Journal of Mathematical Economics 3 (1976) 185–196. © North-Holland Publishing Company

ABSOLUTE APPROXIMATIONS TO EQUILIBRIUM IN MARKETS WITH NON-CONVEX PREFERENCES*

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Received September 1974, revised version received August 1975

It is by now a well-known result that non-convexity can be 'cured' in large markets. There exist approximations to core allocations and competitive equilibria, in these approximations the excess demand is made small relative to the size of the economy. This work is concerned with absolute approximations, that is, making the excess demand itself arbitrarily small.

1. Introduction

Section 2 in this paper defines a convexification process of a preference order and attaches economic meaning to it. Although this convexification is slightly different from the one used in the works of Starr (1969), Arrow–Hahn (1971) and Henry (1972), it serves for their purposes and it can be proved to be continuous whereas they have to assume it.

Section 3 proves the existence of absolute approximations to equilibrium in replicas of a given market, and the fourth section does the same for markets with indivisible goods. Proofs of the theorems are given in section 5.

2. Convexification of a preference order

A consumer with a non-convex utility function u, has in some sense higher expectations than a 'convex' consumer: suppose the consumer expects to take part in a competitive market where prices exist, and expects to be able to maximize his utility subject to his budget constraints. Let $E(x)$ denote the lowest utility level that he expects to achieve under these conditions. If u is not quasi-concave, then for some $x : E(x) > u(x)$, whereas for a quasi-concave $u : E(x) = u(x)$ for all x. E, it will be shown, is the convexification of u.

*This work is part of a Ph.D. thesis done under the supervision of Professor M. Yaari, Hebrew University, Jerusalem, and whom the author wishes to thank for instructive remarks and ideas.

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Let u be the consumer's utility function defined on his consumption space C, assume that:

(1) $0 \in C \subseteq E^n_+$.
(2) C is closed and convex.
(3) u is continuous on C.
(4) For all $x \in C$, $u(x) \geqq u(0) = 0$.
(5) u is locally insatiable.

Let u^* be the dual utility function and let u^{**} be the dual to u^*, defixed by:

Definition. (i) For every $0 < p \in E^n_+$, $Y > 0$:
$$u^*(p, Y) = \text{Sup}\,\{u(x)/px \leqq Y, x \in C\}.$$
(ii) For every $x \in E^n_+$:
$$u^{**}(x) = \text{Inf}\,\{u^*(p, Y)/0 < p \in E^n_+,\ Y \geqq 0, px \leqq Y\}.$$

Note that, though u is defined in C, u^{**} is defined on E^n_+. u^{**} is finite since for every x, $u^{**}(x) \leqq u^*(p, px)$ for some $p \gg 0$ and $u^*(p, px)$ is finite for such p.

Theorem 1. (i) *$u(x) \leqq u^{**}(x)$, for all $x \in E^n_+$.*

(ii) *For every $x, y \in E^n_+$, $x \leqq y$:*
$$u^{**}(x) \leqq u^{**}(y).$$

(iii) *u^{**} is a quasi-concave function.*

Denote:
$$A_k = \{x/x \in C, u(x) \geqq k\},$$
$$B_k = \{x/x \in C, u(x) \leqq k\},$$
$$C_k = A_k \cap B_k,$$
$$A^{**}_k = \{x/x \in E^n_+, u^{**}(x) \geqq k\},$$
$$B^{**}_k = \{x/x \in E^n_+, u^{**}(x) \leqq k\}.$$

By Theorem 1, $A^{**}_k \supseteq \text{conv}\,A_k$ and A^{**}_k is convex. The next theorem proves that u^{**} is the convexification of u, and discloses the difference between this convexification and the one used by Starr, for which $A^{**}_k = \text{conv}\,A_k$.

Theorem 2. For every k, A^{**}_k is a closed set and:
$$A^{**}_k = E^n_+ + \text{conv}\,A_k = E^n_+ + \text{conv}\,C_k.$$

To complete the basic properties of u^{**}:

Theorem 3. (i) *u^{**} is continuous.*

(ii) *For all $x, y \in E^n_+$, if $x \gg y$ then:*
$u^{**}(x) > u^{**}(y)$.

This generalizes Starr's convexification as here we get the continuity of u^{**} as a result and there is no need to assume it (Starr's spannability assumption).

The main theorem needed for the proof of the existence of approximations to equilibrium is the following:

Theorem 4. Let $\bar{x}$ maximize u^{**} subject to the budget constraint $px \leqq Y$, $x \in E^n_+$, then x can be represented as:

$$\bar{x} = \sum_{i=1}^{n+1} \lambda_i x^i + \mu,$$

where

$$\lambda_i \geqq 0, \qquad \sum \lambda_i = 1, \qquad \mu \in E^n_+.$$

$x^i \in C$ and x^i maximize u subject to the same budget constraint. Moreover: $u^{**}(\bar{x}) = u(x^i)$ and $p\mu = 0$.

To prove the existence of approximations to equilibrium in a market M or in replicas of M we shall need to have an equilibrium in M^{**} – the convexification of M.

Let

$$M = \langle u_i, w^i \rangle_i, \qquad i \leqq m,$$

be an exchange market where C_i is the i's consumer's consumption set, u_i his utility and $w^i \in C_i$ his initial endowment. Define a market $M^{**} = \langle u_i^{**}, w^i \rangle$, where u_i^{**} is the convexification of u_i obtained by the method described in this section.

Since by our assumptions u_i^{**} turns out to be only weakly monotonic ($x \gg y \Rightarrow u_i^{**}(x) > u_i^{**}(y)$) we shall need further assumptions to ensure the existence of equilibrium in M^{**}. This could be achieved by means of one of the following assumptions: It can be assumed that $w^i \gg 0$ for all i, or that M^{**} has McKenzie's irreducibility property, or we could make the strong assumption that u_i^{**} is strictly monotonic ($x > y \Rightarrow u_i^{**}(x) > u_i^{**}(y)$). The last assumption can be reduced to the assumption that u_i is defined on all of

E^n_+, $E^n_+ = C$, and strictly monotonic; furthermore, in this case our convexification coincides with Starr's theorem:

Theorem 5. (i) *If* $C = E^n_+$ *and* u *is strictly monotonic, so is* u^{**}.

(ii) *If* u^{**} *is strictly monotonic then for all* k:

$$A_k^{**} = \operatorname{conv} A_k.$$

It will, therefore, be assumed that the convexified economy M^{**} has a competitive equilibrium.

Before we present the results about absolute approximation it remains to be shown that this convexification can be used to obtain Starr's results.

Let M be a market, as before, and let p, $\bar{x}^i$ be a competitive equilibrium of M^{**}, by Theorem 4:

$$\bar{x}^i = \sum_j \lambda_{ij} x^{ij} + \mu^i,$$

where $\lambda_{ij} \geqq 0$, $\sum_j \lambda_{ij} = 1$, $\mu^i \in E^n_+$ and x^{ij} maximizes u_i subject to the constraint $px \leqq pw^i$.

Applying Shapley–Folkman's theorem [appendix of Starr (1969)] to the allocation $\{\bar{x}^i - \mu^i\}_i$, one gets an allocation $\{y^i\}_i$, where for every i, y^i is one of the x^{ij} and such that

$$\sum(\bar{x}^i - \mu^i) = \sum y^i + \mathrm{o}(1/m)$$

(m = the number of traders in M), $\{y^i\}_i$ is, therefore an optimal allocation and an approximation to the feasible sub-allocation $\{\bar{x}^i - \mu^i\}_i$, and the goods disposed of $\sum \mu^i$ are free goods, since, by Theorem 4, $p\mu^i = 0$.

3. Absolute approximations to competitive equilibrium

Let $M = \langle u_i, w^i \rangle$ be a market as before, M^r will denote the market obtained from M by replicating M, r times in the Debreu–Scarf (1963) sense. $\{y^i\}_i$ is an absolute ε-approximation to sub-allocation of M if there exists a sub-allocation $\{x^i\}_i$, $\sum x^i \leqq \sum w^i$ such that $\|\sum x^i - \sum w^i\| < \varepsilon$.

$\{y^i\}_i$ is optimal if there exist prices p such that y^i maximizes u_i subject to the constraint $px \leqq pw^i$.

$\{y^i\}_i$ is absolute ε-approximation to equilibrium in M if it is optimal and ε-approximation to sub-allocation of M.

It will be proved that for every ε there is a sequence of integers r for which there is an ε-approximation to equilibrium in M^r, these r's are equally distributed among the integers, i.e., the distances between r's are bounded.

We shall need the following number theory definitions and theorems concerning simultaneous approximations to real numbers by rational numbers:

Let $\varepsilon, \lambda_1, \ldots, \lambda_k$ be positive real numbers, an integer b is an ε-solution to the approximation problem if there exist integers a_i such that:

$$|b\lambda_i - a_i| < \varepsilon, \qquad 1 \leqq i \leqq k.$$

Theorem 6. Let $\varepsilon, \lambda_1, \ldots, \lambda_k$ be positive real numbers, there exists a sequence $b_1 < b_2 < b_3 < \ldots$ of ε-solutions for the approximation problem such that $\limsup (b_{i+1} - b_i) < \infty$.

Proof for a more general case can be found in Cassels (1957, p. 53).

Using this result we can derive our main theorem:

Theorem 7. For every exchange market M, whose utility functions satisfy Assumptions 1–5, and every $\varepsilon > 0$ there exists a sequence of integers $r_1 < r_2 < \ldots$ such that $\limsup (r_{i+1} - r_i) < \infty$ *and such that there exists an ε-approximation to equilibrium in M^{r_i}.*

Proof. Let $M = \langle u_i, w^i \rangle$, $1 \leqq i \leqq m$. Let M^{**} be the convexification of M and let p, $\{\bar{x}^i\}_i$ be a competitive equilibrium of M^{**}. By Theorem 4:

$$\bar{x}^i = \sum_{j=1}^{n+1} \lambda_{ij} x^{ij} + \mu^i,$$

where $\lambda_{ij} \geqq 0$, $\sum_j \lambda_{ij} = 1$, $\mu^i \in E_+^n$ and x^{ij} maximizes u_i under the constraint $px \leqq pw^i$, and $p\mu^i = 0$. Consider only the terms for which $\lambda_{ij} > 0$.

Let $B = \max \{\|x^{ij}\| / i, j\}$, choose $\delta > 0$ such that:

$$\delta m B(n+1) < \varepsilon \quad \text{and} \quad \delta(n+1) < 1.$$

Consider the approximation problem:

$$\delta, \{\lambda_{ij}\}_{i,j}, \qquad i \leqq m, \quad j \leqq n+1, \quad \lambda_{ij} > 0.$$

Let r be a solution to this problem then there exist integer a_{ij}:

$$|r\lambda_{ij} - a_{ij}| < \delta.$$

Since $\delta(n+1) < 1$:

$$\left| r - \sum_j a_{ij} \right| \leqq \sum_j |r\lambda_{ij} - a_{ij}| < \delta(n+1) < 1,$$

hence

$$r = \sum_j a_{ij}.$$

Define the following allocation in M^r: Let a_{ij} of the r consumers of type i have the bundle x^{ij}. Clearly this is an optimal allocation. It is also an ε-approximation to sub-allocation:

$$\left\| r \sum_i (\bar{x}^i - \mu^i) - \sum_i \sum_j a_{ij} x^{ij} \right\| = \left\| \sum_i \sum_j (r\lambda_{ij} - a_{ij}) x^{ij} \right\|$$

$$\leq \delta m B(n+1) < \varepsilon.$$

By Theorem 6 the solutions to the approximation problem form a sequence $r_1 < r_2 < r_3 < \ldots$ for which a T exists such that $r_{i+1} - r_i < T$. Q.E.D.

4. Absolute approximation to equilibria in markets with indivisible goods

For ε which is less than the minimal indivisible unit in a market with indivisible goods, an ε-absolute approximation to equilibrium will be a feasible optimal allocation – an equilibrium. Though because of indivisibility, this allocation will have other strange properties.

Assume, for simplicity, that the consumption set is $L \subseteq E^n_+$, the integers lattice in E^n_+ (any set closed under addition whose projection on each axis is a discrete set could do). To enable proper convexification of the preference order some assumption has to be made.

Let $x, y \in E^n_+$, define $y^* < x$ if $y_i < x_i$ for all i s.t. $x_i > 0$.

This relation is the extension of $\ll$ to the boundary of E^n_+.

Assume that the consumer's preference order $\prec$ on L has the following properties:

(1) For every $x, y \in L, y \prec x$ there exists $z \in L$, $z \sim y$, $z^* < x$. ($\sim$ is the indifference relation derived from $\prec$.)

(2) For every $x \in L$ there exists $y \in L$, $x \prec y$.

The first assumption makes L a well-ordered set so that there is a utility function u, representing the preference order, whose range is the non-negative integers (Assumption 2). Assume $u(0) = 0$.

Define A_k for integers k, as in section 2, and $A_k^{**} = E^n_+ + \text{conv}\, A_k$, as A_k is closed so is A_k^{**}. (See proof of Theorem 2.)

Let $D_x = \{y \in E^n_+ / y *< x\}$ and define:

$$I_k = \{x \in A_k^{**} / A_k^{**} \cap D_x = \emptyset\}.$$

It is readily verified that:

(1) For all $k \neq h$, $I_k \cap I_h = \emptyset$.

(2) If $x^{k_i} \in I_{k_i}$, then $\{x^{k_i}\}$ has no limit unless $k_i = 1$ for all sufficiently large i's.

(3) If $x \in I_k \cap L$, then $u(x) = k$.

(4) If $x \in A_k^{**}$, then $x = \sum \lambda_i x^i + \mu$ where $\lambda_i > 0, \sum \lambda_i = 1$, $\mu \in E^n_+$, $x^i \in A_k \cap I_k$.

Using the I_k's as a framework for a utility function we want to fill the spaces between them with indifference curves that together will define a function v. This has to be done in a special way: crucial points in the new indifference curves – points that span the curve, will be above points of L in the closest I_k below this curve. To do this rigorously define:

x is an exposed point of a set $A \subseteq E^n$ if x is not of the form $\sum \lambda_i x^i + \mu$ where $\lambda_i > 0, \sum \lambda_i = 1$, $\mu \in E^n_+$, $x \neq x^i \in A$. Denote the set of exposed points of A by $e(A)$.

Theorem 8. (i) *For all* $A \subseteq E^n_+$,

$$E^n_+ + \text{conv}\, A = E^n_+ + \text{conv}\, e(A).$$

(ii) *If* $x \in e(I_k)$, *then* $x \in L$ *and* $u(x) = k$.

(iii) *There exists a continuous quasi-concave weakly monotonic function* v *defined on* E^n_+ s.t.

(1) *for* $x \in L$, $v(x) \geqq u(x)$;

(2) *for integers* k, $I_k = \{x \in E^n_+ / v(x) = k\}$;

(3) *for* $h = k + \lambda$, k *integer*, $0 \leqq \lambda < 1$ *and* $x \in e(\{y / v(y) = h\})$; *there exist* $y \in L$, $y *< x$ s.t. $u(y) = k$.

The proof of this theorem is quite intuitive but rather cumbersome, a detailed sketch of the proof can be found in the last section.

Starting with a market $M = \langle u_i, w^i \rangle$ with the utility functions u_i defined on L and satisfying Assumptions 1, 2, one can convexify the market to obtain $M^{**} = \langle v_i, w^i \rangle$.

Assume that M^{**} has a competitive equilibrium p, $\{\bar{x}^i\}$ (analysis of this assumption was carried out in section 3).

If $v_i(\bar{x}^i) = k$ is an integer, then $\bar{x}^i = \sum_j \lambda_{ij} x^{ij} + \mu^i$, $x^{ij} \in L$, $u_i(x^{ij}) = k$, $M^i \in E^n_+$, $\lambda_{ij} > 0, \sum \lambda_{ij} = 1, p\mu^i = 0$. By the proof of Theorem 4, $px^{ij} \leqq pw^i$.

If $v_i(\bar{x}^i) = k+\lambda$, k an integer, $0 < \lambda < 1$, then, by Theorem 8, $\bar{x}^i = \sum \lambda_{ij} x^{ij} + \mu^i$, $x^{ij} \in L$, $u_i(x^{ij}) = k$, $\mu^i \in E^n_+$, $\lambda_{ij} > 0, \sum \lambda_{ij} = 1$, k is the maximum of u_i in the budget set and $px^{ij} \leqq pw^i$. Generally, $p\mu^i$ in this case is *not* zero.

Given an $\varepsilon > 0$ one can find ε-absolute approximation to equilibrium in M^r, for a sequence of r's whose distances are bounded (Theorem 6). If ε is small enough then an ε-approximation to a sub-allocation of $r \cdot \sum w^i$ is itself a sub-allocation (because the minimal distance of two points of L is 1).

So, for infinite number of r's evenly distributed among the integers, there is an equilibrium in M^r. This equilibrium, however, does not satisfy Walras' law since the goods disposed of (μ^i) are not necessarily of zero price value. One suspects that this equilibrium is not efficient in some sense.

Consider, therefore, the game theoretic properties of this equilibrium. The usual proof holds and this allocation is in the core, i.e., it is not blocked by any coalition – provided blocking is defined as improving the state of *all* members of the coalition, But the possibility of increasing the utility of *part* of the members is not ruled out, so the equilibrium is not necessarily Pareto-optimal. To justify this unpleasant result one can take into account the lack of incentive for a trader to take part in a coalition which is not likely to improve his situation.

5. Proofs of the theorems

Theorem 1

(i) By definition of u^*: $u(x) \leqq u^*(p, Y)$ for $px \leqq Y$, and since $u^{**}(x)$ is the infimum of the right-hand side: $u(x) \leqq u^{**}(x)$.

(ii) Trivial, since $py \leqq Y$ implies $px \leqq Y$.

(iii) Let $0 < \lambda < 1$, $x, y \in E^n_+$. There exists a sequence p_k, Y_k s.t. $p_k(\lambda x + (1-\lambda)y) \leqq Y_k$ and $u^{**}(\lambda x + (1-\lambda)y) = \lim u^*(p_k, Y_k)$. For all k, either $p_k x \leqq Y_k$ or $p_k y \leqq Y_k$, choose a subsequence k' for which, say, $p_{k'} x \leqq Y_{k'}$. By definition of u^{**}: $u^{**}(x) \leqq u^*(p_{k'} Y_{k'})$, hence $\min\{u^{**}(x), u^{**}(y)\} \leqq u^{**}(x) \leqq u^{**}(\lambda x + (1-\lambda)y)$.

Q.E.D.

Theorem 2

The proof will follow these stages:

(i) Let $B \subseteq E^n_+$ be a closed set, then $\overline{\text{conv}\, B} \subseteq E^n_+ + \text{conv}\, B$ (bar denotes closure).

(ii) B as in (i), then: $E^n_+ + \text{conv}\, B$ is closed.

(iii) $E^n_+ + \text{conv } A_k = E^n_+ + \text{conv } C_k$.

(iv) $A^{**}_k = E^n_+ + \text{conv } A_k$.

(i) Let $y \in \overline{\text{conv } B}$, then $y = \lim y^i$, $y^i \in \text{conv } B$, $y^i = \sum_j \lambda_{ij} b^{ij}$, $\sum_j \lambda_{ij} = 1$, $\lambda_{ij} \geqq 0$, $b^{ij} \in B$. The number of terms in each sum can be assumed, by Caratheodory's theorem, to be $\leqq n+1$. By a diagonalizing process it can be assumed that, for each $j \leqq n+1$,

$$\lambda_{ij} \underset{i}{\rightarrow} \lambda_j \geqq 0, \quad b^{ij} \underset{i}{\rightarrow} b^j \in B, \quad \|b^{ij}\| \underset{i}{\rightarrow} \infty.$$

Moreover, $\sum_j \lambda_j = 1$ and if $\lambda_j > 0$ then $\lim b^{ij} \in B$. Denote $P = \{j/\lambda_j > 0\}$, then:

$$y = \sum_{j \in p} \lambda_j b^j + \mu,$$

where $\mu = \lim \sum_{j \notin P} \lambda_{ij} b^{ij}$, clearly $\mu \in E^n_+$. Hence $y \in E^n_+ + \text{conv } B$.

(ii) Let y be in the closure of $E^n_+ + \text{conv } B$, then $y = \lim y^i$, $y^i \in E^n_+ + \text{conv } B$, $y^i = \mu^i + x^i$, $\mu^i \in E^n_+$, $x^i \in \text{conv } B$. By choosing a subsequence i',

$$\mu^{i'} \rightarrow \mu \in E^n_+, \qquad x^{i'} \rightarrow x \in \overline{\text{conv } B}.$$

But by (i): $x = z + \bar{\mu}$, $z \in \text{conv } B$ $\bar{\mu} \in E^n_+$. Hence:

$$y = \mu + x = (\mu + \bar{\mu}) + z \in E^n_+ + \text{conv } B.$$

(iii) Clearly: $E^n_+ + \text{conv } C_k \subseteq E^n_+ + \text{conv } A_k$. Let $y \in E^n_+ + \text{conv } A_k$, $y = \sum_i \lambda_i x^i + \mu$, $x^i \in A_k$, if $u(x^i) > k$, then, since the consumption set is convex and u is continuous on it, there is $y^i < x^i$ $y^i \in C_k$. Hence

$$y = \sum \lambda_i y^i + (\mu + \sum \lambda_i (x^i - y^i)) \in E^n_+ + \text{conv } C_k.$$

(iv) By Theorem 1 (i): $A^{**}_k \geqq A_k$, (iii): $A^{**}_k \geqq \text{conv } A_k$. (vi): $A^{**}_k \geqq E^n_+ + A^{**}_k \supseteq E^n_+ \text{conv } A_k$.

It remains, therefore, to prove that:

$$E^n_+ + \text{conv } A_k \geqq A^{**}_k.$$

Assume that $y \in A^{**}_k - (E^n_+ + \text{conv } A_k)$, then for all $z \leqq y : z \notin E^n_+ + \text{conv } A_k$. Denote $D_y = \{z/z \leqq y\}$, D_y is compact and convex, and $D_y \cap (E^n_+ + \text{conv } A_k) = \emptyset$. By (ii), $E^n_+ + \text{conv } A_k$ is closed (A_k is closed by the continuity of u) and convex (sum of convex sets), hence D_y, $E^n_+ + \text{conv } A_k$ can be strictly separated by a hyperplane defined by p, Y:

$$pz \geqq Y+\varepsilon > Y \geqq px,$$

for all $z \in E^n_+ + \text{conv } A_k$, $x \in D_y$. By rotating p slightly, it is possible to separate the two sets by a vector $q \gg 0$. Define q, for example, by its coordinates:

$$q_i = \begin{cases} p_i, & p_i > 0, \\ \dfrac{\varepsilon}{2n\|y\|}, & p_i = 0. \end{cases}$$

$u^*(q, Y+\varepsilon/2) < k$, as for every $x \in C$ s.t. $qx \leqq Y+\varepsilon/2$, $x \notin E^n_+ + \text{conv } A_k$ and therefore $x \notin A_k$, that is: $u(x) < k$ and since $q \gg 0$, $u^*(q, Y+\varepsilon/2)$ is actually obtained and is less than k. But this implies $u^{**}(y) < k$ which contradicts the assumption $y \in A^{**}_k$. Hence

$$A^{**}_k = E^n_+ + \text{conv } A_k.$$

Q.E.D.

Theorem 3

(i) To prove that u^{**} is continuous it is sufficient to prove that B^{**}_k is a closed set. Let $y^i \to y$, $y^i \in B^{**}_k$ and suppose $u^{**}(y) = h > k$. $y \in A^{**}_h$ and by Theorem 2, y can be represented as $y = \sum \lambda_j x^j + \mu$ where $x^i \in C_h$, by the proof of Theorem 2(iii) it is possible to choose x^i such that for all x in the interval $(0, x^i) \subseteq C: u(x) < h$. Choose $\bar{x}^j$ in the interval $(0, x^j)$ such that $u(\bar{x}^j) = (k+h)/2$, denote: $\bar{y} = \sum \lambda_j \bar{x}^j + \mu$ then, for a sufficiently large i, $y^i \geqq \bar{y}$, but $u^{**}(\bar{y}) \geqq (k+h)/2$, hence $u^{**}(y^i) > (k+h)/2 > k$, in contradiction to the assumption: $y^i \in B^{**}_k$. Hence B^{**}_k is closed.

(ii) Let $u^{**}(y) = k$, then $y = \sum \lambda_i y^i + \mu$, $y^i \in A_k$, since u is locally insatiable there are $\bar{y}^i$ arbitrarily close to y^i such that $u(\bar{y}^i) > u(y^i)$. Choose the $\bar{y}^i$ such that if $\bar{y} = \sum \lambda_i \bar{y}^i + \mu$ then $\bar{y} \leqq x$, this is possible as $y \ll x$. Since $\min u(\bar{y}^i) > k$ then $u^{**}(\bar{y}) > k$, but by Theorem 1, $u^{**}(x) \geqq u^{**}(\bar{y}) > k = u^{**}(y)$. Q.E.D.

Theorem 4

Let $u^{**}(\bar{x}) = k$, choose a representation of $\bar{x}$ as $\bar{x} = \sum \lambda_i x^i + \mu$ where $x^i \in C_k$ (Theorem 2). By the local insatiability of u, $Y \leqq px^i$, since $p\bar{x} = Y$ (local insatiability of u^{**}, Theorem 3), and if $px^i < Y$ there would be an $\bar{x}^i$, $p\bar{x}^i \leqq Y$ for which $k = u(x^i) < u(\bar{x}^i) \leqq u^{**}(\bar{x}^i)$ contradicting the maximality of $\bar{x}$ in the budget set. $p\mu \geqq 0$, therefore $px^i = Y$ for all i, and therefore $p\mu = 0$. x^i maximizes u in the budget set for, $u^{**} \geqq u$. Q.E.D.

Theorem 5

(i) If $y \leqq x$, let $u^{**}(y) = k$ and $y = \sum \lambda_i y^i + \mu$, $y^i \in C_k$, denote $z = x - y \geqq 0$, then $x = \sum \lambda_i(y^i + z) + \mu$, and since u is strictly monotonic $u(y^i + z) > k$ and $u^{**}(x) > k = u^{**}(y)$.

(ii) It is sufficient to show that conv $A_k \supseteq E^n_+ +$conv A_k. Let $y \in E^n_+$conv A_k, $u^{**}(y) = h \geqq k$, then $y = \sum \lambda_i x^i + \mu$, $x^i \in A_h$. If $\mu > 0$, then $u^{**}(y) = u^{**}(\sum \lambda_i x^i + \mu) > u^{**}(\sum \lambda_i x^i) \geqq h$ contradicting our assumption, therefore in this representation $\mu = 0$, i.e., $y \in$ conv $A_h \subseteq$ conv A_k. Q.E.D.

Theorem 8

(i) is a well-known result in convex sets theory. (ii) is trivial. The proof of (iii) follows the following lines: A close look at the proofs of the theorems of section 2 reveals that the assumptions about C and u can, in fact, be weakened. C does not have to be convex and u has to have specific properties of continuity and insatiability only at the exposed points of its indifference curves. As these axioms have no particularly interesting economic interpretation they were not brought in their weak form in the second section.

(1′) $C \subseteq E^n_+$ is closed.
(2′) u is continuous function defined on C, with $u(x) \geqq u(0) = 0$.
(3′) For every exposed point of u : x (i.e., exposed point of one of the indifference curves), there is a sequence x^i, $x^i \to x$, $x^i \, {*}{<} \, x$.
(4′) u is locally insatiable at every exposed point.

Starting with the I_k's we'll introduce segments connecting I_k and I_{k+1} and define a continuous $\bar{u}$ on these segments and the I_k's.

For all k, $e(I_k)$ is a finite set. [Prove that every subspace contains only finite number of points of $e(I_k)$ by induction on the dimension of the subspace.]

Let $x \in e(I_{k+1})$, then $u(x) = k+1$ and there is $y \in L$, $y \, {*}{<} \, x$, $u(y) = k$; y is clearly above I_k. Connect x to y by a segment.

Let $x \in e(I_k)$ then the line $x + \lambda(1, \ldots, 1)$, $\lambda > 0$, will eventually intersect I_{k+1}, take the segment of this line from x to I_{k+1}.

Now, define $\bar{u}$ on these segments continuously and monotonic, assuming values from k to $k+1$, and on I_k as k. Do this for all k (some of these segments may intersect but as there is only a finite number of them there is no difficulty in defining $\bar{u}$).

$\bar{u}$ and the set of I_k's and the segments satisfy the weak axioms 1′–4′ and can therefore be connexified by the process described in section 2. Define $v = \bar{u}^{**}$. Q.E.D. ($\pm$)

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