if necessary through renegotiations.

Ex-ante Efficiency

Given the subgame-perfect continuation of the game (at date 3/2) characterized above, our program is to derive the Nash equilibrium at date 1/2, where the two agents choose their equilibrium investment levels for a fixed price tuple (p_0, p_1). After calculating this equilibrium, we ask if there are optimal prices to be implemented at date 0 which induce efficient investment levels.

For reasons of comparison, we first formulate a benchmark model in which a social planner maximizes welfare with respect to a and e. The planner faces the same veil of uncertainty about the subsequent states of the world as the procurement agency and the private contractor. We obtain the following maximization problem:

$$\text{maximize}_{a,e} \mathscr{W} = \rho(e)\,\pi(a)\,[\bar{v}-\underline{c}] + (1-\rho(e))\,\pi(a)\,[\bar{v}-\bar{c}] + \rho(e)\,(1-\pi(a))\,[\underline{v}-\underline{c}] - \mu(a) - \psi(e). \tag{7}$$

We assume the existence of a unique interior solution, which is described by the first-order conditions

[10] For a more accurate representation of this stage of the game, see Section IV.

[11] Recall the lexicographic preference ordering of the government in Section II.

$$\rho'(e^*)[\underline{v}(1-\pi(a^*))+\pi(a^*)\bar{c}-\underline{c}] = \psi'(e^*), \tag{8}$$

$$\pi'(a^*)[\bar{v}-(1-\rho(e^*))\bar{c}-\rho(e^*)\underline{v}] = \mu'(a^*). \tag{9}$$

The outcome of the benchmark model has to be compared with the Nash equilibrium of the procurement agency and the private contractor resulting from their respective maximizations at date 1/2. The agency is a welfare maximizer. Therefore, its optimization problem is identical to that of a social planner. Accordingly, *ex-ante* efficiency will be achieved if and only if the private firm chooses a welfare-optimal investment level. When will this be the case? At date 1/2, the private contractor is interested in maximizing his expected profit

$$\begin{aligned}\Pi = {} & \pi(a)\rho(e)[p^{\mathrm{T}}(\omega)-p_0-\underline{c}]+\pi(a)(1-\rho(e))[p^{\mathrm{T}}(\omega)-p_0-\bar{c}] \\ & +(1-\pi(a))\rho(e)[p^{\mathrm{T}}(\omega)-p_0-\underline{c}]+p_0-\psi(e),\end{aligned} \tag{10}$$

where $p^{\mathrm{T}}(\omega)$ is the realized trade price in the state of the world ω. In order to facilitate the solution of the problem, we proceed by transforming (10) into a more tractable form. As can easily be verified, $(p_1-p_0)^*>\underline{c}$ is necessary in order to attain positive incentives for the seller's investments. Suppose not: in this case the firm would always receive a net payoff of p_0, either because there is no trade or because it receives a trade price $p^{\mathrm{T}}=p_0+c$ which also guarantees a net payment of p_0. However, if the firm receives p_0, regardless of whether it invests or not, it will always choose $e=0$ in order to minimize investment costs $\psi(e)$. Since *ex-ante* efficiency requires a positive investment, $(p_1-p_0)^*>\underline{c}$ follows immediately. Given the above reasoning, e can rewrite the private contractor's expected profit in the following way:

$$\begin{aligned}\Pi = {} & \pi(a)\rho(e)[p_1-p_0-\underline{c}]+\pi(a)(1-\rho(e))[p^{\mathrm{T}}(\bar{c})-p_0-\bar{c}] \\ & +(1-\pi(a))\rho(e)[p_1-p_0-\underline{c}]+p_0-\psi(e),\end{aligned} \tag{11}$$

where p^{T} in (11) has been replaced by p_1 in the low-cost case and by $p^{\mathrm{T}}(\bar{c})$ in the high-cost case. This leads to the modified first-order condition

$$\rho'(e)[p_1-p_0-\underline{c}-\pi(a)(p^{\mathrm{T}}(\bar{c})-p_0-\bar{c})] = \psi'(e). \tag{12}$$

The resulting investment is not necessarily welfare optimal. Whether a welfare-optimal private investment is achieved depends on the price difference (p_1-p_0). It is easy to show that the chosen investment level of the firm is unique (for any government agency's investment level a) and is a strictly positive function of this contracted price difference. Therefore, we have to ask whether there is an optimal *ex-ante* contracted price tuple that fulfills both the benchmark first-order condition (8) and the Nash-equilibrium condition (12). Equating the terms in brackets in (8) and (12), we obtain that welfare-optimal investments are guaranteed if the prices (p_0,p_1) are chosen according to

$$p_1 - p_0 - \pi(a^*)\,(p^{\mathrm{T}}(\bar{c}) - p_0) = \underline{v}(1 - \pi(a^*)). \tag{13}$$

Let us now characterize $p^{\mathrm{T}}(\bar{c})$: suppose that $(p_1 - p_0)^* > \bar{c}$ which would imply that renegotiations never occur under the optimal solution, even in the high-cost state. In this case (13) could be written as

$$(p_1 - p_0)\,(1 - \pi(a^*)) = \underline{v}(1 - \pi(a^*)) \quad \Leftrightarrow \quad (p_1 - p_0) = \underline{v}, \tag{14}$$

which yields a contradiction to the assumption $(p_1 - p_0)^* > \bar{c}$. Hence, renegotiation necessarily occurs in the high-cost state; accordingly, $p^{\mathrm{T}}(\bar{c}) = p_0 + \bar{c}$ and the welfare-optimal price tuple is characterized by

$$(p_1 - p_0)^* = \underline{v}(1 - \pi(a^*)) + \pi(a^*)\bar{c}. \tag{15}$$

It can directly be seen from (15) that $\underline{v} < (p_1 - p_0)^* < \bar{c}$. The first-best price tuple is chosen in such a way that renegotiation is anticipated for the case of $(\bar{v}, \bar{c})$. This renegotiation leads to a trade price $p^{\mathrm{T}} = p_0 + \bar{c}$ which is higher than originally contracted. Note that the firm derives no rents when nature draws high project-completion costs, whereas in the case of low costs it receives a positive "production rent" $\underline{v}(1 - \pi(a^*)) + \pi(a^*)\bar{c} - \underline{c}$.

Summarizing, we have shown that in our simple setting there is a unique contract $(p_1 - p_0)^*$ which induces efficient specific investments of both agents. Moreover, *ex-post* efficiency (trade iff $v \geq c$) is achieved, if necessary by renegotiation. Since *ex-ante* and *ex-post* efficiency are achieved, a first-best result is obtained. The first-best price tuple necessarily features the occurrence of renegotiations in some states of nature. If renegotiation occurs, it always leads to a higher trade price than originally contracted. This justifies soft budget constraints as a rational government policy.

IV. The General Model

We now provide a general characterization of the *ex-ante* optimal contract. As we will show, the upward-renegotiation result of the preceding example carries over to any arbitrary choice of parameters. While it is not hard to see why any renegotiation will result in an upward renegotiation of the initial trade price, it remains to be shown that renegotiation necessarily must occur in any possible setting, which means that the optimal price difference is characterized by $(p_1 - p_0)^* < \bar{c}$. Instead of only two realizations of benefits and costs, let us now assume many possible realizations which can be ordered as follows:

$$\underline{v} = v_1 < \ldots < v_i < \ldots < v_I = \bar{v}; \qquad I \geq 2, \tag{16}$$

$$\bar{c} = c_1 > \ldots > c_j > \ldots > c_J = \underline{c}; \qquad J \geq 2. \tag{17}$$

At date 1 nature draws the realized values v_i and c_j from the above lists of deterministic variables. The probability that a particular v_i or c_j is drawn depends on the respective investments which, for convenience, are normalized to the zero-one interval. Following Hart and Moore (1988) we specify probabilities of v_i and c_j, respectively:

$$\pi_i(a) = a\pi_i^+ + (1-a)\pi_i^-, \tag{18}$$

$$\rho_j(e) = e\rho_j^+ + (1-e)\rho_j^-. \tag{19}$$

Here π^+ and π^- are probability distributions over $(v_1,\dots,v_I)$ and π_i^+/π_i^- is increasing in i (monotone likelihood ratio property). Analogously, ρ^+ and ρ^- are probability distributions over $(c_1,\dots,c_J)$ and ρ_j^+/ρ_j^- is increasing in j. Moreover, in order to guarantee unique interior solutions, we assume the investment cost functions to be convex in their arguments and $\psi(0) = \mu(0) = \psi'(0) = \mu'(0) = 0$, $\psi'(1) = \mu'(1) = \infty$. According to the linear-distributions-function condition (LDFC) presented in (18) and (19), a particular choice of investment determines a linear combination of two probability distributions, for instance π^+, π^-. Because of the monotone likelihood ratios (which imply first-order stochastic dominance) both agents prefer the "better" distribution (π^+, ρ^+) which they achieve more easily by higher investment. This implies that higher investments increase expected utility and reduce expected costs, respectively. Note that the second derivatives of the probabilities $\pi_i(a)$, $\rho_j(e)$ vanish since the LDFC is characterized by constant first derivatives $\pi_i' = \pi_i^+ - \pi_i^-$ and $\rho_j' = \rho_j^+ - \rho_j^-$. For later reference, also note that $\Sigma_{j=1,\dots,J}\rho_j' = \Sigma_{i=1,\dots,I}\pi_i' = 0$.

Ex-post Efficiency

We now show that at date 3/2 a positive decision on project completion is taken by the agents if trade is efficient, that is if and only if $v_i \geq c_j$ given any realizations v_i and c_j. Furthermore, we argue that the only possible renegotiations refer to an increase in the initially contracted trade price. Consider the parties' objectives: the government trades only if this is efficient and it cannot be induced to an inefficient trade by the seller's offering a lower price than originally contracted. The profit-maximizing seller, on the other hand, will credibly reject trade if this diminishes his *ex-post* profit as compared to his default option payoff p_0. In this case, if trade is efficient, the government will use its power to open renegotiations and will offer a new trade price p^T which makes the firm just indifferent between trade and no trade. (The agency will not offer a higher trade price because of its lexicographic preference ordering.) Hence, the following cases can be distinguished:[12]

[12] This distinction is our counterpart to proposition 1 in Hart and Moore (1988).

(a) if $v_i < c_j$, the firm receives p_0 because the government agency does not agree to trade ($q = 0, p = p_0$).
(b) if $v_i \geq c_j$ and $p_1 - p_0 \geq c_j$, there is trade without renegotiation and the private contractor receives p_1 ($q = 1, p = p^T = p_1$).
(c) if $v_i \geq c_j$ and $p_1 - p_0 < c_j$, there is trade only after renegotiation; in this case, the government agency offers a trade price $p_0 + c_j$ under which the firm (weakly) agrees to trade ($q = 1, p = p^T = p_0 + c_j$).

In all of these cases, *ex-post* efficiency is obtained, in case (c) via renegotiation. Because of the Coase theorem, one should have expected *ex-post* efficiency as we deal with a (constrained) bargaining game under complete information. The specific result of only upward renegotiation, however, rests on our welfare-maximizing buyer setting.

Ex-ante Efficiency

We now examine date 1/2 and consider the investment choices at given prices (p_0, p_1). Since the procurement agency maximizes welfare, we can forgo the explicit presentation of its optimization. Given first-best effort of the private contractor, in a Nash equilibrium it will choose its investments in such a way that *ex-ante* efficiency is obtained. Hence, the main problem is the achievement of the welfare-optimal e^* of the private contractor. For this purpose we start from the private firm's maximization problem

$$\text{maximize}_e \sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a)\,\rho_j(e) \max\{p_1 - p_0 - c_j, 0\} + p_0 - \psi(e). \tag{20}$$

Note that (20) is twice continuously differentiable and concave in e. Given our assumptions, this optimization[13] leads to the following necessary and sufficient first-order conditions for a unique maximum:

$$\sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a)\,\rho_j' \max\{p_1 - p_0 - c_j, 0\} + \psi'(e^N) \qquad \forall a. \tag{21}$$

One can show that the Nash equilibrium investment $e^N = \tilde{e}(a, p_1 - p_0)$ for each a is strictly increasing in its second argument, the price difference ($p_1 - p_0$). Note that only this price difference is relevant for the investment equilibrium level; the absolute values of both prices are irrelevant in this respect. The resulting investment should be welfare optimal. Hence, to achieve *ex-ante* efficiency, e^N must be equivalent to the first-best investments of the firm as obtained by derivation of the benchmark welfare function (5) with respect to e:

[13] We assume that the participation constraint of the firm — nonnegative expected profits at date 0 — is fulfilled by an appropriate choice of p_0.

$$\sum_{i}\sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*)\,\rho_j'(v_i - c_j) = \psi'(e^*). \tag{22}$$

Since $\psi(e)$ is monotonically increasing in e, a necessary and sufficient condition for achieving *ex-ante* efficiency in a Nash-equilibrium is a price difference $(p_1 - p_0)^*$ which equates the left-hand sides of (21) and (22), that is

$$\sum_{i}\sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*)\,\rho_j'(v_i - c_j) = \sum_{i}\sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*)\,\rho_j' \max\{(p_1 - p_0)^* - c_j, 0\}. \tag{23}$$

Given this condition for first-best investments of the firm, the welfare-maximizing government agency will choose investments $a^N = a^*$, which supports (e^*, a^*) as a Nash equilibrium. In the following we use the efficiency condition (23) in order to provide a precise characterization of those production rewards $(p_1 - p_0)^*$ which induce a first best. For this purpose it is convenient to distinguish between the cases of overlapping and non-overlapping distributions of benefits and costs.

Let us start with the most interesting case and suppose the existence of overlapping distributions,[14] that is, there is *ex-ante* uncertainty of the *ex-post* desirability of the project. An optimal price difference exists and can be characterized as shown in the following two steps.

STEP 1: (a^*, e^*) is a Nash equilibrium of the game if there is an optimal price difference $(p_1 - p_0)^*$ which fulfills (23) given that $a = a^*$ has been chosen by the agency. Now consider the left-hand side (LHS) of (23). For any possible investment decision of the procurement agency, the LHS has a unique value which — due to the monotone likelihood ratio property — is a continuous and strictly increasing function of a. Given the welfare-optimal decision of a^* of the government agency, the LHS of (23) has a positive constant value. Hence, we must find a production reward $(p_1 - p_0)^*$ for which the RHS is equal to this constant. First, note that for any a, in particular for a^*, the RHS is continuous and (by the MLRP) strictly increasing in the price difference $(p_1 - p_0)$ as soon as it exceeds $\underline{c}$ — although not everywhere differentiable.[15] Using monotonicity, in order to prove the existence of a unique optimal price difference we have to find values of $(p_1 - p_0)$ which lead to a RHS which falls short of, respectively exceeds, the LHS. We start by considering $(p_1 - p_0) = \underline{c}$. In this case $\max\{p_1 - p_0 - c_j, 0\}$ can never be positive and we can conclude

[14] Formally, this case occurs if $\exists v_i, c_j, c_{j+1} : c_j < v_i < c_{j+1}$.

[15] The derivative of the RHS w.r.t. $(p_1 - p_0)$ has a finite number of jumps occurring when the max-operator becomes positive in one more event (i.e., when $(p_1 - p_0)$ passes the "next" c_j).

RHS $= 0 <$ LHS, that is, underinvestment occurs. Next, let us examine $(p_1 - p_0) = \bar{v}$. Since in this case $\bar{v} > \Sigma_i \pi_i(a) v_i$, RHS $>$ LHS and hence overinvestment results.[16] Obviously, from the intermediate-value theorem, there must be a unique $(p_1 - p_0)^*$ which ensures the identity (23) and generates *ex-ante* efficiency.

The argument of step 1 is similar to the proof of Nöldeke and Schmidt's (1995) main proposition. Note, however, that due to their option-contract assumption they can directly infer that $(p_1 - p_0)^* < \bar{c}$ while step 1 provides the weaker statement $(p_1 - p_0)^* < \bar{v}$.[17]

STEP 2: We prove $(p_1 - p_0)^* < \bar{c}$ by contradiction. Suppose that $(p_1 - p_0) \geq \bar{c}$ which implies that the max-operator on the RHS can be neglected. Now add $\sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*) \rho_j' c_j$ to both sides of (23), and rewrite (23) as

$$\sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*) \rho_j' v_i = \sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*) \rho_j' (p_1 - p_0). \tag{24}$$

Consider an arbitrary benefit $v_{i'}$ drawn by nature. Now distinguish between two cases:

(i) $v_{i'} \geq \bar{c}$. In this case, for any possible cost realization c_j, production is efficient and hence will always occur. Since the changes in probabilities of cost realizations add up to zero if all possible c_j are taken into consideration, that is $\sum_{\substack{j=1,\ldots,J \\ v'_i \geq c_j}} \rho_j' = 0$, we can state

$$\sum_j \pi_{i'}(a^*) \rho_j' v_{i'} = \sum_j \pi_{i'}(a^*) \rho_j' (p_1 - p_0) = 0. \tag{25}$$

Clearly, what is valid for one particular $v_{i'}$, also holds for all other elements of the set $\{v_i : v_i \geq \bar{c}\}$. Summarizing, for all utility realizations which exceed the highest possible production costs, the RHS and LHS of (25) have the same zero value.

(ii) We have to examine a typical element $v_{i'}$ of the complementary set $\{v_i : v_i < \bar{c}\}$. Recall that the LHS and the RHS of (24) differ only in

---

[16] This also proves that $(p_1 - p_0)^* < \bar{v} < \bar{c}$ if the underlying overlapping distributions are characterized by $\bar{c} > \bar{v}$.

[17] $(p_1 - p_0) > \bar{c}$ must trivially hold in the case of option contracts: since the seller is made the residual claimant to his cost savings in *every* state of the world, i.e., even in states where trade is not efficient, choosing a price difference as high as the highest cost realization must result in overinvestment.

the expressions v_i (LHS) and $p_1 - p_0$ (RHS). Since trade cannot be realized for all possible cost realizations, by the monotone likelihood-ratio assumption $\sum_{\substack{j=1,\dots,J \\ v'_i \geq c_j}} \rho'_j > 0$. This and our claim $p_1 - p_0 \geq \bar{c} [> v'_i]$ guarantee that for all elements $v_{i'}$ of the considered set, the value of the RHS exceeds the corresponding value of the LHS.

The assumption of overlapping distributions ensures that both case (i) and case (ii) are to be considered when summing up over all possible realized benefits. Hence, for $(p_1 - p_0) \geq \bar{c}$ we have:

$$\sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*)\,\rho'_j v_i < \sum_i \sum_{\substack{j \\ v_i \geq c_j}} \pi_i(a^*)\,\rho'_j (p_1 - p_0). \tag{26}$$

This, however, is a contradiction to (24) and our claim $(p_1 - p_0)^* \geq \bar{c}$. Hence, we can state the following proposition:

Proposition 1. *If the distributions of benefits and costs overlap, there is a unique ex-ante contracted production reward $(p_1 - p_0)^*$ implementing the first-best outcome. This optimal price difference is characterized by max*$\{\underline{c}, \underline{v}\} < (p_1 - p_0)^* < \min\{\bar{c}, \bar{v}\}$.

Proof: STEP 1 demonstrates the existence and uniqueness of the first-best production reward and characterizes $\underline{c} < (p_1 - p_0)^* < \bar{v}$. In STEP 2 the validity of $(p_1 - p_0)^* < \bar{c}$ is shown. The (tedious, but straightforward) proof of $(p_1 - p_0)^* > \underline{v}$ can be sent to the reader on request.

Let us further characterize the optimal price difference in the case of non-overlapping distributions, that is if there is no *ex-ante* uncertainty of the *ex-post* completion of the project. Obviously, if $\bar{v} < \underline{c}$, from an *ex-ante* view-point the procurement project is not desirable and hence will never get started. The opposite case deserves more interest:

Proposition 2. *If $\underline{v} \geq \bar{c}$, there is a continuum of production rewards inducing a first-best result. They are characterized by $(p_1 - p_0)^* \geq \bar{c}$.*

Proof: Note that (ii) in STEP 2 can be ruled out. Furthermore, (25) holds for all possible combinations of v_i and c_j if and only if $(p_1 - p_0)^* \geq \bar{c}$. Hence, the efficiency condition (23) is satisfied for all values $(p_1 - p_0)^* \geq \bar{c}$. We must now show that any $(p_1 - p_0) < \bar{c}$ does not establish a first-best result by slightly reducing $(p_1 - p_0)$ below $\bar{c}$, the marginal utility of the firm to invest is reduced by $(-\Sigma_i\pi_i(a^*)\,\rho_J'((p_1-p_0)-c_J))>0$. Since we know from STEP 1 that e^N is monotonically increasing in p_1-p_0, the result is established.

The following theorem combines the results and elaborates their economic content:

Theorem 1. *In government procurement, there is a solution to the hold-up problem entailing a basic dichotomy:*

(a) If there is no ex-ante uncertainty of project completion, the set of optimal contracts never induces renegotiation of the initial prices in any state of nature.
(b) Under ex-ante uncertainty, the unique optimal production reward features (i) renegotiation in some states of nature independent of the underlying cost and benefit distributions and (ii) this renegotiation always increases the ex-ante contracted trade price.

Hence, soft budget constraints in government contracting can be rationalized if there is a positive ex-ante probability of the project's shutdown.

The intuition for the no-uncertainty result should be clear. If the project is desirable in all states of the world ($\underline{v}\geq\bar{c}$), the indirect externality between the firm and the government agency caused by the uncertainty of project completion vanishes. Hence, the government can guarantee a welfare-optimal investment level of the firm by making it the residual claimant to its own cost savings in all states of the world. Since the government never insists on renegotiation if trade is efficient, such a contract clearly induces optimal investments.

In the uncertainty case, on the other hand, there is no such simple interpretation: obviously, one can imagine settings where renegotiation occurs in some states of nature. Our result is stronger, however, since it states that $(p_1-p_0)^*<\bar{c}$ independent of the distribution and the values of benefits and costs. Hence, in any possible setting with *ex-ante* uncertainty, there is upward renegotiation in some states of nature. While it should be obvious (and is proven in step 1) that choosing a price difference as large as the highest benefit realization must lead to overinvestment of the seller — and accordingly $(p_1-p_0)^*$ must be smaller than $\bar{v}$ — there is no immediate intuition for our result. The reason is that for all benefits which exceed the highest possible costs, the firm's incentives are independent of the price difference as long as no renegotiations occur, i.e., if $(p_1-p_0)\geq\bar{c}$ is chosen. For lower benefit realizations, on the other hand, this choice would result in overinvestment since the actual costs enabling trade are lower than the price difference. Accordingly, our result follows. Finally, note that this characterization by no means depends on our assumption on the parties' *ex-post* bargaining strength; it would still hold if the firm had any degree of bargaining power.[18]

It was argued in the introduction that the government is interested in extracting the firm's expected rents when it starts a procurement project, as long as this is compatible with the realization of allocative efficiency by $(p_1 - p_0)^*$. If the supplier has no informational advantages over the government at the contracting date, $\Pi = 0$ and according to the definition of expected profit in our general model (20), the payment p_0 amounts to

$$p_0^* = \psi(e^*) - \sum_i \sum_{\substack{j \\ v_i > c_j}} \pi_i(a^*)\,\rho_j(e^*)\max\{p_1 - p_0 - c_j, 0\}. \tag{27}$$

This shows that the no-trade price is lower than the relationship-specific investment costs. The far-right term in (27) expresses the expected "production rent" earned by the firm. hence, one can see immediately that the investment expenditure must exceed the optimal payment p_0. In extreme cases it can even taken a negative value. It is not the innovation stage but the production stage which is profitable for the private contractor. If the procurement agency knows the supplier's investment-cost function $\psi(e)$ the implementation of this optimal price p_0^* creates no problem. If this does not hold, in general there will be a tradeoff between efficiency and rents.

V. Summary

We have shown that in a public-procurement model there exist incomplete contracts which implement the first best. Renegotiation takes place if trade is efficient but the private contractor is not willing to complete the project because the *ex-ante* contracted trade price is too low. In such a case the welfare-optimizing procurement agency will (and should) offer renegotiation which leads to a higher trade price. This is a rational justification of soft budget constraints.

In our setting the optimal contract inevitably leads to renegotiation in some states of nature if there is *ex-ante* uncertainty about the subsequent desirability of project completion. It is interesting to note that this result is independent of the characteristics of the underlying probability distributions. If there is no uncertainty, the result changes drastically. In this case the optimal contract requires that the supplier become the residual claimant to his cost savings in all states of nature. Hence, renegotiation never

[18] Since in this case the firm's production rent would increase, the optimal *ex-ante* price difference $(p_1 - p_0)^*$ had to be even lower than under the assumption of full bargaining power of the government.

occurs. This dichotomy is in accordance with empirical evidence where the upward renegotiation of an *ex-ante* fixed trade price is observed only if uncertain projects requiring innovation are considered.

The outcome of this paper furthermore supports the common belief that, in order to give firms innovation incentives, potential rents must accrue at the production stage. This holds even if the government from a welfare point of view is interested in extracting the contractor's expected profit. In the words of Rogerson (1989), a "prize" has to be paid to the firm in order to enhance innovative activity.

Appendix

In this appendix we consider a welfare function which reflects the costs of raising public funds. As a benchmark, let us derive the first best.[19] First, the *ex-post* efficient decisions are

$$q^* = 1 \quad \Leftrightarrow \quad v_i \geq c_j(1+\lambda), \tag{A1}$$

$$q^* = 0 \quad \Leftrightarrow \quad v_i < c_j(1+\lambda) \tag{A2}$$

where λ refers to the shadow price of public funds. Second, the *ex-ante* efficient investments are given by the (unique) maximizers of the following program:

$$\text{maximize } \mathscr{W} = \sum_i \sum_{\substack{j \\ v_i \leq c_j(1+\lambda)}} \pi_i(a)\,\rho_j(e)\,[v_i - c_j(1+\lambda)] - (1+\lambda)\,(\psi(e) + \mu(a)). \tag{A3}$$

Accordingly, the necessary and sufficient conditions for *ex-ante* efficient investments of the parties are implicitly determined by

$$\mathscr{W}_e(e^*,a^*) = 0 \quad \Leftrightarrow \quad \sum_i \sum_{\substack{j \\ v_i \leq c_j(1+\lambda)}} \pi_i(a^*)\,\rho_j'(e^*)\,(v_i - c_j(1+\lambda)) = (1+\lambda)\,\psi_e(e^*) \tag{A4}$$

and

$$\mathscr{W}_a(e^*,a^*) = 0 \quad \Leftrightarrow \quad \sum_i \sum_{\substack{j \\ v_i \leq c_j(1+\lambda)}} \pi_i'(a^*)\,\rho_j(e^*)\,(v_i - c_j(1+\lambda)) = (1+\lambda)\,\mu_a(a^*). \tag{A5}$$

If the government can credibly commit to neglect the shadow costs of the firm's *ex-post* profits, it will try to extract the seller's profits via the *ex-ante* choice of the no-trade price p_0. Under this commitment, after date 0 it will behave as in the first-best benchmark, that is its investment and renegotiation behavior is influenced only by the shadow costs of production and specific investments and not by the shadow costs of the firm's *ex-post* profits. Note that the *ex-ante* optimal contract under commitment inducing first-best investment decisions is qualitatively identical to that in the case of vanishing shadow costs. The optimal

[19] More precisely, we calculate the optimal decisions given this second-best setting.

contracted price difference, of course, will be lower since the marginal benefit of production is decreased relative to the first-best setup of negligible shadow costs of public funds. Under non-commitment, the government agency *ex post* agrees to trade if and only if

$$v_i \geq c_j + \lambda(p_1^{\mathrm{T}} - p_0). \tag{A6}$$

Besides the usual case of an upward renegotiation, under particular circumstances it is now possible that a downward renegotiation occurs. Suppose trade is efficient but (A6) does not hold under the initial trade price (which implies that $(p_1 - p_0) > c$, that is, the seller agrees to trade under the initial prices). Employing the Hart–Moore renegotiation game, in this case the seller holds all of the bargaining power in renegotiations and reduces the trade price so as to hold the procurement agency indifferent between trade and no trade. Accordingly, the *ex-post* realized trade price becomes

$$p_1^{\mathrm{T}} = \begin{cases} p_1 & \text{if } (v_i - c_j)/\lambda \geq p_1 - p_0 \geq c_j \\ p_0 + c & \text{if } p_1 - p_0 < c_j \leq (v_i - c_j)/\lambda \\ p_0 + (v_i - c_j)/\lambda & \text{if } p_1 - p_0 > (v_i - c_j)/\lambda \geq c_j. \end{cases} \tag{A7}$$

Given any *ex-ante* contracted price tuple and inserting the subgame-perfect continuation of the game, at date 1/2 the optimization approach of the agency is

$$\text{maximize}_a\ \mathcal{G} = \sum_i \sum_{\substack{j \\ v_i \geq c_j(1+\lambda)}} \pi_i(a)\,\rho_j(e) \max\{v_i - c_j - \lambda \max\{p_1 - p_0, c\}, 0\}$$
$$- \lambda p_0 - \psi(e) - \mu(a). \tag{A8}$$

Choosing $p_1 - p_0 \leq \underline{c}$ yields efficient investments of the government agency in the subgame-perfect equilibrium of the game. Moreover, since a^N *is decreasing in* $(p_1 - p_0)$, increasing the initially contracted price difference above $\underline{c}$ induces underinvestment of the government agency.[20] Since the firm will never invest if $p_1 - p_0 \leq \underline{c}$ has been contracted, we observe that both-sided efficient investments in general are unfeasible under non-commitment.

Now, we examine whether one-sided efficient investments of the supplier can be attained. In this case, the firm's optimization approach at date 1/2 is

$$\text{maximize}_e\ \Pi = \sum_i \sum_{\substack{j \\ v_i \geq c_j(1+\lambda)}} \pi_i \rho_j(e) \max\left\{\min\left\{p_1 - p_0 - c_j, \frac{v_i - c_j(1+\lambda)}{\lambda}\right\}, 0\right\}$$
$$+ p_0 - \psi(e). \tag{A9}$$

[20] To be more precise, this statement does not hold if $\underline{c} < \underline{v}$. In this case, efficient investments of the government require the weaker condition $p_1 - p_0 \leq \underline{v}$. In a continuous version of the model, of course, this remark has no relevance.

Let us evaluate whether there are initial prices which support first-best investments. First, consider $p_1 - p_0 \leq \underline{c}$. Under this *ex-ante* price difference, the seller does not invest in relationship-specific assets. Now, assume $(p_1 - p_0) \geq (\bar{v} - \underline{c})/\lambda$. Under this specification, there is downward renegotiation of *ex-ante* contracted prices in every state where trade is efficient and the firm's objective becomes

$$\text{maximize}_e\ \Pi = \sum_i \sum_{\substack{j \\ v_i \geq c_j(1+\lambda)}} \pi_i(a)\,\rho_j(e)\,\frac{v_i - c_j(1+\lambda)}{\lambda} + p_0 - \psi(e). \tag{A10}$$

Comparing the efficiency condition (A4) and the first-order condition of program (A10), one immediately arrives at an overinvestment result. Since $de^N/d(p_1 - p_0) \geq 0$ due to the MLRP, applying the intermediate-value theorem we can conclude that one-sided efficient investments of the firm can be guaranteed for any possible λ by an *ex-ante* contract in the interval $\underline{c} < (p_1 - p_0)^* < (\bar{v} - c)/\lambda$. Interestingly, compared to the Hart–More result derived for a self-interested buyer, two different features arise: first, it is no longer valid that there is downward renegotiation in *every* state of the world under the optimal contract; second, even if trade is realized with certainty (i.e., $\underline{v} \geq \bar{c}$), the *ex-ante* contract must be chosen such that renegotiation occurs in some states of the world. The intuition for both results lies in the fact that the firm underestimates the true social production (and investment) costs; accordingly, making it the residual claimant in all states of the world would induce overinvestment.

References

Aghion, P., Dewatripont, M. and Rey, P.: Renegotiation design with unverifiable information. *Econometrica 62*, 257–282, 1994.
Chung, T.-Y.: Incomplete contracts, specific investments and risk sharing. *Review of Economic Studies 58*, 1031–42, 1991.
Hart, O. and Moore, J.: Incomplete contracts and renegotiation. *Econometrica 56*, 755–85, 1988.
Kovacic, W. E.: Defense contracting and extensions to price caps. *Journal of Regulatory Economics 3*, 219–40, 1991.
Laffont, J.-J. and Tirole, J.: *A Theory of Incentives in Procurement and Regulation.* MIT Press, Cambridge, MA, 1993.
Nöldeke, G. and Schmidt, K. M.: Option contracts and renegotiation: A solution to the hold-up problem. *Rand Journal of Economics 26*, 163–79, 1995.
Rogerson, W. P.: Profit regulation of defense contractors and prizes for innovation. *Journal of Political Economy 97*, 1285–305, 1989.
Tirole, J.: Procurement and renegotiation. *Journal of Political Economy 94*, 235–59, 1986.
Williamson, O. E.: *The Economic Institutions of Capitalism.* Free Press, New York, 1985.

First version submitted August 1994;
final version received April 1995.

ELSEVIER Journal of Public Economics 66 (1997) 55–71

JOURNAL OF PUBLIC ECONOMICS

Conspicuous consumption, snobbism and conformism

Giacomo Corneo[a,*], Olivier Jeanne[b]

[a]*Department of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany*
[b]*CERAS-Ecole Nationale des Ponts et Chaussèes, 28 Rue des Saints Pères, 75007 Paris, France*

Received 30 October 1995; accepted 31 August 1996

Abstract

We develop a model in which consumers purchase a conspicuous good in order to signal high income and thereby achieve greater social status. In equilibrium, the signalling value of conspicuous consumption depends, in an identifiable way, on the number of consumers, and consumer behaviour is characterized by either snobbism or conformism. The market demand curve for the conspicuous good may exhibit a positive slope if consumers are conformist. We derive some unconventional policy implications concerning the taxation of luxuries and the voluntary provision of public goods. © 1997 Elsevier Science S.A.

Keywords: Consumption externalities; Status-seeking; Signalling

JEL classification: D11; D82; H23

1. Introduction

In many circumstances the decision of consumers to purchase a good cannot adequately be explained by the intrinsic utility derived from consuming it. Rather, its rationale may be found in what the purchase of the good symbolizes to others. A prominent example of this type of behaviour, studied in the seminal works of Rae (1834) and Veblen (1922), is when consumers purchase a good in order to advertise their wealth and thereby achieve greater social status. As highlighted by Frank (1985a), (1985b), the quest for social status by means of conspicuous

*Corresponding author.

PII S0047-2727(97)00016-9

consumption may cause serious inefficiencies in the form of downward distortions in individual demands for nonpositional goods.[1] In a recent article, Ireland (1994) has formalized these insights in a full-fledged signalling model of conspicuous consumption. Bagwell and Bernheim (1996) have extended his analysis to a model in which both the quantity and the quality of the conspicuous good are variable.

The present paper investigates the presence of bandwagon and snob effects under conspicuous consumption. These effects were first highlighted by Leibenstein (1950). The bandwagon effect describes a situation in which the demand for the good increases because others are buying the same good. The snob effect is the opposite: market demand decreases because others are purchasing the good. The present paper concentrates on the case in which the conspicuous good is indivisible.[2] The indivisibility assumption has also been adopted by Becker (1991), Karni and Levin (1994), Navon et al. (1995) to study the properties of market demand in the presence of a bandwagon effect. While these papers merely assume that the utility derived from consuming the good is an increasing function of the total number of buyers, we endogenize this relationship in a signalling model. We show that the occurrence of a snob or a bandwagon effect depends, in an identifiable way, on how social norms allocate status on the basis of relative income. In essence, two types of incentives for conspicuous consumption may be distinguished: the desire not to be identified with the poor, and the desire to be identified with the rich. If social norms allocate status in such a way that the first type of incentives predominates, a bandwagon effect arises. Otherwise, a snob effect appears.

We show that the market demand for a conspicuous good might be upward-sloping. This possibility is often pointed out by marketing scholars [e.g., Gaedeke and Tootelian (1983)]. Economists typically associate this phenomenon with markets where the price conveys a signal about the quality of the good [e.g., Milgrom and Roberts (1986)]. While a signalling effect is also at work in the present approach, the price of the good signals the quality of the consumer rather than that of the good. In some circumstances, a price increase triggers such an increase of the signalling value of the conspicuous good that its market demand grows. Interestingly, it is the desire to avoid social ostracism, rather than the search for prestige, which may lead to an upward-sloping demand curve.

Our analysis suggests some unconventional implications for tax policy. We show that taxing the conspicuous good might turn to enlarge its market demand, with negative consequences for the welfare of all individuals. This may occur even in situations where prohibiting the consumption of the conspicuous good would constitute a Pareto-improvement with respect to the laissez-faire. This possibility

[1] In a similar vein, Moffit (1983) has pointed out the distortions arising from the stigma effects induced by participation in welfare programmes.

[2] The indivisibility assumption is natural for a number of goods which may be classified as conspicuous, like luxury cars, swimming pools, domestic servants.

is not acknowledged by previous models of conspicuous consumption, in which introducing taxation at the margin always increases social welfare [Kolm (1972); Ireland (1994)]. Finally, we argue that the quest for status by individuals may overcome the problem of free riding on the provision of public goods.

In the next section we set up the formal model, determine when bandwagon, snob and Veblen effects arise, and establish a necessary condition for the existence of an upward-sloping demand curve. In Section 3 some policy implications of the model are presented. Section 4 provides concluding remarks. Appendix A works out an example illustrating the main insights of the paper.

2. The model

2.1. Assumptions

The economy is populated by a continuum of consumers indexed by $r \in [0,1]$. Consumers are ordered according to their level of income, y_r, in the following way: for all r and r' in $[0,1]$, $r < r' \Leftrightarrow y_r > y_{r'}$. Hence r represents the individual's rank in the income hierarchy. Income is exogenously distributed according to a continuous density function.[3]

There are two consumption goods: an observable good, referred to as the conspicuous good, and an unobservable good, used as the numéraire. Neither the individual's income nor his consumption of the numéraire good is visible to spectators. The only thing that the public observes is the individual's consumption of the conspicuous good. The conspicuous good is indivisible and consumers do not buy more than one unit of it.

Individuals have identical preferences, summarized by the following utility function:

$$U_r = u(c_r) + v(\delta_r) \tag{1}$$

where c_r is the consumption of the numéraire good and δ_r is a dummy variable that takes value 0 if the individual does not purchase the conspicuous good and 1 if he does. The function $u(\cdot)$ is continuously differentiable, strictly increasing and concave, and satisfies $\lim_{c \to +\infty} u'(c) = 0$.

The consumer's valuation of the conspicuous good is given by the function $v(\cdot)$. For the sake of clarity, we adopt the original Veblenian assumption that conspicuous consumption yields no intrinsic utility. The sole purpose of conspicuous consumption is to send a message about the individual's rank in the income distribution. In turn, individuals derive utility from the inference made by others

[3] Robson (1992) shows in a gambling model in which utility depends on income rank that in a stable income distribution there cannot exist a positive measure of individuals having the same level of income.

about their rank.[4] All individuals have access to the same information: the purchasing behaviour with respect to the conspicuous good of each other individual, and the way in which income is distributed in the population. Thus the message formulated by conspicuous consumption is identically interpreted by all observers. Formally,

$$v(\delta_r) = E(a(r)/\delta_r) \tag{2}$$

where $E(./\delta)$ is the expectation conditional on the observation of δ, and $a(r)$ is a continuous function defined on the unit interval, with finite lower and upper bound, referred to as the rank utility. The suggested interpretation of Eq. (2) is in terms of social status. Following sociologists, we may define social status as a general claim to deference [e.g., Coleman (1990)]. In economic terms, an individual's status may be called a socially provided private good. Each individual has a certain fixed amount of a special good – say, deference – that he allocates to others according to some social norm. This norm is represented in our model by the function of income ranks $a(r)$. In turn, the norm may be taken as mirroring societal values that characterize the community in which the individuals interact. The relationship between social deference and relative income may depend on various factors, such as the perceived legitimacy of the origin of personal income, ideas about how private income generation contributes to social welfare, religious beliefs linking financial success with predestination or compliance with religious norms.[5] More standard interpretations of Eq. (2) are also possible. One could monetize the signalling effect through a market transaction occurring under incomplete information. Similarly to Spence's (1974) model of the labour market, individuals might want to signal a high income because it is correlated with their privately known productive ability. Alternatively, one could set up a matching game in which an individual's equilibrium payoff depends on others' beliefs about his rank in the income distribution. For example, Cole et al. (1992) have argued that marriage opportunities depend on the income rank of the individual.

Individuals choose δ_r in order to maximize their utility subject to the budget constraint:

$$c_r + p\delta_r \leq y_r \tag{3}$$

where p is the price of the conspicuous good. Notice how this decision problem differs from the standard one in consumer theory. As the consumption set contains a conspicuous good, the way in which an individual's preferences are optimized depends on how his consumption choice affects others' beliefs about him. Insofar

[4] Clark and Oswald (1996) provide recent econometric evidence in support of the dependence of utility on relative income.

[5] Fershtman and Weiss (1993) discuss the economic impact of culture viewed as a factor determining how status is allocated. For a comparative study of the relationship between personal income and social deference see, e.g., Lipset (1967).

as inferences made by the public depend on the market outcome, the consumers' decision problems are no longer independent. In fact, the appropriate equilibrium notion here is that of signalling equilibrium: an action function (mapping types to actions) and an inference function (mapping actions to inferences about type) such that actions are optimal given inferences, and inferences can be deduced from the action function using Bayes's rule.

2.2. Conformist versus snobbish behaviour

Individuals purchase the conspicuous good if its signalling value $s \equiv v(1) - v(0)$ is large enough. As the maximization of Eq. (1) readily shows, the condition for individual r to purchase the good is:

$$s \geq u(y_r) - u(y_r - p) \tag{4}$$

Suppose that the signalling value is strictly positive and finite. Eq. (4) implies that an individual buys the good if and only if his income is larger than a threshold level, which depends on both the price of the good and its signalling value. This threshold level, denoted by $\bar{y}$ (s, p), is defined as follows:

$$s = u(\bar{y}(s,p)) - u(\bar{y}(s,p) - p) \quad \text{if} \quad s \leq u(p) - u(0) \tag{5}$$

$$\bar{y}(s,p) = p \quad \text{if} \quad s \geq u(p) - u(0) \tag{6}$$

Thus, $\bar{y}$ (s, p) is either the income which makes the individual indifferent between purchasing the good or not, or it is the minimal income which is necessary to afford the good. It follows that the conspicuous good separates the population into two groups, a group of rich who choose to afford it, and a group of poor who prefer to abstain from it. The number of individuals who buy the good may be written as:

$$n = N(\bar{y}(s,p)) \equiv D(s,p) \tag{7}$$

where $N(y)$ denotes the number of individuals in the economy with income higher than y. As $N(\cdot)$ is decreasing, the number of consumers, i.e., the demand for the conspicuous good, is decreasing with the price of the good and increasing with its signalling value.

We now turn to the formation of inferences about types. We shall focus on semi-separating equilibria, i.e., situations in which $n \in]0,1[$.[6] The owners of the conspicuous good send a simple message, which is that they belong to the n richest part of the population, or equivalently that their rank is better than n. Using Bayes's rule, the signalling value of the good, which is the difference between the

[6] The model also admits pooling equilibria. The analysis of these equilibria is conducted in Corneo and Jeanne (1995), available from the authors upon request.

average rank utility of those who purchase it and those who do not, may be written as:

$$s = \frac{\int_0^n a(r)dr}{n} - \frac{\int_n^1 a(r)dr}{1-n} = \frac{1}{n(1-n)} \int_0^n [a(r) - \bar{a}]dr \tag{8}$$

where $\bar{a}$ is the average rank utility over the whole population. Notice that if $a(\cdot)$ is strictly decreasing, s is strictly positive as previously supposed. However, the signalling value of the good may remain positive even if the rank utility is increasing on some interval. Hereafter we shall assume that the form of $a(\cdot)$ ensures that the signalling value is strictly positive for $n \in]0,1[$.

An interesting feature of Eq. (8) is that the utility of the conspicuous good can be expressed as a function of the number of consumers:

$$s = \sigma(n) \tag{9}$$

where $\sigma(\cdot)$ is defined by:

$$\sigma(n) = \frac{1}{n(1-n)} \int_0^n [a(r) - \bar{a}]dr \tag{10}$$

This property departs from the orthodox model of consumer demand, in which the utility of goods is exogenously defined with tastes.[7] There is a formal correspondence between the signalling utility of the conspicuous good and the rank utility function. Function $\sigma(\cdot)$ is determined by $a(\cdot)$ through equation Eq. (10). Conversely, any continuous and differentiable function $\sigma(\cdot)$ defined on the unit interval can be rationalized by a rank utility of the form:

$$a(r) = \bar{a} + (1 - 2r)\sigma(r) + r(1 - r)\sigma'(r) \tag{11}$$

We can now analyze how the signalling value of the good varies with the number of consumers. Following Leibenstein (1950), we shall say that consumer behaviour is conformist[8] if the utility of the conspicuous good grows when it is more widely consumed, and conversely that consumer behaviour is snobbish if the utility from purchasing the good is enhanced by its rarity. Formally, conspicuous consumption is said to be conformist (snobbish) when $\sigma(\cdot)$ is strictly increasing (decreasing).

Conformism and snobbism are neither exogenous characteristics of the good nor

[7] This property makes the conspicuous good formally similar to a network good, e.g., telecommunications systems. However, in the case of a network good the impact of the number of consumers on the utility is directly determined by the technological properties of the good.

[8] Leibenstein (1950) used the term bandwagon to describe conformism.

the consequence of individual tastes for imitation or distinction. They are determined by the form of the rank utility, i.e., by the social norm governing the allocation of status to income groups. As such norms can widely differ across communities, identical economic fundamentals can be consistent with very different consumption patterns. In order to gain some insight about the relationship between social norms and consumer behaviour, let us consider the class of quadratic rank utilities:

$$a(r) = a_0 - a_1 r - a_2 r^2 \tag{12}$$

which are associated with the signalling value:

$$\sigma(n) = \frac{a_1}{2} + \frac{a_2}{3}(n+1) \tag{13}$$

In this case $\sigma(\cdot)$ is linear, and it is increasing with n if and only if a_2 is positive. Hence conspicuous consumption is conformist (snobbish) if and only if the preferences on ranks are concave (convex). The intuition is the following. Concavity of the rank utility means that it is more costly to lose one position in the hierarchy when one is ranked low than when one is ranked high. When the number of consumers increases, the status of nonconsumers rapidly diminishes, i.e., the signalling value of the conspicuous good increases. In essence, societal values induce a fear of being identified with the poor, which expresses itself in the form of conformist behaviour. The logic of snobbish behaviour is the converse one. Convexity of the rank utility means that the demand for the conspicuous good is motivated by the hope of being identified with the rich. The signalling value of conspicuous consumption is independent of the number of consumers only when the marginal rank utility is constant.

2.3. Equilibrium market demand

If $n \in]0,1[$, the equilibrium market demand for the conspicuous good is entirely determined by Eqs. (7) and (9). The former describes the classical impact of the price on demand: given that the utility of the good amounts to s, the demand is decreasing with the price. The latter equation introduces the original feature of conspicuous consumption, which is that the utility of the good depends on the number of consumers. This modifies the properties of the demand function, or equivalently its inverse, the price function $p(n)$. We establish the following facts:

Proposition 1. The price function $p(n)$ is uniquely defined on $]0,1[$ and continuous.

Proof. The price function $p(\cdot)$, if it exists, must satisfy:

$$\text{for all } n \in]0,1[, \quad y_n = \bar{y}(\sigma(n), p(n)) \tag{14}$$

Let us consider a given $n \in]0,1[$. First, let us assume that $\sigma(n) \geq u(y_n) - u(0)$. Then we are in the conditions under which equation (6) applies, so that $p(n) = y_n$, which is a continuous function of n by assumption.

Second, let us assume that $\sigma(n) < u(y_n) - u(0)$. Then we are in the conditions under which equation (5) applies, so that:

$$\sigma(n) = u(y_n) - u(y_n - p(n)) \tag{15}$$

The r.h.s. of Eq. (15) increases, strictly and continuously, from 0 to $u(y_n) - u(0) > \sigma(n)$ when $p(n)$ increases from 0 to y_n. Hence there is one unique $p(n) \in]0, y_n[$satisfying Eq. (15). The function $p(\cdot)$ thus defined is continuous because of the continuity of $u(\cdot)$ and $\sigma(\cdot)$. QED

Proposition 2. Assume that consumer behaviour is snobbish.
Then:

1. *the price function is decreasing in]0,1[;*
2. *if out-of-equilibrium beliefs are passive,*[9] *there exists a price $\hat{p}$ such that market demand is strictly positive for some $p > \hat{p}$, while it is nil if $p \leq \hat{p}$.*

Proof:

1. According to the proof of Proposition 1, $p(n)$ is defined either by $p(n) = y_n$, in which case it is decreasing with n, or by equation Eq. (15). In the latter case, it is easy to infer from Eq. (15), and the fact that $\sigma(n)$ and y_n are decreasing with n, that $p(n)$ is decreasing with n. The reason is that, starting from equality, an increase in n raises the r.h.s. and lowers the l.h.s. of Eq. (15), so that a decrease in $p(n)$ is required to restore equality.
2. Denote $\hat{p} = \lim_{n \to 1} p(n)$. As $p(n)$ is decreasing in $]0,1[$, $p \leq \hat{p}$ implies that $\forall n \in]0,1[$, $p < p(n)$. Hence, if an equilibrium exists at price p, it cannot be a semi-separating one. It must be a pooling equilibrium in which everybody consumes the good ($n = 1$), or nobody does ($n = 0$). If the public makes passive conjectures, the social status obtained off the equilibrium path equals $\bar{a}$ and the only possible equilibrium is the one in which nobody consumes. In fact, nobody would pay a strictly positive price for the conspicuous good if this brings no increase in status. QED

[9] This means that individuals retain their prior after observing out-of-equilibrium actions. The idea is that social perceptions are characterized by some inertia: the observation of behaviour off the equilibrium path is interpreted by the public as a mistake, so that their beliefs are unchanged.

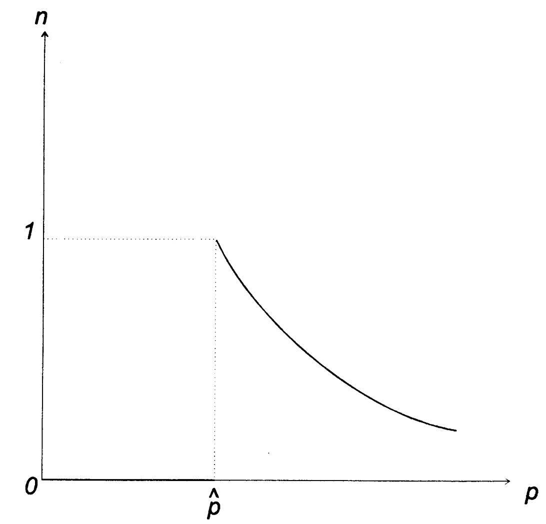


Fig. 1. The demand for the conspicuous good in the snobbish case.

Fig. 1 depicts the demand curve for a conspicuous good under the assumptions of Proposition 2. The demand is nought as long as the price is lower than a critical level $\hat{p}$, then it jumps up to almost the whole population, before it decreases with the price. When the price is lower than $\hat{p}$ everybody would buy the good in an equilibrium in which it has some signalling value. But the very fact that everybody owns the conspicuous good kills its signalling value. Hence the only possible equilibrium entails that nobody buys the good.

Proposition 2 implies that the demand curve can be upward sloping only if consumer behaviour is conformist. If the signalling value of the good is fixed, a demand increase must be associated with a price reduction: this corresponds to the classical effect of the price on demand. It follows that for demand to be increasing with the price, the classical effect must be offset by a change in the signalling value of the good: the increase in the number of consumers has to increase the signalling value by an amount that more than compensates the classical effect. This implies that consumer behaviour must be conformist.[10]

Conspicuous consumption has also been associated with the existence of Veblen effects. By definition, a Veblen effect occurs if the willingness to pay for the good is increasing with its price. In the context of our model, the willingness to pay for the conspicuous good is entirely determined by its signalling value, so that the Veblen effect is present if and only if the signalling value is increasing with the

[10] In the appendix we work out an example in which conformism gives rise to an upward-sloping demand curve.

price. Combining Eqs. (7) and (9) gives us the equilibrium relationship between the signalling value of the conspicuous good and its price:

$$s = \sigma(D(s,p)) \tag{16}$$

which implicitly defines s as a function of p. The following result states the conditions under which this function is increasing.

Proposition 3. The Veblen effect arises in the two following cases:

1. *Consumer behaviour is snobbish;*
2. *Consumer behaviour is conformist and the demand for the conspicuous good is upward-sloping.*

Proof: Differentiating Eq. (16) yields:

$$\left[1 - \sigma'\left(\frac{\partial D}{\partial s}\right)\right]\mathrm{d}s = \sigma'\left(\frac{\partial D}{\partial p}\right)\mathrm{d}p \tag{17}$$

Since $\partial D/\partial s>0$ and $\partial D/\partial p<0$, $\mathrm{d}s/\mathrm{d}p>0$ is possible either if (i) $\sigma'<0$, i.e., consumer behaviour is snobbish; or if (ii) $\sigma'(\partial D/\partial s)>1$. In the second case, σ' has to be positive, so that consumer behaviour has to be conformist. Moreover, differentiating $n=D(\sigma(n),p)$, it easily follows that $\sigma'(\partial D/\partial s)>1$ is equivalent to $\mathrm{d}n/\mathrm{d}p>0$, i.e. the demand for the conspicuous good is upward-sloping. QED

The result that snobbism generates a Veblen effect is easy to interpret. We know from Proposition 1 that in this case the demand for the conspicuous good is downward-sloping. Hence increasing the price reduces the number of consumers, which is precisely what is required to increase the signalling value of the conspicuous good in the snobbish case. Interestingly, the Veblen effect may also arise when consumer behaviour is conformist, provided that the demand for the conspicuous good is increasing with its price. In this case, raising the price of the conspicuous good increases both the number of consumers and its signalling value. In fact, the only case of non-existence of the Veblen effect is when consumer behaviour is conformist and the demand curve is downward-sloping.[11]

[11] Bagwell and Bernheim (1996) define the Veblen effect as the fact that rich individuals are ready to pay a higher price for a functionally equivalent good. They show, in a model where the conspicuous good is perfectly divisible, that such a Veblen effect does not ordinarily arise. This is because with a divisible good the rich prefer to signal high income by consuming large amounts of the good, rather than by overpaying it.

3. Policy implications

3.1. Luxury taxes

As the previous section has made clear, status-seeking generates a consumption externality. We now consider to which extent this consumption externality can be corrected by public policies like the prohibition or the taxation of the conspicuous good. Since the utilities are not quasi-linear, it is useful to distinguish the Pareto-properties of these policies from their impact on social welfare, that we shall define as the sum of individuals' utilities: $SW=\int_0^1 U_r \mathrm{d}r$. We shall study the impact of these policies starting from a laissez-faire situation, in which an equilibrium quantity n^* of the conspicuous good is competitively supplied at its marginal cost γ. Our first result relates to the complete prohibition of the good, i.e., switching from $n=n^*$ to $n=0$.

Proposition 4. Starting from laissez-faire, prohibiting the consumption of the conspicuous good always increases social welfare, and constitutes a Pareto-improvement if and only if:

$$u(y_0) - u(y_0 - \gamma) \geq (1 - n^*)\sigma(n^*). \tag{18}$$

Proof: The laissez-faire social welfare is

$$SW = \int_0^{n^*} u(y_r - \gamma)\mathrm{d}r + \int_{n^*}^{1} u(y_r)\mathrm{d}r + \bar{a} \tag{19}$$

If the conspicuous good is not marketed, social welfare is $SW=\int_0^1 u(y_r)dr+\bar{a}$, which is unambiguously larger than social welfare in Eq. (19).

The welfare of nonconsumers is unambiguously improved by the prohibition of the conspicuous good because their consumption of numéraire good remains unchanged and their status utility increases. For those who consume the conspicuous good, the utility change is given by

$$u(y_r) - u(y_r - \gamma) + \bar{a} - \frac{\int_0^{n^*} a(i)\mathrm{d}i}{n^*} \tag{20}$$

For these individuals there is both an increase in the utility derived from consumption of the numéraire and a decrease in the utility derived from social status. Forbidding the conspicuous good induces a Pareto-improvement if and only if the first effect dominates for each individual. As the gain in intrinsic utility is least for the individual with the highest income, this condition is

$$u(y_0) - u(y_0 - \gamma) \geq \frac{\int_0^{n*} [a(i) - \bar{a}]di}{n*} \tag{21}$$

which can be rewritten as in Eq. (18). QED

As aggregate social status is constant, status-seeking by means of conspicuous consumption is a zero-sum game.[12] The production of the conspicuous good unambiguously reduces social welfare, since it consumes some numéraire without yielding any increase in the aggregate status utility. Prohibiting the conspicuous good increases social welfare by suppressing this source of inefficiency. Furthermore, if the richest individual is not too rich, so that he cares enough about the consumption of the numéraire good, even he benefits from the prohibition of the conspicuous good. In this case, the prohibition improves not only social welfare, but also the welfare of all individuals.

We now turn to the welfare implications of introducing a specific tax t per unit of conspicuous good. In the spirit of Ireland (1994), we posit that the tax revenue is equally distributed in a lump-sum way among the individuals who pay the tax.

Proposition 5. Assume $a'(r)<0$. Starting from laissez-faire, a marginal increase of the tax on the conspicuous good constitutes a Pareto-improvement (Pareto-worsening) if and only if $p'(n^*)<0$ ($p'(n^*)>0$).

Proof. Setting a tax t and making a transfer of size t to the n richest individuals implements an equilibrium market demand of n, characterized by:

$$u(y_n + t) - u(y_n - \gamma) = \sigma(n) \tag{22}$$

which implies that n is locally decreasing (increasing) with t if and only if $p'(n^*)<0$ ($p'(n^*)>0$).

In such an equilibrium, the utility of individual r can be written as:

$$U_r^*(n) = \text{Sup}\{u(y_r) + \bar{a} - n\sigma(n), u(y_r - \gamma) + \bar{a} + (1-n)\sigma(n)\} \tag{23}$$

Using Eq. (10) one obtains that ($\mathrm{d}\, n\sigma(n)/\mathrm{d}n>0$) and ($\mathrm{d}(1-n)\sigma(n)/\mathrm{d}n)<0$ if $a'(r)<0$, so that $\forall r$, $(\mathrm{d}U_r^*/\mathrm{d}n)<0$. Suppose that $p'(n^*)<0$. Then increasing the tax lowers n, which increases the welfare of all individuals. The converse applies if $p'(n^*)>0$. QED

That taxing conspicuous consumption may be Pareto-worsening is an unconventional implication of our model. To the extent that conspicuous consumption is socially wasteful, one would rather expect its taxation to be efficient, which is the

[12] The aggregate social status is equal to $(1-n)v(0)+nv(1)=(1-n)\,\{\int_n^1 a(r)dr/1-n\}+n\int_0^n a(r)dr/n=\bar{a}$.

result usually obtained in the literature [Bagwell and Bernheim (1996); Ireland (1994), Kolm (1972)]. In the present model, taxation can produce the opposite effect. When market demand is upward-sloping, taxing the good increases the number of consumers, which makes everybody worse off.

In order to illustrate the paradoxical properties of our model, the Appendix shows a specification in which taxing conspicuous expenditures makes everybody worse off, even though forbidding conspicuous consumption would yield a Pareto-improvement. In this context, quantitative restrictions on conspicuous consumption constitute a Pareto-improving policy, for which tax policy provides no good substitute.[13] From this point of view, it is worthwhile noting that sumptuary laws, setting quantitative limitations to various forms of conspicuous consumption, as for clothes and food, have long been in vigour in many European countries and in Japan, where they existed until the last century [Berry (1994)].

3.2. Voluntary provision of public goods

While it is a theoretical fact that under laissez-faire there is underprovision of public goods, voluntary donations to charity and other forms of private provision of public goods abound in the real world. An explanation of this phenomenon which is suggested by our model is that voluntary donations often take the form of conspicuous expenditures, by means of which individuals can signal high relative wealth and thereby obtain greater social status. As an example one may think of the widespread custom in classical Greece and the Roman Empire, by which wealthy citizens used to offer religious monuments, popular entertainments, and sacrifices to the deity in the name of the whole city [Veyne (1976)]. Nowadays, organizations like Rotary and Lions offer concerts and dinners at a price considerably above the market price and devote the profit in support of charitable activities. These meetings are organized in order to make socially visible who is contributing and who is not. This provides an opportunity to signal high income and obtain social recognition, which might explain why individuals are ready to pay a price above the market clearing level.[14]

It is easy to extend our model to voluntary contributions to a public good. Rewrite the utility function Eq. (1) as:

$$U_r = u(c_r) + v(\delta_r) + z(G) \tag{24}$$

where δ_r is the decision to contribute a given amount for the provision of the public good, and $z(G)$ is the utility derived from consuming the amount G of the

[13] Subsidizing conspicuous consumption is not equivalent to quantitative restrictions because it changes the form of the signalling game.

[14] Glazer and Konrad (1996) present extensive empirical evidence in support of the hypothesis of conspicuous giving for signalling purposes.

public good. Assume that p is the contribution which is demanded from individuals and that G depends on the aggregate level of contributions. Insofar as individual contributions have a negligible impact on the amount of the public good, equilibrium behaviour is exactly as described in the previous section. Clearly, status-seeking may lead to overprovision or underprovision of the public good. Furthermore, the case of an upward-sloping demand curve points out an unconventional possibility: that the number of voluntary contributors increases with the fee that they are requested to pay.

4. Concluding remarks

We have developed a model in which an individual's status depends on public perceptions about his relative income, and the individual chooses his consumption pattern by trading off the gain in status obtained by impressing the public with the loss in the consumption of commodities that are intrinsically more useful. Our analysis has focussed on the case of an indivisible conspicuous good, thereby adding to the analysis of Bagwell and Bernheim (1996) and Ireland (1994) in the same area. An attractive feature of the indivisible case is that it lends itself nicely to the analysis of bandwagon and snob effects. It makes possible to characterize the signalling value of the conspicuous good as a single variable that is a well-defined function of the number of consumers. By contrast, in the case of a perfectly divisible conspicuous good, studied for example by Ireland (1994), all individuals consume the good in different quantities, and the marginal signalling value of the good depends on the quantity consumed.

We have shown that bandwagon and snob effects can be related to the social norm that governs the allocation of status. Furthermore, the market demand curve for the conspicuous good may exhibit unusual properties; in particular, it may be upward-sloping if consumer behaviour is conformist. Conventional tax policies might be unwarranted under conspicuous consumption: taxing conspicuous expenditures may turn to enlarge the market for these goods, and diminish the welfare of everybody. Conspicuous spending may be socially desirable when it takes the form of a gift to the community: the desire to achieve status may overcome the problem of free riding on the provision of public goods.

The present approach could be extended to analyze the role of status in the area of competition policy, where unconventional policy implications might also be expected. In our model, when consumers are snobbish, a monopoly may be socially desirable insofar as it produces less than the competitive supply. On the other hand, if consumers are conformist and the demand curve is upward-sloping, a monopoly will produce more than a competitive industry and thereby reduce social welfare. The form of competition might also determine whether the conspicuous good is marketed at all. As shown by Proposition 2, there are circumstances in which a conspicuous good without intrinsic utility is purchased only if its price is high enough. If the marginal cost of supplying the good is lower than the critical price, a perfectly competitive industry will not be viable while the conspicuous good will be marketed by a monopoly.

A natural extension of the present analysis would be to consider the interaction of several competing indivisible conspicuous goods. If we consider, for example, the case in which two conspicuous goods are supplied at different prices, the population will be partitioned into three distinct classes: those who consume the expensive conspicuous good, those who consume the cheap one, and the non-consumers. The market demand for conspicuous consumption will exhibit the same kind of externalities as those studied in this paper, so that the essence of our results should carry through. It would be interesting to develop such a model, however, in order to study the endogenous supply of conspicuous goods by competing producers.

Acknowledgements

We are grateful to Douglas Bernheim, Dieter Bös, Philippe Jehiel, George Mailath and two anonymous referees for many helpful comments. We also benefitted from suggestions by participants at the First Meeting of the Research Group on the Economics of Trophies (IDEI, Toulouse, March 1995), and at seminars held in Bonn, Konstanz, Linz, Milan, Paris, Prague. Financial support from the Deutsche Forschungsgemeinschaft, SFB 303 at the University of Bonn is gratefully acknowledged.

Appendix A

A numerical example

We present a specification of the model in which: (i) the demand for the conspicuous good is increasing with its price; (ii) prohibiting conspicuous consumption is Pareto-improving, but (iii) taxing the conspicuous good is Pareto-worsening. The specification is as follows:

$$\sigma(n) = \frac{n}{3} \tag{25}$$

$$u(c) = c\left(1 - \frac{c}{2}\right) \tag{26}$$

$$y_r = 1 - \frac{r}{3} \tag{27}$$

$$\gamma = 1/2 \tag{28}$$

Plugging the adopted specification in Eqs. (5)–(7) we obtain that equilibrium market demand is related to the price through:

$$n = \frac{3}{2}\frac{p^2}{1-p} \tag{29}$$

for $p \in]0, \frac{\sqrt{7}-1}{3}[$. Hence, demand is a well defined and increasing function of the price.

Under laissez-faire, $p = \gamma = 1/2$, so that the number of consumers is $n^* = 3/4$. Is it Pareto-improving to forbid the good? It is not difficult to see that Eq. (18) is satisfied by this specification, so that forbidding the conspicuous good increases the utility of all individuals.

We now examine the impact of a specific tax t which is redistributed to the n richest individuals. Using Eq. (22), the number of consumers is characterized by:

$$n = \frac{3}{4}(1 + 2t) \tag{30}$$

where $t < 1/6$. Hence, increasing the tax rate raises the number of consumers. Using Eq. (23) one can easily verify that a marginal increase of the tax is Pareto-worsening.

References

Bagwell, L.S., Bernheim, D.B., 1996. Veblen effects in a theory of conspicuous consumption. American Economic Review 86, 349–373.

Becker, G., 1991. A note on restaurant pricing and other examples of social influences on price. Journal of Political Economy 99, 1109–1116.

Berry, C., 1994. The Idea of Luxury: A Conceptual and Historical Investigation, Cambridge Univ. Press, New York.

Clark, A.E., Oswald, A.J., 1996. Satisfaction and comparison income, Journal of Public Economics, 61, 359–381.

Cole, H.L., Mailath, G.J., Postlewaite, A., 1992. Social norms, savings behavior, and growth. Journal of Political Economy 100, 1092–1125.

Coleman, J., 1990. Foundations of Social Theory, Harvard Univ. Press, Cambridge.

Corneo, G., Jeanne, O., 1995. Conformism and snobbism in a signaling model of conspicuous consumption, working paper Ceras 95-04.

Fershtman, C., Weiss, Y., 1993. Social status, culture and economic performance. Economic Journal 103, 964–969.

Frank, R.H., 1985. The demand for unobservable and other nonpositional goods. American Economic Review 75, 101–116.

Frank, R.H., 1985b. Choosing the Right Pond, Oxford Univ. Press, New York.

Gaedeke, R.M., Tootelian, D.H., 1983. Marketing, West Publishing, St. Paul, MN.

Glazer, A., Konrad, K., 1996. A signalling explanation for charity, American Economic Review, 86, 1019–1928.

Ireland, N.J., 1994. On limiting the market for status signals. Journal of Public Economics 53, 91–110.
Karni, E., Levin, D., 1994. Social attributes and strategic equilibrium: A restaurant pricing game. Journal of Political Economy 102, 822–840.
Kolm, S.-Ch., 1972. La taxation de la consommation ostentatoire, Revue d'Economie Politique, 65–79.
Leibenstein, H., 1950. Bandwagon, snob, and Veblen effects in the theory of consumers' demand. Quaterly Journal of Economics 64, 183–207.
Lipset, S.M., 1967. Value patterns, class, and the democratic polity: the United States and Great Britain. In: Bendix, R., Lipset, S.M. (Eds.), Class, Status and Power, Routledge and Kegan Paul, London.
Milgrom, P., Roberts, J., 1986. Price and advertising signals of product quality. Journal of Political Economy 94, 796–821.
Moffit, R., 1983. An economic model of welfare stigma. American Economic Review 73, 1023–1035.
Navon, A., Shy, O., Thisse, J.-F., 1995. Product differentiation in the presence of bandwagon effects, mimeo, Ceras, Paris.
Rae, J., 1834. The Sociological Theory of Capital, MacMillan, New York.
Robson, A.J., 1992. Status, the distribution of wealth, private and social attitudes to risk. Econometrica 60, 837–857.
Spence, A.M., 1974. Market Signaling, Information Transfer in Hiring and Related Processes, Harvard University Press, Cambridge.
Veblen, T., 1922. The Theory of the Leisure Class. An Economic Study of Institutions, George Allen Unwin, London. (First published, 1899).
Veyne, P., 1976. Le Pain et le Cirque, Seuil, Paris.

Econometrica, Vol. 59, No. 6 (November, 1991), 1525–1549

EMPIRICAL EVIDENCE ON THE LAW OF DEMAND

BY WOLFGANG HÄRDLE, WERNER HILDENBRAND, AND MICHAEL JERISON[1]

A sufficient condition for market demand to satisfy the Law of Demand is that the mean of all households' income effect matrices be positive definite. We show how this mean income effect matrix can be estimated from cross section data under metonymy, an assumption about the distribution of households' characteristics. The estimation procedure uses the nonparametric method of average derivatives. Income effect matrices estimated this way from U.K. family expenditure data are in fact positive definite. This result can be explained by a special form of heteroskedasticity in the data: households' demands are more dispersed at higher income levels.

KEYWORDS: Law of demand, income effect, average derivatives, nonparametric estimation, metonymy.

1. INTRODUCTION

WHEN GENERAL EQUILIBRIUM MODELS are used to make comparative static predictions they cease to be general. This is necessarily so. Without a specific structure of the demand and supply system one cannot expect any definite comparative static results. However, in most analyses, conclusions depend upon structure imposed either by aggregating consumers into a single representative, or by assuming restrictive forms for utility or production functions. Such analyses therefore deal with special cases. The present paper considers an alternative way of imposing structure on a general equilibrium model. It considers sufficient conditions for the multimarket version of the "Law of Demand" in a consumption sector; cf. Hicks (1956). The sufficient conditions are a hybrid, combining standard theoretical restrictions with restrictions that do not come from a theoretical model. The latter restrictions can, under certain conditions, be tested and we provide such a test using U.K. family expenditure data.

The Law of Demand concerns effects of price changes when households' budgets (total expenditures) are fixed. It is a condition referring to a counterfactual, asking how mean demand would differ if prices were different. As such it cannot generally be tested using time series data. If the observation period were long enough to reveal significant price variation, it would probably also show changes in households' budgets, preferences, and demographic characteristics. Our analysis describes a way of relating the Law of Demand to cross section data.

[1] Supported by the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 der Universität Bonn. We thank Kurt Hildenbrand, Rosa Matzkin, and Thomas Stoker for stimulating discussion, Sigbert Klinke and Berwin Turlach for computing assistance. We also thank Robert Porter and three referees for comments on earlier versions of the paper. We especially thank Whitney Newey for formulating a statistical test of the metonymy assumption.

The Law of Demand is essentially equivalent to negative definiteness of the Jacobian matrix of price derivatives of mean demand. Note that this is much stronger than the requirement that demand for a good be downward sloping with respect to its own price. The Jacobian matrix can be decomposed into a mean of individual Slutsky substitution matrices and a mean of income effect matrices. Standard theory implies that the former matrix is negative semidefinite, but says nothing about the latter. A sufficient condition for the Law of Demand is positive definiteness of the mean income effect matrix. However, for a single consumer, the income effect matrix cannot be positive definite. It can be positive semidefinite, but only in the restrictive case of homothetic preferences. Hildenbrand (1983) and Hildenbrand and Hildenbrand (1986) have shown that when households have identical demand functions, dispersion in the income distribution contributes to the positive definiteness of the mean income effect matrix. In this paper we show that dispersion in tastes can also help. In particular, if the Engel curves of different consumers spread out at higher income levels, the income effect matrix is likely to be positive definite. This type of spreading of demands, a special form of heteroskedasticity is well supported by the expenditure data examined below. Our cross section estimate of the mean income effect matrix is indeed positive definite.

Our estimation procedure is nonparametric. Such nonparametric estimates are ordinarily less efficient than parametric ones when the parametric forms are known. However, the functional forms of the households' demands are in fact not known and cannot be accurately estimated from our data given that they differ across households. The potential advantage of parametric estimation is likely to turn out to be a disadvantage if the hypothesized parametric family is misspecified. More important, even this *potential* advantage is illusory in our framework. We estimate a matrix of derivatives averaged over households, and for these average derivatives, nonparametric estimates achieve the same rate of convergence as parametric ones; c.f. Stoker (1986) and Härdle and Stoker (1989).

There is another subtler reason for avoiding assumptions about functional form. Suppose we assume that households of a particular type have identical demand functions with a form commonly used in empirical analysis. The Engel curves for such demands are quite smooth, i.e. do not wiggle much. It can be shown that if the distribution of the households' budgets is sufficiently dispersed, then the mean income effect matrix is positive semidefinite; c.f. Chiappori (1985) and Grodal and Hildenbrand (1989). The sufficient degree of budget dispersion depends on the form of the Engel curves but for most commonly used forms it is not large, and the dispersion in our data is larger. Thus by assuming one of the standard functional forms for household Engel curves one effectively obtains the Law of Demand by assumption (with no further restrictions on households' demands). Among the standard forms we have in mind are polynomials of degree less than 5 or the forms proposed by Leser (1963). The nonparametric approach permits us to relax an assumption that is clearly too strong since it implies the conclusion we are investigating.

The paper proceeds as follows. In Section 2 we present a model of a large consumption sector. We define the Law of Demand and the mean income effect matrix and show how a closely related matrix can be estimated using cross section data. In Section 3 we discuss the latter matrix, estimated using the method of average derivatives. The estimation procedure is described in the Appendix.

2. THE LAW OF DEMAND AND THE MEAN INCOME EFFECT MATRIX

2.1. *A Sufficient Condition for the Law of Demand*

We consider a group (population) of households. Each household spends its exogenously given budget (total expenditure), $b \geqslant 0$, on the demand for l consumption goods. The consumption behavior of a household is described by an *individual demand function* $f\colon (p,b) \mapsto f(p,b) \in \mathbb{R}^l_{+}$, where $p \in \mathbb{R}^l_{++}$ denotes the vector of prices of the l commodities. By definition we have $p \cdot f(p,b) = b$ for all price vectors p. In empirical literature, demand is commonly treated as a function of current budget and prices although household decisions during the period of observation depend on expectations about conditions after the period. The above formulation is appropriate if the household has preferences for goods during the period that are separable from later consumption, or alternatively if the household faces a binding constraint on borrowing and the budget is fixed in advance. More generally, the household could face a longer term budget constraint, and price changes could affect the total expenditure b during the observation period. The Law of Demand concerns the effect of price changes with b held fixed, and this effect can be induced by changing prices *and the long run budget* by the proper amount. Then long run optimization does not imply the usual Slutsky conditions for the short run demand function f, but as noted below, we will not need to assume that all households' demands satisfy the Slutsky conditions.

Typically, different households may have different individual demand functions f and different budgets b. The class of all admissible individual demand functions f is denoted by $\mathscr{F}$. For example, $\mathscr{F}$ might be the class of demand functions which are generated by all (or a certain subset of) strictly convex and continuous (or smooth) preference relations on $\mathbb{R}^l_{+}$ or, more generally, the class of all demand functions which satisfy the Weak Axiom of Revealed Preference. It will be convenient in the following to label the demand functions in $\mathscr{F}$ by an index α (we then write $f^{\alpha}(p,b)$) with $f^{\alpha}(\cdot,\cdot) \neq f^{\alpha'}(\cdot,\cdot)$ if $\alpha \neq \alpha'$. The index set $\mathscr{A}$ may be a finite set, any subset of Euclidian space or, more generally, any metric space. We shall assume that $f^{\alpha}(p,b)$ depends continuously on the index α. (This representation of $\mathscr{F}$ entails no loss of generality since we can always choose $\mathscr{F}$ itself as an index set.)

With this notation every household i is described by a pair $(b_i, \alpha_i) \in \mathbb{R}_{+} \times \mathscr{A}$, that is to say, by its budget b_i and its demand function f^{α_i}. A population of households is described by a *joint distribution* of budgets b and individual demand functions f. Let μ be any probability measure on the space of consumption characteristics $\mathbb{R}_+ \times \mathscr{A}$. The *mean demand* F of a consumption sector described by the distribution μ is then defined by

$$p \mapsto F(p) = \int_{\mathbb{R}_+ \times \mathscr{A}} f^{\alpha}(p,b)\, d\mu \in \mathbb{R}^l_+.$$

We say that the *Law of Demand* holds in the consumption sector μ if the mean demand function F is *monotone*, i.e.,

$$(p-q)\cdot(F(p)-F(q)) < 0$$

for every $p, q \in \mathbb{R}^l_{++}$ with $p \neq q$. This says that for any two different price vectors p and q, the vector $(p-q)$ of price changes and the vector $(F(p)-F(q))$ of corresponding demand changes point in opposite directions. Thus, in particular, every partial demand curve is downward sloping. There is no need here to emphasize the importance and the implications of the Law of Demand (see, for example, Hicks (1956, p. 59)).

The Law of Demand holds trivially if all individual demand functions f are monotone in p for every given budget b. The standard example for this case is the set of demand functions which are derived from homothetic preferences. For a general characterization of utilities or preferences which lead to monotone demand functions we refer to Mitjuschin and Polterovich (1978) or Kannai (1989). Another case where one obtains the Law of Demand quite easily is given by a consumption sector with a decreasing density of budgets and a common demand function which satisfies the Weak Axiom of Revealed Preference (Hildenbrand (1983)). These cases, however, are examples; they cannot be considered satisfactory foundations for the Law of Demand.

In this paper we shall proceed as follows; in a first step we derive, under suitable assumptions on the individual demand functions, a *sufficient condition* for the monotonicity of the mean demand function F. There is no reason to suppose that this sufficient condition is implied by any reasonable restriction on the individual consumption characteristics and/or assumptions on the distribution μ. Then, in a second step, we develop for this sufficient condition, under suitable assumptions on the distribution μ, an empirical test based on cross-section data.

We assume from now on that the individual demand functions in $\mathscr{F}$ are continuously differentiable in prices and budget. It is well-known that the differentiable mean demand function F is monotone if the Jacobian matrix

$$\partial F(p) = \left(\partial_{p_j} F_k(p)\right)_{j,k=1,\ldots,l}$$

is negative definite for every $p \in \mathbb{R}^l_{++}$. Define the Slutsky (substitution) matrix of the demand function $f^{\alpha}(p,b)$ by

$$S(p,b,\alpha) = \partial_p f^{\alpha}(p,b) + \partial_b f^{\alpha}(p,b) f^{\alpha}(p,b)^T$$

where $f^{\alpha}(p,b)$ and $\partial_b f^{\alpha}(p,b)$ are column vectors and the superscript T

denotes the transpose. For the Jacobian matrix of the mean demand function F we then obtain

$$\partial F(p) = \bar{S}(p) - \bar{M}(p),$$

where

$$\bar{S}(p) = \int_{\mathbb{R}_+ \times \mathscr{A}} S(p, b, \alpha)\, d\mu \qquad \text{(mean Slutsky matrix)}$$

and

$$\bar{M}(p) = \int_{\mathbb{R}_+ \times \mathscr{A}} \partial_b f^\alpha(p, b) f^\alpha(p, b)^T\, d\mu$$

(mean income effect matrix).

Consequently, a sufficient condition for the monotonicity of the mean demand function F is that the mean Slutsky matrix $\bar{S}$ is negative semidefinite and the mean income effect matrix $\bar{M}$ is positive definite. If one is willing to accept the hypothesis that individual demand functions $f(p, b)$ are either derived from preference maximization or, more generally, satisfy the Weak Axiom of Revealed Preference, then it is well-known that every individual Slutsky matrix $S(p, b, \alpha)$, and hence the mean Slutsky matrix $\bar{S}(p)$, is negative semidefinite.

Of course such hypotheses are made throughout the theoretical and empirical literature. As noted above, they could be problematic when the consumers' time horizon is longer then the observation period. There is little empirical evidence concerning whether individual demands satisfy the revealed preference axioms. Battalio, et. al. (1973) describe individual consumer expenditure data in which violations of the Strong Axiom are fairly common but are small in a well-defined sense. Even if some consumers violate the Weak Axiom slightly, their effect on the Slutsky matrix $\bar{S}$ can be counterbalanced by other consumers who satisfy the axiom.

In conclusion, assuming that the mean Slutsky matrix $\bar{S}(p)$ is negative semidefinite, a sufficient condition for monotonicity of F is that the mean income effect matrix $\bar{M}(p)$ is positive definite. This property does not follow from an assumption on "rational" individual behavior. Our goal is to develop a better understanding of the class of consumption sectors μ that lead to a positive definite mean income effect matrix $\bar{M}(p)$. For the remainder of the paper we fix the price vector p and omit it as an argument.

2.2. *The Mean Income Effect Matrix for Metonymic Consumption Sectors*

The mean income effect matrix $\bar{M}$ cannot be estimated directly. In this section we describe a closely related matrix A, that can be estimated from cross section data. Note that the matrix $\bar{M}$ is positive definite if and only if the symmetrized matrix

$$M = \bar{M} + \bar{M}^T$$

has this property. The matrix M is given by

$$M = \left(\int_{\mathbb{R}_+ \times \mathscr{A}} \partial_b \left(f_j^\alpha(b) \cdot f_k^\alpha(b) \right) d\mu \right)_{j,k=1,\dots,l}.$$

To simplify notation, let $g_{jk}(b,\alpha) = f_j^\alpha(b) \cdot f_k^\alpha(b)$. We call the matrix $G(b,\alpha) = (g_{jk}(b,\alpha))$ the product matrix of the demand function f^α at expenditure level b. Thus, in matrix notation,

$$M = \int_{\mathbb{R}_+ \times \mathscr{A}} \partial_b G(b,\alpha)\, d\mu.$$

In order to define a matrix A which will be shown to be related to the matrix M and which can be estimated from cross section data we need the following properties of the distribution μ on $\mathbb{R}_+ \times \mathscr{A}$.

(i) The marginal distribution of budgets is absolutely continuous, i.e., there exists a density for the budget distribution, which we denote by ρ. In addition we shall assume that the density ρ is smooth.

(ii) Let $\mu|b$ denote the conditional distribution of α given the budget level b and consider the functions

$$\bar{f}_j(b) = \int_{\mathscr{A}} f_j^\alpha(b)\, d\mu|b \qquad (j = 1,\dots,l)$$

and

$$\bar{g}_{jk}(b) = \int_{\mathscr{A}} f_j^\alpha(b) \cdot f_k^\alpha(b)\, d\mu|b \qquad (j,k = 1,\dots,l).$$

We shall assume that the statistical Engel curve $\bar{f}_j(\cdot)$ and the conditional mean product function $\bar{g}_{jk}$ are continuously differentiable.

Let $\overline{G}(b)$ be the matrix with components $\bar{g}_{jk}$ and define the matrix A by

$$A = \int_{\mathbb{R}_+} \left(\partial_b \overline{G}(b) \right) \rho(b)\, db.$$

This matrix can be estimated from cross section data since the element a_{jk} of A is the *average derivative* of the regression function $b \mapsto \int_{\mathscr{A}} g_{jk}(b,\alpha)\, d\mu|b$. For details we refer to the Appendix.

The matrices M and A are closely related. Indeed, since

$$M = \int_{\mathbb{R}_+} \left[\int_{\mathscr{A}} \partial_b G(b,\alpha)\, d\mu|b \right] \rho(b)\, db,$$

they are in fact identical, if for every b,

$$(*) \qquad \int_{\mathscr{A}} \partial_b G(b,\alpha)\, d\mu|b = \partial_b \int_{\mathscr{A}} G(b,\alpha)\, d\mu|b,$$

i.e., the $\mu|b$ conditional mean of the derivatives of $f_j(b,\alpha) \cdot f_k(b,\alpha)$ is equal to the derivative of the conditional mean $\int_{\mathscr{A}} f_j(b,\alpha) f_k(b,\alpha)\, d\mu|b$. Thus, in particular, if the conditional distribution $\mu|b$ of individual demand functions does not depend on the budget level b (i.e. μ is a product measure), then $M=A$.

The case in which $M=A$ is particularly interesting since it permits the estimation of the symmetric mean income effect matrix M from cross section data. This motivates the following definition.

DEFINITION: A distribution μ of households' characteristics (b,α) with properties (i) and (ii) is called *metonymic* if $M=A$, which is impled by $(*)$.

To obtain a better understanding of the metonymy assumption we shall now clarify the general relationship between the two matrices M and A. For this it is helpful to imagine a Gedanken experiment in which the initially given household budgets are perturbed. Households with initial budget b will be called b-households. The derivative $\partial_b G(b,\alpha)$ in the expression for M is determined by comparing the product matrix of b-households to their product matrix when their budgets change. The derivative $\partial_b \overline{G}(b)$ in the definition of A is determined by comparing the mean product matrix for a different set of households. Define

$$\tilde{G}(b,\beta) = \int_{\mathscr{A}} G(\beta,\alpha)\, d\mu|b,$$

the mean product matrix that b-households would have if their budgets were changed to β. Then we obtain

$$M = A - U$$

where

$$U = \int \left[\partial_1 \tilde{G}(b,b)\right] \rho(b)\, db.$$

($\partial_1 \tilde{G}$ denotes the partial derivative of $\tilde{G}$ with respect to the first argument.) Metonymy requires that the matrix U vanish. The left-hand side of $(*)$ is $\partial_2 \tilde{G}(b,b)$ and the right-hand side is $\partial_1 \tilde{G}(b,b) + \partial_2 \tilde{G}(b,b)$. Thus the equality $(*)$, which is equivalent to $\partial_1 \tilde{G}(b,b) = 0$, implies that $U=0$. Note that for a product measure μ the mapping $\tilde{G}(b,\beta)$ is constant in its first argument, hence the matrix U vanishes. The property $(*)$ is weaker since it only requires that the partial derivative of $\tilde{G}$ with respect to the first argument is zero on the diagonal $b=\beta$. Metonymy is weaker still, requiring only that the integral U be zero.

Roughly speaking, under the condition $(*)$ the distribution of demands by β-households can be used to represent what the corresponding distribution for b-households would look like if their budgets changed to β, for β near b. We will make this more precise. Define

$$\tilde{f}(b,\beta) = \int_{\mathscr{A}} f^{\alpha}(\beta)\, d\mu|b$$

and

$$\tilde{C}(b,\beta) = \tilde{G}(b,\beta) - \tilde{f}(b,\beta)\tilde{f}(b,\beta)^T,$$

respectively, the mean demand and the covariance matrix of the demands by b-households whose budgets are changed to β. By the budget identity we have $G(\beta,\alpha)p = \beta f^\alpha(\beta)$, so $(*)$ implies

$$0 = \partial_1 \tilde{G}(b,\beta)p = \partial_b \int_{\mathscr{A}} \beta f^\alpha(\beta)\, d\mu | b$$

where the derivatives are evaluated at $b = \beta$. Thus $(*)$ implies

$(*.1)$ $\quad \partial_1 \tilde{f}(b,b) = 0,$

and by definition of $\tilde{C}$,

$(*.2)$ $\quad \partial_1 \tilde{C}(b,b) = 0.$

These conditions say that the mean demand and the covariance matrix of demands by $(b + \Delta b)$-households are essentially equal respectively to what the mean demand and covariance for the b-households would be if their budgets expanded by Δb. Conditions $(*.1)$ and $(*.2)$ together imply $(*)$ and hence are equivalent to $(*)$. Thus a distribution μ satisfying $(*)$ looks locally like a product measure at least in so far as its first and second conditional moments are concerned. In fact, if the individual demand functions are homogeneous of degree zero then $\tilde{f}(b,\beta)$ is independent of b.

In summary: Let the individual demand functions in $\mathscr{F}$ be *continuously differentiable* and satisfy the *Weak Axiom of Revealed Preference*. If μ is a *metonymic distribution* on $\mathbb{R}_+ \times \mathscr{A}$, then a sufficient condition for the mean demand

$$F(p) = \int_{\mathbb{R}_+ \times \mathscr{A}} f^\alpha(p,b)\, d\mu$$

to be monotone is that the matrix A be positive definite.

Given the importance of the metonymy assumption it is worthwhile considering an example in which it is violated. Let the consumption sector have a finite number of household types. All households of the same type α are assumed to have the same demand function f^α. The types of households might be identified by demographic characteristics such as the number of household members, their ages, etc. Among the households with budget b, the fraction that are of type f^α will be denoted by $\nu_\alpha(b)$. If μ is a product measure, then the functions $\nu_\alpha(\cdot)$ are constant. On the other hand for certain demographic characteristics these functions cannot be assumed constant. In our example we obtain for the matrix U:

$$U = \sum_\alpha \int \left(f^\alpha(b) f^\alpha(b)^T\right) \nu'_\alpha(b) \rho(b)\, db.$$

The matrix U may be positive or negative definite or indefinite. The example shows that it might well happen that metonymy is not satisfied for the whole population but that after appropriate stratification the subpopulations satisfy it.

The violation of metonymy poses no problem in the above example. If the household types can be identified, then the mean income effect matrix for the entire population can be calculated from the corresponding matrices of the various household types. More generally, we can consider the case in which the population is partitioned into subgroups that each satisfy metonymy. The mean income effect matrix is then a weighted average of the average derivative A matrices of the subgroups. To be more precise, let v_i be the fraction of the population in subgroup i and let μ_i be the (conditional) distribution of household characteristics within that subgroup. The average derivative matrix for subgroup i is

$$A_i = \int_{\mathbb{R}_+} (\partial_b \overline{G}_i(b)) \rho_i(b)\, db$$

where $\overline{G}_i(b)$ has jk component

$$\int_{\mathscr{A}} f_j^\alpha(b) \cdot f_k^\alpha(b)\, d\mu_i | b$$

and where $\rho_i(b) = \int_{\mathscr{A}} d\mu_i | b$. Metonymy for subgroup i implies that the matrix A_i equals the subgroup's symmetrized mean income effect matrix

$$M_i = \int_{\mathbb{R}_+ \times \mathscr{A}} \partial_b G(b, \alpha)\, d\mu_i.$$

Since $\mu = \Sigma_i v_i \mu_i$, the symmetrized mean income effect matrix for the entire population is $M = \Sigma v_i M_i = \Sigma v_i A_i$. So the matrix M can be estimated by estimating the average derivative matrices A_i for all the subgroups. In this case, metonymy for the entire population can be tested by comparing A to $\Sigma v_i A_i$. If they are not equal, the population or some subgroup must violate metonymy. A statistical test based on estimates of A and A_i is described and carried out in the Appendix. Whitney Newey has pointed out that average derivatives can be computed conditioning on any covariates of the households' demands. The tests based on stratification are simply special cases of such conditioning.

We conclude this section with a brief discussion of the matrix A. In order to isolate the factors that contribute to its positive definiteness, it is useful to compare A to the income effect matrix estimated by Hildenbrand and Hildenbrand (1986). In a consumption sector described by the distribution μ on $\mathbb{R}_+ \times \mathscr{A}$, the *statistical Engel curve* is defined by the function

$$b \mapsto \int_{\mathscr{A}} f^\alpha(p, b)\, d\mu | b = \bar{f}(p, b).$$

Hildenbrand and Hildenbrand (1986) estimate the symmetrized mean income effect matrix of $\bar{f}$, i.e., the matrix

$$B = \int \partial_b \left(\bar{f}(p,b) \bar{f}(p,b)^T \right) \rho(b)\, db.$$

This matrix turns out to be "approximately" positive definite. More precisely, the matrix B is typically ill-conditioned; some eigenvalues are very small in magnitude (positive or negative), however the larger eigenvalues are always positive. It is easy to imagine consumption sectors for which the matrix B is singular. For example, if ρ is the uniform distribution on the interval $[0, \beta]$, then $B = \bar{f}(\beta)\bar{f}(\beta)^T$, which is a positive semidefinite matrix of rank one. Under appropriate assumptions on the form of the statistical Engel curves one can show, as mentioned above, that the matrix B is always positive semidefinite provided the variance of the budget distribution is sufficiently large (for details see Chiappori (1985) and Grodal and Hildenbrand (1989)).

The matrix B differs from the above matrix A by the average derivative of a conditional covariance matrix. To see this, we note that the jk component of the conditional covariance matrix $C(b)$ of the demands of b-households is

$$\operatorname{cov}_{\mu|b}\left(f_j^\alpha(b), f_k^\alpha(b) \right) = \int_{\mathscr{A}} f_j^\alpha(b) f_k^\alpha(b)\, d\mu|b - \bar{f}_j(b) \bar{f}_k(b).$$

Hence we obtain

$$A = B + V$$

where

$$V = \int \partial_b C(b) \rho(b)\, db$$

is the average derivative of the conditional covariance matrix $C(b)$. Note that $C(b)p = 0$ and hence, $Vp = 0$, so V is singular.

The jth diagonal component of $C(b)$ is the variance of the demands for good j by b-households. The magnitude of the jth diagonal component of V measures the heteroskedasticity of the households' demands for good j since it is an average derivative with respect to b of the conditional variances of demands for good j. In a typical cross-section, demand for each good is heteroskedastic (variance increases with total expenditure b), so the diagonal components of V are strictly positive.

Positive semidefiniteness of the matrix V means roughly that on average the dispersion in consumer demands rises with the size of the budget b. A closely related type of increasing dispersion was shown by Jerison (1982) to be the weakest Engel curve restriction ensuring that mean demand satisfies the Weak Axiom (see also Freixas and Mas-Colell (1987)). Increasing dispersion has a simple geometric representation. Given a budget b, the dispersion of the b-households' demands for, say, the first m goods is measured by the principal minor matrix $\hat{C}(b)$ formed from $C(b)$ by deleting its last $l - m$ rows and columns. When $\hat{C}(b)$ is nonsingular, this demand dispersion can be represented geometrically. There is a unique ellipsoid (called the *ellipsoid of concentration*) centered at the origin in $\mathbb{R}^m$ such that a uniform distribution over the ellipsoid has the variance-covariance matrix $\hat{C}(b)$. The ellipsoid consists of the set of x satisfying

$$x \cdot \hat{C}(b)^{-1} x = m + 2;$$

cf. Cramér (1946, Ch. 22). The ellipsoid gives a simple description of the form of the dispersion of the b-households' demands for the m goods. Larger variances correspond to a larger ellipsoid. A strong form of increasing dispersion can be represented by nested ellipsoids, with the ellipsoid at budget b contained in the one at $\beta > b$. The formal requirement for this is that $x \cdot \hat{C}(\beta)^{-1}x \leqslant lm + 2$ for each x with $x \cdot \hat{C}(b)^{-1}x \leqslant lm + 2$. This is equivalent to the positive semidefiniteness of $\hat{C}(b)^{-1} - \hat{C}(\beta)^{-1}$, which is equivalent to positive semidefiniteness of $\hat{C}(\beta) - \hat{C}(b)$, c.f. Dhrymes (1984, Prop. 65, p. 76). This last condition implies that the matrix of derivatives $\partial_b C(b)$ is positive semidefinite, so the corresponding principal minor matrix of V is also positive semidefinite. Note that the matrix $\hat{C}(b)$ cannot be taken to be $C(b)$ in the argument above since the latter matrix is singular with $C(b)p = 0$. However if $C(b)$ has maximal rank $l - 1$ then $\hat{C}(b)$ can be taken to be its leading principal minor matrix of order $l - 1$. This principal minor is positive definite and hence nonsingular. If the ellipsoids of concentration for the first $l - 1$ goods are nested, then as above $\hat{C}(\beta) - \hat{C}(b)$ is positive for $\beta > b$. But this implies that $C(\beta) - C(b)$ is positive semidefinite and hence V also. (To see this, note that any l-vector x can be written as $v + \lambda p$, where λ is a scalar and the last component of v is 0. Then $x \cdot [C(\beta) - C(b)]x = u \cdot [\hat{C}(\beta) - \hat{C}(b)]u \geqslant 0$, where u is obtained from v by removing its last component.) Thus, for V to be positive semidefinite it is sufficient but not necessary that the ellipsoids of concentration for the first $l - 1$ goods be nested, expanding with the budget level. Sections of estimated ellipsoids projected on the plane are illustrated in Figure 4 below.

3. EMPIRICAL EVIDENCE

In this section we present estimates of the matrix A for various populations, along with other empirical evidence that will help in interpreting the results.

3.1. *The Variables and Data*

We consider expenditures on nine commodity aggregates:

1. Housing (HOU)
2. Fuel, light and power (FUE)
3. Food (FOO)
4. Clothing and footwear (CLO)
5. Durable household goods (DUR)
6. Services (SER)
7. Transport (TRA)
8. Other goods, and miscellaneous (OGM)
9. Alcohol and tobacco (ATO)

by each sampled household in the U.K. Family Expenditure Surveys (FES) from

1969 to 1983. Each year the expenditures of approximately 7000 households are reported. For details concerning the samples and commodity classification, see Family Expenditure Survey (1968–1983), Kemsley, Redpath, and Holmes (1980), and Schmidt (1989). In order to interpret the results, it is convenient to normalize the mean budget and the price indices of all the commodity aggregates to equal 1. This is legitimate since the estimation of a given A matrix involves observations from a single period. The demand for a good by a particular household is therefore the household's expenditure on the good divided by the mean budget for the whole population.

3.2. *Estimates of A*

The procedure for estimating A by the method of average derivatives is described in the Appendix. The estimate $\hat{A} = (\hat{a}_{jk})$ is symmetric, and is positive definite if all of its eigenvalues are strictly positive. Table I contains the smallest and largest eigenvalues of $\hat{A}$ estimated from the entire FES sample in each of the years 1969–1983. These eigenvalues are all strictly positive, so the matrices are positive definite.

The ratio of the largest to the smallest eigenvalue in Table I is never greater than 200. So the estimated matrices are well conditioned and their positive definiteness cannot be attributed to numerical (rounding) errors. In order to interpret the magnitudes of the eigenvalues in Table I it is helpful to consider the components of $\hat{A}$. Tables IIa and IIb show the components of the 1969 and 1983 $\hat{A}$ matrices multiplied by 100.

The diagonal components of $\hat{A}$ yield estimated bounds on the own price elasticities of demand. To see this, recall that $\partial_p F = \bar{S} - \bar{M}$. Under the assumption that the mean substitution matrix $\bar{S}$ is negative semidefinite, the own price effect $\partial F_j / \partial p_j$ is bounded above by the jth diagonal component of $-\bar{M}$. Under

TABLE I

MINIMAL AND MAXIMAL EIGENVALUES OF $\hat{A}$.

Year	Sample Size	$\lambda_{min} \times 100$	$\lambda_{max} \times 100$
1969	7007	0.31	25
1970	6391	0.24	25
1971	7238	0.31	25
1972	7017	0.28	25
1973	7125	0.26	24
1974	6694	0.29	24
1975	7201	0.33	24
1976	7203	0.29	24
1977	7198	0.26	24
1978	7001	0.20	24
1979	6777	0.14	23
1980	6943	0.28	24
1981	7525	0.18	23
1982	7428	0.20	24
1983	6973	0.13	23

TABLE IIA

$\hat{A} \times 100$ FOR 1969.

HOU	FUE	FOO	CLO	DUR	TRA	SER	OGM	ATO
2.91	0.86	3.84	1.75	1.40	2.91	1.65	1.33	1.47
	0.74	2.03	0.92	0.60	1.50	0.80	0.67	0.85
		10.03	4.24	2.73	6.54	3.56	3.13	4.10
			3.53	1.27	2.60	1.58	1.41	1.71
				4.10	1.64	0.96	0.94	1.10
					8.84	2.56	2.11	2.56
						3.74	1.26	1.39
							1.75	1.22
								3.00

TABLE IIB

$\hat{A} \times 100$ FOR 1983.

HOU	FUE	FOO	CLO	DUR	TRA	SER	OGM	ATO
5.12	1.18	4.21	1.76	1.87	4.13	2.63	2.01	1.67
	0.52	1.53	0.67	0.68	1.45	0.94	0.72	0.63
		6.48	2.66	2.40	5.29	3.31	2.77	2.56
			2.34	1.03	2.24	1.47	1.28	1.05
				4.23	2.04	1.29	1.16	0.94
					8.86	3.23	2.36	2.12
						5.62	1.53	1.33
							2.42	1.10
								1.89

metonymy, $A = \overline{M} + \overline{M}^T$, so this diagonal component is $a_{jj}/2$, and the own price elasticity ε_j of demand for good j satisfies

$$\varepsilon_j(p) \equiv \left| \frac{p_j}{F_j(p)} \frac{\partial F_j(p)}{\partial p_j} \right| \geqslant \frac{a_{jj}}{2} \frac{p_j}{F_j(p)}.$$

Since we normalized prices to equal 1 and divided each household's demand by the mean budget, the mean demand $F_j(p)$ equals the budget share for good j for the entire consumption sector. The estimate of $a_{jj}/2F_j(p)$ is an estimated lower bound on the magnitude of the jth own price elasticity, the bound due to income effects. The set of estimated bounds is given in Table III for 1969 and 1983.

The eigenvalues of A yield similar bounds for the effects of price changes on the demand for certain composite commodities. Let λ be an eigenvalue of A

TABLE III

LOWER BOUNDS FOR OWN PRICE ELASTICITIES, 1969 AND 1983.

Year	HOU	FUE	FOO	CLO	DUR	TRA	SER	OGM	ATO
1969	0.12	0.06	0.19	0.20	0.35	0.33	0.22	0.11	0.16
1983	0.15	0.04	0.15	0.17	0.32	0.31	0.27	0.15	0.12

and let v be the corresponding unit length eigenvector. Consider the composite commodity formed by weighting the commodities by the components of v. Mean demand for this composite commodity at the price vector p is $v \cdot F(p)$. When prices change in the direction v, the directional derivative of demand for the composite derivative is

$$v \cdot \partial F(p) v = v \cdot \bar{S} v - v \cdot \overline{M} v$$
$$\leqslant -v \cdot \overline{M} v = -\tfrac{1}{2} v \cdot M v,$$

and this last term under metonymy is $-\lambda/2$. For a discrete price change, say from q to $p = q + tv$, the effect on demand is $F(p) - F(q) \approx t\partial F(q)v$, so the effect on demand for the composite commodity is

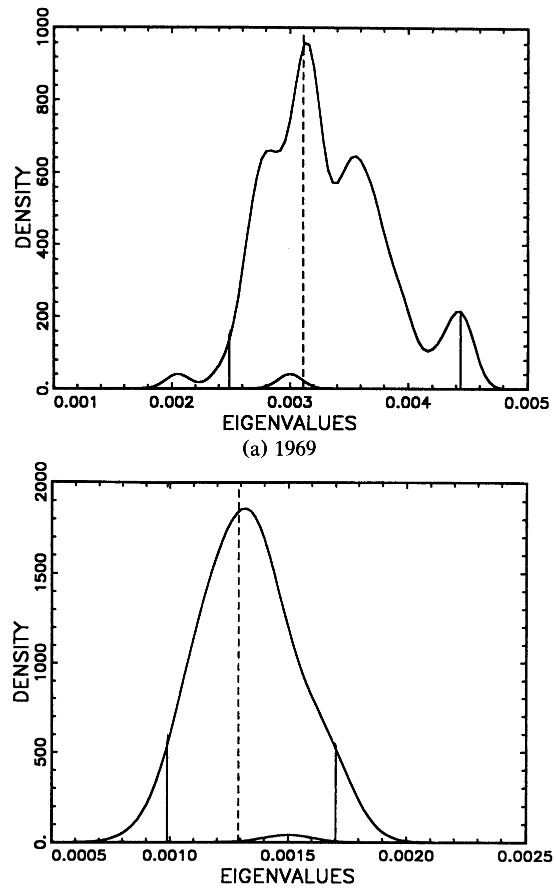
$$(p-q)(F(p) - F(q)) \approx tv \cdot \partial F(q) v = -t\frac{\lambda}{2}.$$

Table I shows that in each year the maximal eigenvalue λ is near 0.2. This implies that if prices change from q to p in the direction of the eigenvector corresponding to λ, then the term $(p-q)(F(p) - F(q))$ is bounded above by $-(.1)|p-q|$.

3.3. *Sensitivity of Estimates*

Computation of the estimate of A involves estimating ρ, the density of households' budgets, using a kernel estimator. The smoothness of this estimator is controlled by a "bandwidth" parameter. A second parameter is used to delete observations at which the estimate of ρ is very small. (See Härdle and Stoker (1989) for discussion of these parameters.)

The estimated components and eigenvalues of A are not very sensitive to the choice of bandwidth and cut-off parameters. Variations in these parameters never overturn the positive definiteness of the estimated $\hat{A}$. Concerning sampling variation, there is to our knowledge no theory of the distribution of eigenvalues of a matrix with correlated random components. However, one gets an idea of the distribution of the estimated minimum eigenvalue of $\hat{A}$ by considering the sample distribution of minimum eigenvalues computed from bootstrap estimates of $\hat{A}$. One selects randomly (with replacement) n observations from the original sample, and estimates A using the constructed bootstrap sample. Figure 1a, b shows smoothed kernel density functions for the smallest eigenvalues of the matrices estimated in this way from 100 bootstraps of the 1969 and 1983 samples. All the eigenvalues computed from the bootstrap samples were strictly positive. The Appendix contains an argument relating the bootstrap distributions to the sampling distribution of minimum eigenvalues. An elaborated theory can be found in Härdle and Hart (1989).



(b) 1983

FIGURE 1.—Estimated smallest eigenvalue kernel density functions from bootstrapping.

3.4. *Subpopulations*

The metonymy condition is more plausible the more "homogeneous" the population. For this reason we tested the positive definiteness of the matrix A for subgroups of the population, considering stratifications by age and occupation of the household head, and household composition. Table IV lists the smallest eigenvalues of the estimates of $A \times 100$ for each age group. Nearly all of the estimated matrices are positive definite and most of the others belong to the age group 80–89 with the smallest sample size.

The sum of the A matrices for the subgroups, weighted by the sample size provides an alternative estimate for M, and the minimum eigenvalue of this estimate is bounded below by the sum of the eigenvalues for the subgroups, weighted by sample size. These weighted sums are strictly positive for all years.

TABLE IV

MINIMAL EIGENVALUES OF $\hat{A}$ FOR THE STRATA "AGE."

	20–29		30–39		40–49		50–59		60–69		70–79		80–89	
Year	n	$\lambda_{min}\times 100$	n	$\lambda_{min}\times 100$	n	$\lambda_{min}\times 100$	n	$\lambda_{min}\times 100$	n	$\lambda_{min}\times 100$	n	$\lambda_{min}\times 100$	n	$\lambda_{min}\times 100$
1969	825	0.30	1275	0.25	1380	0.34	1292	0.29	1310	0.20	706	0.29	198	−0.30
1970	874	−0.49	1106	0.19	1216	0.20	1125	0.21	1192	0.47	659	0.44	190	−0.23
1971	980	0.24	1245	0.26	1336	0.31	1307	0.36	1309	0.30	820	0.13	209	0.85
1972	998	0.11	1244	0.15	1268	0.17	1299	0.42	1239	0.38	750	0.55	186	0.11
1973	1003	0.14	1180	0.30	1167	−0.29	1309	0.47	1354	0.68	844	0.50	229	0.12
1974	912	0.16	1211	0.25	1109	0.28	1179	0.91	1248	0.16	775	0.35	227	−0.76
1975	1034	0.49	1296	0.75	1173	0.37	1217	0.20	1348	0.19	828	−0.20	264	−0.37
1976	1026	0.17	1270	0.16	1140	0.24	1244	0.16	1332	0.36	905	0.83	249	−0.96
1977	991	0.14	1361	0.29	1174	0.19	1216	0.15	1282	0.16	888	0.29	246	0.29
1978	940	−0.13	339	0.78	1103	0.87	1268	0.15	1220	0.32	832	0.57	252	−0.91
1979	957	0.75	1313	0.10	1079	−0.15	1143	0.18	1078	0.13	903	0.11	260	−0.90
1980	912	0.62	1416	0.69	1107	0.74	1170	0.16	1169	0.62	851	0.61	285	−0.16
1981	918	0.13	1594	0.10	1212	0.20	1229	0.27	1290	0.19	973	0.22	271	0.34
1982	987	0.45	1533	0.56	1201	0.85	1225	0.70	1194	0.42	939	0.63	295	−0.19
1983	898	0.78	1451	0.75	1147	0.44	1089	0.14	1170	0.33	927	0.50	254	−0.11

Thus the minimal eigenvalues of the weighted sum of the subpopulation matrices are positive also. The weighted sums of these subpopulation matrices are statistically different from the A matrix estimated from the entire population. However, this difference is not large in magnitude; see the Appendix.

Similar results obtain for the stratifications by occupation in Table V and by household composition in Table VI. The categories for the latter stratification

TABLE V

MINIMAL AND MAXIMAL EIGENVALUES OF $\hat{A}$ FOR THE STRATA "PROFESSION"

	Pensioneer			Worker			Self-employed			Others		
Year	n	$\lambda_{min}\times 100$	$\lambda_{max}\times 100$	n	$\lambda_{min}\times 100$	$\lambda_{max}\times 100$	n	$\lambda_{min}\times 100$	$\lambda_{max}\times 100$	n	$\lambda_{min}\times 100$	$\lambda_{max}\times 100$
1969	1200	0.19	26	3193	0.33	25	529	0.13	23	2085	0.38	25
1970	1127	0.49	25	2899	0.16	26	486	0.15	24	1879	0.25	25
1971	1332	0.13	24	3102	0.39	25	580	0.27	24	2224	0.32	26
1972	1282	0.41	25	3065	0.20	26	468	0.14	22	2202	0.26	25
1973	1422	0.33	24	3010	0.26	25	492	0.09	21	2201	0.34	24
1974	1343	0.39	24	2735	0.11	25	561	0.50	23	2055	0.21	24
1975	1521	0.33	25	2901	0.35	25	497	0.11	24	2282	0.44	23
1976	1568	0.70	25	2951	0.22	25	454	−2.12	23	2230	0.32	25
1977	1567	0.34	25	2884	0.24	25	506	0.14	22	2241	0.23	24
1978	1529	0.62	25	2764	0.15	26	434	0.27	23	2274	0.14	24
1979	1565	0.13	24	2567	0.11	23	429	−0.90	23	2216	0.18	24
1980	1584	0.46	26	2571	0.46	26	462	0.12	23	2326	0.16	25
1981	1774	0.16	24	2659	0.15	24	564	0.09	20	2528	0.22	24
1982	1725	0.52	26	2474	0.02	25	491	−0.16	22	2737	0.14	24
1983	1719	0.46	24	1982	0.04	24	509	0.24	22	2763	0.10	24

TABLE VI

MINIMAL EIGENVALUES OF $\hat{A}$ FOR THE STRATA "HOUSEHOLD TYPE"

	1M		1F		1A + 1		2A		2A + 1		2A − 2		2A + 3		2A + + 3	
Year	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$	n	$\lambda_{min} \times 100$
1969	334	0.15	777	0.27	101	0.00	2120	0.22	723	0.31	839	0.12	322	0.30	206	0.27
1970	307	0.13	752	0.12	132	0.02	1909	0.22	621	0.07	787	0.12	339	0.04	168	0.07
1971	365	0.15	863	0.24	157	0.18	2209	0.22	695	0.12	832	0.08	359	0.10	194	0.07
1972	373	0.06	820	0.05	143	−0.01	2118	0.28	735	0.14	831	0.11	362	0.18	189	−0.01
1973	410	0.40	909	0.20	175	0.00	2196	0.14	796	0.20	858	0.15	410	0.23	212	0.14
1974	368	0.19	881	0.03	200	0.02	2075	0.38	664	0.06	872	0.31	392	0.21	203	0.21
1975	400	0.02	1020	0.45	185	0.04	2139	0.29	668	0.19	1025	0.35	373	0.15	204	0.17
1976	476	0.11	985	0.05	240	0.13	2277	0.42	668	0.27	961	0.25	354	0.06	168	0.15

are:

1 male (1 M)	2 adults + 1 child (2A + 1)
1 female (1 F)	2 adults + 2 children (2A + 2)
1 adult + 1 child (1A + 1)	2 adults + 3 children (2A + 3)
2 adults (2A)	2 adults + more than 3 children (2A + + 3)

For all stratifications, the only negative eigenvalues occur in small subpopulations.

3.5. *Further Evidence*

The estimates presented above support the hypothesis that the cross section matrix A is positive definite. Rather than present a theory consistent with such a result we will discuss further evidence that makes the above estimates more understandable. The jk component of A was shown in Section 2 to be the average derivative of the regression function $\bar{g}_{jk}$ that associates with each budget level b the average of the products of demands for goods j and k by households with budget b.

The larger the diagonal components of A the more likely is the matrix positive definite. Kernel estimates of the functions $\bar{g}_{jj}$ for 1969 are shown in Figure 2, where the index j runs over the commodity aggregates food, fuel, and transport. Estimates of $\bar{g}_{jk}$ for cross products of the same commodities ($j \neq k$) are shown in Figure 3. The household budgets and demands have been normalized, so the unit on the horizontal axis is the mean budget.

All the curves have positive slopes. What is important is that the slopes of the cross product curves are sufficiently small compared with the slopes of the corresponding (own) product curves. For example, consider the curves for food and fuel in Figures 2 and 3. The distribution of household budgets is concentrated on the interval from 0 to twice the mean budget and we can see that the

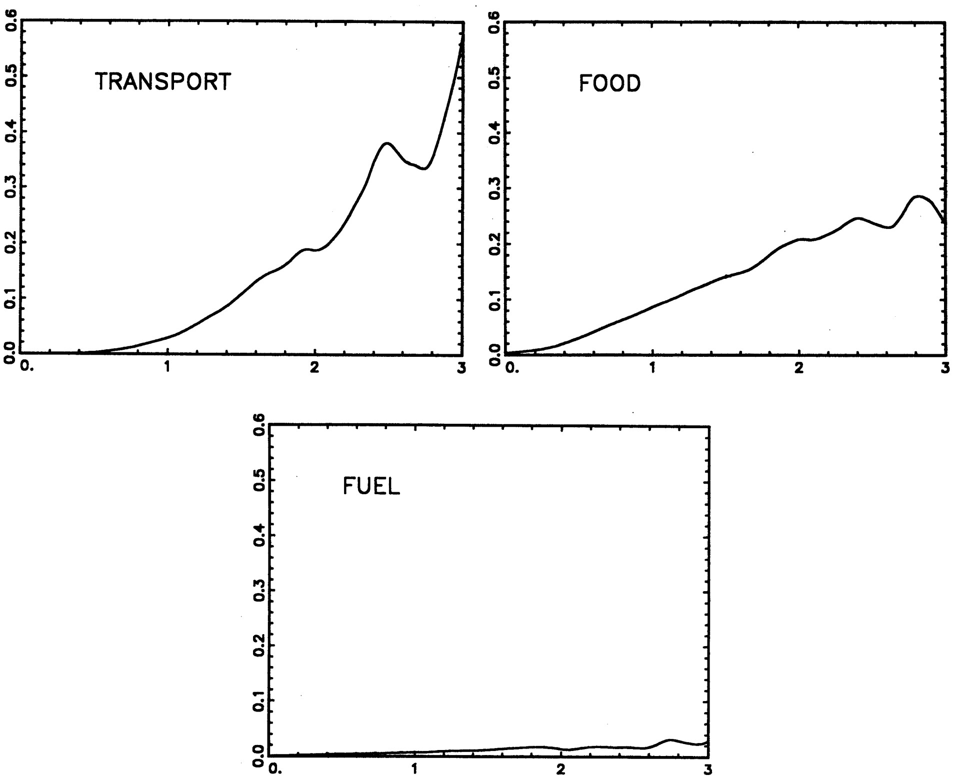


FIGURE 2.—Mean product functions $\bar{g}_{jj}$ for 1969. The unit on the horizontal axis is total expenditure divided by its mean.

slopes of the food, fuel, and food-fuel cross product curves are approximately .1, .01, and .02 respectively. These are essentially the values appearing in the 2×2 minor matrix for food and fuel in Table IIA, and this minor matrix is positive semidefinite. The graphs of $\bar{g}$ for other commodity aggregates have shapes and slopes similar to the ones shown here.

As discussed in Section 2, the positive semidefiniteness of $\hat{A}$ can be better understood by comparing it to the matrix of income effects of the cross section (statistical) Engel curve estimated by Hildenbrand and Hildenbrand (1986). The difference between these two matrices is the matrix V, the average derivative of the conditional covariance matrix. The V matrices estimated from the entire sample for the years 1969–83 are all positive semidefinite. By construction $V_p = 0$ so V cannot be positive definite. However all the estimated matrices V are positive definite on the space orthogonal to p. Unlike the product matrices, they are nearly dominant diagonal. The matrix estimates for 1969 and 1983 are shown in Table VIIa, b.

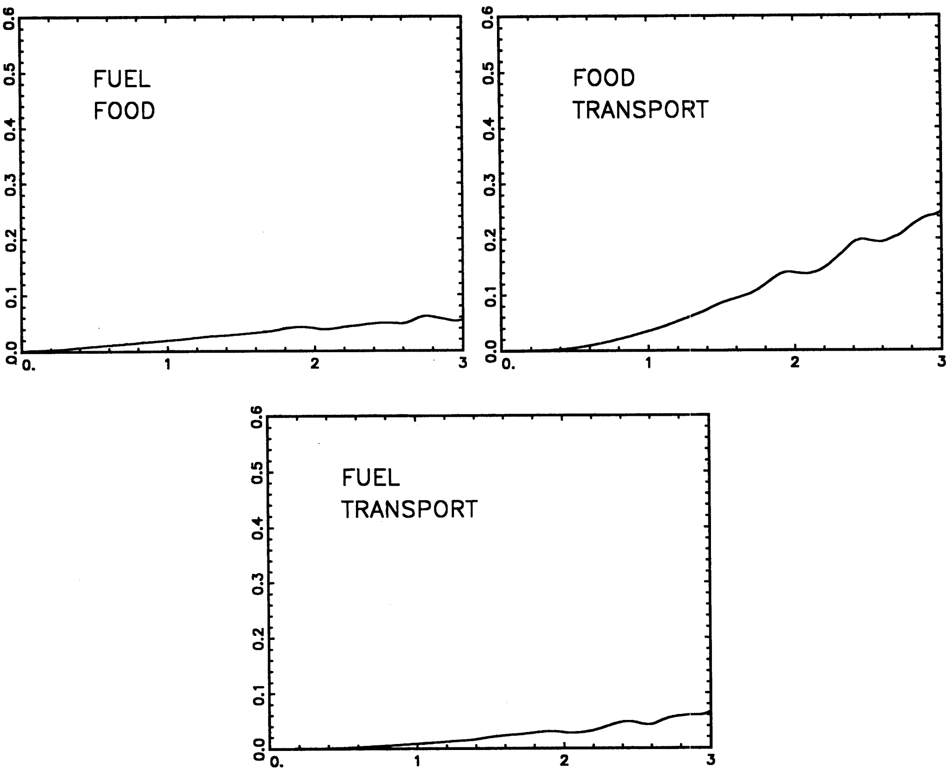


FIGURE 3.—Mean cross product functions $\bar{g}_{jk}$ for 1969. The unit on the horizontal axis is total expenditure divided by its mean.

The matrices for all the years are quite similar. Since the matrices are symmetric by definition, they have 45 components which can vary independently. All the components remain of the same order of magnitude during the sample period, and only two change sign. The spectrum of eigenvalues is also quite stable over time. For example, the eigenvalues vary by less than 30 percent. The strong positive definiteness of the estimates of V on the orthogonal component of p can be explained along lines suggested in Section 2. Positivity of the diagonal components follows from the heteroskedasticity of the households' demand for each good. This is sufficient to make V nearly dominant diagonal because the conditional correlations of households' demands for pairs of goods are rather small (generally below .2 in magnitude) and do not vary systematically with total expenditure.

Kernel estimates of the conditional covariance matrices $C(b)$ for budget levels of 0.5, 1, 1.5, and 2 times the mean budget have been computed using 1983 data and a bandwidth equal to 0.2 (see Appendix). As discussed in Section

TABLE VII

a. ENTRIES OF V FOR 1969.

HOU	FUE	FOO	CLO	DUR	TRA	SER	OGM	ATO
1.09	0.00	−0.15	−0.22	−0.10	−0.25	−0.06	−0.07	−0.25
	0.31	0.01	−0.05	−0.08	−0.11	−0.04	−0.02	−0.04
		0.98	0.02	−0.34	−0.52	−0.22	−0.03	0.18
			1.62	−0.26	−0.71	−0.24	−0.03	−0.08
				2.39	−0.70	−0.46	−0.16	−0.29
					3.66	−0.54	−0.33	−0.44
						1.94	−0.13	−0.24
							0.86	−0.14
								1.30

b. ENTRIES OF V FOR 1983.

HOU	FUE	FOO	CLO	DUR	TRA	SER	OGM	ATO
1.42	0.06	−0.14	−0.24	−0.24	−0.37	−0.14	−0.16	−0.21
	0.14	0.02	−0.03	−0.05	−0.08	−0.03	−0.02	−0.02
		0.82	0.12	−0.30	−0.35	−0.36	0.05	0.14
			1.19	−0.23	−0.40	−0.35	0.00	−0.04
				2.76	−0.75	−0.76	−0.21	−0.18
					3.43	−0.76	−0.35	−0.29
						2.94	−0.25	−0.23
							1.02	−0.05
								0.89

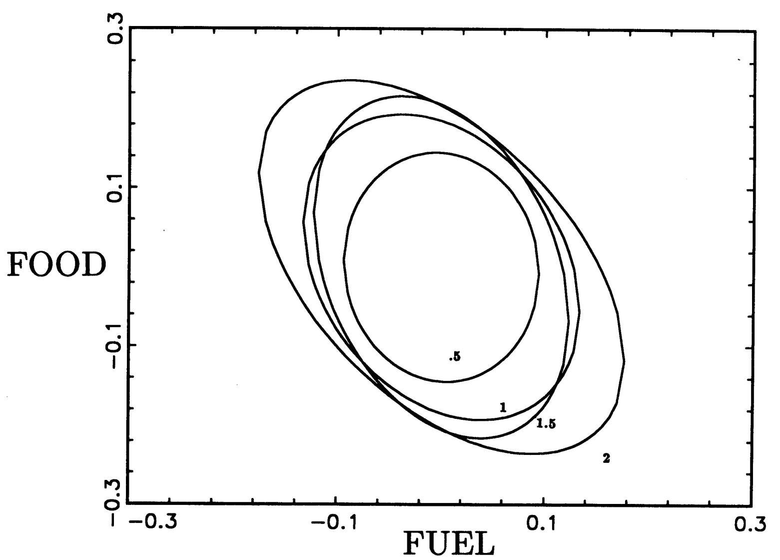


FIGURE 4.—Ellipses of concentration for 1983 at budget levels 0.5, 1.0, 1.5, 2.0 times the mean budget.

2, these matrices determine ellipses of concentration for each pair of goods. (The coordinates of the ellipsoid that correspond to the other goods are set equal to zero.) These ellipses are not always nested, but are nearly so. Figure 4 shows the ellipses for food and fuel. The conditional variances of demands for nearly all goods are larger for β-households than for b-households when $\beta > b$. The only exception is for fuel with $b = 1$ and $\beta = 1.5$. On average, the dispersion of the consumers' demands clearly increases with the budget level.

CORE, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium,
Department of Economics, Universitat Bonn, Adenauerallee 24-26, W-5300 Bonn 1, Germany,
and
Department of Economics, State University of New York, Albany, NY 12222, USA

Manuscript received September, 1988; final revision received February, 1991.

APPENDIX

Estimation of A

In this section, we describe the procedure used to estimate the matrix

$$A = \int_{\mathbb{R}_+} \left(\partial_b \overline{G}(b)\right) \rho(b)\, db.$$

The data consist of households' expenditures on each of the 9 commodity aggregates during a given period.

We normalize the prices of all commodity aggregates to be 1. A household's demand for a good is then equal to its expenditure on the good. The characteristics (b_i, α_i) of a randomly sampled household i have the distribution μ. The mean budget in the sample is denoted $\bar{b}$. We consider a fixed pair of goods j and k, and define $X_i = b_i/\bar{b}$ and $Y_i = f_j^{\alpha_i}(p, b_i) f_k^{\alpha_i}(p, b_i)/(\bar{b})^2$. Then we can interpret X_i as the budget of household i and Y_i as the jk component of the household's product matrix when the mean budget is normalized to 1. Since $\bar{b}$ is a sample mean, the pairs (X_i, Y_i) are correlated for different households. However, since the sample is large, the correlation is slight, and we will ignore it, treating the (X_i, Y_i) as i.i.d. These random variables have a distribution induced by μ, and the regression function is denoted $m(x) = E(Y_i | X_i = x)$. The jk component of A is then $\delta \bar{b}$, where

$$\begin{aligned}\delta &= E_X m'(X) \\ &= \int m'(x) \rho(x)\, dx\end{aligned}$$

is the average derivative of m. By construction, the sum of the components of $f^{\alpha_i}(p, b_i)$ is b_i, and the b_i variables are distributed with compact support. Thus the distribution of (X_i, Y_i) has compact support.

Our approach to estimation of the average derivative δ is based on the simple observation that if ρ vanishes at the boundary of its support, then partial integration gives

$$\delta = \int m(x) L(x) \rho(x)\, dx$$

with

$$L = -d \log \rho / dx = -\rho'/\rho. \tag{4.1}$$

Since $L(\cdot)$ is not known we have to estimate it. We use the kernel technique and estimate the

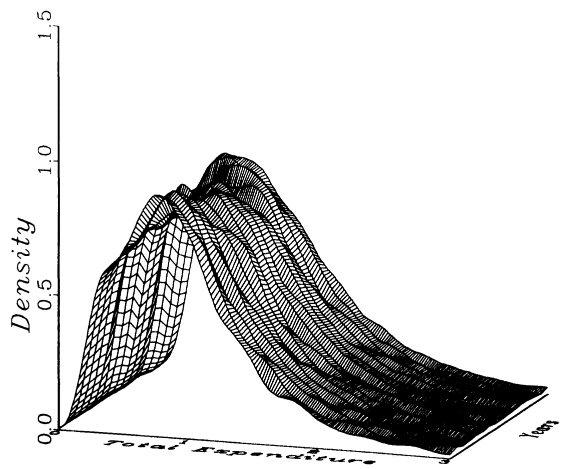


FIGURE 5.—The estimated densities of total expenditure $\hat{\rho}_h(x)$, 1968–1983.

density function $\rho(x)$ by a Rosenblatt-Parzen kernel density estimator

$$\hat{\rho}_h(x) = n^{-1} \sum_{i=1}^{n} K_h(x - X_i) \tag{4.2}$$

where $K_h(\cdot) = h^{-1}K(\cdot/h)$ is a kernel function with bandwidth h. We use a quartic kernel, $K(u) = (15/16)(1-u^2)^2$ for $|u| \leqslant 1$; see Härdle (1990). Figure 5 shows the estimated density functions $\hat{\rho}_h$ for the entire sample period.

From the estimates $\hat{\rho}_h(x)$ we obtain as an approximation to $L(x)$ the ratio $\hat{L}_h(x) - \hat{\rho}'_h(x)/\hat{\rho}_h(x)$. (To avoid a zero denominator in low density regions we compute this only for budgets in the interval from 0.1 to 3 times the mean budget.) The Average Derivative Estimator $\hat{\delta}$ is then defined as

$$\hat{\delta} = n^{-1} \sum_{i=1}^{n} Y_i \hat{L}_h(X_i). \tag{4.3}$$

The argument in Härdle and Stoker (1989) yields the following theorem.

AVERAGE DERIVATIVE ESTIMATION THEOREM: *There exists a sequence of bandwidths* $h_n \to 0$ *with corresponding average derivative estimator* $\hat{\delta}$, *defined in* (4.3) *such that* $\sqrt{n}(\hat{\delta} - \delta)$ *has a limiting Normal distribution with mean* 0 *and variance* σ^2, *where*

$$\sigma^2 = \operatorname{var}[m'(X) + (Y - m(X))L(X)]. \tag{4.4}$$

This version of the theorem can be proved by modifying the proof of Härdle and Stoker (1989) slightly to allow for nonnegative kernels. The $\sqrt{n}$ rate of convergence is remarkable in that all the components of δ are nonparametrically estimated without any structural assumptions on ρ and m. Thus, although nonparametric estimation typically exhibits slower rates of convergence, the specific structure of the average derivative functional makes it possible to achieve the rate of convergence that is typical for parametric problems.

The computations for the A-matrix have been performed with a variety of values for the bandwidth h. All of the results reported in Section 3 use $h = 0.2$ (i.e. two tenths of the mean budget). This is the optimal value of h minimizing the mean square error (MSE) of (4.3). Härdle, Hart, Marron, and Tsybakov (1991) analyzed this mean square error and showed that there exist constants C_1 and C_2 such that $\text{MSE} = \sigma^2 n^{-1} + C_1 n^{-2} h^{-3} + C_2 h^4$. From this expression a "plug-in" estimate for the optimal h can be derived. The optimization of the kernel function for Average Derivative Estimation has been considered by Mammitzsch (1989) who showed that the Quartic kernel used in our studies is optimal.

In order to estimate the variability of the average derivative estimates we used the sample based terms given in Härdle and Stoker (1989, formula (3.6)),

$$\hat{r}_{hi} = \hat{L}_h(X_i)Y_i + n^{-1}\sum_{j=1}^{n}\left[K_h'(X_i - X_j) - K_h(X_i - X_j)\hat{L}_h(X_j)\right]\frac{Y_j}{\hat{\rho}_h(X_j)}. \tag{4.5}$$

The sample variance of these terms approximates the variance given in (4.4). The formula (4.5) is based on a linearization of the average derivative estimator in (4.3). The fact however that we used a fixed smoothing parameter for the whole range of income created high estimated variances for the entries of the $\hat{A}$ matrices. This becomes evident from Figure 5 which shows the estimated densities of total expenditure over time: at the far end (near the value of total expenditure 3.0) the estimate $\hat{\rho}_h(x)$ is very small. Therefore the score function L, although we used the cutoff technique described in Härdle and Stoker (1989), must become rather unstable. To overcome this difficulty we could, of course, use a varying bandwidth $h = h(x)$ but this is still an open problem.

An alternative method of measuring the standard error of the average derivatives is to compute the interquartile range (or F-spread) of the terms $\hat{r}_{hi}$ in (4.5). The F-spreads (times 100) for the diagonal elements of the A matrix of 1983 for instance are

$$(2.3, 0.6, 3.8, 1.9, 2.2, 7.4, 2.6, 1.0, 1.1).$$

The variances (times 100) of the terms $\hat{r}_{hi}$ for these diagonal elements are

$$(19.4, 2.14, 11.9, 11.0, 29.5, 32.4, 36.7, 7.1, 6.9).$$

The variances are much larger than the F-spreads because the distributions are highly skewed. For normal data the standard deviation is 1.39 times the F-spread.

Using these measures of variation we can consider the question of metonymy of the full population and each subclass defined by stratification. As an example we consider the age strata. Metonymy requires that A equal the weighted average of the A_i matrices estimated from the strata; see Section 2. For simplicity we consider the comparison of the diagonal elements. The weighted average matrix had the following diagonal elements in 1983:

$$(4.82, 0.46, 4.13, 1.81, 3.23, 7.29, 5.17, 1.41, 1.08).$$

As a first step one could treat these diagonal elements as given and apply a t test for each element. However, this procedure is inadequate because the two matrices are computed from the same data. The resulting correlation is accounted for in the following test suggested by Whitney Newey. Let ξ denote the vector of elements of A. Then

$$T = n(\hat{\xi}_1 - \hat{\xi}_2)\cdot\hat{\Sigma}^{-1}(\hat{\xi}_1 - \hat{\xi}_2)$$

is an asymptotic chi-square statistic for the difference between the stratified and unstratified estimates of A. Here $\hat{\xi}_1$ is the vector of components of $\hat{A}$, $\hat{\xi}_2$ is the vector of components of the weighted average of $\hat{A}_i$ estimates from the strata and $\hat{\Sigma}$ denotes a consistent variance estimator for the difference. Formula (4.5) can be used to calculate T:

$$\sqrt{n}\left(\hat{\xi}_j - \xi\right) \approx \sum_{i=1}^{n} r_{hi}^{(j)}/\sqrt{n}, \qquad j = 1, 2,$$

where $r_{hi}^{(j)}$ denotes the vector of terms in (4.5) for the stratified and unstratified case. The covariance matrix of the difference can be estimated by

$$\hat{\Sigma} = n^{-1}\sum_{i=1}^{n}\left(r_{hi}^{(1)} - r_{hi}^{(2)}\right)\left(r_{hi}^{(1)} - r_{hi}^{(2)}\right)^T.$$

We performed this test for the diagonal of A and obtained the value of $T = 0.046$ for the year 1983. The other years had T values in the range 0.03 to 0.1. So the hypothesis that the matrices are equal cannot be rejected.

Bootstrapping the Distribution of the Smallest Eigenvalue of A

The distribution of the smallest eigenvalue of $\hat{A}$ is asymptotically normal, as is seen below in Theorem A. In the context of estimating covariance matrices similar asymptotic normality have been derived. To our knowledge such a result for general random matrices is not available. In the following presentation we follow the paper by Härdle and Hart (1991). A column vector of 0's and a $k \times k$ identity matrix will be denoted, respectively, 0 and I. The eigenvalues of A are $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, while those of $\hat{A}$ are $\hat{\lambda}_1 < \hat{\lambda}_2 < \cdots < \hat{\lambda}_k$. $C = [c_{ij}]$ will denote a $k \times k$ matrix with typical element c_{ij}. For any $k \times k$ symmetric matrix C, $u\,\mathrm{vec}(C)$ is the $k(k+1)/2$ component column vector $(c_{11}, \ldots, c_{1k}, c_{22}, \ldots, c_{2k}, \ldots, c_{kk})'$. Let V denote the asymptotic covariance matrix of $u\,\mathrm{vec}(\hat{A})$.

THEOREM A: *Define $A_{ij}(\lambda_1)$ to be the cofactor of the ijth element of $A - \lambda_1 I$. Let $B = 2[A_{ij}(\lambda_1)] - \mathrm{diag}(A_{11}(\lambda_1), \ldots, A_{kk}(\lambda_1))$, and let $D(x) = |A - xI|$. Then*

$$\sqrt{n}\left(\hat{\lambda}_1 - \lambda_1\right) \xrightarrow{\mathrm{D}} N\left(0, \sigma_1^2\right),$$

where

$$\sigma_1^2 = \frac{u\,\mathrm{vec}(B)' V u\,\mathrm{vec}(B)}{\left(D'(\lambda_1)\right)^2}.$$

Although an estimator $\hat{V}$ of V can be constructed to use this result for testing $\lambda_1 > 0$ the procedure for doing so will be quite complicated. Therefore a bootstrap approximation to the distribution of $\sqrt{n}(\hat{\lambda}_1 - \lambda_1)$ seems to be an attractive alternative. The bootstrap we used resamples from the data $\{(b_i, f_j^{\alpha_i}(p, b_i))\}_{i=1}^n$ for a given year. More precisely n new observations are sampled with replacement. The bootstrap sample determines for each pair of goods a pair (X_i^*, Y_i^*) defined the same way as (X_i, Y_i).

To define the bootstrap distribution P^* of the smallest eigenvalue we have to compute A^*, the matrix $\hat{A}$ computed from a bootstrap sample (X_i^*, Y_i^*). Now calculate λ_1^*, the smallest eigenvalue of A^*. Repeated sampling allows one to approximate the bootstrap distribution P^* of $(\lambda_1^* - \hat{\lambda}_1)$ and then to conduct a test of the relevant hypothesis. Theorem B in Härdle and Hart (1991) shows, in fact, that the bootstrap distribution of $\sqrt{n}(\lambda_1^* - \hat{\lambda}_1)$ is asymptotically close to that of $\sqrt{n}(\hat{\lambda}_1 - \lambda_1)$.

A bootstrap test can now be conducted as follows. One determines an interval $[-B^*, C^*]$ from the bootstrap distribution of $\hat{\lambda}_1^* - \hat{\lambda}_1$ which has probability, say, .95. Then one computes a confidence interval for λ_1 as $[\hat{\lambda}_1 - C^*, \hat{\lambda}_1 + B^*]$. The hypothesis of positive definiteness is rejected if $\hat{\lambda}_1 - C^* > 0$. (Of course, the nominal level of this one-sided test is .025.)

REFERENCES

BATTALIO, R. G. ET AL. (1973): A Test of Consumer Demand Theory Using Observations of Individual Consumer Purchases. *Western Economic Journal*, 11, 411–428.

CHIAPPORI, P. A. (1985): "Distribution of Income and the 'Law of Demand'," *Econometrica*, 53, 109–127.

CRAMÉR, H. (1946): *Mathematical Methods of Statistics*. Princeton: Princeton University Press.

DHRYMES, P. J. (1984): *Mathematics for Econometrics*. New York: Springer-Verlag.

FAMILY EXPENDITURE SURVEY, ANNUAL BASE TAPES (1968–1983): Department of Employment, Statistics Division, Her Majesty's Stationary Office, London, 1968–1983. The data utilized in this paper were made available by the ESRC Data Archive at the University of Essex.

FREIXAS, X., AND A. MAS-COLELL (1987): "Engel Curves Leading to the Weak Axiom in the Aggregate," *Econometrica*, 55, 515–531.

GRODAL, B., AND W. HILDENBRAND (1989): "Statistical Engelcurves, Income Distribution and the Law of Demand," SFB 303, Universität Bonn DP No. A-108. To appear in *Aggregation, Consumption and Trade: Essays in Honor of H. S. Houthakker*, ed. by L. Phlips and L. D. Taylor. Dordrecht: Kluwer Academic Publishers, 1992.

HÄRDLE, W. (1990): *Applied Nonparametric Regression*. Econometric Society Monograph Series 19. Cambridge: Cambridge University Press.

HÄRDLE, W., AND J. HART (1991): "A Bootstrap Test for Positive Definiteness of Income Effect Matrices," to appear in *Econometric Theory*.

HÄRDLE, W., J. HART, J. S. MARRON, AND A. B. TSYBAKOV (1991): "Choice of Smoothing Parameters for Average Derivative Estimation," to appear in the *Journal of the American Statistical Association*.

HÄRDLE, W., AND T. M. STOKER (1989): "Investigating Smooth Multiple Regression by the Method of Average Derivatives," *Journal of the American Statistical Association*, 84, 986–995.
HICKS, J. R. (1956): *A Revision of Demand Theory*. Oxford: Clarendon Press.
HILDENBRAND, K., AND W. HILDENBRAND (1986): *On the Mean Income Effect: A Data Analysis of the U.K. Family Expenditure Survey*. Contributions to Mathematical Economics, in Honor of Gérard Debreu, ed. by W. Hildenbrand and A. Mas-Colell. Amsterdam: North Holland, 247–268.
HILDENBRAND, W. (1983): "On the Law of Demand," *Econometrica*, 51, 997–1019.
JERISON, M. (1982): "The Representative Consumer and the Weak Axiom when the Distribution of Income is Fixed," SUNY Albany, DP 150.
KANNAI, Y. (1989): "A Characterization of Monotone Individual Demand Functions," *Journal of Mathematical Economics*, 18, 87–94.
KEMSLEY, W. F., R. D. REDPATH, AND M. HOLMES (1980): *Family Expenditure Survey Handbook*. London: Her Majesty's Stationary Office.
LESER, C. E. (1963): "Forms of Engel Functions," *Econometrica*, 31, 594–703.
MAMMITZSCH, V. (1989): "Asymptotically Optimal Kernels for Average Derivative Estimation," manuscript, also given as an IMS Lecture, Davis, California, June, 1989.
MITJUSCHIN, L. G., AND J. POLTEROVICH (1978): "Criteria for Monotonicity of Demand Functions" (in Russian), *Ekonomika i Matematicheski Metody*, 14, 122–128.
SCHMIDT, H. (1989): "Family Expenditure Survey—Methodology, and Data Used in Microeconomic Demand Analysis," SFB 303, Universitat Bonn, DP No. A231.
STOKER, T. M. (1986): "Consistent Estimation of Scaled Coefficients," *Econometrica*, 54, 1461–1481.

ELSEVIER Journal of Mathematical Economics 31 (1999) 81–109

JOURNAL OF Mathematical ECONOMICS

Demand aggregation under structural stability

W. Hildenbrand [a,*], A. Kneip [a,b,1]

[a] *Rheinische Friedrich-Wilhelms-Universität Bonn, Department of Economics, Lennéstraße 37, D-53113 Bonn, Germany*

[b] *C.O.R.E., 34 Voie du Roman Pays, B-1348 Louvain-La-Neuve, Belgium*

Received 2 October 1997; accepted 15 July 1998

Abstract

The goal of this paper is to model the mean (aggregate) consumption expenditure of a large and heterogeneous population of households. The aggregation process is based on assumptions of how the income distribution and the composition of the population evolves over time (structural stability). It is shown that the change in the aggregate consumption expenditure ratio can be decomposed into an effect of changing income dispersion, an effect of income growth, an effect of price-inflation and an effect of changing composition of the population. © 1999 Elsevier Science S.A. All rights reserved.

JEL classification: D 11; D 31; C 43; E 10

Keywords: Aggregate consumption expenditure; Aggregation; Structural stability; Income distribution; Distribution of household attributes

1. Introduction

The goal of this paper is to model the change over time of *mean consumption expenditure* C_t of a *large and heterogeneous population* H_t of house holds:

$$C_t = \frac{1}{\#H_t} \sum_{h \in H_t} c_t^h \tag{1}$$

where c_t^h denotes the consumption expenditure of household h in current prices

* Corresponding author. Tel.: +49-228-73-92-42; fax: +49-228-73-79-40; e-mail: with2@econ2.uni-bonn.de

[1] Tel.: +32-10-47-43-30; fax: +32-10-47-30-32; e-mail: kneip@stat.ucl.ac.be.

PII: S0304-4068(98)00058-5

during period t on all commodities that belong to a certain consumption category, such as food, housing or non-durables.

The starting point of any analysis of aggregation across households is a model of individual household behaviour.

To concentrate in this introduction on the essential we start directly from a *micro-relation* $c(x,\chi)$

$$c_t^h = c\left(x_t^h, \chi_t^h\right), \quad h \in H_t \tag{2}$$

where x_t^h denotes disposable *income* in period t of household h and $\chi_t^h = (\chi_{c,1}^h, \chi_{t,2}^h, \ldots)$ denotes a vector of *household characteristics* that are used as explanatory variables in the underlying model of household behaviour (e.g. preferences). We do not explicitly mention prices and interest rates in the introduction; this amounts to assuming that they do not change over time.

The population of households in period t is described by the joint distribution μ_t of household income x^h and characteristics χ^h across the population H_t. Given the micro-relation (2), one obtains for mean consumption expenditure

$$C_t = \int c(x,\chi) \mathrm{d}\mu_t. \tag{3}$$

Thus, given the micro-relation c, C_t is a function of μ_t; $C_t = C(\mu_t)$.

The distribution μ_t, however, is not a useful explanatory variable for mean consumption expenditure, because it is a far too detailed description of the population. The goal of aggregation theory [2] is to simplify the function $C(\mu_t)$ by reducing the entire distribution μ_t to certain relevant characteristics of μ_t, such as mean or dispersion. Obviously, such a simplification – even if one is satisfied with an approximation to $C(\mu_t)$ – is only possible if one restricts the way in which the distribution μ_t changes over time and/or if one appropriately specifies the micro-relation.

In order to illustrate this point we give a simple example. If the Engel curve of the population

$$x \mapsto \int c(x,\chi) \mathrm{d}\mu_t | x = :\bar{c}_t(x)$$

is time-invariant, i.e., $\bar{c}_t = \bar{c}$ (definitely an unrealistic assumption), then Eq. (3) becomes

$$C_t = \int \bar{c}(x) \rho_t(x) \mathrm{d}x$$

where ρ_t denotes the density of the income distribution in period t.

Thus, if $\bar{c}$ is linear, then mean consumption expenditure $C_t = \bar{c}(X_t)$, and hence depends only on mean income X_t without any restriction on the changes in the income distributions.

[2] There is a large literature on aggregation starting with Antonelli in 1886. For a general discussion of the various aspects of aggregation theory we recommend Malinvaud (1993).

On the other hand, if changes in the income distributions are restricted to proportional changes in household income, hence the relative income distribution is time-invariant, say equal to ρ^*, then one obtains $C_t = \int \bar{c}(X_t \cdot \xi)\rho^*(\xi)\mathrm{d}\xi$. Thus, mean consumption expenditure depends only on mean income X_t without any restriction on the Engel curve $\bar{c}$.

In this paper we want to avoid, as far as possible, any assumption on the micro-relation (other than being smooth in the relevant variables, e.g., income). Thus, the micro-relation is merely a notation; it just specifies the set of explanatory variables for consumption expenditure on the household level. To achieve the desired simplification of $C(\mu_t)$ we must therefore restrict the evolution over time of the distribution μ_t.

Since some of the household characteristics—that are explanatory variables in the micro-relation—are *unobservable*, we consider in addition to household characteristics also *observable* household *attributes*, such as age and employment status or household size. Household characteristics that are observable may be listed among attributes as well. Household income and attributes are used to stratify the population.

Then we obtain

$$C_t = \int \left[\int c(x,\chi) \mathrm{d}\mu_t|(x,a) \right] \mathrm{d}\nu_t$$

that is to say, we first consider mean consumption expenditure of the subpopulation consisting of all households with income x and attribute profile a and then we average over the subpopulations, i.e., we integrate with respect to the joint distribution ν_t of income and attributes.

In Section 2 we model the changes over time of the conditional distribution $\mu_t|(x, a)$ of household characteristics (Hypothesis 1) and of the joint distribution ν_t of household income and attributes (Hypotheses 2 and 3).

It is our goal to 'explain' the observed changes over time of C_t by changes in the observable income-attribute distribution ν_t. Such an 'explanation' is only satisfactory if changes in C_t are not attributed to changes in the unobservable distributions $\mu_t|(x, a)$. Therefore we must somehow link changes in $\mu_t|(x, a)$ to changes in ν_t. This is achieved by Hypothesis 1, which is called 'structural stability [3] of household characteristics with respect to household attributes'. In the special case where the conditional distribution $\mu_t|(x, a)$ does not depend on x, the hypothesis simply expresses that the distribution $\mu_t^X|a$ of household characteristics across all households with attribute profile a changes very slowly over time such that the distributions $\mu_s^X|a$ and $\mu_t^X|a$ can be considered as identical for periods s and t that are not too far apart from each other (local time-invariance).

[3] The idea of "structural stability" is borrowed from Malinvaud (1981), chapter 2.3 and Malinvaud (1993), section 10.

Hypothesis 2 describes how the income distributions are allowed to change over time. Since we want to allow for changing income dispersion (e.g., changing Gini-coefficient) we cannot rely on the simple assumption of time-invariance of the relative income distribution. Obviously, the actual evolution of household income is more complex than just a proportional change. No single assumption can exactly describe the complex evolution of income. We have chosen the simple hypothesis of local time-invariance of the standardized log income distribution (Hypothesis 2). Of course, this hypothesis should only be considered as an approximation to the complex actual changes of income distributions. The descriptive accuracy of Hypothesis 2 is illustrated in Figs. 1 and 2.

Finally, we model how the attribute distributions are allowed to change over time. Hypothesis 3 expresses that the income-conditioned attribute distribution $\nu_s|x_s$ in period s is 'approximately' equal to the income-conditioned attribute distribution $\nu_t|x_t$ in period t for two periods s and t that are close to each other, provided the income levels x_s and x_t are in the same percentile position (quantile) in the income distribution in period s and t, respectively.

As in the case of Hypothesis 2, this Hypothesis should be interpreted as an approximation, capturing the main tendency of the actual very complex change of attributes. The empirical content of Hypothesis 3 is illustrated in Figs. 3–5.

The propositions in Section 3 are based on a strong version of Hypothesis 3. It is assumed that the difference between the attribute distributions $\nu_s|x_s$ and $\nu_t|x_t$

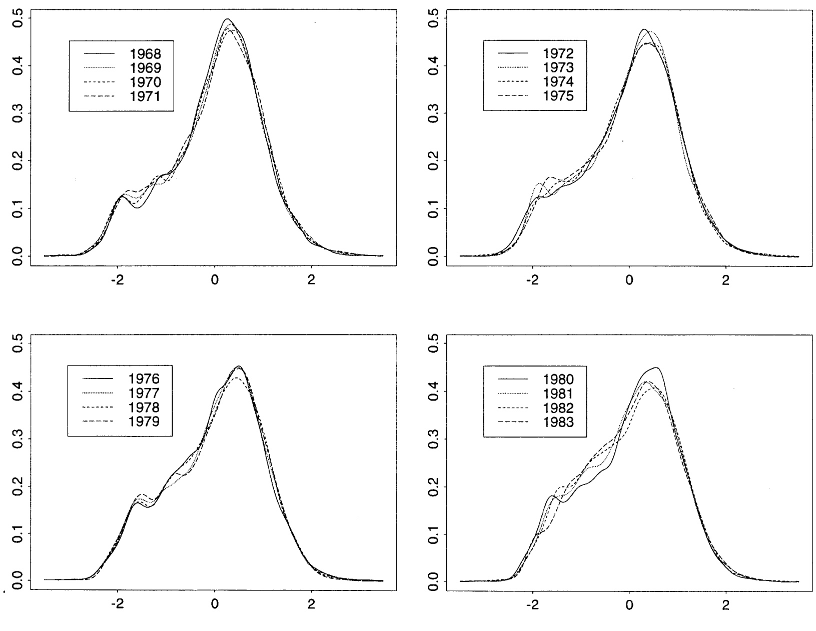


Fig. 1. Kernel density estimates of the standardized log income distribution; Her Majesty's Stationary Office, total population.

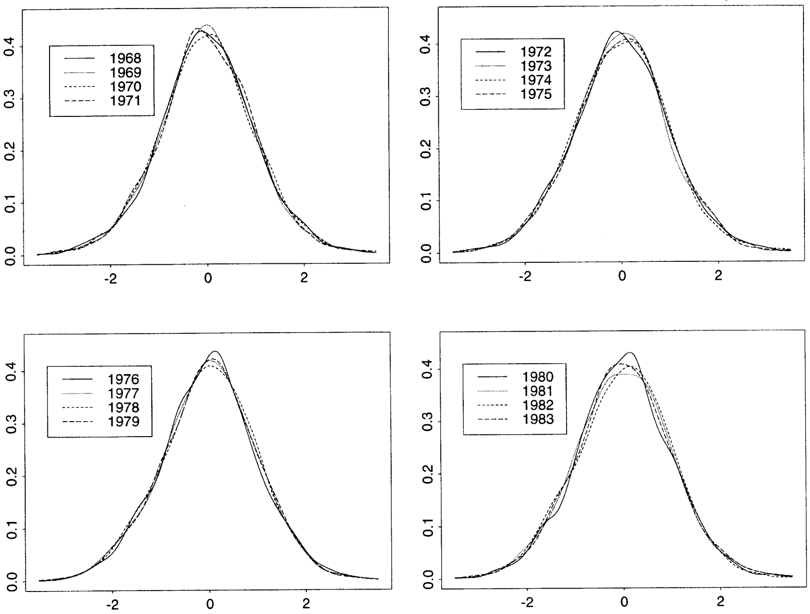


Fig. 2. Kernel density estimates of the standardized log income distribution: Her Majesty's Stationary Office, subpopulation of full-time employed head of household.

can be neglected (Hypothesis 3^+). This requires, of course, that the periods s and t are close to each other.

We then derive for the change of the mean consumption expenditure ratio a first-order approximation (Proposition 2):

$$C_t/X_t - C_s/X_s = \alpha_s \log\frac{\sigma_t}{\sigma_s} + \beta_s \log\frac{X_t}{X_s} + O\left(\max\left\{\left(\log\frac{\sigma_t}{\sigma_s}\right)^2, \left(\log\frac{X_t}{X_s}\right)^2\right\}\right)$$

where the coefficients α_s and β_s are determined by the micro-relation c and the distribution μ_s and σ_s is a measure of income dispersion.

Consequently, neglecting second-order terms, the change in the mean consumption expenditure ratio is the sum of two terms: the effect of the changing income dispersion, $\alpha_s \log(\sigma_t)/(\sigma_s)$, and the effect of mean income growth, $\beta_s \log(X_t)/(X_s)$.

The sign of the coefficients α_s and β_s are, of course, important. For example, a negative α_s implies that increasing income dispersion, (hence $\log(\sigma_t)/(\sigma_s)$ is positive), decreases the mean consumption expenditure ratio.

In Section 3 we show what kind of information on the micro-relation and the distribution μ_s is required to infer the sign or magnitude of the coefficients α_s and β_s. It turns out that no assumption on the micro-relation alone determines the sign of the coefficient α_s.

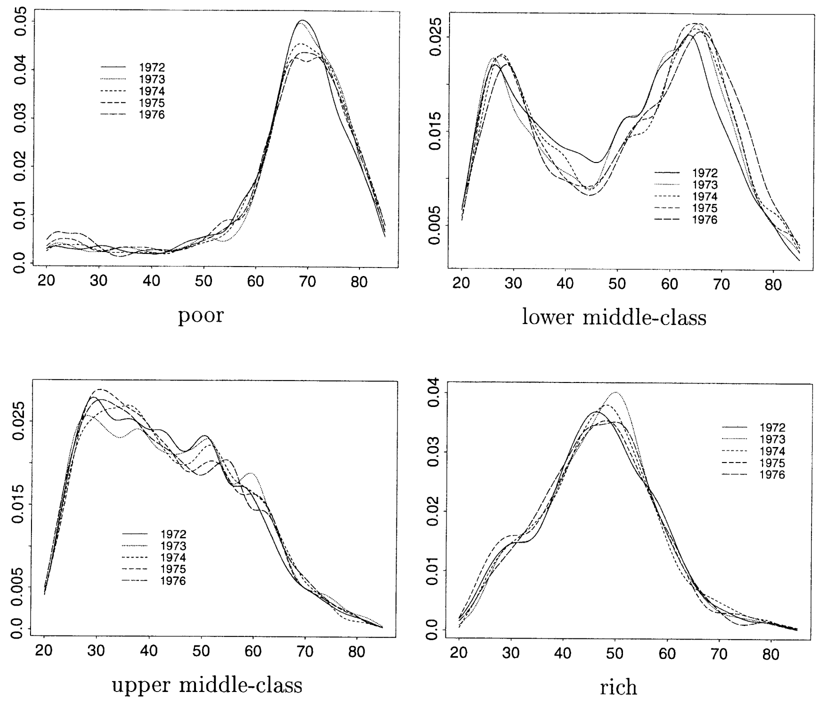


Fig. 3. Estimates of age distributions of head of household conditioned on four quantile interval positions: 'poor' (0–16%), 'lower middle-class' (17–50%), 'upper middle-class' (50–84%), and 'rich' (85–100%). Her Majesty's Stationary Office, total population.

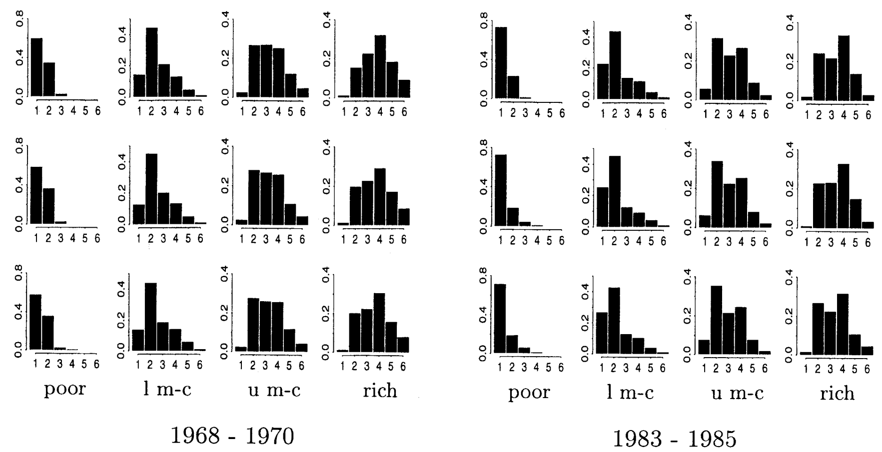


Fig. 4. Estimates of the distribution of number of persons in household conditioned on four quantile interval positions: 'poor', 'lower middle-class', 'upper middle-class', and 'rich'.

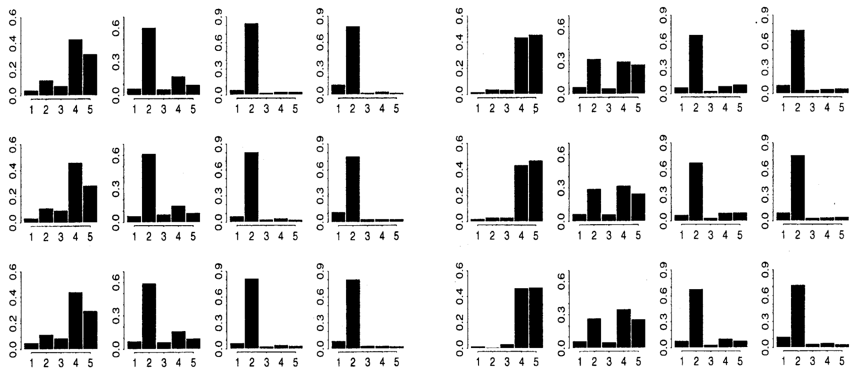


Fig. 5. Estimates of the distribution of employment status of head of household conditioned on four quantile interval positions: 'poor', 'lower middleclass', 'upper middle-class', and 'rich'. Self employed 1, full-time employed 2, retired 3, unoccupied 4 and others 5.

In certain circumstances, that are explained in Section 3, one can estimate the coefficients α_s and β_s from cross-section data in period s. In this case one does not need the micro-relation to determine α_s and β_s; only the actual household consumption expenditure in period s is needed.

In Section 4 we extend the approximation of Section 3 to the case where the difference of the attribute distributions $\nu_s | x_s$ and $\nu_t | x_t$ is not negligible.

2. Notation, definitions and the modelling methodology

The goal of this analysis is to explain or more modestly to model the change over time of aggregate consumption expenditure $C_{t,K}$ on a certain consumption category K (such as food, housing and non-durables) in current prices of a large and heterogeneous population. What are the relevant explanatory variables for modelling the change in $C_{t,K}$?

Every analysis that accounts for aggregation over economic agents must begin with a model of individual behavior. Consequently, the starting point of our analysis is a *micro-demand function* which relates household h's commodity demand vector $z_t^h \in \mathbb{R}^l$ in period t to the price system $p_t \in \mathbb{R}^l$, the interest rate r_t, the disposable income x_t^h and certain theoretical household characteristics $\chi_t^h = (\chi_{1,t}^h, \chi_{2,t}^h, \dots)$. The consumption expenditure on the consumption category $K \subset \{1, \dots, l\}$ is then defined by

$$c_{t,K}^h = \sum_{i \in K} p_{t,i} z_{t,i}^h \left(p_t, r_t, x_t^h, \chi_t^h \right) = : c_K \left(p_t, r_t, x_t^h, \chi_t^h \right) \tag{4}$$

We shall drop in the following the index K indicating the particular consumption category K that is considered.

The nature of these household characteristics χ^h depends on the specification of the micro-model of individual behavior. Typically, some of the household characteristics are unobservable. If individual behavior is modelled as an intertemporal decision problem (forward-looking households) then, the micro-relation depends also on past information, for example, past income $x_{t-1}^h, \dots$ and past prices $p_{t-1}, \dots$ since this information is needed to predict future income and future prices. In the present paper, however, we do not treat this general case, since it greatly complicates the analysis. We start from a micro-relation (1) defined by a function c which is the same for all households; households differ, however, in income $x^h \in \mathbb{R}_+$ and in characteristics $\chi^h \in \chi$. We assume that space of household characteristics χ can always be considered as a metric space.

In addition to the *theoretical household characteristics* we shall consider *observable household attributes*, such as age, employment status, household size and composition etc. Such a profile of household attributes is denoted by $a = (a_1, a_2, \dots, a_n)$ which takes values in a set $\mathscr{A}$, a subset in $\mathbb{R}^n$.

Household characteristics which are *observable* belong also to the set of attributes (e.g., stock of financial assets). Typically, there are household attributes which are not household characteristics, that is to say, the micro-model is not formulated in terms of these attributes.

A household h is described by his income x^h, his profile χ^h of characteristics and his profile a^h of attributes; thus by a point in $\mathbb{R}_+ \times \chi \times \mathscr{A}$.

The population of households in period t is then described by a *joint distribution* ω_t of (x, χ, a) over $\mathbb{R}_+ \times \chi \times \mathscr{A}$.

The following notation for relevant marginal and conditional distributions derived from ω_t is used:

μ_t on $\mathbb{R}_+ \times \chi$	the joint distribution of income x and household characteristics χ
μ_t^χ on χ	the marginal distribution of household characteristics
ν_t on $\mathbb{R}_+ \times \mathscr{A}$	the joint distribution of income x and household attributes a
ν_t^a on $\mathscr{A}$	the marginal distribution of household attributes
ρ_t on $\mathbb{R}_+$	the density of the income distribution
$\mu_t\|(x,a)$ on χ	the conditional distribution of household characteristics given (x,a)
$\mu_t\|x$ on χ	the conditional distribution of household characteristics given income x
$\nu_t\|x$ on $\mathscr{A}$	the conditional distribution of household attributes given income x

Given the micro-relation (1), *mean consumption expenditure* C_t in period t is defined by

$$C_t = \int_{\mathbb{R}_+ \times \chi} c(p_t, r_t, x, \chi) \mathrm{d}\mu_t = C(p_t, r_t, \mu_t) \tag{5}$$

which is equal to

$$\begin{aligned} C_t &= \int_{\mathbb{R}_+\times\mathscr{A}} \left\{ \int_X c(p_t,r_t,x,\chi)\,\mathrm{d}\mu_t|(x,a) \right\} \mathrm{d}\nu_t \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathscr{A}} \left\{ \int_X c(p_t,r_t,x,\chi)\,\mathrm{d}\mu_t|(x,a) \right\} \mathrm{d}\nu_t|x \right] \rho_t(x)\,\mathrm{d}x. \end{aligned} \tag{6}$$

Let $\bar{c}_t\,(p_t,r_t,x,a) := \int_X c(p_t,r_t,x,\chi)\mathrm{d}\mu_t|(x,a)$ denote the regression function of consumption expenditure vs. income x and household attribute a and $\bar{c}_t(p_t,r_t,x) := \int_X c(p_t,r_t,x,\chi)\mathrm{d}\mu_t|x$ denote the regression function of consumption expenditure vs. income x, that is to say, the *Engel curve* in period t. With this notation we obtain

$$C_t = \int_{\mathbb{R}_+\times\mathscr{A}} \bar{c}_t(p_t,r_t,x,a)\,\mathrm{d}\nu_t$$

and

$$C_t = \int_{\mathbb{R}_+} \tilde{c}(p_t,r_t,x)\,\rho_t(x)\,\mathrm{d}x.$$

2.1. Prototype example

There are two distributions τ_1 and τ_2 on χ. There is a function π_t from $\mathbb{R}_+$ into [0,1] and ρ_t denotes the density of an income distribution. We consider one attribute which can take two values; $\mathscr{A} = \{a^1,a^2\}$.

Define

$$\mu_t|(x,a^1) := \tau_1,\quad \mu_t|(x,a^2) := \tau_2$$

$$\mu_t|x := \pi_t(x)\tau_1 + (1-\pi_t(x))\tau_2$$

$$\nu_t|x := \begin{cases} a^1 & \text{with probability} \quad \pi_t(x) \\ a^2 & \text{with probability} \quad (1-\pi_t(x)) \end{cases}$$

$$\nu_t^a := \begin{cases} a^1 & \text{with probability} \quad \int \pi_t(x)\,\rho_t(x)\,\mathrm{d}x \\ a^2 & \text{with probability} \quad \int (1-\pi_t(x))\,\rho_t(x)\,\mathrm{d}x \end{cases}$$

$$\bar{c}_t(p_t,r_t,x,a^i) = \int_X c(p_t,r_t,x,\chi)\,\mathrm{d}\tau_i, \qquad i = 1,2$$

$$\tilde{c}_t(p_t,r_t,x) = \pi_t(x)\bar{c}_t(p_t,r_t,x,a^1) + (1-\pi_t(x))\bar{c}_t(p_t,r_t,x,a^2)$$

Note that in this example, income x and household characteristics χ are independently distributed if one conditions on household attributes. Consequently

$$\partial_x\tilde{c}_t(p_t,r_t,x,a^i) = \int_X [\partial_x c(p_t,r_t,x,\chi)]\,\mathrm{d}\mu_t|(x,a^i).$$

However,

$$\partial_x \tilde{c}_t(p_t, r_t, x) \neq \int_{\mathcal{X}} \partial_x c(p_t, r_t, x, \chi) \mathrm{d}\mu_t | x.$$

The evolution over time of μ_t is determined by the evolution of ρ_t and π_t. Note that without specific assumptions on the evolution of ρ_t and π_t the marginal distribution $\mu_t^{\mathcal{X}}$ of household characteristics is not time-invariant.

In order to model the change over time of mean consumption expenditure

$$C_t = \int_{\mathbb{R}_+ \times \mathscr{A}} \left[\int_{\mathcal{X}} c(p_t, r_t, x, \chi) \mathrm{d}\mu_t | (x, a) \right] \mathrm{d}\nu_t \tag{7}$$

we have to formulate hypotheses on the *change over time* of (1) the conditional distribution $\mu_t|(x,a)$ of household characteristics given income x and attribute profile a, or alternatively, the regression function $\bar{c}_t(p_t, r_t, x, a)$. (2) the joint distribution ν_t of income and attributes.

2.2. Structural stability of household characteristics

Hypotheses on the change over time of the conditional distribution $\mu_t|(x,a)$ are delicate. Any hypothesis on the change of the distribution $\mu_t|(x,a)$ is purely theoretical, that is to say, speculative, since $\mu_t|(x,a)$ describes a distribution of unobservable household characteristics.

It is our goal to 'explain' the observed changes over time of C_t by changes in the observable distribution ν_t (thus, in particular, invariance of ν_t, p_t and r_t must imply invariance of C_t). Therefore we must link possible changes in $\mu_t|(x,a)$ with changes in ν_t. Otherwise ν_t cannot serve as a satisfactory explanatory variable since changes in C_t can then always be attributed to unobservable changes in $\mu_t|(x,a)$. Of course, it might turn out that the observed changes in C_t cannot be 'explained' by the observed changes in ν_t since the set $\mathscr{A}$ of attributes is not sufficiently comprehensive.

The distribution $\mu_t^{\mathcal{X}}$ of household characteristics of the whole population may change over time, yet this change should be caused by a change in the distribution ν_t of income and attributes. The simplest way to achieve this is to *postulate that the distribution $\mu_t^{\mathcal{X}}|a$ of characteristics of all households with attribute profile a is time-invariant* or, at least, changes very slowly over time (local time-invariance). This postulate is based on the idea that households typically keep their characteristics if their attribute profile does not change over time.

In the case where the conditional characteristic distribution $\mu_t|(x,a)$ *does not depend on x* (i.e., within the subpopulation of all households with attribute profile a, income x and household characteristics χ are independently distributed), the above postulate can serve as a definition of 'structural stability' of household characteristics with respect to a set of household attributes. In this case the *regression function* $\bar{c}_t(\cdot,\cdot,\cdot,a)$, defined by

$$\bar{c}_t(p,r,x,a) = \int_X c(p,r,x,\chi)\,\mathrm{d}\mu_t|(x,a)$$

is time-invariant; a property which usually is assumed in applied micro-econometrics (e.g., Jorgenson et al., 1982; Blundell et al., 1993 and Stoker, 1993).

We would like to define 'structural stability' also in the case where $\mu_t|(x,a)$ might depend on x. Then, with household income and prices changing over time, it does not seem to us meaningful to postulate time-invariance of $\mu_t|(x,a)$, since x denotes nominal income. Rather one should condition on 'real' income or on quantiles in the income distribution.

Definition: The income levels x_s and x_t in period s and period t, respectively, *are in the same quantile position* in the income distribution in period s and t, respectively, if

$$\int_0^{x_s} \rho_s(x)\,\mathrm{d}x = \int_0^{x_t} \rho_t(x)\,\mathrm{d}x.$$

The following heuristic argument motivates the definition of structural stability that is basic for our analysis.

Consider two periods s and t and the income densities ρ_s and ρ_t. For periods that are close to each other one expects a *high positive association* between household's income in period s and the later period t. If this association were perfect then households would remain in the same quantile position in the income distributions ρ_s and ρ_t. Furthermore, if *households whose attribute profile does not change* in going from period s to period *t keep their characteristics* then the distributions of household characteristics $\mu_s|(x_s,a)$ and $\mu_t|(x_t,a)$ will approximately be the same if x_s and x_t are in the same quantile position in ρ_s and ρ_t, respectively.

Hypothesis 1: structural stability of household characteristics

Structural stability of household characteristics with respect to the set A of household attributes is defined by the following property:

> if the periods s and t are close to each other and if the income levels x_s and x_t are in the same quantile position in the income distribution of the whole population in period s and t, respectively, then, for every $a \in \mathscr{A}$,
>
> $$\mu_s|(x_s,a) = \mu_t|(x_t,a).$$

Structural stability implies that the *distribution of household characteristics of all households with attribute profile a is locally time-invariant.* This justifies the label 'structural stability' (in the sense of time-invariance). This property, in turn, implies structural stability if $\mu_t|(x,a)$ does not depend on x, that is to say, conditioned on household attribute profile a, income x and household characteristics χ are independent. We remark that the prototype example is structurally stable.

2.3. The change over time of the distribution ν_t

In formulating hypotheses on the change over time of the observable distribution ν_t we have to face the following well-known empirical facts: (1) ν_t describes a multi-variate distribution whose *components are not independently distributed*, that is to say, the joint distribution ν_t of income and attributes is not the product of its marginals (e.g., the distributions of income, age, household size, etc.). Typically there is a high correlation between income and the various household attributes. (2) There is *no* satisfactory a priori given *functional form* (determined up to some few parameters!) for the distribution ν_t and even for its marginals.

For example, the shape and the change over time of income distributions are the outcome of many different forces some of which are operating in different directions. Furthermore, the shape and the change over time of income distributions depend on the notion of income (e.g., 'disposable income'), on the units over which the distribution is defined (e.g., 'household') and on the population (e.g., 'full-time employed household head').

By 'modelling the change over time' of the distribution ν_t we do not mean 'to predict the evolution of ν_t' but to suitably restrict the possible changes, that is, we want a 'parametrization of the *transition*' from ν_s in period s to ν_t in period t.

To explain this 'parametrization of the transition' we consider first the case of income distributions.

Example: time-invariance of the relative income distribution.

The *relative* income distribution in period t is obtained by dividing the income x_t^h of every household in the population by mean income X_t. If ρ_t and ρ_t^* denote the density of the income and relative income distribution, respectively, then one obtains

$$\rho_t*(\xi) = X_t \rho_t(X_t \cdot \xi). \tag{8}$$

Time-invariance of ρ_t^* then implies that for two periods s and t

$$\rho_t(x) = \frac{X_s}{X_t}\rho_s\left(\frac{X_s}{X_t}\cdot x\right). \tag{9}$$

Thus, the *transition* from ρ_s to ρ_t is parametrized by mean income X_t, that is to say, if one knows ρ_s, hence X_s, and X_t then ρ_t is determined.

Time-invariance of the relative income distribution cannot serve as a suitable hypothesis for our analysis. Indeed, it implies that the income dispersion, measured, for example, by the Gini-coefficient or the standard deviation of log income, remains constant over time. It is however a well-established empirical fact (e.g., Gottschalk and Smeeding, 1997) that for most countries the income dispersion changes over time, even though, for some countries, the change is very slow. To take into account a changing income dispersion we consider the following

Hypothesis 2: Time-invariance of the standardized log income distribution.

Instead of income x consider the logarithm of income, log x. Let m_t and σ_t denote the mean and standard deviation, respectively, of the distribution of log x in period t.

The *standardized log income distribution* is then defined as the distribution of

$$\frac{1}{\sigma_t}(\log(x) - m_t).$$

The density of this distribution is denoted by ρ_t^+. The relationship between the densities ρ_t and ρ_t^+ is given by

$$\rho_t^+(z) = y_t \sigma_t \exp(\sigma_t \cdot z)\, \rho_t(y_t \exp(\sigma_t \cdot z)) \tag{10}$$

where $y_t = \exp(m_t)$. Note, y_t is equal to the median of the income distribution ρ_t if log x is symmetrically distributed. Furthermore, if income were log-normally distributed (i.e., log x has a normal distribution) then ρ_t^+ is the normal distribution with mean 0 and variance 1.

Time-invariance of ρ_t^+ then implies for two periods s and t

$$\rho_t(x) = \frac{\sigma_s}{\sigma_t} \cdot \frac{y_s}{y_t^{\sigma_s/\sigma_t}} \cdot x^{(\sigma_s/\sigma_t)-1} \rho_s\left(\frac{y_s}{y_t^{\sigma_s/\sigma_t}} \cdot x^{(\sigma_s/\sigma_t)}\right). \tag{11}$$

It is not hard to show that the transition can also be parametrized by mean income X_t and σ_t;

$$\rho_t(x) = \sigma_s/\sigma_t \cdot \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t}\right)^{\sigma_s/\sigma_t} \cdot x^{(\sigma_s/\sigma_t)-1} \cdot \rho_s\left(\left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x\right)^{\sigma_s/\sigma_t}\right) \tag{12}$$

where $m_s(\sigma) = \int x^{\sigma} \rho_s(x) \mathrm{d}x$.

Thus, the *transition* from ρ_s to ρ_t is parametrized by (X_t, σ_t), that is to say, if one knows ρ_s, hence X_s and σ_s, and X_t, σ_t then ρ_t is determined.

It follows from Eq. (12) that x_s and x_t are in the same percentile position of ρ_s and ρ_t, respectively, if

$$x_s = \varphi(x_t) \tag{13}$$

where the function φ is defined by $\varphi(x) := \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x\right)^{\sigma_s/\sigma_t}$

Remark: Naturally, time-invariance of the standardized log income distribution will never hold exactly, even for periods s and t that are close to each other. Hypothesis 2 should be considered as an *approximation* to the actual complex change in the short-run. It is important to remark that the income distributions can be estimated and therefore one can decide whether the hypothesis satisfactorily captures the main tendency of the actual change. Since we need the income distribution only to compute the mean (integral) it might be sufficient to know ρ_t approximately provided the regression function that we want to integrate, is sufficiently regular.

It might well be that alternative hypotheses will be found that yield a better approximation. The standardized log transformation is particularly simple, yet our approach can be adapted to any other transformation of income distributions that leads to time-invariance. Given ρ_s, Eq. (12) defines a parametrization of ρ_t in terms of X_t, σ_t. This parametrization of income distribution is 'mean-scaled' in the sense of Lewbel (1990) and Lewbel (1992).

Figs. 1 and 2 show kernel density estimates of the standardized log income distribution based on data from the UK Family Expenditure Survey. [4]

Next we have to model the change over time of the distribution of household attributes. The modelling approach is formally analogous to the one in Section 2.2 in the case of distributions of household characteristics. There is, however, an important difference; distributions of attributes are observable. For example, it is a well-established empirical fact that the attribute distribution $\nu_t|x$ of all households with income x *depends quite strongly on the income level* x and the distribution $\nu_t|x$ is *not time-invariant* (e.g., see Figs. 3–5).

The heuristic argument preceding the definition of structural stability in Section 2.2 suggests the following

Hypothesis 3: For two periods s and t that are close to each other the income-conditioned attribute distribution $\nu_s|x_s$ is 'approximately' equal to the income-conditioned attribute distribution $\nu_t|x_t$ if x_s and x_t are in the same quantile position in ρ_s and ρ_t, respectively.

This hypothesis is based on the view that household attributes change relatively slowly as compared with income. We shall consider two versions of Hypothesis 3; we shall assume in Section 3 that the difference $\nu_s|x_s - \nu_t|x_t$ is negligible and in Section 4 that the difference is 'small', in a sense to be explained later.

To illustrate the empirical content of Hypothesis 3 we show in Figs. 3–5 estimates of the age distribution of head of household, the distribution of household size and the distribution of employment status of head of household, respectively, conditioned on four quantile positions: 'poor', 'lower middle-class', 'upper middle-class', and 'rich'.

[4] Material from the Family Expenditure Survey is Crown Copyright; has been made available by the Office for National Statistics through the Data Archive; and has been used by permission. Neither the ONS nor the Data Archive bear any responsibility for the analysis or interpretation of the data reported here.

3. Aggregation under structural stability of household attributes

In this section we explore the implications of Hypothesis 1, structural stability of household characteristics, Hypothesis 2, time-invariance of the standardized log income distributions, and a strong version of Hypothesis 3, which is

Hypothesis 3$^+$: Structural stability of household attributes.

If x_s and x_t are in the same quantile position in the income distributions of period s and t, respectively, then

$$\nu_s | x_s = \nu_t | x_t$$

Hypothesis 3$^+$ is very restrictive; it will be modified later. The hypothesis implies that the distribution of household attributes is time-invariant. Yet the distributions of age, household size etc. that are estimated from time series of cross-section data are not time-invariant, even though they change over time very slowly (see Hildenbrand et al. (1998)).

Proposition 1: *Hypothesis 1, 2 and 3$^+$ imply*

$$C_t = \int_{\mathbb{R}_+ \times \chi} c\left(p_t, r_t, \frac{X_t}{m_s(\sigma_t/\sigma_s)} \cdot x^{\sigma_t/\sigma_s}, \chi \right) \mathrm{d}\mu_s = : K_s(p_t, r_t, X_t, \sigma_t) \tag{14}$$

that is to say, given the micro-relation c and the distribution μ_s in period s, then mean consumption expenditure C_t in period t is a function K_s in p_t, r_t, X_t and σ_t.

Proof: By Hypothesis 2 we obtain

$$\rho_t(x) = \sigma_s/\sigma_t \cdot \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \right)^{\sigma_s/\sigma_t} \cdot x^{(\sigma_s/\sigma_t)-1} \rho_s \left(\left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x \right)^{\sigma_s/\sigma_t} \right) \tag{15}$$

Hypothesis 1, structural stability of household characteristics, and Hypothesis 3$^+$, structural stability of household attributes, imply

$$\mu_t | x = \mu_s | \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t} \cdot x \right)^{\sigma_s/\sigma_t} \tag{16}$$

We now substitute Eqs. (15) and (16) into the definition of C_t and obtain with the notation $\sigma = \sigma_t / \sigma_s$

$$C_t = \int_{\mathbb{R}_+} \left[\int_X c(p_t, r_t, x, \chi) \mathrm{d}\mu_s | \left(\frac{m_s(\sigma)}{X_t} \cdot x \right)^{1/\sigma} \right] \frac{1}{\sigma} \left(\frac{m_s(\sigma)}{X_t} \cdot x \right)^{1/\sigma}$$
$$\times \frac{1}{x} \rho_s \left(\left(\frac{m_s(\sigma)}{X_t} x \right)^{1/\sigma} \right) \mathrm{d}x.$$

The substitution $\xi = \left(\frac{m_s(\sigma)}{X_t} \cdot x \right)^{1/\sigma}$, hence $x = \frac{X_t}{m_s(\sigma)} \xi^{\sigma}$ and $\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{X_t}{m_s(\sigma)} \cdot \sigma \cdot \xi^{\sigma - 1}$, leads to

$$C_t = \int_{\mathbb{R}_+} \left[\int_X c\left(p_t, r_t, \frac{X_t}{m_s(\sigma)} \cdot x^{\sigma}, \chi \right) \mathrm{d}\mu_s | x \right] \rho_s(x) \mathrm{d}x.$$

Q.E.D.

Proposition 1 shows that $C_t = K_s(p_t, r_t, X_t, \sigma_t)$ is determined by the micro-relation c—which we did not specify up to now—and the distribution μ_s—which is only partially observable. In the following we want to approximate the integral in Eq. (14) by a simple expression in the variables p_t, r_t, X_t and σ_t.

The simplest approximation of $K_s(p_t, r_t, X_t, \sigma_t)$ which comes into mind is a first-order Taylor expansion in the variables p, r, X and σ at the values p_s, r_s, X_s and σ_s. However, even for periods s and t which are close to each other, say $t = s + 1$, the change $X_t - X_s$ of mean income (or the price change $p_t - p_s$) will typically not be *very* small, for example, mean income might well increase by 10% or even more. Consequently, to obtain a satisfactory approximation of C_t we have either to take a higher-order Taylor expansion—which will complicate the analysis—or we look for a suitable non-linear first-order approximation which is a satisfactory approximation even for values of the variables p, r, X, σ that are not very near to p_s, r_s, X_s, σ_s. The choice of such an approximation requires, of course, some knowledge of the shape of the function that we want to approximate. For details we refer to the Remark after the proof of Proposition 2.

Proposition 2: *Hypothesis 1, 2 and 3^+ imply for a smooth micro-relation c.*

$$C_t/X_t = C_s/X_s + \alpha_s \log\frac{\sigma_t}{\sigma_s} + \beta_s \log\frac{X_t}{X_s} + \sum_{i=1}^{l} \gamma_s^i \log\frac{p_{t,i}}{p_{s,i}} + \eta_s \log\frac{r_t}{r_s} + O$$
$$\times \left(\max\left\{ \left(\log\frac{\sigma_t}{\sigma_s} \right)^2, \left(\log\frac{X_t}{X_s} \right)^2, \left(\log\frac{p_{t,i}}{p_{s,i}} \right)^2, \left(\log\frac{r_t}{r_s} \right)^2 \right\} \right). \quad (17)$$

The coefficients α_s, β_s, γ_s^i and η_s are defined by

$$\alpha_s = \partial_\sigma \left[\frac{1}{m_s(\sigma)} \int_{\mathbb{R}_+ \times \chi} x^\sigma \, \partial_x c(p_s, r_s, x, \chi) \mathrm{d}\mu_s \right]_{\sigma = 1}$$

$$\beta_s = \frac{1}{X_s} \left(\int_{\mathbb{R}_+ \times \chi} x \partial_x c(p_s, r_s, x, \chi) \mathrm{d}\mu_s - C_s \right)$$

$$\gamma_s^i = \frac{1}{X_s} \int_{\mathbb{R}_+ \times \chi} \partial_\lambda [c(p_{s,1}, \dots, \lambda \cdot p_{s,i}, \dots p_{s,l}, r_s, x, \chi)]_{\lambda = 1} \mathrm{d}\mu_s$$

$$\eta_s = \frac{1}{X_s} \int_{\mathbb{R}_+ \times \chi} \partial_\lambda [c(p_s, \lambda r_s, x, \chi)]_{\lambda = 1} \mathrm{d}\mu_s .$$

We emphasize that the coefficients are defined by the micro-relation and the distribution μ_s in period s; hence they can be interpreted as behavioral parameters that depend on the composition of the population (see Section 3.1 for details).

Proof: With the notation $p_i := p_{t,i}/p_{s,i}$, $(i = 1, \dots, l)$, $r := r_t/r_s$, $X := X_t/X_s$, $\sigma := \sigma_t/\sigma_s$ and

$$w(p_t, r_t, \xi, \chi) := \frac{1}{\xi} c(p_t, r_t, \xi, \chi)$$

we obtain from Proposition 1 that

$$C_t/X_t = \int_{\mathbb{R}_+ \times \chi} \frac{x^\sigma}{m_s(\sigma)} \cdot w\left(p_1 \cdot p_{s,1}, \dots, p_l \cdot p_{s,l}, r \cdot r_s, X \cdot \frac{X_s}{m_s(\sigma)} \cdot x^\sigma, \chi \right) \mathrm{d}\mu_s . \tag{18}$$

Given p_s, r_s, X_s, σ_s and μ_s this integral defines a function f_s in $(p_1, \dots, p_l, r, X, \sigma)$ with $f_s(p_1, \dots, p_l, r, X, \sigma) = C_t/X_t$ and $f_s(1, \dots, 1) = C_s/X_s$.

We now take a first-order Taylor expansion of the function f_s in $\log p_1 =: \tilde{p}_1$, $\dots$, $\log p_l =: \tilde{p}_l$, $\log r =: \tilde{r}$, $\log X =: \tilde{X}$ and $\log \sigma =: \tilde{\sigma}$ at $(\tilde{p}, \tilde{r}, \tilde{X}, \tilde{\sigma}) = (0, \dots, 0)$.

That is to say, we take a usual first order Taylor expansion of the function

$$\left(\tilde{p}_1, \dots, \tilde{p}_l, \tilde{r}, \tilde{X}, \tilde{\sigma} \right) \mapsto f_s\left(\exp \tilde{p}_1, \dots, \exp \tilde{p}_l, \exp \tilde{r}, \exp \tilde{X}, \exp \tilde{\sigma} \right) \tag{19}$$

at $(\tilde{p}_1, \dots, \tilde{\sigma}) = (0, \dots, 0)$. For a justification of proceeding in this way see the remark after the proof.

Thus we obtain

$$f_s(p_1,\ldots,p_l,r,X,\sigma) \approx f_s(1,\ldots,1) + \alpha_s \log \sigma + \beta_s \log X + \sum_{i=1}^{l} \gamma_s^i \log p_i + \eta_s \log r$$

which is the claimed approximation. The coefficients $\alpha_s,\ldots,\eta_s$ are defined as partial derivatives of the function (16) at $(0,\ldots,0)$.

By definition

$$\alpha_s := \partial_{\tilde{\sigma}}[f_s(1,\ldots,1,\exp\tilde{\sigma})]_{\tilde{\sigma}=0} = \partial_\sigma[f_s(1,\ldots,1,\sigma)]_{\sigma=1}.$$

The definition of the function f_s by Eq. (18) then implies

$$\sigma_s = \partial_\sigma\left[\int_{\mathbb{R}_+\times\chi} \frac{x^\sigma}{m_s(\sigma)} w\left(p_s,r_s,\frac{X_s}{m_s(\sigma)}\cdot x^\sigma,\chi\right)\mathrm{d}\mu_s\right]_{\sigma=1}.$$

Define $g(\sigma) = (x_\sigma)/(m_s(\sigma))$. Then we obtain

$$\alpha_s = \int_{\mathbb{R}_+\times\chi} \partial_\sigma[g(\sigma)w(p_s,r_s,g(\sigma)X_s,\chi)]_{\sigma=1}\mathrm{d}\mu_s.$$

We now compute the derivative under the integral.

$$\begin{aligned}\partial_\sigma[g(\sigma)w(p_s,r_s,g(\sigma)X_s,\chi)]_{\sigma=1} &= g'(\sigma)|_{\sigma=1}\cdot(w(p_s,r_s,x,\chi) + x\partial_x w(p_s,r_s,x,\chi))\\ &= g'(\sigma)|_{\sigma=1}\cdot\partial_x c(p_s,r_s,x,\chi)).\end{aligned}$$

Consequently, $\alpha_s = \int \partial_\sigma[\frac{x^\sigma}{m_s(\sigma)}\cdot\partial_x c(p_s,r_s,x,\chi)]_{\sigma=1}\mathrm{d}\mu_s$ which is equal to the expression claimed in Proposition 2.

By definition

$$\beta_s := \partial_{\tilde{X}}\left[f_s(1,\ldots,1,\exp\tilde{X},1)\right]_{\tilde{X}=0} = \partial_x[f_s(1,\ldots,1,X,1)]_{x=1}.$$

The definition of the function f_s by Eq. (18) implies

$$\begin{aligned}\beta_s &= \frac{1}{X_s}\partial_x\left[\int xw(p_s,r_s,X\cdot x,\chi)\mathrm{d}\mu_s\right]_{x=1}\\ &= \frac{1}{X_s}\int x^2\partial_x w(p_s,r_s,x,\chi)\mathrm{d}\mu_s.\end{aligned}$$

Since $x^2\partial_x w(p_s,r_s,x,\chi) = x\cdot\partial_x c(p_s,r_s,x,\chi) - c(p_s,r_s,x,\chi)$ we obtain for the coefficient β_s the expression as claimed in Proposition 2.

By definition

$$\gamma_s^i = \partial_{\tilde{p}_i}[f_s(1,\ldots,\exp\tilde{p}_i,\ldots,1)]_{\tilde{p}_i=0}$$

$$\gamma_s^i = \partial_{p_i}[f_s(1,\ldots,p_i,\ldots,1)]_{p_i=1}$$

The definition of the function f_s then implies

$$\gamma_s^i = \frac{1}{X_s}\partial_\lambda\left[\int c(p_{s,i},\ldots,\lambda\cdot p_{s,i},\ldots,r_s,x,\chi)\mathrm{d}\mu_s\right]_{\lambda=1}.$$

The coefficient η_s is obtained analogously. Q.E.D.

Remark: The macro-relation (14) in Proposition 2 has been obtained as a first-order approximation. The term $(\log(\sigma_t)/(\sigma_s))^2$ can safely be assumed as negligible, since income dispersion typically changes very slowly. Yet the term $(\log(X_t)/(X_s))^2$, even for $t=s+1$, might well be non negligible. For example, mean income growth of $\log(X_{s+1})/(X_s)\approx 0.1$ is possible.

Whether the approximation (14) in Proposition 2 is also a good approximation on a larger domain where second-order terms, such as $(\log(X_t)/(X_s))^2$, are not negligible depends on the magnitude of the second derivative of the function defined by Eq. (19) in the proof of Proposition 2. For example, can we assume that the second derivative $\partial^2_{\tilde{X}} f_s(1,\ldots,1, \exp\tilde{X}, 1)$ is small for $\tilde{X}$ in a reasonable large domain around zero?

It follows from the definition of the function f_s by Eq. (19) that

$$\partial_{\tilde{X}} f_s(1,\ldots,1,\exp\tilde{X},1) = \frac{1}{X_s}\int_{\mathbb{R}\times A} x\left[\int \partial_{\tilde{X}} w(p_s,r_s,x\cdot\exp\tilde{X},\chi)\mathrm{d}\mu_s|(x,a)\right]\mathrm{d}\nu_s.$$

If we assume now that $\mu_s|(x,a)$ does not depend on the income level x, then we obtain

$$\partial_{\tilde{X}} f_s(1,\ldots,1,\exp\tilde{X},1) = \frac{1}{X_s}\int_{\mathbb{R}_+\times A} x\partial_{\tilde{X}}\overline{w}(p_s,r_s,x\cdot\exp\tilde{X},a)\mathrm{d}\nu_s \tag{20}$$

where $\overline{w}(p_s,r_s,\xi,a) := \int_\chi w(p_s,r_s,\xi,\chi)\mathrm{d}\mu_s|(\xi,a)$ denotes the mean income share of all households with income ξ and attribute profile a.

The mean income share curve

$$\xi\mapsto\overline{w}(p_s,r_s,\xi,a)$$

can be estimated from cross-section data. The shape of the estimates depends on the consumption category for which we consider consumption expenditure (e.g., food, clothing, etc) and on the attribute profile a. However, it turns out that all estimates have a common property: $\xi\cdot\partial_\xi\overline{w}(p_s,r_s,\xi,a)$ changes only very slowly with the income level ξ, that is to say, on relatively large income-intervals $\xi\cdot\partial_\xi\overline{w}(p_s,r_s,\xi,a)$ is approximately constant (not over the whole income domain).[5]

[5] The assumption that $\xi\cdot\partial_\xi\overline{w}(p,\xi,a)$ is constant in ξ amounts to assuming that $\overline{w}(p,\xi,a)=\alpha(p,a)+\beta(p,a)\log\xi$. This specification of the functional form of $\overline{w}$, first used by Working (1943), is often made in the literature.

This property implies by Eq. (20) that the derivative $\partial_{\tilde{X}} f_s(1,\ldots, 1, \exp \tilde{X},1)$ does not depend sensitively on $\tilde{X}$ and, hence, that the second derivative is quite small for $\tilde{X}$ in a relatively large domain around zero. Thus, the particular first-order approximation in Proposition 2 is based on the above claimed property of the mean income share curve.

By an analogous argument one can show—using results of Kneip (1998)—that 'sufficient heterogeneity' of the population implies that the approximation (14) of Proposition 2 is a good approximation even if the terms $(\log p_{t,i}/p_{s,i})^2$ are not negligible.

3.1. Discussion of the coefficients

The coefficient α_s: by definition of α_s in Proposition 2 we obtain

$$\alpha_s = \partial_\sigma \left[\frac{1}{m_s(\sigma)} \int_{\mathbb{R}_+} x^\sigma \cdot \mathrm{MMP}_s(x) \rho_s(x) \mathrm{d}x \right]_{\sigma=1}$$

where $\mathrm{MMP}_s(x) := \int_X \partial_x c(p_s, r_s, x, \chi) \mathrm{d}\mu_s | x$, i.e., the mean marginal propensity to consume of all households with income x.

The sign of α_s depends on the form of the function $\mathrm{MMP}_s(x)$. Indeed, it follows from the Lemma in the Appendix A that for every density ρ_s the coefficient

$$\alpha_s \le 0 (\ge 0) \text{ if } \mathrm{MMP}_s(x) \text{ is a decreasing (increasing) function in } x.$$

More generally, $\alpha_s \le 0$ if for all z in a neighborhood of 1,

$$\min_{0 \le x \le z} \mathrm{MMP}_s(x) \ge \max_{z \le x} \mathrm{MMP}_s(x).$$

If the individual household propensity to consume $\partial_x c(p_s, r_s, x, \chi)$ is decreasing in x, i.e., $c(p_s, r_s, x, \chi)$ is a concave function in x—which is frequently postulated for the micro-model—then it does not necessarily follow that the $\mathrm{MMP}_s(x)$ is also decreasing in x since the conditional distribution $\mu_s | x$ depends on x.

For example, let $\mu_s | x$ be equal to χ^1 with probability $\pi(x)$ and χ^2 with probability $1 - \pi(x)$. Let $c(p_s, r_s, x, \chi^i)$ be linear in x and $\partial_x c(p_s,r_s,x,\chi^1) \neq \partial_x c(p_s,r_s,x,\chi^2)$. Then the $\mathrm{MMP}(x)$ can be decreasing, increasing or not be monotone at all depending on the function $\pi(x)$.

Fig. 6 shows three examples of the function $\pi(x)$.

Fig. 7 shows for each function π the mean marginal propensity to consume $\mathrm{MMP}(x)$.

Thus we have shown that the sign of α_s does not only depend on suitable properties of the micro-model but also on the composition of the population. In particular, linearity in income on the household level does not necessarily imply that the distribution effect of income is positive or negative nor that it can be

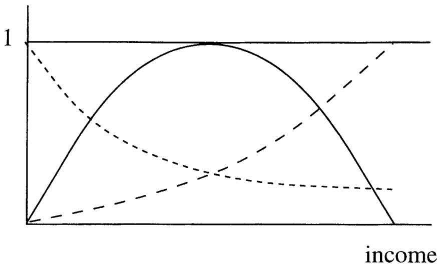


Fig. 6.

neglected. The magnitude of the coefficient α_s depends not only on the form of $\mathrm{MMP}_s(x)$,but also on the income distribution ρ_s; in particular, the magnitude of α_s depends on the dispersion σ_s. This is best illustrated by an example: let $\mathrm{MMP}_s(x) = d_s + b_s \log x$ and ρ_s a log normal density with parameters (m_s, σ_s). Then one can explicitly compute the coefficient α_s and obtains $\alpha_s = 2\sigma_s^2 \cdot b_s$.

How can one estimate the coefficient α_s? If one conditions on household attributes one might expect (or assume) that the conditional distribution $\mu_s|(x, a)$ does not depend very sensitively on x, more precisely,

$$\int_X \partial_x c(p_s, r_s, x, \chi) \mathrm{d}\mu_s|(x,a) \approx \partial_x \int_X c(p_s, r_s, x, \chi) \mathrm{d}\mu_s|(x,a)$$

$$\int_X \partial_x c(p_s, r_s, x, \chi) \mathrm{d}\mu_s|(x,a) \approx \partial_x \tilde{c}_s(p_s, r_s, x, a) \qquad (\bigstar)$$

If this condition is satisfied one can define the following proxi for α_s:

$$\tilde{\alpha}_s := \partial_\sigma \left[\frac{1}{m_s(\sigma)} \int_{\mathbb{R}_+} x^\sigma \left[\int_A \partial_x \tilde{c}_s(p_s, r_s, x, a) \mathrm{d}\nu_s | x \right] \rho_s(x) \mathrm{d}x \right]_{\sigma=1}.$$

Indeed, if condition $(\bigstar)$ holds with equality, then $\alpha_s = \overline{\alpha}_s$. The important point now is that $\overline{\alpha}_s$ *can be estimated from cross-section data in period s*. For

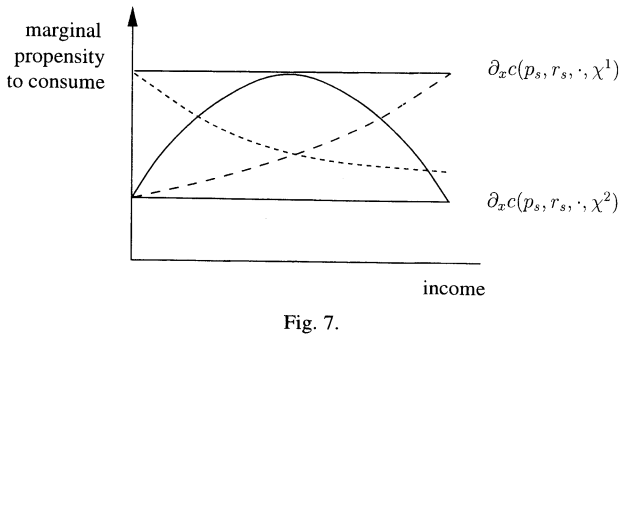


Fig. 7.

estimating $\bar{\alpha}_s$ one does not need a micro-model c, only the actual consumption decisions in period s are needed.

We have estimated the coefficient $\bar{\alpha}_t$ (stratification with respect to household size and age in 5×8 disjoint groups) using the data from the U.K. Family Expenditure Survey. The mean value of $\bar{\alpha}_t$ for the years 1968–1986 is -0.036 for consumption expenditure on non-durables; -0.009 for food and $+0.006$ for services. The method of estimation and further details can be found in Hildenbrand and Kneip (1998).

The coefficient β_s: One obtains from the definition of β_s that

$$\beta_s = \frac{1}{X_s} \int_{\mathbb{R}_+} \left[\int_{\mathcal{X}} (x \partial_x c(p_s, r_s, x, \chi) - c(p_s, r_s, x, \chi)) \mathrm{d}\mu_s | x \right] \rho_s(x) \mathrm{d}x.$$

Hence, the coefficient β_s is negative (positive) if in average

$$\int_{\mathcal{X}} (x \partial_x c(p_s, r_s, x, \chi) - c(p_s, r_s, x, \chi)) \mathrm{d}\mu_s | x$$

is negative (positive).

Note that the smaller the individual household marginal propensity to consume, $\partial_x c(p_s, r_s, x, \chi)$, the larger is $-\beta_s$. In the extreme case, where $\partial_x c(p_s, r_s, x, \chi) \approx 0$ one obtains $\beta_s \approx -(C_s / X_s)$.

As in the case of the coefficient α_s one can define a proxi $\bar{\beta}_s$ for the coefficient β_s which can be estimated from cross-section data in period s;

$$\bar{\beta}_s := \frac{1}{X_s} \int_{\mathbb{R}_+} x \left[\textstyle\int_A \partial_x [\bar{w}_s(p_s, r_s, \lambda x, a]_{\lambda = 1} \mathrm{d}\nu_s | x \right] \rho_s(x) \mathrm{d}x$$

$$\bar{\beta}_s = \frac{1}{X_s} \int_{\mathbb{R}_+} x \left[\textstyle\int_A \partial_x \bar{c}_s(p_s, r_s, x, a) \mathrm{d}\nu_s | x \right] \rho_s(x) \mathrm{d}x - \frac{C_s}{X_s}.$$

As before, $\bar{\beta}_s$ is a proxi for β_s provided condition (★) is satisfied. For estimating $\bar{\beta}_s$ one does not need a micro-relation.

Estimates of the coefficient $\bar{\beta}_s$ (stratification with respect to household size and age in 5×8 disjoint groups) for the years 1968–1986 yield a mean value of -0.242 for consumption expenditure on non-durables; -0.152 for food and $+0.022$ for services. For details see Hildenbrand and Kneip (1998).The coefficients γ_s^i and η_s: the interpretation of the coefficient γ_s^i and η_s can be short. By definition they depend on how, on average, households react to changes in prices or interest rate.

4. Aggregation under slowly changing attribute distributions

The last section was based on the hypothesis of structural stability not only of household characteristics (that is, Hypothesis 1) but also of household attributes (that is, Hypothesis 3^+). As mentioned already in Section 3 this last hypothesis is in contradiction with empirical facts: indeed, the distributions of household attributes typically change over time; this change, however, is quite slow (see Hildenbrand et al. (1998)). Therefore we base this section on Hypothesis 3 in a less restrictive version than in Section 3.

The distribution μ_t in period t has been decomposed in the conditional distribution of household characteristics

$$\mu_t|(x,a),$$

the conditional distribution of household attributes

$$\nu_t|x$$

and the income density

$$\rho_t(x).$$

Hypothesis 2, i.e., the time-invariance of the standardized log income distribution, implies (see Eqs. (11) and (12) of Section 2)

$$\rho_t(x) = \varphi'(x)\rho_s(\varphi(x)),$$

where $\varphi(x) := \left(\frac{m_s(\sigma_t/\sigma_s)}{X_t}\cdot x\right)^{\sigma_s/\sigma_t}$.

Hypothesis 1, i.e., structural stability of household characteristics, expresses that

$$\mu_t|(x,a) = \mu_s|(\varphi(x),a).$$

Instead of assuming $\nu_t|x = \nu_s|\varphi(x)$ as in Section 3, we now assume Hypothesis 3, that is, for periods s and t that are close to each other the difference

$$\nu_t|x - \nu_s|\varphi(x)$$

is 'small', but not necessarily negligible. We shall explain in the sequel (see the approximation (18)) in which sense the difference $\nu_t|x - \nu_s|\varphi(x)$ should be small.

By definition

$$C_t = \int_{\mathbb{R}_+}\left[\int_{\mathscr{A}}\left\{\int_{\chi} c(p_t,r_t,x,\chi)\mathrm{d}\mu_t|(x,a)\right\}\mathrm{d}\nu_t|x\right]\rho_t(x)\mathrm{d}x.$$

Since by assumption $\mu_t|(x,a) = \mu_s|(\varphi(x),\ a)$ and $\rho_t(x) = \varphi'(x)\rho_s(\varphi(x))$ we obtain

$$C_t = \int_{\mathbb{R}_+}\left[\int_{\mathscr{A}}\left\{\int_{\chi} c(p_t,r_t,x,\chi)\mathrm{d}\mu_s|(\varphi(x),a)\right\}\mathrm{d}\nu_t|x\right]$$
$$\times\varphi'(x)\rho_s(\varphi(x))\mathrm{d}x.$$

Substituting $\nu_t|x = \nu_s|\varphi(x) + (\nu_t|x - \nu_s|\varphi(x))$ yields by Proposition 1

$$C_t = \int_{\mathbb{R}_+\times\chi} c(p_t,r_t,\varphi^{-1}(x),\chi)\mathrm{d}\mu_s + A \qquad (21)$$

where

$$A := \int_{\mathbb{R}_+}\left[\int_{\mathscr{A}}\left\{\int_X c(p_t, r_t, x, \chi)\,d\mu_s|(\varphi(x), a)\right\}d(\nu_t|x - \nu_s|\varphi(x))\right]$$
$$\times \varphi'(x)\rho_s(\varphi(x))\,dx.$$

For the first term on the right hand side of Eq. (21) we have given an approximation in Proposition 2 of Section 3. We shall now develop an approximation for the second term, that is to say, for the integral which is denoted by A.

Substituting $\xi = \varphi(x)$ with $\varphi^{-1}(\xi) = \frac{X_t}{m_s(\sigma_t/\sigma_s)}\xi^{\sigma_t/\sigma_s}$ yields

$$A = \int_{\mathbb{R}_+}\left[\int_{\mathscr{A}}\left\{\int_X c\left(p_t, \frac{X_t}{m_s(\sigma_t/\sigma_s)}\cdot x^{\sigma_t/\sigma_s}, \chi\right)d\mu_s|(x,a)\right\}d\left(\nu_t|\frac{X_t}{m_s(\sigma_t/\sigma_s)}\right.\right.$$
$$\left.\left.\cdot x^{\sigma_t/\sigma_s} - \nu_s|x\right)\right]\rho_s(x)\,dx.$$

To simplify the notation we assume that the set $\mathscr{A}$ of attribute profiles is *finite*. Then we denote

$$\left(\nu_t\left|\frac{X_t}{m_s(\sigma_t/\sigma_s)}\cdot x^{\sigma_t/\sigma_s} - \nu_s\right|x\right)\{a\} = :\Delta(x,a).$$

Hypothesis 3 then says that $|\Delta(x,a)|$ is small for all income levels and $a \in \mathscr{A}$

$$A = \int_{\mathbb{R}_+}\left[\sum_{a\in A}\left\{\int_X c\left(p_t, \frac{X_t}{m_s(\sigma_t/\sigma_s)}\cdot x^{\sigma_t/\sigma_s}, \chi\right)d\mu_s|(x,a)\right\}\Delta(x,a)\right]$$
$$\times \rho_s(x)\,dx.$$

We now define $\bar{A}$ by

$$\bar{A} = \int_{\mathbb{R}_+}\left[\sum_{a\in A}\left\{\frac{X_t}{X_s}\bar{c}_s(p_s, x, a)\right\}\Delta(x,a)\right]\rho_s(x)\,dx.$$

The only difference between A and $\bar{A}$ is the expression in the bracket $\{\dots\}$. In the following we want to argue that $\bar{A}$ can be considered as an approximation of A. For this we have to show under what circumstances the two expressions in the bracket $\{\dots\}$ are 'approximately' equal.

We always assume that the two periods s and t are close to each other, say $t = s + 1$ or $s + 2$.

First we assume that income dispersion changes very slowly. Then we assume $\sigma_s = \sigma_t$ since s and t are close to each other. This implies that the brace $\{\dots\}$ of A is equal to

$$\int_X c\left(p_t, \frac{X_t}{X_s}x, \chi\right) \mathrm{d}\mu_s|(x,a). \tag{a}$$

Second we assume that the demand function on the household level is homogenous of degree zero in (p, x). Then we obtain for the micro-relation of expenditure

$$c(p,x,\chi) = \frac{1}{\lambda} c(\lambda p, \lambda x, \chi), \qquad \lambda > 0.$$

Hence (a) becomes

$$\int_X \frac{X_t}{X_s} c\left(\frac{X_s}{X_t} p_t, x, \chi\right) \mathrm{d}\mu_s|(x,a) = \frac{X_t}{X_s} \bar{c}_s\left(\frac{X_s}{X_t} p_t, x, a\right). \tag{b}$$

Third, we assume that price changes are proportional and that real income growth is very small. Thus we obtain $(X_s/X_t) p_t = p_s$.

Under the above three assumption one obtains $A = \bar{A}$. Certainly, none of these assumptions will hold exactly, particularly the third assumption. However, they might hold approximately which implies that

$$\begin{aligned} \mathrm{d}(x,a) :&= \left\{ \int_X c\left(p_t, \frac{X_t}{m_s(\sigma_t/\sigma_s)} \cdot x^{\sigma_t/\sigma_s}, \chi\right) \mathrm{d}\mu_s|(x,a) \right\} \\ &\quad - \left\{ \frac{X_t}{X_s} \bar{c}(p_s, x, a) \right\} \end{aligned}$$

is small, though not negligible.

Since $A - \bar{A} = \int_{\mathbb{R}_+} [\sum_{a \in \mathscr{A}} \mathrm{d}(x,a) \cdot \Delta(x,a)] \rho_s(x) \mathrm{d}x$ and since the product of the two small terms $d(s, a)$ and $\Delta(x, a)$ is very small (negligible) we conclude that $\bar{A}$ is a satisfactory approximation of A. Thus, in this section the term 'approximate' in Hypothesis 3 has to be interpreted as implying $A \approx \bar{A}$.

To evaluate $\bar{A}$ we assume that the set of household attributes is finite $\mathscr{A} = \{a^1, \dots, a^m\}$ and use the following notation:

$$\nu_s|x\{a^i\} = :\nu_s^i(x) \text{ and } \nu_s\{\mathbb{R}_+ \times a^i\} = :\nu_s^i.$$

Furthermore we make the following

Hypothesis 4: For every x_s and x_t that are in the same quantile position in period s and t, respectively,

$$\frac{\nu_t^i(x_t)}{\nu_s^i(x_s)} \approx \frac{\nu_t^i}{\nu_s^i}$$

Let us recall, Hypothesis 3 is used to justify the approximation $A \approx \bar{A}$ and Hypothesis 4 is a technical assumption which allows to evaluate $\bar{A}$.

Indeed,

$$\bar{A} = \frac{X_t}{X_s} \int_{\mathbb{R}_+} \left[\sum_{i=1}^{m} \bar{c}_s(p_s, r_s, x, a^i) \left(\nu_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right) - \nu_s^i(x) \right) \right] \rho_s(x) \mathrm{d}x.$$

Since $\Sigma_{i=1}^{m} \nu_t^i(\xi) = 1$ and $\Sigma_{i=1}^{m} \nu_s^i(x) = 1$ we obtain

$$\sum_{i=l}^{m} \tilde{c}_s(p_s, r_s, x,) \left(v_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right) - \nu_s^i(x) \right) = 0.$$

Hence

$$\bar{A} = \frac{X_t}{X_s} \int_{\mathbb{R}_+} \left[\sum_{i=1}^{m} [\bar{c}_s(p_s, r_s, x, a^i) - \tilde{c}_s(p_s, r_s, x)] \left[\nu_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right) - \nu_s^i(x) \right] \right] \rho_s(x) \mathrm{d}x$$

Now

$$\left[\nu_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right) - \nu_s^i(x) \right] = \left[\frac{\nu_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right)}{\nu_t^i} \cdot \nu_t^i - \frac{\nu_s^i(x)}{\nu_s^i} \cdot \nu_s^i \right]$$

$$\left[\nu_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right) - \nu_s^i(x) \right] = \frac{\nu_s^i(x)}{\nu_s^i} (\nu_t^i - \nu_s^i)$$

$$\left[\nu_t^i \left(\frac{X_t}{m_s(\sigma)} \cdot x^{\sigma} \right) - \nu_s^i(x) \right] = \nu_s^i(x) \left(\frac{\nu_t^i}{\nu_s^i} - 1 \right).$$

Thus,

$$\bar{A} = X_t \cdot \sum_{i=1}^{m} \lambda_s^i \left(\frac{\nu_t^i}{\nu_s^i} - 1 \right)$$

where

$$\lambda_s^i = \frac{1}{X_s} \int_{\mathbb{R}_+} [\bar{c}_s(p_s, r_s, x, a^i) - \tilde{c}_s(p_s, r_s, x)] \nu_s^i(x) \rho_s(x) \mathrm{d}x.$$

Note that the coefficient λ_s^i can be estimated from cross-section data.

The coefficient λ_s^i measures to what extent the Engel curve of the subpopulation consisting of all households with attribute a^i, i.e., $\bar{c}_s(p_s, r_s, \cdot, a^i)$ differs (on average) from the Engel curve of the whole population, i.e., $\tilde{c}_s(p_s, r_s, \cdot)$. Consequently,

$$\lambda_s^i\left(\frac{\nu_t^i}{\nu_s^i} - 1\right)$$

describes the effect (on C_t/X_t) of the changing composition of the population with respect to the attribute a^i.

In summary, in this section we extended Proposition 2 and derived the following approximation:

$C_t/X_t - C_s/X_s$	change in the aggregate consumption ratio
$\approx \alpha_s \log(\sigma_t/\sigma_s)$	effect of the changing income dispersion
$+\beta_s \log(X_t/X_s)$	effect of mean (nominal) income growth
$+\sum_{i=1}^{l}\gamma_s^i \log(p_{t,i}/p_{s,i})$	effects of price-inflation
$+\eta_s \log(r_t/r_s)$	effect of interest rate changes
$+\sum_{i=1}^{m}\lambda_s^i((\nu_t^i/\nu_s^i) - 1)$	effect of the changing distribution of attributes

Acknowledgements

We would like to thank K. Utikal for his collaboration in the statistical analysis of the U.K. Family Expenditure Survey. The figures in this paper have been prepared by him. We would also like to thank the participants of the Bonn-Workshop on Aggregation (June 1997) for their critical comments and helpful suggestions. We are particularly grateful to R. Blundell, J.M. Grandmont, M. Jerison, A. Lewbel and T. Stoker. Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn and Max–Planck–Forschungspreis is gratefully acknowledged.

Appendix A

Lemma: *For every continuous and decreasing function v of $\mathbb{R}_+$ into $\mathbb{R}$ and every density ρ on $\mathbb{R}_+$ such that*

$$A(\sigma) := \frac{\int x^{\sigma} v(x)\rho(x)\mathrm{d}x}{\int x^{\sigma}\rho(x)\mathrm{d}x}$$

is defined on an open interval around $\sigma = 1$ if follows that

$$\partial_\sigma[A(\sigma)]_{\sigma=1} \le 0$$

Proof: In order to prove the assertion of the lemma it suffices to show that for all σ with $0 < \sigma \leq 1$,

$$\frac{\int x^{\sigma} v(x)\rho(x)\mathrm{d}x}{\int x^{\sigma}\rho(x)\mathrm{d}x} \geq \frac{\int x v(x)\rho(x)\mathrm{d}x}{\int x\rho(x)\mathrm{d}x} \tag{1a}$$

Let $m(\sigma) := \int x^{\sigma}\rho(x)\mathrm{d}x$ and $z(\sigma) := \left(\frac{m(1)}{m(\sigma)}\right)^{\frac{1}{1-\sigma}}$. Then, since $\sigma < 1$,

$$\frac{x^{\sigma}}{m(\sigma)} \geq \frac{x}{m(1)} \qquad \text{if } 0 \leq x \leq z(\sigma) \tag{2a}$$

and

$$\frac{x^{\sigma}}{m(\sigma)} \leq \frac{x}{m(1)} \qquad \text{if } z(\sigma)x. \tag{3a}$$

Since $\int \frac{x^{\sigma}}{m(\sigma)}\rho(x)\mathrm{d}x = \int \frac{x}{m(1)}\rho(x)\mathrm{d}x = 1$ one obtains

$$\int_0^{z(\sigma)} \left(\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right)\rho(x)\mathrm{d}x = -\int_{z(\sigma)}^{\infty} \left(\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right)\rho(x)\mathrm{d}x,$$

and by relations (2) and (3),

$$\int_0^{z(\sigma)} \left|\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right|\rho(x)\mathrm{d}x = \int_{z(\sigma)}^{\infty} \left|\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right|\rho(x)\mathrm{d}x, \tag{4a}$$

Relation (1) holds, if and only if for all σ with $0 < \sigma \leq 1$ one has

$$R(\sigma) := \int \left(\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right) v(x)\rho(x)\mathrm{d}x \geq 0. \tag{5a}$$

However, Eq. (8) is an immediate consequence of Eq. (7) and the assumption on v. Indeed,

$$\begin{aligned} R(\sigma) &= \int_0^{z(\sigma)} \left(\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right) v(x)\rho(x)\mathrm{d}x \\ &\quad + \int_{z(\sigma)}^{\infty} \left(\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right) v(x)\rho(x)\mathrm{d}x \\ &\geq \left[\min_{0 \leq x \leq z(\sigma)} v(x)\right] \cdot \int_0^{z(\sigma)} \left|\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right|\rho(x)\mathrm{d}x \\ &\quad - \left[\max_{z(\sigma) \leq x < \infty} v(x)\right] \\ &\quad \cdot \int_{z(\sigma)}^{\infty} \left|\frac{x^{\sigma}}{m(\sigma)} - \frac{x}{m(1)}\right|\rho(x)\mathrm{d}x = 0 \end{aligned}$$

Q.E.D.

References

Blundell, R., Pashardes, P., Weber, G., 1993. What do we learn about consumer demand patterns from micro data?. The American Economic Review 83, 570–597.

Gottschalk, P., Smeeding, T., 1997. Cross-national comparisons of earnings and income inequality. Journal of Economic Literature 35, 633–687.

Her Majesty's Stationary Office. Family Expenditure Survey. Her Majesty's Stationary Office. Annual Report.

Hildenbrand, W., Kneip, A., 1998. To appear.

Hildenbrand, W., Kneip, A., Utikal, K.J., 1998. A non-parametric analysis of the distribution of household income and attributes. To appear in Revue de Statistique Appliquée.

Jorgenson, D.W., Lau, L., Stoker, T., 1982. The transcendental logarithmic model of aggregate consumer behavior. In: Basman, R., Rhodes, G. (Eds.), Advances in Econometrics. JAI Press, Greenwich, CT.

Kneip, A., 1998. Behavioral heterogeneity and structural properties of aggregate demand. In this issue.

Lewbel, A., 1990. Income distribution and aggregate money illusion. Journal of Econometrics 43, 35–42.

Lewbel, A., 1992. Aggregation with log–linear models. Review of Economic Studies 59, 635–642.

Malinvaud, E., 1981. Theorie Macro-Economique. Dunod, Paris.

Malinvaud, E., 1993. A framework for aggregation theories. Ricerche Economiche 47, 107–135.

Stoker, T.M., 1993. Empirical approaches to the problem of aggregation over individuals. Journal of Economic Literature 31 (4), 1827–1874.

STOCHASTIC ANALYSIS AND APPLICATIONS, 12(1), 41-63 (1994)

ON THE EXISTENCE AND CHARACTERIZATION OF ARBITRAGE–FREE MEASURES IN CONTINGENT CLAIM VALUATION

Norbert Christopeit[1] and **Marek Musiela**[2]

[1] Institut für Ökonometrie und Operations Research, Universität Bonn, Adenauerallee 24–42, D–5300 Bonn, Germany, e–mail: OR337@dbnuor1.bitnet

[2] The University of New South Wales, P.O.Box 1, Kensington, New South Wales, Australia, 2033, Telex:AA26054

ABSTRACT

In this paper, necessary and sufficient conditions for existence of equivalent martingale measures in semimartingale models for the pricing of contingent claimes are derived.

1. INTRODUCTION

Since the pioneering work of Black and Scholes [1] and Merton [12], financial mathematics underwent a major change. It was realized that the pricing of options or, more generally, of derivative securities, should be based on non–arbitrage considerations rather than on preference–related concepts such as expected values. Briefly, the idea is to replicate a derivative security by a dynamical portfolio strategy whose initial value will then give the "fair" price of the contingent claim. In particular due to the work of Harrison and Kreps [7] and Harrison and Pliska [8] it became apparent that semimartingale theory provided a natural framework for the analysis of financial markets. Basically, the ingredients for a theory of, say, option pricing are a price process $X = (X_t)$ for some risky asset (a stock, an exchange rate, etc.) as well as the price of some riskless asset (bond, savings account). The latter may be taken equal to 1 (by discounting the risky asset, if necessary). The problem is then to find a dynamical portfolio strategy $(\varphi, \psi) = (\varphi_t, \psi_t)$ s.t. its value process $V_t = \varphi_t X_t + \psi_t$ coincides at the expiration date T with the contingent claim to be priced. The initial amount V_0 will then give the fair or non–arbitrage price π of the contingent claim. For a nice exposition of these principles cf. [6].

This is the point where *equivalent martingale measures* come into the picture. π may most conveniently (and without calculating a duplicating strategy) be calculated by evaluating the expected value of V_T under a probability measure which is equivalent to the basic reference measure and under which X is a martingale. Moreover, from a more theoretical point of view, the existence of such a measure is intimately related to the non–existence of arbitrage opportunities and thereby to the uniqueness of the valuation procedure sketched above. Therefore a problem of both practical and theoretical importance is to

a) obtain a criterion of whether a given model for the price process X allows for an equivalent martingale measure or not;

b) provide a method of constructing such a measure.

These two question are addressed in the present paper. The main result is a simple test for existence of an equivalent martingale measure.

Some notations used in the sequel. All processes are assumed to be defined on the time interval $[0, T]$ and some basic filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, P)$. Localizing stopping times are assumed to satisfy $\tau_n = T$ eventually with probability one. In other words, all spaces $\mathcal{M}, \mathcal{M}_{loc}$, etc. may be considered as obtained from the corresponding spaces on $[0, \infty)$ by stopping at time T.
$\mathcal{M}(P)$ = space of P–martingales.
$\mathcal{M}_0(P)$ = space of P–martingales null at zero.

$\mathcal{M}_0^2(P)$ = space of square–integrable P–martingale null at zero.
$\mathcal{M}_{loc}(P), \mathcal{M}_{0,loc}(P), \mathcal{M}_{0,loc}^2(P)$ are obtained by localization. As a short hand notation, we shall often write $\mathcal{M}$ for $\mathcal{M}(P)$ and $\tilde{\mathcal{M}}$ for $\mathcal{M}(\tilde{P})$ and similarly for the other spaces.
FV_0 = space of regular right continuous (rrc) adapted processes of finite variation null at zero.
lb$\mathcal{P}$ = space of locally bounded predictable processes.
$\Lambda^p(M) = \{\varphi = (\varphi_t) : \varphi \text{ predictable}, E(\int_o^T |\varphi_t|^p d\langle M\rangle_t) < \infty\}$

$\Lambda^p_{loc}(M) = \{\varphi \text{ locally in } \Lambda^p(M)\} = \{\varphi = (\varphi_t) : \varphi \text{ predictable},$
$P(\int_o^T |\varphi_t|^p d\langle M\rangle_t < \infty) = 1\}$
(for $M \in \mathcal{M}_{0,loc}^2(P)$, $p \geq 1$).
The stochastic integral (if defined) will be denoted as

$$(\varphi \cdot M)_t = \int_0^t \varphi_s dM_s.$$

2. SOME PRELIMINARY RESULTS

We start with a price process $X = (X_t)_{0\leq t\leq T}$, which is supposed to be a semimartingale on some filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ satisfying the "usual conditions". X has a decomposition

$$X = X_0 + A + M \tag{2.1}$$

with $A \in FV_0, M \in \mathcal{M}_{0,loc}$. Let now $\tilde{P}$ be any probability measure on $\mathcal{F}_T$ which is equivalent to P and let

$$Z_T = \frac{d\tilde{P}}{dP} \tag{2.2}$$

denote the density of $\tilde{P}$ with respect to P and Z a rrc process s.t.

$$Z_t = E(Z_T|\mathcal{F}_t) \quad \text{a.s.} \tag{2.3}$$

Then Z is a ui P–martingale and

$$E(Z_T) = 1.$$

Moreover, by mutual absolute continuity,

$$P(\inf_t Z_t > 0) = 1$$

(cf. [10], III.3.5), which means that Z is strictly positive and Z_-^{-1} is locally bounded. We may then define a process $Y = (Y_t)_{0 \le t \le T}$ by

$$Y_t = \int_0^t \frac{dZ_s}{Z_{s-}}. \tag{2.4}$$

Y will then be a local P–martingale and satisfy the equation

$$Z_t = 1 + \int_0^t Z_{s-} dY_s. \tag{2.5}$$

Moreover, by strict positivity of Z,

$$\Delta Y_t > -1 \text{ for all } t. \tag{i}$$

As is well known (cf. [4]), the solution of (2.5) is given by

$$Z = \mathcal{E}(Y), \tag{2.6}$$

where $\mathcal{E}(Y)$ denotes the exponential

$$\mathcal{E}(Y)_t = \exp\left[Y_t - \frac{1}{2}\langle Y^c\rangle_t\right] \prod_{0 \le s \le t} (1 + \Delta Y_s) e^{-\Delta Y_s}. \tag{2.7}$$

(2.7) is well defined for any semimartingale Y and is a strictly positive supermartingale as well as a local martingale for all $Y \in \mathcal{M}_{0,loc}$ satisfying (i). It is a ui martingale if and only if

$$E\left[\mathcal{E}(Y)_T\right] = 1 \tag{ii}$$

holds.

Conversely, if we start with some $Y \in \mathcal{M}_{0,loc}$ satisfying (i) and (ii) defining Z by (2.6) and

$$d\tilde{P} = Z_T dP$$

will provide us with an equivalent probability measure $\tilde{P}$.

Proposition 1. *There exists a 1–1 correspondence between equivalent probability measures $\tilde{P}$ and P–local martingales Y satisfying (i) and (ii), which is given by*

$$d\tilde{P} = Z_T dP, \quad Z = \mathcal{E}(Y),$$

where Z is a rrc version of the ui martingale

$$Z_t = E(Z_T|\mathcal{F}_t).$$

Y and Z are determined from one another as the unique solutions of the stochastic differential equations

$$Z_t = 1 + \int_0^t Z_{s-} dY_s$$

and

$$Y_t = \int_0^t \frac{dZ_s}{Z_{s-}}.$$

Let now $Y, Z, \tilde{P}$ be as in Proposition 1. Then $\tilde{P}$ is an *equivalent martingale measure*, i.e.

$$X \in \mathcal{M}_{loc}(\tilde{P}),$$

if and only if

$$XZ \in \mathcal{M}_{loc}(P)$$

(cf. [4], 13.10). By partial integration,

$$\begin{aligned} X_t Z_t &= X_0 Z_0 + \int_0^t X_{s-} dZ_s + \int_0^t Z_{s-} dX_s + [X, Z]_t \\ &= X_0 Z_0 + V_t + N_t \end{aligned} \tag{2.8}$$

with a finite variation part

$$V_t = \int_0^t Z_{s-} dA_s + [X, Z]_t = \int_0^t Z_{s-} d(A_s + [X, Y]_s)$$

and a local martingale part

$$N_t = \int_0^t X_{s-} dZ_s + \int_0^t Z_{s-} dM_s.$$

Note that

$$V \in \mathcal{M}_{0,loc} \Longleftrightarrow A + [X,Y] \in \mathcal{M}_{0,loc}.$$

Since

$$\begin{aligned}[X,Y]_t &= [A,Y]_t + [M,Y]_t \\ &= \sum_{s\le t} \Delta A_s \Delta Y_s + [M,Y]_t,\end{aligned}$$

we find that

$$XZ \in \mathcal{M}_{loc}(P) \Longleftrightarrow A + [M,Y] + \sum_{s\le \cdot} \Delta A_s \Delta Y_s \in \mathcal{M}_{0,loc}. \tag{2.9}$$

Sometimes it may be advantageous to give a characterization in terms of $\mathcal{M}_{0,loc}(\tilde{P})$. To this end, transform (2.8) to the form

$$X_t = X_0 Z_t^{-1} + V_t Z_t^{-1} + N_t Z_t^{-1}. \tag{2.10}$$

Again by [4], 13.10, $X_0 Z^{-1}$ and NZ^{-1} are local martingales under $\tilde{P}$. Since $V \in FV_0$ and $Z^{-1} \in \tilde{\mathcal{M}}_{loc}, VZ^{-1}$ is a semimartingale and

$$\begin{aligned}V_t Z_t^{-1} &= \int_0^t V_{s-} dZ_s^{-1} + \int_0^t Z_s^{-1} dV_s \\ &= \int_0^t V_{s-} dZ_s^{-1} + \int_0^t Z_s^{-1} Z_{s-} dA_s + \int_0^t Z_s^{-1} d[X,Z]_s.\end{aligned}$$

The first integral on the right handside is a local $\tilde{P}$–martingale. Since

$$Z^{-1} = Z_-^{-1} - Z^{-1}\Delta Y, \quad Z = Z_-(1 + \Delta Y), \tag{2.11}$$

$$\begin{aligned}\int_0^t Z_s^{-1} Z_{s-} dA_s &= A_t - \int_0^t Z_s^{-1} Z_{s-} \Delta Y_s dA_s \\ &= A_t - \sum_{s\le t} \frac{Z_{s-}}{Z_s} \Delta Y_s \Delta A_s \\ &= A_t - \sum_{s\le t} \frac{\Delta Y_s}{1+\Delta Y_s} \Delta A_s,\end{aligned} \tag{2.12}$$

$$\int_0^t Z_s^{-1} d[X,Z]_s = [M,Y]_t + [A,Y]_t - \int_0^t Z_s^{-1} \Delta Y_s d[X,Z]_s$$

$$
\begin{aligned}
&= [M,Y]_t + [A,Y]_t - \sum_{s\le t} Z_s^{-1}\Delta Y_s \Delta X_s \Delta Z_s \\
(2.13) \qquad &= [M,Y]_t + [A,Y]_t - \sum_{s\le t} \frac{Z_{s-}}{Z_s}\Delta Y_s^2 \Delta A_s \\
&\qquad - \sum_{s\le t} \frac{Z_{s-}}{Z_s}\Delta Y_s^2 \Delta M_s \\
&= \langle M^c, Y^c\rangle_t + \sum_{s\le t}\left(1 - \frac{Z_{s-}}{Z_s}\Delta Y_s\right)\Delta Y_s \Delta A_s \\
&\qquad + \sum_{s\le t}\left(1 - \frac{Z_{s-}}{Z_s}\Delta Y_s\right)\Delta Y_s \Delta M_s \\
&= \langle M^c, Y^c\rangle_t + \sum_{s\le t}\frac{\Delta Y_s}{1+\Delta Y_s}\Delta A_s + \sum_{s\le t}\frac{\Delta Y_s}{1+\Delta Y_s}\Delta M_s,
\end{aligned}
$$

where again use has been made of the relations (2.11) and

$$1 - \frac{Z_-}{Z}\Delta Y = \frac{Z_-}{Z}.$$

Consequently, (2.10)–(2.13) gives us a decomposition of the semimartingale X in the form

$$X_t = X_0 + \tilde{A}_t + \tilde{M}_t,$$

where

$$\tilde{A}_t = A_t + \langle M^c, Y^c\rangle_t + \sum_{s\le t}\frac{\Delta Y_s}{1+\Delta Y_s}\Delta M_s \qquad (2.14)$$

is of finite variation and

$$\tilde{M}_t = X_0(Z_t^{-1} - 1) + \int_0^t V_{s-} dZ_s^{-1} + N_t Z_t^{-1}$$

is in $\tilde{\mathcal{M}}_{0,loc}$.

Gathering the results obtained so far we arrive at

Proposition 2. *$\tilde{P}$ is an equivalent martingale measure*

$$\iff A + [M,Y] + \sum_{s\le \cdot} \Delta A_s \Delta Y_s \in \mathcal{M}_{0,loc}(P)$$

$$\iff A + \langle M^c, Y^c \rangle + \sum_{s \leq \cdot} \frac{\Delta Y_s}{1 + \Delta Y_s} \Delta M_s \in \mathcal{M}_{0,loc}(\tilde{P}).$$

Corollary 1. *Suppose that X is a special semimartingale and $|\Delta M| \leq c$ for some constant c. Let (2.1) be its canonical decomposition (i.e. the unique decomposition in which A is predictable). Then*

$$\tilde{P} \text{ is an equivalent martingale measure} \iff A + \langle M, Y \rangle = 0.$$

Proof. Since A is predictable, $\sum_{s \leq t} \Delta A_s \Delta Y_s$ is a local martingale (cf. [4], 12.8). Moreover, $[M, Y]$ has locally integrable variation (cf. [10], III 3.14), so that its dual predictable projection $[M, Y]^p$ exists and is equal to $\langle M, Y \rangle$. Hence

$$V = A + [M, Y] + \sum_{s \leq t} \Delta A_s \Delta Y_s = A + \langle M, Y \rangle + N$$

with $N \in \mathcal{M}_{0,loc}$ and

$$V \in \mathcal{M}_{0,loc} \iff A + \langle M, Y \rangle \in \mathcal{M}_{0,loc}.$$

But since $A + \langle M, Y \rangle$ is predictable and of finite variation, it is a local martingale if and only if it is zero (cf. [14]).

□

Actually, instead of requiring boundedness of ΔM, it suffices to require that $\langle M, Y \rangle$ exists (e.g. if $[M, Y]$ is of locally integrable variation).

This result may also be obtained directly by writing

$$X = X_0 + A + \langle M, Y \rangle + (M - \langle M, Y \rangle)$$

and noting that $M - \langle M, Y \rangle$ is a local $\tilde{P}$–martingale by Girsanov's theorem (cf. [4], 13.19). Similar results can be found in [3].

Corollary 2. *Under the assumptions of Corollary 1, there is a 1–1 correspondence — given in the same way as in Proposition 1 — between equivalent martingale measures $\tilde{P}$ and local P–martingales Y satisfying*

(i) $$\Delta Y > -1,$$

(*ii*) $$E[\mathcal{E}(Y)_T] = 1,$$

(*iii*) $$A + \langle M, Y\rangle = 0.$$

Occasionally, we shall denote this $1-1$ correspondence by

$$\Pi : \mathcal{Y} \longrightarrow \mathbb{P},$$

where $\mathcal{Y} = \{Y \in \mathcal{M}_{0,loc} : Y \text{ satisfies (i) – (iii)}\}$ and $\mathbb{P}$ denotes the set of equivalent martingale measures.

3. MAIN RESULTS

We are now in the position to derive necessary and sufficient conditions for the existence of an equivalent martingale measure. Introduce, for $M \in \mathcal{M}^2_{0,loc}$, the following subspaces of $\mathcal{M}^2_{0,loc}$:

$$[[M]] = \{h \cdot M : h \in \Lambda^2_{loc}(M)\},$$

$$[[M]]^{\perp} = \text{orthogonal complement of } [[M]] = \{N \in \mathcal{M}^2_{0,loc} : \langle M, N\rangle = 0\}.$$

We shall say that M has the *martingale representation property* (MRP) if $[[M]]^{\perp} = 0$. Since, by localizing the corresponding decomposition of $\mathcal{M}^2_0$ (cf. [4], 9.17), every $Y \in \mathcal{M}^2_{0,loc}$ may be represented (uniquely) as

$$Y = h \cdot M + N \quad \text{with} \quad h \in \Lambda^2_{loc}(M), \quad N \in [[M]]^{\perp},$$

M has the MRP if and only if $[[M]] = \mathcal{M}^2_{0,loc}$.

Lemma 1. *Let $|\Delta M| \le c$ and $Y \in \mathcal{M}_{0,loc}$. Then the measure $d\langle M, Y\rangle dP$ is locally absolutely continuous w.r. to $d\langle M\rangle dP$ on the predictable σ–field, i.e. there exists a sequence of stopping times τ_n s.t. for every nonnegative bounded predictable h*

$$E\int_0^{\tau_n} h_s d\langle M\rangle_s = 0 \Longrightarrow E\int_0^{\tau_n} h_s d\langle M, Y\rangle_s = 0. \tag{3.1}$$

Proof. Note that $\langle M, Y\rangle$ is well defined (cf. proof of Corollary 1). If Y were in $\mathcal{M}^2_{0,loc}$, $\langle Y\rangle$ would be defined, too, and (3.1) would be an easy consequence of the Kunita–Watanabe inequality. Namely, if τ_n are reducing stopping times for $\langle M\rangle + \langle Y\rangle$, say

$$\langle M\rangle_{\tau_n} + \langle Y\rangle_{\tau_n} \le n,$$

(cf. [4], 11.48), then

$$\begin{aligned} E\int_0^{\tau_n} |h_s| d|\langle M, Y\rangle|_s &\le \left(E\int_0^{\tau_n} |h_s| d\langle M\rangle_s\right)^{\frac{1}{2}} \cdot \left(E\int_0^{\tau_n} |h_s| d\langle Y\rangle_s\right)^{\frac{1}{2}} \\ &\le K n^{\frac{1}{2}} \left(E\int_0^{\tau_n} |h_s| d\langle M\rangle_s\right)^{\frac{1}{2}} \end{aligned}$$

(with K a bound for h).

In case Y is not in $\mathcal{M}^2_{0,loc}$, $[Y]$ is not locally integrable and hence $\langle Y\rangle$ is not defined. Let τ_n be stopping times reducing $\langle M\rangle + |\langle M, Y\rangle|$ to bounded processes, and consider the local L^2–martingale

$$N = h \cdot M.$$

Since, by optional stopping, $E(N^2_\tau) \le E(\langle N\rangle_\tau)$ for every stopping time τ, (3.1) implies that

$$E(N^2_{t\wedge\tau_n}) \le E(\langle N\rangle_{t\wedge\tau_n}) = E[(h \cdot \langle M\rangle)_{t\wedge\tau_n}] = 0$$

for all t and hence

$$N_{t\wedge\tau_n} = 0 \quad \text{a.e. for all } t.$$

By regularity of paths,

$$N_{t\wedge\tau_n} = 0 \quad \text{for all } t \tag{3.2}$$

with probability one. Since $|\Delta N| \le a$ for some constant a, $\langle N, Y\rangle$ is well defined and

$$\langle N, Y\rangle_t = \int_0^t h_s d\langle M, Y\rangle_s$$

(cf. [10], (3.18)). On the other hand,

$$\langle N^{\tau_n}, Y\rangle = \langle N, Y\rangle^{\tau_n}$$

(where, for any process X, X^τ denotes the stopped process $X^\tau_t = X_{t\wedge\tau}$), hence, since $N^{\tau_n} = 0$ by (3.2), $\langle N, Y\rangle^{\tau_n} = 0$, i.e.

$$\langle N, Y\rangle_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} h_s d\langle M, Y\rangle_s = 0 \quad \text{for all } t$$

with probability one.

□

Proposition 3. *Let X be as in Corallary 1. (i) Suppose there exists an equivalent martingale measure $\tilde{P} = \Pi(Y)$ with $Y \in \mathcal{Y}$. Then A is absolutely continuous with respect to $\langle M\rangle$, i.e. there exists $\varphi \in \Lambda^1_{loc}(M)$ s.t. with probability one*

$$A_t = \int_0^t \varphi_s d\langle M\rangle_s \quad \textit{for all } t. \tag{3.3}$$

(ii) Conversely, if there exist a $\varphi \in \Lambda^2_{loc}(M)$ satisfying (3.3) and $N \in [[M]]^\perp$ such that $Y = -\varphi \cdot M + N$ satisfies (i) and (ii) of Corollary 2, then

$$d\tilde{P} = Z_T dP, \quad Z = \mathcal{E}(Y)$$

defines an equivalent martingale measure $\tilde{P}$, and $Y \in \mathcal{Y} \cap \mathcal{M}^2_{0,loc}$.

Proof. Note first that, since M has bounded jumps, it is locally square integrable; hence the predictable quadratic variation $\langle M\rangle$ is well defined. So is the stochastic integral $Y = \varphi \cdot M$ for $\varphi \in \Lambda^2_{loc}(M)$, and $Y \in \mathcal{M}^2_{0,loc}$.

(i) Let τ_n be as in Lemma 1. Define measures on the predictable σ-field $\mathcal{P}$ by

$$\mu_n(C) = E\int_0^{\tau_n} 1_C(s,\omega) d\langle M\rangle_s,$$

$$\nu_n(C) = E\int_0^{\tau_n} 1_C(s,\omega) d\langle M, Y\rangle_s.$$

By Lemma 1, ν_n is on $\mathcal{P}$ absolutely continuous with respect to μ_n and we may write

$$\nu_n(C) = E\int_0^T 1_C \bar{\varphi}^{(n)}_s d\langle M\rangle_s$$

for some predictable $\bar{\varphi}^{(n)} \in \Lambda^1(M)$.

Note that the $\bar{\varphi}^{(n)}$ may be chosen in such a way that

$$\bar{\varphi}^{(n+1)}_t(\omega) = \bar{\varphi}^{(n)}_t(\omega) \text{ on } [0, \tau_n].$$

Since $\tau_n = T$ finally, this gives us a predictable process $\bar{\varphi}$ defined by

$$\bar{\varphi}_t = \lim_{n\to\infty} \bar{\varphi}_t^{(n)}$$

s.t.

$$E\int_0^{\tau_n} |\bar{\varphi}_t| d\langle M\rangle_t < \infty \tag{3.4}$$

and

$$E\int_0^{\tau_n} 1_C d\langle M, Y\rangle = E\int_0^{\tau_n} 1_C \bar{\varphi} d\langle M\rangle \quad \text{for all } C \in \mathcal{P}.$$

Hence $A + \langle M, Y\rangle = 0$ implies that

$$E\int_0^{\tau_n} 1_C dA = -E\int_0^{\tau_n} 1_C \bar{\varphi} d\langle M\rangle \quad \text{for all } C \in \mathcal{P}.$$

In particular, for $C = (s,t] \times F,\ F \in \mathcal{F}_s \quad (s < t)$,

$$E\left\{(A_{t\wedge\tau_n} - A_{s\wedge\tau_n})1_F\right\} = -E\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} \bar{\varphi}_s d\langle M\rangle_s \cdot 1_F\right),$$

saying that $L_t^n = A_{t\wedge\tau_n} + \int_0^{t\wedge\tau_n} \bar{\varphi}_s d\langle M\rangle_s$ is a martingale. Since L^n is of finite variation and predictable, $L^n = 0$, i.e.

$$A_{t\wedge\tau_n} = -\int_0^{t\wedge\tau_n} \bar{\varphi}_s d\langle M\rangle_s \quad \text{a.e..}$$

and finally, with $\varphi = -\bar{\varphi}$,

$$A_t = \int_0^t \varphi_s d\langle M\rangle_s \quad \text{a.e. for all } t.$$

Eventually, by right continuity,

$$P\left(A_t = \int_0^t \varphi_s d\langle M\rangle_s \quad \text{for all } t\right) = 1.$$

Moreover, from (3.4),

$$\int_0^{\tau_n} |\varphi_s| d\langle M\rangle_s < \infty \quad \text{a.e. for all } n.$$

whence

$$P\left(\int_0^T |\varphi_s| d\langle M\rangle_s < \infty\right) = 1.$$

(ii) For $Y = -\varphi \cdot M + N$

$$\langle M, Y\rangle_t = -\int_0^t \varphi_s d\langle M\rangle_s = -A_t \quad \text{for all } t,$$

i.e. Y satisfies (iii) of Corollary 2.

□

The necessary part (i) is useful as a test of whether a given semimartingale $X = X_0 + A + M$ is a local martingale under a suitable absolutely continuous change of measure: A should be absolutely continuous with respect to $\langle M\rangle$.

On the other hand, (ii) provides a method of constructing equivalent martingale measures. Note, however, the gap between Λ^1_{loc} and Λ^2_{loc} in (i) and (ii), resp., which makes it, in general, impossible to give a complete description of all equivalent martingale measures in terms of densities $\frac{dA}{d\langle M\rangle}$.

As an example for the use of (i) as a test for the existence of an equivalent martingale measure, let X be reflected Brownian motion:

$$X_t = |x + B_t|.$$

Then, by Tanaka's formula, X has the representation

$$X_t = x + l_t + W_t$$

with a standard Brownian motion W and Brownian local time l. But l is a.e. singular with respect to $\langle W\rangle_t = t$, hence there exists no equivalent martingale measure.

Corollary 3. *Let X be as in Corollary 1.*
(i) A necessary condition for the existence of an equivalent martingale measure $\tilde{P} = \Pi(Y)$ with $Y \in \mathcal{Y} \cap \mathcal{M}^2_{0,loc}$ is that A is absolutely continuous w.r. to $\langle M\rangle$ with

$$\varphi = \frac{dA}{d\langle M\rangle} \in \Lambda^2_{loc}(M). \tag{3.5}$$

(ii) This condition is also sufficient, provided there exists $N \in [[N]]^\perp$ s.t. $Y = -\varphi \cdot M + N$ satisfies (i) and (ii) of Corollary 2. In this case $d\tilde{P} = \mathcal{E}(Y)_T dP$ defines an equivalent martingale measure.

Proof. By virtue of the decomposition of $\mathcal{M}^2_{0,loc}$ w.r. to stable subspaces,

$$Y = h \cdot M + N, \quad h \in \Lambda^2_{loc}(M), \quad N \in [[M]]^{\perp},$$

implying

$$\langle M, Y \rangle = h \cdot \langle M \rangle = -A.$$

On the other hand, by Proposition 3,

$$A = \varphi \cdot \langle M \rangle.$$

Hence $h = -\varphi \quad d\langle M \rangle dP$ a.e., i.e. $\varphi \in \Lambda^2_{loc}(M)$.

□

Corollary 4. *Let X be as in Corollary 1 and (3.5) hold. Then the subclass*

$$\mathbb{P}' = \Pi(\mathcal{Y} \cap \mathcal{M}^2_{0,loc})$$

of equivalent martingale measures consists of all $\tilde{P}$ s.t.

$$d\tilde{P} = Z_T dP, \qquad Z = \mathcal{E}(Y),$$

where Y is of the form

$$Y = -\varphi \cdot M + N \quad \text{for some} \quad N \in [[M]]^{\perp} \tag{3.6}$$

and satisfies (i) and (ii).

If moreover, M has the MRP, then either $|\mathbb{P}'| = 1$ with single element

$$d\tilde{P} = \mathcal{E}(-\varphi \cdot M)_T dP$$

or $\mathbb{P}' = \emptyset$, according to whether $Y = -\varphi \cdot M$ satisfies (i) and (ii) or not.

Proof. By the proof of Corollary 3, every $Y \in \mathcal{M}^2_{0,loc}$ may be decomposed as (3.6) (since $h \cdot M = -\varphi \cdot M$), hence every $\tilde{P} \in \mathbb{P}'$ is of the form indicated.

□

The following example shows that $\mathbb{P}'$ may be void even under the assumption of Corollary 4.

Example. Consider φ_t as in Exercise 5.18 of [11], i.e.

$$\varphi_t = \begin{cases} -\frac{2}{(1-t)^2} W_t 1_{[t \leq \tau]} & , t < 1, \\ 0 & , t = 1, \end{cases}$$

where $\tau = \inf\{0 \leq t \leq 1 : t + W_t^2 = 1\}$ (and W is standard Brownian motion). Then $\varphi \in \Lambda^2_{loc}(W)$, but

$$E\{\mathcal{E}[\varphi \cdot W]_1\} < 1.$$

Hence, for $X = X_0 + A + W$ with $dA_t = -\varphi_t dt$, the assumptions of Cor. 4 are fulfilled, but $Y = \varphi \cdot W$ does not satisfy (ii). Consequently, since W has the MRP, $\mathbb{P}' = \emptyset$. Actually, since in this case all local martingales (w.r. to the filtration $(\mathcal{F}_t) = (\bar{\mathcal{F}}_t^W)$) are continuous and hence in $\mathcal{M}^2_{loc}$, $\mathbb{P} = \emptyset$ will hold.

Next we give an example in which the Λ^1–test turns out positive and only the Λ^2–test shows that $\mathbb{P}' = \emptyset$.

Example: Consider the process

$$X_t = X_0 \exp(\mu t + \sigma B_t),$$

where B is a Brownian bridge (null at $T = 1$). X may be thought of as modelling the price fluctuations of a pure discount bond whose value at maturity date T is fixed (cf. [5]). Since B may be written

$$B_t = -\int_0^t \frac{B_s}{1-s} ds + W_t \quad \text{on } [0,1)$$

with a Brownian motion W, application of Ito's formula shows that

$$dX_t = X_t \left[\left(\mu + \frac{\sigma^2}{2} - \frac{\sigma B_t}{1-t} \right) + \sigma dW_t \right].$$

Hence X is a semimartingale on $[0,1)$ with

$$A_t = \int_0^t X_s \left(\mu + \frac{\sigma^2}{2} - \frac{\sigma B_s}{1-s} \right) ds.$$

Actually, as is shown in [13],

$$\int_0^1 \frac{|B_s|}{1-s} ds < \infty \quad \text{a.e.,} \tag{3.7}$$

so that A_t is finite on $[0,1]$ and X is actually a semimartingale on $[0,1]$. In addition, (3.7) shows that

$$\varphi = \frac{dA}{d\langle M\rangle} \in \Lambda^1_{loc}(W).$$

On the other hand,

$$\int_0^1 \left|\frac{B_s}{1-s}\right|^2 ds = \infty \quad \text{a.e.} \tag{3.8}$$

(3.8) may easiest be seen by first noting that a different representation for B is

$$B_t = (1-t)\int_0^1 \frac{dW}{1-s}$$

(integrate by parts). Hence, by time change,

$$\frac{B_t}{1-t} = \int_0^t \frac{dW_s}{1-s} = \tilde{W}\left(\frac{t}{1-t}\right)$$

with a new Brownian motion $\tilde{W}$, and

$$\int_0^1 \left(\frac{B_t}{1-t}\right)^2 dt = \int_0^1 \tilde{W}\left(\frac{t}{1-t}\right)^2 dt = \int_0^\infty \frac{\tilde{W}(u)^2}{(1+u)^2} du.$$

By partial integration,

$$\int_0^T \frac{\tilde{W}(u)^2}{(1+u)^2} du = -\frac{\tilde{W}(T)^2}{1+T} + \int_0^T \frac{du}{1+u} + 2\int_0^T \frac{\tilde{W}(u)}{1+u} d\tilde{W}(u). \tag{3.9}$$

Suppose now that

$$\int_0^\infty \frac{\tilde{W}(u)^2}{(1+u)^2} du < \infty \tag{3.10}$$

on some set C of positive measure. Then the martingale

$$M_t = \int_0^t \frac{\tilde{W}(u)}{1+u} d\tilde{W}(u)$$

converges a.e. on C. Moreover, by the law of the iterated logarithm,

$$\frac{\tilde{W}(T)^2}{1+T} = 0(1)\log_2 T \quad as \quad T \to \infty.$$

Hence, on C, the rhs of (3.9) behaves as

$$0(1) \cdot (1 + \log_2 T) + \log T,$$

contradicting (3.10). Consequently, (3.8) must be true, implying

$$\varphi \notin \Lambda^2_{loc}(W).$$

Hence, by Corollary 3, $\mathbb{P}' = \emptyset$. If the underlying filtration is such that all local martingales are continuous, e.g. $(\mathcal{F}_t) = (\bar{\mathcal{F}}_t^W)$ then also $\mathbb{P} = \emptyset$ will hold. (cf also [2]).

Actually, our main interest is in $\mathbb{P}$ instead of $\mathbb{P}'$.

Theorem 1. *Let $\tilde{P}$ be an equivalent martingale measure s.t. $X \in \mathcal{M}^2_{loc}(\tilde{P})$. Then we have the following equivalence:*

$$X \text{ has the MRP w.r. to } \tilde{P} \iff |\mathbb{P}| = 1.$$

Proof. Theorems 11.2 and 11.3, together with Corollary 11.4 and Proposition 4.13 (or rather local versions thereof on $[0, T]$) in [9] give rise to the following chain of equivalences:

$[[X]] = \tilde{\mathcal{M}}^2_{0,loc} \iff [[X]]^{\perp} = 0$

$\iff$ all local L^2–martingales N orthogonal to X (i.e. s.t. $XN \in \tilde{\mathcal{M}}_{loc}$) are trivial

(since in $\tilde{\mathcal{M}}^2_{0,loc}$ orthogonality is equivalent to $\langle X, N\rangle = 0$)

$\iff$ all local martingales in $\mathcal{H}^{\infty}_{0,loc}(\tilde{P})$ (cf. below) orthogonal to X are trivial

(by Prop.(4.13))

$\iff$ $\tilde{P}$ is an extremal point of $\mathbb{P} = \mathbb{P}(X)$

(Theor. 11.2)

$\iff$ $\mathbb{P} = \{\tilde{P}\}$

(by Cor. 11.4, since all $Q \in \mathbb{P}$ are equivalent to $\tilde{P}$).

□

For the sake of completeness, let us have a brief look at what happens beyond L^2–theory (i.e. without the assumption $X \in \mathcal{M}^2_{loc}(\tilde{P})$). Introduce, for $\tilde{P} \in \mathbb{P}$, $1 \leq p \leq \infty$, the spaces

$$L^p(X) = \{h \text{ predictable} : \tilde{E}\{(h^2 \cdot [X])_T^{p/2}\} < \infty\},$$

$$\mathcal{H}_0^p(\tilde{P}) = \{L \in \mathcal{M}_0(\tilde{P}) : \|L_T^*\|_p < \infty\}$$

endowed with the norm $\|L\|_{\mathcal{H}^p} = \|L_T^*\|_p$, (where $L_t^* = \sup_{s \leq t} |L_s|$ and $\|\cdot\|_p$ denotes norm in $L^p(\Omega, \mathcal{F}, \tilde{P})$) together with their localizations $L^p_{loc}(X)$ and $\mathcal{H}^p_{0,loc}(\tilde{P})$.

Note that, for $X \in \tilde{\mathcal{M}}^2_{loc}$,

$$\tilde{E}(h^2 \cdot [X])_T = \tilde{E}(h^2 \cdot \langle X \rangle)_T$$

(by properties of the dual predictable projection), hence $L^2(X) = \Lambda^2(X)$. Then the stochastic integrals $h \cdot X$ are well defined for $h \in L^p(X)$ (resp. $h \in L^p_{loc}(X)$) and are in $\mathcal{H}_0^p(\tilde{P})$ (resp. $\mathcal{H}^p_{0,loc}(\tilde{P})$) (cf. Jacod, chapter II.), so that we may consider the spaces

$$\mathcal{L}^p(X) = \{h \cdot X : h \in L^p(X)\} \subset \mathcal{H}_0^p(\tilde{P})$$

(and their local versions $\mathcal{L}^p_{loc}(X)$). Note that, for $X - X_0 \in \mathcal{M}^2_{0,loc}(\tilde{P})$ $(= \mathcal{H}^2_{0,loc}(\tilde{P}))$, $[[X]] = \mathcal{L}^2_{loc}(X)$.

Using the results from [9] cited above, we have the following string of implications:

$\mathcal{L}^p(X) = \mathcal{H}_0^p(\tilde{P})$ (for some $1 \leq p < \infty$)

$\Longrightarrow$ each $N \in \mathcal{H}_0^q$, $\frac{1}{p} + \frac{1}{q} = 1$, orthogonal to X is zero ("condition C_q") $(1 < q \leq \infty)$

(Theor. (4.11))

$\Longleftrightarrow$ for all $1 \leq r \leq \infty$, each $N \in \mathcal{H}_0^r$ orthogonal to X is zero

(by equivalence of conditions C_p for all p, cf. (4.13))

$\Longleftrightarrow$ $\tilde{P}$ is an extremal point of $\mathbb{P}$

((11.2))

$\iff \mathbb{P} = \{\tilde{P}\}$.

If, in addition, $X - X_0 \in \mathcal{H}^p_{0,loc}(\tilde{P})$, then the first implication may be replaced by an equivalence (cf. (4.11) (b)).

Hence, we arrive at the following generalization of Theorem 1.

Theorem 2. *Let $\tilde{P} \in \mathbb{P}$. If $\mathcal{L}^p(X) = \mathcal{H}^p_0(\tilde{P})$ for some $1 \leq p < \infty$, then $|\mathbb{P}|| = 1$. Conversely, if $X - X_0 \in \mathcal{H}^p_{0,loc}(\tilde{P})$, then $|\mathbb{P}|| = 1$ implies $\mathcal{L}^p(X) = \mathcal{H}^p_0(\tilde{P})$.*

For $p = 1$, this is just the result obtained in [8] (noting that $\mathcal{H}^1_{0,loc}(\tilde{P}) = \mathcal{M}_{0,loc}(\tilde{P})$, cf. [9], Proposition (2.38)).

Remark. The representation property $\mathcal{L}^p(X) = \mathcal{H}^p_0(\tilde{P})$ may be replaced by its local version

$$\mathcal{L}^p_{loc}(X) = \mathcal{H}^p_{0,loc}(\tilde{P})$$

This equivalence follows, in the trivial direction, by localization. For the other direction, if $L \in \mathcal{H}^p_0(\tilde{P})$ has the representation

$$L = h \cdot X \qquad \text{with } h \in L^p_{loc}(X),$$

it follows from the Burkhölder–Davis–Gundy inequality that

$$\tilde{E}\{(h^2 \cdot [X])^{p/2}_T\} = \tilde{E}\{[L]^{p/2}_T\} \leq C \cdot \|L\|^p_{\mathcal{H}^p} < \infty$$

i.e. $h \in L^p(X)$.

Let us now come back to the situation considered in the Corollaries above. We shall first deal with the case where M has the MRP.

Lemma 2. *Let X be as in Corollary 1 and $\varphi = dA/d\langle M\rangle \in \Lambda^2_{loc}(M)$. Then, if M has the MRP, $\mathbb{P} = \mathbb{P}'$.*

Proof. Note that, since M has bounded jumps, if is locally bounded and hence in $\mathcal{H}^\infty_{0,loc}(P)$. Hence, by the string of equivalence leading to Theorem 2 (with X resp. $\tilde{P}$ replaced by M resp. P),

$$\begin{aligned} M \text{ has the MRP } &\iff \mathcal{L}^2_{loc}(M) = \mathcal{H}^2_{0,loc}(P) \\ &\iff \mathcal{L}^1_{loc}(M) = \mathcal{M}_{0,loc}(P). \end{aligned}$$

Suppose $\tilde{P} \in \mathbb{P}$, with

$$\tilde{P} = \Pi(Y), \qquad Y \in \mathcal{Y}.$$

Then Y may be written as stochastic integral

$$Y = h \cdot M \quad \text{with } h \in L^1_{loc}(M).$$

Since

$$\langle M, Y \rangle = h \cdot \langle M \rangle$$

(cf. Lemma 3 below), and, on the other hand,

$$\langle M, Y \rangle = -A = -\varphi \cdot \langle M \rangle$$

with $\varphi \in \Lambda^2_{loc}(M)$, it follows that $h = -\varphi \quad d\langle M \rangle dP-$a.e, hence $h \in \Lambda^2_{loc}(M)$ and $Y \in \mathcal{M}^2_{0,loc}(M)$.

□

Lemma 3. *For M, Y as in Lemma 2 and its proof,*

$$\langle M, Y \rangle = h \cdot \langle M \rangle.$$

Proof. (Actually, though Y is not in $\mathcal{M}^2_{0,loc}$, this result should be standard, but we have not found a reference). Consider the local martingale

$$Y' = |h| \cdot M.$$

Since M has bounded jumps, $[M, Y'] = |h| \cdot [M]$ is of locally integrable variation (cf. [10], III. 3.14). Hence there exists a localizing sequence τ_n of stopping times (note $\tau_n = T \quad$ a.e. finally) s.t.

$$E\left(|h| \cdot [M]\right)_{\tau_n} = E\left(|h| \cdot \langle M \rangle\right)_{\tau_n} < \infty$$

(by elementary properties of the dual predictable projection $\langle M \rangle$). Consequently, $|h| \cdot \langle M \rangle$ is locally integrable, equivalently, by predictability,

$$(|h| \cdot \langle M \rangle)_T < \infty \quad a.e.$$

On the other hand, by elementary properties of dual predictable projections,

$$E\left(\eta h^{+}\cdot[M]\right)=E\left(\eta h^{+}\cdot\langle M\rangle\right)_{T},$$
$$E\left(\eta h^{-}\cdot[M]\right)_{T}=E\left(\eta h^{-}\cdot\langle M\rangle\right)_{T}$$

for all nonnegative predictable processes η. Since the integrals $h^{\pm}\cdot[M]$ and $h^{\pm}\cdot\langle M\rangle$ are well defined and of locally integrable variation, this means that

$$\left(h^{\pm}\cdot[M]\right)^{p}=h^{\pm}\cdot\langle M\rangle,$$

whence

$$\langle M,Y\rangle=[M,Y]^{p}=(h\cdot[M])^{p}=h\cdot\langle M\rangle.$$

□

The second part of the Corollary 4 may now be sharpened to

Theorem 3.

Let X and φ be as in Lemma 2. Then, if M has the MRP, either $\mathbb{P}=\emptyset$ or $|\mathbb{P}|=1$, according to whether $Y=-\varphi\cdot M$ satisfies (i) and (ii) or not. If $|\mathbb{P}|=1$, then the sole element $\tilde{P}\in\mathbb{P}$ is given by

$$d\tilde{P}=\mathcal{E}(Y)_{T}dP,$$

and X has the MRP w.r. to $\tilde{P}$.

Theorem 3 gives the complete answer for the case

A) $\varphi=dA/d\langle M\rangle\in\Lambda^{2}_{loc}(M)$ and
a) M has the MRP.
In the case where
b) M does not have the MRP,
it may either happen that
b1) $\tilde{Y}=-\varphi\cdot M$ satisfies (i) and (ii),
in which case $\tilde{P}=\Pi(\tilde{Y})$ is in $\mathbb{P}'$ (cf. Corollary 3 or 4), in particular $\mathbb{P}\neq\emptyset$, or
b2) $\tilde{Y}$ does not satisfiy (i) and (ii),
then examples with discontinuous martingales show that $\mathbb{P}'$ may be nonvoid. Actually, these examples are built around violation of condition (i). In cases, however, where all local martingales are continuous (as on filtrations generated by Brownian motions), (i) is trivially satisfies and the crucial condition for $Y=\tilde{Y}+N$, $N\in[[M]]^{\perp}$, to define an equivalent martingale measure becomes (ii), which may be written (by orthogonality)

$$E[\mathcal{E}(\tilde{Y})_T \cdot \mathcal{E}(N)_T] = 1. \tag{3.11}$$

An interesting and, to the authors' knowledge, still open question arising in this context is whether there can exist $N \perp M$ s.t. (3.11) holds, despite the fact that

$$E[\mathcal{E}(\tilde{Y})_T] < 1$$

(since $\tilde{Y}$ should violate (ii)). If the answer is yes, a second question arising is if any N archieving (3.11) must necessarily have

$$E[\mathcal{E}(N)_T] = 1$$

or not.

For the case
B) $\varphi = dA/d\langle M\rangle \in \Lambda^1_{loc}(M)$ (not in Λ^2_{loc}!),
$\mathbb{P}' = \emptyset$ by virtue of Corollary 3. Hence, in situations where all local martingales involved are continuous, also $\mathbb{P} = \emptyset$.

ACKNOWLEDGMENT

This work was supported by Deutsche Forschungsgemeinschaft under SFB 303.

REFERENCES

[1] Black, F. and M. Scholes : The Pricing of Options and Corporate Liabilities. Journal of Political Economy 81 (1973) 637–659.

[2] Cheng, S.: On the Feasibility of Arbitrage-based Option Pricing when Stochastic Bond Price Processes are Involved. Columbia University, Preprint, 1991.

[3] El Karoui, N. and M.C. Quenez: Programmation Dynamique et Evaluation des Actifs Contingents en Marche Incomplet. Working Paper, 1991.

[4] Elliott, R. J. : Stochastic Calculus and Applications. Springer–Verlag, New York 1982.

[5] Elliott, R. J. and P. E. Kopp: Equivalent Martingale Measures for Bridge Processes. Stochastic Analysis and Applications 9 (1991) 429-444.

[6] Föllmer, H.: Probability Aspects of Options. Discussion Paper, 1991.

[7] Harrison, J. M. and D. M. Kreps: Martingales and Arbitrage in Multiperiod Securities Markets. Journal of Economic Theory 20 (1979) 381–408.

[8] Harrison, J. M. and S. R. Pliska: Martingales and Stochastic Integrals in the Theory of Continuous Trading. Stochastic Processes and their Applications 11 (1981) 215–260.

[9] Jacod, J.: Calcul stochastique et problèmes de martingales. Lecture Notes in Mathematics 714. Springer–Verlag, New York 1979.

[10] Jacod, J. and A. N. Shiryayev: Limit Theorems for Stochastic Processes. Springer–Verlag, New York 1991.

[11] Karatzas, I. and S. E. Shreve: Brownian Motion and Stochastic Calculus. Springer–Verlag, New York 1988.

[12] Merton, R. C.: Theory of Rational Option Pricing. Bell Journal of Economics and Management Science 4 (1973) 141–183.

[13] Protter, P.: Stochastic Integration and Differential Equations. Springer Verlag, New York 1991.

[14] Rogers, L. and D. Williams: Diffusions, Markov Processes and Martingales, vol. 2. Wiley, New York 1987.

Chapter 12

HEDGING OF NON-REDUNDANT CONTINGENT CLAIMS

HANS FÖLLMER

Eidgenössische Technische Hochschule, Zurich, Switzerland

DIETER SONDERMANN

University of Bonn, Bonn, W. Germany

1. Introduction

Financial economics has undoubtedly achieved some of its most striking results in the theory of option pricing, starting with the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). Another approach based on arbitrage methods was introduced independently by Ross; for basic results along this line cf. Cox and Ross (1976a). A further important development is due to Harrison and Kreps (1979): They analyzed the valuation of contingent claims in terms of martingale theory and thus clarified the mathematical structure of the problem.

There is, however, what Hakansson (1979, p. 722) called *The Catch 22 of Option Pricing*: "A security can be unambiguously valued by reference of the other securities in a perfect market if and only if the security being valued is redundant in that market." Indeed all preference-independent valuation formulas assume that the asset to be valued is *attainable*, i.e., that it can be perfectly duplicated by a dynamically adjusted portfolio of the existing assets. "But if this is the case, the option adds nothing new to the market and no social welfare can arise – the option is perfectly redundant .... So we find ourselves in the awkward position of being able to derive unambiguous values only for redundant assets and unable to value options which do have social value." [Hakansson (1979), p. 723)].[1]

[1] Of course, as Hakansson points out, one can still use arbitrage arguments to put bounds on the value of a non-redundant asset as done, e.g., in Harrison and Kreps (1979) and Egle and Trautmann (1981).

In this paper, our purpose is to extend the martingale approach of Harrison and Kreps (1979) to contingent claims which are *non-redundant*. We are less concerned here with valuation formulas than with how to use the existing assets for an optimal hedge against the claim. To this end we introduce a class of admissible portfolio strategies which generate a given contingent claim at some terminal time T. Due to the underlying martingale assumptions, the expected terminal cost does not depend on the specific choice of the strategy. It is therefore natural to look for admissible strategies which *minimize risk* in a sequential sense. We show in Section 3 that this problem has a unique solution where the risk is reduced to what we call the intrinsic risk of the claim. This risk-minimizing strategy is *mean-self-financing*, i.e., the corresponding cost process is a martingale. A claim is attainable if and only if its intrinsic risk is zero. In that particular case, the risk-minimizing strategy becomes *self-financing*, i.e., the cost process is constant, and we obtain the usual arbitrage value of the claim. In Section 4 we study the dependence on the hedger's subjective beliefs: It is shown how the strategy changes under an absolutely continuous change of the underlying martingale measure.

In Section 5 we illustrate our results by computing explicitly the intrinsic risk and the risk-minimizing strategy for a call option where the underlying stock price follows a two-sided jump process. Contrary to jump processes already studied in the literature [see, e.g., Cox and Ross (1976b)], this model is not complete, and a typical call is non-redundant.

It is a pleasure to thank M. Schweizer who worked out a large part of the first example in Section 5; cf. Schweizer (1984).

2. *Basic definitions*

Let $(\Omega, \mathscr{F}, P^*)$ be a probability space, and let $(\mathscr{F}_t)_{0 \le t \le T}$ denote a right-continuous family of σ-algebras contained in $\mathscr{F}$; $\mathscr{F}_t$ is interpreted as the collection of events which are observable up to time t. A *stochastic process* $Z = (Z_t)_{0 \le t \le T}$ is given by a measurable function Z on $\Omega \times [0, T]$. Z is called *adapted* if Z_t is $\mathscr{F}_t$-measurable for each $0 \le t \le T$; it is called *predictable* if it is measurable with respect to the σ-algebra $\mathscr{P}$ on $\Omega \times [0, T]$ which is generated by the adapted processes with left-continuous paths. We refer to Metivier (1982) for further details.

The evolution of *stock prices* will be described by a stochastic process $X = (X_t)_{0 \le t \le T}$ which is adapted and whose paths are right-continuous with limits X_{t-} from the left.[2] The process $Y = (Y_t)_{0 \le t \le T}$ of *bond prices* is fixed to

[2] For simplicity we only consider the one-dimensional case. The extension to an n-dimensional stock process $X = (X^1, \dots, X^n)$ is straightforward if the components of X are mutually orthogonal; see, e.g., Schweizer (1984). For difficulties which can arise otherwise, see Müller (1984).

be $Y_t = 1$.[3] We assume that P^* is a *martingale measure* in the sense of Harrison and Kreps (1979); i.e., we assume

$$\mathrm{E}^*[X_T^2] < \infty, \tag{1}$$

and

$$X_t = \mathrm{E}^*[X_T | \mathscr{F}_t], \qquad 0 \leq t \leq T, \tag{2}$$

where $\mathrm{E}^*[\cdot | \mathscr{F}_t]$ denotes the conditional expectation under P^* with respect to the σ-algebra $\mathscr{F}_t$. This means that X is a *square-integrable martingale* under P^*.[4] Let $\langle X \rangle = (\langle X \rangle_t)_{0 \leq t \leq T}$ be the corresponding *Meyer process*, i.e., the unique predictable process with $\langle X \rangle_0 = 0$ and right-continuous increasing paths such that $X^2 - \langle X \rangle$ is a martingale;[5] cf. Metivier (1982). We denote by P_X^* the finite measure on $(\Omega \times [0, T], \mathscr{P})$ given by

$$P_X^*[A] = \mathrm{E}^*\left[\int_0^T I_A(t, \omega)\, \mathrm{d}\langle X \rangle_t(\omega)\right],$$

and by $L^2(P_X^*)$ the class of predictable processes Z which, viewed as $\mathscr{P}$-measurable functions on $\Omega \times [0, T]$, are square-integrable with respect to P_X^*. Two such processes will be considered as equal if they coincide P_X^*-almost surely.

A *trading strategy* will be of the form $\varphi = (\xi, \eta)$ where $\xi = (\xi_t)_{0 \leq t \leq T}$ and $\eta = (\eta_t)_{0 \leq t \leq T}$ describe the successive amounts invested into the stock and into the bond. Thus,

$$V_t \equiv \xi_t X_t + \eta_t \tag{3}$$

is the *value of the portfolio* at time t. We need the following technical assumptions.

Definition 1. $\varphi = (\xi, \eta)$ is called a *strategy* if
(a) ξ is a predictable process, and $\xi \in L^2(P_X^*)$,
(b) η is adapted,
(c) $V \equiv \xi X + \eta$ has right-continuous paths and satisfies $V_t \in L^2(P^*)$, $0 \leq t \leq T$.

[3]As shown by Harrison and Kreps (1979), there is no loss of generality in making this assumption, since their method of standardizing the bond process to unity allows also stochastic interest rates.

[4]In this paper we leave aside problems of viability, i.e., we only consider martingale measures and do not study the case where the underlying probability distribution is only assumed to admit an equivalent martingale measure. For the relationship between viability, absence of arbitrage opportunities, and the existence of an equivalent martingale measure, we refer to Harrison and Kreps (1979), Harrison and Pliska (1981), and Müller (1984).

[5]E.g., if X is the Brownian Motion with variance σ^2 then $\langle X \rangle_t$ equals $\sigma^2 t$.

Condition (a) allows to calculate the *accumulated gain* obtained from the stock price fluctuation up to time t as the stochastic integral

$$\int_0^t \xi_s \,\mathrm{d} X_s, \qquad 0 \leq t \leq T. \tag{4}$$

For fixed t, the gain has expectation $\mathrm{E}^*[\int_0^t \xi_s \,\mathrm{d} X_s] = 0$ and variance

$$\mathrm{E}^*\left[\left(\int_0^t \xi_s \,\mathrm{d} X_s\right)^2\right] = \mathrm{E}^*\left[\int_0^t \xi_s^2 \,\mathrm{d}\langle X\rangle_s\right]. \tag{5}$$

Viewed as a stochastic process, (4) defines a square-integrable martingale with right-continuous paths. The *accumulated cost* of the strategy up to time t can now be defined as

$$C_t \equiv V_t - \int_0^t \xi_s \,\mathrm{d} X_s. \tag{6}$$

$V = (V_t)_{0 \leq t \leq T}$ and $C = (C_t)_{0 \leq t \leq T}$ are adapted processes with right-continuous paths; they are called the *value process* and the *cost process*.

Remark 1. Consider a *simple strategy* where stock trading only occurs at finitely many times, $0 \leq t_0 < \cdots < t_n \leq T$, i.e.,

$$\xi_t(\omega) = \sum_i \alpha_i(\omega) I_{(t_i, t_{i+1}]}(t), \tag{7}$$

where α_i is $\mathscr{F}_{t_i}$-measurable. Equation (7) means that the amount ξ_t is fixed just before the portfolio is actually changed, in accordance with predictability. Since

$$\int_0^t \xi_s \,\mathrm{d} X_s = \sum_{j<i} \alpha_j\left(X_{t_{j+1}} - X_{t_j}\right) + \alpha_i\left(X_t - X_{t_i}\right),$$

for $t \in (t_i, t_{i+1}]$, the cost process (6) is given by

$$C_t = \eta_t + \alpha_i X_{t_i} - \sum_{j<i} \alpha_j\left(X_{t_{j+1}} - X_{t_j}\right), \tag{8}$$

for $t \in (t_i, t_{i+1}]$. Since η is only assumed to be adapted, not necessarily predictable, the value of η_t can be fixed after observing the situation at time t. In particular, η can be used to keep the value process V on a certain desired level V^*. For the class of mean-self-financing admissible strategies introduced below, this level V^* will not depend on the specific choice of ξ. In that case we can justify definition (6), which is intuitive for simple strategies, by a passage to the limit. Indeed, for any predictable process $\xi \in L^2(P_X^*)$ there is a sequence of simple predictable processes ξ^n of the form (7) which converges to ξ in $L^2(P_X^*)$, and (5) together with a maximal inequality for martingales implies that

$$\sup_{0 \le t \le T} |C_t^n - C_t| = \sup_{0 \le t \le T} \left| \int_0^t \xi^n \, \mathrm{d}X - \int_0^t \xi \, \mathrm{d}X \right|$$

converges to zero in $L^2(P^*)$.

Definition 2. A strategy $\varphi = (\xi, \eta)$ is called *mean-self-financing* if the corresponding cost process $C = (C_t)_{0 \le t \le T}$ is a martingale.

Remark 2. A strategy $\varphi = (\xi, \eta)$ is called *self-financing* if the cost process has constant paths, i.e., if

$$C_t \equiv C_0, \qquad P^*\text{-a.s.}, \qquad 0 \le t \le T. \tag{9}$$

Any self-financing strategy is clearly mean-self-financing. For a self-financing strategy, the value process is of the form

$$V_t = C_0 + \int_0^t \xi_s \, \mathrm{d}X_s, \qquad 0 \le t \le T, \tag{10}$$

hence a square-integrable martingale. Self-financing strategies are the key tool in the analysis of option pricing in "complete" security markets; cf. Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), and Müller (1984). But in many situations security markets are *incomplete* in the sense that there may not be any self-financing strategy which allows to realize a pre-assigned terminal value $V_T = H$. This is the reason why we introduce the broader concept of a mean-self-financing strategy. As stated in the following lemma, the value process of a mean-self-financing strategy is again a martingale. But in general we cannot expect that this martingale can be represented as a stochastic integral with respect to X as in (10).

Lemma 1. A strategy is mean-self-financing if and only if its value process is a square-integrable martingale.

Proof. The properties of a strategy as defined in Definition 1 imply that the process of accumulated gains

$$\int_0^t \xi_s \, \mathrm{d} X_s, \qquad 0 \le t \le T,$$

is a square-integrable martingale, and that V_t, $0 \le t \le T$, is square-integrable. Thus Lemma 1 is clear from (6). □

3. *The intrinsic risk of contingent claims*

Let us fix a *contingent claim* $H \in \mathscr{L}^2(P^*)$. For example, H could be a call option of the form $H = (X_T - C)^+$.

Definition 3. A strategy is called *admissible* (with respect to H) if its value process has terminal value

$$V_T = H, \qquad P^*\text{-a.s.} \tag{11}$$

For any admissible strategy $\varphi = (\xi, \eta)$, the terminal cost is given by

$$C_T = H - \int_0^T \xi_s \, \mathrm{d} X_s. \tag{12}$$

In particular, the expected value,

$$\mathrm{E}^*[C_T] = \mathrm{E}^*[H],$$

does not depend on the specific choice of the strategy as long as it is admissible. We are now going to analyse which admissible strategies have minimal risk in a suitable sense. As a first step in that direction, let us determine all admissible strategies which

$$\text{minimize the variance } \mathrm{E}^*\left[(C_T - \mathrm{E}^*[H])^2\right]; \tag{13}$$

the second step will consist in replacing (13) by a sequential procedure.

In view of (13), let us write the claim H in the form

$$H = \mathrm{E}^*[H] + \int_0^T \xi_s^* \, \mathrm{d} X_s + H^*, \tag{14}$$

with $\xi^* \in L^2(P_X^*)$ where $H^* \in L^2(P^*)$ has expectations zero and is orthogonal to the space $\{\int_0^t \xi_s \, \mathrm{d}X_s | \xi \in L^2(P_X^*)\}$ of stochastic integrals with respect to X; cf. Metivier (1982) for the existence and uniqueness of this representation.

Theorem 1. An admissible strategy $\varphi = (\xi, \eta)$ *has minimal variance*

$$\mathrm{E}^*\left[(C_T - \mathrm{E}^*[H])^2\right] = \mathrm{E}^*\left[(H^*)^2\right], \tag{15}$$

if and only if $\xi = \xi^*$.

Proof. For an admissible strategy $\varphi = (\xi, \eta)$ we have, by (14),

$$\begin{aligned} C_T &= H - \int_0^T \xi_s \, \mathrm{d}X_s \\ &= \mathrm{E}^*[H] + \int_0^T (\xi_s^* - \xi_s) \, \mathrm{d}X_s + H^*. \end{aligned}$$

Since H^* is orthogonal to the stochastic integral on the right side, we obtain

$$\begin{aligned} \mathrm{E}^*\left[(C_T - \mathrm{E}^*[H])^2\right] &= \mathrm{E}^*\left[\left(\int_0^T (\xi_s^* - \xi_s) \, \mathrm{d}X_s\right)^2\right] + \mathrm{E}^*\left[(H^*)^2\right] \\ &= \mathrm{E}^*\left[\int_0^T (\xi_s^* - \xi_s)^2 \, \mathrm{d}\langle X \rangle_s\right] + \mathrm{E}^*\left[(H^*)^2\right]. \end{aligned}$$

Thus, the minimum $\mathrm{E}^*[(H^*)^2]$ is assumed if and only if $\xi = \xi^*$ in $L^2(P_X^*)$. □

So far we can draw no conclusion concerning the process $\eta = (\eta_t)_{0 \le t \le T}$, except that it must make the strategy admissible, i.e.,

$$\eta_T = H - \xi_T X_T. \tag{16}$$

Example 1. One natural idea is to use a *self-financing* strategy during the interval $[0, T)$, and to make up the balance at the end, i.e., to put

$$\eta_t \equiv \mathrm{E}^*[H] + \int_0^t \xi_s^* \, \mathrm{d}X_s - \xi_t^* X_t, \qquad 0 \le t \le T, \tag{17}$$

in addition to (16) so that

$$\begin{aligned} C_t &= \mathrm{E}^*[H] && \text{for} \quad 0 \le t < T, \\ &= \mathrm{E}^*[H] + H^* && \text{for} \quad t = T. \end{aligned} \tag{18}$$

This strategy would indeed realize the minimal variance $E^*[(C_T - E^*[H])^2] = E^*[(H^*)^2]$ in (15).

We are now going to show that a sharper formulation of problem (13) determines a unique admissible strategy $\varphi^* = (\xi^*, \eta^*)$ which has *minimal risk* in a sequential sense, and which will be different from the strategy considered in Example 1.

Consider any strategy $\varphi = (\xi, \eta)$. Just before time $t < T$ we have accumulated the cost C_{t-}. The strategy tells us how to proceed at and beyond time t. In particular, it fixes the present cost C_t and determines the remaining cost $C_T - C_t$. Let us measure the *remaining risk* by

$$R_t^\varphi = E^*\left[(C_T - C_t)^2 | \mathscr{F}_t\right]. \tag{19}$$

In view of (19), we might want to compare φ to any other strategy $\tilde{\varphi}$ which coincides with φ at all times $< t$ and which leads to the same terminal value V_T. Let us call such a $\tilde{\varphi}$ an *admissible continuation* of φ at time t.

Definition 4. A strategy φ is called *risk-minimizing* if φ at any time minimizes the remaining risk, i.e., for any $0 \leq t < T$, we have

$$R_t^\varphi \leq R_t^{\tilde{\varphi}}, \qquad P^*\text{-a.s.}, \tag{20}$$

for every admissible continuation $\tilde{\varphi}$ of φ at time t.

Remarks 3. (1) Any self-financing strategy φ is clearly risk-minimizing since $R_t^\varphi \equiv 0$.

(2) Suppose that $\varphi = (\xi, \eta)$ is a risk-minimizing strategy which is also admissible. Then φ is in particular a solution of problem (13). In fact, (20) with $t = 0$ implies that φ minimizes

$$E^*\left[(C_T - C_0)^2\right] = E^*\left[(C_T - E^*[C_T])^2\right] + (E^*[C_T] - C_0)^2.$$

Thus, ξ minimizes the variance of C_T and this implies $\xi = \xi^*$ according to Theorem 1. In addition we obtain the condition

$$\eta_0 = C_0 - \xi_0^* X_0 = E^*[H] - \xi_0^* X_0.$$

The sequential version of this second fact will be provided by Theorem 2 below.

Lemma 2. An admissible risk-minimizing strategy is mean-self-financing.

Proof. Consider a strategy $\varphi = (\xi, \eta)$ and a fixed time $0 \leq t_0 \leq T$. Define $\tilde{\eta}_t = \eta_t$, $t < t_0$, and

$$\tilde{\eta}_t \equiv \tilde{C}_t + \int_0^t \xi_s \, \mathrm{d}X_s - \xi_t X_t, \qquad t_0 \leq t \leq T,$$

where $(\tilde{C}_t)_{0 \leq t \leq T}$ denotes a right-continuous version of the martingale

$$\tilde{C}_t = \mathrm{E}^*[C_T | \mathscr{F}_t], \qquad 0 \leq t \leq T.$$

Then $\tilde{\varphi} \equiv (\xi, \tilde{\eta})$ is an admissible continuation of φ at time t_0, and its remaining cost is given by

$$\tilde{C}_T - \tilde{C}_{t_0} = (C_T - C_{t_0}) + (C_{t_0} - \tilde{C}_{t_0}).$$

This implies

$$\mathrm{E}^*\left[(C_T - C_{t_0})^2 | \mathscr{F}_{t_0}\right] = \mathrm{E}^*\left[(\tilde{C}_T - \tilde{C}_{t_0})^2 | \mathscr{F}_{t_0}\right] + (\tilde{C}_{t_0} - C_{t_0})^2.$$

Thus, φ is risk-minimizing only if $C_{t_0} = \tilde{C}_{t_0}$ P^*-a.s. for any $t_0 \leq T$, i.e., if φ is mean-self-financing. □

In order to formulate our final result, let us denote by $V^* = (V_t^*)$ a right-continuous version of the square-integrable martingale

$$V_t^* = \mathrm{E}^*[H | \mathscr{F}_t], \qquad 0 \leq t \leq T. \tag{21}$$

To the representation (14) of the claim H corresponds the following sequential representation of V^*:

$$V_t^* = V_0^* + \int_0^t \xi_s^* \, \mathrm{d}X_s + N_t^*, \tag{22}$$

where $N_t^* = \mathrm{E}^*[H^* | \mathscr{F}_t]$ is a square-integrable martingale with zero expectations which is orthogonal to X in the following sense.

Remark 4. Two square-integrable martingales M_1 and M_2 are called *orthogonal* if their product $M_1 M_2$ is again a martingale, and this is equivalent to the condition

$$\langle M_1, M_2 \rangle \equiv \tfrac{1}{2}(\langle M_1 + M_2 \rangle - \langle M_1 \rangle - \langle M_2 \rangle) = 0. \tag{23}$$

In view of (26) below the process $R^* = (R_t^*)$, defined as a right-continuous version of

$$R_t^* = \mathrm{E}^*\left[(N_T^* - N_t^*)^2 | \mathscr{F}_t\right] = \mathrm{E}^*[\langle N^* \rangle_T | \mathscr{F}_t] - \langle N^* \rangle_t, \tag{24}$$

will be called the *intrinsic risk process* of the claim H. The expectation $E^*[R_0^*]$ coincides with the minimal variance calculated in (15); let us call it the *intrinsic risk* of the claim.

Theorem 2. There exists a unique admissible strategy φ^* which is risk-minimizing, namely

$$\varphi^* = (\xi^*, V^* - \xi^* X). \tag{25}$$

For this strategy, the remaining risk at any time $t \leq T$ is given by

$$R_t^{\varphi^*} = R_t^*, \qquad P^*\text{-a.s.} \tag{26}$$

Proof. (1) Since the value process of the strategy (25) is given by the martingale V^* in (21), the strategy is admissible. If φ is an admissible continuation of φ^* at time t, then its cost process satisfies

$$C_T - C_t = \int_t^T (\xi_s^* - \xi_s)\, \mathrm{d}X_s + N_T^* - N_t^* + V_t^* - V_t,$$

due to (22). The orthogonality of X and N^* implies

$$\begin{aligned} E^*\left[(C_T - C_t)^2 | \mathscr{F}_t\right] = E^*\left[\int_t^T (\xi_s^* - \xi_s)^2\, \mathrm{d}\langle X\rangle_s | \mathscr{F}_t\right] \\ + R_t^* + (V_T^* - V_t)^2, \end{aligned} \tag{27}$$

and in particular (26). This shows that φ^* is risk-minimizing.

(2) Let $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$ be any admissible strategy which is risk-minimizing. This implies $\xi = \xi^*$, as pointed out in Remark 3. By Lemma 2, $\tilde{\varphi}$ is mean-self-financing, and so its value process $\tilde{V}$ is a martingale. Since $\tilde{\varphi}$ is admissible, $\tilde{V}$ must coincide with the martingale V^* defined in (21), and this implies $\eta^* = V^* - \xi^* X$. Thus, a risk-minimizing admissible strategy is uniquely determined by (25). □

As a special case of Theorem 2 we obtain the following characterization of attainable contingent claims [cf. Harrison and Kreps (1979)].

Corollary 1. The following statements are equivalent:
(1) *The risk-minimizing admissible strategy φ^* is self-financing.*
(2) *The intrinsic risk of the contingent claim H is zero.*
(3) *The contingent claim H is attainable, i.e.,*

$$H = E^*[H] + \int_0^T \xi_s^*\, \mathrm{d}X_s, \qquad P^*\text{-a.s.} \tag{28}$$

Proof. φ^* is self-financing if and only if the remaining risk at any time is given by $R_t^{\varphi^*} = 0$. By (26), this is equivalent to $R_t^* = 0$, $0 \leq t \leq T$, and this means that the intrinsic risk is zero. Remark 4 shows that $R_0^* = 0$ is equivalent to the condition $N_t^* = 0$, $0 \leq t \leq T$, in the representation (22), and this is equivalent to (28). □

4. Changing the measure

Let us now see how the risk-minimizing strategy is affected by an absolutely continuous change of the underlying martingale measure.

Let P be any martingale measure which is absolutely continuous with respect to P^*. Thus, the process X is again a square-integrable martingale under P. Let us also assume that our contingent claim $H \in L^2(P^*)$ is again square-integrable under P. Then the representation (22) and Theorem 2, applied to P instead of P^*, show that the risk-minimizing strategy under P is given by $\varphi = (\xi, V - \xi X)$, with

$$V_t = \mathrm{E}[H|\mathscr{F}_t] = V_0 + \int_0^t \xi_s \, \mathrm{d}X_s + N_t. \tag{29}$$

Let us now describe how ξ is related to ξ^*. In order to simplify the exposition we add the technical assumption

$$\xi^* \in L^2(P_X). \tag{30}$$

While X is again a martingale under P, the martingale property of (N_t^*) in (22) may be lost. In general, we have the Doob decomposition

$$N^* = M + A, \tag{31}$$

where $M = (M_t)$ is a martingale under P and $A = (A_t)$ is a predictable process with $A_0 = 0$ and with right-continuous paths of bounded variation; cf. Metivier (1982). Let us introduce the predictable processes ξ^M and ξ^A defined by

$$\langle M, X \rangle_t = \int_0^t \xi_s^M \, \mathrm{d}\langle X \rangle_s \quad \text{and} \quad \langle M^A, X \rangle_t = \int_0^t \xi_s^A \, \mathrm{d}\langle X \rangle_s,$$
$$0 \leq t \leq T,$$

where M^A denotes a right-continuous version of the martingale

$$M_t^A \equiv \mathrm{E}[A_T|\mathscr{F}_t], \qquad 0 \le t \le T.$$

Theorem 3. The risk-minimizing strategy under P is given by $\varphi = (\xi, V - \xi X)$ *with*

$$\xi = \xi^* + \xi^M + \xi^A, \tag{32}$$

and

$$V_t = V_t^* + M_t^A - A_t, \qquad 0 \le t \le T. \tag{33}$$

Proof. Consider the representation (14) of the contingent claim, i.e.,

$$H = \mathrm{E}^*[H] + \int_0^T \xi_s^* \, \mathrm{d}X_s + H^*, \tag{34}$$

P^*-a.s., hence P-a.s.. Since

$$H^* = N_T^* = M_T + A_T, \qquad P\text{-a.s.},$$

(30) and (34) imply $\mathrm{E}[H] = \mathrm{E}^*[H] + \mathrm{E}[A_T]$, hence

$$\begin{aligned} H &= \mathrm{E}[H] + \int_0^T \xi_s^* \, \mathrm{d}X_s + M_T + A_T - \mathrm{E}[A_T] \\ &= \mathrm{E}[H] + \int_0^T (\xi^* + \xi^M + \xi^A)_s \, \mathrm{d}X_s + \tilde{N}_T, \qquad P\text{-a.s.}, \end{aligned} \tag{35}$$

where we put

$$\tilde{N}_t \equiv M_t + M_t^A - \int_0^t (\xi^M + \xi^A)_s \, \mathrm{d}X_s - \mathrm{E}[A_T], \qquad 0 \le t \le T.$$

By the definition of ξ^M and ξ^A,

$$\begin{aligned} \langle \tilde{N}, X \rangle_t &= \langle M, X \rangle_t + \langle M^A, X \rangle_t - \int_0^t (\xi^M + \xi^A)_s \, \mathrm{d}\langle X \rangle_s \\ &= 0, \qquad 0 \le t \le T, \end{aligned}$$

i.e., $\tilde{N}$ is orthogonal to X. Thus, (32) follows from (35), and $\tilde{N}$ coincides with the orthogonal martingale N introduced in (29). Moreover, (35) implies that

$$V_t^* + M_t^A - A_t = \mathrm{E}^*[H] + \int_0^t \xi_s^* \, \mathrm{d}X_s + M_t + M_t^A$$
$$= \mathrm{E}[H] + \int_0^t (\xi^* + \xi^M + \xi^A)_s \, \mathrm{d}X_s + \tilde{N}_t,$$
$$0 \le t \le T,$$

is a right-continuous version of the martingale

$$V_t = \mathrm{E}[H | \mathscr{F}_t], \qquad 0 \le t \le T,$$

and this determines $\eta = V - \xi X$. □

Corollary 2. If both M and M^A are orthogonal to X, then we have $\xi = \xi^*$.

Proof. By Remark 4 we get $\langle M, X\rangle = \langle M^A, X\rangle = 0$, hence $\xi^M = \xi^A = 0$ P_X-a.s.. □

Remarks 5. (1) If X is a martingale with *continuous paths*, then $\langle M, X\rangle$ can be evaluated pathwise as a quadratic variation and coincides with $\langle N^*, X\rangle = 0$ P^*-a.s., hence P-a.s. This implies $\xi^M = 0$ P_X-a.s., hence

$$\xi = \xi^* + \xi^A. \tag{36}$$

(2) If P is a martingale measure in the stricter sense that it also preserves the martingale property of N^*, then we have $A = 0$, hence

$$\xi = \xi^* + \xi^M, \tag{37}$$

and

$$V_t = V_t^*. \tag{38}$$

If X has continuous paths then we can conclude, due to step (1), that the risk-minimizing strategy is completely preserved.

(3) An example in Section 5 will show that $\xi = \xi^*$ may occur even if the martingale property of N^* is lost under P.

5. *Two examples*

We illustrate the preceding results by two examples where the process X is a two-sided jump process. In both cases, the stock process X will be defined in terms of two independent Poisson processes N^+ and N^- with parameter

$\lambda^* > 0$ on some probability space $(\Omega, \mathscr{F}, P^*)$; $(\mathscr{F}_t)_{0 \le t \le T}$ will denote the smallest right-continuous family of σ-algebras which makes N^+ and N^- adapted.

Remark 6. The underlying stochastic model can be characterized as follows: The paths of $N^\pm$ are right-continuous and piecewise constant with jumps of size 1, and under P^* the two processes

$$M_t^\pm \equiv N_t^\pm - \lambda^* t, \qquad 0 \le t \le T, \tag{39}$$

are square-integrable martingales with

$$\langle M^\pm \rangle_t = \lambda^* t, \qquad \langle M^+, M^- \rangle = 0. \tag{40}$$

It is also well-known that M^+ and M^- form a *basis*, i.e., any square-integrable martingale with respect to P^* and $(\mathscr{F}_t)$ is of the form

$$M_t = M_0 + \int_0^t \xi_s^+ \, \mathrm{d}M_s^+ + \int_0^t \xi_s^- \, \mathrm{d}M_s^-, \tag{41}$$

where $\xi^\pm$ is the unique predictable process in $L^2(P^* \times \mathrm{d}t)$ such that

$$\langle M, M^\pm \rangle_t = \lambda^* \int_0^t \xi_s^\pm \, \mathrm{d}s. \tag{42}$$

In our *first example* we suppose that the stock process X is of the form

$$X = x_0 + N^+ - N^- = x_0 + M^+ - M^-. \tag{43}$$

Thus, X is a square-integrable martingale with

$$\langle X \rangle_t = 2\lambda^* t, \qquad 0 \le t \le T, \tag{44}$$

whose paths are piecewise constant with jumps of size ± 1.

Now let $H \in \mathscr{L}^2(P^*)$ be a contingent claim of the form

$$H = h(X_T). \tag{45}$$

The Markov property of X implies that the value process is of the form

$$V_t^*(\omega) = \mathrm{E}^*[H | \mathscr{F}_t](\omega) = v^*(X_t(\omega), t), \tag{46}$$

with

$$
\begin{aligned}
v^*(x,t) &= \sum_{y \in Z} h(x+y) P^*[(N_T^+ - N_t^+) - (N_T^- - N_t^-) = y] \\
&= \sum_{y \in Z} h(x+y) \sum_{\substack{k,l \geq 0 \\ k-l=y}} \mathrm{e}^{-2\lambda^*(T-t)} \frac{[\lambda^*(T-t)]^{k+l}}{k!l!}.
\end{aligned}
\tag{47}
$$

Putting

$$
\Delta^{\pm}(x,t) \equiv v^*(x \pm 1, t) - v^*(x,t), \tag{48}
$$

we obtain the following proposition.

Proposition 1. The risk-minimizing strategy (25) is determined by

$$
\xi_t^* = \frac{\Delta^+ - \Delta^-}{2}(X_{t-}, t), \tag{49}
$$

and the intrinsic risk process is given by

$$
R_t^* = \frac{\lambda^*}{2} \mathrm{E}^*\left[\int_t^T (\Delta^+ + \Delta^-)^2 (X_{s-}, x)\, \mathrm{d}s \,|\, F_t\right]; \tag{50}
$$

note that R_t^* can be calculated explicitly in the manner of (47).

Proof. (1) The process $Z \equiv M^+ + M^-$ is a square-integrable martingale with $\langle Z \rangle_t = 2\lambda^* t$, and Z is orthogonal to X since

$$
X \cdot Z = (M^+ - M^-)(M^+ + M^-) = (M^+)^2 - (M^-)^2
$$

is a martingale due to (39). In the representation (41), the basis (M^+, M^-) can thus be replaced by the basis (X, Z). In particular, we can represent the process $V_t^* = \mathrm{E}^*[H|\mathscr{F}_t]$, $0 \leq t \leq T$, in the form

$$
V_t^* = V_0^* + \int_0^t \xi_s^* \, \mathrm{d}X_s + \int_0^t \zeta_s^* \, \mathrm{d}Z_s, \tag{51}
$$

where ξ^* and ζ^* are determined by the equations

$$
\langle V^*, X \rangle_t = 2\lambda^* \int_0^t \xi_s^* \, \mathrm{d}s, \tag{52}
$$

and

$$\langle V^*, Z\rangle_t = 2\lambda^* \int_0^t \zeta_s^* \, ds. \tag{53}$$

By (22) and (24), the risk process is given by

$$\begin{aligned} R_t^* &= E^*\left[\int_t^T (\zeta_s^*)^2 \, d\langle Z\rangle_s | \mathscr{F}_t\right] \\ &= 2\lambda^* \, E^*\left[\int_t^T (\zeta_s^*)^2 \, ds | \mathscr{F}_t\right]. \end{aligned} \tag{54}$$

(2) In order to calculate ξ^* and ζ^* via (52) and (53), we introduce the quadratic variation processes

$$[V^*, X]_t \equiv \sum_{0 \le s \le t} (\Delta V^*)_s (\Delta X)_s,$$

and

$$[V^*, Z]_t \equiv \sum_{0 \le s \le t} (\Delta V^*)_s (\Delta Z)_s,$$

where we put

$$(\Delta X)_t(\omega) \equiv X_t(\omega) - X_{t-}(\omega), \quad \text{etc.}$$

The process $\langle V^*, X\rangle$ can be characterized as the unique predictable process such that $[V^*, X] - \langle V^*, X\rangle$ is a martingale; cf. Metivier (1982). But since

$$\begin{aligned} [V^*, X]_t &= \sum_{0 \le s \le t} \left(\Delta^+(X_{s-}, s)\Delta_s N^+ - \Delta^-(X_{s-}, s)\Delta_s N^-\right) \\ &= \int_0^t \Delta^+(X_{s-}, s) \, dM_s^+ - \int_0^t \Delta^-(X_{s-}, s) \, dM_s^- \\ &\quad + \lambda^* \int_0^t (\Delta^+ - \Delta^-)(X_{s-}, s) \, ds, \end{aligned}$$

we see that

$$\langle V^*, X\rangle_t = \lambda^* \int_0^t (\Delta^+ - \Delta^-)(X_{s-}, s) \, ds,$$

and this together with (52) implies (49). Since

$$[V^*, Z]_t = \sum_{0 \le s \le t} (\Delta^+(X_{s-}, s)\Delta_s N^+ + \Delta^-(X_{s-}, s)\Delta_s N^-),$$

we obtain in the same way

$$\zeta_t^* = \frac{\Delta^+ + \Delta^-}{2}(X_{t-}, t), \tag{55}$$

and this implies (50) due to (54). □

Remarks 7. (1) To replace P^* by an equivalent martingale measure P means that we replace λ^* by any $\lambda > 0$. Under P, the process $Z = M^+ + M^-$ is now of the form $Z = N + B$ with

$$N_t = N_t^+ + N_t^- - 2\lambda t, \qquad B_t = 2(\lambda - \lambda^*)t.$$

The martingale M is orthogonal to X and this implies $\xi^M = 0$ since $M_t = \int_0^t \zeta_s^* \, \mathrm{d}N_s$ is also orthogonal to X. Thus, (32) reduces to

$$\xi = \xi^* + \xi^A, \tag{56}$$

just as in the continuous case (36) with $A_t = 2(\lambda - \lambda^*)\int_0^t \zeta_s^* \, \mathrm{d}s$.

(2) By (56), the risk-minimizing strategy remains unchanged if and only if the martingale

$$M_t^A = 2(\lambda - \lambda^*)\,\mathrm{E}\left[\int_0^T \zeta_s^* \, \mathrm{d}s | \mathscr{F}_t\right]$$

is orthogonal to X. Consider, for example, the special case $H = X_T^2$. Since

$$\mathrm{E}^*[X_T^2 | \mathscr{F}_t] = X_t^2 + 2\lambda^*(T - t),$$

we obtain $\Delta^{\pm}(x, t) = \pm 2x + 1$, hence $\zeta_t^*(\omega) \equiv 1$ by (55). In particular, M^B is orthogonal to X, and the risk-minimizing strategy,

$$\xi_t(\omega) = 2X_{t-}(\omega), \tag{57}$$

does not depend on the specific choice of $\lambda > 0$. The value process $V_t = X_t^2 + 2\lambda(T - t)$ *does* depend on λ, and so does the intrinsic risk process $R_t = 2\lambda(T - t)$.

In our *second example* we assume that the security process X is governed by the stochastic differential equation

$$\mathrm{d}X = \delta X_{-}(\mathrm{d}N^{+} - \mathrm{d}N^{-}), \tag{58}$$

with some $\delta > 0$. For a given initial value $x_0 > 0$, (58) implies

$$X_t = x_0(1+\delta)^{N_t^+}(1-\delta)^{N_t^-}, \qquad 0 \leq t \leq T. \tag{59}$$

By (58), X is a square-integrable martingale with

$$\langle X \rangle_t = 2\delta^2\lambda \int_0^t X_{s-}^2 \,\mathrm{d}s, \qquad 0 \leq t \leq T.$$

The value process V^* associated to the contingent claim (45) is again of the form (46), now with the function

$$v^*(x,t) = \sum_{k,l \geq 0} h\left(x(1+\delta)^k(1-\delta)^l\right) \mathrm{e}^{-2\lambda^*(T-t)} \frac{[\lambda^*(T-t)]^{k+l}}{k!\,l!}. \tag{60}$$

Defining $\Delta^{\pm}$ as before in (48), we obtain the following proposition.

Proposition 2. The risk-minimizing strategy (24) is determined by

$$\xi_t^* = \frac{(\Delta^+ - \Delta^-)(X_{t-}, t)}{2\delta X_{t-}}, \tag{61}$$

and the intrinsic risk process is given by

$$R_t^* = 2\lambda\, \mathrm{E}^*\left[\int_t^T (\zeta_s^*)^2 \,\mathrm{d}s \,|\, \mathscr{F}_t\right] \tag{62}$$

where

$$\zeta_t^* = \frac{(\Delta^+ + \Delta^-)(X_{t-}, t)}{2\delta Z_{t-}}, \tag{63}$$

and

$$Z_t = (1+\delta)^{N_t^+ + N_t^-}\,\mathrm{e}^{-2\delta t}. \tag{64}$$

Proof. We have $\mathrm{d}X = \delta X_-(\mathrm{d}M^+ - \mathrm{d}M^-)$. The process Z defined in (64) is a solution of

$$\mathrm{d}Z = \delta Z_-(\mathrm{d}M^+ + \mathrm{d}M^-),$$

and this implies that X and Z are two orthogonal martingales which may be used as a basis. Proceeding exactly as in the proof of Proposition 1, we obtain

$$\langle V^*, X\rangle_t = \int_0^t \delta X_{s-}(\Delta^+ - \Delta^-)(X_{s-}, s)\,\mathrm{d}s,$$

and

$$\langle V^*, Z\rangle_t = \int_0^t \delta Z_{s-}(\Delta^+ + \Delta^-)(X_{s-}, s)\,\mathrm{d}s,$$

hence (61) and (63). □

References

Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, 637–654.

Cox, J. and S. Ross, 1976a, A survey of some new results in financial option pricing theory, The Journal of Finance 31, 383–402.

Cox, J. and S. Ross, 1976b, The valuation of options for alternative stochastic processes, Journal of Financial Economics 3, 145–166.

Egle, K. and S. Trautmann, 1981, On preference-dependent pricing of contingent claims, in: H. Göppl and R. Henn, eds., Geld, Banken und Versicherungen (Athenäum-Verlag, Königstein i.T.) 400–416.

Hakansson, N., 1979, The fantastic world of finance: Progress and the free lunch, Journal of Financial and Quantitative Analysis (Proceedings Issue) 14, 717–734.

Harrison, J. and D. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory 20, 381–408.

Harrison, J. and S. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, Stochastic Processes and their Applications 11, 215–260.

Harrison, J. and S. Pliska, 1983, A stochastic calculus model of continuous trading: Complete markets, Stochastic Processes and their Applications 15, 313–316.

Merton, R., 1973, Theory of rational option pricing, Bell Journal of Economics and Management Science 4, 141–183.

Metivier, M., 1982, Semimartingales: A Course on Stochastic Processes (de Gruyter, Berlin/New York).

Müller, S., 1984, Arbitrage und die Bewertung von contingent claims, Ph.D. thesis (University of Bonn, Bonn).

Schweizer, M., Varianten der Black–Scholes-Formel, Diplomarbeit (ETH Zürich, Zürich).

THE JOURNAL OF FINANCE • VOL. LII, NO. 1 • MARCH 1997

Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates

KRISTIAN R. MILTERSEN, KLAUS SANDMANN, and DIETER SONDERMANN

ABSTRACT

We derive a unified model that gives closed form solutions for caps and floors written on interest rates as well as puts and calls written on zero-coupon bonds. The crucial assumption is that simple interest rates over a fixed finite period that matches the contract, which we want to price, are log-normally distributed. Moreover, this assumption is shown to be consistent with the Heath–Jarrow–Morton model for a specific choice of volatility.

CLOSED FORM SOLUTIONS FOR interest rate derivatives, in particular caps, floors, and bond options, have been obtained by a number of authors for Markovian term structure models with normally distributed interest rates or alternatively log-normally distributed bond prices (see, for example, Jamshidian (1989, 1991a); Heath, Jarrow, and Morton (1992); Brace and Musiela (1994); Geman, El Karoui, and Rochet (1995)). These models support Black–Scholes type formulas most frequently used by practitioners for pricing bond options and swaptions. Unfortunately, these models imply negative interest rates with positive probabilities, and hence they are not arbitrage free in an economy with opportunities for riskless and costless storage of money. Briys, Crouhy, and Schöbel (1991) apply the Gaussian framework to derive closed form solutions for caps, floors, and European zero-coupon bond options. To exclude the influence of negative forward rates on the pricing of zero-coupon bond options, they introduce an additional boundary condition. As shown by Rady and Sandmann (1994) these pricing formulas are only supported by a term structure model with an absorbing boundary for the forward rate at zero, where the absorbing probability is not negligible, which for a term structure model is a quite problematic assumption.

* Miltersen is from Odense Universitet, Denmark. Sandmann is from Johannes-Gutenberg Universität, Mainz, and Sondermann is from Rheinische Friedrich-Wilhelms-Universität, Bonn, Germany. An earlier version of this article, Sandmann, Sondermann, and Miltersen (1994), was presented at the *Seventh Annual European Futures Research Symposium* in Bonn and at Norwegian School of Economics and Business Administration, Bergen, Norway. This version of the article was presented at the Isaac Newton Institute at Cambridge University during the Financial Mathematics Program. We are grateful to an anonymous referee and to Simon Babbs, Darrell Duffie, Claus Munk, Sven Rady, Erik Schlögl, and the two CBOT discussants, Chris Veld and Frans de Roon, for comments and assistance. Financial support from the Danish Natural and Social Science Research Councils for the first author and from the Deutsche Forschungsgemeinschaft at Rheinische Friedrich-Wilhelms-Universität Bonn for all three authors is gratefully acknowledged.

Alternatively, modeling log-normally distributed interest rates avoids the problems of negative interest rates. However, as shown by Morton (1988) and Hogan and Weintraub (1993), these rates explode with positive probability, implying zero prices for bonds and hence also arbitrage opportunities. Furthermore, so far, no closed form solutions are known for these models.

As observed by Sandmann and Sondermann (1994), the problems of exploding interest rates result from an unfortunate choice of compounding period of the interest rates modeled, namely the continuously compounding rate. Assuming that the continuously compounded interest rate is log-normally distributed results in "double exponential" expressions, i.e., the exponential function is itself an argument of an exponential function, thus giving rise to infinite expectations of the accumulation factor and of inverse bond prices under the martingale measure. The problem disappears as shown in Sandmann and Sondermann (1994) if, instead of assuming that the continuously compounded interest rates are log-normally distributed, one assumes that simple interest rates over a fixed finite period are log-normally distributed. In practice, interest rates, both spot and forward, are quoted as simple rates per annum (yearly), even if the finite period is different from one year, for example, three months. Moreover, effective annual rates[1] are calculated and used as *the* benchmark for comparing simple rates over different finite periods. Hence, simple interest rates over finite periods are directly observable in the market and form a natural starting point for modeling the term structure. We are aware of two alternative approaches that are similar to our approach and also avoid the problem of exploding rates: (i) Musiela (1994) models instantaneous forward rates with noncontinuous compounding as log-normal and finds the corresponding dynamics of the continuously compounding rates. (ii) Ho, Stapleton, Subrahmanyam, and Thanassoulas (1994) model "bankers discount" rates as log-normal. However, this latter approach implies negative bond prices with positive probability.

The main result of this article is a *unified* model that provides closed form solutions for interest rate caps and floors as well as puts and calls written on zero-coupon bonds within the context of a log-normal interest rate model. These solutions coincide with modifications of the Black–Scholes formula. In particular, for caps and floors with payment periods of the same length as the fixed period of the underlying simple interest rates we obtain the Black formula often used by market practitioners *without* making the unrealistic assumption that forward rates are independent of the accumulation process.[2] Thus, in this case our model supports market practice. For call and put options on zero-coupon bonds, our derived closed form solution matches the formula

[1] By effective annual rates we mean the annually compounded rate which yields the same return as the original rate compounded appropriately.

[2] Hull (1993, p.375).

derived in Käsler (1991).[3] Käsler (1991) derives the formula using no-arbitrage arguments on two bond prices only. In this article, we provide a supporting no-arbitrage term structure model. Moreover, the log-normal assumption is shown to be consistent with the Heath–Jarrow–Morton model for a specific choice of volatility structure.

The article is organized as follows. Section I presents the model. Solutions for interest rate derivatives are then derived in Section II. The relation to the Heath–Jarrow–Morton model is found in Section III, and a discussion of the limitations of the model is found in the conclusion, Section IV. Finally, some proofs are deferred to the Appendix.

I. A Model for Simple Forward Rates over a Fixed Period

Let $P(t, T)$ denote the price, at date t, of a (default-free) zero-coupon bond that pays one dollar at maturity date T. Let $f(t, T, \alpha)$ denote the simple *forward rate* at date t over a fixed period of length α prevailing at date t for the future time interval $[T, T + \alpha]$. That is,

$$P(t, T + \alpha) = P(t, T)\frac{1}{1 + \alpha f(t, T, \alpha)} . \tag{1}$$

The limit case $\alpha = 0$ corresponds to the continuously compounding forward rate. This rate has to be treated as a special case in the following way:

$$f(t, T, 0) \equiv \lim_{\alpha \to 0} f(t, T, \alpha),$$

deduced using l'Hospital's rule from the following definition of a continuously compounded forward rate:

$$f(t, T, 0) = -\frac{(\partial/\partial T)P(t, T)}{P(t, T)} .$$

Consider at date t an agreement between two parties to sell or buy the zero-coupon bond with maturity $T + \alpha$ at the future date T, which is known as a forward contract. The forward price $F(t, T, \alpha)$ of the contract is defined as the fixed price which the buyer agrees to pay at date T for the bond with maturity $T + \alpha$, such that the value of the forward contract at date t is zero. No-arbitrage implies

$$F(t, T, \alpha) = \frac{P(t, T + \alpha)}{P(t, T)} = \frac{1}{1 + \alpha f(t, T, \alpha)} . \tag{2}$$

[3] This formula is published in Käsler's Ph.D. dissertation written in German. The formula appears in the English manuscript Rady and Sandmann (1994), which is a comparative study of different bond based no-arbitrage models.

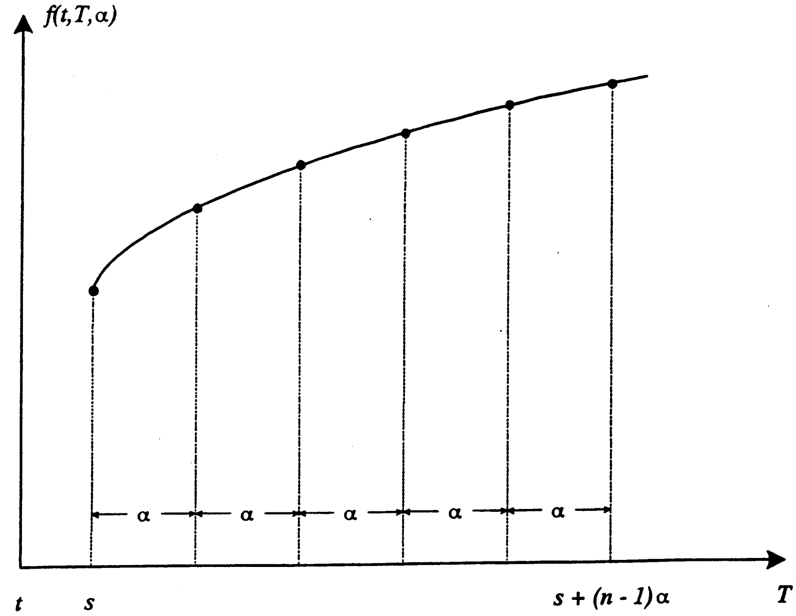


Figure 1. Curve of the simple forward rates with fixed period length α.

Note that, at each date t, bond prices, forward prices, and forward rates are related by

$$P(t, s + n\alpha) = P(t, s) \prod_{i=0}^{n-1} \frac{P(t, s + (i+1)\alpha)}{P(t, s + i\alpha)}$$

$$= P(t, s) \prod_{i=0}^{n-1} \frac{1}{1 + \alpha f(t, s + i\alpha, \alpha)}, \tag{3}$$

for $n = 1, \ldots,$ and $s \in [t, t + \alpha)$, where, for pure simplicity, we have chosen the same fixed period length α for each interval. Figure 1 shows the points on the forward rate curve used to price the bond with maturity $s + n\alpha$ in the above situation.

In our model, the stochastic behavior of the term structure of interest rates is determined by the simple forward rates. We assume that, at each date t, we observe several simple forward rates that are different with respect to the length and position of their compounding interval. The set of simple forward rates, at date t, can be expressed by the set of their compounding intervals

$\{[T_i, T_i + \alpha_i]\}_{i\in\mathbb{I}}$, where $T_i < T_j$, for $i, j \in \mathbb{I}$ such that $i < j$, $\alpha_i > 0$ for $i \in \mathbb{I}$, and $\mathbb{I}$ is an ordered index set that may be infinite.

In practice, this set of intervals is determined by the observed simple forward rates and therefore by maturities of existing bonds or interest rate futures. We model the processes of the simple forward rates in this set as log-normal diffusions,[4] i.e.,

$$df(\,\cdot\,, T, \alpha)_t = \mu(t, T, \alpha)f(t, T, \alpha)dt + \gamma(t, T, \alpha)f(t, T, \alpha)dW_t, \tag{4}$$

where $\{f(\cdot, T, \alpha)_t\}$ is initiated using the term structure of interest rates observable at date zero

$$f(0, T, \alpha) = \frac{1}{\alpha}\left(\frac{P(0, T)}{P(0, T + \alpha)} - 1\right).$$

No arbitrage implies that the set of forward rate processes, satisfying the above log-normal assumption, is restricted to those processes that cannot replicate each other. The mathematical reason is that the sum of log-normally distributed variables is itself not log-normally distributed. From the economic point of view, out of all traded simple forward rates we have to choose a nonredundant subset of processes that can be modeled as log-normal diffusions. We are free in the choice of this subset. For instance we can choose the forward rates given by three month Eurodollar futures contracts.[5] In this case T_i would correspond to the settlement date of the i-th contract, and $\alpha_i = T_{i+1} - T_i$ to the length of the period covered by this contract. Therefore, purely in order to simplify the notation, we sometimes take the length of the compounding period α as fixed.

The existence of a unique nonnegative solution of the stochastic differential equation, SDE (4), is proven (under suitable regularity conditions)[6] in Brace, Gatarek, and Musiela (1995).

To complete the model, consider the situation where SDE (4) is satisfied for all T and one fixed α. Then we have not only specified the stochastic model for the simple forward rates with compounding period α, but simultaneously we

[4] Under the usual regularity conditions we can extend this to a multidimensional Wiener process. Similar closed form solutions can be derived in this situation. For simplicity of exposition we are concentrating on the one-dimensional case. The Heath, Jarrow, and Morton (1992) model use the continuously compounded forward rates as a starting point, whereas our modeling assumption is based on the simple forward rates. The relationship between both approaches is discussed in Section III.

[5] A three month Eurodollar futures quote of 94.47 corresponds to a three month forward LIBOR rate of 5.53 (compare Hull (1993, p.99)). This is common market practice, i.e., the market neglects the stochastic effect of margin payments and thus the difference between a futures contract and a forward rate agreement (FRA) based on the futures quote. With reference to the discussion in Section II, it is an assumption of the model that the underlying interest rate is default free.

[6] Taking up the ideas from our model, Brace, Gatarek, and Musiela (1995, Theorem 2.1) have shown existence for a bounded and (piecewise) continuous volatility function $\gamma : \mathbb{R}_+^2 \to \mathbb{R}$. The existence of a solution to the SDE (4) under the original probability measure is well-known. The question solved in their article is the existence under the equivalent martingale measure.

have determined the stochastic model for all rates with any compounding period (including the continuously compounding rates) through the bond prices. That is, bond prices are calculated using equation (3). Given the bond prices, forward rates can be calculated with any different compounding period than the chosen α using equation (1). However, the domain of this stochastic description is only the time interval $[t + \alpha, \infty)$; compare Figure 1. That is, for rates with shorter compounding periods, β, than α, we have not determined the stochastic model of simple forward rates with compounding period β (including the continuously compounding rates, i.e., $\beta = 0$) in the time interval $[t, t + \alpha - \beta]$.

Using Itô's lemma on the forward price process from equation (2) gives

$$\begin{aligned} \mathrm{vol}(dF(\,\cdot\,, T, \alpha)_t) &= -F(t, T, \alpha)^2 \alpha \gamma(t, T, \alpha) f(t, T, \alpha) dW_t \\ &= -F(t, T, \alpha)(1 - F(t, T, \alpha))\gamma(t, T, \alpha) dW_t, \end{aligned} \tag{5}$$

where we are only calculating the diffusion part of the Itô processes in this article, since we know from Harrison and Kreps (1979) and Harrison and Pliska (1981) that the drift part will not play any role for the pricing of contingent claims. For that purpose, we have introduced the obvious notation "vol." That is, for the Itô process

$$dX_t = \xi(X_t, t)dt + \delta(X_t, t)dW_t \quad \text{we define} \quad \mathrm{vol}(dX_t) \equiv \delta(X_t, t)dW_t.$$

II. Closed Form Solutions for Interest Rate Derivatives

In this section, we focus on the arbitrage price of interest rate derivatives. More precisely, we consider two special interest rate derivatives: interest rate caps and floors, and European debt options where the underlying security is a zero-coupon bond. Since the construction of the underlying term structure model is very closely related to the Black–Scholes model, we should expect similar pricing formulas for these derivatives within our model.

Caps and floors are special types of options where a nominal interest rate is the underlying security. The underlying interest rate could be, for example, the three or six month London Inter Bank Offered Rate (LIBOR). A cap is an insurance against upward movements in the interest rate, and a floor is an insurance against downward movements in the interest rate. Let $\{r_t\}$ be a nominal interest rate process with compounding period α, for instance, for $\alpha = \frac{1}{4}$ the process $\{r_t\}$ is the quoted three-month LIBOR. It is an assumption of the model that the underlying interest rate, $f(\cdot, \cdot, \alpha)$, is default-free since it is used for pricing default-free bonds. In practice, the LIBOR is based on a "replenished" AA rate and, hence, not default-free. However, (i) assuming that the short position of the cap or floor contract has the same credit quality as the one on which LIBOR is based, and (ii) modeling the default risk as in Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1994), the same formulas apply with the volatility process adjusted to include the default spread on LIBOR. As it is shown in Duffie (1994), the volatility of the credit spread and of the default-free rate simply adds together to give the volatility of the defaultable rate. This result also applies to our model, SDE (4), with appropriate dynamics of the default risk. Duffie and Singleton (1994) then show that options, etc., written on defaultable interest rates can be priced using standard option pricing techniques, such as valuing expectations under an equivalent martingale measure or solving partial differential equations (PDE)s, by (i) simply substituting the default-free volatility with the volatility of the defaultable rate and (ii) using the defaultable rate as the short rate in the option pricing model.[7]

Let $t_0 < t_1 < \cdots < t_N$ be a set of dates and define $\alpha_i = t_{i+1} - t_i$. A cap contract with level L, face value V, underlying nominal interest rate process $\{r_t\}$, and payment dates $t_1, \ldots, t_N$ is defined by the payoff at all dates t_{i+1}

$$V\alpha_i[r_{t_i} - L]^+ \equiv V\alpha_i \max\{r_{t_i} - L, 0\},$$

if payments are made in arrear. A cap with one payment date is called a *caplet*. Since all caps are portfolios of caplets, we concentrate on pricing a caplet. Clearly, $r_{t_i} = f(t_i, t_i, \alpha_i)$. Since this rate is known at date t_i, the payoff of a caplet at date t_{i+1} is also known at date t_i, hence the present value of this payoff, at date t_i, is equal to

$$\begin{aligned}
P(t_i, t_{i+1})V\alpha_i[f(t_i, t_i, \alpha_i) - L]^+ \\
&= \frac{V}{1 + \alpha_i f(t_i, t_i, \alpha_i)}[1 + \alpha_i f(t_i, t_i, \alpha_i) - (1 + \alpha_i L)]^+ \\
&= V\left[1 - \frac{1 + \alpha_i L}{1 + \alpha_i f(t_i, t_i, \alpha_i)}\right]^+ \\
&= V(1 + \alpha_i L)\left[\frac{1}{1 + \alpha_i L} - F(t_i, t_i, \alpha_i)\right]^+. \qquad (6)
\end{aligned}$$

The floor is just the opposite contract, and the present value at date t_i is given by

$$V(1 + \alpha_i L)\left[F(t_i, t_i, \alpha_i) - \frac{1}{1 + \alpha_i L}\right]^+. \qquad (7)$$

The payoff of a cap or a floor, at each date t_i, is equivalent to $V(1 + \alpha_i L)$ times the payoff of a European put option or a European call option, respectively, with exercise date t_i, exercise price $K = 1/(1 + \alpha_i L)$, and a zero-coupon bond with maturity $t_{i+1} = t_i + \alpha_i$ as the underlying security. Thus, the arbitrage price of a cap or a floor is equal to the arbitrage price of a portfolio of European put options or European call options, respectively.

[7] We are indebted to Darrell Duffie for pointing this out to us.

PROPOSITION 1: *Consider a European call option with exercise price K and exercise date T written on a zero-coupon bond with maturity date $T+\alpha$. If the simple forward rate process $\{f(t, T, \alpha)\}_{t\in[0,T]}$ is log-normally distributed, i.e. satisfies SDE (4), then the arbitrage price is*

$$\begin{aligned}\text{Call} &= P(t, T+\alpha)N(e_1) - KP(t, T)N(e_2) - KP(t, T+\alpha)(N(e_1) - N(e_2)) \\ &= (1-K)P(t, T+\alpha)N(e_1) - K(P(t, T) - P(t, T+\alpha))N(e_2),\end{aligned} \tag{8}$$

with

$$e_{1,2} = \frac{1}{\sigma(t, T, \alpha)}\left(\ln \frac{P(t, T+\alpha)(1-K)}{(P(t, T) - P(t, T+\alpha))K} \pm \frac{\sigma^2(t, T, \alpha)}{2}\right),$$

$$\sigma^2(t, T, \alpha) = \int_t^T \gamma^2(s, T, \alpha)\, ds,$$

where $N(\cdot)$ denotes the standard normal distribution.

Proof: The proof consists of two steps. First, we consider a self-financing portfolio strategy on the bond market that duplicates the payoff of the European call option. The resulting partial differential equation will be solved in the second step.

Assume that there exists a self-financing portfolio strategy $(\phi^1, \phi^2) \equiv \{(\phi_t^1, \phi_t^2)\}_{t\in[0,T]}$ on the bond market with value process $V \equiv \{V_t\}_{t\in[0,T]}$. Then the dynamics of the value process, V, is according to Itô's lemma

$$dV_t = \phi_t^1 dP(t, T+\alpha) + \phi_t^2 dP(t, T) \qquad \text{with} \quad V_T = [P(T, T+\alpha) - K]^+.$$

By no arbitrage, the value process of the call option is then equal to the value process of the portfolio strategy. Consider instead the forward value process $\hat{V}$ defined by

$$\hat{V}_t \equiv \frac{V_t}{P(t, T)} = \phi_t^1 F(t, T, \alpha) + \phi_t^2.$$

If the portfolio strategy (ϕ^1, ϕ^2) is self-financing on the bond market we derive by Itô's lemma that

$$d\hat{V}_t = \phi_t^1 dF(\cdot, T, \alpha)_t \qquad \text{with} \quad \hat{V}_T = [P(T, T+\alpha) - K]^+ = [F(T, T, \alpha) - K]^+.$$

Thus, ϕ_t^1 can be interpreted as the number of T forward contracts to hold at date t committing us to buy, at time T, a zero-coupon bond with maturity $T + \alpha$. Define the forward price of the call option as

$$\hat{c}(t, F(t, T, \alpha)) \equiv \frac{1}{P(t, T)} \text{ Call.}$$

By no arbitrage, the forward value of the portfolio strategy is equal to the forward price of the call option. This corresponds exactly to a change of measure from the martingale measure to the T forward risk adjusted measure with the zero-coupon bond $P(\cdot, T)$ as numeraire (compare Geman (1989), Jamshidian (1991b), or Geman, El Karoui, and Rochet (1995)). That is,

$$\hat{V}_t = \phi_t^1 F(t, T, \alpha) + \phi_t^2 = \hat{c}(t, F(t, T, \alpha))$$

implying that

$$\begin{aligned} d\hat{V}_t &= \phi_t^1 dF(\,\cdot\,, T, \alpha)_t = d\hat{c}(\,\cdot\,, F(\,\cdot\,, T, \alpha))_t \\ &= \hat{c}_t(t, F)dt + \tfrac{1}{2}\hat{c}_{FF}(t, F)d\langle F(\cdot)\rangle_t + \hat{c}_F(t, F)dF. \end{aligned}$$

In particular, the self-financing portfolio strategy is determined by $\phi_t^1 = \hat{c}_F(t, F(t, T, \alpha))$ and $\phi_t^2 = \hat{c}(t, F(t, T, \alpha)) - \phi_t^1 F(t, F(t, T, \alpha))$. Furthermore, the forward price process of the call option is a solution of the PDE

$$\hat{c}_t(t, F(t, T, \alpha)) + \tfrac{1}{2}\hat{c}_{F,F}(t, F(t, T, \alpha))d\langle F(\,\cdot\,, T, \alpha)\rangle_t = 0, \tag{9}$$

with boundary condition $\hat{c}(T, F(T, T, \alpha)) = [F(T, T, \alpha) - K]^+$. From equation (5) the process of the quadratic variation of the forward price is known. Hence,

$$\begin{aligned} \hat{c}_t(t, F(t, T, \alpha)) + \tfrac{1}{2}\hat{c}_{F,F}(t, F(t, T, \alpha))\gamma^2(t, T, \alpha)F^2(t, T, \alpha)& \\ \cdot (1 - F(t, T, \alpha))^2 dt = 0.& \end{aligned} \tag{10}$$

The solution of the PDE (10) follows the presentation in Rady and Sandmann (1994). For completeness, we give the outline of the proof in the Appendix. Q.E.D.

Note that the self-financing portfolio strategy is given by

$$\begin{pmatrix} \phi_t^1 \\ \phi_t^2 \end{pmatrix} = \begin{pmatrix} (1 - K)N(e_1) + KN(e_2) \\ -KN(e_2) \end{pmatrix},$$

where ϕ_t^1 (ϕ_t^2) is the number of bonds with maturity $T + \alpha$ (T) to hold at date t. The equivalent hedge on the forward market, at any date $0 \le t \le T$, consists of holding a long position of ϕ_t^1 forward contracts with forward price $F(t, T, \alpha) = P(t, T + \alpha) \cdot P(t, T)^{-1}$ and holding $\hat{V}_t = V_t \cdot P(t, T)^{-1}$ bonds with maturity T. As shown in the proof of Proposition 1 this strategy is self-financing, duplicates the call value V_t, and requires no cash transactions between the start of the option and its settlement date. Using put-call parity, the value for the put option is

$$\text{Put} = K(P(t,T) - P(t,T+\alpha))N(-e_2) - (1-K)P(t,T+\alpha)N(-e_1), \quad (11)$$

where $N(\cdot)$, e_1, e_2, and σ are as defined in Proposition 1. For the put option, the self-financing portfolio strategy on the bond market is given by $1 - \phi_t^1$ and $1 - \phi_t^2$, respectively.

We have written two versions of the closed form solutions in equation (8). The first version has three terms, whereas the first two terms look similar to the Black–Scholes formula (but note that e_1 and e_2 are not the usual arguments of $N(\cdot)$), and then there is a third correction term. The second version is in structure a Black–Scholes formula, where $(1 - K)P(t, T + \alpha)$ should be interpreted as the price of the underlying security and $K(P(t, T) - P(t, T + \alpha))$ as the present value of the exercise price.

A closed form solution of the type equation (8) was first derived by Käsler (1991) under specific assumptions for the two underlying zero-coupon bonds. A discussion of this model relative to other bond price based models[8] can be found in Rady and Sandmann (1994). In a pure probabilistic framework for zero-coupon bonds, Rady (1995) has recently derived the same pricing formulas for zero-coupon bond options using the change of measure technique. This approach is based on the fact that under T forward risk adjusted measure, Q_T, the forward price of the call option is a martingale; i.e.,

$$\hat{c}(t,x) = E_{Q_T}[[F(T,T,\alpha) - K]^+ | F(t,T,\alpha) = x].$$

The main difficulty is to determine the transition density $q(v; t, x)$. Given the solution of the PDE (10) (see the Appendix) the transition density for $v \in]0, 1[$ is equal to

$$q(v;t,x) \equiv \frac{1}{\sqrt{2\pi\sigma^2(t)}} \cdot \frac{x}{v^2(1-v)} \cdot \exp\left(-\frac{[\ln((1-v)/v) - \ln((1-x)/x) + \tfrac{1}{2}\sigma^2(t)]^2}{2\sigma^2(t)}\right) \quad (12)$$

with

$$\sigma^2(t) = \int_t^T \gamma^2(u, T, \alpha)\, du.$$

[8] By the term bond price based model we understand a model in which the bond prices are the exogeneously given basic underlying variables. This is in contrast to the model of this article—and many other models—where the exogeneously given basic underlying variable is the term structure of interest rates from which the bond prices are then given endogeneously.

Applying the substitution $u = (1 - v)/v$ with $du = -(1/v^2)\,dv$ and $(u + 1)/u = 1/(1 - v)$ yields that

$$\int_0^1 q(v; t, x)\,dv$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(t)}} \cdot x \cdot \frac{1+u}{u} \cdot \exp\left(-\frac{[\ln(u) - \ln((1-x)/x) + \tfrac{1}{2}\sigma^2(t)]^2}{2\sigma^2(t)}\right) du = 1,$$

which implies that $q(v; t, x)$ is indeed a density. Furthermore, the forward price $F(t, T, \alpha)$ is a martingale under Q_T; i.e. $x = E_{Q_T}[F(T, T, \alpha) \mid F(t, T, \alpha) = x]$, and the PDE (10) is satisfied by the function $\hat{c}(t, x) = \int_0^1 [v - K]^+ q(v; t, x)dv$. Since x is equal to the forward price at time t; i.e. $x = 1/[1 + \alpha f(t, T, \alpha)]$ the substitution $\rho = \ln[(1 - v)/\alpha v]$ yields

$$\hat{c}(t, x) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \int_{-\infty}^{+\infty} \left[\frac{1}{1 + \alpha e^{\rho}} - K\right]^+ \frac{(1 + \alpha e^{\rho})}{(1 + \alpha f)} \cdot \exp\left\{-\frac{(\rho - \ln f + \tfrac{1}{2}\sigma^2(t))^2}{2\sigma^2(t)}\right\} d\rho.$$

The solution can be calculated in the same way as the one in the Appendix.[9] We can now apply Proposition 1 and equation (11) to the pricing of interest rate caps and floors.

Proposition 2: *Consider a cap with interest rate level L, face value V, and (arrear) payment dates $t_1, \ldots, t_N$. Define $\alpha_i = t_{i+1} - t_i$, for $i = 0, \ldots, N - 1$. If the simple forward rate processes $\{f(t, t_i, \alpha_i)\}_{t \in [0, t_i]}$ satisfy the SDE (4), then the arbitrage price of the cap at date $t \le t_0 = t_1 - \alpha_0$ is*

$$\text{Cap} = V \sum_{i=0}^{N-1} \alpha_i P(t, t_{i+1})(f(t, t_i, \alpha_i)N(d_1(t, t_i, \alpha_i)) - LN(d_2(t, t_i, \alpha_i))), \tag{13}$$

[9] This is not only an alternative way to prove Proposition 1. The arbitrage price of any European type contingent claim with Borel measurable payoff function $h(\cdot)$ at time T on the zero coupon bond with maturity $T + \alpha$ is determined by

$$P(t, T + \alpha) \cdot E_{Q_T}[h(F(T, T, \alpha)) | F(t, T, \alpha) = x] = P(t, T + \alpha) \cdot \int_0^1 h(v) q(v; t, x)\,dv.$$

Hence, the pricing of such claims is reduced to the (numerical) calculation of a one-dimensional integral equation.

with

$$d_{1,2}(t, s, \alpha) = \frac{1}{\sigma(t, s, \alpha)}\left(\ln \frac{f(t, s, \alpha)}{L} \pm \frac{\sigma^2(t, s, \alpha)}{2}\right),$$

$$\sigma^2(t, s, \alpha) = \int_t^s \gamma^2(u, s, \alpha)\, du,$$

where $N(\cdot)$ denotes the standard normal distribution.

Proof: By arbitrage, we can reduce the problem to the pricing of a caplet with arrear payment date $t_{i+1} = t_i + \alpha_i$. From equation (6), the payoff of the caplet is equivalent to $V(1 + \alpha_i L)$ times the payoff of a European put option with exercise price $1/(1 + \alpha_i L)$, where the underlying security is a zero-coupon bond with maturity date t_{i+1}. Then, according to equation (11), the arbitrage price of the caplet is

$$\begin{aligned}
\text{Caplet} = V(1 + \alpha_i L)\Bigg(&\frac{1}{1 + \alpha_i L}(P(t, t_i) - P(t, t_{i+1}))N(-e_2) \\
&- \left(1 - \frac{1}{1 + \alpha_i L}\right)P(t, t_{i+1})N(-e_1)\Bigg) \\
= VP(t, t_{i+1})&\left(\left(\frac{P(t, t_i)}{P(t, t_{i+1})} - 1\right)N(-e_2) - \alpha_i L N(-e_1)\right) \\
= VP(t, t_{i+1})&(\alpha_i f(t, t_i, \alpha_i)N(-e_2) - \alpha_i L N(-e_1)),
\end{aligned} \tag{14}$$

with

$$\begin{aligned}
-e_2 &= \frac{-1}{\sigma(t, t_i, \alpha_i)}\left(\ln \frac{P(t, t_{i+1})(1 - 1/(1 + \alpha_i L))}{(P(t, t_i) - P(t, t_{i+1}))[1/(1 + \alpha_i L)]} - \frac{\sigma^2(t, t_i, \alpha_i)}{2}\right) \\
&= \frac{-1}{\sigma(t, t_i, \alpha_i)}\left(\ln \frac{\alpha_i L}{\alpha_i f(t, t_i, \alpha_i)} - \frac{\sigma^2(t, t_i, \alpha_i)}{2}\right) = d_1(t, t_i, \alpha_i)
\end{aligned}$$

By summing the respective caplets, this yields the pricing formula for a cap. Q.E.D.

The hedge strategy for each caplet is determined by a long and a short position on the bond market. For the caplet with payment date t_{i+1} the hedge strategy for $t \le t_i$ is equal to a short position ϕ_t^1 in the bond with maturity t_{i+1} and a long position ϕ_t^2 in the bond with maturity t_i, with

$$\begin{pmatrix} \phi_t^1 \\ \phi_t^2 \end{pmatrix} = \begin{pmatrix} -V[N(d_1(t, t_i, \alpha_i)) + \alpha_i L N(d_2(t, t_i, \alpha_i))] \\ VN(d_1(t, t_i, \alpha_i)) \end{pmatrix}.$$

An equivalent hedge on the forward market is to hold a short position of ϕ_t^1 forward contracts settled at t_i on the bond maturing at $t_i + \alpha_i$, and to invest the caplet premium into bonds with maturity t_i. Again, no cash transactions are needed between 0 and t_i.

A similar proof gives, under the same conditions as in Proposition 2, that the price of a floor is

$$\text{Floor} = V \sum_{i=0}^{N-1} \alpha_i P(t, t_{i+1})(LN(-d_2(t, t_i, \alpha_i)) - f(t, t_i, \alpha_i)N(-d_1(t, t_i, \alpha_i))), \tag{15}$$

with $d_{1,2}$ and σ are as defined in Proposition 2.

The formulas (13) and (15) coincide with the formulas frequently used by market practioners for the pricing of caps and floors. So far, there was no convincing theoretical argument justifying this practice. In particular, it was an open question whether there exists a consistent term structure model in which these formulas are valid, ruling out arbitrage across different maturities.

The proof of Proposition 2 uses the relationship between a caplet and a European zero coupon put option. Alternatively, one can calculate the arbitrage price of a caplet directly under the T forward risk adjusted measure Q_T. The transition probability $q(v; t, x)$, where x is the time t forward price is given by equation (12). With $x = 1/(1 + \alpha f)$ and the substitution $\rho = \ln[(1 - v)/\alpha v]$ this implies:

$$\begin{aligned}
\text{Caplet} &= P(t, t_i)V(1 + \alpha_i L)E_{Q_T}\left[\left[\frac{1}{1 + \alpha L} - F(T, T, \alpha)\right]^+ \middle| F(t, T, \alpha) = x\right] \\
&= P(t, t_i)V\int_{-\infty}^{+\infty}\left[1 - \frac{1 + \alpha L}{1 + \alpha e^\rho}\right]^+ \frac{1 + \alpha e^\rho}{1 + \alpha f} \exp\left\{-\frac{(\rho - \ln f + \tfrac{1}{2}\sigma^2)^2}{2\sigma^2}\right\} d\rho \\
&= \frac{P(t, t_i)V\alpha}{1 + \alpha f}\int_{\ln L}^{+\infty} (e^\rho - L)\exp\left\{-\frac{(\rho - \ln f + \tfrac{1}{2}\sigma^2)^2}{2\sigma^2}\right\} d\rho \\
&= P(t, t_{i+1})V\alpha[fN(d_1) - LN(d_2)]
\end{aligned}$$

where, for simplicity, we have set

$$f \equiv f(t, T, \alpha), \qquad \sigma^2 \equiv \int_t^T \gamma^2(u, T, \alpha)\, du \qquad \text{and} \quad d_{1,2} \equiv \frac{1}{\sigma}\left(\ln\left(\frac{f}{L}\right) \pm \frac{\sigma^2}{2}\right).$$

However, under Q_T the simple forward rate $f(t, T, \alpha)$ is not a martingale since

$$E_{Q_T}[f(T, T, \alpha)|F(t, T, \alpha) = x] = f(t, T, \alpha) \cdot \frac{1 + \alpha f(t, T, \alpha)e^{\sigma^2(t,T,\alpha)}}{1 + \alpha f(t, T, \alpha)}. \tag{16}$$

In order to clarify the relation between the different forward rates and to strengthen the intuition behind these formulas we pick up a suggestion made by a referee of this journal. It was observed by the referee (and by Brace et al. (1995)) that a probabilistic proof of the cap pricing formula—without recurrence to the PDE (10) and the put option formula (11)—may be obtained as follows: "show that under an appropriate change of measure the caplet formula (14) is the expected value of its payoff at maturity under this measure. . . . This involves identifying the change of measure; explicitly identifying the distribution under the change of measure; and explicitly identifying how to evaluate the integral. . . . Given expression (4), show the form of equation (4) under the change of measure. Show how to compute the expression for the cap."[10] Let

$$df(t, T, 0) = \mu(t, T, f)dt + \eta(t, T, f)dW_t \tag{17}$$

be the corresponding Heath, Jarrow, and Morton (1992) model for the continuously compounding forward rates (compare Section III). Denote by P^* the risk-neutral probability measure of this model, and by $E^*[\cdot|\mathcal{F}_t]$ the conditional expectation under P^* with respect to the σ-algebra $\mathcal{F}_t$ at date t (compare Heath, Jarrow, and Morton (1992)). Then $f(t, t, 0)$ is the continuously compounding spot rate. It is well-known that under the risk-neutral measure P^* the value $c(t)$ of the caplet (with face value one dollar), at any date $t \leq T$, is given by

$$c(t) = E^*\left[\exp\left\{-\int_t^{T+\alpha} f(u, u, 0)\, \mathrm{d}u\right\}\cdot \alpha \cdot [f(T, T, \alpha) - L]^+ |\mathcal{F}_t\right]. \tag{18}$$

This expression can be evaluated by a change of measure from P^* to the $(T + \alpha)$ forward risk adjusted measure $Q_{T+\alpha}$

$$Q_{T+\alpha} = (P(0, T + \alpha) \cdot \beta(T + \alpha))^{-1} P^*,$$

where $\beta(T + \alpha) = \exp \int_0^{T+\alpha} f(u, u, 0)\, \mathrm{d}u$ is the accumulation factor up to date $T + \alpha$ (see for example Geman (1989), Jamshidian (1991a) or Geman, El Karoui, and Rochet (1995)). Denoting by $E_{T+\alpha}[\cdot|\mathcal{F}_t]$ the conditional expectation under $Q_{T-\alpha}$, the expression (18) transforms to

$$c(t) = P(t, T + \alpha)E_{T+\alpha}[\alpha[f(T, T, \alpha) - L]^+|\mathcal{F}_t].$$

According to the following Lemma, $f(t, T, \alpha)$ is a log-normal martingale under $Q_{T+\alpha}$. Hence Black's formula gives immediately

$$c(t) = \alpha \cdot P(t, T + \alpha)\{f(t, T, \alpha) \cdot N(d_1) - L \cdot N(d_2)\}$$

with $d_{1,2}$ as defined in Proposition 2.

[10] Quotations from a referee's report on an earlier version of this article.

LEMMA 3: *The forward rate process equation (4) satisfies the stochastic differential equation*

$$\frac{df(\,\cdot\,, T, \alpha)_t}{f(t, T, \alpha)} = \gamma(t, T, \alpha)dW_{T+\alpha}(t),$$

where $W_{T+\alpha}(t)$ is a Wiener process under the $(T + \alpha)$ forward measure $Q_{T+\alpha}$. (Proof: see the Appendix.)

III. The Corresponding Term Structure Model for the Continuously Compounding Interest Rates

We want to show how to specify the volatility of the Heath–Jarrow–Morton model such that this model will give the crucially needed log-normal simple forward rates. In the Heath–Jarrow–Morton model the continuously compounding forward rate, $\{f(t, T, 0)\}_{t\in[0,T]}$, for $T \in [0, \tau]$, is the basic modeling element. This process is modeled as an Itô process in the following way

$$df(\,\cdot\,, T, 0)_t = \mu(t, T, f(t, T, 0))dt + \eta(t, T, f(t, T, 0))dW_t.$$

The relation between the continuously compounding forward rates and the simple forward rates over a fixed period α is given by

$$\frac{1}{1 + \alpha f(t, T, \alpha)} = F(t, T, \alpha) = \exp\left(-\int_T^{T+\alpha} f(t, s, 0)ds\right) \qquad t \le T.$$

On the first hand, define $Y(t, T, \alpha) = -\ln F(t, T, \alpha)$ then

$$\frac{\partial}{\partial T} Y(t, T, \alpha) = f(t, T + \alpha, 0) - f(t, T, 0). \tag{19}$$

On the other hand, $Y(t, T, \alpha) = \ln(1 + \alpha f(t, T, \alpha))$, therefore,

$$\frac{\partial}{\partial T} Y(t, T, \alpha) = \frac{1}{1 + \alpha f(t, T, \alpha)}\, \alpha f_T(t, T, \alpha) = F(t, T, \alpha)\alpha f_T(t, T, \alpha), \tag{20}$$

where $f_T(t, T, \alpha)$ denotes $(\partial/\partial T)f(t, T, \alpha)$. Combining equations (19) and (20) yields

$$f(t, T + \alpha, 0) - f(t, T, 0) = \alpha F(t, T, \alpha)f_T(t, T, \alpha). \tag{21}$$

Solving the simple difference equation (21) gives

$$f(t, s + n\alpha, 0) = f(t, s, 0) + \sum_{i=0}^{n-1} \alpha F(t, s + i\alpha, \alpha) f_T(t, s + i\alpha, \alpha), \qquad s \in [t, t + \alpha), \quad (22)$$

with initial condition

$$\int_t^{t+\alpha} f(t, s, 0)\, ds = \ln(1 + \alpha f(t, t, \alpha)). \quad (23)$$

This is compatible with our earlier findings in Section I; that is, when specifying the Itô process of the simple forward rates with fixed period length α, we do not specify the continuously compounding interest rates in the time interval $[t, t + \alpha]$. So any (nonnegative) value of the continuously compounding forward rate in that interval, fulfilling the initial condition (23), is valid, because the continuously compounded rates specified in the time interval $[t + \alpha, \tau]$ by equation (22) take care of integrating up to the right bond prices. Again, the reader is referred to Figure 1 to get the intuition.

To find the volatility of the corresponding continuously compounding interest rates we just have to find $\mathrm{vol}(dX(\cdot, T, \alpha)_t)$, where

$$X(t, T, \alpha) = \alpha f_T(t, T, \alpha) F(t, T, \alpha)$$

and then use equation (22). We already know $\mathrm{vol}(dF(\cdot, T, \alpha)_t)$ from equation (5). Moreover, using the Itô process description from equation (4) and a result from Fernique et al. (1983, Chapter 2), $\mathrm{vol}(df_T(\cdot, T, \alpha)_t)$ can be calculated as

$$\begin{aligned}\mathrm{vol}(df_T(\cdot, T, \alpha)_t) &= \frac{\partial}{\partial T}(f(t, T, \alpha)\gamma(t, T, \alpha))dW_t \\ &= (f_T(t, T, \alpha)\gamma(t, T, \alpha) + f(t, T, \alpha)\gamma_T(t, T, \alpha))dW_t. \quad (24)\end{aligned}$$

Finally, using Itô's lemma and the relation $\alpha \cdot f_T(t, T, \alpha) \cdot F(t, T, \alpha) = 1 - F(t, T, \alpha)$

$$\begin{aligned}\mathrm{vol}(dX(\cdot, T, \alpha)_t) &= \alpha(-f_T(t, T, \alpha)F(t, T, \alpha)(1 - F(t, T, \alpha))\gamma(t, T, \alpha) \\ &\quad + F(t, T, \alpha)(f_T(t, T, \alpha)\gamma(t, T, \alpha) + f(t, T, \alpha)\gamma_T(t, T, \alpha)))dW_t \\ &= (\alpha f_T(t, T, \alpha)F^2(t, T, \alpha)\gamma(t, T, \alpha) \\ &\qquad + (1 - F(t, T, \alpha))\gamma_T(t, T, \alpha))dW_t \\ &= (-F_T(t, T, \alpha)\gamma(t, T, \alpha) + (1 - F(t, T, \alpha))\gamma_T(t, T, \alpha))dW_t \\ &= \frac{\partial}{\partial T}((1 - F(t, T, \alpha))\gamma(t, T, \alpha))dW_t. \quad (25)\end{aligned}$$

Using Itô's lemma on equation (22) and the result of equation (25) yields the volatility of the corresponding continuously compounding forward rate model. However, it should be emphasized that this volatility process of the Heath–Jarrow–Morton model is state dependent.

IV. Conclusion

Under the assumption of log-normality of simple rates, we have derived intuitive closed form solutions for pricing and hedging caps and floors in a consistent arbitrage-free term structure model of the Heath–Jarrow–Morton type. These formulas support common market practice to price caplets by a naive application of Black's formula on interest rates. However, there is another common market practice that is inconsistent with the caplet formula: namely, to apply Black's formula also to the pricing of bond options and swaptions. We have shown how the formula for calls and puts on zero coupon bonds has to be modified in order to be consistent with the caplet formula.

However, our model does not strictly support the application of the Black type formulas to all interest rate derivatives in large portfolios. Recall the assumption of log-normality of the α_i simple forward rates, where we are free to choose the interval lengths α_i. In practice, one would choose these intervals according to the most liquid markets for forward rates; for instance first model the three-month forward rates using either the Eurodollar futures—or the FRA—market for the first two or three years, then space the α_i according to market information on longer forward rates. But, if for example, two consecutive three-month rates are log-normal, the six-month rate for the same period cannot be log-normal at the same time. Thus, if we have priced the two three-month caplets consistently, the caplet formula for the six-month caplet must be considered with caution, since Black's formula will not give the exact arbitrage-free price. But the Heath–Jarrow–Morton solution to our term structure model is valid for *all* forward rates and provides the exact no-arbitrage price for any composed or interpolated rate. However, this solution may require numerical techniques.

In general, if some of the needed forward rate processes are replicable, there are two ways to proceed: either (i) one can solve the corresponding PDE by numerical procedures, or (ii) one can inconsistently assume that the true underlying simple interest rate process is log-normally distributed. Surely, at first glance (ii) is questionable. However, the question that arises is: how big is the mispricing if, in spite of the inconsistency, one uses the closed form solutions to price different options? Further research is needed to measure the size of this problem. This mispricing should be counterbalanced with the extra calculations needed to do numerical procedures.

The second problem is analogous to the problem of using the Black–Scholes formula on individual assets simultaneously while using the Black–Scholes formula on an arithmetic index of the same assets. An inconsistency problem that practitioners do not care much about, because the magnitude of this problem is smaller than many other theoretically inconsistent problems of using the Black–Scholes formula.

Appendix

Solution of the PDE (10):

Proof: Given the assumptions of Proposition 1, we have to solve the PDE (10) on $[0, T] \times (0, 1)$ where, for simplicity, we omit the period length α, i.e.,

$$\hat{c}_t(t, x) + (1/2)\gamma^2(t, T)x^2(1 - x)^2\hat{c}_{xx}(t, x) = 0$$

with

$$\hat{c}(T, x) = [x - K]^+, \qquad x \in [0, 1],$$

where $\hat{c}(t, x)$ is the date T forward value of the option contract. This problem is transformed by introducing the new time variable

$$s = s(t, T) = \int_t^T \gamma^2(r, T)\, dr$$

and the new space variable

$$z = \ln \frac{x}{1 - x} \Leftrightarrow x = \frac{1}{1 + \exp(-z)}$$

and, finally, setting $\hat{c}(t, x) = a(z)b(s)h(s, z)$. The idea is now to choose differentiable functions $a(\cdot)$ and $b(\cdot)$ in such a way that any solution $h(\cdot, \cdot)$ of the heat equation yields a solution $\hat{c}(\cdot, \cdot)$ of the original PDE. As shown in Rady and Sandmann (1994) this can be done by setting

$$a(z) \equiv \frac{1}{\exp(z/2) + \exp(-z/2)} \qquad \text{and} \quad b(s) = e^{-s/8}.$$

That is,

$$\hat{c}(t, x) = \frac{1}{\exp(z/2) + \exp(-z/2)}\, e^{-s/8} h(s, z).$$

The transformed problem on $[0, T] \times \mathbb{R}$ is

$$\frac{1}{2} h_{zz} - h_s = 0, \qquad \text{with} \quad h(0, z) = (e^{z/2} + e^{-z/2})\left[\frac{1}{1 + \exp(-z)} - K\right]^+.$$

This equation is known as the Heat Equation (compare for example Merton (1973, p. 225) or McKean (1965)) with the solution

$$h(s,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(0, z+\rho\sqrt{s})e^{-\rho^2/2}d\rho$$

$$= \frac{1}{\sqrt{2\pi}}\int_{(1/\sqrt{s})(\ln[K/(1-K)]-z)}^{\infty} (e^{(z+\rho\sqrt{s})/2} + e^{-(z+\rho\sqrt{s})/2}) \cdot \left(\frac{1}{1+\exp(-(z+\rho\sqrt{s}))} - K\right)e^{-\rho^2/2}d\rho$$

$$=(1-K)I_1 - KI_2,$$

with

$$I_1 = \frac{1}{\sqrt{2\pi}}\int_{(1/\sqrt{s})(\ln[K/(1-K)]-z)}^{\infty} e^{(z+\rho\sqrt{s})/2}e^{-\rho^2/2}\,d\rho = e^{z/2}e^{s/8}N\left(\frac{1}{\sqrt{s}}\left(z+\ln\frac{1-K}{K}+\frac{s}{2}\right)\right),$$

$$I_2 = \frac{1}{\sqrt{2\pi}}\int_{(1/\sqrt{s})(\ln[K/(1-K)]-z)}^{\infty} e^{-(z+\rho\sqrt{s})/2}e^{-\rho^2/2}\,d\rho = e^{-z/2}e^{s/8}N\left(\frac{1}{\sqrt{s}}\left(z+\ln\frac{1-K}{K}-\frac{s}{2}\right)\right).$$

Therefore,

$$\hat{c}(t,x) = \frac{\exp(-s/8)}{\exp(z/2)+\exp(-z/2)}h(s,z)$$

$$= (1-K)\underbrace{\frac{\exp(z/2)}{\exp(z/2)+\exp(-z/2)}}_{=x}N(e_1) - K\underbrace{\frac{\exp(-z/2)}{\exp(z/2)+\exp(-z/2)}}_{=1-x}N(e_2)$$

and since

$$P(t, T+\alpha) = P(t,T)F(t,T,\alpha) = P(t,T)\frac{1}{1+\alpha f(t,T,\alpha)},$$

the spot arbitrage price of the European call option is

$$\text{Cal} = P(t,T)\hat{c}(t,F(t,T,\alpha)).$$

Q.E.D.

Proof of Lemma 3

Proof: From the Heath–Jarrow–Morton model we know that

$$\frac{dP(\cdot,T)_t}{P(t,T)} = f(t,t,0)dt - \delta(t,T)dW_t,$$

where W is a Wiener process under P^* and

$$\delta(t, T) = \int_t^T \eta(t, u)\, du.$$

Using $P(t, T)$ as numeraire (i.e., dollars delivered at date T), Girsanov's theorem implies that

$$W_T(t) = W(t) + \int_0^t \delta(u, T)\, du$$

is a Wiener process under Q_T (compare for example Duffie (1992) or Karatzas and Shreve (1988)). Itô's lemma applied to the T forward price process $F(t, T, \alpha) = P(t, T + \alpha)/P(t, T)$ gives

$$\begin{aligned}\frac{dF(\,\cdot\,, T, \alpha)_t}{F(t, T, \alpha)} &= (\delta(t, T) - \delta(t, T + \alpha))(dW(t) + \delta(t, T)dt)\\ &= -\int_T^{T+\alpha} \eta(t, u)\, du\, dW_T(t)\end{aligned} \tag{A1}$$

and by comparison of vol(dF) with equation (5)

$$\int_T^{T+\alpha} \eta(t, u)\, du = (1 - F(t, T, \alpha)) \cdot \gamma(t, T, \alpha).$$

Since

$$f(t, T, \alpha) = \frac{1}{\alpha}\left(\frac{1}{F(t, T, \alpha)} - 1\right),$$

Itô's lemma in connection with equation (A1) implies

$$\begin{aligned}df &= \frac{1}{\alpha}\left(-\frac{1}{F^2}\, dF + \frac{1}{F^3}\, d\langle F\rangle\right)\\ &= \frac{1}{\alpha F}\left(\int_T^{T+\alpha} \eta(t, u)\, du\, dW_T + \left(\int_T^{T+\alpha} \eta(t, u)\, du\right)^2 dt\right)\\ &= \frac{1}{\alpha F}\int_T^{T+\alpha} \eta(t, u)\, du\, dW_{T+\alpha} = \frac{1 - F}{\alpha f}\, \gamma(t, T, \alpha)\, dW_{T+\alpha}\\ &= f \cdot \gamma(t, T, \alpha)\, dW_{T+\alpha}.\end{aligned}$$

Q.E.D.

REFERENCES

Brace, Alan, Dariusz Gatarek, and Marek Musiela, 1995, The market model of interest rate dynamics, *Mathematical Finance*, forthcoming.

Brace, Alan, and Marek Musiela, 1994, A multifactor Gauss Markov implementation of Heath, Jarrow, Morton, Working paper, University of New South Wales, Sydney, Australia.

Briys, Eric, Michel Crouhy, and Rainer Schöbel, 1991, The pricing of default-free interest rate cap, floor, and collar agreements, *The Journal of Finance* 46, 1879–1892.

Duffie, James Darrell, 1992, *Dynamic Asset Pricing Theory* (Princeton University Press, Princeton, New Jersey).

Duffie, James Darrell, 1994, Forward rate curves with default risk, Working paper, Stanford University, Stanford, California.

Duffie, James Darrell, Mark Schroder, and Costis Skiadas, 1994, Recursive valuation of defaultable securities and the timing of resolution of uncertainty, Working paper, Stanford University, Stanford, California.

Duffie, James Darrell, and Kenneth J. Singleton, 1994, Econometric modeling of term structures of defaultable bonds, Working paper, Stanford University, Stanford, California.

Fernique, Xavier, P. Warwick Millar, Daniel W. Stroock, and Michel Weber, 1983, *Ecole d'Eté de Probabilités de Saint-Flour XI - 1981*, volume 976 of *Lecture Notes in Mathematics* (Springer-Verlag, Berlin, Germany).

Geman, Hélyette, 1989, L'Importance de la probabilité "forward neutie" dans une approache stochastique des taux d'Intérêt, Working paper, ESSEC.

Geman, Hélyette, Nicole El Karoui, and Jean-Charles Rochet, 1995, Changes of numeraire, changes of probability measure and option pricing, *Journal of Applied Probability* 32, 443–458.

Harrison, Michael J., and David M. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* 20, 381–408.

Harrison, Michael J., and Stanley R. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* 11, 215–260.

Harrison, Michael J., and Stanley R. Pliska, 1983, A stochastic calculus model of continuous trading: Complete markets, *Stochastic Processes and their Applications* 15, 313–316.

Heath, David, Robert Jarrow, and Andrew J. Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation, *Econometrica* 60, 77–105.

Ho, Thomas S., Richard C. Stapleton, Marti G. Subrahmanyam, and Constantine Thanassoulas, 1994, A two-factor model for the risk management of interest rate derivatives, Working paper, Lancaster University, Lancaster, United Kingdom.

Hogan, Michael, and Keith Weintraub, 1993, The log-normal interest rate model and Eurodollar futures, Working paper, Citibank, New York.

Hull, John, 1993, *Options, Futures, and other Derivative Securities*, 2nd ed. (Prentice-Hall, Inc., Englewood Cliffs, New Jersey).

Jamshidian, Farshid, 1989, An exact bond option formula, *The Journal of Finance* 44, 205–209.

Jamshidian, Farshid, 1991a, Bond and option evaluation in the Gaussian interest rate model, *Research in Finance* 9, 131–170.

Jamshidian, Farshid, 1991b, Forward induction and construction of yield curve diffusion models, *The Journal of Fixed Income* 1, 62–74.

Karatzas, Ioannis, and Steven E. Shreve, 1988, *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate Texts in Mathematics* (Springer-Verlag, New York).

Käsler, Joachim, 1991, *Optionen auf Anleihen*, PhD thesis, Universität Dortmund, Germany.

McKean, Henry P., Jr., 1965, Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics, *Industrial Management Review* 6, 32–39.

Merton, Robert C., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, Robert C., 1990, *Continuous-Time Finance* (Basil Blackwell Inc., Padstow, Great Britain).

Miltersen, Kristian R., 1993, An empirical study of the term structure of interest rates, *Scandinavian Journal of Management* 9, 29–46.

Morton, Andrew J., 1988, Arbitrage and martingales, Working paper, College of Engineering, Cornell University, Ithaca, New York.

Musiela, Marek, 1994, Nominal annual rates and log-normal volatility structure, Working paper, University of New South Wales, Sydney, Australia.

Rady, Sven, 1995, Option pricing with natural boundaries and a quadratic diffusion term, *Finance and Stochastics*, forthcoming.

Rady, Sven, and Klaus Sandmann, 1994, The direct approach to debt option pricing, *The Review of Futures Markets* 13, 461–514.

Sandmann, Klaus, and Dieter Sondermann, 1993, A term structure model and the pricing of interest rate derivatives, *The Review of Futures Markets* 12, 391–423.

Sandmann, Klaus, and Dieter Sondermann, 1994, On the stability of log-normal interest rate models, Working paper, University of Bonn, Bonn, Germany.

Sandmann, Klaus, Dieter Sondermann, and Kristian R. Miltersen, 1994, Closed form term structure derivatives in a Heath–Jarrow–Morton model with log-normal annually compounded interest rates, In *Proceedings of the Seventh Annual European Futures Research Symposium Bonn*, pp 145–165 (Chicago Board of Trade, Chicago, Ill).

Unraveling in Guessing Games: An Experimental Study

By ROSEMARIE NAGEL *

Consider the following game: a large number of players have to state simultaneously a number in the closed interval [0, 100]. The winner is the person whose chosen number is closest to the mean of all chosen numbers multiplied by a parameter p, where p is a predetermined positive parameter of the game; p is common knowledge. The payoff to the winner is a fixed amount, which is independent of the stated number and p. If there is a tie, the prize is divided equally among the winners. The other players whose chosen numbers are further away receive nothing.[1]

The game is played for four rounds by the same group of players. After each round, all chosen numbers, the mean, p times the mean, the winning numbers, and the payoffs are presented to the subjects. For $0 \leq p < 1$, there exists only one Nash equilibrium: all players announce zero. Also for the repeated supergame, all Nash equilibria induce the same announcements and payoffs as in the one-shot game. Thus, game theory predicts an unambiguous outcome.

The structure of the game is favorable for investigating whether and how a player's mental process incorporates the behavior of the other players in conscious reasoning. An explanation proposed, for out-of-equilibrium behavior involves subjects engaging in a finite depth of reasoning on players' beliefs about one another. In the simplest case, a player selects a strategy at random without forming beliefs or picks a number that is salient to him (*zero-order belief*). A somewhat more sophisticated player forms *first-order beliefs* on the behavior of the other players. He thinks that others select a number at random, and he chooses his best response to this belief. Or he forms *second-order beliefs* on the first-order beliefs of the others and maybe nth order beliefs about the $(n - 1)$th order beliefs of the others, but only up to a finite n, called the n-depth of reasoning.

The idea that players employ finite depths of reasoning has been studied by various theorists (see e.g., Kenneth Binmore, 1987, 1988; Reinhard Selten, 1991; Robert Aumann, 1992; Michael Bacharach, 1992; Cristina Bicchieri, 1993; Dale O. Stahl, 1993). There is also the famous discussion of newspaper competitions by John M. Keynes (1936 p. 156) who describes the mental process of competitors confronted with picking the face that is closest to the mean preference of all competitors.[2] Keynes's game, which he considered a Gedankenexperiment, has $p = 1$. However, with $p = 1$, one cannot distinguish between different steps of reasoning by actual subjects in an experiment.

There are some experimental studies in which reasoning processes have been analyzed in ways similar to the analysis in this paper. Judith Mehta et al. (1994), who studied behavior in two-person coordination games, suggest that players coordinate by either applying depth of reasoning of order 1 or by picking a focal point (Thomas C. Schelling, 1964), which they call "Schelling salience." Stahl and Paul W. Wilson (1994) analyzed behavior in symmetric 3×3 games and concluded that subjects were using depths of reasoning of orders 1 or 2 or a Nash-equilibrium strategy.

* Department of Economics, Universitat Pompeu Fabra, Balmes 132, Barcelona 08008, Spain. Financial support from Deutsche Forschungsgemeinschaft (DFG) through Sonderforschungsbereich 303 and a postdoctoral fellowship from the University of Pittsburgh are gratefully acknowledged. I thank Reinhard Selten, Dieter Balkenborg, Ken Binmore, John Duffy, Michael Mitzkewitz, Alvin Roth, Karim Sadrieh, Chris Starmer, and two anonymous referees for helpful discussions and comments. I learned about the guessing game in a game-theory class given by Roger Guesnerie, who used the game as a demonstration experiment.

[1] The game is mentioned, for example, by Hervé Moulin (1986), as an example to explain rationalizability, and by Mario H. Simonsen (1988).

[2] This metaphor is frequently mentioned in the macroeconomic literature (see e.g., Roman Frydman, 1982).

Both of these papers concentrated on several one-shot games. In my experiments, the decisions in first period indicate that depths of reasoning of order 1 and 2 may be playing a significant role. In periods 2–4, for $p < 1$, I find that the modal depth of reasoning does not increase, although the median choice decreases over time.[3] A simple qualitative learning theory based on individual experience is proposed as a better explanation of behavior over time than a model of increasing depth of reasoning. This is the kind of theory that Selten and Joachim Buchta (1994) call a "learning direction theory," which has been successfully applied in several other studies.

Other games with unique subgame-perfect equilibria that have been explored in the experimental literature include Robert Rosenthal's (1981) "centipede game," a market game with ten buyers and one seller studied experimentally by Roth et al. (1991), a public-goods-provision game studied by Vesna Prasnikar and Roth (1992), and the finitely repeated prisoner's dilemma studied experimentally by Selten and Rolf Stoecker (1986). In the experimental work on the centipede game by Richard McKelvey and Thomas Palfrey (1992) and on the prisoner's dilemma supergame, the outcomes are quite different from the Nash equilibrium point in the opening rounds, as well as over time. While the outcomes in Roth et al. (1991), Prasnikar and Roth (1992), and my experiments are also far from the equilibrium in the opening round, they approach the equilibrium in subsequent rounds. Learning models have been proposed to explain such phenomena (see e.g., Roth and Ido Erev, 1995).

I. The Game-Theoretic Solutions

For $0 \leq p < 1$, there exists only one Nash equilibrium at which all players choose 0.[4] All announcing 0 is also the only strategy combination that survives the procedure of infinitely repeated simultaneous elimination of weakly dominated strategies.[5] For $p = 1$ and more than two players, the game is a coordination game, and there are infinitely many equilibrium points in which all players choose the same number (see Jack Ochs [1995] for a survey). For $p > 1$ and $2p < M$ (M is the number of players), all choosing 0 and all choosing 100 are the only equilibrium points. Note that for $p > 1$ there are no dominated strategies.[6] The subgame-perfect equilibrium play (Selten, 1975) does not change for the finitely repeated game.

II. A Model of Boundedly Rational Behavior

In the first period a player has no information about the behavior of the other players. He has to form expectations about choices of the other players on a different basis than in subsequent periods. In the subsequent periods he gains information about the actual behavior of the others and about his success in earlier periods. Therefore, in the analysis of the data I make a distinction between the first period and the remaining periods.

[3] This kind of unraveling is similar to the naturally occurring phenomena observed by Alvin E. Roth and Xiaolin Xing (1994) in many markets in which it is important to act just a little earlier in time than the competition.

[4] Assume that there is an equilibrium at which at least one player chooses a positive number with positive probability. Let k be the highest number chosen with positive probability, and let m be one of the players who chooses k with positive probability. Obviously, in this equilibrium p times the mean of the numbers chosen is smaller than k. Therefore, player m can improve his chances of winning by replacing k by a smaller number with the same probability. Therefore no equilibrium exists in which a positive number is chosen with positive probability.

[5] Numbers in $(100p, 100]$ are weakly dominated by $100p$; in the two-player game, 0 is a weakly dominant strategy. The interpretation of the infinite iteration process might be: it does not harm a rational player to exclude numbers in the interval $(100p, 100]$. If this player also believes that all other players are rational, he consequently believes that nobody will choose from $(100p, 100]$, and therefore he excludes $(100p^2, 100]$; if he thinks that the others believe the same, $(100p^3, 100]$ is excluded, and so on. Thus, 0 remains the only nonexcluded strategy based on common knowledge of rationality. If choices were restricted to integers, all choosing 1 is also an equilibrium.

[6] It is straightforward to show that all choosing 0 and all choosing 100 are equilibria: it does not pay to deviate from 0 (100) if all other players choose 0 (100) and the number of players is sufficiently large. There is no other symmetric equilibrium since with a unilateral small increase a player improves his payoff. Also, other asymmetric equilibria or equilibria in mixed strategies cannot exist for analogous reasons, as in the case $p < 1$.

The model of first-period behavior is as follows: a player is strategic of degree 0 if he chooses the number 50. (This can be interpreted as the expected choice of a player who chooses randomly from a symmetric distribution or as a salient number à la Schelling [1960]). A person is strategic of degree n if he chooses the number $50p^n$, which I will call iteration step n. A person whose behavior is described by $n = 1$ just makes a naive best reply to random behavior.[7] However, if he believes that the others also employ this reasoning process, he will choose a number smaller than $50p$, say $50p^2$, the best reply to all other players using degree-1 behavior. A higher value of n indicates more strategic behavior paired with the belief that the other players are also more strategic; the choice converges to the equilibrium play in the limit as n increases.

For periods 2–4, the reasoning process of period 1 can be modified by replacing the initial reference point $r = 50$ by a reference point based on the information from the preceding period. A natural candidate for such a reference point is the mean of the numbers named in the previous period. With this initial reference point, iteration step 1, which is the product of p and the mean of the previous period, is similar to Cournot behavior (Antoine A. Cournot, 1838) in the sense of giving a best reply to the strategy choices made by the others in the previous period (assuming that the behavior of the others does not change from one period to the next).[8]

I can also consider "anticipatory learning," in which an increase in iteration steps is expected of the other players. Specifically, one can ask whether, with increasing experience, higher and higher iteration steps will be observed. I will show, however, that the modal frequency, polled over all sessions, remains at iteration step 2 in all periods. In Section V-C a quite different adjustment behavior is examined, which does not involve anything similar to the computation of a best reply to expected behavior. Instead of this, a behavioral parameter—the adjustment factor—is changed in the direction indicated by the individual experience in the previous period.

III. The Experimental Design

I conducted three sessions with the parameter $p = 1/2$ (sessions 1–3), four sessions with $p = 2/3$ (4–7), and three sessions with $p = 4/3$ (8–10).[9] I will refer to these as $1/2$, $2/3$, or $4/3$ sessions, respectively. A subject could participate in only one session.

The design was the same for all sessions: 15–18 subjects were seated far apart in a large classroom so that communication was not possible. The same group played for four periods; this design was made known in the written instructions. At each individual's place were an instruction sheet, one response card for each period, and an explanation sheet on which the subjects were invited to give written explanations or comments on their choices after each round. The instructions were read aloud, and questions concerning the rules of the game were answered.[10]

After each round the response cards were collected. All chosen numbers, the mean, and

[7] If the mean choice of the others is 50, the number that really comes nearest to p times the mean is a little lower since this player's choice also influences p times the mean. My interpretation of iteration step 1 is comparable to the definition of secondary salience introduced by Mehta et al. (1994) or the level-1 type in Stahl and Wilson (1994).

[8] Actually, Cournot behavior in response to an assumed mean choice $\bar{x}_{-i}$ of the other players would not lead to p times the mean, but to

$$p\frac{M-1}{M-p}\bar{x}_{-i}$$

where M is the number of players. However, there is no indication that subjects try to compute this best reply. Moreover, for M between 15 and 18, the number of subjects in my experiments, the difference between this best reply and p times the mean is not large.

[9] I use $p = 1/2$, because it reduces calculation difficulties. With $p = 2/3$, I am able to distinguish between the hypothesis that a thought process starts with the reference point 50 and the game-theoretic hypothesis that a rational person will start the iterated elimination of dominated strategies with 100. For $p > 1$, $p = 4/3$ is used to analyze behavior. There are no sessions with $p = 1$; this game is similar to a coordination game with many equilibria, which has already been studied experimentally (e.g., John Van Huyck et al., 1990).

[10] A copy of the instructions used in the experiment may be obtained from the author upon request.

the product of p and the mean were written on the blackboard (the anonymity of the players was maintained). The number closest to the optimal number and the resulting payoffs were announced. The prize to the winner of each round was 20 DM (about $13). If there was a tie, the prize was split between those who tied. All other players received nothing.[11] After four rounds, each player received the sum of his gains of each period and an additional fixed amount of 5 DM (approximately $3) for showing up. Each session lasted about 45 minutes, including the instruction period.

IV. The Experimental Results

The raw data can be found in Nagel (1993) and are also available from the author upon request. Whereas I use only nonparametric tests in the following sections, Stahl (1994) applies parametric tests to these data and confirms most of the conclusions.

A. *First-Period Choices*

Figure 1 displays the relative frequencies of all first-period choices for each value of p, separately. The means and medians are also given in the figure. All but four choices are integers. No subject chose 0 in the $^2/_3$ and $^1/_2$ sessions, and only 6 percent chose numbers below 10. In the $^4/_3$ sessions, only 10 percent chose 99, 100, or 1. Thus, the sessions with different parameters do not differ significantly with respect to frequencies of equilibrium strategies and choices near the equilibrium strategies. *Weakly dominated choices*, choices larger than $100p$, were also chosen infrequently: in the $^1/_2$ sessions, 6 percent of the subjects chose numbers greater than 50 and 8 percent chose 50; in the $^2/_3$ sessions, 10 percent of the chosen numbers were greater than 67, and 6 percent were 66 or 67. From these results

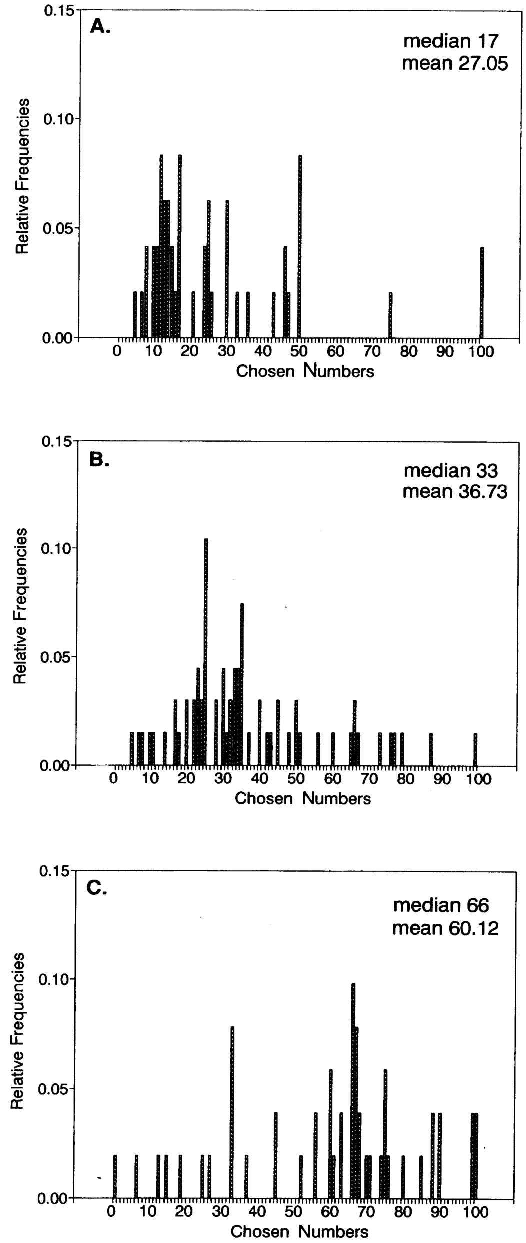


FIGURE 1. CHOICES IN THE FIRST PERIOD: A) SESSIONS 1–3 ($p = {}^1/_2$); B) SESSIONS 4–7 ($p = {}^2/_3$); C) SESSIONS 8–10 ($p = {}^4/_3$)

[11] This all-or-nothing payoff structure might trigger unreasonable behavior by some subjects which in turn impedes quicker convergence. In John Duffy and Nagel (1995), the behavior in p-times-the-median game was studied, in an effort to weaken the influence of outliers. While first round behavior in both the mean- and median-treatments was not significantly different, fourth round choices in p-times-the-median game were slightly lower than those in p-times-the-mean game. Changes in the payoff structure, for example, negative payoffs to losers, might affect the evolution of behavior on the guessing game in a different way.

one might infer that either dominated choices are consciously eliminated or reference points are chosen that preclude dominated choices. For $p > 1$, dominated strategies do not exist. Apart from the similarities just mentioned, there are noticeable differences between the distributions of choices in sessions with different values of p. When p was increased, the mean of the chosen numbers was higher. I can reject the null hypothesis that the data from the $^1/_2$ and $^2/_3$ sessions are drawn from the same distribution, in favor of the alternative hypothesis that most of the chosen numbers in the $^1/_2$ sessions tend to be smaller, at the 0.001 level of statistical significance, according to a Mann-Whitney U test. The same holds for a test of the data from the $^2/_3$ sessions against those of the $^4/_3$ sessions: the chosen numbers in the former tend to be smaller than those in the latter; the null hypothesis is rejected at the 0.0001 level. This result immediately suggests that many players do not choose numbers at random but instead are influenced by the parameter p of the game.

I also tested whether the data exhibit the structure suggested by the simple model given in Section III, that is, taking 50 as an initial reference point and considering several iteration steps from this point ($50p^n$). Figure 1 shows that the data do not correspond exactly to these iteration steps. However, are the data concentrated around those numbers? In order to test this possibility, I specify *neighborhood intervals* of $50p^n$, for which n is 0, 1, 2, ... Intervals between two neighborhood intervals of $50p^{n+1}$ and $50p^n$ are called *interim intervals*. I use the geometric mean to determine the boundaries of adjacent intervals. This approach captures the idea that the steps are calculated by powers of n. The interim intervals are on a logarithmic scale approximately as large as the neighborhood intervals, if rounding effects are ignored.[12]

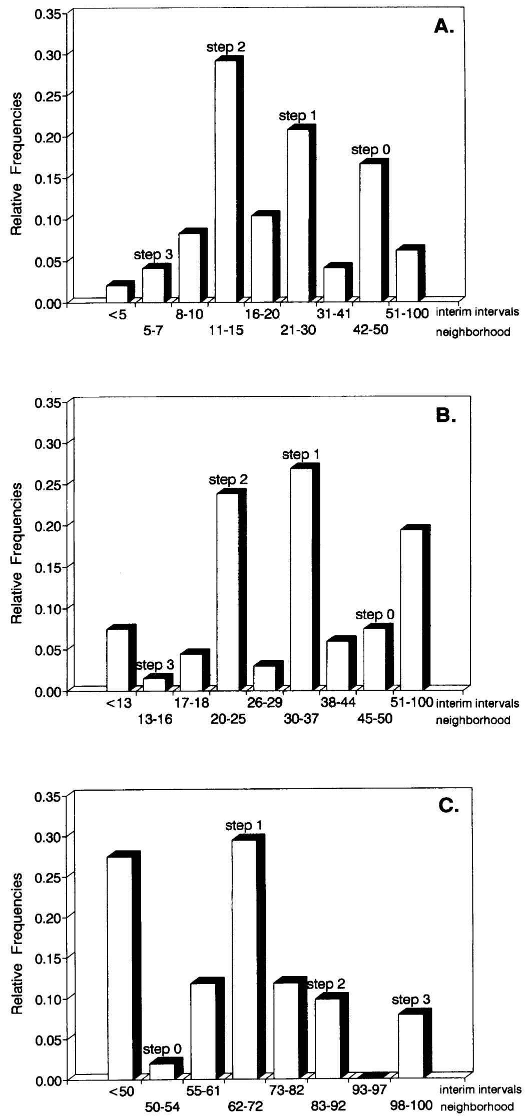


FIGURE 2. RELATIVE FREQUENCIES OF CHOICES IN THE FIRST PERIOD ACCORDING TO THE INTERVAL CLASSIFICATION WITH REFERENCE POINT 50: A) SESSIONS 1–3 ($p = {}^1/_2$); B) SESSIONS 4–7 ($p = {}^2/_3$); C) SESSIONS 8–10 ($p = {}^4/_3$)

Figure 2 shows the number of observations in each of these neighborhood and interim intervals for the respective sessions. The neighborhood and interim intervals are stated on the horizontal axis. Note the similarity between Figure 2A and Figure 2B. In the $^1/_2$ and $^2/_3$

[12] In general, the neighborhood interval of $50p^i$ has the boundaries $50p^{i+1/4}$ and $50p^{i-1/4}$, rounded to the nearest integers, since mostly integers were observed. Note that the neighborhood of $50p^0$ is bounded from the right side by 50 for $p < 1$ and bounded to the left side by 50 for $p > 1$. (The results we present would not change if we had included a right-hand-side neighborhood for $p < 1$, or a left-hand-side neighborhood for $p > 1$).

sessions, almost 50 percent of the choices are in the neighborhood interval of either iteration step 1 or 2, and there are few observations in the interim interval between them. In all sessions only 6–10 percent are at step 3 and higher steps (the aggregation of the two left-hand columns in Figure 2A and Figure 2B [$p = {}^1\!/_2$ and $p = {}^2\!/_3$], respectively, and the right-hand column of Figure 2C [$p = {}^4\!/_3$]). (Choices above 50 in the $^1\!/_2$ and $^2\!/_3$ sessions and choices below 50 in the $^4\!/_3$ sessions are graphed only in aggregate.) The choices are mostly below 50 in the $^1\!/_2$ and $^2\!/_3$ sessions and mostly above 50 in the $^4\!/_3$ sessions; this difference is statistically significant at the 1-percent level, based on the binomial test.

To test whether there are significantly more observations within the neighborhood intervals than in the interim intervals I consider only observations between step 0 and step 3. Hence, the expected proportion within the neighborhood interval under the null hypothesis (that choices are randomly distributed between interim and neighborhood intervals) is then the sum of the neighborhood intervals divided by the interval between step 0 and step 3. Note that this is a stronger test than taking the entire interval 0–100. The one-sided binomial test, taking into consideration the proportion of observations in the neighborhood intervals, rejects the null hypothesis in favor of the hypothesis that the pooled observations are more concentrated in the neighborhood intervals (the null hypothesis is rejected at the 1-percent level, both for the $^1\!/_2$ sessions and for the $^2\!/_3$ sessions; it is rejected at the 5-percent level for the $^4\!/_3$ sessions).[13,14]

Note that over all $^1\!/_2$ sessions, the optimal choice (given the data) is about 13.5, which belongs to iteration step 2, which is also where we observe the modal choice, with nearly 30 percent of all observations. Over all $^2\!/_3$ sessions, the optimal choice is about 25, which also belongs to iteration step 2 with about 25 percent of all observations, the second-highest frequency of a neighborhood interval. Thus, many players are observed to be playing approximately optimally, given the behavior of the others.

B. *The Behavior in Periods 2, 3, and 4*

To provide an impression of the behavior over time, Figures 3–5 show plots of pooled data from sessions with the same p for each period; the plots show the choices of each subject from round t to $t + 1$. In the $^1\!/_2$ and $^2\!/_3$ sessions, 135 out of 144 (3 transition periods $\times$ 48 subjects) and 163 out of 201 observations (3×67 subjects), respectively, are below the diagonal, which indicates that most subjects decrease their choices over time. In all sessions with $p < 1$, the medians decrease over time (see Table 1); this is also true for the means except in the last period of the $^1\!/_2$ sessions. In the $^4\!/_3$ sessions, the reverse is true: 133 out of 153 observations (3×51 subjects) are above the diagonal and the medians increase and are 100 in the third and fourth periods. Thus from round to round, the observed behavior moves in a consistent direction, toward an equilibrium. (It is this movement that is reminiscent of the unraveling in time observed in many markets by Roth and Xing [1994].)

In the $^1\!/_2$ sessions, more than half of the observations were less than 1 in the fourth round. However, only three out of 48 chose 0. In the $^2\!/_3$ sessions, only one player chose a number less than 1. On the other hand, in the $^4\!/_3$ sessions, 100 was already the optimal choice in the second period, being chosen by 16 percent of the subjects; and in the third and fourth period, it was chosen by 59 percent and 68 percent, respectively. Thus, for the $^4\!/_3$ sessions I conclude that the behavior of the majority of the subjects can be simply described as the best reply (100) to the behavior observed in the previous period. (Some of the subjects who deviated from this behavior argued that they tried to influence the mean [to bring it

[13] Since the iterated elimination of dominated strategies starts the reasoning process at 100, I also tested whether that initial reference point would structure the data in a coherent way for the different parameters. For $p = {}^4\!/_3$ all iteration steps collapse into 100; thus spikes cannot be explained. For the $^2\!/_3$ sessions the data are not only concentrated around $100 \times (2/3)^n$. On the other hand, the pattern of the $^1\!/_2$-session data is similar to the pattern in Figure 3A, except that step n becomes step $n + 1$. Hence, 100 is not a plausible initial reference point for most subjects.

[14] The written comments of the subjects also seem to support our model. For details see Nagel (1993).

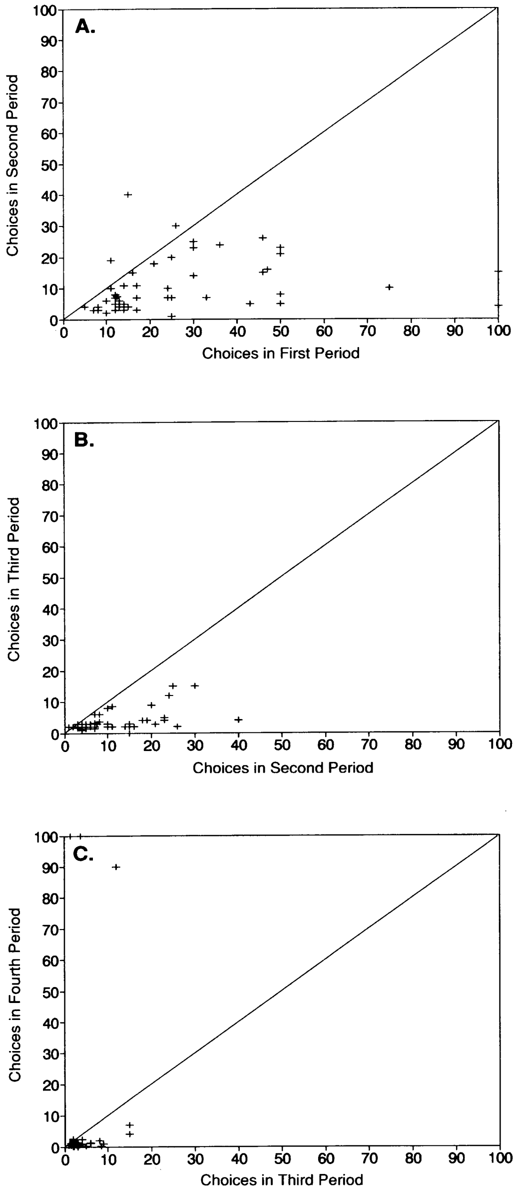


FIGURE 3. OBSERVATIONS OVER TIME FOR SESSIONS 1–3 ($p = 1/2$): A) TRANSITION FROM FIRST TO SECOND PERIOD; B) TRANSITION FROM SECOND TO THIRD PERIOD; C) TRANSITION FROM THIRD TO FOURTH PERIOD

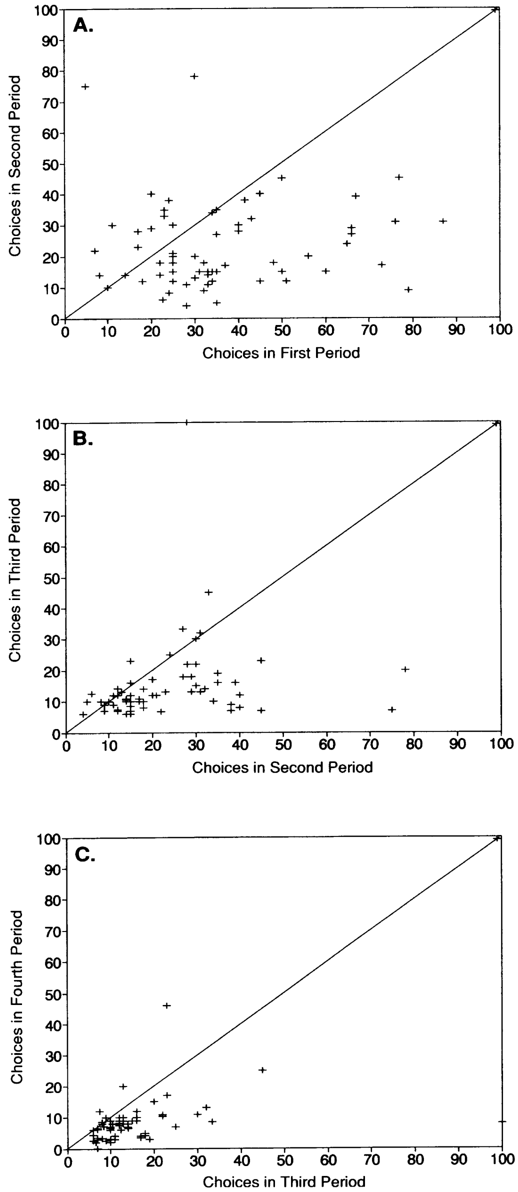


FIGURE 4. OBSERVATIONS OVER TIME FOR SESSIONS 4–7 ($p = 2/3$): A) TRANSITION FROM FIRST TO SECOND PERIOD; B) TRANSITION FROM SECOND TO THIRD PERIOD; C) TRANSITION FROM THIRD TO FOURTH PERIOD

down again] or wrote that the split prize was too small to state the obvious right answer.)

The adjustment process toward the equilibrium in the $1/2$ and $2/3$ sessions is quite different from that in the $4/3$ sessions. Zero is never the best reply in the $1/2$ and $2/3$ sessions, given the actual strategies. Instead the best reply is a moving target that approaches 0. The adjustment process is thus more complicated. Comparing Figures 3 and 4, one can see that the

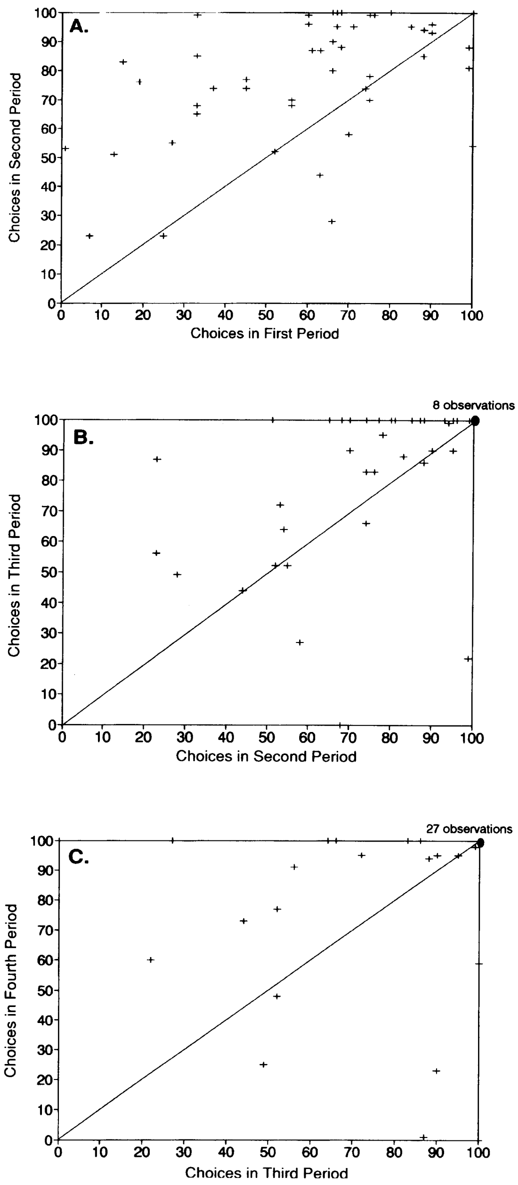


FIGURE 5. OBSERVATIONS OVER TIME FOR SESSIONS 8–10 ($p = \frac{4}{3}$): A) TRANSITION FROM FIRST TO SECOND PERIOD; B) TRANSITION FROM SECOND TO THIRD PERIOD; C) TRANSITION FROM THIRD TO FOURTH PERIOD

choices in $\frac{1}{2}$ sessions converge faster toward 0 than those in $\frac{2}{3}$ sessions. However, the reason might be that the initial distribution is at lower choices in the former sessions than in the latter. Therefore, to investigate whether the actual choices decrease faster for $p = \frac{1}{2}$ than for $p = \frac{2}{3}$, I define a rate of decrease of the means and medians from period 1 to period 4 within a session by

$$(1a) \quad w_{\text{mean}} = \frac{(\text{mean})_{t=1} - (\text{mean})_{t=4}}{(\text{mean})_{t=1}}$$

$$(1b) \quad w_{\text{med}} = \frac{(\text{median})_{t=1} - (\text{median})_{t=4}}{(\text{median})_{t=1}}.$$

The rates of decrease of the single sessions are shown in Table 1, in the last lines of panels A and B. The rates of decrease of the session medians in the $\frac{1}{2}$ sessions are higher than those in the $\frac{2}{3}$ sessions, and the difference is statistically significant at the 5-percent level (one-tailed), based on a Mann-Whitney U test. There is no significant difference in the rates of decrease of the means. The median seems more informative than the mean, since the mean may be strongly influenced by a single deviation to a high number. Thus, I conclude that the rate of decrease depends on the parameter p.

Analyzing the behavior in the first period, I found some evidence that $r = 50$ was a plausible initial reference point. Below, I classify the data of each of the subsequent periods according to the reference point r (mean of the previous period) and iteration steps n: rp^n. Numbers above the mean are aggregated to "above mean$_{t-1}$" (see Table 2).[15]

As was the case for the first-period behavior, one cannot expect that exactly these steps are chosen. Grouping the data of the subsequent periods and sessions in the same way as in the first period, namely, in neighborhood intervals of the iteration steps and interim in-

[15] The chosen numbers tend to be below the mean of the previous period, and the difference is significant at the 5-percent level for all $\frac{1}{2}$ and $\frac{2}{3}$ sessions and all periods $t = 2$–4, according to the binomial test. The same test does not reject the null hypothesis for p times the mean of the previous period, for periods 2 and 3. In the fourth period the chosen numbers are significantly (at the 1-percent level) below $p \times r$, in six out of seven sessions. Note that if I had analyzed the data starting from reference point "naive best reply of the previous period" ($p \times r$), instead of starting from the mean, step n would become step $n - 1$, and all choices above the naive best reply would be aggregated to one category.

TABLE 1—MEANS AND MEDIANS OF PERIODS 1–4, AND RATE OF DECREASE FROM PERIOD 1 TO PERIOD 4

A. *Sessions with* $p = \frac{1}{2}$:

	Session 1		Session 2		Session 3	
Period	Mean	Median	Mean	Median	Mean	Median
1	23.7	17	33.2	30	24.2	14
2	10.9	7	12.1	10	10.2	6
3	5.3	3	3.8	3.3	2.4	2.1
4	8.1	2	13.0	0.57	0.4	0.33
Rate of decrease:[a]	0.66	0.88	0.61	0.98	0.98	0.97

B. *Sessions with* $p = \frac{2}{3}$:

	Session 4		Session 5		Session 6		Session 7	
Period	Mean	Median	Mean	Median	Mean	Median	Mean	Median
1	39.7	33	37.7	35	32.9	28	36.4	33
2	28.6	29	20.2	17	20.3	18	26.5	20
3	20.2	14	10.0	9	16.7	10	16.7	12.5
4	16.7	10	3.2	3	8.3	8	8.7	8
Rate of decrease:[a]	0.58	0.7	0.92	0.91	0.75	0.71	0.76	0.76

[a] Rate of decrease w from period 1 to 4 (see formula 1).

tervals between two steps, I find no significant difference between the frequencies of observations in the neighborhood intervals and the frequencies of observations in the interim intervals. Note also that as the mean decreases, the interval between two steps becomes rather small.[16] However, I would like to know within which iteration steps the numbers are located in the different periods; therefore, I divide the interval between steps i and $i + 1$ geometrically into two intervals.[17]

Parts A and B of Table 2 present the frequencies of observations for each iteration step, pooled over the $\frac{1}{2}$ and $\frac{2}{3}$ sessions, respectively. I also state the mean area of each iteration step over all sessions, separately for each period. In most sessions and periods, at least 80 percent of the observations remain within the bounds of iteration step 0 and iteration step 3, with the modal frequency (30 percent or more) at iteration step 2 when the previous period's mean is the reference point.[18] In fact, in periods 1–3, the best reply is within step 2 in at least five of the seven sessions. Within the neighborhood of the mean of the previous period (step 0) there are only a few observations, and those frequencies decrease in the $\frac{2}{3}$ sessions. The frequency of choices around iteration step 1, corresponding to the Cournot process, also declines to less than 15 percent in the third and fourth periods. The frequencies with more than three steps are below 10 percent, except in period 4 of the $\frac{1}{2}$ sessions. I interpret these results to mean that there is no support for the hypothesis of increasing depth of reasoning, since there is no tendency for the majority of the subjects to increase the depth of reasoning beyond step

[16] Most of the subjects just mentioned in their comments that the mean will decrease. There were less precise calculations than in the first period.

[17] If one normalizes the mean of the previous period to 1, the boundaries of step n are $(p^{n+1/2}, p^{n-1/2}]$. As in period 1, step 0 has its right-hand boundary at 1. Table 2 reports the unnormalized length (called "area") of an iteration step. For example, for $p = \frac{1}{2}$, in period 2, the area of numbers above the mean is 73, since on average, over all $\frac{1}{2}$ sessions the mean of the previous period (r) is 27.

[18] This corresponds to what we called the anticipatory learning process in Section II. Hence, one might infer that a substantial proportion of subjects believe that the average behavior in period t will be around p times the mean of period $t - 1$.

TABLE 2—RELATIVE FREQUENCIES AND AREAS OF PERIODS 2–4 ACCORDING TO THE STEP-MODEL FOR AGGREGATED DATA

	Period 2		Period 3		Period 4	
Classification	Relative frequency	Area	Relative frequency	Area	Relative frequency	Area
A. *Sessions 1–3* ($p = {}^1/_2$):						
Higher steps	4.2	2.4	4.2	1.0	20.8	0.3
Step 3	25.0	2.4	12.5	1.0	22.9	0.3
Step 2	31.3	4.9	60.4	2.0	29.2	0.7
Step 1	27.0	9.6	12.5	3.9	14.5	1.4
Step 0	2.1	7.9	4.1	3.2	4.2	1.1
Above mean_{t-1}	10.4	73.0	6.3	88.9	8.3	96.2
All	100.0	100.0	100.0	100.0	100.0	100.0
B. *Sessions 4–7* ($p = {}^2/_3$):						
Higher steps	7.5	8.9	1.5	5.8	7.5	3.8
Step 3	11.9	4.4	17.9	2.9	25.3	1.9
Step 2	31.3	6.7	46.2	4.3	47.8	2.9
Step 1	20.9	10.0	16.4	6.5	10.4	4.3
Step 0	14.9	6.7	7.5	4.4	3.0	2.9
Above mean_{t-1}	13.4	63.3	10.5	76.1	6.0	84.1
All	100.0	100.0	100.0	100.0	100.0	100.0

2.[19] In the next section I describe the observed behavior from period 2 to period 4 in a different way—by a qualitative learning-direction theory. This theory might explain why the modal frequency of depth of reasoning does not increase.

C. *Adjustment Process Due to Individual Experience (for $p < 1$)*

So far, I have categorized behavior into classes based on the deviation from the mean of the previous period. I now analyze individual adjustments due to individual experience. There are two possible experiences due to payoffs a player obtained:

(i) the player gained a share or all of the prize in the previous period, because his announcement was closest to the product of p and the mean; or

(ii) he earned nothing in the previous period, because his chosen number was either *below* or *above p* times the mean (and not the closest to it).

Since there are only a few winners in each period, data on having chosen the winning number are scarce. Therefore, I exclude those choices that led to a positive payoff (19 out of 144 [13 percent] in the $^1/_2$ sessions and 23 out of 201 [9 percent] in the $^2/_3$ sessions) and propose a simple qualitative learning theory for the change of behavior after having faced zero payoffs.[20]

For this purpose, I introduce a parameter called the adjustment factor:

$$(2)\quad a_{it} = \begin{cases} \dfrac{x_{it}}{50} & \text{for } t = 1 \\[2ex] \dfrac{x_{it}}{(\text{mean})_{t-1}} & \text{for } t = 2, 3, 4 \end{cases}$$

[19] In periods 2 and 3, step 2 is the modal choice in six out of seven sessions; in period 4, this holds in four sessions, and in three sessions, the modal choice is step 3, tied with step 2 or 4.

[20] Stahl (1994) compares several learning models. I apply only one learning model, a kind of model that has been successfully used in different experimental settings (see e.g., Selten and Buchta, 1994).

where x_{it} is the number chosen by player i in period t. Hence, a_{it} is the relative deviation from the mean of the previous period $t-1$; the mean is the initial reference point. The adjustment factor for period 1 is the choice in period 1 divided by 50, where 50 is the initial reference point, as mentioned in Subsection IV-A. The retrospective "optimal" adjustment factor in period t is defined as the optimal deviation from the mean of period $t-1$ that leads to p times the mean of period t:

$$(3)\quad a_{\mathrm{opt},t} = \begin{cases} \dfrac{x_{\mathrm{opt},t}}{50} = \dfrac{p \times (\mathrm{mean})_t}{50} & \text{for } t = 1 \\[2ex] \dfrac{x_{\mathrm{opt},t}}{(\mathrm{mean})_{t-1}} = \dfrac{p \times (\mathrm{mean})_t}{(\mathrm{mean})_{t-1}} & \text{for } t = 2, 3, 4. \end{cases}$$

The idea of this simple learning-direction theory is that in an *ex post* reasoning process a player compares his adjustment factor a_t with the optimal adjustment factor $a_{\mathrm{opt},t}$. In the next period he most likely adapts in the direction of the optimal adjustment factor. Thus, he reflects which deviation from the previous initial reference point would have been better:

$$(4)\qquad \text{if } a_t > a_{\mathrm{opt},t} \Rightarrow a_{t+1} < a_t$$
$$\text{if } a_t < a_{\mathrm{opt},t} \Rightarrow a_{t+1} > a_t.$$

In words, if he observed that his chosen number was above p-times the mean in the previous period (i.e., his adjustment factor was higher than the optimal adjustment factor), then he should decrease his rate; if his number was below p times the mean (i.e., his adjustment factor was lower than the optimal adjustment factor), he should increase his adjustment factor.

Figure 6 shows the changes of behavior due to experience from period to period, pooled over all $^1/_2$ sessions (Fig. 6A–C) and over all $^2/_3$ sessions (Fig. 6D–F). The bars within each histogram sum to 100 percent. The two left-hand bars in a histogram depict the relative frequencies after the experience that the adjustment factor was higher than the optimal adjustment factor, and the two right-hand bars show the frequencies when the factor was lower. The striped bars show the frequencies of *increased* adjustment factors, and the white bars show the frequencies of *decreased* adjustment factors from period t to period $t+1$.

In each session, pooling the data over the three transition periods, the majority of behavior (between 67 percent and 78 percent, with a mean of 73 percent over all sessions) is in accordance with the learning-direction theory. Thus, taking each session as an independent observation, the null hypothesis that experience is irrelevant can be rejected at the 1-percent level, based on the binomial test. One may also ask whether the frequencies of decreases in adjustment factors independent of experience are higher than the frequencies of increases.[21] In each session, a majority of subjects decrease the factor; however, the percentage who do so is only between 51 percent and 69 percent, with a mean of 58 percent for all sessions. Comparing the two findings, in each session the frequency in accordance with the learning-direction theory is higher than the frequency of decreases, independent of experience. I interpret this result as indicating that the learning theory provides a better explanation than the hypothesis of decreasing adjustment factor.

The theory of adjustment due to experience is similar to the findings on changes of behavior in other experimental studies. Gerard P. Cachon and Colin Camerer (1991) studied behavior in a coordination game, the so-called *median-effort-game*. They mention that a player who observed that he was below the median in the previous period would most likely increase his effort level and vice versa. Over time, the median effort level remains constant and does not converge to the efficient equilibrium. Also, in Michael Mitzkewitz and Nagel (1993) a simple learning theory related to ours is studied in a completely different setting, with similar results. Selten and Stoecker (1986) analyzed in great detail the influence of experience on end-effect behavior in finite

[21] This question is related to increasing steps of reasoning.

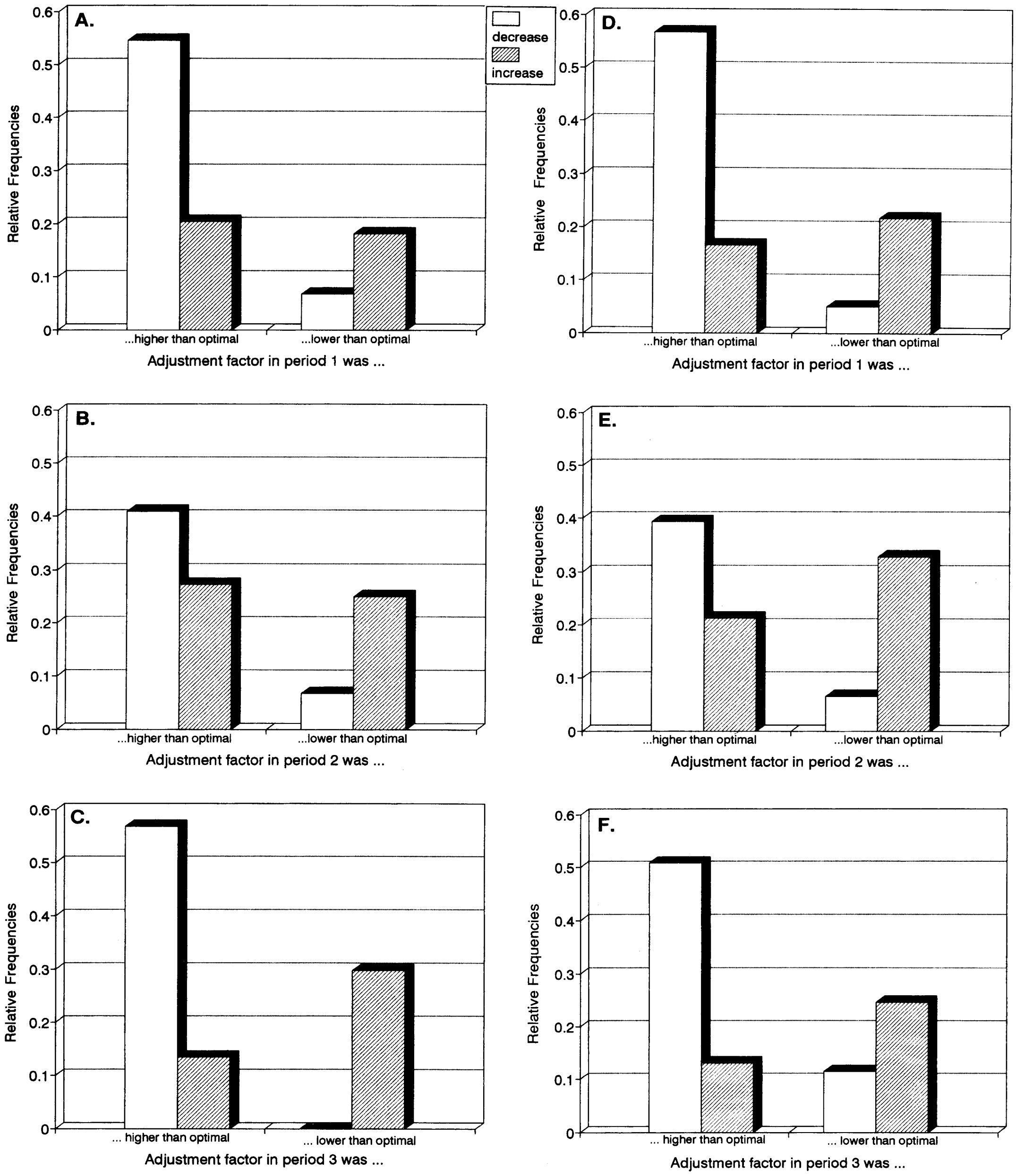


FIGURE 6. RELATIVE FREQUENCIES OF CHANGES IN ADJUSTMENT FACTORS DUE TO INDIVIDUAL EXPERIENCE IN THE PRECEDING PERIOD: A) $p = 1/2$, TRANSITION FROM FIRST TO SECOND PERIOD; B) $p = 1/2$, TRANSITION FROM SECOND TO THIRD PERIOD; C) $p = 1/2$, TRANSITION FROM THIRD TO FOURTH PERIOD; D) $p = 2/3$, TRANSITION FROM FIRST TO SECOND PERIOD; E) $p = 2/3$, TRANSITION FROM SECOND TO THIRD PERIOD; F) $p = 2/3$, TRANSITION FROM THIRD TO FOURTH PERIOD

prisoner's-dilemma supergames. Thus for different games, similar kinds of adjustment processes have been used to explain behavior. However, the dynamics of the behavior can be quite different: in some games there is a convergence toward an equilibrium, whereas in others, the adjustment process may not lead to an (efficient) equilibrium.

V. Summary

My analysis of behavior in an abstract game leads me to believe that the structure of the game is favorable for the study of thought processes of actual players. In the first period the behavior deviates strongly from game-theoretic solutions. Furthermore, the distribution of the chosen numbers over the [0, 100] interval in sessions with different parameters were significantly different. I have proposed a theory of boundedly rational behavior in which the "depths of reasoning" are of importance. The results indicate that, starting from initial reference point 50, iteration steps 1 and 2 play a significant role, that is, most of the observations are in the neighborhood of $50p$ or $50p^2$, independent of the parameter p. This result accounts for the difference of the distributions of the chosen numbers for different parameter values p.

Thus, the theory of boundedly rational behavior for the first period deviates in several ways from the game-theoretic reasoning:

(i) I suggested that the "reference point" or starting point for the reasoning process is 50 and not 100. The process is driven by iterative, naive best replies rather than by an elimination of dominated strategies.
(ii) The process of iteration is finite and not infinite.
(iii) I apply the same theory for $p > 1$ and $p < 1$, whereas game-theoretic reasoning is different for those parameter sets.

Over time the chosen numbers approach an equilibrium or converge to it. In the $^4/_3$ sessions, the choice 100 is the best reply in all periods but the first. In the third and fourth period more than 50 percent of the subjects choose this strategy. In the sessions with $p < 1$, there is a moving target, which approaches zero. I apply the theory of first-round behavior also to the subsequent periods 2–4, using as the initial reference point the mean of the previous period. I find that the modal choices are around iteration step 2, and the majority of observations remain below step 3. In most sessions, the best reply is within step 2 in periods 1–3. I cannot accept the hypothesis of increasing iteration steps, and I suggest that another explanation of the observed behavior may be more adequate for periods 2–4. I propose a qualitative learning-direction theory which predicts that a subject tends to increase his adjustment factor in the direction of the optimal adjustment factor if it was below the optimal one and tends to decrease the adjustment factor if it was above the optimal one. A similar kind of simple learning theory has been applied successfully in other experiments.

REFERENCES

Aumann, Robert. "Irrationality in Game Theory," in Partha Dasgupta, Douglas Gale, Oliver Hart, and Eric Maskin, eds., *Economic analysis of markets and games*. Cambridge, MA: MIT Press, 1992, pp. 214–27.

Bacharach, Michael. "Backward Induction and Beliefs about Oneself." *Synthese*, June 1992, *91*(3), pp. 247–84.

Bicchieri, Cristina. *Rationality and coordination*. Cambridge: Cambridge University Press, 1993.

Binmore, Kenneth. "Modeling Rational Players, Part I." *Economics and Philosophy*, October 1987, *3*, pp. 179–214.

______. "Modeling Rational Players, Part II." *Economics and Philosophy*, April 1988, *4*(3), pp. 9–55.

Cachon, Gerard P. and Camerer, Colin. "The Sunk Cost Fallacy, Forward Induction, and Behavior in Coordination Games." Working paper, University of Pennsylvania, 1991.

Cournot, Antoine A. "Recherches sur les Principes Mathématiques de la Théorie de Richesses." 1838. Translated by N. T. Bacon, *Researches into the mathematical principles of the theory of wealth*. London: Hafner, 1960.

Duffy, John and Nagel, Rosemarie. "On the Robustness of Behavior on Experimental Guessing Games." Mimeo, University of Pittsburgh, 1995.

Frydman, Roman. "Towards an Understanding of Market Processes: Individual Expectations, Learning, and Convergence to Rational Expectations Equilibrium." *American Economic Review*, September 1982, *72*(4), pp. 652–68.

Keynes, John M. *The general theory of interest, employment and money*. London: Macmillan, 1936.

McKelvey, Richard and Palfrey, Thomas. "An Experimental Study of the Centipede Game." *Econometrica*, July 1992, *60*(4), pp. 803–36.

Mehta, Judith; Starmer, Chris and Sudgen, Robert. "The Nature of Salience: An Experimental Investigation of Pure Coordination Games." *American Economic Review*, June 1994, *84*(3), pp. 658–73.

Mitzkewitz, Michael and Nagel, Rosemarie. "Experimental Results on Ultimatum Games with Incomplete Information." *International Journal of Game Theory*, 1993, *22*(2), pp. 171–98.

Moulin, Hervé. *Game theory for social sciences*. New York: New York Press, 1986.

Nagel, Rosemarie. "Experimental Results on Interactive, Competitive Guessing." Discussion Paper No. B-236, University of Bonn, 1993.

Ochs, Jack. "Coordination Problems," in John Kagel and Alvin E. Roth, eds., *Handbook of experimental economics*. Princeton, NJ: Princeton University Press, 1995, pp. 195–252.

Prasnikar, Vesna and Roth, Alvin E. "Considerations of Fairness and Strategy: Experimental Data from Sequential Games." *Quarterly Journal of Economics*, August 1992, *107*, pp. 865–88.

Rosenthal, Robert. "Games of Perfect Information, Predatory Pricing, and the Chain Store Paradox." *Journal of Economic Theory*, August 1981, *25*(1), pp. 92–100.

Roth, Alvin E. and Erev, Ido. "Learning in Extensive-Form Games: Experimental Data and Simple Dynamic Models in the Intermediate Term." *Games and Economic Behavior*, January 1995, *8*(1), pp. 164–212.

Roth, Alvin E.; Prasnikar, Vesna; Okuno-Fujiwara, Masahiro and Zamir, Shmuel. "Bargaining and Market Behavior in Jerusalem, Ljubljana, Pittsburgh, and Tokyo: An Experimental Study." *American Economic Review*, December 1991, *81*(5), pp. 1068–95.

Roth, Alvin E. and Xing, Xiaolin. "Jumping the Gun: Imperfections and Institutions Related to the Timing of Market Transactions." *American Economic Review*, September 1994, *84*(4), pp. 992–1044.

Schelling, Thomas C. *The strategy of conflict*. Cambridge, MA: Harvard University Press, 1960.

Selten, Reinhard. "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games." *International Journal of Game Theory*, 1975, *4*(1), pp. 25–55.

———. "Anticipatory Learning in Two-Person Games," in R. Selten, ed., *Game equilibrium models I*. Berlin: Springer Verlag, 1991, pp. 98–154.

Selten, Reinhard and Buchta, Joachim. "Experimental Sealed Bid First Price Auction with Directly Observed Bid Functions." Discussion Paper No. B-270, University of Bonn, 1994.

Selten, Reinhard and Stoecker, Rolf. "End Behavior in Sequences of Finite Prisoner's Dilemma Supergames: A Learning Theory Approach." *Journal of Economic Behavior and Organization*, March 1986, *7*(1), pp. 47–70.

Simonsen, Mario H. "Rational Expectations, Game Theory, and Inflationary Inertia," in P. W. Anderson, Kenneth J. Arrow, and David Pines, eds., *The economy as an evolving complex system*, Vol. 5. Redwood City, CA: Addison-Wesley, 1988, pp. 205–41.

Stahl, Dale O. "The Evolution of Smart$_n$ Players." *Games and Economic Behavior*, October 1993, *5*(4), pp. 604–17.

———. "Rule Learning in a Guessing Game." Working paper, University of Texas, 1994.

Stahl, Dale O. and Wilson, Paul W. "Experimental Evidence on Players' Models of Other Players." *Journal of Economic Behavior and Organization*, December 1994, *25*(3), pp. 309–27.

Van Huyck, John B.; Battalio, Raymond C. and Beil, Richard O. "Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure." *American Economic Review*, March 1990, *80*(1), pp. 234–48.

Econometrica, Vol. 65, No. 3 (May, 1997), 517–555

DUOPOLY STRATEGIES PROGRAMMED BY EXPERIENCED PLAYERS

BY REINHARD SELTEN, MICHAEL MITZKEWITZ, AND GERALD R. UHLICH

The strategy method asks experienced subjects to program strategies for a game. This paper reports on an application to a 20-period supergame of an asymmetric Cournot duopoly. The final strategies after three programming rounds show a typical structure. Typically, no expectations are formed and nothing is optimized. Instead of this, fairness criteria are used to determine cooperative goals, called "ideal points." The subjects try to achieve cooperation by a "measure-for-measure policy," which reciprocates movements towards and away from the ideal point by similar movements. A strategy tends to be more successful the more typical it is.

KEYWORDS: Duopoly, strategy method, computer tournament.

1. INTRODUCTION

AFTER 150 YEARS SINCE COURNOT (1838) the duopoly problem is still open. An empirically well supported duopoly theory has not yet emerged. Field studies meet the difficulty that cost functions, demand functions, and prices are often insufficiently observable. Game playing experiments permit the control of these basic data. However, only plays are observed and strategies remain hidden. Usually, any given play of a duopoly supergame can be the result of a great multitude of strategy pairs.

More than 20 years ago, one of the authors described a method of experimentation which makes strategies observable (Selten (1967)). This procedure, called the "strategy method," first exposes a group of subjects to the repeated play of a game, and then asks them to design strategies on the basis of their experiences. The strategy method was applied to an oligopoly situation with investment and price variation (Selten (1967)). In view of the special character of the dynamic oligopoly game investigated there, the issue of cooperation which will be important in the paper did not arise in this earlier study. Here we are concerned with a much more basic duopoly situation, namely a finite supergame of an asymmetric Cournot duopoly. Asymmetry is essential for this study, because we are interested in whether and how cooperation can evolve in such situations.

Cournot's quantity variation model is the most popular one in the oligopoly literature. Many theories have been developed in this framework. Supergames of the Cournot model have also been explored in the newer game-theoretical literature (e.g., Friedman (1977), Radner (1980), Abreu (1986), Segerstrom (1988)). Therefore, it seems to be interesting to apply the strategy method to a supergame of an asymmetric Cournot duopoly.

Infinite supergames cannot be played in the laboratory. Attempts to approximate the strategic situation of an infinite game by the device of a supposedly fixed stopping probability are unsatisfactory since a play cannot be continued beyond the maximum time available. The stopping probability cannot remain fixed but must become one eventually. Therefore, we decided to base our study on a finite supergame. The experimental literature shows that apart from the end effect there seems to be no significant behavioral difference between infinite and sufficiently long finite supergames (Stoecker (1983), Selten and Stoecker (1986)).

Our subjects were participants of a seminar who first gained experience in playing a 20-period supergame in the Bonn laboratory of experimental economics which is equipped with a network of personal computers. After having gained experience with the game, the participants had to program strategies. These strategies were played against each other in computer tournaments. The participants had the opportunity to improve their strategies in the light of their experience in such tournaments.

Our evaluation will mainly concern the strategies programmed for the final computer tournament. We shall only shortly report on some interesting phenomena observed in the initial game playing rounds and the intermediate tournaments.

The first step in the evaluation of the final tournament strategies was a classification according to structural properties. These properties, called "characteristics," were suggested by a close look at the strategies. We found 13 characteristics, all of which are present in the majority of cases to which they can be applied. A typical structure of a strategy emerges from these characteristics. The programs usually distinguish among an initial phase, a main phase, and an end phase. The initial phase consists of the first one to four periods with outputs depending only on the number of the period. In the main phase, outputs were made dependent on the opponent's previous outputs. By the initial phase the strategies try to prepare cooperation with the opponent to be reached in the main phase. In an end phase of the last one to four periods cooperation is replaced by noncooperative behavior.

Typically, the participants tried to approach the strategic problem in a way which is very different from that suggested by most oligopoly theories. These theories almost always involve the maximization of profits on the basis of expectations on the opponent's behavior. It is typical that the final tournament strategies make no attempt to predict the opponent's reactions and nothing is optimized. Instead of this, a cooperative goal is chosen by fairness considerations and then pursued by an appropriate design of the strategy. Cooperative goals take the form of "ideal points." An ideal point is a pair of outputs at which a player wants to achieve cooperation with his opponent. Such ideal points guide the behavior in the main phase. A move of the opponent towards the player's ideal point usually leads to responses which move the player's output in the direction of his ideal point. Similarly, a move of the opponent away from the ideal point is usually followed by a response which shifts the output away from the ideal point. We refer to this kind of behavior as a "measure-for-measure policy."

The fairness criteria underlying the selection of ideal points are different for different participants, but in most cases not completely arbitrary. Measure-for-measure policies for the effectuation of ideal points may be quite different in detail, but they are all based on the same general idea.

On the basis of the 13 characteristics which express structural properties common to most of the strategies we have constructed a measure of typicity which is applied both to characteristics and strategies. The typicity of a strategy is proportional to the sum of the typicities of its characteristics and the typicity of a characteristic is porportional to the sum of the typicities of the final tournament strategies with this characteristic. It was an unexpected result of our investigation that there is a highly significant positive correlation between the typicity of a final tournament strategy and its success in the final tournament. Moreover, it turned out that for each of the 13 characteristics separately those strategies which have it are more successful than those which do not have it.

In order to get a better insight into the implications of the typical structure of final tournament strategies, we constructed a family of "simple typical strategies." In these strategies the details left open by the 13 characteristics are filled in the simplest possible way. The behavior in the main phase is described by a piecewise-linear continuous reaction function.

Two game-theoretical requirements on simple typical strategies are discussed: "conjectural equilibrium conditions" and "stability against short-run exploitation." These requirements impose restrictions on the ideal points. The first requirement is rarely satisfied but the second one is fulfilled by the vast majority of the ideal points used in final strategies. This condition also turns out to be of descriptive value for the profit combinations reached in the last tournament.

We do not claim that our results are transferable to real duopolies. First of all, it is doubtful whether a supergame of the Cournot duopoly is a realistic description of duopolistic markets. Nevertheless, the structure of behavior in such supergames is of great theoretical interest. Our results throw a new light on the duopoly problem posed in this framework. The choice of an ideal point by fairness consideration combined with the pursuit of this cooperative goal by a measure-for-measure policy constitutes a surprisingly simple approach which avoids optimization and the prediction of the opponent's behavior. The connection between typicity and success in the final tournament shows that this approach is not only simple and practicable but also advisable in the pursuit of high profits.

The participants of our seminar did not develop their strategy programs independently of each other. Interaction in the game playing rounds and the preliminary tournaments was unavoidable. It cannot be completely excluded that our results are due to a cultural evolution which might have a different outcome in a different experimental group. One application of the strategy method alone is not sufficient to establish a firm basis for far-reaching behavioral conclusions.

The tit-for-tat strategy which did so well in Axelrod's tournaments (Axelrod (1984)) is the natural consequence of the transfer of the strategic approach emerging from this study to the prisoner's dilemma supergame. There one finds only one reasonable ideal point, namely the cooperative choice taken by both players, and only one measure-for-measure policy fitting this cooperative goal, namely tit-for-tat.

More recently, a paper by Fader and Hauser (1988) reports on programs written for two symmetric price triopolies. The players had no opportunity to play the games before writing their strategies and submitted a program only once for each of both models. Fader and Hauser classified strategies according to "features," but it cannot be said that a typical structure emerges from this classification. Perhaps the lack of a typical structure is due to the fact that in comparison to our students the participants of the tournaments were much less experienced with their task. Maybe it is necessary to provide the opportunity to gain extensive game-playing experience and to permit repeated program revisions after preliminary tournaments in order to obtain strategies which show a typical structure.

Nevertheless, this study shows that strategies based on the measure-for-measure principle are very successful against the strategies submitted. The agreement of our findings with those of Axelrod and of Fader and Hauser confirms our impression that the pursuit of ideal points by measure-for-measure policies is more than the accidental result of an isolated study.

The model and the experimental procedure are described in Sections 2 and 3. Then the results of the game playing rounds and the results of the tournaments are discussed in Sections 4 and 5. The evaluation of the strategies programmed for the final tournament begins with Section 6. There the 13 characteristics are introduced and explained in detail. The strategic approach underlying typical strategies is discussed. Section 7 is devoted to the connection between typicity and success. A family of simple typical strategies is introduced in Section 8 as an idealization of the general pattern observed in the programmed strategies. Theoretical properties of these strategies are discussed and game-theoretic stability conditions are compared with the data of the final tournament. Section 9 looks at the implications of our results for duopoly theory. A summary of our findings is given in Section 10.

2. THE MODEL

The experiment is based on a fixed nonsymmetric Cournot duopoly with linear cost and demand functions. Strategies had to be programmed for the 20-period supergame of this Cournot duopoly. The duopolists were fully informed about cost and demand functions, the length of the supergame, and the opponent's decisions in past periods. The decision variable of duopolist i in period t is the quantity $x_i(t)$ for $i = 1,2$ and $t = 1,\ldots,20$. Quantities must be chosen from nonnegative real numbers. The costs $C_1(t)$ and $C_2(t)$ of duopolists 1 and 2 and the price $p(t)$ in period t are given as follows:

$$C_1(t) = 9820 + 9x_1(t), \qquad x_1(t) \geq 0,$$
$$C_2(t) = 1260 + 51x_2(t), \qquad x_2(t) \geq 0,$$
$$p(t) = \max\,(0; 300 - x_1(t) - x_2(t)).$$

TABLE I

SOME THEORETICAL POINTS IN THE SOURCE GAME

Concept	Player 1's Output	Player 2's Output	Price	Player 1's Profit	Player 2's Profit
Cournot	111.0	69.0	120.0	2501.0	3501.0
Monopoly of player 1	145.5	0.0	154.5	11350.3	−1260.0
Monopoly of player 2	0.0	124.5	175.5	−9820.0	14240.3
Stackelberg with player 1 as leader	166.5	41.3	92.3	4041.1	441.6
Stackelberg with player 2 as leader	93.8	103.5	102.7	−1030.9	4096.1
Nash product maximum	86.8	49.5	163.7	3615.0	4313.5
Pareto optimum A of Figure 1	79.1	56.1	164.8	2503.8	5124.2
Pareto optimum B of Figure 1	94.8	42.7	162.5	4731.8	3501.0

The supergame payoff of each duopolist is the sum of his profits over all twenty periods.

Table I and Figure 1 show some theoretical features of the Cournot duopoly described above. The row "Nash product maximum" presents the output combination which maximizes the Nash product with the Cournot solution as fixed threat point. Point A in Figure 1 is the Pareto optimum which yields Cournot equilibrium profits for player 1. Analogously, B is the Pareto optimum which yields Cournot profits for player 2. Figure 1 shows that the model is quite asymmetric. Even point A is below the 45-degree line.

3. EXPERIMENTAL PROCEDURE

The experiment was performed in a seminar lasting over the whole summer semester 1987 at Bonn University, Federal Republic Germany. The subjects were 24 students of economics in the third or fourth year with some knowledge of micro- and macroeconomics and some experience with computer programming, but without special training in price theory and game theory. No introduction in these fields was given in the seminar and no references to the relevant literature was supplied. The seminar was organized in five plenary sessions, three rounds of game playing, and three computer tournaments of programmed strategies.

Plenary sessions: In the first plenary session the participants were informed about the organization of the seminar and the model presented in Section 2,

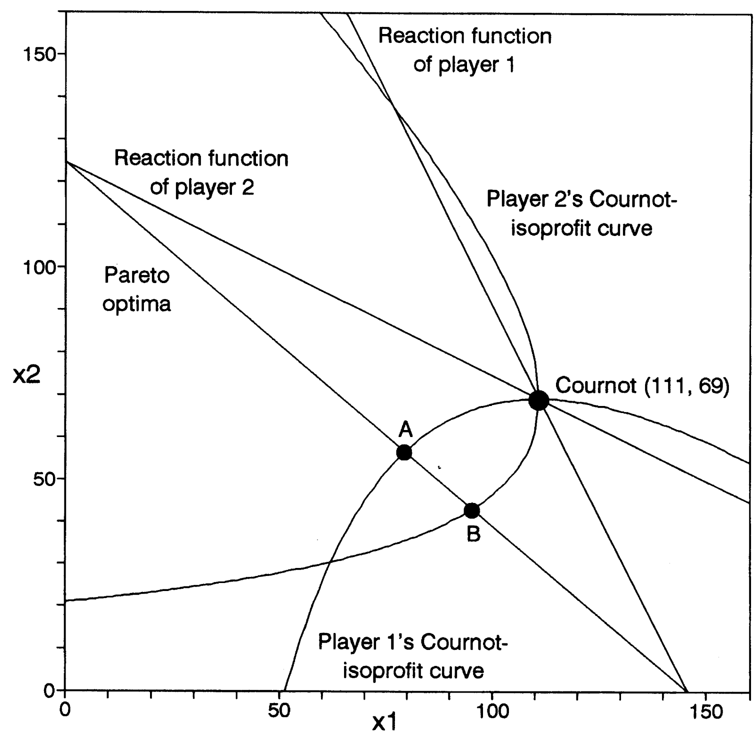


FIGURE 1.—Graphical representation of theoretical features of the one-period Cournot duopoly.

but, of course, without the theoretical features. Moreover, an introduction to the programming of strategies in PASCAL was given. It was not necessary to explain more than an excerpt of PASCAL, since strategies were conceived as subroutines in a game program.[1]

The participants had the task first to gain experience by three rounds of playing the 20-period supergame and then to program strategies for both players in the 20-period supergame. They were told that their objective should be to attain a sum of profits as high as possible in a final tournament in which the strategies of all participants compete against each other. Final strategies had to be documented and reasons had to be given for the decisions embodied in the strategies.

The second plenary session took place after two rounds of game playing. The results of these games were presented, but in a way which left players anonymous. The participants were asked to comment on their experiences.

[1]The Pascal source code of the students' strategies is available on request.

Each of the three tournaments was followed by one plenary session. Results were presented and students received printouts of the games in which their own strategies were involved. Opponents remained unidentified. The participants were encouraged to discuss strategic problems.

In the last of the five plenary sessions, the most successful participant explained his strategy. Anonymity was not completely preserved in this final plenary session at the end of the seminar.

Game playing rounds: Twenty-two subjects played three 20-period supergames against changing anonymous opponents, two subjects played only two supergames. The subjects were visually isolated from each other in cubicles containing computer terminals. The players interacted only by their decisions via the computer network. The decision time for each period was limited to three minutes. One week passed between one supergame and the next one. In this time the participants had the opportunity to reflect on their experiences. Each subject played with each of both cost functions at least once.

Strategy programming: After the game playing rounds the students had to program strategies in PASCAL for the 20-period supergame. Every student had to write a pair of strategies, one for each player of the supergame. We shall refer to this pair as the student's strategy. PC-owners could program at home, but all participants had the opportunity to develop their strategies at the Bonn laboratory of experimental economics with our technical assistance. A special program called "trainer" could be used by the students to play against their own programmed strategies. The "trainer" was a valuable tool for the development of strategies. No restrictions were imposed on strategies. Decisions could depend on the whole previous history of the play.

Computer tournaments: At three fixed dates the students had to hand in a programmed strategy. In the first two tournaments all workable strategies submitted at this date competed with each other. In the third tournament the last workable strategy of each participant was used. Each of the 24 students succeeded in writing at least one workable strategy.

The tournament program proceeded as follows: Let n be the number of workable strategies. Each of the n strategies played against all others in the role of both players. Payoff sums for player 1 and player 2 were computed on the basis of the $n - 1$ games played in the concerning role.

The procedure has the consequence that for each pair of strategies and each assignment of player roles, two supergames are simulated even if the payoff summation for one strategy makes use only of one of these games. Since sometimes random decisions are used in strategy programs, both games may be different. Altogether, $2n(n - 1)$ supergames were simulated in a tournament. The success of a strategy can be measured for the roles of both players separately by the corresponding payoff sums. The sum of these two measures is a measure for the overall success of a strategy in a tournament. This measure of overall success was the goal variable in the tournament. Strategies were ranked according to the measure of overall success, but also for the success of both player roles separately. Each participant received period-by-period printouts of all $2(n-1)$ games underlying the computation of his success measure. Moreover, all participants received lists of success measures, but without identification of the other writers of strategy programs. On the basis of this information the students could try to improve their strategy programs from one tournament to the next one.

Motivation: In view of the length of the experiment, it was not possible to provide an appropriate financial incentive. Presumably, money payoffs in the framework of a student seminar are not legal anyhow. The students were told that their grades would strongly depend on their success in the last tournament. It was emphasized that the absolute payoff sum rather than the rank was important in this respect. We had the impression that for almost all participants the task itself provided a high intrinsic motivation.

4. RESULTS OF THE GAME PLAYING ROUNDS

In this section we give a brief summary of the results of the game playing rounds. The games served the purpose to provide experiences which could be used in the development of strategy programs. Of course, it is plausible to assume that the subjects were intrinsically motivated by the game payoffs, but it is also possible that some of the behavior in these games was exploratory rather than directly payoff-oriented. Nevertheless, it is interesting to look at the results of the game playing rounds. However, our discussion will not be very detailed because our main interest is in the investigation of the final strategy programs.

First game playing round: Although the participants had been informed one week in advance about the structure of the game, their behavior seemed to be confused. Figure 2 shows the supergame payoffs of the 11 groups (two partici-

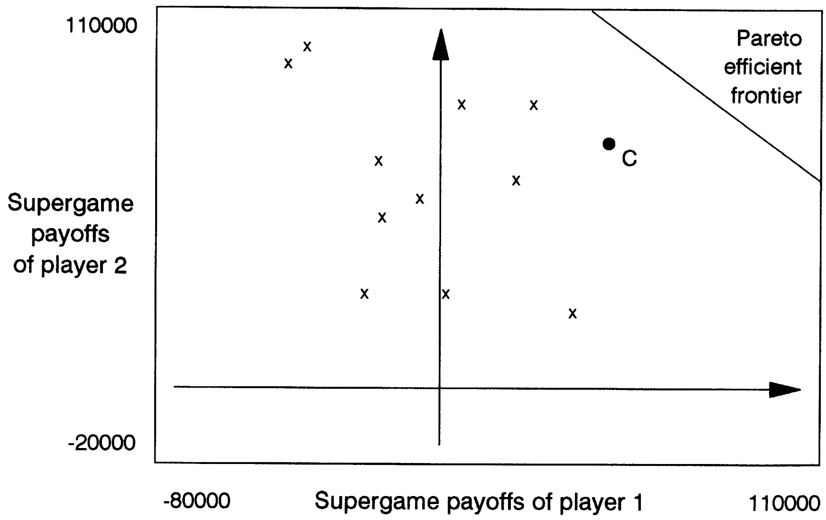


FIGURE 2.—Supergame payoff pairs in the first game playing round.

pants were absent). The repeated Cournot solution (point C in the diagram) yields 50020 for player 1 and 70020 for player 2. It never happened that *both* players achieved at least their Cournot payoffs. Furthermore, in all 11 cases the sum of both supergame payoffs was below the sum of the Cournot payoffs. In seven cases both players earned less than the Cournot payoffs. The role of player 1 (low variable and high fixed costs) was relatively less successful than the role of player 2. In the mean, subjects in the role of player 1 earned 79% of the Cournot gross profit (gross profit is profit plus fixed costs), whereas the corresponding figure for player 2 is 91%. The correlation coefficient between the payoffs of the two players within the groups is $-.36$. This suggests that some players succeeded to exploit their opponents. Figure 2 also shows part of the Pareto efficient frontier.

Second game playing round: The results of the second game playing round are shown in Figure 3. Here, two groups reached a Pareto improvement over the Cournot payoffs. In one group both players supplied the Cournot outputs in almost all periods. "It's the best thing you can do," they commented afterwards. In the remaining nine groups, both players sustained a loss in comparison with the Cournot solution. The mean gross profits of subjects in the role of player 1 was higher than in the first game playing round (87% of the Cournot gross profit), but the mean gross profit of subjects in the role of player 2 was lower than in the first game playing round (77% of the Cournot gross profit). The mean joint profit of both players was only slightly improved compared with the first game playing round.

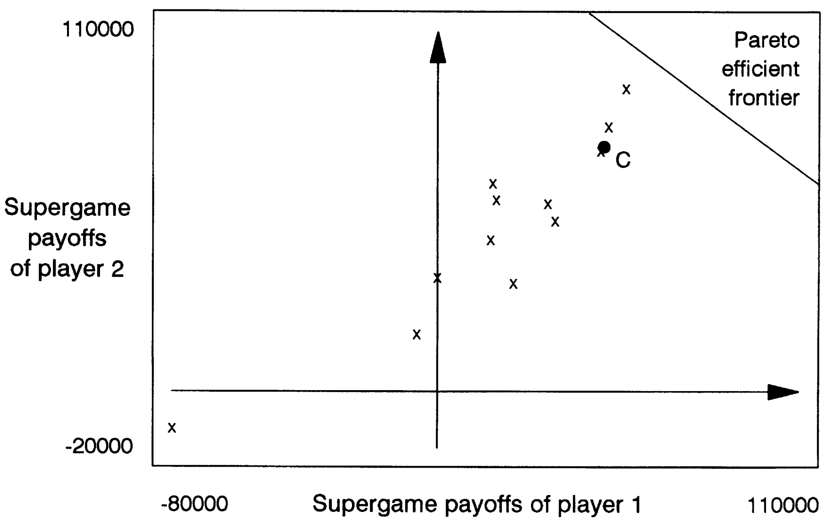


FIGURE 3.—Supergame payoff pairs in the second game playing round.

There is one striking difference to the first game playing round. In the second game playing round, the correlation coefficient between both players' payoffs is now $+.91$. This suggests that in the second game playing round the aggressiveness of both players shows a stronger coordination than in the first one. Even if most of the subjects did not yet succeed to play the game well, they seemed to have learned something about the power relationship in the game.

Third game playing round: This round shows an enormous improvement in mean payoffs (Figure 4). Now, subjects in the role of player 1 achieved 101% of the Cournot gross profit and the corresponding figure for those in the role of player 2 is 107%. Eight of the twelve groups succeeded to obtain Pareto improvements over the Cournot payoffs. One group reached a result almost at the Pareto efficient frontier. This group was the only one among those with Pareto improvements over the Cournot payoffs which did not show an end effect. The end effect consists in the breakdown of cooperation in the last periods of the supergame. It is clear that payoffs at the Pareto efficient frontier cannot be achieved if an end effect occurs.

The correlation coefficient between the payoffs of both players is $+.72$ in the third game playing round. In this respect, the third game playing round is similar to the second one.

It is clear that most of the subjects had learned to cooperate in the supergame in the third game playing round. The results of the three game playing rounds are not dissimilar to those obtained in other experimental studies where finite supergames were repeatedly played against changing anonymous opponents (Stoecker (1983), Selten and Stoecker (1986)). Subjects tend to learn to cooperate but they also learn to exhibit end effect behavior.

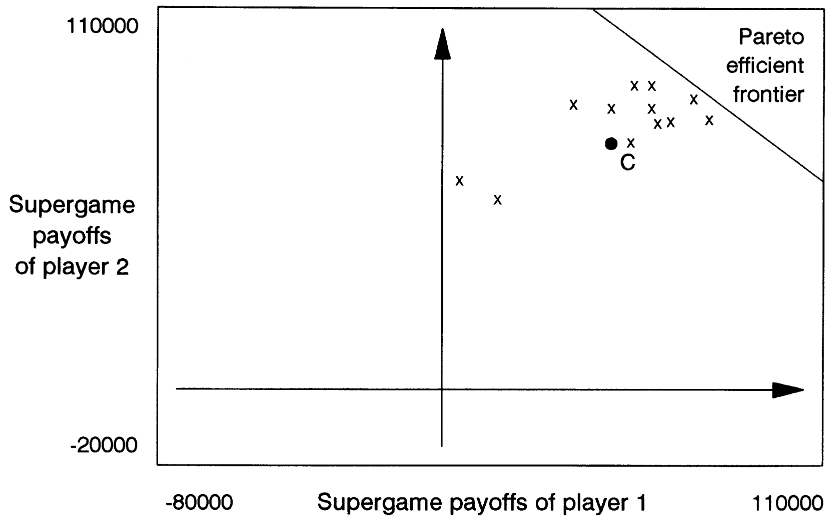


FIGURE 4.—Supergame payoff pairs in the third game playing round.

5. RESULTS OF THE TOURNAMENTS

In the following section we shall discuss the results of the tournaments without giving a detailed account of the strategies used. The typical structure of the strategies of the final tournament will be examined in the next section.

First tournament: Two weeks after the third game playing round the participants had to hand in a programmed strategy for the supergame. Unfortunately, 4 of the 24 strategies had to be excluded from the first tournament since programming errors like dividing by zero or taking the root of a negative number prevented the execution of these programs. The outcome of the first tournament is presented in Figure 5. The significance of the points in Figure 5 is not the same as in Figures 2, 3, and 4. A point now shows the combination of mean payoffs achieved by one participant's strategy in both player roles. Moreover, a larger scale has been chosen. One of the 20 strategies competing in tournament 1 is not shown in Figure 5 since it achieved a very bad result, namely (−6484, +58178), which is outside the scope of the drawing. We omitted this point in order to be able to present the results of all three tournaments with the same scale without losing the distinguishability of different points.

The participant with the omitted bad result programmed a strategy which supplied the respective Stackelberg leader output each period regardless of the behavior of the other player. Only a few times he succeeded in forcing his opponent to the Stackelberg follower position. In most cases his "aggressive" behavior was punished by high opponent's outputs.

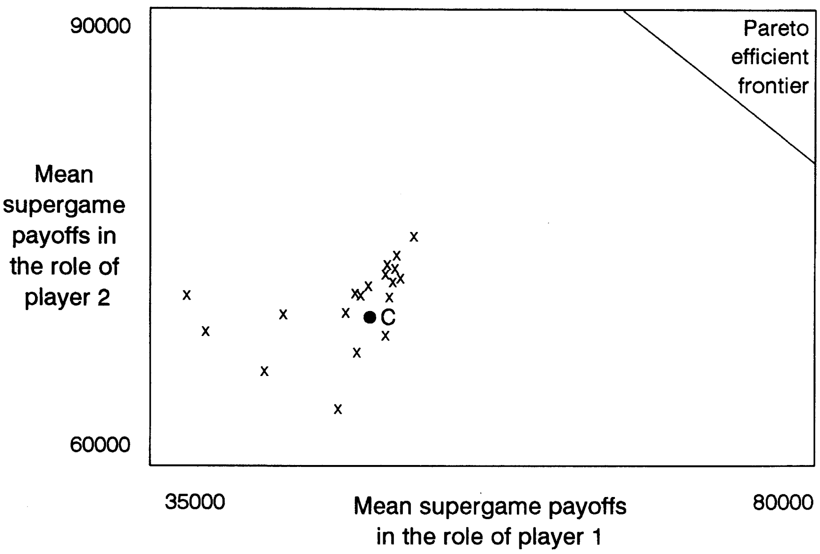


FIGURE 5.—Mean supergame payoffs for both player roles in the first tournament. Each "X" refers to one participant.

The mean gross profit over the whole simulation was 98% of the Cournot gross profit for the role of player 1 and 99.8% of the Cournot gross profit for the role of player 2. The mean performance is inferior to the third game playing round. Maybe the subjects did not yet succeed sufficiently well to mold their game playing intuition into computer programs.

Second tournament: Within three weeks after the first tournament the participants had the opportunity to improve their strategies. Unfortunately, this time only 16 participants presented workable strategies. In the same way as in Figure 5, the results are shown in Figure 6. One point, namely (23860, 63691) is omitted in Figure 6. Each of the other 15 subjects achieved results higher than Cournot payoffs in both player roles. The mean gross profit was now 104% of Cournot gross profit for Player 1 and 109% of Cournot gross profit for player 2. This is a considerable improvement in comparison with the first tournament. It must be admitted, however, that the comparison with the first tournament is difficult in view of the smaller number of workable strategies. Moreover, the result of the second tournament is also influenced by a "conspiracy" of two subjects represented by the two points nearest the right border of Figure 6. In the first period both participants used special outputs specified up to many decimal places in an unusual way. With the help of this code they recognized each other when they played together in the tournament. They then played in the remaining periods the output combination that maximizes joint profits. In order to prevent this type of behavior in the final tournament, we replaced the 8th digit behind the decimal point of each output decision by a random number.

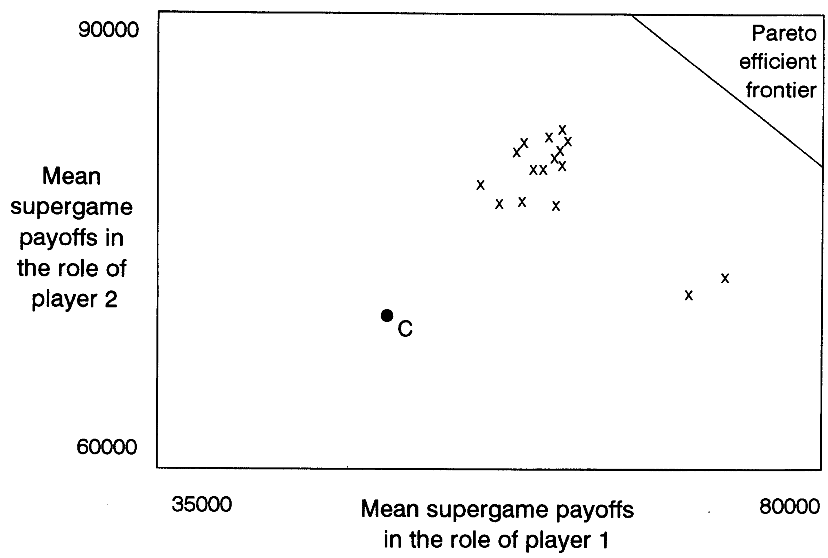


FIGURE 6.—Mean supergame payoffs for both player roles in the second tournament. Each "X" refers to one participant.

This has a negligible influence on the computation of profits. In the plenary session after the second tournament, we announced that similar conspiracies will be prevented in the future. We did not observe any attempt to conspire in the final tournament.

Third tournament: After two more weeks the final strategies had to be turned in. Again four participants did not succeed to program a workable strategy. Fortunately, each of these participants had completed at least one workable strategy in the two preceding tournaments. The last workable strategy entered the final tournament.

A superficial examination of the programs revealed that one strategy consisted of two sequences of fixed outputs for every period, one sequence for each player. The numbers varied unsystematically from period to period. The seminar paper of this student loosely described a completely different strategy which was much more reasonable. Obviously, this student wanted to avoid investing time and effort into the programming of the strategy described in his paper. The irregularity of the output sequences served the purpose of hiding the discrepancy between the program and its description in the seminar paper. Obviously, the programmed strategy cannot be taken seriously and therefore has been excluded from the third tournament for the purposes of this paper.

The mean gross profit was 105% of the Cournot gross profit for player 1 and 111% of the Cournot gross profit for player 2. These figures are only slightly higher than those of the second tournament. Figure 7 shows the results of the third tournament. Computations of standard deviations of mean payoffs confirm the visual impression that the points in Figure 7 are more strongly concentrated than those in Figure 6.

In 983 of the 1012 supergames simulated in the third tournament, the payoffs of *both* players were greater than their Cournot payoffs. In this sense, we can speak of successful cooperation in 97.1% of all cases. It is also worth mentioning that in none of the remaining 29 supergames did *both* players obtain smaller payoffs than their Cournot payoffs.

In the third game playing round only eight out of twelve supergames resulted in payoffs which were greater than the corresponding Cournot payoffs for both players. The comparison with the results of the third tournament shows that the final programmed strategies tend to be much more cooperative than the behavior in the third game playing round. This suggests that the learning process which began with the three game playing rounds was continued in the three tournaments. The results of the third tournament do not seem to be very different from that which could be expected as the outcome of spontaneous game playing after a comparable amount of experience.

6. THE STRUCTURE OF PROGRAMMED STRATEGIES

In this section we shall concentrate our attention on the structure of the final strategies. We shall not be concerned with the success of the strategies. For the reasons which have been discussed in Section 5 (third tournament), one of

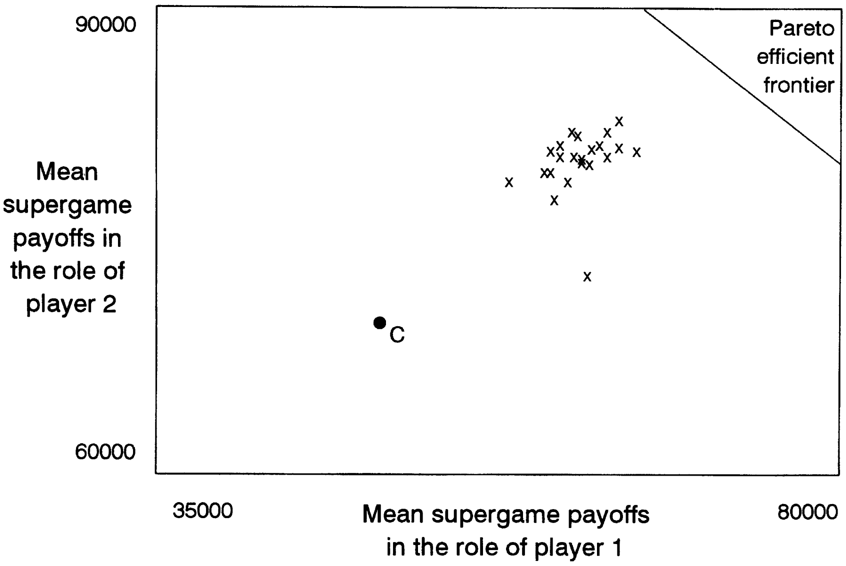


FIGURE 7.—Mean supergame payoffs for both player roles in the third tournament. Each "X" refers to one participant.

the programs will not be considered here. The remaining 23 programs and the underlying ideas expressed in the seminar papers are the basis of the evaluation of structural properties.

A preliminary examination of the strategies and the seminar papers conveyed the impression of a typical structure which is more or less present in almost all programs. Most programs deviate from this typical structure in some respects but the degree of conformity is remarkable.

Usually a program distinguishes three phases of the supergame: an initial phase, a main phase, and an end phase. The initial phase consists of one to four periods and the end phase is formed by the one to four last periods. The main phase covers the periods between the initial phase and the end phase. Different methods of output determination are used in the three phases. The initial phase is characterized by fixed outputs which do not depend on the behavior of the opponent. In the main phase the decisions are responsive to previous developments with the purpose to establish cooperation. In the end phase decisions are guided by the attempt to maximize short-run payoffs.

Different strategies approach the decision problems of the three phases in different ways, but nevertheless a typical structure emerges in this respect, too. In order to describe similarities and differences among the 23 strategies, we introduce 13 *characteristics*. A characteristic is a property of a strategy whose presence or absence can be objectively determined by the examination of a program and its description in the seminar paper. In some cases our characteristics are indicators of strategic ideas underlying the program; in other cases the characteristics directly refer to the structure of decision rules. We shall distinguish characteristics concerning general principles and the three phases of the supergame.

All characteristics are typical in the sense that they are present in the majority of all strategies to which they can be meaningfully applied. Characteristic 7 is meaningful only if Characteristic 6 holds, too, and Characteristics 12 and 13 presuppose that the strategy has an end phase. These three characteristics are present only in the majority of all relevant cases. All other characteristics hold for the majority of all final strategies.

6.1. *General Principles*

The first three characteristics are indicators of general principles underlying the typical approach to the problem of writing a strategy program.

CHARACTERISTIC 1: *No prediction.*

Many oligopoly theories proceed from the assumption that a player has a method to predict his opponent's behavior and tries to optimize against his predictions. The predictions may involve reactions to own output changes and the payoff maximization may be long-term rather than short-term. In the final tournament, only 5 of 23 strategies involved any predictions of the opponent's behavior.

In the first two tournaments, predictions were more widespread. Subjects tried to obtain a satisfactory payoff against the predicted output of the opponent in the next period. Several subjects who initially wrote programs involving predictions later expressed the opinion that it is useless to try to predict the opponent's behavior. It seems to be more important to react in a way which indicates willingness to cooperate and resistance to exploitation.

The fact that the absence of any predictions is a typical feature of final strategies seems to be of great significance, precisely because it is in contrast with most oligopoly theories.

CHARACTERISTIC 2: *No random decisions.*

At the beginning of the seminar we observed that several students preferred to build random decisions into their strategies. They motivated this by the belief that a deterministic strategy could possibly be outguessed and exploited by the opponent. In the course of the seminar, most of them learned that in an attempt to achieve cooperation, it is important to signal one's intentions. It may be preferable to be outguessed by the opponent. Cooperation requires reliability and random decisions may be counterproductive in this respect. Twenty-two of the 23 final strategies never make a random decision.

CHARACTERISTIC 3: *Non-integer outputs.*

It is natural that real persons playing at computer terminals use mostly integer outputs. This was actually the case in the game playing rounds. Usually, a programmed strategy employs functions which make the output dependent on previous quantities. In general, the values obtained are not integers. However, four of the final strategies did not specify such functions but rather made case distinctions; for each case a different integer output or integer output change was prescribed. Since only relatively few cases are distinguished, this way of programming output decisions is less flexible than the specification of a function. In this light, Characteristic 3 is an indicator of smoothness and flexibility of the response pattern.

6.2. *The Initial Phase*

Two characteristics describe the typical behavior in the initial phase.

CHARACTERISTIC 4: *Fixed outputs for at least the first two periods.*

If no randomization takes place the first period is always a fixed output. Therefore, Characteristic 4 is almost equivalent to a nontrivial initial phase where fixed outputs are chosen. Ten strategies make their decision for the second period dependent on their opponent's choice in the first period, but 13 strategies have fixed amounts for more than one period. The length of the initial phase with fixed outputs is two periods for seven strategies, three periods for four strategies, and four periods for two strategies. Twelve of the 13 strategies with nontrivial initial phases play successively reduced outputs. The participants explained this behavior as a signal of their willingness to cooperate. If one's own output is a response to that of the opponent too early, an unsatisfying decision of the opponent in the first period could lead to an aggressive reaction of oneself in the second period that again could annoy the opponent and so forth, so that no cooperation might evolve over the 20 periods. Some subjects observed such unfavorable oscillations in the printouts of the first two tournaments.

CHARACTERISTIC 5: *The last fixed output decision is at least 8% below the Cournot quantity of the concerning player.*

The percentage by which the last fixed output in the initial phase is below the Cournot output can be regarded as a rough measure of a strategy's initial cooperativeness. A Pareto optimum is reached if both players' outputs are about 24.5% below the Cournot output. The criterion of the 8% limit of Characteristic 5 goes roughly a third of the way towards this Pareto optimum. Admittedly, it is arbitrary to measure cooperativeness by percentages of the Cournot output and to fix the limit at exactly 8%. Characteristic 5 is present in 13 of the 23 strategies. If the limit were increased to 10%, only a minority of 10 strategies would meet the criterion.

6.3. *The Main Phase*

The decision rules for the main phase are the most important part of a strategy program. Characteristics 6 to 11 concern the main phase. The rules given there do not apply to the initial phase and the end phase. This will not be mentioned explicitly in the text of the characteristics.

Typically participants approached the problem of decision making in the main phase by first looking at the question of where cooperation should be achieved. They tried to find an output combination which gives higher profits than Cournot profits to both players and can be considered as a reasonable compromise between the interests of both players. An output combination of this kind which guides the decisions in the main phase will be called an "ideal point." Ideal points are usually not far away from Pareto optimality. They are often based on considerations of equity which will be described below. Some participants used different ideal points for the roles of both players.

In Characteristic 6 we shall speak of "decisions guided by ideal points." With these words we want to express that the strategy program makes explicit use of an ideal point in order to determine the next output as a function of the past history. This can be done in many ways. One possible method connects the ideal point and the Cournot point by a straight line segment in the quantity or profit space. The next output then matches the opponent's last output on the line segment as long as the opponent's last output is in the range where this is possible.

CHARACTERISTIC 6: *Decisions are guided by one or two ideal points.*

The property expressed by Characteristic 6 holds for 18 of the 23 final strategies. Twelve strategies use only one ideal point for both players, whereas 6 strategies specify different ideal points for the two player roles.

Table II gives an overview over the equity concepts underlying the construction of ideal points as far as such concepts could be identified on the basis of the seminar papers. The reasons for the choice of 10 of the 24 ideal points are at least partially unclear. To some extent ideal points were adapted to the learning experience of the first two tournaments and thereby shifted away from equity concepts.

The participants who based their ideal points on equity considerations often did not correctly compute the intended ideal points. They rarely used analytical methods but rather relied on more or less systematic numerical search. The values used instead of the correct ones are given in the footnotes below Table II.

The concept described by the first row of Table II looks at equal profit increases in comparison to Cournot profits as a fair compromise. The Pareto optimum corresponding to this idea is the intended ideal point. The concept of the second row requires profit increases proportional to Cournot profits at a Pareto optimal point.

TABLE II

CONCEPTS UNDERLYING IDEAL POINTS

Concept	Quantities		Number of Strategies
	Player 1	Player 2	
Maximal equal absolute additional profits compared to Cournot profits	85.61	50.50	3[a]
Maximal profits proportional to Cournot profits	84.37	51.56	2[b]
Profit monotonic quantity reduction along the straight line through the intersections of both Cournot-isoprofit curves	86.53	49.71	2
Profit monotonic quantity reduction proportionally to Cournot quantities	89.73	55.77	1[c]
Maximal equal profits	89.70	47.01	2[d]
Prominent numbers	85.00	50.00	2
	90.00	50.00	2
Unclear	—	—	10

[a]Approximated by (85.50) in all three cases.
[b]Approximated by (86.53) and (84.33, 51.55).
[c]Approximated by (89.76, 55.80).
[d]In one case approximated by (89.0, 46.5).

The third and fourth rows of Table II involve a procedure referred to as *profit monotonic quantity reduction.* Along a prespecified positively inclined straight line through the Cournot point in the quantity diagram, quantities are gradually reduced as long as both profits are increased in this way. The output combination reached by the procedure is the ideal point. In the case of row 3 of Table II, the prespecified straight line connects the intersections of both Cournot isoprofit curves. In the case of row 4 the prespecified straight line connects the Cournot point and the origin.

The concept of row 3 yields a Pareto optimum even if Pareto optimality is not a part of the underlying idea. Contrary to this, profit monotonic quantity reduction proportional to Cournot quantities yields an ideal point which is not even approximately Pareto optimal.

The concept of maximal equal profits determines the Pareto optimum where both profits are equal. Obviously, this ideal point does not only depend on variable costs but also on fixed costs. The same is true for maximal profits proportional to Cournot profits. Two of the ideal points classified as unclear also were based on equal profits but without an attempt towards maximization.

Some participants chose pairs of prominent quantities as ideal points. Roundness in the sense of divisibility by 5 seems to be the prominence criterion. More detailed discussions of prominence in the decimal system can be found in the literature (Schelling (1960), Albers and Albers (1983), Selten (1987)).

Figure 8 shows the ideal points used by the final strategies. The ideal points are given as quantity combinations. In the quantity diagram the Cournot-isoprofit curves of the two players enclose a lens-shaped area. The ideal points used by final strategies are in a relatively small area in the middle of this lens. The mean of all ideal points is located at (87.02, 49.43). This combination is almost Pareto optimal.

Characteristics 7 to 11 are described as rules to be followed by a programmer of a strategy.

CHARACTERISTIC 7: *If your opponent has chosen an output below his output specified by your ideal point, then choose your ideal point quantity in the next period.*

If a strategy is based on two ideal points then the words "your ideal point" refer to the ideal points for the concerning player role. The interpretation of Characteristic 7 is simple. If your opponent is even more cooperative than required by your ideal point, then there is no reason to deviate from your own ideal point quantity. Ten of the 18 final strategies based on ideal points have this characteristic. However, some other strategies increase the output in the situation of Characteristic 7 in order to test the opponent's willingness to cooperate at a point more favorable for oneself.

The remaining characteristics will be applicable to strategies without ideal points, too. Even if a strategy is not based on an ideal point, it may still involve a

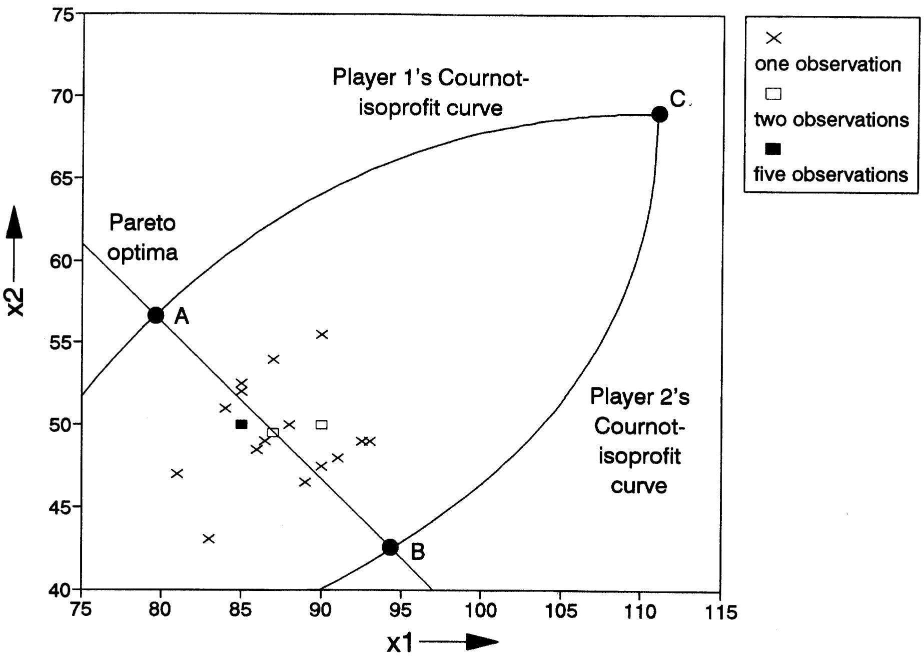


FIGURE 8.—Ideal points used by final strategies.

measure which permits an interpersonal comparison of cooperativeness. Thus a strategy may look at the profit difference achieved at the beginning of the main phase as a standard reference. Both players are judged to be equally cooperative if this profit difference is attained.

CHARACTERISTIC 8: *If your opponent has chosen an output above his Cournot quantity, then in the next period choose your Cournot quantity.*

Twelve of the 23 strategies obey this rule. The other strategies do not use more rigorous methods of punishment; instead, if they realize that their opponent plays permanently above his Cournot quantity, they abandon the idea of punishment after some periods and reduce their own output below their Cournot quantity to increase their profits. Such strategies run the danger of becoming exploitable by attempts to establish Stackelberg leadership. Characteristic 8 on the one hand avoids excessive aggressiveness and on the other hand provides protection against exploitative opponents.

CHARACTERISTIC 9: *If your opponent has chosen his Cournot quantity, then in the next period choose a quantity not higher than your Cournot quantity and 5% at most below your Cournot quantity.*

It can be seen with the help of Figure 8 that Characteristic 9 limits the response to the opponent's choice of his Cournot quantity to a relatively small interval. Sixteen of the 23 final strategies satisfy the requirement of Characteristic 9. Among these 16 strategies there are 10 which respond to Cournot quantities by Cournot quantities. The remaining 6 strategies want to indicate their willingness to cooperate by a slightly smaller output. Of course, the number of 5% in Characteristic 9 is to some extent arbitrary.

The following two Characteristics 10 and 11 apply to situations in which the following four conditions hold.

(i) The last period was a period of your main phase.

(ii) Up to now you always followed your strategy.

(iii) In the last period your opponent's output was below his Cournot output.

(iv) If you have an ideal point (for the relevant player role), then your opponent's output was above his output in your ideal point.

CHARACTERISTIC 10: *Suppose that conditions (i), (ii), (iii) and (iv) hold. If in the last period your opponent has raised his output, then your decision raises your output to a quantity below your Cournot output.*

CHARACTERISTIC 11: *Suppose that conditions (i), (ii), (iii), and (iv) hold. If in the last period your opponent has lowered his output, then your decision lowers your output. If you have an ideal point, then your new output remains above your ideal point output.*

To illustrate Characteristics 10 and 11, let us give an example: Consider a strategy which in the main phase matches the opponent's last output on a straight line between the Cournot point and an ideal point in the quantity space, of course, only as long as the opponent's last output was in the relevant range. A strategy of this kind satisfies Characteristics 10 and 11. However, it is necessary to impose condition (i) since in the first period of the main phase matching on the line may require an increase of output even if the opponent has lowered his output.

As long as condition (ii) is satisfied matching on the line in later periods of the main phase will move in the right direction. Conditions (iii) and (iv) make sure that Characteristics 10 and 11 apply only in the relevant range.

Both characteristics can be satisfied for strategies not based on a line between the Cournot point and an ideal point in any space. They may even be satisfied for strategies without ideal points. Thus a strategy's response may be guided by the criterion of a profit difference equal to that at the Cournot point without any regard to Pareto optimality. Two of the final strategies were of this kind.

Fourteen final strategies have Characteristic 10. The number of final strategies with Characteristic 11 is also 14, but only 11 final strategies have both characteristics.

6.4. *The End Phase*

A strategy with an end phase has a special method of output determination for the last one to four periods. Attempts towards cooperation which are typical for the main phase are not continued in the end phase. Instead of this, short-run profit goals are pursued.

Only 2 of the 23 final strategies do not have an end phase. One of these 2 strategies was typical in many other respects but the other was the most atypical. This atypical strategy tries to estimate response functions of the opponent and then computes the output decision by an elaborate approximative method for the solution of the dynamic program of maximizing expected profits for the remainder of the game. Even if something like an end effect is automatically produced by the dynamic programming approach, no end phase is present here since the method of output determination is always the same.

CHARACTERISTIC 12: *The strategy has an end phase of at least two periods.*

Characteristic 12 is shared by 11 of the 21 final strategies with end phases. Ten of these strategies planned an end effect only for the last period.

CHARACTERISTIC 13: *The strategy has an end phase and specifies the Cournot output of the relevant player as the output for all periods of the end phase.*

This characteristic is present in 12 final strategies. Other strategies sometimes optimized short-run profits against the opponent's last output or approached the Cournot output in several fixed steps.

6.5. *The Strategic Approach Underlying Typical Strategies*

A typical strategy does not try to optimize against expectations on the opponent's behavior (Characteristic 1). The strategic problem is not viewed as an optimization problem but rather as a bargaining problem. The first question to be answered concerns the point where cooperation should be achieved. Of course, cooperation should be favorable for oneself but it also must be acceptable for the opponent. A failure to reach cooperation is expected to lead to Cournot behavior. Therefore, cooperation requires that both players obtain more than their Cournot profits. Ideal points are constructed as reasonable offers of cooperation within these constraints. Various kinds of fairness considerations but also prominence (divisibility by five) and prior experience may influence the selection of ideal points.

After the choice of an ideal point the question arises as to how cooperation at this point or in its neighborhood can be achieved. It is necessary to indicate one's willingness to cooperate there and to show that one is not going to accept less favorable terms.

A decreasing sequence of outputs in the initial phase is a natural signal indicating cooperativeness. In the main phase a typical strategy evaluates the cooperativeness of the opponent's last output and responds by an output of a similar degree of cooperativeness according to some criterion. The response may depend on whether the opponent decreased or increased his output. If there is such a difference, it is natural to respond more aggressively to the same output after an increase.

One may say that main-phase behavior is guided by a principle of "measure for measure." Small changes of the opponent's output lead to small reactions and big changes cause big reactions.

Many oligopoly theories are based on the idea that a player anticipates the reaction of his opponent in order to maximize his profits. Contrary to this, a strategy based on an ideal point and a response rule guided by the principle "measure for measure" does not involve any anticipation of the opponent's reactions. The aim is to exert influence on the opponent rather than to adapt to his behavior. In order to achieve this aim one's own behavior has to provide a clear indication of one's own intentions. If the implied offer of cooperation is reasonable, one can hope that the aim will be reached. A response guided by the principle "measure for measure" protects against attempts to exploit one's own cooperativeness and rewards cooperative moves of the other player.

Of course, cooperation breaks down in the end phase. The strategies have been written for the 20-period supergame. This game permits only one subgame perfect equilibrium path, namely Cournot outputs in every period. The participants were aware of the backward induction argument which came up in the discussions of the plenary sessions. They accepted the idea that cooperation must break down in the last periods but as the strategies show they did not accept the full force of the backward induction argument. An explanation of this phenomenon is given elsewhere (Selten (1978a)).

7. TYPICITY AND SUCCESS

All characteristics are typical for the final strategies in the sense that they are present in the majority of the cases to which they are applicable. Of course, they are not all equally typical. Some appear in more of the final strategies than others. Moreover, the extent to which a characteristic is typical should not only be judged by the number of strategies with this characteristic, but also by the extent to which these strategies are typical. In the following, we shall construct a measure of typicity applicable to both characteristics and strategies which tries to do justice to these considerations.

The measure of typicity assigns a real number to each characteristic and to each strategy. The sum of the typicities of all 13 characteristics is normed to 1. The measure of typicity can be thought of as the outcome of an iterative procedure. At the beginning, all characteristics have the same typicity 1/13. Then, in each step, first a new typicity is computed for each strategy as the sum of the typicities of its characteristics. Afterwards, a new typicity for each characteristic is computed as proportional to the sum of the typicities of the strategies with this characteristic. The sum of the typicities of all characteristics is again normed to 1.

In order to give a more precise mathematical definition of our measure, it is necessary to introduce some notation. The typicity of characteristic i is denoted by c_i and s_j stands for the typicity of strategy j. The symbol c is used for the column vector with the components $c_1, \ldots, c_{13}$ and s denotes the column vector with the components $s_1 \cdots s_{23}$. Let A be the 13×23-matrix with entries a_{ij} as follows: $a_{ij} = 1$ if strategy j has characteristic i, and $a_{ij} = 0$ otherwise. In our case c and s are uniquely determined by the following equations.

$$c = \alpha A s,$$
$$s = A^T c,$$
$$\sum_{i=1}^{13} c_i = 1,$$

where A^T is the transpose of A and $1/\alpha$ is the greatest eigenvalue of AA^T. It is a consequence of elementary facts of linear algebra that the iterative process described above converges to vectors c and s which can be described as the solution of this system of equations.

Table III shows which strategy has which characteristics. The rows correspond to the 13 characteristics and the columns to the 23 final strategies. The strategies have been numbered according to the success in the final tournament. Strategy 1 is the most successful one, strategy 2 the second most successful one, etc. A black mark indicates that the strategy corresponding to the column has the characteristic corresponding to the row.

Obviously, the black marks in Table III describe the matrix A. A black mark corresponds to an entry 1 and the absence of a black mark corresponds to an entry 0. The typicities of the characteristics are given at the right margin and the typicities of the strategies can be found at the bottom of Table III.

TABLE III

TYPICITY OF CHARACTERISTICS AND STRATEGIES[a]

Characteristics	Strategies																						Typicity
1	—	—	—	—	—	—		—	—	—	—		—	—	—		—	—	—		—		— .0917
2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—		—	— .1062
3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—				—	—	.1005
4	—	—		—			—				—	—	—	—	—		—	—			—		.0609
5	—	—	—	—		—	—			—	—	—		—	—	—	—						.0710
6	—	—	—	—	—	—	—	—	—	—				—	—	—	—	—	—		—		— .0927
7	—	—	—	—	—				—						—	—					—		— .0545
8	—		—		—	—		—	—			—	—	—		—			—		—		.0653
9	—	—	—		—	—		—	—	—		—	—	—	—	—		—			—	—	.0851
10	—		—		—		—	—	—	—	—	—	—			—					—	—	— .0729
11	—	—	—			—	—	—	—	—		—	—			—	—					—	— .0749
12	—	—	—	—		—	—				—		—			—	—	—					.0591
13	—		—					—	—	—	—	—		—		—	—		—		—		.0652
Ranking of success	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
Ranking of typicity	1	5	2	17	14	8	16	7	4	6	18	12	11	9	15	3	10	19	22	23	13	21	20
Typicity	1.0000	.7966	.9391	.6366	.6689	.7465	.6382	.7545	.8090	.7602	.6275	.7020	.7166	.7386	.6626	.8474	.7222	.4957	.4211	.1062	.6888	.4396	.4929

[a] The Spearman rank correlation coefficient between typicity and success of strategies is $r_s = .619$.

The table also shows the ranking of success in the final tournament and the ranking of typicity of the 23 strategies. The Spearman rank correlation coefficient between success and typicity is $+.619$. This value is significant at the 1% level (two-tailed test).

It is an unexpected phenomenon that *there is a strongly significant positive correlation between the typicity and the success of final strategies*. In principle, the opposite relationship would also seem to be possible. It is not inconceivable that typical characteristics reflect nothing else than typical mistakes. However, in our case the characteristics seem to embody advisable strategic principles. Maybe the positive correlation between typicity and success is the result of the learning process which produced the final strategies.

For each characteristic the mean success rank of those strategies which have it is smaller than the mean success rank of those which do not have it. This shows that *each of the characteristics separately is positively connected to the success in the final tournament.* In this sense all 13 characteristics are favorable structural properties of a strategy.

Our judgments of the advisability of the characteristics must be understood relative to the strategies developed by the participants of our experiment. We cannot exclude the possibility that a very atypical strategy can be found which turns out to be very good in a tournament against the 23 final strategies. In fact, the participant who wrote a strategy with success rank 20 firmly believes that this approximative dynamic programming approach based on an estimated response function of the opponent can be improved to a degree which will make it superior to all final strategies in a tournament against them. We doubt that this is the case. The difficulty with the dynamic programming approach is the problem of forming a correct estimate of the opponent's behavior. A best response to a wrong prediction can have disastrous consequences.

Admittedly, our experiment does not really justify strong conclusions since the final strategies have not been developed independently of each other. Perhaps a different picture of a typical strategy would emerge in a repetition of the experiment. Nevertheless, the results reported in this section seem to be of considerable significance for the further development of oligopoly theory.

8. A FAMILY OF SIMPLE TYPICAL STRATEGIES

The 13 characteristics do not completely determine a strategy. Many details are left open. In this section we shall construct a family of strategies which are typical in the sense that they have all 13 characteristics and the missing details are furnished in a particularly simple way. The members of the family differ only by the pair of ideal points used for both player roles. The special case of only one ideal point is not excluded.

For our family of simple typical strategies we shall discuss the question of what happens if two strategies with different ideal points play against each other. This exercise conveys some insight into the strategic properties implied by the 13 characteristics. We shall also look at the question of what is a reasonable choice of ideal points. For this purpose we have determined that member of the family which did best in a tournament against 22 of the final strategies. (The only strategy which involved random decisions was eliminated in order to avoid time consuming Monte-Carlo simulation.)

8.1. *Description of the Simple Typical Strategies*

The ideal points are described by output pairs u and v, one for each player role:

Ideal point for the role of player 1: $u = (u_1, u_2)$.

Ideal point for the role of player 2: $v = (v_1, v_2)$.

The first components of the vectors u and v denote player 1's output and the second stands for player 2's output. As mentioned above, the special case $u = v$ is not excluded. We also introduce the following notation for the output combination in the Cournot equilibrium of the underlying duopoly.

Cournot equilibrium: $c = (c_1, c_2)$.

We now can describe the decision $x_i(t)$ specified by the *simple typical strategy* with ideal points u and v. The following conditions (i) and (ii) have been imposed on the ideal points:

(i) The ideal points u and v are Pareto superior to the Cournot equilibrium.

(ii) $u_1 \leq .92c_1$ and $v_2 \leq .92c_2$.

Condition (ii) is necessary to make the specification of the initial phase compatible with Characteristic 5.

Initial phase:

$$x_1(t) = \frac{t}{3}u_1 + \frac{3-t}{3}c_1,$$

$$x_2(t) = \frac{t}{3}v_2 + \frac{3-t}{3}c_2 \qquad \text{for} \quad t = 1,2,3.$$

Main phase:

$$x_1(t) = \begin{cases} u_1 & \text{for } x_2(t-1) \leq u_2, \\ c_1 & \text{for } x_2(t-1) \geq c_2, \\ u_1 + \dfrac{c_1 - u_1}{c_2 - u_2}(x_2(t-1) - u_2) & \text{otherwise;} \end{cases}$$

$$x_2(t) = \begin{cases} v_2 & \text{for } x_1(t-1) \leq v_1, \\ c_2 & \text{for } x_1(t-1) \geq c_1, \\ v_2 + \dfrac{c_2 - v_2}{c_1 - v_1}(x_1(t-1) - v_1) & \text{otherwise.} \end{cases}$$

End phase:

$$x_i(t) = c_i \quad \text{for } i = 1,2 \text{ and } t = 19,20.$$

The initial phase can be thought of as a sequence of three equal "concessions" moving from the Cournot output c_i to the ideal point output u_1 or v_2 respectively. The first period already makes the first concession. Obviously, the initial phase satisfies Characteristic 4 which requires at least two periods. Characteristic 5 is satisfied since u_1 and v_2 are not greater than $.92c_1$ and $.92c_2$, respectively.

Characteristic 6 requires that the strategy make use of ideal points. Obviously, this is the case for our family of simple typical strategies.

We now turn our attention to the equation for the main phase. The upper line on the right-hand side secures Characteristic 7. The middle line is in agreement with Characteristics 8 and 9. Characteristic 9 concerns the special case $x_j(t-1) = c_j$ and permits a response $x_i(t)$ up to 5% lower than c_i. As has been pointed out before, the majority of these final strategies which conformed to Characteristic 9 specified a response of exactly c_i. Therefore, this response can be considered as typical.

The lower line on the right-hand side of the equation for the main phase is a very simple version of the principle "measure for measure." The last output of the opponent is matched by the corresponding output on the straight line which connects the ideal point and the Cournot point in the quantity space. Obviously, this has the consequence that Characteristics 10 and 11 are present in the strategies of our family.

The end phase has two periods and, therefore, conforms to Characteristic 12. The output in the end phase is always c_i, as required by Characteristic 13.

The strategies of our family also have the Characteristics 1, 2, and 3. In accordance with Characteristic 1, no attempt is made to predict the opponent's behavior and to optimize against this prediction. As required by Characteristic 2, the strategies are completely deterministic. In the main phase the strategies permit a continuum of possible responses and therefore have Characteristic 3.

8.2. *Simple Typical Strategies Playing Against Each Other*

Consider a play of the 20-period supergame where each of both players uses a member of the family described above as his strategy. Let u and v be the ideal points of the strategy of player 1. Similarly, let u^* and v^* be the ideal points of the strategy of player 2. Actually, only u and v^* are of interest here since we have fixed the player roles.

The behavior in the main phase can be described by two "reaction functions," r and r^*:

$$r(x_2) = \begin{cases} u_1 & \text{for } x_2 \le u_2, \\ c_1 & \text{for } x_2 \ge c_2, \\ u_1 + \dfrac{c_1 - u_1}{c_2 - u_2}(x_2 - u_2) & \text{otherwise;} \end{cases}$$

$$r^*(x_1) = \begin{cases} v_2^* & \text{for } x_1 \le v_1^*, \\ c_2 & \text{for } x_1 \ge c_1, \\ v_2^* + \dfrac{c_2 - v_2^*}{c_1 - v_1^*}(x_1 - v_1^*) & \text{otherwise.} \end{cases}$$

The development of the play in the main phase is given by the following equations:

$$x_1(3) = u_1,$$

$$x_2(3) = v_2^*,$$

$$x_1(t) = r(x_2(t-1)) \quad \text{for } t = 4,\dots,18,$$

$$x_2(t) = r^*(x_1(t-1)) \quad \text{for } t = 4,\dots,18.$$

Figure 9 shows four examples for the development of this system of difference equations. In Figures 9a and 9b the path of output combinations moves towards the Cournot equilibrium. In Figure 9c the path stays at (u_1, v_2^*) for $t = 3,\dots,18$.

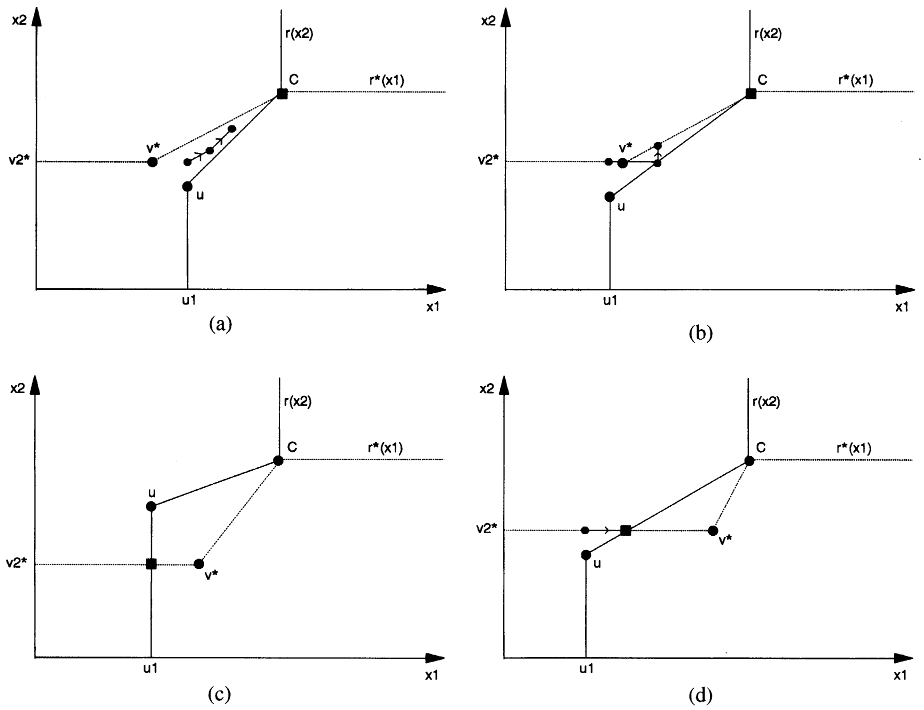


FIGURE 9.—Simple typical strategies playing against each other. Four examples with different ideal point pairs u and v^*.

In Figure 9d we have

$$x_1(t) = r(v_2^*) \quad \text{for } t = 4,\ldots,18,$$
$$x_2(t) = v_2^* \quad \text{for } t = 4,\ldots,18.$$

We shall speak of a *conflict case* if the output combination path moves towards the Cournot equilibrium, and of an *agreement case* if the output path becomes stationary in periods 4 to 18.

It can be seen without difficulty that a conflict case is obtained whenever the Cournot output combination is the only common point of r and r^*. All other cases are agreement cases. An agreement case is also characterized by the condition that player 2's ideal point is not above the straight line through the Cournot point and player 1's ideal point. This is the case if and only if player 1's ideal point is not below the straight line through the Cournot point and player 2's ideal point. From what has been said it follows that an agreement case is obtained if and only if the following inequality holds:

$$\frac{c_2 - v_2^*}{c_1 - v_1^*} \geq \frac{c_1 - u_2}{c_1 - u_1}.$$

Figure 9 will illustrate the consequences of this condition: In the special case in which both ideal points are Pareto optimal, an agreement case is reached if each player does not ask for more than the other player will grant him. The ideal points are like bargaining offers. The less one asks for oneself and the more one grants to the other player, the better are the chances for agreement.

In view of the condition for an agreement case it seems to be quite reasonable to specify two different ideal points for the two player roles in such a way that player 1's ideal point is more favorable for player 2 and vice versa. However, those 6 participants who specified two different ideal points did this in a way which leads to a conflict case if the strategy plays against itself. In each player role these subjects wanted more for themselves than they would grant to the other player if he were in this role.

It can be seen without difficulty that the condition which distinguishes agreement cases from conflict cases does not depend crucially on the special way in which our simple typical strategies specify the initial phase. As long as at the end of the initial phase both outputs are below the respective Cournot outputs, the output combination path moves towards the Cournot point in a conflict case and towards stationary cooperation in an agreement case.

8.3. *The Best Ideal-Point Selection Against the Final Strategies*

It is interesting to ask the question of what is the best selection of ideal points within the family of simple typical strategies defined above in a tournament against the final strategies. Actually, we simulated tournaments only against 22 of the final strategies since we omitted the only strategy which uses random choices. The best choice of ideal points turned out to be as follows:

$$u = (89.4, 55.6),$$

$$v = (86.6, 50.4).$$

Both components of u are greater than the corresponding components of v, but if this strategy plays against itself an agreement case is obtained; the quantity combination (89.4, 52.6) is played in periods 4 to 18.

The ideal point (86.6, 50.4) is nearly Pareto optimal whereas $u = (89.4, 55.6)$ is relatively far from the Pareto optimal line. However, $u = (89.4, 55.6)$ has the advantage that it yields agreement cases against all ideal points which have been specified for the role of player 2 by those of the 22 participants who used ideal points. This is due to the fact that $u_2 = 55.6$ is rather large.

The ideal point $v = (86.6, 50.4)$ does yield conflict cases against some of the ideal points specified for the role of player 1 by participants. These ideal points for player 1 are too aggressive to make it worthwhile to reach agreement with them by a more generous ideal-point choice which, of course, would diminish payoffs against other strategies.

The simple typical strategy with $u = (89.4, 55.6)$ and $v = (86.6, 50.4)$ is not only the best among its family but is also the winner of the tournament against the 22 final strategies. This seems to indicate that the way in which the simple typical strategies fill in the details left open by the 13 characteristics is not an unreasonable one. One may say that the structure of these strategies provides an appropriate idealized image of typical behavior of experienced strategy programmers, at least as far as our experiment is concerned.

8.4. *Game-Theoretic Properties of Simple Typical Strategies*

The 20-period supergame has only one subgame perfect equilibrium point. In this equilibrium point both players always choose their Cournot quantities regardless of the previous history. If both players use simple typical strategies of the family described above the resulting strategy pair is always a disequilibrium, simply because it would be advantageous to deviate in the fourth last period.

Game theoretically there is a fundamental difference between finite and infinite supergames. It is known from the experimental literature that this difference seems to have little behavioral relevance. In sufficiently long finite experimental supergames cooperation is possible until shortly before the end, even if the source game has only one equilibrium point (Stoecker (1983), Selten and Stoecker (1986)). If one wants to connect finite supergame behavior with game-theoretical equilibrium notions, one has to take the point of view that the players behave as if they were in an infinite supergame.

It is shown in another paper of one of the authors that it is possible to construct equilibrium points for the infinite supergame of our duopoly model based on the main phase of our simple typical strategies (Mitzkewitz (1988)). In these equilibrium points both players have the same ideal point. This ideal point is chosen in the first period of the game; later the strategies respond to the previous period as specified by the reaction functions r and r^*. Under certain conditions which have to be imposed on the ideal points, equilibrium points are obtained in this way. However, these equilibrium points are not subgame perfect. This is a consequence of a result in the literature which shows that equilibria where output continuously depends on the opponent's last-period output only cannot be subgame perfect unless the Cournot output is specified regardless of the previous history (Stanford (1986), Robson (1986)). Mitzkewitz (1988) shows that an appropriate modification of the main phase of the simple typical strategies yields subgame perfect equilibrium points for a wide range of ideal points.

Among the newer game-theoretical literature on the duopoly problem we have only found one paper which shows some similarities with the approach taken here (Friedman and Samuelson (1988)).

8.5. *Reasonable Conditions for Ideal Points*

One may ask the question whether it is possible to impose reasonable restrictions on the choice of ideal points in our simple typical strategies. A strategy programmer who considers an ideal point for one of the player roles will probably explore what happens if his opponent uses the same ideal point for the opposite player role. Therefore, it is natural to focus on the case in which both opponents use the same ideal point $u = (u_1, u_2)$ for both player roles.

Suppose player 1 knows that player 2 plays a simple typical strategy as defined above with the ideal point $u = (u_1, u_2)$. Suppose that for some output x_1 the profit $G_1(x_1, r^*(x_1))$ is greater than $G_1(u_1, u_2)$. Then player 1 has a better alternative than to agree to player 2's ideal point (u_1, u_2). This consideration and an analogous one for player 2 lead to the following conditions:

$$G_1(u_1, u_2) = \max_{x_1} G_1(x_1, r^*(x_1)),$$

$$G_2(u_1, u_2) = \max_{x_2} G_2(r(x_2), x_2).$$

We refer to these two equations as "conjectural equilibrium conditions" since there is an obvious relationship to conjectural oligopoly theories (see Selten (1980)).

Another reasonable condition on ideal points is connected to the possibility of attempts of short-run exploitation. Suppose that a player deviates just once from the ideal point and then returns to cooperation at the ideal point. It should not be possible to improve profits in this way. This leads to the following conditions:

$$2G_1(u_1, u_2) = \max_{x_1} [G_1(x_1, u_2) + G_1(u_1, r^*(x_1))],$$

$$2G_2(u_1, u_2) = \max_{x_2} [G_2(u_1, x_2) + G_2(r(x_2), u_2)].$$

We refer to these equations as "stability against short-run exploitation." In our numerical case the conjectural equilibrium conditions imply stability against short-run exploitation, but this is not the case for all possible parameter values.

As has been explained in subsection 8.4 it will be shown elsewhere (Mitzkewitz (1988)) that subgame perfect equilibrium points for the infinite supergame can be constructed on the basis of the reaction functions (but with memory also of the own behavior) embodied in the main phase of simple typical strategies if certain conditions on the ideal point are satisfied. These conditions are nothing else than the conjectural equilibrium conditions and the stability against short-run exploitation.

Perhaps it is also of interest that only one Pareto optimal point satisfies the conjectural equilibrium conditions, namely the point described in the third row of Table II: profit monotonic quantity reduction along the straight line through the intersections of both Cournot-isoprofit curves (see Mitzkewitz (1988)). It is tempting to look at this ideal point as distinguished among others by its special theoretical properties. In the final strategies it has been employed twice. However, as can be seen in Table II, other ideal points based on different principles have proved to be at least as attractive to the participants.

Figure 10 shows the ideal points used in final strategies of the participants and the restrictions imposed by the conjectural equilibrium conditions (the smaller lens-shaped area) and by stability against short-run exploitation (the greater lens-shaped area). The equations for these curves will be discussed elsewhere

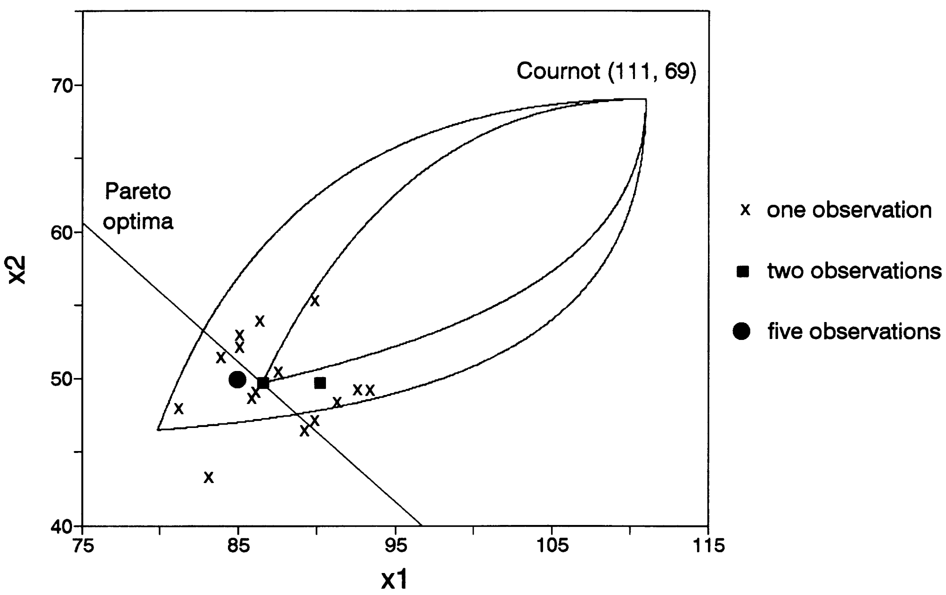


FIGURE 10.—The set of ideal points satisfying the conjectural equilibrium conditions (smaller lens), the set of ideal points stable against short-run exploitations (greater lens), and the ideal points used in final strategies.

(Mitzkewitz (1988)). Only 4 of the 24 ideal points satisfy the conjectural equilibrium conditions, but 21 of the ideal points are stable against short-run exploitation.

Obviously, the participants were not concerned about the conjectural equilibrium conditions. Maybe a violation of these conditions is not perceived as a serious danger since in the case of an optimization of the other player along one's own reaction function, cooperation will still be reached, even if the resulting output levels are higher than in the ideal point.

Some strategies which were not yet the final ones contained attempts at short-run exploitation. Most participants seemed to be aware of this possibility since the "trainer"-program enables them to play against their own strategy. They were able to check short-run exploitability without analytical computations. Of course, such numerical checks will sometimes fail to reveal the right answer. Maybe it is of interest in this connection that two of the three ideal points without stability against short-run exploitation are very near to the corresponding area in Figure 10.

8.6. *Stability against Short-Run Exploitation and Outcomes of Plays in the Final Tournament*

In the tournament among 23 final strategies (including the strategy with random choices) 1012 plays were simulated (two plays for each strategy pair). Table IV shows the distribution of the pairs of total profits in the 1012 plays. The inner cells of the table correspond to profit intervals of four thousand for both players.

The curve superimposed on this table is connected to stability against short-run exploitations. The curve encloses all profit pairs which can be reached by plays in which the same ideal point with the property of stability against short-run exploitation is played in all 20 periods. We call the region enclosed by this curve the "exploitation stability region."

Consider two simple typical strategies whose ideal points are stable against short-run exploitation. Whenever such strategies are played against each other, the resulting profit combination of the 20-period supergame must be in the exploitation stability region, regardless of whether the ideal points of both players are equal or not. However, the set of all profit combinations which can be reached in this way is a proper subset of the exploitation stability region. This is due to the behavior in the initial phase and the end phase. The exploitation stability region can be obtained by pairs of modified simple typical strategies, strategies in which the initial phase and the end phase are of different length, but the behavior in the main phase remains the same.

In the final tournament 983 (97.1%) of the 1012 plays resulted in total profit combinations in the exploitation stability region. In those few total profit combinations outside the exploitation stability region, one of both profits is below the corresponding Cournot profit.

TABLE IV

SUPERGAME PROFIT PAIRS IN THE FINAL TOURNAMENT AND THE EXPLOITATION STABILITY REGION.

Total Profits of Player 2 (in thousands)	<46	46-50	50-54	54-58	58-62	62-66	66-70	70-74	74-78	78-82	82-86	86-90	90-94	94-98	>98
>106	–	–	–	–	–	–	–	–	–	–	–	–	–	–	
102-106	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
98-102	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
94-98	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
90-94	2	–	1	2	–	4	–	–	–	–	–	–	–	–	–
86-90	–	–	–	5	9	86	48	–	–	–	–	–	–	–	–
82-86	–	4	4	11	36	146	93	10	2	–	–	–	–	–	–
78-82	2	–	1	25	61	60	66	35	14	–	–	–	–	–	–
74-78	4	–	17	75	37	15	8	9	9	–	–	–	–	–	–
70-74	6	2	33	37	10	8	2	4	–	–	–	–	–	–	–
66-70	–	–	3	–	–	–	–	–	–	–	–	–	–	–	–
<66	–	–	–	–	–	–	2	–	–	–	–	–	–	–	4

Total Profits of Player 2 (in thousands)

Cournot Profits fo Player 2

Pareto-efficient frontier

Cournot Profits of Player 1

Total Profits of Player 1 (in thousands)

The evidence of Figure 10 and Table IV strongly suggests that stability against short-run exploitation has some relevance for the prediction of outcomes of plays between strategies written by experienced players.

9. IMPLICATIONS FOR DUOPOLY THEORY

The results presented in this paper suggest a new view of the duopoly problem. Traditional duopoly theories and game-theoretical approaches rely heavily on optimization ideas. Usually, a duopolist is assumed to optimize against expectations on his opponent's behavior. Contrary to this, it is typical for the strategies programmed by the experienced players in our experiment that no expectations are formed and nothing is optimized.

The approach to the duopoly problem suggested by our results can be described as the "active pursuit of a cooperative goal." First, one has to answer the question of where one wants to cooperate. The goal of cooperation is made precise by the concept of an ideal point. The ideal point should be a reasonable compromise between both players' interests; otherwise, one cannot hope to achieve cooperation. Concepts of fairness such as those listed in Table II are the basis for judgments on the reasonableness of compromises.

It is well known in the experimental literature that considerations of fairness have a strong influence on observed behavior. Many of the empirical and experimental phenomena can be subsumed under an equity principle (Selten (1978b)). Further literature can be found there and in a newer paper which contains many illustrative examples (Kahneman, Knetsch, and Thaler (1986)). Fairness considerations also have been proved to be useful in the explanation of behavior in duopoly experiments (Friedmann (1970), Selten and Berg (1970)).

Once an ideal point has been chosen one has to determine a policy for its effectuation. Formally, an effectuation policy may be described by a reaction function as in the simple typical strategies of Section 8. However, contrary to conjectural oligopoly theory, such reaction functions are not to be interpreted as hypotheses on the opponent's behavior. Effectuation policies are more like reinforcement schedules which serve the purpose to guide the opponent's behavior rather than to optimize against it.

The typical structure of an effectuation policy is based on the principle of measure for measure. This principle requires an interpersonal comparison of the degree of cooperativeness of the players' actions. The degree of cooperativeness measures the nearness to the ideal point. The response matches the opponent's last action according to this measure.

A player who plays the dynamic game may try to learn how to do best against his opponent's behavior. A player who does this takes a "learning approach." It is also possible to take a "teaching approach," which means that one behaves in a way which induces the other player to conform to one's own goals.

It seems to be very difficult to design a reasonable strategy which takes the learning approach. One participant tried to do this in a sophisticated way. His strategy involved an approximate intertemporal optimization against statistical estimates of his opponent's strategy. His success rank was 20. As Table III shows, his strategy has only one of the thirteen characteristics, namely the absence of random decisions. Obviously, the optimization attempt, of this participant failed badly. The reason for this lies in the difficulty of forming an accurate estimate of the opponent's behavior on the basis of relatively few observations.

The difficulties connected to the learning approach point in the direction of a teaching approach. Of course, somebody who takes the teaching approach does not necessarily expect that the other player takes a learning approach. The other player may very well take a teaching approach, too. This will not lead to difficulties if both players pursue compatible cooperative goals. However, if the opponent tries to adapt to my strategy, this should not endanger my cooperative goal.

Maybe in a very long supergame of thousands of periods, a good strategy would involve both, teaching and learning, but within 20 periods not much can be learned which still can be used within this time. Real duopoly situations rarely are analogous to very long supergames. Maybe a relatively short supergame more adequately captures the decision problem of managers who want to be successful within a foreseeable time.

The new view of the duopoly problem emerging from our results may be described by the slogan "fairness and firmness." One must first choose a fair goal of cooperation and then devise an effectuation policy which shows one's willingness to cooperate and firmly communicates resistance to unfair behavior.

As we have seen, the requirement of stability against short-run exploitation seems to be a restriction obeyed by the participants' choices of ideal points, even if their effectuation policies were not exactly the same as those of the simple typical strategies. It is clear that one should not give rise to the possibility of being exploited. Moreover, in the case in which the other player selects one's own ideal point, he should not be exploitable. This criterion of stability against short-run exploitation is in good agreement with our data.

It is clear that the theory of fairness and firmness can be easily transferred to different contexts, e.g. price-variation duopoly supergames. The tit-for-tat strategy which was the winner of Axelrod's contests (1984) is in harmony with the fairness-and-firmness theory. In the prisoner's dilemma the choice of an ideal point is not an issue. In view of the symmetry of the situation there is only one natural cooperative goal. Since there are only two choices available, measure for measure cannot mean anything else than tit-for-tat.

It must be admitted that no strong conclusions can be drawn from our data since the final strategies cannot be regarded as statistically independent observations. The participants interacted in game playing rounds and tournaments. Moreover, there was some verbal communication, even if the participants seemed to be reluctant to reveal the principles underlying their strategies.

More studies similar to the investigation presented here are necessary to establish the empirical relevance of the fairness-and-firmness theory. It should also be kept in mind that the final strategies of our participants are the result of a long experience with the game situation. It is quite possible that real duopolists have much less experience with their strategic situation and therefore do not achieve the same extent of cooperation. The experimental literature shows that only after a considerable amount of experience, subjects learn to cooperate (Stoecker (1980), Friedman and Hogatt (1980), Alger (1984, 1986), Benson and Faminow (1988)).

It would be wrong to assert that there is no difference between a programmed strategy and spontaneous behavior. The strategy method cannot completely reveal the structure of spontaneous behavior. However, it seems to be plausible that somebody who writes a strategy program is guided by the same motivational forces which would influence his spontaneous behavior. Of course, a strategy program is likely to be more systematic. Obviously this is an advantage from the point of view of theory construction.

10. SUMMARY OF RESULTS

1. Mean profits increased from one game playing round to the next.

2. The correlation between both player profits was negative in the first game playing round and became positive in the second and the third game playing round. This can be interpreted as a growth of understanding of the strategic situation.

3. Mean profits increased from one computer tournament to the next. In the final tournament 97.1% of all plays had profits above Cournot profits for both players.

4. Typically, a strategy program for the final tournament distinguishes among an initial phase, a main phase, and an end phase. Outputs independent of the opponent's previous behavior are specified for the initial phase of one to four periods. In the main phase the strategies aim at a cooperation with the opponent. Noncooperative behavior characterizes an end phase of one to four periods.

5. Typical structural features of strategies programmed for the final tournament can be described by 13 characteristics. These characteristics imply a strategic approach which begins with the selection of a cooperative goal described by an "ideal point." (A different ideal point may be chosen for each player role.) Cooperation at the ideal point is then pursued by a "measure-for-measure policy." If the opponent moves towards the ideal point or away from it, the response of a measure-for-measure policy is of similar force in the same direction. In the end phase a typical strategy always chooses Cournot outputs.

6. Typically, no predictions about the opponent's behavior are made and nothing is optimized.

7. The extent to which a strategy or a characteristic is typical can be measured by an index of typicity. There is a highly significant positive rank correlation between the index of typicity and the success of a strategy in the final tournament.

8. For each of the 13 characteristics separately those final strategies which have this characteristic have a higher average success rank than those which do not have it.

9. Ideal points are often based on various fairness considerations (see Table II).

10. A family of "simple typical strategies" has been introduced as an idealized description of the structure implied by the 13 characteristics. The simple typical strategy which performed best against the final tournament strategies was determined by a computer simulation. This "best" simple typical strategy is also the winner in the tournament against the final strategies.

11. Two game-theoretical requirements for simple typical strategies impose restrictions on ideal points. One of these restrictions, the "conjectural equilibrium conditions," is rarely satisfied by the ideal points in the final strategies.

However, most of these ideal points satisfy the weaker restriction of "stability against short-run exploitation."

12. An "exploitation stability region" for profit combinations reached in the supergame can be derived from the requirement of stability against short-run exploitation. The profit combinations of all plays in the final tournament in which both players received more than their Cournot profits are in the exploitation stability region. These are 97.1% of all plays in the final tournament.

Universität Bonn, Wirtschaftstheoretische Abteilung I, Adenauerallee 24-42, D-53113 Bonn, Germany.

Manuscript received October, 1990; final revision received April, 1996.

REFERENCES

ABREU, D. (1986): "Extremal Equilibria of Oligopolistic Supergames," *Journal of Economic Theory*, 39, 191–225.

ALBERS, W., AND G. ALBERS (1983): "On the Prominence Structure of the Decimal System," in *Decision Making Under Uncertainty*, ed. by R. W. Scholz. Amsterdam: Elsevier, pp. 271–287.

ALGER, D. (1984): "Equilibria in the Laboratory: Experiments with Oligopoly Markets where Goods Are Made to Order," Working Paper No. 121, Bureau of Economics, Federal Trade Commission, Washington.

——— (1986): *Investigating Oligopolies within the Laboratory*. Washington: Bureau of Economics, Federal Trade Commission.

AXELROD, R. (1984): *The Evolution of Cooperation*. New York: Basic.

BENSON, B. L., AND M. D. FAMINOW (1988): "The Impact of Experience on Prices and Profits in Experimental Duopoly Markets," *Journal of Economic Behavior and Organization*, 9, 345–365.

COURNOT, A. (1938): *Recherches sur les Principes Mathematiques de la Theorie des Richesses*. Paris: Hachette.

FADER, P. S., AND J. R. HAUSER (1988): "Implicit Coalitions in a Generalized Prisoner's Dilemma," *Journal of Conflict Resolution*, 32, 553–582.

FRIEDMANN, J. W. (1970): "Equal Profits as a Fair Division," in *Beiträge zur Experimentellen Wirtschaftsforschung*, Band 2, ed. by H. Sauermann. Tübingen: J. C. B. Mohr, pp. 19–32.

——— (1977): *Oligopoly and the Theory of Games*. Amsterdam: North Holland.

FRIEDMANN, J. W., AND A. C. HOGATT (1980): *An Experiment in Noncooperative Oligopoly Research in Experimental Economics*, Vol. 1, Supplement 1. Greenwich: JAI Press.

FRIEDMANN, J. W., AND L. SAMUELSON (1988): "Subgame Perfect Equilibrium with Continuous Reaction Functions," Mimeo, University of Bielefeld, Center of Interdisciplinary Research.

KAHNEMAN, D., J. L. KNETSCH, AND R. H. THALER (1986): "Fairness and the Assumptions of Economics," *Journal of Business*, 59, 285–300.

MITZKEWITZ, M. (1988): "Equilibrium Properties of Experimentally Based Duopoly Strategies," Working Paper No. B-107, University of Bonn.

RADNER, R. (1980): "Collusive Behavior in Noncooperative Epsilon-Equilibria of Oligopolies with Long but Finite Lives," *Journal of Economic Theory*, 22, 136–154.

ROBSON, A. (1986): "The Existence of Nash Equilibria in Reaction Functions for Dynamic Models of Oligopoly," *International Economic Review*, 27, 539–544.

SCHELLING, T. C. (1960): *The Strategy of Conflict*. Cambridge: Harvard University Press.

SEGERSTROM, P. S. (1988): "Demons and Repentance," *Journal of Economic Theory*, 45, 32–52.

SELTEN, R. (1967): "Die Strategiemethode zur Erforschung des eingeschränkt rationalen Verhaltens im Rahmen eines Oligopolexperiments," in *Beiträge zur Experimentellen Wirtschaftsforschung*, ed. by H. Sauermann. Tübingen: J. C. B. Mohr, pp. 136–168.

——— (1970a): "The Chain Store Paradox," *Theory and Decision*, 9, 127–159.

——— (1978b): "The Equity Principle in Economic Behavior," in *Decision Theory and Social Ethics: Issues in Social Choice*, ed. by H. W. Gottinger and W. Leinfellner. Dordrecht: Reidel, pp. 289–301.

——— (1980): "Oligopoltheorie," in *Handwörterbuch der Wirtschaftswissenschaft*, ed. by W. Albers et al. Stuttgart-New York: Fischer, pp. 667–678.

——— (1987): "Equity and Coalition Bargaining in Experimental Three-person Games," in *Laboratory Experimentation in Economics*, ed. by A. E. Roth. Cambridge: Cambridge University Press, pp. 42–98.

SELTEN, R., AND C. C. BERG (1970): "Drei experimentelle Oligopolspielserien mit kontinuierlichem Zeitablauf," in *Beiträge zur Experimentellen Wirtschaftsforschung*, Band 2, ed. by H. Sauermann. Tübingen: J. C. B. Mohr, pp. 162–221.

SELTEN, R., AND R. STOECKER (1986): "End Behavior in Finite Prisoner's Dilemma Supergames," *Journal of Economic Behavior and Organization*, 7, 47–70.

STANFORD, W. G. (1986): "Subgame Perfect Reaction Function Equilibria in Discounted Duopoly Supergames are Trivial," *Journal of Economic Theory*, 39, 226–232.

STOECKER, R. (1980): *Experimentelle Untersuchung des Entscheidungsverhaltens im Bertrand-Oligopol.* Bielefeld: Pfeffer.

——— (1983): "Das Erlernte Schlußverhalten—eine Experimentelle Untersuchung," *Zeitschrift für die Gesamte Staatswissenschaft*, 139, 100–121.

REAL EXCHANGE RATES WITHIN AND BETWEEN CURRENCY AREAS: HOW FAR AWAY IS EMU?

Jürgen von Hagen and Manfred J. M. Neumann*

Abstract—The renewed quest for a European monetary union raises the question: Is Europe ready for a common currency? We compare the conditional variance and the persistence of real exchange rate shocks within the German monetary union and between Germany and eight European countries to assess the viability of a monetary union in Europe. The results suggest a "Europe of Two Speeds": A core union among Germany, her smaller neighbors and France would be viable today. Further reduction of real exchange rate variability is needed, in contrast, between these countries and Denmark, Italy, and the U.K. Alternatively, monetary union should be postponed until further adjustment has occurred. Such a waiting period would neither require nor benefit much from further tightening of the current EMS.

I. Currency Areas and Real Exchange Rates

THE project of a European Monetary Union (EMU) as laid down in the Maastricht revisions of the Treaty of Rome raises the question, which of the economies of the European Community (EC) are ready for a common currency?[1] The main economic cost of EMU results from the loss of nominal exchange rate flexibility as an instrument for real exchange rate (RER) adjustment between regions exposed to asymmetric shocks. With sticky goods and factor prices, RER adjustment is achieved more swiftly and easily, if nominal exchange rates rather than regional price levels change. Consequently, the cost of EMU is the larger, the more often RER changes are required.

Traditional optimal-currency literature suggests structural criteria such as factor mobility (Mundell, 1961), trade integration (McKinnon, 1963), and regional production patterns (Kenen, 1969) to assess a region's readiness for monetary union. More recent literature points to the importance of flexible nominal exchange rates in the presence of strongly asymmetric shocks to or pronounced differences in shock-absorbing mechanisms of the countries pondering monetary unification (e.g., Marston, 1984, von Hagen and Fratianni, 1991). A common problem with these criteria is that they are hard to verify empirically. Data on factor movements and trade and production patterns are too sparse and aggregated to indicate the importance of nominal exchange rate flexibility. Empirical results indicating a decreasing degree of asymmetry of the stochastic shocks hitting the potential member countries of the EMU (Cohen and Wyplosz, 1989; Fratianni and von Hagen, 1990; Weber, 1990) must be regarded with caution in this context, since the nature of the underlying shocks cannot be identified.[2]

Vaubel (1976) proposed to assess the conditions for a common currency on the basis of the empirical variance of RER rather than the magnitude of potential *sources* of RER variance. Subsequently Mussa (1986) and Meltzer (1986) used RER variance for empirical characterizations of exchange rate regimes. Observing a stable RER between two currency areas would suggest that shocks demanding RER adjustment are small, and, consequently, that the cost of giving up nominal exchange rate flexibility would be small. Conversely, the observation of large RER instability would indicate the superiority of flexible nominal exchange rates.

Using the RER variance criterion raises the question of what is "large" and what is "small." Vaubel (1976, 1978), and others, use observed RER variances within existing currency areas as a yardstick. Eichengreen (1990) and Poloz (1990) compare RER variances among EC countries and regions of the United States and Canada. However, this standard of comparison is unsatisfactory, because of the large structural differences

Received for publication February 19, 1992. Revisions accepted for publication April 30, 1993.

* University of Mannheim, Indiana University School of Business, and CEPR; and University of Bonn, respectively.

Helpful comments from two anonymous referees are gratefully acknowledged. This research was funded in part by the Deutsche Forschungsgemeinschaft, SFB 303.

[1] Estimates of economic benefits from EMU have been presented by the European Commission (1990). For a critical discussion of these estimates see Minford and Rastogi (1990).

[2] Bayoumi and Eichengreen (1992) use a "structural" VAR to assess the asymmetry of shocks to the EMS countries. There, identification of supply and demand shocks depends critically on the assumption that demand shocks have no lasting effects on output and all supply shocks are permanent. Furthermore, their analysis identifies some foreign demand shocks domestically as output supply shocks (Neumann, 1993).

between the North American and the European economies and the likely possibility that Europe and North America were exposed to different economic shocks in the past.

In this paper, we take RER variance within an existing monetary union in Europe as the standard to assess RER variance among European currency areas. Specifically, we compare the variance of RER, i.e., the variance of regional relative price levels, among six West German *Länder* with the RER variance of the same six *Länder* with several European countries. Our standard of comparison thus is much less biased by differences in shocks or economic structures. Since unexpected variation of relative prices is generally thought to matter more than expected variation, we also extend the approach by considering conditional RER variance, i.e., the variance of unexpected RER fluctuations, instead of observed variance. Furthermore, we investigate the persistence of RER fluctuations and the potential role of monetary policy coordination to reduce RER variance in the wake of EMU.

Our paper proceeds as follows. Section II describes the data. Section III presents the empirical results. Section IV summarizes the main findings and offers some conclusions.

II. The Data

Our data consist of monthly seasonally unadjusted series of prices and exchange rates from January 1973 to November 1989. All prices are consumer price indexes. To reduce the dimension of our analysis, we use price indexes for 6 of the 12 German *Länder*: Bavaria, Baden-Württemberg, Berlin, Hesse, Northrhine-Westphalia, and the Saarland. They are representative for the Federal Republic during the sample period in the sense that Berlin and the Saarland represent the small and peripheral *Länder*, Bavaria and Baden-Württemberg the large and rapidly growing *Länder*, and Hesse and Northrhine-Westphalia the large *Länder* of sluggish growth in the 1970s and 1980s. Consumer price indexes are cost-of-living indexes for an average dependent-worker household with a working head-of-family. In addition, we use the consumer price indexes for Austria, Belgium, Denmark, France, Italy, Luxembourg, the Netherlands, and the United Kingdom.

With the exception of Austria, these countries and Germany formed the "European Snake" in 1972, an arrangement limiting nominal spot rate fluctuations to plus/minus 2.25% around adjustable central parities. The United Kingdom left the Snake as early as 1972; France did not participate in it between January 1974 and July 1975, and finally opted out altogether in March 1976. Since March of 1979, Belgium, Denmark, France, Italy, Luxembourg, the Netherlands and Germany have participated in the European Exchange Rate Mechanism (ERM), which limits nominal spot exchange rate fluctuations to plus/minus 2.25% (plus/minus 6% for Italy) around adjustable central parities. Meanwhile, Austria has fixed the Schilling's exchange rate with the Deutsche Mark since the early 1970s. The United Kingdom did not participate in the ERM during the sample period and serves us as a reference case to consider the impact of the exchange rate arrangements; Italy left the ERM in September of 1992. Belgium and Luxembourg maintained a currency union throughout the sample period. Nevertheless, we include both countries in our analysis, since the markedly different economic structures in these two countries still allow sizeable relative price variation.

We distinguish four subsamples in our analysis: the pre-EMS period of 1973–1978, the early EMS period of 1979–1982, an intermediate EMS period of 1983–1986, and the late EMS period of 1987–1989. The definition of the separate EMS periods accounts for important changes in the functioning of the system (e.g., von Hagen, 1991a). The early period is one of little monetary policy coordination and frequent realignments of central parities, the intermediate period is characterized by greater convergence of monetary policies but yet recurrent realignments, while after January 1987 no further realignment occurred during the sample period.

III. Empirical Analysis of Real Exchange Rate Variability

A. *Conditional Variances of Real Exchange Rates*

Let $p_{i,t}$ be the logarithm of German *Land* i's CPI in period t, $P_{k,t}$ the logarithm of country k's CPI in period t, and $s_{k,t}$ the logarithmic nominal exchange rate between k's currency and the DM. Nominal exchange rates are calculated on the

TABLE 1.—STANDARD DEVIATIONS OF REAL EXCHANGE RATE SHOCKS
(average over six German *Länder*, 1/1000)

Period	BA	BE	BW	HE	NW	SA	A	B	DK	F	I	L	NL	UK
					Monthly Real Exchange Rate Changes									
1973–78	16	21	16	17	17	19	51	90	123	167	225	95	96	217
1979–82	11	17	12	13	12	13	36	91	86	101	84	96	44	204
1983–86	12	20	11	12	11	13	31	45	60	55	66	47	31	165
1987–89	7	10	8	10	7	7	21	21	43	28	59	21	24	126
					Quarterly Real Exchange Rate Changes									
1973–78	26	34	24	31	27	27	100	182	187	379	492	163	142	414
1979–82	24	28	18	20	18	20	71	208	185	209	172	189	102	465
1983–86	15	28	14	15	15	18	53	71	96	109	119	84	45	379
1987–89	9	14	8	10	8	8	34	36	60	43	111	34	44	175

Note: BA: Bavaria, BE: Berlin, BW: Baden-Württemberg, HE: Hesse, NW: Northrhine-Westphalia, SA: Saarland, A: Austria, B: Belgium, DK: Denmark, F: France, I: Italy, L: Luxembourg, NL: Netherlands, UK: United Kingdom.

basis of monthly averages. We define the RER between *Land* i and country k as

$$Q_{ki,t} = P_{kt} + s_{k,t} - p_{i,t}, \tag{1}$$

and the RER between two *Länder* i and j as

$$q_{ji,t} = p_{j,t} - p_{i,t}, \tag{2}$$

since the nominal exchange rate between the latter two is one.

Consumer price indexes across regions differ significantly in their seasonal patterns, reflecting, e.g., different preference structures, regulatory provisions, and supply conditions. To eliminate the impact of such differences in seasonal patterns on our measures of RER variation, we regress the observed changes in RER on a set of 12 monthly dummies D_m:

$$\Delta Q_{ik,t} = \sum_{m=1,12} \beta_m D_m + R^1_{ki,t} \tag{3a}$$

$$\Delta q_{ij,t} = \sum_{m=1,12} \beta_m D_m + r^1_{ji,t} \tag{3b}$$

and use the residuals, $R^1_{ki,t}$ and $r^1_{ji,t}$, as seasonally adjusted RER changes.[3] In order to compare variances of high- and low-frequency RER changes, we analyze these monthly data together with non-overlapping, quarterly changes in seasonally adjusted RER, denoted by $R^3_{ki,t}$ and $r^3_{ji,t}$, respectively.[4]

[3] The seasonal adjustment regressions were performed individually for each subperiod defined above, to allow for changes in the seasonal patterns over time.

[4] The quarterly series are computed by aggregating over the monthly series as follows: $R^3_{ki,t} = \Sigma_{m=0,2} R_{ki,3t-m}$ and $r^3_{ji,t} = \Sigma_{m=0,2} r_{ji,3t-m}$.

To derive the unexpected component of these variables, we regress them on their own lags. For the monthly series, six lags were sufficient to obtain residuals with no autocorrelation; for the quarterly series, two lags were sufficient in all cases. Each regression used interactive dummies on the lag terms to allow for parameter changes between the subsamples.[5]

Call the residuals of these regressions RER shocks, denoted by $U^h_{ki,t}$ for RER between the six German *Länder* and the European countries and $u^h_{ji,t}$ for RER among the six *Länder*, where h indicates monthly ($h = 1$) or quarterly ($h = 3$) data. Computing the conditional standard deviations (STD) of these RER shocks and averaging over the six German *Länder* yields our measures of conditional RER variance[6]:

$$V^h_k = (1/6) \sum_{i=1,6} \left[\mathrm{var}\left(U^h_{ki,t}\right)\right]^{1/2}, \tag{4a}$$

$$v^h_j = (1/6) \sum_{i=1,6} \left[\mathrm{var}\left(r^h_{ji,t}\right)\right]^{1/2}, \tag{4b}$$

$$h = 1, 3.$$

Table 1 reports these measures. Consider the variability of monthly RER shocks, first. In the first subsample, STDs of RER shocks between the six German *Länder* and the European coun-

[5] Autocorrelation of the residuals was tested using LaGrange multiplier tests for all series.

[6] Note that (4a) is not defined to be a measure of general RER variance in the potential EMU. (4a) and (4b) focus on the bilateral RER among the six *Länder* and between these and other potential member regions of EMU to improve comparability of the variances. The focus on bilateral RER with German *Länder* may be justified by the fact that the vast majority of intra-European trade relations involves Germany as a partner.

tries are much larger than those among the *Länder*. Those countries which maintained an exchange rate arrangement with Germany in the mid-1970s had STDs ranging between 2.5 and 5 times larger than intra-German STDs. France, Italy and the United Kingdom had STDs of RER shocks between 8 and 11 times larger. By the late 1980s, in contrast, conditional real exchange rate variability had declined dramatically.

Table 1 indicates that, by the end of the 1980s, the countries maintaining an exchange rate arrangement with Germany can be divided into two groups. RER shocks between the six German *Länder* and Austria, Belgium, Luxembourg, and the Netherlands have STDs comparable to STDs of RER shocks between the *Länder* in the 1970s, though still about twice as large as the STDs of RER shocks between the *Länder* in the late 1980s. The STD of monthly RER shocks between the six German *Länder* and France is larger but close to those of this group. This suggest that a monetary union among these countries and Germany today would exhibit a variance of regional relative price shocks comparable to that existing within Germany in the 1970s, and, therefore, can be regarded as sustainable. Yet, since the 1970s were a period of economic turmoil, one may add as a note of caution, that the performance of such a monetary union might be exposed to considerable strain and, with less factor mobility than within Germany, might need some fiscal mechanisms to equilibrate regional imbalances, such as a provision for inter-regional redistribution of tax revenues, or a sufficiently large Community budget directed at this purpose.[7]

RER shocks between the six German *Länder* and Denmark and Italy have STDs in the late 1980s comparable to those of the first group in the mid-1980s, and still substantially larger than STDs within Germany. This suggests a need for further convergence before a viable EMU including these countries can be formed.

The U.K., which chose not to participate in the ERM during the 1980s, experienced a remarkably different performance during the sample period. Its conditional RER variance with the six German *Länder* remained the same until the mid-1980s. Only in the late 1980s was there a sizeable decline in the STD of monthly RER shocks. This improvement may reflect in part the British policy of "shadowing the Mark" during that period. Conditional RER variance between the U.K. and the German *Länder* still remains many times larger than the variance within Germany or between the six German *Länder* and the other countries. As a result, a EMU including Britain would demand much higher variation of regional relative prices than a union excluding Britain, and much higher than the regional relative price variation we observe within Germany.

Turning to quarterly RER shocks, the differences between the STDs of RER shocks among the six German *Länder* and the STDs of RER shocks between the latter and the European countries were much larger, both in absolute and in relative terms, compared to the STDs of monthly RER shocks. That is, RER behavior between the national monetary unions differed more significantly from RER behavior within the German monetary union with respect to long-run than with respect to short-run variability. During the 1980s, long-run variability declined together with short-run variability. But the differences between STDs of RER shocks within Germany and between our German *Länder* and other countries remain much larger at the quarterly frequency in the late 1980s.

The table indicates that, for quarterly RER shocks, only Austria, Belgium, Luxembourg, and, less so, France, and the Netherlands come close to STDs prevailing within Germany in the late 1970s. Assuming that high-frequency RER changes reflect predominantly nominal and financial market shocks whereas low-frequency changes reflect real shocks, this evidence suggests that asymmetric real shocks remain relatively more important between the European countries than within the German monetary union. In conclusion, even a monetary union among Germany, Austria, Belgium, Luxembourg, France and the Netherlands would exhibit much larger RER variability at low frequencies than the German monetary union.

The difference between the results for the United Kingdom and the other countries suggests that exchange rate arrangements contributed significantly to the reduction of conditional RER

[7] For example, the Delors Report called for a Community budget of 5% to 7% of GDP, but suggested that closer coordination of budgetary policies could be used instead, if a centralized budget was politically not feasible. For a study of this issue in the United States, see von Hagen (1991b).

TABLE 2.—TESTS FOR CONSTANT VARIANCES

Sample	BA	BE	BW	HE	NRW	SA	A	B	DK	F	I	L	NL	UK
							Monthly Conditional Variances							
II/I	.019	.086	.031	.042	.016	.020	.099	.930	.078	.009	.005	.94	.011	.660
III/I	.007	.330	.002	.004	.0008	.003	.008	.053	.0008	.0001	.0001	.088	.0002	.068
IV/I	.0008	.002	.0006	.002	.001	.0002	.002	.028	.0005	.0001	.0003	.053	.0005	.007
IV/III	.03	.004	.150	.270	.03	.015	.15	.003	.037	.047	.54	.003	.32	.082
							Quarterly Conditional Variances							
II/I	.190	.350	.160	.065	.090	.140	.300	.690	.850	.068	.059	.560	.110	.570
III/I	.02	.29	.025	.003	.013	.022	.06	.11	.01	.002	.009	.09	.0004	.64
IV/I	.01	.03	.01	.002	.005	.003	.04	.11	.008	.003	.019	.055	.001	.059
IV/III	.14	.039	.11	.17	.066	.035	.24	.091	.22	.036	.82	.021	.81	.068

Note: Entries are geometric averages of marginal significance levels of tests for constant conditional variances between the relevant samples. Tests used are White's test for heteroskedasticity. Samples are I: 1973–78, II: 1979–82, III: 1983–86, IV: 1987–89.

variance.[8] But the observation that STDs generally decreased both among the German *Länder* and between these and other European economies during the sample period leaves the possibility that the declining conditional RER variances in the ERM were due, to some extent, to common shocks hitting the ERM economies rather than exchange rate policies. To explore this issue, we calculate the relative percentage changes of the conditional RER variances in each subperiod compared to the previous one. Among the German *Länder*, the STD of monthly (quarterly) RER shocks fell by 28% (25%) on average in the early 1980s, by 0.1% (20%) in the mid-1980s, and by 28% (46%) in the late 1980s. The STD of monthly (quarterly) RER shocks between the *Länder* and the countries participating in the ERM fell on average by 34% (22%) in the early 1980s, by 36% (49%) in the mid-1980s, and by 28% (40%) in the late 1980s. This indicates a specific ERM effect on conditional RER variability only several years after the system had started and operating mostly in the mid-1980s.

Table 2 reports the results of statistical tests to see whether or not the changes in conditional RER variances are significant. To economize on space, we report the (geometric) means of the marginal significance levels for rejecting constant variances averaged over the six *Länder*, based on White's heteroskedasticity test. Almost all of the observed variance reductions of monthly RER shocks between the late 1970s and the early, mid, and late 1980s are statistically significant. Exceptions are only Belgium, Luxembourg and the United Kingdom in the second subsample, and Austria and Italy in the last subsample when compared to the mid-1980s.

The results are more mixed with regard to quarterly RER shocks. Most variance reductions in the second subsample, both for shocks among the six *Länder* and for shocks between these and the European countries were not statistically significant. Notable exceptions are France and Italy, both non-members of the Snake for most of the first subperiod. While the variance reductions of quarterly RER shocks of the last subsample are significant when compared with the first subsample with only one exception (Belgium), they are not significant for shocks among the six *Länder*, nor for shocks between the *Länder* and Austria, Denmark, Italy, and the Netherlands, when the last sample is compared to the mid-1980s, Belgium being a borderline case. This, again, is consistent with the notion that the specific EMS effect operated mostly in the mid-1980s.

Finally, we compute the average variances of RER shocks in the period of 1987 to 1989 as a fraction of the corresponding variances in the period of 1973 to 1978 for an easier comparison of the total variance reductions at the monthly and quarterly frequencies. Among the German *Länder*, the relative variance reduction was larger at the quarterly frequency. The European countries, again, fall into three groups. Similar to the German *Länder*, the relative variance reduction between these and Austria, France and the United Kingdom was larger at the lower fre-

[8] Note, however, that the different performance of the United Kingdom may also be to some extent due to the fact that the United Kingdom is the only exporter of crude oil in our group of counties, and that crude oil prices were subject to large changes during the sample period.

TABLE 3.—FIRST-ORDER AUTOCORRELATION OF MONTHLY REAL EXCHANGE RATE CHANGES

Sample	BA	BE	BW	HE	NW	SA	A	B	DK	F	I	L	NL	UK
					Average Autocorrelation Coefficient									
1973–78	−.15	−.08	−.16	−.02	−.09	−.12	.23	.40	.03	.38	.36	.30	.22	.33
1979–82	−.27	−.11	−.26	−.02	−.09	−.32	.11	.25	.23	.28	.39	.25	.35	.42
1983–86	−.24	−.06	−.10	−.19	−.25	−.16	−.21	−.01	.12	.31	.13	.01	.18	.31
1987–89	−.31	−.04	−.36	−.47	−.24	−.23	−.29	−.04	.13	.01	.12	.12	.05	.17
					Number of Significant Coefficients[a]									
1973–78	1	0	3	0	1	1	6	6	0	6	6	6	6	6
1979–82	3	1	2	2	4	4	0	4	0	6	6	4	6	6
1983–86	3	0	0	2	2	1	1	0	0	6	0	0	0	6
1987–89	3	0	3	4	3	1	3	0	0	0	0	0	0	0

[a] Based on t-test and 10% significance levels.

quency of RER fluctuations. The opposite holds for the relative variance reduction between the *Länder* and the Netherlands. Finally, the relative reductions were similar at both frequencies between the German *Länder* and Belgium, Denmark, Italy, and Luxembourg. This suggests that asymmetric real shocks to the economies in the first group have been attenuated more strongly than asymmetric nominal shocks, while the variances of asymmetric real and nominal shocks to the economies in the third group seem to have decreased to similar degrees.

B. *Persistence of Real Exchange Rate Changes*

Besides the variance of RER shocks, the persistence of RER fluctuations is a further, relevant characteristic of currency areas. The more economically integrated regions are, the less persistent should changes in RER be, as commodity arbitrage by interregional trade as well as factor movements would quickly eliminate relative price differentials. The observation of very persistent RER fluctuations, in contrast, would indicate a relatively low degree of economic integration.

Table 3 reports the average first-order autocorrelation coefficient of monthly, seasonally adjusted RER changes and the number of significant coefficients observed in each subsample as simple measures of persistence.[9] A negative average autocorrelation coefficient and a large number of significant coefficients indicate a tendency of RER changes to be reverted over time. This is clearly the case for Bavaria, Hesse, Baden-Württemberg, and Northrhine-Westphalia. The two more peripheral *Länder*, Berlin and the Saarland, in contrast, behave quite differently. In the mid- and late 1980s, Berlin has no significant correlation, the Saarland has only one. Thus, RER fluctuations are more persistent between the *Länder* at the periphery and the rest of Germany's currency union than among the larger *Länder*.

RER fluctuations between the *Länder* and the European countries were characterized by considerable persistence in the 1970s. With the exception of Denmark, all countries had significant, positive autocorrelation coefficients. This pattern changed drastically in the 1980s. By the end of this decade, RER fluctuations between the six German *Länder* and Austria show a self-reverting tendency over time similar to the RER changes among the large German *Länder*. In the mid and late 1980s, Belgium, Denmark, Italy, Luxembourg, and the Netherlands all resemble the peripheral *Länder* of Germany: RER fluctuations have no significant autocorrelation left. Finally, France and the United Kingdom joined this group in the late 1980s. In sum, the general reduction of the persistence of RER changes has improved the conditions for EMU.

[9] Alternatively, we regressed the monthly RER changes on their first four own lags and added the regression coefficients to obtain a second measure of persistence. Between the German *Länder* the resulting measures are uniformly negative and close to minus one, indicating the expected tendency of reversion over time. Between the *Länder* and the European countries, the sum of the lag coefficients turned from positive to zero or negative during the 1980s, with Austria, Belgium, Denmark, Italy, Luxembourg, and the Netherlands all having results similar to the peripheral German *Länder*. Thus, the results from using the alternative measure confirm the results from using the simple autocorrelation coefficients.

TABLE 4.—REAL EXCHANGE RATE VARIABILITY WITH COORDINATED MONETARY POLICIES
(standard deviations of monthly relative RER changes averaged over 6 *Länder*)
(1/1000)

	A	B	DK	F	I	L	NL	UK
				1973–1978				
Estimated Std	36	82	89	56	127	96	25	226
ΔStd	15	8	34	111	98	1	71	9
				1979–1982				
Estimated Std	22	82	50	6	68	84	29	232
ΔStd	14	9	36	35	16	12	15	28
				1983–1986				
Estimated Std	31	26	48	50	65	32	19	155
ΔStd	0	19	12	5	1	15	12	10
				1987–1989				
Estimated Std	21	18	33	12	54	15	19	122
ΔStd	0	3	10	16	5	6	6	4

Note: Estimated Std is the standard deviation of monthly RER shocks estimated from the steady-state solution of model (5) assuming perfect policy coordination. ΔStd is the difference between this estimate and the standard deviation shown in table 1 for the relevant period.

C. *The Role of Monetary Policy Coordination*

The EMS was formed in 1978 to create a "zone of monetary stability" in Europe. This goal included reducing the contribution of uncoordinated monetary policies in the region to RER variability.[10] Since advocates of EMU emphasize the need for further monetary policy coordination in the EC (e.g., European Commission, 1990), one may ask, how important was the lack of perfect policy coordination as a source of RER variability in the past, and what will a further strengthening of coordination contribute to stabilizing RER in the region?

To answer these questions, we wish to estimate the effect of uncoordinated monetary policies on the variance of RER shocks. Following ARCH methodology, we model and estimate the variance of monthly RER shocks as a function of an indicator of policy discoordination. Let $H_{ki,t} = (U^1_{ki,t})^2$ be the square, monthly RER shock between *Land i* and country k. Let $M_{Gk,t} = (\Delta \ln M1_{G,t} - \Delta \ln M1_{k,t})^2$ be the squared difference of seasonally adjusted monthly growth rates of the money supply M1 between Germany and country k, and let $I_{Gk,t} = (\Delta I_{G,t} - \Delta I_{k,t})^2$ be the squared difference in the month-to-month changes in money market rates between the two countries. Since, with perfect monetary policy coordination, money growth and interest rate changes should be the same or very similar, we use M and I as two variables indicating the imperfectness of monetary policy coordination between Germany and country k. That is, a larger M or I indicates less policy coordination, $M = 0$ or $I = 0$ indicate perfect coordination. To isolate the contribution of incomplete monetary policy coordination we estimate the following regression model explaining the variance of RER shocks by its own past and the two indicators of policy coordination:

$$H_{ki,t} = \alpha_0 + \alpha(L)H_{ki,t-1} + \beta_1(L)M_{Gk,t} + \beta_2(L)I_{Gk,t} + v_t, \quad (5)$$

where $\alpha(\cdot)$, $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are polynomials of maximum order four in the lag operator L with parameters restricted to be positive and where $\alpha(\cdot)$ has no roots inside the unit circle.[11] To allow for changes in the model parameters over time,

[10] See von Hagen (1991a) for a review.

[11] This regression can be interpreted as the variance part of a generalized ARCH model. Restricting the parameter values is necessary to assure that the estimated conditional variances of RER shocks are positive even when large monetary or interest rate shocks occur. This does not exclude the possibility that monetary policy stabilized RER fluctuations. See Weiss (1986) for a discussion.

we estimate this model separately for each of our four subsamples. The steady state solution of (5) setting $M_{Gk,t} = I_{Gk,t} = 0$, $\alpha_0/(1-\alpha(1))$, then yields an estimate of the variance of RER shocks under perfect policy coordination.[12]

Table 4 reports these estimates. In the 1970s, the variance of RER shocks between the German *Länder* and Austria, Denmark, France, Italy, and the Netherlands could all have been reduced substantially by perfect monetary policy coordination. The corresponding relative reductions in the variance of monthly RER shocks amount to about 90% for France and the Netherlands, and between 48% and 68% for Austria, Italy and Denmark. This suggests that monetary policy was the dominating source of short-run RER shocks between these countries and the six German *Länder*. Other sources of shocks apparently dominated the conditional variance RER between the German *Länder* and Belgium, Luxembourg and the United Kingdom. In the late 1980s, monetary policy discoordination seems to have only minor importance as a source of RER shocks between the German *Länder* and Austria, Belgium, Luxembourg, and the Netherlands. This is consistent with the common view that monetary policies in these countries have been following Bundesbank policies very closely in the past decade, resulting in a very high degree of policy coordination. In contrast, monetary policy discoordination still contributed a sizeable part to the variance of RER shocks between the German *Länder* and Denmark and France in the late 1980s.

Table 4 conveys three main messages. First, in the 1970s and early 1980s, lack of monetary policy coordination was indeed an important source of RER variability in the EC. Second, closer policy coordination in the EMS seems to have contributed significantly to reducing the variance of RER shocks. Finally, judging from the late 1980s, a further strengthening of monetary policy coordination in the transition to EU will not contribute much to reduce conditional RER variance except between Germany and France and Germany and Denmark. On the other hand, the recent exit of the United Kingdom and Italy from the ERM, and the deterioration of policy coordination it implies, are likely to raise RER variability between these countries and Germany in the 1990s.

[12] For reasons of practicability of the empirical analysis, we neglect the possibility of regional differences in money growth due to differences in money demand which may still arise in a monetary union, e.g. due to differences in real income growth. The main problem with this procedure is that the transition to EMU may produce changes in the parameters of the model in addition to letting $M_{Gi,t}$ and $I_{Gi,t}$ be zero.

IV. Conclusions

We have studied the conditions for a common currency in Europe using a comparison of the variability of real exchange rate shocks within Germany and between Germany and a number of European countries. We find that the variability of real exchange rate shocks has declined dramatically both within Germany and between Germany and the European countries in our sample during the 1980s, both at monthly and quarterly frequencies. Exchange rate arrangements in the ERM and between Germany and Austria seem to have contributed to the variance reduction, but they were not the only source. Furthermore, the persistence of real exchange rate changes between Germany and its EMS partners and Austria has become very similar to the persistence of real exchange rate movements between the peripheral and the more central *Länder* of Germany. Thus, the conditions for entering an EMU have clearly improved.

Our results suggest that Austria, Belgium, France, Luxembourg, and the Netherlands today have achieved a sufficiently low conditional real exchange rate variance for EMU, a variance comparable to that prevailing among the German *Länder* in the 1970s. A common currency for these five countries and Germany would result in regional price level variation similar to regional price variation in Germany during that period. From this perspective, it would certainly be a viable arrangement.

In contrast, conditional real exchange rate variance between Germany and Denmark, Italy, and the U.K. remains much higher than within the German monetary union. It is important to note that, except to some extent for Denmark, this higher variance does not seem to be the consequence of imperfect monetary coordination between Germany and these countries. Our results suggest instead that the higher remaining variance results mainly from sources other than monetary policy, such as asymmetric real supply and demand shocks. A premature monetary union including Germany and these countries may, therefore, put excessive strain on the participating economies by demanding large regional price level variation to facilitate real exchange rate changes. Thus, a union with these countries could experience large regional imbalances and require mechanisms for regional stabilization. If national budget deficits are limited, as the Maastricht Agreement stipulates, coordinated spending policies or an elaborate scheme of interregional redistribution of tax revenue could become necessary.

In conclusion, the empirical evidence on real exchange rates favors a "Europe of Two Speeds," with Germany, Austria, Belgium, France, Luxembourg, and the Netherlands forming the initial core of the monetary union, and the remaining countries joining after having reached a comparable degree of real exchange rate variability. If this solution is deemed politically unattractive, our results indicate that a large EMU might be premature and a postponement until further economic adjustment and convergence have occurred may be advisable. From the point of view of reducing conditional real exchange rate variance, such a waiting period would neither require nor be likely to benefit much from closer monetary policy coordination beyond the current arrangements in the EMS.

REFERENCES

Delors Report, "Report on Economic and Monetary Union in the European Community," Committee for the Study of Economic and Monetary Union, Luxembourg: Office for Official Publications of the European Communities (1989).

Eichengreen, Barry, "Is Europe an Optimum Currency Area?" CEPR Discussion Paper 478 (1990).

Eichengreen, Barry, and Tamim Bayoumi, "Shocking Aspects of European Monetary Unification," Working paper, University of California Center for German and European Studies (1992).

European Commission (Commission of the European Communities), "One Market—One Money," *European Economy* 44 (1990).

Fratianni, Michele, and Jürgen von Hagen, "The European Monetary System Ten Years After," in Allan H. Meltzer (ed.), *Unit Roots, Investment, and Other Essays*, Carnegie Rochester Conference Series on Public Policy 32 (Amsterdam: North-Holland, 1990).

Kenen, Peter, "The Theory of Optimum Currency Areas: An Eclectic View," in Robert A. Mundell and Alexander Swoboda (eds.), *Monetary Problems in the International Economy* (Chicago: University of Chicago Press, 1969).

Marston, Richard, "Exchange Rate Unions as an Alternative to Flexible Rates: The Effects of Real and Monetary Disturbances," in John F. O. Bilson and Richard C. Marston (eds.), *Exchange Rate Theory and Practice* (Chicago: University of Chicago Press, 1984).

MacDougall Report, "Report of the Study Group on the Role of Public Finance in European Integration," Brussels: European Communities, 1977.

McKinnon, Ronald, "Optimum Currency Areas," *American Economic Review* 53 (1963), 17–25.

Meltzer, Allan H., "Size, Persistence, and Interrelation of Nominal and Real Shocks," *Journal of Monetary Economics* 17 (1986), 161–194.

Minford, Patrick, and Anupam Rastogi, "The Price of EMU," Working Paper 90/07, Dept. of Economics and Accounting, University of Liverpool (1990).

Mundell, Robert A., "The Theory of Optimum Currency Areas," *American Economic Review* 51 (1961), 657–665.

Mussa, Michael, "Nominal Exchange Rate Regimes and the Behavior of Real Exchange Rates: Evidence and Implications," in Karl Brunner and Allan H. Meltzer (eds.), *Real Business Cycles, Real Exchange Rates, and Actual Policies*. Carnegie Rochester Conference Series 25 (Amsterdam: North-Holland, 1986).

Neumann, Manfred J. M., "The Gold Standard, Bretton Woods and Other Monetary Regimes: Comment on Bordo," Federal Reserve Bank of St. Louis *Review* 73 (2) (1993).

Poloz, Stephen S., "Real Exchange Rate Adjustment between Regions in a Common Currency Area," mimeo, Bank of Canada (1990).

Vaubel, Roland, "Real Exchange Rate Changes in the European Community: The Empirical Evidence and Its Implications for European Currency Unification," *Weltwirtschaftliches Archiv* 112 (1976), 429–470.

______, "Real Exchange Rate Changes in the European Community: A New Approach to the Determination of Optimum Currency Areas," *Journal of International Economics* 8 (1978), 319–339.

von Hagen, Jürgen, "Policy Coordination in the European Monetary System," in Dominik Salvatore and Michele Fratianni (eds.), *Handbook of Monetary Policy in Developed Economies* (Greenwood, 1991a).

______, "Fiscal Arrangements in a Monetary Union: Evidence from the US," in Don Fair and Christian de Boissieux (eds.), *Fiscal Policy, Taxes, and the Financial System in an Increasingly Integrated Europe* (Deventer: Kluwer, 1991b).

von Hagen, Jürgen, and Michele Fratianni, "Monetary Policy Coordination in the EMS with Stochastic and Structural Asymmetries," in Clas Wihlborg, Michele Fratianni and Thomas D. Willett (eds.), *Financial Regulations and Monetary Arrangements after 1992* (Amsterdam: North-Holland, 1991).

Weber, Axel A., "European Economic and Monetary Union and Asymmetries and Adjustment Problems in the EMS: Some Empirical Evidence," *European Economy* 44, special edition (2) (1991).

Weiss, Andrew A., "Asymptotic Theory for ARCH Models: Estimation and Testing," *Econometric Theory* 2 (1986), 107–131.

Reprinted from Journal of Economic Theory
All Rights Reserved by Academic Press

Vol. 78, No. 1, January 1998
Printed in Belgium

Why Imitate, and If So, How?

A Boundedly Rational Approach to Multi-armed Bandits[1]

Karl H. Schlag

Economic Theory III, University of Bonn, Adenauerallee 24-26, 53113 Bonn, Germany

Received December 23, 1994; revised June 19, 1997

Individuals in a finite population repeatedly choose among actions yielding uncertain payoffs. Between choices, each individual observes the action and realized outcome of *one* other individual. We restrict our search to learning rules with limited memory that increase expected payoffs regardless of the distribution underlying their realizations. It is shown that the rule that outperforms all others is that which imitates the action of an observed individual (whose realized outcome is better than self) with a probability proportional to the difference in these realizations. When each individual uses this best rule, the aggregate population behavior is approximated by the replicator dynamic. *Journal of Economic Literature* Classification Numbers: C72, C79, D83. © 1998 Academic Press

1. INTRODUCTION

Imitation, as opposed to innovation, is the act of copying or mimicking the action of others. Imitation is a commonly observed behavior of human decision making.[2] We ask why individuals should imitate, and what sort of imitation rule they should adopt. First we identify a uniquely optimal individual rule and then derive implications for societies where each individual uses this rule. Optimality is determined according to two different perspectives: that of a boundedly rational individual and that of a social planer. Both approaches lead to the same unique prescription of how to choose future actions:

[1] This paper developed out of earlier unpublished work (Schlag, 1994). The author wishes to thank Dirk Bergemann, Jonas Björnerstedt, Georg Nöldeke, Larry Samuelson, Avner Shaked, a referee and an associate editor for helpful comments. Financial support from the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn is gratefully acknowledged.

[2] Many recent models of social learning consider individuals who select future actions by imitating others (e.g., Banerjee [1]; Björnerstedt and Weibull [4]; Cabrales [7]; Ellison and Fudenberg [9]; Gale *et al.* [11]; Helbing [12]; Hofbauer [13]; Rogers [17]).

0022-0531/98 $25.00

• follow an imitative behavior, i.e., change actions only through imitating others

• never imitate an individual that performed worse than you

• imitate an individual that performed better with a probability that is proportional to how much better this individual performed.

Rules meeting these three criteria are called *Proportional Imitation Rules* herein. When each individual in a large society adopts this optimal rule then the stochastic process governing learned choices throughout society is approximated in the short run by the replicator dynamic (Taylor [25]).

The basic decision problem is modelled as a *multi-armed bandit*. An individual must repeatedly choose an action from a finite set of actions A. Actions yield uncertain payoffs. Payoffs are realized independently, their distribution has finite support, and belongs to a bounded interval $[\alpha, \omega]$. Multi-armed bandits have wide application in economics and behavioral sciences; the arm chosen can be, e.g., choice of technology or managerial structure within industries, setting prices under uncertain demand, or visit of a restaurant of uncertain quality.[3]

In our model, identical individuals belong to a finite population in which, periodically, new individuals replace (some) existing ones. Each individual is equally likely to be replaced regardless of prior durations. Individuals in the population face, independently and repeatedly, the same multi-armed bandit. Individuals do not know the probability distributions governing payoffs realized by the arms. Instead, they gather information from each other in the following way. On entry an individual observes the previous choice and realized payoff of the individual replaced. Before each payoff realization each individual observes (or *samples*) the previous choice and realized payoff of one other individual. Sampling is independent of actions or realized payoffs.

In the classical multi-armed bandit setting, an individual has infinite memory and constantly updates a subjective prior over possible payoff distributions (Rothschild [18]). We restrict attention to simpler individual behavior by assuming that an individual forgets all information she acquired before the last payoff realization. Hence, the *behavioral rule*, the rule determining an individual's next choice, is a function of the payoffs achieved and actions taken both that individual (or by the replaced individual) and by the individual sampled in the previous round.[4] Each individual must commit to a behavioral rule before entering the population.

[3] (Ellison and Fudenberg [9]; Schmalensee [24])

[4] We ignore the issue of which action individuals choose at the beginning of time when there is no one to replace.

We will determine which of these rules is 'optimal' from two distinct perspectives.

The first approach (formalized in Section 5.1) assumes boundedly rational individuals. Here individuals are myopic, only interested in how rules perform upon first encounter of the bandit. Thus, the entering individual acts as if she were to exit the population after one round. The description of the action which each individual in the population chooses in a given round is called a *population state*; *entry state* is the population state at an individual's entry. The performance of a behavioral rule depends on the entry state and the payoff distribution of the commonly experienced multi-armed bandit. It also depends on the realization of 'objective'[5] uncertainty, i.e., point of entry, sample and payoff. Individuals are assumed to be risk neutral towards objective uncertainty, i.e., lotteries induced by the realization of objective uncertainty are compared based on expected payoff.[6]

One feasible behavioral rule, *Never Switch*, is to forever pull the arm last chosen by the individual you replaced. The expected payoff of this rule will reflect the information accumulated in the population about the bandit. Some population states and bandits may exist in which other rules perform better (worse) than Never Switch. Classic decision theory (Savage [21]) demands that an individual determines an estimate (a subjective prior) of the likelihood of each bandit and population state and then selects a rule that maximizes subjective expected payoffs. Our boundedly rational approach does not utilize subjective priors. We assume that an individual wants to perform well in each situation, in particular, never worse than the 'baseline' rule Never Switch. Hence, individuals restrict attention to *improving* rules that sometimes yield higher, and never lower, (objective) expected payoff than Never Switch upon first encounter of the bandit in any entry state and any bandit (with action set A that yields payoffs in $[\alpha, \omega]$).

In our second approach to selecting behavioral rules (formalized in Section 5.2) a social planner determines a rule for common use that yields the best performance for the entire population. For a specific multi-armed bandit, a rule is called *payoff increasing* if it generates a population dynamic where average expected payoffs will weakly increase (i.e., not decrease) over time for each initial state. This concept is compatible with the evolutionary game theory literature (e.g. Weibull [26]) where similar conditions on population dynamics are postulated.

Our social planner limits attention to rules that are *payoff increasing in each multi-armed bandit* with action set A that yields payoffs in $[\alpha, \omega]$.

[5] Savage [21] distinguishes between objective and subjective (or personal) uncertainty.

[6] Risk neutrality is assumed for simplicity. More general risk preferences can be incorporated as follows: individuals observe payoffs, translate them into utilities and then apply their rule to the utilities.

Uncertainty regarding the payoff distribution of the bandit, or rare, unobservable changes in the payoff distributions during an individual's life time motivate this criterion. An explicit analytic justification for the social planner's objective can be found in an evolutionary model of Björnerstedt and Schlag [3] where rare mutations affect rules and payoff distributions.

A first result establishes that a behavioral rule is improving if and only if it is payoff increasing in each bandit. Hence, both the boundedly rational individual and the social planner will select among improving rules. In fact, when an individual receives a rule from the social planner her expected payoff calculated a priori to her entry will weakly increase over time.

Simple improving rules are easily found, e.g., the rule Never Switch and the self-explanatory rule *Always Switch.* Our first goal is to characterize the entire set of improving rules. A first lemma shows that improving rules are imitating, i.e., an individual using an improving rule changes actions only through imitating others. The main theorem (Theorem 1) completes the characterization. Thereby, an imitating rule is improving if and only if, when two individuals using different actions happen to sample each other, the difference in the probabilities of switching is *proportional* to the difference in their realized payoffs—the individual realizing the lower payoff being more likely to switch. This relationship between switching probabilities and realized payoffs results from the linear structure of taking expectations. There are many rules with this property, e.g., *Proportional Imitation Rules* as defined above. On the other hand, the rule, *Imitate if Better*[7], which only (and always) allows imitation of individuals with higher payoff than self is *not* improving. We also show that improving rules would perform just like Never Switch were the set of obtainable payoffs not bounded.

The severe restrictions on the switching behavior of improving rules simplifies selection among them dramatically. Under various criteria and for either bounded rationality or social planning we find the same (unique) rule to be optimal. This rule is a Proportional Imitation Rule with a specific proportionality constant that depends on the payoff interval $[\alpha, \omega]$ (see Theorem 2).

Next we make some predictions about a large population in which individuals use our optimal rule and sample randomly and independently. Here, the stochastic process governing the choices made in the population over time can be approximated in the short run by a discrete version of the replicator dynamic (Taylor [25]). In particular, for any initial state in which each action is present, with probability arbitrarily close to one, provided the population is sufficiently large, most individuals will be choosing an expected payoff maximizing action after a finite number of rounds.

[7] (Ellison and Fudenberg [9]; Malawski [14])

In a further section we consider a more general two population random matching scenario. In each round two types of individuals are matched to play a normal form game. Selection of a behavioral rule using generalizations of the previous concepts yields the same optimal rule. In a large population under random and independent sampling with each individual using the optimal rule, short run adjustment is again approximated by the discrete replicator dynamic.

The paper is organized as follows. In Sections 2 and 3 the basic payoff realization and sampling scenario are introduced. Feasible behavioral rules are presented in Section 4. Section 5 contains two alternative approaches to selecting a behavioral rule, each leading to the condition of improving. In Section 6 we present a first lemma on improving rules. In Section 7 this lemma is used to illustrate why Imitate if Better is not improving. Section 8 contains the main theorem completely characterizing improving rules. In Section 9 we select an optimal rule. Section 10 deals with the implications of optimal behavior for aggregate population adjustment. In Section 11 previous findings are generalized to a game playing scenario. Section 12 contains a discussion. The Appendix contains a corollary on improving rules.

2. THE PAYOFF REALIZATION SCENARIO

In the following three sections we describe a dynamic process of choosing actions, sampling and updating. First we establish how payoffs are realized. Let W be a finite population (or set) of N individuals, $N \geqslant 2$. In a sequence of rounds, each individual in the population must choose an action (or arm) from a finite set of actions A, $|A| \geqslant 2$. Choosing action i yields an uncertain payoff drawn from a given probability distribution P_i with *finite* support in $[\alpha, \omega]$, where α and ω, $\alpha < \omega$, are exogenous parameters. π_i denotes the expected payoff generated by choosing action i, i.e., $\pi_i = \sum_x xP_i(x)$, $i \in A$. Payoffs are realized independently of all other events. The tuple $\langle A, (P_i)_{i \in A} \rangle$, which specifies the set of actions together with a payoff distribution for each action, will be called a *multi-armed bandit* (or game against nature). $\mathcal{G}(A, [\alpha, \omega])$ denotes the set of all such multi-armed bandits.[8]

Let A, $[\alpha, \omega]$ and N be fixed throughout the rest of the paper. A *population state* $s \in A^W$ in a given round t is the description of the action which each individual chooses in round t. Let $m_i = m_i(s)$ denote the number of

[8] Alternatively, one might say that each arm can be one of an infinite number of types, the true type of an arm i being associated with a specific underlying payoff distribution P_i. The set of feasible types of arm i is then the set of probability distributions with finite support on $[\alpha, \omega]$. In our notation, a multi-armed bandit is the *realization* of a type for each arm, denoted by $\langle A, (P_i)_{i \in A} \rangle$, not the *collection* of feasible types of each arm, denoted by $\mathcal{G}(A, [\alpha, \omega])$.

individuals choosing the action i in state s, i.e., $m_i = |\{c \in W : s(c) = i\}|$ $(i \in A)$. Let $\Delta(A)$ be the set of probability distributions on A. For a given state s let $p \in \Delta(A)$ denote the probability distribution that is associated with randomly selecting an individual and observing the action she is choosing, i.e., $p_i = m_i/N$ for $i \in A$. The set of all such probability distributions will be denoted by $\Delta^N(A)$, i.e., $p \in \Delta^N(A)$ and $i \in A$ implies $N \cdot p_i \in \mathbb{N} \cup \{0\}$. Given this notation, the *average expected payoff* in the population in state s, $\bar{\pi}(s)$, is given by $\bar{\pi}(s) = \sum_i p_i \pi_i$.

Individuals do not remain in the population W forever. Periodically a new individual appears who randomly replaces one of the individuals in the population; replacement occurs after a payoff realization, $1/N$ is the probability of replacing a given individual, the replaced individual exits the population. It will not be necessary for the analysis that follows to explicitly specify the process governing when new individuals appear. An individual's *entry state* is the population state of the round in which this individual enters the population.

3. INFORMATION ABOUT OTHERS

An entering individual learns the last choice and payoff realized by the individual she replaces.

Once in the population an individual receives information about the play of other individuals according to the following sampling scenario. After a round of payoff realization, each individual meets (or samples) one other individual from the population and receives the following information. When individual c samples individual d $(c, d \in W)$, then individual c observes the action d used and the payoff d achieved in the last round without observing the identity of d. Sampling does not depend on realized payoffs nor on the population state and occurs independent of previous events. Who gets to sample whom with which probability is determined by the *sampling procedure*. Given $Z = \{f \in W^W : f(c) \neq c \; \forall c \in W\}$, let $f \in Z$ denote the event in which individual c samples individual $f(c)$, $c \in W$. A *sampling procedure* is an exogenously given distribution z over the events $f \in Z$, i.e., $z \in \Delta(Z)$.

For $c, d \in W$, $c \neq d$, let $c \rightsquigarrow d$ denote the event of c sampling d and let $\Pr(c \rightsquigarrow d)$ denote the probability of this event, i.e., $\Pr(c \rightsquigarrow d) = \sum_{f : f(c) = d} z(f)$. In the following we will restrict attention to *symmetric sampling procedures*, i.e., for any $c, d \in W$, $c \neq d$, the probability of c sampling d is the same as vice versa, i.e., $\Pr(c \rightsquigarrow d) = \Pr(d \rightsquigarrow c)$.

Symmetric sampling procedures can have a variety of different characteristics, e.g., regarding the way information is obtained. One may want to assume that individuals exchange information. In our setting this means that individuals sample each other. Here, $c \rightsquigarrow d$ is the same event as $d \rightsquigarrow c$ for each $c, d \in W$, $c \neq d$. We also allow for settings in which individuals obtain information without necessarily revealing their own. Such a situation arises when individuals *sample independently*, i.e., when $\Pr(c \rightsquigarrow d \cap d \rightsquigarrow c) = \Pr(c \rightsquigarrow d) \cdot \Pr(d \rightsquigarrow c)$ for all $c, d \in W$, $c \neq d$.

Symmetric sampling procedures may differ according to the number of different samples an individual may obtain. E.g., a symmetric sampling procedure obtains from the following story. Individuals are located on a circle. Each individual randomly samples with equal probability among her $2m$ closest neighbors (m to the left, m to the right, $m \leqslant N/2$). In the extreme case, *random sampling*, each individual randomly samples (with equal probability) from the entire population, i.e., $\Pr(c \rightsquigarrow d) = 1/(N-1)$ for $c, d \in W$, $c \neq d$.

4. BEHAVIORAL RULES

A *behavioral rule* is the formal description of how an individual chooses her next action as a function of past experience. We focus on behavior of individuals entering after the very first round of the model. How the first N individuals choose their actions is not modelled. Choice will depend on (i) individual knowledge, (ii) information and memory, and (iii) tools available.

(i) An individual knows the action set A, the set of feasible payoffs $[\alpha, \omega]$ but not the underlying payoff distributions in the bandit. She knows the symmetric sampling procedure and the entry and exit mechanism.

(ii) An individual forgets all information she obtained prior to the previous round. She does not condition play on the current round number. In particular, in her first encounter of the bandit, she treats the previous choice and realized payoff of the individual she replaced as if it were her own.[9]

(iii) An individual has access to a randomizing device that generates independent events.

The extreme limitation posed by (ii) is a means to focus on simple behavior. Given the above assumptions, a *behavioral rule* F is characterized by a function

$$F\colon A \times [\alpha, \omega] \times A \times [\alpha, \omega] \rightarrow \Delta(A), \tag{1}$$

[9] Relaxing this assumption will have no effect on optimal behavior.

where $F(i, x, j, y)_k$ is the probability of choosing action k in the next round after previously choosing action i, receiving payoff x and sampling an individual who chose action j and received payoff y.

Given a behavioral rule F and a multi-armed bandit in $\mathscr{G}(A, [\alpha, \omega])$, let F_{ij}^k be the probability of playing action k after playing action i and sampling an individual using action j calculated a priori to realization of payoffs $(i, j, k \in A)$, i.e.,

$$F_{ij}^k = \sum_{x, y} F(i, x, j, y)_k \, P_i(x) \, P_j(y). \tag{2}$$

$(F_{ij}^k)_{i, j, k \in A}$ are called *induced switching probabilities.*

One of the simplest behavioral rules, *Never Switch*, is the rule F that satisfies $F(i, x, j, y)_i = 1$ for $i, j \in A$ and $x, y \in [\alpha, \omega]$. An opposite behavior is exhibited by the rule, *Always Switch*, where $F(i, x, j, y)_j = 1$ for $i, j \in A$ and $x, y \in [\alpha, \omega]$. A more plausible rule seems to be to act according to *Imitate if Better* (Ellison and Fudenberg [9]; Malawski [14]), i.e., use the rule F given by $F(i, x, j, y)_j = 1$ if $y > x$ and $F(i, x, j, y)_i = 1$ if $y \leqslant x$. The three rules described above belong to the class of behavioral rules that are based on imitation, i.e., either the individual does not change actions or she switches to the action used by the individual she sampled. More generally, we call a behavioral rule F *imitating* if $F(i, x, j, y)_k = 0$ when $k \notin \{i, j\}$ $(x, y \in [\alpha, \omega])$. The *Proportional Imitation Rule*, is the imitating rule F where there exists $\sigma \in (0, 1/(\omega - \alpha)]$ such that $F(i, x, j, y)_j = 0$ if $y \leqslant x$ and $F(i, x, j, y)_j = \sigma(y - x)$ if $y > x$, $i \neq j$ and $x, y \in [\alpha, \omega]$. The associated constant σ is called the *switching rate.*[10]

5. SELECTION OF RULES

Each individual must commit to a behavioral rule before she enters the population. The major part of our analysis is concerned with finding an optimal rule. We present two alternative scenarios (or approaches) for determining the notion of optimality.

5.1. *A Boundedly Rational Approach*

In the first scenario we consider boundedly rational individuals. Here, individuals are myopic and evaluate rules according to performance in their first encounter of the bandit. This performance depends on entry state and payoff distributions $(P_i)_{i \in A}$ of the commonly experienced bandit. It

[10] Switching behavior as displayed by Proportional Imitation Rule appears in papers by Cabrales [7] and Helbing [12], the former intuitively justifying such behavior through uniformly distibuted costs for switching actions.

also depends on realization of, 'objective' (*sensu* Savage [21]) uncertainty implicit in the model, i.e., point of entry, sample and payoff. Individuals are assumed to be risk neutral towards objective uncertainty. Thus, if an individual were to know entry state and payoff distributions of the bandit, comparing two given rules she would prefer the one yielding higher expected payoff in her first encounter. In the following, the term 'performance' refers to objective expected payoff in the first encounter.

Given the entry state and the multi-armed bandit encountered the selected behavioral rule might perform poorly and it might perform excellently. We assume that an individual prefers a rule that performs well whatever circumstances she enters into. This criterion does not make sense until we calibrate the measurement of performance. Following the rule Never Switch means to perform as well as the average individual performed in the previous round. Thus, the expected payoff of this rule will reflect the information accumulated in the population about the bandit. Never Switch will be our a baseline for analyzing the performance of a rule. Hence, "to perform well" will mean to perform at least as good as Never Switch. Rules that perform at least as good as Never Switch under any circumstances will be called improving, a condition to be formalized in the following.

Consider an individual that is about to enter in round t a population in state s^t. Remember that an entering individual adapts all attributes of the individual she replaces. Hence, the individual's expected payoff in round $t+1$, denoted by $E_{F,s^t}\pi'$, is given by

$$E_{F,s^t}\pi' = \frac{1}{N}\sum_{c\in W}\ \sum_{d\in W\setminus\{c\}} \Pr(c \rightsquigarrow d) \sum_{r\in A} F^r_{s^t(c)\,s^t(d)}\pi_r.$$

The expected payoff of Never Switch in round $t+1$ is equal to the average expected payoff in the population in round t, $\bar{\pi}(s^t)$. Let $EIP_F(s^t)$ denote the difference between the performance of F and Never Switch, i.e.,

$$EIP_F(s^t) = E_{F,s^t}\pi' - \bar{\pi}(s^t). \tag{3}$$

$EIP_F(s)$ is called the *expected improvement* under F in state s. The behavioral rule F is called *improving* (given A and $[\alpha,\omega]$) if $EIP_F(s) \geqslant 0$ for any state $s \in A^W$ and any multi-armed bandit in $\mathcal{G}(A,[\alpha,\omega])$.[11] The improving rule F is *degenerate* if $EIP_F(s) = 0$ for any state $s \in A^W$ and any multi-armed bandit in $\mathcal{G}(A,[\alpha,\omega])$.

[11] The concept of improving is very closely related to the concept of *absolute expediency* defined by Sarin [19] in a slightly different context. Applied to our model, an absolutely expedient rule is an improving rule with the property that the expected improvement is strictly positive whenever not each action currently used in the population achieves the same expected payoff. As such this concept leads to a refinement of improving rules.

5.2. *A Social Planer's Approach*

In this alternative scenario, a social planner determines the behavioral rule each individual follows. Equal treatment of the identical individuals calls for the social planner to prescribe the same rule to each individual. The aim of the planner is to select the rule that is best (to be specified) for society.

When each individual follows the same behavioral rule we obtain a *monomorphic population*. The *initial state* is the population state in the very first round of the model. Given initial state $s^1 \in A^W$, multi-armed bandit $b \in \mathcal{G}(A, [\alpha, \omega])$ and behavioral rule F, the monomorphic population induces a Markov process on A^W that describes the change of the population state over time. If s^t is the population state in round t then the expected proportion of individuals in round $t+1$ using action i, $E_F p'_i(s^t)$, calculated a priori to the payoff realizations in round t, is given by

$$E_F p'_i(s^t) = \frac{1}{N} \sum_{c \in W} \sum_{d \in W \setminus \{c\}} \Pr(c \rightsquigarrow d)\, F^i_{s^t(c)\, s^t(d)}. \tag{4}$$

$E_F \bar{\pi}'(s^t) = \sum_i E_F p'_i(s^t) \cdot \pi_i$ is the average expected payoff in the population in round $t+1$. Notice that

$$E_F \bar{\pi}'(s^t) = E_{F,\, s^t} \pi'. \tag{5}$$

The behavioral rule F is called *payoff increasing* in the bandit b if average expected payoff weakly increases over time for any initial state, i.e., $E_F \bar{\pi}'(s) \geqslant \bar{\pi}(s)$ for any $s \in A^W$.

The social planner finds the population already "in action" when he first prescribes a rule. Lack of information about bandit and current state or rare, unobservable, changes in payoff distributions make the planner prefer a rule that performs well in each situation. Here, the social planer selects among the rules that are payoff increasing in each bandit in $\mathcal{G}(A, [\alpha, \omega])$.

As in the bounded rational setting, as of yet a formal justification why a rule should be payoff increasing in each bandit is missing. The story of a social planner makes it easy to describe the payoff increasing condition. However, this condition also plays an important role without social planner when rules are under selection pressure. Consider a *large* population in which successful rules propagate. If success of a rule is determined by average payoff in a given state then a successful rule must be able to find the expected payoff maximizing action. Otherwise an alternative rule with a bias towards the action maximizing expected payoff will have a selective advantage. At the same time, two rules that are both able to learn which action is best among those present will eventually eliminate selection pressure between them and hence both survive. Consequently, in an evolutionary setting in which bandits are subject to rare, arbitrary, and unobservable changes, it seems that only a rule that is payoff increasing in each bandit can be successful. Björnerstedt and Schlag [3] confirm this intuition in an evolutionary analysis of an infinite population facing our matching and sampling scenario.[12]

5.3. *Comparing Approaches*

Combining (3) and (5), we obtain that

Remark 1. A behavioral rule is improving if and only if it is payoff increasing in any multi-armed bandit belonging to $\mathcal{G}(A, [\alpha, \omega])$.

Boundedly rational individuals modelled in Section 5.1 restrict attention to performance in their first encounter. Thus, for each individual, performance can be calculated independently of rules used by others. How does an individual perceive her performance in later rounds if she receives her rule from a social planner? The social planner prescribes the same improving rule to each individual. Hence, realization of future states does not depend on which individuals are replaced by whom. In any round (and not only the round in which the individual enters) and for any state in this round the individual expects to be equally likely in the position of each of the individuals $c \in W$. Hence,

Remark 2. For any (improving) rule prescribed by the social planner, an individual's expected payoff calculated a priori to her entry weakly increases over time for any entry state and any bandit.

Given above, both approaches select among improving rules. Once we have characterized improving rules it will become clear which improving rule a boundedly rational individual or a social planer will regard as optimal.

6. A FIRST LEMMA

Clearly, the rule Never Switch is improving. Matching being symmetric causes Always Switch to be improving too. Either rule is not a very good candidate for optimal behavior since they both leave average expected payoff constant over time, i.e., each of them is a degenerate improving rule.

The following preliminary result characterizes improving rules in a way that does not depend on the population state. Hereby, a behavioral rule F

[12] Typically an individual that selects a behavioral rule according to a subjective prior will perform worse. In this sense, this is an instance (similar to Robson [16]) in which successful behavior in an evolutionary setting does not comply with the fundaments of rational decision making.

is improving if and only if it is an imitating rule that satisfies the following condition. Consider two individuals choosing different actions, using the same rule F, that sample each other. Then, before observing each other's payoff, the individual with the lower expected payoff is more likely to switch actions.

LEMMA 1. *Let F be a behavioral rule. Then F is improving if and only if F is imitating and for any multi-armed bandit in $\mathcal{G}(A, [\alpha, \omega])$, for any $i, j \in A$, $i \neq j$,*

$$(F_{ij}^j - F_{ji}^i)(\pi_j - \pi_i) \geq 0. \tag{6}$$

The proof of the imitation property is quite intuitive. An individual does not switch to an action she did not observe since she fears this action achieves the lowest and all other actions the highest expected payoff. Notice that imitation remains necessary to ensure the improving condition even after the event of receiving the lowest possible payoff α and sampling an individual who used the same action and also obtained α. This is because it may be that obtaining α is an unlucky event for the own action whereas it is the only outcome for any other action.

Proof. We will first show the "if" statement. Calculating expected improvement for imitating rules yields

$$EIP_F(s) = \frac{1}{N} \sum_{c \in W} \sum_{d \in W \setminus \{c\}} \Pr(c \rightsquigarrow d)\, F_{s(c)\, s(d)}^{s(d)} [\pi_{s(d)} - \pi_{s(c)}].$$

Using the fact that the sampling procedure is symmetric we obtain

$$EIP_F(s) = \frac{1}{N} \sum_{i<j} \left[\sum_{\substack{c:\, s(c) = i \\ d:\, s(d) = j}} \Pr(c \rightsquigarrow d) \right] (F_{ij}^j - F_{ji}^i)(\pi_j - \pi_i), \tag{7}$$

which completes the proof of the "if" statement.

Next we will show that improving rules are imitating. Assume that the behavioral rule F is improving. Let $x, y \in [\alpha, \omega]$, $i, j \in A$ and $r \in A \setminus \{i, j\}$ be such that $F(i, x, j, y)_r > 0$. Consider a multi-armed bandit belonging to $\mathcal{G}(A, [\alpha, \omega])$ with $P_i(x) = P_i(y) = P_i(\omega) = \frac{1}{3}$, $P_j \equiv P_i$ and $P_k(\alpha) = 1$ for all $k \in A \setminus \{i, j\}$. It follows that $\pi_i = \pi_j > \pi_k$. Choose $c, d \in W$ such that $\Pr(c \rightsquigarrow d) > 0$ and consider a population state s such that $s(c) = i$, $s(d) = j$ and $m_i + m_j = N$. Then $F(i, x, j, y)_r > 0$ implies $EIP(s) < 0$ which contradicts the fact that F is improving.

Finally, we will show that an imitating rule F that violates (6) for some $i \neq j$ and some multi-armed bandit in $\mathcal{G}(A, [\alpha, \omega])$ is not improving. Choose again $c, d \in W$ such that $\Pr(c \rightsquigarrow d) > 0$ and consider a population state s such that $s(c)=i$, $s(d)=j$ and $m_i+m_j=N$. Since $(F^j_{ij}-F^i_{ji})(\pi_j-\pi_i)<0$, following (7), $EIP_F(s)<0$ which implies that F is not improving. ▮

In the social planer's approach we restricted attention to rules that are payoff increasing in any bandit. One might wish to impose the following weaker condition. Assume that only those actions are expected to increase that perform at least as well as some other action present, i.e.,

$$\forall s\in A^W,\ i\in A: E_F p'_i(s)\geqslant p_i(s)\Rightarrow \exists c\in W: \pi_i\geqslant \pi_{s(c)}. \tag{8}$$

Our condition (8) is weaker than most necessary conditions postulated in evolutionary game theory for reasonable dynamics in infinite populations.[13] Never-the-less it is sufficient to drive our results.

Remark 3. A behavioral rule induces a monomorphic population dynamic that satisfies (8) in all multi-armed bandits contained in $\mathscr{G}(A,[\alpha,\omega])$ if and only if it is improving.

The statement in Remark 3 is easily verified using the proof of the imitation property in Lemma 1 and the fact that (8) is equivalent to payoff increasing when $|A|=2$.

7. THE DRAWBACK OF IMITATE IF BETTER

Imitate if Better is a plausible rule. In fact, it performs well in multi-armed bandits in which uncertainty is driven solely through idiosyncratic (*sensu* Ellison and Fudenberg [9]) shocks. Consider a multi-armed bandit in $\mathscr{G}(A,[\alpha,\omega])$ with the following properties. There is a probability distribution Q with finite support and mean 0 such that $P_i(x)=Q(x-\pi_i)$ for each $i\in A$ and $x\in[\alpha,\omega]$. Throughout this section, let F denote the rule Imitate if Better. Then

$$F^j_{ij}-F^i_{ji}=\frac{1}{2}\sum_{x,\,y} Q(x)\,Q(y)\begin{bmatrix} F(i,\pi_i+x,j,\pi_j+y)_j-F(j,\pi_j+y,i,\pi_i+x)_i \\ +F(i,\pi_i+y,j,\pi_j+x)_j-F(j,\pi_j+x,i,\pi_i+y)_i\end{bmatrix}$$

and hence, $F^j_{ij}-F^i_{ji}\geqslant 0$ when $\pi_j\geqslant\pi_i$. With (7) it follows that the expected improvement of Imitate if Better is non negative in such multi-armed bandits.

However, we will see that Imitate if Better generates negative expected improvement in some extremely simple multi-armed bandits; it can not distinguish between lucky and certain (or highly probable) payoffs. Let

[13] (E.g., compatibility, also known as payoff monotonicity, and weak compatibility, Friedman [10]; payoff positivity, Weibull [26])

$x \in (\alpha, (\alpha+\omega)/2)$. Consider a multi-armed bandit in which $P_1(x)=1$, $P_2(\alpha)=\lambda$ and $P_2(\omega)=1-\lambda$ for some $0<\lambda<1$. It follows that

$$\pi_2 > \pi_1 \qquad \text{if and only if} \quad \lambda < \frac{\omega - x}{\omega - \alpha}.$$

On the other hand, $F_{12}^2 = 1-\lambda$ and $F_{21}^1 = \lambda$, and hence,

$$F_{21}^1 > F_{12}^2 \qquad \text{if and only if} \quad \lambda > \tfrac{1}{2}.$$

Consequently, when $\frac{1}{2} < \lambda < (\omega - x)/(\omega - \alpha)$ then (6) is violated and hence Imitate if Better is not improving.

8. A COMPLETE CHARACTERIZATION

The fact that being improving is equivalent to being imitating and more likely to imitate an action with a higher expected payoff than vice versa (Lemma 1) is quite intuitive. The difficulty in finding improving rules is that an individual is not able to condition her behavior on expected payoffs but must base her decision on realized payoffs. The following theorem contains the central result of this paper, a somewhat surprising characterization of the set of behavioral rules that are improving. According to this result only switching in a way that "net" switching behavior is linear in payoff differences ensures that an imitating rule is in fact improving. The consequent proof reveals that this strong characterization is due to the linear structure of taking expectations.

THEOREM 1. *The behavioral rule F is improving if and only if*

(i) *F is imitating and*

(ii) *for all* $i, j \in A$, $i \neq j$ *there exists* $\sigma_{ij} = \sigma_{ji} \in [0, 1/(\omega-\alpha)]$ *such that*

$$F(i, x, j, y)_j - F(j, y, i, x)_i = \sigma_{ij}(y - x) \qquad \text{for all} \quad x, y \in [\alpha, \omega]. \tag{9}$$

From (9) we see immediately that Imitate if Better is not improving, confirming our findings from Section 7.

Proof. We will first show that conditions (i) and (ii) are sufficient. Let F be an imitating behavioral rule that satisfies condition (ii). (2) and (9) imply

$$F_{ij}^j - F_{ji}^i = \sigma_{ij}(\pi_j - \pi_i). \tag{10}$$

Together with Lemma 1 it follows that F is improving.

We will now prove the necessity of conditions (i) and (ii). Let F be improving and fix $i, j \in A$ with $i \neq j$. Let $g_{ij}(x, y) := F(i, x, j, y)_j - F(j, y, i, x)_i$ for $x, y \in [\alpha, \omega]$. First we will show that

$$\frac{g_{ij}(x, u)}{u - x} = \frac{g_{ij}(x, z)}{z - x} \qquad \forall u < x < z. \tag{11}$$

Given $u < x < z$, consider a multi-armed bandit where $P_i(x) = 1$, $P_j(u) = \lambda$ and $P_j(z) = 1 - \lambda$, $0 \leqslant \lambda \leqslant 1$. Then $\pi_j > \pi_i$ if and only if $\lambda < (z - x)/(z - u) =: \lambda^*$ where $0 < \lambda^* < 1$. It follows from Lemma 1 that

$$F_{ij}^j - F_{ji}^i = \lambda g_{ij}(x, u) + (1 - \lambda)\, g_{ij}(x, z) \geqslant 0 \qquad \text{if} \quad \lambda < \lambda^* \text{ and} \tag{12}$$

$$\lambda g_{ij}(xc, u) + (1 - \lambda)\, g_{ij}(x, z) \leqslant 0 \qquad \text{if} \quad \lambda > \lambda^* \tag{13}$$

Therefore, $\lambda^* g_{ij}(x, u) + (1 - \lambda^*)\, g_{ij}(x, z) = 0$, which, after simplification, shows that (11) is true.

Since the left hand side in (11) is independent of z, so is the right hand side. Given $x \in (\alpha, \omega)$, let $\sigma_{ij}(x) = (g_{ij}(x, z))/(z - x)$ for some $z > x$. Following (11), $g_{ij}(x, u) = \sigma_{ij}(x) \cdot (u - x)$ for all $u < x$ and $g_{ij}(x, z) = \sigma_{ij}(x) \cdot (z - x)$ for all $z > x$. Hence, for all $x, y \in (\alpha, \omega)$, $x \neq y$,

$$g_{ij}(x, y) = F(i, x, j, y)_j - F(j, y, i, x)_i = \sigma_{ij}(x)(y - x), \tag{14}$$

or equivalently,

$$F(j, y, i, x)_i - F(i, x, j, y)_j = \sigma_{ij}(x)(x - y). \tag{15}$$

Exchanging the variables i and j and the variables x and y in (14) implies

$$F(j, y, i, x)_i - F(i, x, j, y)_j = \sigma_{ji}(y)(x - y). \tag{16}$$

From (15) and (16) it follows that $\sigma_{ij} = \sigma_{ji}$ is a constant. Setting $\lambda = 0$ in (12) it follows that this constant is non-negative. Hence, we have shown (9) for all $x, y \in (\alpha, \omega)$, $x \neq y$.

Looking back at the above proof we see that the explicit values of α and ω did not enter the argument. Hence, (9) holds for all $x, y \in [\alpha, \omega]$, $x \neq y$.

Assume that $g_{ij}(x, x) > 0$ for some $x \in [\alpha, \omega)$. Consider a multi-armed bandit where $P_i(x) = 1 - \lambda$, $P_i(\omega) = \lambda$ and $P_j(x) = 1$, $0 < \lambda < 1$. Then $\pi_j < \pi_i$ and $F_{ij}^j - F_{ji}^i = (1 - \lambda)\, g_{ij}(x, x) + \lambda \sigma_{ij} \cdot (x - \omega) > 0$ for λ sufficiently small contradicts the fact that F is improving. Similarly, $g_{ij}(x, x) < 0$ leads to a contradiction. Hence, $g_{ij}(x, x) = 0$ for all $x \in [\alpha, \omega)$. The proof of $g(\omega, \omega) = 0$ is analogue. This completes the proof of (9).

Finally, $\sigma_{ij}(\omega - \alpha) = g_{ij}(\alpha, \omega) \leqslant F_{ij}^j \leqslant 1$ implies $\sigma_{ij} \leqslant 1/(\omega - \alpha)$. ∎

From (7) and (10) we obtain that

COROLLARY 1. *An improving rule is degenerate if and only if* $\sigma_{ij}=0$ *for all* $i, j \in A$, $i \neq j$.[14]

Thus, non degenerate improving rules induce stochastic behavior. Moreover, since σ_{ij} is bounded above by $1/(\omega-\alpha)$ (Theorem 1), all improving rules would be degenerate if payoffs were not contained in a bounded interval.

In the appendix we include a corollary that gives a more precise characterization of improving rules.

9. SELECTING AMONG IMPROVING RULES

We now proceed with our search for an optimal behavioral rule. First we will show that there are improving rules that perform better than all others. A behavioral rule *F dominates the improving rules* (or short, is a *dominant rule*) if it *always* generates weakly higher expected improvement than any other improving rule, i.e., for any improving rule F', state s and multi-armed bandit in $\mathcal{G}(A, [\alpha, \omega])$, $EIP_F(s) \geqslant EIP_{F'}(s)$ holds. With (3) and (5) it follows that dominant rules are also the rules that maximize the increase in average expected payoffs of a monomorphic population in any given state and bandit among the set of improving rules. Hence, both the boundedly rational individual and the social planer will select a dominant rule if such a rule exists.

Following (7) and (10),

$$EIP_F(s)=\left[\frac{1}{N}\sum_{i<j}\sum_{\substack{c:s(c)=i\\ d:s(d)=j}}\Pr(c\rightsquigarrow d)\right]\sigma_{ij}(\pi_j-\pi_i)^2. \tag{17}$$

Given (17), the expected improvement of an improving rule depends only on the factors $(\sigma_{ij})_{\substack{i,j\in A\\ i\neq j}}$. Hence

PROPOSITION 1. *A behavioral rule is a dominant rule if and only if it is improving and for any* $i\neq j$, $\sigma_{ij}=1/(\omega-\alpha)$.

Next we demonstrate three unique properties of the Proportional Imitation Rule with switching rate $1/(\omega-\alpha)$ (defined in Section 4, denoted

[14] In this context, notice that a rule is absolutely expedient (Footnote 11) if and only if it is improving with $\sigma_{ij}>0$ for all $i, j \in A$, $i \neq j$.

by F^p). These properties will cause us to select it as the unique optimal rule for our model.

THEOREM 2. *F^p is the unique dominant rule that satisfies any one of the following properties.*

(i) *It never prescribes to imitate an action that achieved a lower payoff.*

(ii) *In each multi-armed bandit in $\mathcal{G}(A, [\alpha, \omega])$ and state it minimizes the probability of switching among dominant rules.*

(iii) *For each multi-armed bandit in $\mathcal{G}(A, [\alpha, \omega])$ and current state it minimizes the variance of the average payoff in a monomorphic population in the next round among dominant rules.*

Proof of Theorem 2. Statements (i) and (ii) follow easily from Corollary 2 stated in the appendix since F^p is the unique dominant rule with $g_{ij}(x, y) = -\min\{x, y\}$. Part (iii) follows from part (ii) of Theorem 1 and some easy calculations. ▮

Following (i) in Theorem 2, it can be argued that F^p is the best dominant rule under deterministic payoffs; realized payoffs never decrease when actions yield certain payoffs. (ii) implies that F^p is the dominant rule that changes actions the least number of times. Given (iii), among improving rules, F^p maximizes increase in average expected payoffs using minimal variance. This leads us to conjecture that F^p maximizes the probability that average payoffs realized in a monomorphic population increase over time.

The Proportional Imitation Rule with switching rate $1/(\omega - \alpha)$ is improving. It is dominant (Proposition 1), and hence always performs at least as well as any other improving rule regarding expected improvement. Finally, its unique properties among the dominant rules (Theorem 2) lead us to argue that it is the *optimal rule* in either selection approach. Notice that the optimal rule does not depend on the size of the population N.

Remark 4. One should mention that there is a dominant rule that utilizes less information than the dominant Proportional Imitation Rule. The dominant *Proportional Reviewing Rule* is the imitating rule F where $F(i, x, j, y)_j = (\omega - x)/(\omega - \alpha)$ for $i \neq j$, $i, j \in A$ and $x, y \in [\alpha, \omega]$.[15] It can be easily shown (see Schlag [22] for more details) that the dominant proportional reviewing rule is the *unique* dominant improving rule that does not depend on the sampled individual's payoff.

[15] Björnerstedt and Weibull ([4]) and Gale *et al.* ([11]) both use a variant of this rule in their model, the later interpret it on the basis of random aspiration levels.

10. POPULATION DYNAMICS

In this section we investigate how much individuals learn about the bandit when each of them uses the optimal rule. For this we analyze the stochastic process governing the evolution of the population state over time. Attention is restricted to random and independent sampling in large populations.

First we derive a 'law of large numbers' type of result for a monomorphic population based on an arbitrary behavioral rule. We identify a state $s \in A^W$ with the associated distribution, $p \in \Delta^N(A)$, of actions chosen. For a monomorphic population of size N which is in state $p^N(1) \in \Delta^N(A)$ in round 1, let $p^N(t) \in \Delta^N(A)$ be the random state in round t, $t = 2, 3, \ldots$ Let $\|\cdot\|$ denote the supremums norm. For large populations, our result shows that the stochastic adjustment can be approximated in the short run by the deterministic adjustment that would take place if the size of the population were infinite.

THEOREM 3. *Assume that sampling is random and independent. Assume that each individual is using the rule F. For each $\delta > 0$, $\varepsilon > 0$ and $T \in \mathbb{N}$ there exists $N_0 \in \mathbb{N}$ such that for any population size $N > N_0$ and any initial state $\tilde{p} \in \Delta^N(A)$, the event $\{\|p^N(T) - p(T)\| > \delta\}$ occurs with probability less than ε where $p^N(1) = p(1) = \tilde{p}$ and $(p(t))_{t \in \mathbb{N}}$ satisfies*

$$p_i(t+1) = \sum_{j,r} p_j(t)\, p_r(t)\, F^i_{jr}, \qquad t \in \mathbb{N}. \tag{18}$$

Proof. We will first prove the statement for $T = 2$. Fix $i \in A$ and $\tilde{p} \in \Delta^N(A)$. For $c \in W$ let $w_i(c)$ be the random variable such that $w_i(c) = 1$ if individual c uses action i in round two, otherwise $w_i(c) = 0$. Then

$$\Pr(w_i(c) = 1) = \frac{m_{s(c)} - 1}{N-1} F^i_{s(c)\,s(c)} + \sum_{j \neq s(c)} \frac{m_j}{N-1} F^i_{s(c)\,j}$$

and $p_i^N(2) = (1/N) \sum_{c \in W} w_i(c)$. Since $w_i(c)$ and $w_i(d)$ are independent variables for $c \neq d$ and $VAR(w_i(c)) \leqslant 1$ it follows that $VAR(p_i^N(2)) \leqslant 1/N$. Applying Tschebysheff's inequality we obtain that the event $\{|p_i^N(2) - E[p_i^N(2)]| > \delta/2\}$ occurs with a probability of less than $4/N\delta^2$. Given

$$E[p_i^N(2)] = \frac{N}{N-1} \sum_{j,r} \tilde{p}_j \tilde{p}_r F^i_{jr} - \frac{1}{N-1} \sum_j \tilde{p}_j F^i_{jj}, \tag{19}$$

there exists N_0 such that $4/N\delta^2 < \varepsilon$ and $|E[p_i^N(2)] - \sum_{j,r} \tilde{p}_j \tilde{p}_r F^i_{jr}| < \delta/2$ for $N > N_0$. Then $\{|p_i^N(2) - \sum_{j,r} \tilde{p}_j \tilde{p}_r F^i_{jr}| > \delta\}$ occurs with probability less than ε when $N > N_0$. Since N_0 can be chosen independent of $\tilde{p}$ the proof for $T = 2$ is complete.

We will now prove the statement for $T = 3$ by iterating the proof for $T = 2$. Let $\delta > 0$ and $\varepsilon > 0$ be given. Let $f: \Delta(A) \to \Delta(A)$ be defined by $f(p)_i = \sum_{j,r} p_j p_r F^i_{jr}$, $i \in A$. Let $p^N(t, \tilde{p})$ be the random state in round t given state $\tilde{p} \in \Delta^N(A)$ in round one ($t > 1$). Since f is a continuous function on a compact space there exists $\beta \in (0, \delta/2)$ such that $\|f(w) - f(w')\| < \delta/2$ if $\|w - w'\| < \beta$. Let μ be such that $(1-\mu)^2 = 1 - \varepsilon$. Following the proof for $T = 2$ there exists N_0 such that for $N > N_0$ and $\tilde{p} \in \Delta^N(A)$, $\Pr(\|p^N(2, \tilde{p}) - f(\tilde{p})\| < \beta) > 1 - \mu$. For $N > N_0$ it follows that

$$\begin{aligned}
&\Pr(\|p^N(3, \tilde{p}) - f(f(\tilde{p}))\| \leqslant \delta) \\
&\quad = \sum_{w \in \Delta^N(A)} \Pr(\|p^N(2, w) - f(f(\tilde{p}))\| \leqslant \delta) \cdot \Pr(p^N(2, \tilde{p}) = w) \\
&\quad \geqslant \sum_{w \in \Delta^N(A)\,:\,\|w - f(\tilde{p})\| < \beta} \Pr\left(\|p^N(2, w) - f(w)\| \leqslant \frac{\delta}{2}\right) \cdot \Pr(p^N(2, \tilde{p}) = w) \\
&\quad \geqslant (1-\mu)^2 = 1 - \varepsilon,
\end{aligned}$$

which completes the proof for $T = 3$. The proof for $T > 3$ follows similarly using induction. ∎

Theorem 3 makes a statement about the short run adjustment of large populations. First the time horizon and precision of the approximation is set, then we choose the population size to be sufficiently large. Why is it necessary to keep the time horizon fixed? For any given population size, long run behavior can differ quite dramatically from the behavior of (18).[16] The following is easily verified.

Remark 5. Consider a monomorphic population of size N, based on a non degenerate improving rule, facing a two-armed bandit, i.e., $A = \{1, 2\}$. Typically, $F^2_{12} > 0$ and $F^1_{21} > 0$.[17] Then for any initial interior state (i.e., $0 < p^N_1(1) < 1$), eventually each individual will be playing the same action. There is a positive probability that all individuals will be playing the worse action after a finite number of rounds. This can not happen in an infinite population as we will see below.

Given Theorem 3, understanding adjustment of the infinite population helps understand short run adjustment of a large but finite population. An infinite monomorphic population induces a deterministic process $(p(t))_{t \in \mathbb{N}}$

[16] For further reading, see (Boylan [6]) and Binmore *et al.* [2]).

[17] i.e., unless $\mathrm{Supp}(1) \cap \mathrm{Supp}(2) = \varnothing$ where $\mathrm{Supp}(i) = [\min\{y: P_i(y) > 0\}, \max\{y: P_i(y) > 0\}]$.

that satisfies (18). If the underlying rule F is improving then, using (10), (18) simplifies to

$$p_i(t+1) = p_i(t) + p_i(t) \sum_{j \in A} \sigma_{ij} \cdot p_j(t) \cdot (\pi_i - \pi_j). \tag{20}$$

Consequently, if F is improving with underlying $\sigma_{ij} > 0$ for all $i \neq j$, then in the long run all individuals in an infinite monomorphic population will choose actions achieving maximal expected payoff among those that were initially present, i.e., $\lim_{t \to \infty} p_i(t) = 0$ for $i \notin \arg\max_{j \in A} \{\pi_j, p_j(1) > 0\}$. It is easy to show that the converse of this statement is also true (use Lemma 1 and (18)). In particular, if all individuals in an infinite population use Imitate if Better then eventually they will all be choosing the inefficient action in the bandit from Section 7 if $\frac{1}{2} < \lambda < (\omega - x)/(\omega - \alpha)$ and $p_1(1) \in (0, 1)$.

If F is a dominant improving rule (e.g., the dominant Proportional Imitation Rule), then

$$p_i(t+1) = p_i(t) + \frac{1}{\omega - \alpha} [\pi_i - \bar{\pi}(p(t))] \cdot p_i(t), \tag{21}$$

where $\bar{\pi}(p) = \sum_i \pi_i \cdot p_i$ is the average payoff instate $p \in \Delta(A)$. Hence, if each individual uses the optimal rule then dynamic adjustment of a large but finite population is approximated in the short run by (21)—a discrete version of the replicator dynamic (Taylor [25]) applied to multi-armed bandits.

This leads to the following result about what typically happens in a large population of individuals using the optimal rule (compare to Remark 5). Loosely speaking, it is highly probable that most individuals will choose the best action after some finite time provided all actions are initially present.

Remark 6. Consider a finite population of individuals, using the dominant Proportional Imitation Rule, facing a given bandit in $\mathcal{G}(A, [\alpha, \omega])$. Let $M = \arg\max_{i \in A} \{\pi_i\}$. Then for any $\gamma \in (0, 1/|A|)$, $\delta > 0$ and $\varepsilon > 0$ there exits $T, N_0 \in \mathbb{N}$ such that the event $\{|1 - \sum_{i \in M} p_i^N(T)| > \delta\}$ occurs with probability less than ε given that $N > N_0$ and $p_i^N(1) > \gamma$ for all $i \in A$.

This statement follows directly from Theorem 3 and (21).

11. A GAME PLAYING SETTING

Above we derived optimal behavioral rules for stationary multi-armed bandits. In the following we extend our approach to the classic evolutionary game theoretic model of interacting individuals. Here, individuals are repeatedly randomly matched to play a one shot game. This means that individuals repeatedly face a non-stationary multi-armed bandit where changes in payoff distributions result from changes in play of matched opponents. In order to simplify presentation we restrict attention to two person games. More general results for games with any given finite number of players are easily derived.

Consider two finite, disjoint populations W_1 and W_2, each of size N, referred to as *population one* and *two*. In a sequence of rounds each individual must choose an action and is then matched with an individual from the opposite population. Let A_i be the finite set of actions available to an individual in population i, $i=1,2$. When an individual in population one using action $i \in A_1$ is matched with an individual in population two using action $j \in A_2$, the individual in population k receives an uncertain payoff drawn from the probability distribution P^k_{ij}, $k=1,2$. Payoffs are realized independently. Associating player i to being an individual in population i, the tuple $\langle A_1, A_2, (P^1_{ij})_{\substack{i\in A_1\\ j\in A_2}}, (P^2_{ij})_{\substack{i\in A_1\\ j\in A_2}} \rangle$ defines an *asymmetric two player normal form game.* We restrict attention to the class of asymmetric two player normal form games, $\mathcal{G}(A_1, A_2, [\alpha_1, \omega_1], [\alpha_2, \omega_2])$, in which P^k_{ij} has finite support in $[\alpha_k, \omega_k]$, $i \in A_1$, $j \in A_2$ and $k=1,2$; $\alpha_1 < \omega_1$ and $\alpha_2 < \omega_2$ are given. For a given asymmetric game, let $\pi_1(\cdot, \cdot)$ and $\pi_2(\cdot, \cdot)$ be the bilinear functions on $\Delta(A_1) \times \Delta(A_2)$ where $\pi_k(i, j)$ is the expected payoff to player k when player one is using action i and player two is using action j, i.e., $\pi_k(i, j) = \sum_{\{x \in [\alpha_k, \omega_k] : P_{ij}(x) > 0\}} x P^k_{ij}(x)$, $k=1,2$.

In each round, the *population state* (s_1, s_2) is an element of $(A_1)^{W_1} \times (A_2)^{W_2}$. Individuals are randomly matched in pairs, each individual being equally likely to be matched with each individual of the opposite population. Given the population state s, let $p(s) \in \Delta^N(A_1)$ be the vector of proportions of each action chosen in population one. Similarly let $q(s) \in \Delta^N(A_2)$ be the corresponding expression for population two. Then $\pi_1(i, q(s))$ specifies the expected payoff of an individual in population one using action $i \in A_1$ and $\pi_1(p(s), q(s))$ specifies the average expected payoff in population one in this state. For a given current state, each individual in population one is facing a multi-armed bandit $\langle A_1, (P'_i)_{i \in A} \rangle$ in $\mathcal{G}(A_1, [\alpha_1, \omega_1])$ where $P'_i(x) = \sum_{j \in A} q_j(s) \cdot P^1_{ij}(x)$ for $x \in [\alpha_1, \omega_1]$.

Sampling occurs within each population according to a sampling procedure as described in Section 3.

A *behavioral rule* F for an individual in population k $(k=1,2)$ is a function

$$F\colon A_k \times [\alpha_k, \omega_k] \times A_k \times [\alpha_k, \omega_k] \to \Delta(A_k).$$

Switching probabilities now depend on the population state. For a given behavioral rule F of an individual in population one, the induced switching probabilities $(F^i_{jr}(s))_{i,j,r\in A_1}$ in state $s=(s_1, s_2)\in (A_1)^{W_1}\times (A_2)^{W_2}$ are given by

$$F^i_{jr}(s)=\sum_u \frac{n_u(n_u-1)}{N(N-1)} F(j, \pi_1(j, u), r, \pi_1(r, u))_i$$

$$+\sum_{u\neq v} \frac{n_u n_v}{N(N-1)} F(j, \pi_1(j, u), r, \pi_1(r, v))_i,$$

where $n_k=|\{c\in W_2 : s_2(c)=k\}|$ for $k\in A_2$ $(i, j, r\in A_1)$.

11.1. *Optimal Behavior*

Which behavioral rule should an individual entering into population one use? Consider a boundedly rational individual. In any given state the game appears as a multi-armed bandit. However, in contrast to the multi-armed bandit setting, underlying payoff distributions are no longer stationary. The best choice of an action in the next round depends on how opponents' adjust. We assume that an individual does not anticipate how the play of her opponents changes. Instead, she evaluates performance in her first encounter according to the play of population two in her entry state. Thus, the individual acts as if she were going to face a stationary bandit. Here, as in the multi-armed bandit setting, the Proportional Imitation Rule with switching rate $1/(\omega_1-\alpha_1)$ is the optimal rule.

Consider now a social planer selecting individual behavior, prescribing the same behavior to individuals belonging to the same population. If each individual in population one is using the rule F and s is the population state in round t then the expected proportion of individuals choosing action $i\in A_1$ in round $t+1$, denoted by $E_F p'_i(s)$, is given by

$$E_F p'_i(s)=\frac{1}{N}\sum_{c,d\in W_1, c\neq d} \Pr(c\rightsquigarrow d)\cdot F^i_{s_1(c)s_1(d)}(s), \qquad i\in A_1. \tag{22}$$

We say that F is *expected to induce a weak compatible dynamic in population one* if for each round and state, average expected play in the next round is a better reply to the state of the previous round, i.e., if

$$\sum_{i\in A_1} \pi_1(i, q(s))\cdot E_F p'_i(s)-\pi_1(p(s), q(s))\geqslant 0 \tag{23}$$

holds for all states $s\in (A_1)^{W_1}\times (A_2)^{W_2}$.[18] (23) replaces the 'payoff increasing' condition in Section 5.2.

[18] Definition adapted from the concept of *weak compatibility* for infinite populations (Friedman [10]).

The social planer chooses a rule for individuals in population one that is expected to induce a weak compatible dynamic (in population one) in each asymmetric game in $\mathscr{G}(A_1, A_2, [\alpha_1, \omega_1], [\alpha_2, \omega_2])$. These are precisely the rules that are improving for bandits in $\mathscr{G}(A_1, [\alpha_1, \omega_1])$. Further selection as in the multi-armed bandit setting (maximize left hand side in (23) with minimal variance) reveals the Proportional Imitation Rule with switching rate $1/(\omega_1 - \alpha_1)$ as the unique optimal rule. Symmetric arguments apply to population two.

11.2. *Population Dynamics in Games*

Assume that each individual uses the optimal Proportional Imitation Rule for her population. How does the population state evolve under random and independent sampling? Using the same law of large numbers type of argument as in Theorem 3[19] behavior of a large but finite population is approximated in the short run by the deterministic dynamic $(p^t, q^t)_{t=1,2,3,\dots}$ that satisfies

$$\begin{aligned} p_i^{t+1} &= p_i^t + \frac{1}{\omega_1 - \alpha_1}\,[\pi_1(i, q^t) - \pi_1(p^t, q^t)] \cdot p_i^t, \qquad i \in A_1, \\ q_j^{t+1} &= q_j^t + \frac{1}{\omega_2 - \alpha_2}\,[\pi_2(p^t, j) - \pi_2(p^t, q^t)] \cdot q_j^t, \qquad j \in A_2, \quad t \in \mathbb{N}. \end{aligned} \tag{24}$$

Notice that (24), the two population analogue of (21), is a discrete version of the replicator dynamic defined by Taylor [25].

12. DISCUSSION

In this section we discuss some of our assumptions and relate our work to existing literature.

The central theme of our analysis is the selection of an individual's behavioral or learning rule, the description of what to do whenever a decision must be made. We search for behavioral rules that perform well in each situation. Our notion of performing well leads to the condition of *improving*, a rule performing better than any other improving rule in any situation is called *dominant*. A rule selected among the dominant rules is called *optimal*. For this discussion, let *optimal population adjustment* refer to an infinite population in which each individual uses the optimal rule.

[19] Small adjustments in the proof need to be made (see Schlag [22]) since switching probabilities are no longer (completely) independent due to the fact that individuals are matched in pairs.

An individual's decision is based on the information available about the specific situation. Naturally, different informational assumptions lead to the selection of different behavioral rules.

In our model, individual information is extremely limited—an individual only observes the performance of *one* other individual between rounds. The Proportional Imitation Rule is argued to be the unique optimal rule, optimal population adjustment follows the replicator dynamic. Our model is the first to reveal a derivation of the replicator dynamic from a model in which individual behavior is chosen optimally. Others have been able to construct adaptive rules that lead to the replicator dynamic (Björnerstedt and Weibull [4]; Cabrales [7]; Gale *et al.* [11], Helbing [12]), however they did not choose to analytically justify individual behavior. Axiomatizations of learning rules in slightly different contexts have also lead to the replicator dynamic (Easley and Rustichini [8]; Sarin [19], in combination with the paper by Börgers and Sarin [5]). However, their basic approach differs fundamentally from ours—the former models contain axioms concerning the functional form of a desirable learning rule whereas the selection of rules in our model is based entirely on individual information and induced performance.

The existence of dominant rules in our setting is quite surprising. In a recent investigation we expand our model and assume that an individual samples *two* individuals between rounds (Schlag [23]). Here dominant rules no longer exist. However, we find a simple, optimal, rule (a modification of the Proportional Imitation Rule) that is best at performing better than any improving rule based on a single sample. Optimal population adjustment is described by an aggregate monotone dynamic (Samuelson and Zhang [20]).

When individuals in our setting have perfect information, playing a best response would be the unique dominant rule. Optimal population adjustment becomes trivial in the multi-armed bandit setting; all individuals immediately adapt an action that achieves the highest expected payoff. In the two person game setting (Section 11) optimal adjustment follows a version of the best response dynamic (Matsui [15]). Comparing this result to ours, we see that the replicator dynamic and the best response dynamic compromise extreme points in the class of adjustment dynamics based on individually optimal myopic behavior.

An intermediate case regarding informational assumptions is a scenario where individuals observe expected payoffs of action used and action sampled. Although this assumption is difficult to motivate it is quite popular in the literature (e.g., see Björnerstedt and Weibull [4]; Hofbauer [13]).[20]

[20] Repeated (i.e., finitely many) pulls of the same arm between samples does not generate this situation since unlucky draws will distort information.

Here, Imitate if Better is the unique dominant rule and hence optimal. Optimal population adjustment in the multi-armed bandit setting leads to the state in which each individual chooses the best action among those initially present. We show that the dominant Proportional Imitation Rule has the same property under much less severe informational requirements.

Two alternative justifications for why individuals may choose to imitate under similar circumstances should be mentioned. Rogers [17] presents an example of a changing environment in which individuals imitate in order to evade search costs. The evolutionarily stable proportions of individual learning (i.e., individuals incur a cost and learn the currently best action) and social learning (i.e., individuals imitate without observing payoffs) are computed. Banerjee [1] presents a model in which rational individuals imitate for hope that the observed individual has more information.

Finally, we want to mention Malawski's [14] experiments in the game playing setting of Section 11. In this investigation an imitation hypothesis is refuted due to the high proportion of individuals switching to actions other than the one previously observed (over 30%). The data is partially explained with aspiration level learning, a model that entirely ignores information obtained through sampling. In the mean time, Malawski and Schlag have informally reviewed the data from this experiment and discovered that observations of the performance of others, in fact, differences between others and own performance, does influence switching behavior. An extensive reevaluation of the data from the experiment of Malawski and the conduction of new experiments has therefore been planned.

APPENDIX A: A COROLLARY ON IMPROVING RULES

COROLLARY 2. *Condition* (ii) *in Theorem* 1 *holds if and only if the following conditions holds*:

(ii′) *for all* $i, j \in A$, $i \neq j$, *either* $F(i, x, j, y)_j = F(j, y, i, x)_i$ *for all* $x, y \in [\alpha, \omega]$ *or there exists* $\sigma_{ij} = \sigma_{ji} > 0$ *and a function* $g_{ij}\colon [\alpha, \omega] \times [\alpha, \omega] \to \mathbb{R}$ *such that for* $x, y \in [\alpha, \omega]$,

$$-\min\{x, y\} \leqslant g_{ij}(x, y) \leqslant -\max\{x, y\} + \frac{1}{\sigma_{ij}},$$

$$F(i, x, j, y)_j = \sigma_{ij} \cdot (y + g_{ij}(x, y)) \quad \textit{and}$$

$$F(j, y, i, x)_i = \sigma_{ij} \cdot (x + g_{ij}(x, y)).$$

Proof. The fact that (ii′) implies (ii) follows directly. Conversely, let $i, j \in A$, $i \neq j$ and let F satisfy (ii). If $\sigma_{ij} = 0$ then (ii) implies $F(i, x, j, y)_j = F(j, y, i, x)_i$ for all $x, y \in [\alpha, \omega]$. Assume now that $\sigma_{ij} > 0$. Let $g_{ij}(\cdot, \cdot)$ be defined by $g_{ij}(x, y) = (1/\sigma_{ij}) F(i, x, j, y)_j - y$ $(x, y \in [\alpha, \omega])$. It follows that $-y \leqslant g_{ij}(x, y) \leqslant -y + (1/\sigma_{ij})$ and $F(i, x, j, y)_j = \sigma_{ij} \cdot (y + g_{ij}(x, y))$. Together with (ii) we obtain $F(j, y, i, x)_i = F(i, x, j, y)_j - \sigma_{ij}(y - x) = \sigma_{ij} \cdot (x + g_{ij}(x, y))$. This implies $-x \leqslant g_{ij}(x, y) \leqslant -x + (1/\sigma_{ij})$ which completes the proof of condition (ii′). ∎

REFERENCES

1. A. V. Banerjee, A simple model of herd behavior, *Quart. J. Econ.* **107** (1992), 797–818.
2. K. G. Binmore, L. Samuelson, and R. Vaughan, Musical chairs: Modeling noisy evolution, *Games Econ. Beh.* **11** (1995), 1–35.
3. J. Björnerstedt and K. H. Schlag, "On The Evolution of Imitative Behavior," Discussion Paper No. **B-378**, University of Bonn, 1996.
4. J. Björnerstedt and J. Weibull, Nash equilibrium and evolution by imitation, "The Rational Foundations of Economic Behaviour," Proc. IEA Conference, (K. Arrow *et al.*, Eds.), pp. 155–171, MacMillan, London, 1996.
5. T. Börgers and R. Sarin, "Learning Through Reinforcement and Replicator Dynamics," Discussion Paper No. **93-19**, University College of London, 1993.
6. R. T. Boylan, Laws of large numbers for dynamical systems with randomly matched individuals, *J. Econ. Theory* **57** (1992), 473–504.
7. A. Cabrales, "Stochastic Replicator Dynamics," mimeo, University of California, San Diego, 1993.
8. D. Easley and A. Rustichini, "Choice Without Beliefs," mimeo, Cornell University and C.O.R.E., 1995.
9. G. Ellison and D. Fudenberg, Word-of-mouth communication and social learning, *Quart. J. Econ.* **440** (1995), 93–125.
10. D. Friedman, Evolutionary games in economics, *Econometrica* **59** (1991), 637–666.
11. J. Gale, K. G. Binmore, and L. Samuelson, Learning to be imperfect: The ultimatum game, *Games Econ. Beh.* **8** (1995), 56–90.
12. D. Helbing, Interrelations between stochastic equations for systems with pair interactions, *Physica A* **181** (1992), 29–52.
13. J. Hofbauer, "Imitation Dynamics for Games," University of Vienna, mimeo, 1995.
14. M. Malawski, "Some Learning Processes in Population Games," Inaugural-Dissertation, University of Bonn, 1989.
15. A. Matsui, Best response dynamics and socially stable strategies, *J. Econ. Theory* **57** (1992), 343–362.
16. A. Robson, A biological basis for expected and non-expected utility, *J. Econ. Theory* **9** (1996), 397–424.
17. A. Rogers, Does biology constrain culture? *Amer. Anthropol.* **90** (1989), 819–831.
18. M. Rothschild, A two-armed bandit theory of market pricing, *J. Econ. Theory* **9** (1974), 185–202.
19. R. Sarin, "An Axiomatization of the Cross Learning Dynamic," mimeo, University of California, San Diego, 1993.
20. L. Samuelson and J. Zhang, Evolutionary stability in asymmetric games, *J. Econ. Theory* **57** (1992), 363–391.
21. L. J. Savage, "The Foundations of Statistics," Wiley, New York, 1954.
22. K. H. Schlag, "Why Imitate, and if so, How? Exploring a Model of Social Evolution," Discussion Paper **B-296**, University of Bonn, 1994.

23. K. H. Schlag, "Which One Should I Imitate?," Discussion Paper **B-365**, University of Bonn, 1996.
24. R. Schmalensee, Alternative models of bandit selection, *J. Econ. Theory* **10** (1975), 333–342.
25. P. Taylor, Evolutionarily stable strategies with two types of players, *J. Applied Prob.* **16** (1979), 76–83.
26. J. Weibull, "Evolutionary Game Theory," MIT Press, Cambridge, MA, 1995.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium

Appendix: Documentation

1 Institutes

Institut für Angewandte Mathematik, IAM
Institut für Internationale Wirtschaftspolitik, IIW
Institut für Gesellschafts- und Wirtschaftswissenschaften, IGW
Institut für Ökonometrie und Operations Research, IÖOR
Mathematisches Institut der Universität Köln, MIK

2 Projects

Projects	Title	Director	Period of Funding
Research Area A			
A1	Theoretische Analyse ökonomischer Systeme bei unvollständiger Information	Prof. M. Hellwig, Ph.D.	1985-1987
A1	Theoretische Analyse ökonomischer Systeme bei unvollständiger Information	Prof. Dr. U. Schweizer	1988-1990
A1	Transaktionskosten-Ökonomik	Prof. Dr. U. Schweizer	1991-1999
A2	Informations- und Anreizstrukturen in der öffentlichen Finanzwirtschaft	Prof. Dr. Dr. D. Bös	1985-1999
A3	Verteilung mikroökonomischer Charakteristika und die Struktur von Gleichgewichtsmodellen	Prof. Dr. W. Hildenbrand	1985-1999
Research Area B			
B1	Dynamische stochastische Systeme	Prof. Dr. P. Schönfeld	1985-1987
B1	Ökonometrische Analyse von Informationsstrukturen	Prof. Dr. P. Schönfeld	1988-1990
B1	Ökonometrische Analyse zeitvariabler und rückgekoppelter Systeme	Prof. Dr. P. Schönfeld	1991-1999
B2	Datenanalyse ökonomischer Strukturen	Prof. Dr. H. Wiesmeth	1985-1987
B3	Stochastik der Finanzmärkte	Prof. Dr. D. Sondermann	1985-1999
B4	Experimentelle Wirtschaftsforschung	Prof. Dr. R. Selten	1988-1999
B5	Makroökonomische Institutionen und Strukturen	Prof. Dr. M.J.M. Neumann	1988-1999
B6	Lokale Interaktionssysteme mit beschränkter Information und beschränkter Rationalität	Prof. A. Shaked, Ph.D.	1991-1993
B6	Lernregeln, Evolution und lokale Interaktionssysteme	Prof. A. Shaked, Ph.D.	1994-1999
Research Area C			
C1	Theoretische und prozedurale Konzepte in diskreten Strukturen	Prof. Dr. B. Korte	1985-1990
C2	Geometrie kombinatorischer Strukturen	Prof. Dr. A. Bachem	1985-1987
C2	Geometrie von Algorithmen	Prof. Dr. A. Bachem	1988-1990
C3	Systemsoftware und Implementierung kombinatorischer Algorithmen	Prof. Dr. B. Korte/ Prof. Dr. R.H. Möhring	1985-1987
C3	Systemsoftware und Implementierung kombinatorischer Algorithmen	Prof. Dr. R. Schrader/ Prof. Dr. B. Korte	1988-1990
C3	Service-Projekt: Betreuung der Hardware und Software für den Sonderforschungsbereich	Prof. Dr. B. Korte	1991-1996
C3	Service-Projekt: Betreuung der Hardware und Systemsoftware für den Sonderforschungsbereich, Rechnernetze, Parallelisierung	Prof. Dr. B. Korte	1997-1999
Research Area D	Hierarchische und laterale Koordination	Prof. Dr. H. Albach	1985-1987

Projects	Title	Director	Period of Funding
	wirtschaftlicher Aktivitäten im Unternehmen		
Z	Zentrales Verwaltungsprojekt des Sonderforschungsbereichs	Prof. Dr. W. Hildenbrand	1997-1999

3 Alphabetical List of Members and Co-workers[M)]

Name	Academic Degree	Institute	Project	Period of Funding
Abbink, Klaus	Dipl. Vw.	IGW	B4	4, 5
Ackermann, Michael*	Dr.	IGW	B1	4, 5
Albach, Horst	Prof. Dr.	IGW	D	1
Anhalt, Christoph [M)]	Dipl. Math.	IÖOR	C3	5
Arns, Jürgen	Dipl. Vw.	IGW	A3	5
Baatz, Erlfried* [M)]	Dipl. rer. oec.	IGW	D	1
Bachem, Achim	Prof. Dr.	MIK	C2	1, 2, 3
Balkenborg, Dieter*	Dr.	IGW	B6	3, 4, 5
Belcourt, Tracey	M.Sc.	IGW	A2	4, 5
Bester, Helmut**	Prof. Dr.	IGW	A1	1, 2
Bettzüge, Marc-Oliver*	Dr.	IGW	A3	4, 5
Bös, Dieter	Prof. Dr. Dr.	IGW	A2	1-5
Brackly, Günther*	Dr.	IGW	B1/2	1
Brettschneider, Julia	Dipl. Math.	IAM	B3	4
Breuer, Wolfgang	Prof. Dr.	IGW	A1	5
Broecker, Thorsten*	Dr.	IGW	A1	1, 2
Büchner, Heinz-J.*	Dipl. Vw.	IGW	A2	1
Buchta, Joachim	Dipl. Vw.	IGW	B4	3, 4, 5
Burg, Karl-Heinz*	Dipl. Math.	IGW	D	1
Byg, Torkild*	Dr.	IGW	A1	4
Christopeit, Norbert	Priv. Doz. Dr.	IÖOR	B1/2	1-5
Corneo, Giacomo**	Priv. Doz. Dr. Dr.	IGW	A2	4, 5
Cron, Axel* [M)]	Dipl. Vw.	IÖOR	B1	5
Damme, Eric van	Prof. Dr.	IGW	A1	1, 2
Derigs, Ulrich	Priv. Doz. Dr. Dr.	IÖOR	C1	1
Dudenhausen, Antje	Dipl. Vw.	IGW	B3	5
Ebell, Monique	Dipl. Vw.	IGW	A2	4, 5
Ebert, Udo**	Prof. Dr.	IGW	B3	1, 2
Eckwert, Bernhard	Prof. Dr.	IGW	A3	2
Emons, Winand*	Prof. Dr.	IGW	A1	1
Engel, Joachim	Dr.	IGW	A3	3, 4
Ewerhart, Christian*	Dr.	IGW	A1	5
Faigle, Ulrich**	Priv. Doz. Dr.	IÖOR	C1	1
Folkertsma, Carsten	Dr.	IGW	A3	3
Föllmer, Hans	Prof. Dr.	IAM	B3	3, 4
Franke, Bernd	Dipl. Math.	IGW	D	1
Frey, Rüdiger*	Dipl. Vw.	IGW	B3	4, 5
Funk, Peter* **	Dr.	IGW	A1	3, 4, 5
Gaube, Thomas	Dipl. Vw.	IGW	A2	5
Gerber, Antje	Dipl. Wirt. Math.	IGW	A3	5
Gocke, Clemens	Dipl. Kfm.	IGW	D	1
Gotterbarm, Franz*	Dr.	IÖOR	C1	1
Grüner, Hans-Peter**	Priv. Doz. Dr.	IGW	A1	5
Gyárfás, Garbor* [M)]	Dr.	IGW	A2	5
Hagedorn, Marcus	Dipl. Math.	IGW	A2	5
Hain, Roland	Dipl. Vw.	IGW	B6	5
Haller, Hans	Dr.	IGW	A3	1
Hammerstein, Klaus*	Dipl. Math.	IÖOR	B1/2	1
Hansen, Nico*	Dipl. Vw.	IGW	A2	4, 5
Härdle, Wolfgang**	Prof. Dr.	IGW	A1	1, 2
Heid, Frank*	Dr.	IGW	A3	4, 5
Held, Thomas*	Dr.	IGW	D	1

Name	Academic Degree	Institute	Project	Period of Funding
Hellwig, Martin	Prof. Ph.D.	IGW	A1	1
Hennig-Schmidt, Heike*	Dr.	IGW	A3/B4	1-5
Hens, Thorsten* **	Priv. Doz. Dr.	IGW	A3	3, 4, 5
Herreiner, Dorothea	M.Sc.	IGW	B6	5
Hetzel, Asmus* M)	Dr.	IÖOR	C3	5
Hildenbrand, Kurt	Dr.	IGW	A3	1-5
Hildenbrand, Werner	Prof. Dr.	IGW	A3	1-5
Hochstättler, Winfried*	Dr.	MIK	A3	2, 3, 4
Hoderlein, Stefan	Dipl. Vw.	IGW	A3	5
Hofmeister, Michael	Dr.	MIK	C2	2
Holland, Olaf*	Dr.	IÖOR	C3	1, 2
Holthausen, Cornelia	Dipl. Vw.	IGW	B4	5
Irlenbusch, Bernd	Dipl. Vw., Dipl. Inf.	IGW	B4	4, 5
Irngartinger, Markus	Dipl. Vw.	IGW	A1	5
Janeba, Eckhard*	Dipl. Vw.	IGW	A2	3, 4
Jang, Insong	Ph.D.	IGW	A3	5
John, Reinhard	Dr.	IGW	A3	1-5
Jost, Peter-Jürgen*	Dr.	IGW	A1	2
Kamecke, Ulrich* **	Dr.	IGW	A1	2, 3, 4
Karnbach, Bodo* M)	Dr.	IÖOR	C3	5
Kaul, Ashok	Dipl. Vw.	IGW	A2	5
Kern, Walter* **	Dr.	MIK	C2	1, 2
Keser, Claudia*	Dipl. Vw.	IGW	B4	2, 3
Kessler, Anke*	Dr.	IGW	A2	5
Keuschnigg, Christian	Dr.	IGW	A2	3
Kirchen, Alfons*	Dipl. Math.	IGW	B1/2	1
Kirchkamp, Oliver*	Dipl. Vw.	IGW	B6	4, 5
Klein, Martin**	Dr.	IGW	B5	2, 3
Kneip, Alois**	Priv. Doz. Dr.	IGW	A3	3, 4, 5
Koch, Karl-Josef*	Dr.	IGW	A3	1
Koehl, Jürgen	Dr.	IÖOR	C1	2
Kolmar, Martin	Dr.	IGW	A2	5
Konrad, Kai	Priv. Doz. Dr.	IGW	A3	4
Körner, Heinrich	Dipl. Vw.	IGW	A1	3
Korte, Bernhard	Prof. Dr.	IÖOR	C1	1-5
Korte, Norbert*	Dipl. Inf.	IÖOR	C3	1
Kottmann, Thomas*	Dr.	IÖOR	B1	2
Krelle, Wilhelm	Prof. Dr.	IGW	B1/2	1-5
Krieger, Rolf	Dipl. Phys.	IÖOR	C1	2
Lang, Günter*	Dipl. Vw.	IGW	A2	2, 3, 4
Leininger, Wolfgang**	Prof. Ph.D.	IGW	A1	1, 2
Li, Zhu-Yu	Ph.D.	IGW	A3	4, 5
Look, Stefan*	Dr.	IGW	B3	4, 5
Lotz, Christopher	Dipl. Math.	IGW	B3	5
Lülfesmann, Christoph*	Dr.	IGW	A2	3, 4, 5
Lux, Thomas	Prof. Dr.	IIW	B5	5
Marquardt, Marko*	Dr.	IGW	A2	5
Middendorf, Matthias*	Dr.	MIK	A3	3
Mitzkewitz, Michael	Dipl. Vw.	IGW	B4	2, 3, 4
Mohr, Michael*	Dipl. Math.	IÖOR	B1	2
Möhring, Rolf-H.	Prof. Dr.	IÖOR	C3	1
Moldovanu, Benedict* **	Priv. Doz. Dr.	IGW	B4	2, 3, 4
Müller, Sigrid**	Dr.	IGW	B3	1
Muuss, Karsten M)	Dipl. Inf.	IÖOR	C3	5
Nagel, Rosemarie*	Dr.	IGW	B4	3, 4
Nakamura, Shinichiro M)	Dr.	IGW	B3	1
Nett, Lorenz*	Dr.	IGW	A2	2, 3
Neumann, Manfred J.M.	Prof. Dr.	IGW	A1	1-5
Nicolin, Andreas M)	Dipl. Vw.	IGW	B3	3
Nöldeke, Georg*	Prof. Dr.	IGW	B6	3, 4, 5
Ostmann, Axel	Priv. Doz. Dr.	IGW	B4	2

Name	Academic Degree	Institute	Project	Period of Funding
Peters, Wolfgang* **	Priv. Doz. Dr.	IGW	A2	1, 2, 3, 4
Petersen, Thomas*	Dipl. Math.	IGW	D	1
Pollock, Gregor	Ph.D.	IGW	B6	5
Probst, Daniel*	Lec. rer. pol.	IGW	B6	4, 5
Prömel, Hans-Jürgen**	Prof. Dr.	IÖOR	C3	1, 2, 3, 4
Puppe, Clemens	Prof. Dr.	IGW	B6	5
Reimer, Matthias*	Dipl. Vw.	IGW	B3	3, 4, 5
Rockenbach, Bettina* **	Dr.	IGW	B4	4, 5
Rosenbaum, Anja		IGW	B4	3
Rosenkranz, Stephanie	Dr.	IGW	A1	5
Rütsch, Ursula*	Dipl. Vw.	MIK	A3	3
Rüttermann, Marcus	Dipl. Math.	IAM	B3	3, 4
Sadrieh, Abdolkarim*	Dr.	IGW	B4	3, 4, 5
Salchow, Hans-Jürgen	Dipl. Vw.	IGW	A3	3, 4
Sandmann, Klaus* **	Priv. Doz. Dr.	IGW	B3	1-5
Sarrazin, Hermann	Dipl. Vw.	IGW	B1/2	1
Scheer, Jens-Uwe	Dipl. Math.	IGW	A3	5
Schietke, Jürgen M)	Dipl. Inf.	IÖOR	C3	5
Schils, Rüdiger*	Dipl. Vw.	IGW	A1	5
Schlag, Karl* **	Ph.D.	IGW	B6	3, 4, 5
Schlögl, Eric*	Dr.	IGW	B3	3, 4, 5
Schmachtenberg, Rolf	Dr.	IGW	A3	2
Schmidt, Klaus* **	Priv. Doz. Dr.	IGW	A1	3, 4
Schmitz, Heinz-P.	Dr.	IGW	A3	1, 2
Schmitz, Patrick*	Dipl. Vw.	IGW	A1	5
Schnitzer, Monika* **	Priv. Doz. Dr.	IGW	A1	3, 4, 5
Schönbucher, Philipp*	Dipl. Vw.	IGW	B3	5
Schönfeld, Peter	Prof. Dr.	IÖOR	B1/2	1-5
Schrader, Rainer**	Prof. Dr.	IÖOR	C3	1,2
Schuhmacher, Achim	Dipl. Vw.	IGW	A1	4, 5
Schuhmacher, Frank*	Dr.	IGW	A1/B6	5
Schürger, Klaus	Prof. Dr.	IGW	B3	1-5
Schweizer, Martin**	Prof. Dr.	IGW	B3	2, 3
Schweizer, Urs	Prof. Dr.	IGW	A1	1-5
Schwietzke, Edgar	Dipl. Inf.	IÖOR	C1	2
Selten, Reinhard	Prof. Dr.	IGW	A1	1-5
Shaked, Avner	Prof. Ph.D.	IGW	A1	2, 3, 4, 5
Sippel, Reinhard	Dipl. Vw.	IGW	A3	4, 5
Siwik, Thomas M)	Dipl. Vw.	IGW	A1	5
Sliwka, Dirk	Dipl. Vw.	IGW	A1	5
Sondermann, Dieter	Prof. Dr.	IGW	B3	1-5
Tang, Fang-Fang*	Dr.	IGW	B4	4, 5
Tillmann, Georg**	Dr.	IGW	A2	1, 2
Triesch, Eberhard M)	Prof. Dr.	IÖOR	C3	5
Trockel, Walter	Prof. Dr.	IGW	A3	1
Utikal, Klaus	Ph.D.	IGW	A3	4, 5
v. Hagen, Jürgen*	Prof. Dr.	ZEI	B5	5
v. Weizsäcker, Robert K.* **	Prof. Dr.	IGW	A2	1, 2
Voigt, Bernd	Priv. Doz. Dr.	IÖOR	C1	2
Vygen, Jens M)	Dipl. Math.	IÖOR	C3	5
Wanka, Alfred*	Dr.	MIK	C2	1, 2
Weber, Axel	Prof. Dr.	IGW	B5	4, 5
Weidmann, Jens* M)	Dipl. Vw.	IIW	B5	5
Weihs, Claus*	Dipl. Math.	IGW	B1/2	1
Weimer, Theodor*	Wirt.-Ref.	IGW	D	1
Weissenberger, Edgar M)	Ph.D.	IÖOR	B1/2	1
Werner, Hans-J.	Priv. Doz. Dr.	IÖOR	B1	1-5
Werner, Jan*	Dr.	IGW	B3	1
Wesche, Katrin*	Dr.	IIW	B5	5
Weskamp, Anita*	M.A.	IGW	A1	2
Wessels, Joachim*	Dipl. Vw.	IGW	A1	4, 5

Name	Academic Degree	Institute	Project	Period of Funding
Wiesmeth, Hans	Prof. Dr.	IGW	B2	1
Wiesmeth, Hans	Prof. Dr.	IGW	B3	2, 3
Winter, Eyal	Ph.D.	IGW	A1	2
Yanelle, Marie Odile*	M.Sc.	IGW	A1	1
Zachau, Ulrich	Dr.	IGW	A1	2
Zenner, Markus* M)	Dipl. Math.	IÖOR	B1	5
Zimmermann, Gabriele	Dipl. Vw.	IGW	B3	5
Zimmermann, Hans*	Dipl. Math.	IGW	A2	1
Zühlsdorff, Christian	Dipl. Math.	IGW	B3	5

4 Promotion of Young Scientists

4.1 List of Dissertations and Habilitations

4.1.1 Dissertations

Name	Title	Institution	Project	Year/ Period of Fund-ing
Ackermann, Michael	Die optimale Angleichung der neuen Bundesländer an die Lebensverhältnisse in Westdeutschland	Uni Bonn	B1	1998
Artale, Angelo	Rings in Auctions: An Experimental Approach	Uni Bonn	B4	1996
Baatz, Erlfried	Unternehmensstrategien auf stagnierenden Märkten. Eine Untersuchung in der chemischen Industrie	Uni Bonn	D	1985
Badke, Michael	Eine Theorie des technischen Fortschritts	Uni Bonn	B1	1990
Balkenborg, Dieter	The Properties of Persistent Retracts and Related Concepts	Uni Bonn	B6	1992
Bettzüge, Marc Oliver	Financial Innovation from a General Perspective	Uni Bonn	A3	1997
Bock, K.	Unternehmenserfolg und Organisation. Eine empirische Untersuchung	Uni Bonn	D	I
Brackley, Günter	Zur Geometrie der zweidimensionalen nicht-metrischen Entfaltung	Uni Bonn	B1/2	1985
Brennscheid, Gunnar	Predictive Behavior – An Experimental Study	Uni Bonn	B1	1993
Broecker, Thorsten	On the Role of Active Monitoring in Markets with Asymmetric Information	Uni Bonn	A1	1987
Büchner, Heinz-Jürgen	Ökonomische Theorien zur Steuerhinterziehung	Uni Bonn	A2	1988
Burg, Karl-Heinz	Das Bonner Modell der Firmenentwicklung: Ein ökonometrisches Modell für die deutschen Industrieaktiengesellschaften	Uni Bonn	D	1987
Byg, Torhild	Essays on the Economic Analysis of Provision	Uni Bonn	A1	1996
Claßen, Karl	Determination der Investitionstätigkeit deutscher Unternehmen	Uni Bonn	D	1986
Corte, Christiane	Die Übernahme kommunaler Aufgaben durch private Unternehmen und freie Berufe	Uni Bonn	D	1990
Cron, Axel	Robust Nonparametric Estimation and Prediction in Arch-Related Modells	Uni Bonn	B1	1997
Dahremöller, Axel	Erstellung regionaler und betriebsgrößenbezogener Arbeitsmarktbilanzen – Generierung der Daten und Analyse der unternehmens- und arbeitsplatzbezogenen Fluktuationsvorgänge unter Aufwendung der komponenten-spezifisch erweiterten Shift-Analyse	Uni Bonn	D	1988
Degenhart, Heinrich	Zweck und Zweckmäßigkeit bankaufsichtlicher Eigenkapitalnormen	Uni Bonn	D	1986
Emons, Winand	On the Economic Theory of Warranties	Uni Bonn	A1	1986
English, Thomas	Ein Investitionsplanungsmodell für die materielle	Uni Bonn	D	1988

Name	Title	Institution	Project	Year/ Period of Fund-ing
	Streitkräfteplanung dargestellt am Beispiel der Deutschen Bundeswehr			
Ewerhart, Christian	On Strategic Reasoning and Theories of Rational Behavior	Uni Bonn	B6	1997
Fang-Fang, Tang	Anticipatory Learning in Two-Person Games: An Experimental Study	Uni Bonn	B4	1996
Fischer, Klaus	Hausbankbeziehungen als Instrument der Bindung zwischen Banken und Unternehmen – eine theoretische und empirische Analyse	Uni Bonn	A1	1990
Frey, Rüdiger	Asset Price Volatility and Option Hedging in Imperfect Elastic Markets	Uni Bonn	A1	1996
Friede-Mohr, Christina	Die Entwicklung der Personalkosten	Uni Bonn	D	1987
Funk, Peter	Bertrand and Walras Equilibrium in Large Economies	Uni Bonn	A1	1990
Goecke, Oskar	Eliminationsprozesse in der kombinatorischen Optimierung - ein Beitrag zur Greedoidtheorie	Uni Bonn	C1	1986
Gotterbarm, Franz	Modelle und Optimierungsansätze zur Analyse des kollektiven Bausparens	Uni Bonn	C1	1985
Gürtler, Marc	Die Lebesquesche Optionspreistheorie: Ein Modell zur Bewertung von pfadabhängigen Derivaten	Uni Bonn	A1	1997
Gyárfás, Gábor	Ein Simulationsmodell der Einkommensbesteuerung auf der Grundlage synthetischer Mikrodaten	Uni Bonn	A2	1990
Hammerstein, Klaus	Die Kruskal'sche Stress-Funktion: Theoretische Grundlagen und Verallgemeinerungen für temporale Sequenzen von Querschnittsdaten	Uni Bonn	B1/2	1986
Hansen, Nico	On the Political Economy of Federal Systems	Uni Bonn	A2	1997
Heid, Frank	Nonparametric Modelling of Financial Time Series	Uni Bonn	A3	1998
Held, Thomas	Kurzfristige Preispolitik bei kapitalintensiver Produktion und unterausgelasteten Kapazitäten	Uni Bonn	D	1985
Hennig-Schmidt, Heike	Bargaining in a Videoexperiment – Determinants of Bounded Rational Behavior	Uni Bonn	B4	1998
Hens, Thorsten	Structure of General Equilibrium Models with Incomplete Markets	Uni Bonn	A3	1992
Hetzel, Asmus	Verdrahtung im VLSI-Design: Spezielle Teilprobleme und ein sequentielles Lösungsverfahren	Uni Bonn	C3	IV
Hochstättler, Winfried	Seitenflächenverbände orientierter Matroide	Uni Köln	A3	III
Holland, Olaf	Schnittebenenverfahren für Travelling-Salesman- und verwandte Probleme	Uni Bonn	C1	1987
Holtmann, Michael	Personelle Verflechtungen auf der Leitungsebene im Konzern	Uni Bonn	D	1988
Hunsdiek, Detlef	Unternehmensgründung als Folgeinnovation – Struktur, Hemmnisse und Erfolgsbedingungen der Gründung industrieller und innovativer Unternehmen	Uni Bonn	D	1987
Janeba, Eckhard	International Tax Competition	Uni Bonn	A2	1994
Jost, Peter-Jürgen	On Control in Principal-Agent Relationships	Uni Bonn	A1	1988
Kamecke, Ulrich	Modelling Competitive Matching Markets with Non-Cooperative Game Theory	Uni Bonn	A1	1988
Karnbach, Bodo	Graphisch-interaktive und spezifikationssprachliche Verfahren zur Projektierung von Schaltanlagenbau	Uni Bonn	C3	IV
Kern, Walter	Verbandstheoretische Dualität in kombinatorischen	Uni Bonn	C2	I

Name	Title	Institution	Project	Year/ Period of Funding
	Geometrien und Orientierten Matroiden			
Keser, Claudia	Experimental Duopoly Markets with Demand Inertia: Game-Playing Experiments and the Strategic Method	Uni Bonn	B4	1992
Kessler, Anke	On the Value of Information in Trade Relationships	Uni Bonn	A2	1996
Kirchen, Alfons	Schätzung zeitveränderlicher Strukturparameter in ökonometrischen Prognosemodellen	Uni Bonn	B1	1987
Kirchkamp, Oliver	Evolution and Learning in Spatial Models	Uni Bonn	B6	1996
Koch, Karl-Josef	Consumer Demand and Aggregation	Uni Bonn	A3	1987
Korte, Norbert	Intervallgraphen und Seriationsprobleme	Uni Bonn	C3	I
Kottmann, Thomas	Learning Procedures and Rational Expectations in Linear Models with Forecast Feedback	Uni Bonn	B1	1990
Kuon, Bettina	Two-Person Bargaining Experiments with Incomplete Information	Uni Bonn	B4	1993
Laitenberger, Jörg	Demand Theory in Models of Financial Markets	Uni Bonn	A3	1999
Laitenberger, Marta	Market Imperfection in General Equilibrium Models with Uncertainty	Uni Bonn	A3	1997
Lang, Günther	On Overlapping Generations Models with Productive Capital	Uni Bonn	A2	1995
Leclerc, Mathias	Algorithmen für kombinatorische Optimierungsprobleme mit Paritionsbeschränkungen	Uni Köln	C2	II
Leisen, Dietmar	On Efficient Binomial Option Price Approximations	Uni Bonn	B3	1998
Look, Stefan	Die stochastische Methode der finiten Elemente und Anwendungen bei der Bewertung von Finanzderivaten	Uni Bonn	B3	1999
Lülfesmann, Christoph	Incomplete Contracts and Renegotiation in Long-Term Trade Relationships	Uni Bonn	A2	1996
Malawski, Marcin	Some Learning Processes in Population Games	Uni Bonn	B4	1988
Marquardt, Marko	Theoretische Analyse der Rentenversicherung	Uni Bonn	A2	1999
Middendorf, Mathias	Repräsentierungen von Graphen	Uni Köln	C1	II
Mohr, Michael	Asymptotic Theory of Ordinary Least Squares Estimators in Regression Models with Forecast Feedback	Uni Bonn	B1	1990
Moldovanu, Benedict	Competition and Bargaining in Games and Markets	Uni Bonn	B4	1991
Müller-Brockhausen, Gerd	Probleme von Hypothesentests in kleinen Stichproben und bei Fehlspezifikationen	Uni Bonn	B1/2	1987
Nagel, Rosemarie	Reasoning and Learning in Guessing Games and Ultimatum Games with Incomplete Information – An Experimental Investigation	Uni Bonn	B4	1994
Nett, Lorenz	Public Regulation in an Oligopolistic Market	Uni Bonn	A2	1992
Nöldeke, Georg	Signalling in Markets and Games with Incomplete Information	Uni Bonn	B6	1992
Peitz, Martin	Demand Aggregation and the Theory of Product Differentiation	Uni Bonn	A3	1995
Peters, Wolfgang	Theorie der Renten- und Invaliditätsversicherung	Uni Bonn	A2	1988
Petersen, Thomas	Optimale Anreizsysteme	Uni Bonn	D	1988
Pfeiffer, Frank	Zur Komplexität des disjunkte-Wege-Problems	Uni Bonn	C1	II
Plehn, Jürgen	Über die Existenz und das Finden von Subgraphen	Uni Bonn	C3	III
Probst, Daniel	On Evolution and Learning in Games	Uni Bonn	B6	1996
Rabe, Uwe	Dynamisch-Optimale Gebührenpolitik für einen	Uni Bonn	A2	1988

Name	Title	Institution	Project	Year/ Period of Funding
	neuen Telekommunikationsdienst			
Rafi, Aare	Statistische Analyse ökonometrischer Ungleichgewichtsmodelle	Uni Bonn	B1	1988
Reimer, Matthias	Examining Binomial Option Price Approximations	Uni Bonn	B3	1997
Rey, Michael	Die Geldnachfrage in der Bundesrepublik Deutschland	Uni Bonn	A1	1994
Rieder, Jörg	Gitterstrukturen bei Matroidproblemen	Uni Köln	A3	III
Rütsch, Ursula	Charakterisierung ganzzahliger Gitter über kombinatorischen Strukturen	Uni Köln	A3	III
Ryll, Wolfgang	Experimentation of Litigation and Settlement in a Game with Asymmetric Information	Uni Bonn	B4	1995
Sadrieh, Abdolkarim	The Alternating Double Auction Market – a Game and Experimental Investigation	Uni Bonn	B4	1997
Sandmann, Klaus	Arbitrage und die Bewertung von Zinssatzoptionen	Uni Bonn	B3	1990
Schils, Rüdiger	Entrepreneurship and Economic Activities	Uni Bonn	A1	1998
Schlögl, Erik	Interest Rate Factor Models: Term Structure Dynamics and Derivatives Pricing	Uni Bonn	B3	1997
Schmidt, Klaus M.	Commitment in Games with Asymmetric Information	Uni Bonn	A1	1991
Schmidt, Roland	Risikoprämien an den Devisenmärkten am Beispiel des US-Dollars	Uni Bonn	B5	1994
Schmidt, Wolfgang	Strukturelle Aspekte in der kombinatorischen Optimierung: -Greedoide auf Graphen	Uni Bonn	C1	1986
Schmitz, Heinz-Peter	Die zeitliche Invarianz von Einkommensverteilungen. Eine Analyse der Einkommensverteilungen in Großbritannien	Uni Bonn	A3	1988
Schmitz, Patrick	Investment Incentives under Asymmetric Information and Incomplete Contracts	Uni Bonn	A1	1999
Schnitzer, Monika	Takeovers and Tacit Collusions	Uni Bonn	A1	1991
Schönbucher, Philipp	Credit Risk Modelling and Credit Derivatives	Uni Bonn	B3	2000
Schuhmacher, Frank	The Bayesian Foundations of Interactive Decision Making	Uni Bonn	B6	1996
Schuhmacher, Joachim	Debt Structure and Financial Intermediation	Uni Bonn	A1	1999
Schultes, Dieter	Bestimmungsfaktoren der Geldpolitik der Deutschen Bundesbank	Uni Bonn	A1	1993
Schwärzler, Werner	Signierte Polynome und Mehrgüterflüsse auf Matroiden	Uni Bonn	C3	III
Sliwka, Dirk	On Incentives and the Decentralization of Decisions in Organizations	Uni Bonn	A1	1999
Sommer, Daniel	Valuation of Contingent Claims with Interest and Exchange Rate Risk and the Exogenous Issuing of New Bonds	Uni Bonn	B3	1996
Steger, Angelika	Die Kleitman-Rothschild-Methode	Uni Bonn	C1	II
Stubben, F.	Informationskostenrechnung	Uni Bonn	D	I
Tötsch, Inge	The Influence of the Unemployment Duration on the Reemployment Probabilities	Uni Bonn	A1	1992
Tröger, Thomas	Bounded Rationality and Contracts	Uni Bonn	B6	1999
Uhlich, Gerald Roger	Descriptive Theories of Bargaining	Uni Bonn	B4	1988
Viefers, Ulrich	Forschungs- und Entwicklungsaktivitäten und Unternehmensgröße	Uni Bonn	D	1986
von Hagen, Jürgen	Strategien kurzfristiger Geldmengensteuerung	Uni Bonn	A1	1985

Name	Title	Institution	Project	Year/ Period of Funding
von Thadden, Ernst-Ludwig	Financial Intermediation, Control, and the Investment Horizon	Uni Bonn	A1	1991
von Weizsäcker, Robert K.	Zur Theorie der Verteilung von Arbeitseinkommen	Uni Bonn	A2	1985
Wanka, Alfred	Matroideerweiterungen zur Existenz endlicher LP-Algorithmen, von Hahn-Banach-Sätzen und Polarität in orientierten Matroiden	Uni Bonn	C2	I
Waragai, Tomoki	Unternehmensverhalten im Strukturwandel – eine Methode zur Strukturbruchanalyse und eine Untersuchung des Investitionsverhaltens von Unternehmen in Japan und der Bundesrepublik Deutschland	Uni Bonn	D	1990
Wedekind, E.E.	Interaktive Bestimmung von Aufbau- und Ablauforganisation als Instrument des Informationsmanagements	Uni Bonn	D	I
Weidmann, Jens	Geldpolitik und europäische Währungsintegration: empirische Aspekte der Zinsbestimmung	Uni Bonn	B5	1997
Weihs, Claus	Auswirkungen von Fehlern in den Daten auf Parameterschätzungen und Prognosen	Uni Bonn	B1/2	I
Weimer, Theodor	Das Substitutionsgesetz der Organisation. Eine theoretische Fundierung.	Uni Bonn	D	1987
Weingärtner, Tom	Einkommensverteilung in Frankreich. Eine Evaluierung mittels eines Maßes für Information auf dem Arbeitsmarkt und seine Wirkung auf Angebot und Nachfrage	Uni Bonn	A2	1988
Werner, Jan	Arbitrage and Equilibrium in Economies with Incomplete Markets	Uni Bonn	B3	1985
Wesche, Katrin	Die Geldnachfrage in Europa: Aggregationsprobleme und Empirie	Uni Bonn	B5	1997
Weskamp, Anita	Die Auswirkungen der Sicherungsrechte auf die effiziente Ausgestaltung von Kreditbeziehungen	Uni Bonn	A1	1988
Wessels, Joachim H.	Asymmetric Information and the Design of Optimal Contracts	Uni Bonn	A1	1996
Will, Heide	On the Interdependencies among Information, Organization, and Incentives	Uni Bonn	A1	1998
Wurzel, Eckhard	An Econometric Analysis of Individual Unemployment Duration	Uni Bonn	B5	1992
Yanelle, Marie-Odile	On the Theory of Intermediation	Uni Bonn	A1	1988
Zenner, Markus	Learning to Become Rational in Self-Referential Autoregressive and Non-Stationary Models	Uni Bonn	B1	1996
Zimmermann, Hans-Georg	Privates Sparen versus Sozialversicherung	Uni Bonn	A2	1987

4.1.2 Habilitations

Name	Title	Institution	Project	Year
Bester, Helmut	Non-Cooperative Bargaining and Imperfect Competition	Uni Bonn	A1	1987
Corneo, Giacomo	Economics of Social Status	Uni Bonn	A2	1997
Ebert, Udo	Beiträge zur Wohlfahrtsökonomie - Effizienz und Verteilung	Uni Bonn	B3	1986

Name	Title	Institution	Project	Year
Faigle, Ulrich	Submodular Combinatorial Structures	Uni Bonn	C1	1985
Funk, Peter	The Direction of Technological Change	Uni Bonn	A1	1996
Grüner, Hans-Peter	The Economics of Distributive Politics	Uni Bonn	A1	1999
Härdle, Wolfgang	Applied Nonparametric Regression	Uni Bonn	A3	1988
Hens, Thorsten	General Equilibrium Foundations of Finance	Uni Bonn	A3	1996
Kamecke, Ulrich	Competitive Bidding in English Multi-Object Auctions	Uni Bonn	A1	1995
Kern, Walter	Verfahren der kombinatorischen Optimierung und ihre Gültigkeitsbereiche	Uni Köln	C2	1989
Klein, Martin	Bewertung von Länderrisiken	Uni Bonn	B5	1993
Kneip, Alois	Heterogeneity of Demand Behavior and the Space of Engel Curves	Uni Bonn	A3	1994
Kuon, Bettina	Experimental and Theoretical Contributions to Decision Making in Financial Asset Markets	Uni Bonn	B4	1999
Leininger, Wolfgang	Dynamic Competition and Strategic Behavior	Uni Bonn	A1	1988
Moldovanu, Benedict	Strategic Markets with Externalities	Uni Bonn	B4	1995
Müller, Sigrid	Common Value Auctions: Theory and Applications	Uni Bonn	B3	1992
Peters, Wolfgang	Intergenerative Umverteilung	Uni Bonn	A2	1995
Prömel, Hans Jürgen	Ramsey Theory for Discrete Structures	Uni Bonn	C1	1987
Sandmann, Klaus	Derivative Asset Analysis under Stochastic Interest Rates	Uni Bonn	B3	1996
Schlag, Karl	Justifying Imitation	Uni Bonn	B6	1998
Schmidt, Klaus	Contracts, Competition and the Theory of Reputation	Uni Bonn	A1	1995
Schnitzer, Monika	Solutions to the Sovereign Debt Problem: Countertrade and Foreign Direct Investment	Uni Bonn	A1	1995
Schrader, Rainer	Structured Theory of Discrete Greedy Procedures	Uni Bonn	C1	1987
Schweizer, Martin	Approximating Random Variables by Stochastic Integrals and Applications in Financial Mathematics	Uni Göttingen	B3	1993
Siebe, Wilfried	General Equilibrium with a Continuum of Oligopolies	Uni Bonn	B4	1991
Tillmann, Georg	Equity, Incentives, and Taxation	Uni Bonn	A2	1987
v. Weizsäcker, Robert K.	Bevölkerungsentwicklung, Rentenfinanzierung und Einkommensverteilung	Uni Bonn	A2	1990

4.2 Graduiertenkolleg and Other Activities to Promote Young Researchers within the Sonderforschungsbereich 303

- European Doctoral Program in Quantitative Economics (since 1977)
- Graduiertenkolleg “Interaktive ökonomische Entscheidungen” (since 1991)
- Bonn Graduate School of Economics (since October 1998)

5 Alphabetical List of Guest Researchers
(visit of 2 weeks and more)

Name	Academic Degree	Home institution	Year of Stay
Agastya, Murali	Prof.	University College, London, GB	1995
Ahsan, Syed	Prof.	Concordia University Montreal, Quebec, Kanada	1995

Name	Academic Degree	Home institution	Year of Stay
Aizenman, Josua	Prof.	Dartmouth College, Hanover, N.H., USA	1992
Albin, Peter	Prof.	City University of New York, USA	1991
Anderlini, Luca	Prof.	University of Cambridge, Cambridge, England, GB	1992
Andjiga, Nicolas	Dr.	Université de Yaoundé, Kamerun	1989
Apesteguia, Jose J.		Department of Economics, State University of Navarre, Spanien	1999
Appelbaum, Elie	Prof.	York University, Toronto, Kanada	1994
Applegate, David	Prof.	Carnegie Mellon University, Pittsburgh, USA	1994
		AT&T Bell Laboratories, Murray Hill, USA	1995
Apps, Patricia	Prof.	University of Syndey, Australien	1985
Araujo, Aloisio	Prof.	Instituto de Mathematica, Rio de Janeiro, Brasilien	1989
		IMPA, Rio de Janeiro, Brasilien	1994
Babadjanian, Arkady	Prof. Dr.	Armenische Akademie der Wissenschaften, Erivan, UdSSR	1986
Bergemann, Dirk	Prof.	Princeton University, Princeton, USA	1995
Berninghaus, Siegfried	Dr.	Universität Konstanz	1985, 1986
Bewley, Truman	Prof.	Yale University, New Haven, USA	1988, 1997, 1998
Binmore, Ken	Prof.	University of Michigan, Ann Arbor, USA	1990, 1992, 1993
		University College, London, GB	1995
Bock, K.	Dipl.-Kfm.	Institut für Mittelstandsforschung	1985
Bolton, Gary	Prof.	Pennsylvania State University, University Park, USA	1995
Börgers, Tilman	Prof.	University College London, England, GB	1992
Bottazzi, Jean-Marc	Prof.	Université de Paris I, Frankreich	1994
Breton, Michel le	Prof.	Université d'Aix-Marseille, Frankreich	1991, 1992
Brousseau, Vincent	Dr.	Université de Paris I, Frankreich	1991, 1992
Brown, Donald	Prof.	Stanford University, Stanford, USA	1988, 1989
Brueckner, Jan	Prof.	University of Illinois, Cahampaign, USA	1996
Cabrales, Antonio	Prof.	Universität Pompeu Fabra, Barcelona, Spanien	1997
Canning, David	Prof.	Pembroke College, Cambridge, England, GB	1991
Cao, Ricardo	Dipl.-Math.	Universität Santiago de Compostela, Spanien	1989
Carrol, Ray	Prof.	Texas A&M University, College Station, USA	1989
Cass, David	Prof.	University of Pennsylvania, Philadelphia, USA	1988
Chatterjee, Kalyan	Prof.	Pennsylvania State University, Philadelphia, USA	1992
Chen, Yan	Prof.	University of Michigan, Ann Arbor, USA	1996, 1997
Cho, In-Koo	Prof.	University of Chicago, USA	1990, 1993
Clemenz, Gerhard	Dr.	Universität Wien, Österreich	1985
Cook, William J.	Prof.	Rice University, Houston, Texas, USA	1996, 1997, 1998, 1999
Cornelli, Francesca	Prof.	London School of Economics, GB	1993
Courchene, Tom	Prof.	Queens University, Kingston, Ontario, Kanada	1998
Cox, Dennis D.	Prof.	University of Illinois at Urbana, Champaign, USA	1991
Cripps, Martin	Prof.	University of Warwick, GB	1993
Cunningham, William	Prof.	Carleton University, Ottawa, Kanada	1993
		University of Waterloo, Ontario, Kanada	1994
Debreu, Gérard	Prof.	University of California, Berkeley, USA	1989, 1994, 1995
Dechert, William	Prof.	University of Houston, Texas, USA	1989
Dehez, Pierre	Prof.	European University Institute, Florenz, Italien	1990, 1991
Deneckere, Ray	Prof.	Northwestern University, Evanston, USA	1985
Dierker, Egbert	Prof.	Universität Wien, Österreich	1988), 1989

Name	Academic Degree	Home institution	Year of Stay
Dobrinski, R.	Dr.	Bulgarian Academy of Sciences, Sofia, Bulgarien	1985, 1986
Dow, James	Prof.	London Business School, England, GB	1990
Duecker, Michael J.	Ph.D.	Federal Reserve Bank of St. Louis, USA	1998
Duffie, Darrell	Prof.	Stanford University, Stanford, USA	1988
Dutta, Jayasri	Prof.	Columbia University, New York, USA	1989
Eismont, Oleg	Dr.	Akademie der Wissenschaften der Sowjetunion, Institue for Systems Studies (VNIISI), Moskau, UdSSR	1985
Eshel, Ilan	Prof.	Tel Aviv University, Tel Aviv, Israel	1992, 1993, 1995, 1996, 1997, 1998, 1999
Evstigneev, Igor	Prof.	Central Economics and Mathematics Institute (CEMI), Academy of Sciences, Moskau, Rußland	1991, 1992, 1993, 1994, 1996, 1997, 1998, 1999
Farmer, Roger	Prof.	University of California, Los Angeles, USA	1996
Felli, Leonardo	Prof.	London School of Economics, GB	1993
Fersthmann, Chaim	Prof.	Tel Aviv University, Tel Aviv, Israel	1992
Fischer, Alastair	Prof.	Cambridge University, GB	1995
Fonlupt, Jean	Prof.	Université Paris VI, Frankreich	1994, 1995
		Université Pierre et Marie Curie, Paris, Frankreich	1996
Fraja, Giovanni De	Prof.	York University, GB	1997
Frank, András	Prof.	Eötvös Loránd University, Budapest, Ungarn	1999
Friedman, Daniel	Prof.	University of California, Santa Cruz, USA	1994
Fujishige, Satoru	Prof.	Tsukuba University, Japan	1994
Gajda, Jan	Dr.	University of Lodz, Polen	1985
Gardner, Roy	Prof.	Indiana University, Bloomington, USA	1985, 1986, 1993, 1998
Geanakoplos, John	Prof.	Yale University, New Haven, USA	1988
Gilboa, Itzhak	Prof.	Northwestern University, Evanston, Ill., USA	1995
Girko, Vyacheslav	Prof.	Kiev University, Ukraine	1994
Gordon, Roger	Prof.	University of Michigan, Ann Arbor, USA	1985
Grodal, Birgit	Prof.	University of Copenhagen, Dänemark	1988, 1992
Hagen, Jürgen von	Prof.	Indianan University, Bloomington, USA	1990
Haller, Hans	Prof.	Virginia Polytechnic Institute, Blacksburg, USA	1990
Hammond, Peter	Prof.	Stanford University, Stanford, USA	1988
Härdle, Wolfgang	Prof.	CORE, Louvain-la-Neuve, Belgien	1990, 1991, 1992
Harstad, Ron	Prof.	University of Mississippi, Mississippi, USA	1993, 1995
Hart, Jeff	Prof.	Texas A&M University, College Station, USA	1988
Hauswald, Robert	Prof.	Kelley School of Business, IU, USA	1998
Helmes, Kurt	Prof.	University of Kentucky, Lexington, USA	1988, 1990, 1991, 1993
Herrendorf, Berthold	Ph.D.	Universität Warwick, GB	1996, 1997
Hetzel, Robert	Dr.	Federal Reserve Bank of Richmond, USA	1998
Hildenbrand, Kurt	Dr.	Universität Bonn	1985, 1986
Ishii, Yasunori	Prof.	Yokohama City University, Japan	1991
Islam, Saiful	Dr.	Universität Bielefeld	1985
Janeba, Eckhard	Prof.	Indiana University, Bloomington, USA	1995
Jerison, Michael	Prof.	State University of New York (SUNY), Albany, USA	1986, 1987, 1988, 1989, 1991, 1993, 1994, 1998, 1999
Kandori, Michihiro	Prof.	University of Tokyo, Tokyo, Japan	1992
Kannai, Yakar	Prof.	Weizman Institute of Science, Rehovot, Israel	1999
Kannan, Ravindran	Prof.	Yale University, New Haven, USA	1999
Karni, Edi	Prof.	The John Hopkins University, Baltimore, USA	1989
Kast, Robert D.	Prof.	GREQUE-EHESS, Marseille, Frankreich	1992

Name	Academic Degree	Home institution	Year of Stay
Keuschnigg, Christian	Dr.	Universität Innsbruck, Österreich	1989, 1990
Kirman, Alan P.	Prof.	Universität Aix-Marseille, Frankreich	1985, 1996
Klein, Lawrence	Prof.	University of Pennsylvania, Philadelphia, USA	1987
Kletzer, Ken	Prof.	UC Santa Cruz, USA	1998
Klinger-Monteiro, Paulo	Prof.	Universidade Federal do Rio de Janeiro, Brasilien	1992
Kneip, Alois	Prof.	CORE, Louvain-la-Neuve, Belgien	1994, 1995
Körösi, Gabor	Okl. Közgazda, Economist	Ungarische Akademie der Wissenschaften, Budapest, Ungarn	1985, 1986, 1987
Kramkov, Dimitri	Prof.	Akademie der Wissenschaften, Steklo Institut, Moskau, GUS	1994
Lehmann-Waffenschmidt, Marco	Dr.	Universität Karlsruhe	1987
Lehrer, Ehud	Prof.	Tel Aviv University, Tel Aviv, Israel	1991
Lipman, Barton L.	Prof.	Carnegie Mellon University, Pittsburgh	1991
Liska, Tibor	Dipl.-Math.	Ungarische Akademie der Wissenschaften, Budapest, Ungarn	1986, 1987
Ma, Ching-to Albert	Prof.	Boston University, Massachusetts, USA	1989, 1990
Machina, Mark	Prof.	University of California, San Diego, USA	1989
Magill, Michael	Prof.	University of Southern California, Los Angeles, USA	1988, 1989, 1990, 1992, 1994
Mailath, George	Prof.	University of Pennsylvania, Philadelphia, USA	1992, 1995
Malawski, Marcin	Prof.	Institute of Computer Science, Warschau, Polen	1990
Maret, Isabel	Prof.	European University Institute, Florenz, Italien	1994
Marin, Dalia	Prof.	Institute für höhere Studien, Wien, Österreich	1993
Marjit, Sugata	Prof.	Jadavpur University Calcutta, Indien	1993
		Indian Statistical Institute, Calcutta, Indien	1996, 1997, 1998, 1999
Marron, Steve (J.S.)	Prof.	University of North Carolina, Chapel Hill, USA	1986, 1988
Matzkin, Rosa	Prof.	Yale University, New Haven, USA	1988
Maxwell, John		Indiana University, Blomminton, Indiana, USA	1998
McKenzie, Ken	Prof.	University of Calgary, Kanada	1998
Miltersen, Kristian	Prof.	Odense Universität, Dänemark	1994
Minyi, Yue	Prof.	Institute of Applied Mathematics, Academia Sinica Beijing, China	1997
Monderer, Dov	Prof.	Technion Haifa, Israel	1991
Mori, Akio	Prof.	Kobe Universität, Japan	1985, 1986
Musiela, Marek	Prof.	University of New South Wales, Sydney, Australien	1991, 1996
Myers, Gordon M.	Prof.	University of Essex, Wivenhoe Park, Colchester, GB	1998
Myerson, Roger	Prof.	Northwestern University, Evanston, Ill., USA	1994
Naeve, Jörg	Dr.	Universität Bielefeld	1997
Nerlove, Marc	Prof.	University of Philadelphia, Philadelphia, USA	1989
Neuefeind, Wilhelm	Prof.	Washington University, St. Louis, USA	1988
Nielsen, Joergen	Prof.	Aarhus Universität, Dänemark	1993
Nitzan, Shmuel	Prof.	Bar-Ilan Universität, Ramat Gan, Israel	1993
Nöldeke, Georg	Prof.	Princeton University, Princeton, USA	1993, 1994
Nussbaum, Michael	Dr.	Akademie der Wissenschaften der DDR, Berlin	1989
Okuguchi, Koji	Prof. Dr.	Metropolitan University, Tokio, Japan	1986
O'Neill, Barry	Prof.	York University, Toronto, Kanada	1991
		Yale University, New Haven, USA	1995, 1998, 1999

Name	Academic Degree	Home institution	Year of Stay
Ortmann, Andreas	Prof.	Bowdoin College, Brunswick, USA	1994
Owen, Guillermo	Prof.	Dept. of the Navy, Monterey, USA	1989
		Naval Postgraduate School, Monterey, USA	1990
Padberg, Manfred	Prof.	New York University, USA	1998
Peitz, Martin	Prof.	Universidad de Alicante, Spanien	1996
Pezanis-Christou, Paul	Dr.	University of New South Wales, Sydney, Australien	1998, 1999
Podczeck, Konrad	Dr.	Universität Wien, Österreich	1986, 1987
Polemarchakis, Heraklis	Prof.	Columbia University, New York, USA	1988, 1989
		CORE, Louvain-la-Neuve. Belgien	1992, 1997
Pollock, Gregory	Ph.D.	Phoenix, Arizona, USA	1995
		University College London (UCL), GB	1997
Pope, Robin Elizabeth	Prof.	Australian National University, Canberra, Australien	1996, 1997, 1998, 1999
Postlewaite, Andrew	Prof.	University of Pennsylvania, Philadelphia, USA	1992
Quinzii, Martine	Prof.	University of Southern California, Los Angeles, USA	1988, 1989, 1990
		University of California, Davis, USA	1990, 1992, 1994
Rady, Sven	Prof.	Graduate School of Business, Stanford University, USA	1998
Razin, Assaf	Prof.	Tel Aviv University, Israel	1989
Rees, Ray	Prof.	University of Cardiff, GB	1985
Reichelstein, Stefan	Prof.	Walter A. Haas School of Business, University of California, Berkeley, USA	1997, 1998
Rob, Rafael	Prof.	University of Pennsylvania, Philadelphia, USA	1991
Robson, Arthur	Prof.	University of Western-Ontario, Kanada	1998
Rosenthal, Robert	Prof.	Boston University, USA	1995
Ross, Hermann	Dipl.-Math.	Bonn-IIASA-Research Projekt	1985, 1986
Rossini, Antony	Dipl.-Math.	Rice University, Houston, Texas, USA	1989
Roth, David	Prof.	University of Michigan, Ann Arbor, USA	1993
Rothschild, Michael	Prof.	University of California, San Diego, USA	1989
Rustichini, Aldo	Prof.	Innocenco Gasparini Institute for Economics, Mailand, Italien	1992
Rutkowsky, Marek	Prof.	Warsaw University of Technology, Polen	1996
Sadka, Efraim	Prof.	Tel Aviv University, Israel	1989
Samet, Dov	Prof.	Tel Aviv University, Israel	1995, 1997
Samuelson, Larry	Prof.	University of Wisconsin, Madison, USA	1991, 1992, 1993, 1995
Sandmo, Agnar	Prof.	Norwegian School of Economics and Business Administration, Bergen, Norwegen	1986
Sansone, Emilia	Dr.	Universität Neapel, Italien	1995, 1996
Santos, Manuel	Prof.	Universidad Carlos III, Madrid, Spanien	1992
Sarda, Pascal	Dipl.-Math.	Université Paul Sabatier, Toulouse, Frankreich	1987
Sela, Aner	Dr.	Technion-Israel Institute of Technology, Haifa, Israel	1996
Shafer, Wayne	Prof.	University of Southern California, Los Angeles, USA	1986, 1987, 1988, 1989, 1990
		University of Illinois at Urbana Champaign, USA	1990, 1992
Shepherd, Bruce	Prof.	London School of Economis, GB	1996
Shin, Hyun Song	Prof.	Southampton University, GB	1995
Shiryaev, Albert	Prof.	Akademie der Wissenschaften, Steklo Institute, Moskau, GUS	1994
Sisak-Fekite, Zsuzsanne	Dr.	Budapest, Ungarn	1985
Stacchetti, Ennio	Prof.	University of Michigan, Ann Arbor, USA	1993
Stoker, Thomas M.	Prof.	Massachusetts Institute of Technology, Cambridge, USA	1986, 1987

Name	Academic Degree	Home institution	Year of Stay
Sturmfels, Bernd	Dipl.-Math.	Technische Hochschule Darmstadt	1985
Swinkels, Jeroen	Prof.	Washington University, St. Louis, USA	1998
Székeli, István	Dr.	Karl-Marx-University of Economics, Budapest, Ungarn	1985, 1986
Taksar, Michael	Prof.	State University of New York at Stony Brook, New York, USA	1992, 1999
Thomas, Jonathan	Prof.	University of Warwick, GB, and Tilburg University, Niederlande	1991
Thorlund-Petersen, Lars	Prof.	Norwegian School of Economics and Business Administration, Bergen, Norwegen	1985
Tsybakov, Alexander	Prof.	CORE, Louvain-la-Neuve, Belgien	1992
Uhlig, Harald	Prof.	Princeton University, Princeton, USA	1992, 1993
Vasil`ev, Valery	Dr.	Akademie der Wissenschaften der USSR, Institute for Systems Studies (VNIISI), Moskau, UdSSR	1985
	Prof.	Novosibirsk State University, UdSSR	1991
Vaughan, Richard N.	Prof.	University College London, London, England, GB	1992
Villemeur, Etienne Billette de	M.A.	European University Institute, Florenz, Italien und Université de Cergy-Pontoise, Frankreich	1998
Vincent, Daniel	Prof.	Northwestern University, Evanston, USA	1991
Vind, Karl	Prof.	University of Copenhagen, Dänemark	1988, 1992
Virkkunen, Virpi	Dipl. rer. oec.	Wirtschaftshochschule Helsinki, Finnland	1986
Vives, Xavier	Prof.	CSIC, Barcelona, Spanien	1997
Vriend, Nick J.	Prof.	European University Institute, Florenz, Italien	1993
Weber, Shlomo	Prof.	University of Haifa, Israel	1985
		York University, Toronto, North York, Ontario, Kanada	1989, 1990, 1991, 1992
Weiss, Ernst August	Priv.-Doz. Dr.	Bonn	1986, 1987
Wenzel, Walter	Dr.	Universität Bielefeld	1992
Werner, Jan	Prof.	University of Minnesota, Minneapolis, USA	1988, 1989, 1992, 1993, 1995, 1996, 1999
White, Michelle	Prof.	University of Michigan, Ann Arbor, USA	1985
Wieland, Volker	Dr.	Federal Reserve Board, USA	1998
Wildasin, David E.	Prof.	Indiana University, Bloomington, USA	1990
Willassen, Yngve	Prof.	University of Oslo, Norwegen	1997
Winter, Eyal	Dr.	Hebrew University, Jerusalem, Israel	1988, 1989
		University of Pittsburgh, USA	1991, 1992
	Prof.	Hebrew University Jerusalem, Israel	1993
Wolinsky, Asher	Prof.	Northwestern University, Evanston, USA	1990, 1997
Wooders, Myrna	Prof.	York University, Toronto, Kanada	1989, 1991, 1992
Yaari, Menachem	Prof.	Hebrew University, Jerusalem, Israel	1989, 1992
Younes, Yves	Prof.	C.E.P.R.E.M.A.P., Paris, Frankreich	1988
Zame, William	Prof.	State University of New York, Buffalo, USA	1988, 1989
Zandt, Timothy van	Prof.	Princeton University, Princeton, USA	1992
Zink, Helmut	Dr.	University of Berne, Schweiz	1985

6 International Cooperation

International cooperation other than research visits (see also Section 5)

- **Promotion of Young Researchers**
  - **Exchange of Ph.D.-students within the European Doctoral Program in Quantitative Economics**
  - Ecole des Hautes Etudes en Sciences Sociales, Paris, France

- London School of Economics, UK
- Universitat Pompeu Fabra, Barcelona, Spain
- Université Catholique de Louvain, Belgium
- **International seminar for young researchers**
- Bi-annual Bonn-Aarhus-Seminar for PhD-students in finance, Bonn and Aarhus

- **International research networks**
  - EU- Human Capital and Mobility Program network „Fiscal Implications of European Integration“, participating universities: Universität Bonn, Germany, Université Catholique de Louvain, Belgium, University of Essex, UK, Institute of Economic Analysis, Bellaterra, Spain, Université des Sciences Sociales de Toulouse, France, Keele University, UK, The Economic Social Research Institute, Dublin, Ireland, Department of Economic Science, Bologna, University of Liège, Belgium, Copenhagen Business School, Denmark.
  - EU-SPES Project "Economic Policy in Equilibrium", participating universities: Universität Bonn, Germany; Université Catholique de Louvain, Belgium; Copenhagen University, Denmark.
  - EU-TMR research network ENDEAR (European Network for the Development of Experimental Economics and its Application to Research on Institutions and Individual Decision Making) connecting experimental laboratories in Amsterdam, Barcelona, Berlin, Bonn, Jerusalem, Vienna, and York.

- **Other joint projects**
  - establishment of RatioLab, Hebrew University of Jerusalem, Israel
  - Game Theoretic Aspects of Multilateral Bargaining, German Israeli Fund
  - Public Enterprise Economics, German Israeli Fund

7 Total Grant