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EXISTENCE AND COMPUTATION OF MIXED STRATEGY NASH EQUILIBRIUM FOR 3-FIRMS LOCATION PROBLEM

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Location problems of firms on a closed interval were introduced by Hotelling [3] and later investigated by Eaton & Lipsey [2]. Firms sell a homogeneous product at a fixed price, customers distributed along the interval buy one unit each from the firm nearest to them and firms aim to maximize the number of their customers. It is well known that where the customers are evenly distributed along the interval and the number of firms is other than 3 the location problem has a pure strategy Nash equilibrium (see [2]).

The case of 3 firms is peculiar in that no pure strategy Nash equilibrium exists, for in any situation one of the extreme firms (not located between the other two) can increase its revenue by either moving towards or away from the other two firms. The existence of mixed strategies is not obvious because the payoff functions are discontinuous whenever two or more firms are located at the same point.

In a recent paper Dasgupta and Maskin [1] investigate a class of games, all derived from economic problems and all with some discontinuities in their payoff functions. They demonstrate that the discontinuities in the payoff functions do not prevent the existence of mixed strategies Nash equilibria in the games investigated. In particular, location problems of the type described earlier have such "weak" discontinuities and hence possess a mixed strategy Nash equilibrium. Although their proof is constructive, offering a method of approaching these equilibria as a limit of equilibria of finite games, the equations involved very quickly become unmanageable. In this note we compute a mixed strategy Nash equilibrium for the 3 firms location problem.

Let the customers be evenly distributed along [0, 1] then a mixed strategy is a probability measure over [0, 1]. The symmetry of the problem suggests that the solution will be doubly symmetric in that all firms choose the same mixed strategy and that this strategy is symmetric around $\frac{1}{2}$ (symmetric locations are given the same probabilities). Dasgupta and Maskin confine their analysis to doubly symmetric solutions and prove that the strategies in any doubly symmetric equilibrium are atomless.

Given the symmetric nature of the game and following the analysis of Dasgupta and Maskin we shall confine ourselves here to such doubly symmetric equilibria. We demonstrate here that *the 3 firms location problem has a unique doubly symmetric mixed strategy equilibrium*, given by:

$$(*) \qquad g(x) = \begin{cases} 2 & \frac{1}{4} \le x \le \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

This implies that firms in equilibrium avoid the extreme quartiles and choose locations in the remaining half with equal probability.

Let two firms choose their location according to a symmetric (atomless) density probability function $f(x)$; we compute the payoff function for the third firm. For an equilibrium $f(x)$ the payoff function of the third firm assumes a single value over the support of f and a lower value outside the support. We assume without loss of generality that the support of f is a subset of $[\alpha, 1-\alpha]$ for $0 \le \alpha < \frac{1}{2}$.

Define:

$$Q(x) = \int_\alpha^x f(t)\, dt$$

$$R(x) = \int_x^{1-\alpha} f(t)\, dt = 1 - Q(x)$$

Then:

$$Q'(x) = f(x) = -R'(x)$$

Let the third firm locate itself at z, its payoff there is given by:

$$(1) \qquad A(z) = 2\int_\alpha^z f(x)Q(x)\left(1 - \frac{z+x}{2}\right) dx + 2\int_\alpha^z \int_z^{1-\alpha} f(x)f(y)\frac{y-x}{2}\, dy\, dx + 2\int_z^{1-\alpha} f(x)R(x)\,\frac{x+z}{2}\, dx.$$

The first integral represents payoff when the two firms are located to the left of z, and the last integral when both firms are to the right of z, the second integral corresponds to the case of one firm on each side of z.

Differentiating w.r.t. z:

$$(2) \qquad A'(z) = 2f(z)Q(z)(1-z) - \int_\alpha^z f(x)Q(x)\, dx + f(z)\int_z^{1-\alpha} f(y)(y-z)\, dy - f(z)\int_\alpha^z f(x)(z-x)\, dx - 2f(z)R(z)z + \int_z^{1-\alpha} f(x)R(x)\, dx.$$

Since:

$$R + Q \equiv 1, \int f(x)Q(x)\, dx = \tfrac{1}{2}Q^2(x), \int_{\alpha}^{1-\alpha} xf(x)\, dx = \tfrac{1}{2},$$

equation (2) becomes:

$$(3) \qquad A'(z) = (\tfrac{1}{2} - Q(z)) + f(z)[3(\tfrac{1}{2} - z) - 2(\tfrac{1}{2} - Q(z))].$$

We are looking for a function $Q(z)$ (with $Q'(z) = f(z)$) such that $A'(z) = 0$ whenever $f(z) \neq 0$. Under the transformation

$$h(z) = \frac{\tfrac{1}{2} - Q(z)}{\tfrac{1}{2} - z}$$

the differential equation $A'(z) = 0$ becomes:

$$(4) \qquad h'(2h - z)(\tfrac{1}{2} - z) = 2h(h - 2)$$

This is a simple separable equation whose solution is implicitly given by:

$$(5) \qquad h^3(h - 2) = K(\tfrac{1}{2} - z)^{-4},$$

where K is a constant.

Let $K = 0$, this defines one acceptable solution for h: $h \equiv 2$ (the other solution, $h \equiv 0$ implies $Q \equiv \frac{1}{2}$ and is therefore ruled out).

Hence

$$Q(z) = 2(z - \tfrac{1}{2}) + \tfrac{1}{2} = 2(z - \tfrac{1}{4})$$

which is the $Q(z)$ derived from the distribution $g(x)$ defined in (*). To show that this is an equilibrium we need to demonstrate that the payoff outside the support is lower than in it. For $z \leq \frac{1}{4}, f(z) \equiv 0$ and the payoff at z is:

$$A(z) = 2 \int_{1/4}^{3/4} f(x)R(x)\left(\frac{x + z}{2}\right) dx$$

This is an increasing function of z and since $A(z)$ is continuous and symmetric in z, the payoff is highest in the support.

We now demonstrate that no other solution of the differential equation is suitable as an equilibrium. Let $Q(z)$ and $h(z)$ be the solutions of (4) for $K \neq 0$. Let z be the first point for which $Q(z) = \frac{1}{2}$, then z is the limit of points in the support. If $z < \frac{1}{2}$ (or $z > \frac{1}{2}$) then by the definition of h: $h(z) = 0$ but since (5) holds at that point, $h(z)$ cannot be zero, hence $z = \frac{1}{2}$. Rearranging the terms in (5):

$$[h(z)(\tfrac{1}{2} - z)]^3(h(z) - 2)(\tfrac{1}{2} - z) = K.$$

By the definition of $h(z)$ this becomes:

$$(\tfrac{1}{2} - Q(z))^3(2z - Q(z) - \tfrac{1}{2}) = K.$$

Taking the limit as $z \to \frac{1}{2}$, the left hand side converges to 0, i.e. $K = 0$. Hence (5) represents a solution to (4) only for $K = 0$.

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