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HUMAN ENVIRONMENT AS A LOCAL PUBLIC GOOD*

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Individuals of different types form groups, thereby creating a human environment which contributes to their utility. Trade takes place within a group. The paper proves the existence of a stable partition of the individuals and then presents an example showing how this stable partition varies with the overall composition of the society.

1. Introduction

By forming groups, individuals create a human environment which can be considered a local public good attracting or repelling other individuals. This public good is of a very special nature in that it has not been there before the group formed and in that it is not there to be enjoyed by anyone else. The presence of another individual changes the group, thereby changing the human environment.

This aspect of human environment seems to be missing from most models of local public goods and the theory of clubs. In these models individuals choose among a fixed number of cities or clubs, each of which offers a predetermined bundle of public goods. Although some models of the theory of clubs allow congestion to influence the individuals' utility function, they neither treat the neighbourhood, the presence of other individuals as an important factor, nor do they allow the formation of a new group by dissatisfied individuals (only the marginal individual can wander between existing cities or clubs).

This paper attempts to fill this gap by allowing individuals to be sensitive to their human environment and enabling them to form their own environment — thereby leaving it to the model to determine the number and variety of groups formed.

There could be various ways in which the coalition enters the utility functions of its members. The most general one is where individuals care

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about the detailed structure of the group they are in, e.g. they require a complete list of their neighbours. Gale and Shapley (1962) investigate a special case of this, where only coalitions of two individuals of opposite sex (marriage) are allowed to form.

In this general context Drèze and Greenberg (1977) prove that no stable coalition structure exists for some examples.

Schelling (1978, p. 175) introduces the idea that the group has certain characteristics which are a function of the members' characteristics, e.g. the age of the members 'creates' the average age which is a characteristic of the group.

In this model we have a finite number of types (a continuum of players of each type). Each player has an endowment of private goods. Individuals are allowed to congregate and trade within their group. A group is characterized by the distribution of types in it. Thus members of a group derive utility from the goods allocated to them as a result of trading within their group and from the group itself via its composition.

This structure excludes the discussion of space and congestion, as the size of a coalition is unimportant here: coalitions with the same distribution of types can achieve the same utility levels for their members, regardless of their size.

The aim of this paper is to show that there exists a stable partition of the individuals, in the sense that no coalition can block by achieving higher utility levels for its members than in the partition.

A stable partition when it exists does not necessarily treat individuals of the same type equally, thus we may end up with 'unemployment'. Some individuals are involved in a coalition with other types, while others of the same type are left on their own. Section 3 analyses a simple example of this model and its possible economic significance.

2. The model

Let $T \subseteq [0,1]$ be the set of agents. There is a finite number m of types, so $T = \bigcup_{i \leq m} T_i$, where T^i is the set of agents of type i, let $\mu(T^i) = \varepsilon_i > 0$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots . \varepsilon_m)$.

For a coalition $S \subseteq T$ s.t. $\mu(S) > 0$, let $\boldsymbol{\alpha}(S) = (\alpha_1(S), \ldots, \alpha_m(S))$ denote the composition of types in S, and $\alpha_i(S) = \mu(S \cap T^i)/\mu(S)$ the distribution of types in S. Individual of type i has an initial endowment of private goods $\boldsymbol{e}^i \in R^n_+$.

Agents form coalitions and are allowed to trade within their coalition, they then derive utility from the private goods allocated to them and from the coalition itself — the human neighbourhood. The way in which this last factor enters the utility function is through the distribution of types in the coalition. Thus, if a coalition S formed and $\boldsymbol{a}(S)$ is a feasible S-allocation, then individual $s \in S$ of type i derives the utility $u_i(\boldsymbol{\alpha}(S), \boldsymbol{a}(s))$.

It is an important feature of the model that anything that a coalition can achieve can also be achieved by a coalition with the same composition, regardless of the size of the coalition. Neither $\boldsymbol{\alpha}(S)$ nor the allocation $\boldsymbol{a}$ depend on the size of S.

Let $S^1,\dots,S^k$ be a partition of T and $\boldsymbol{a}^1(\cdot),\boldsymbol{a}^2(\cdot),\dots,\boldsymbol{a}^k(\cdot)$ feasible allocations for $S^1,\dots,S^k$, respectively $S^1,\dots,S^k,\boldsymbol{a}^1,\dots,\boldsymbol{a}^k$ is a stable partition-allocation, if no coalition S exists with a feasible S-allocation $\boldsymbol{a}(\cdot)$ s.t. almost every individual in S scores higher utility in S than in the original coalition he belonged to (one of $S^1,\dots,S^k$). Note that in the cooperative game without side payments derived from this structure a stable partition-allocation is in the core.

Theorem. *Let $u_i(\boldsymbol{\alpha},\boldsymbol{a})$ be continuous for $\boldsymbol{\alpha}$ in the m-simplex, $\boldsymbol{a}\in R^n_+$, and quasi-concave in $\boldsymbol{a}$, then there exists a stable partition-allocation.*[1]

The proof closely follows Scarf's (1967) classical proof of existence of core in N person games. However, this theorem is not a special case of Scarf's mainly because the players in this game are different from the classical game. It is groups of identical agents that play in a limiting process a similar role to Scarf's players.

Before presenting the proof of the theorem we require some preliminary concepts and lemmas. The proofs of the lemmas are in section 4.

The first lemma states that the game in question is balanced in some sense:

Lemma 1. *Let $\boldsymbol{\alpha}^1,\dots,\boldsymbol{\alpha}^k$ be a set of distributions of types (i.e., $\boldsymbol{\alpha}^h$ is in m-simplex) and let $\delta_1,\dots,\delta_k$ be non-negative s.t. $\sum_{h=1}^k \delta_h\boldsymbol{\alpha}^h=\boldsymbol{\varepsilon}$, then there exists a partition of $T:R^1,\dots,R^k$ with $\boldsymbol{\alpha}^h=\boldsymbol{\alpha}(R^h)$, $1\leqq h<k$.*

Quasi-concavity of $u(\boldsymbol{\alpha},\boldsymbol{a})$ in $\boldsymbol{a}$ enables us to assume without any loss of generality that in all the partition-allocations involved the allocation of private goods within each coalition satisfies the equal treatment property. (Note that this does not mean that consumers of the same type obtain the same utility level when in different coalitions.) If quasi-concavity is not assumed (see footnote 1) we can restrict the allocations to assume only a finite uniformly bounded number of values.

For a given partition-allocation $R^1,\dots,R^k,\boldsymbol{a}^1(\cdot),\dots,\boldsymbol{a}^k(\cdot)$ define $\boldsymbol{u}(R^1,\dots,R^k,\boldsymbol{a}^1,\dots,\boldsymbol{a}^k)$ by its components

$$u_i(R^1,\dots,R^k,\boldsymbol{a}^1,\dots,\boldsymbol{a}^k)=\min\{u_i(\boldsymbol{\alpha}(R^h),\boldsymbol{a}^h(s))/s \text{ of type } i,\ \alpha_i(R^h)>0,\ 1\leqq h\leqq k\}.$$

[1]The quasi-concavity of $u_i(\boldsymbol{\alpha},\boldsymbol{a})$ in $\boldsymbol{a}$ can be dispensed with at a cost of some modification of the proof. However, since the main interest of this paper is in obtaining a stable partition and not so much in the allocation attached to it, I chose to assume quasi-concavity.

$u_i(R^1,\dots,R^k,\boldsymbol{a}^k)$ is the lowest utility score of an individual of type i in this partition-allocation.

Lemma 2. For every partition-allocation $R^1,\dots,R^k,\boldsymbol{a}^1(\cdot),\dots,\boldsymbol{a}^k(\cdot)$, there exists a partition-allocation $S^1,\dots,S^l,\boldsymbol{b}^1(\cdot),\dots,\boldsymbol{b}^l(\cdot)$, with $l\leqq m$, and $\boldsymbol{u}(R^1,\dots,R^k,\boldsymbol{a}^1,\dots,\boldsymbol{a}^k)\leqq\boldsymbol{u}(S^1,\dots,S^l,\boldsymbol{b}^1,\dots,\boldsymbol{b}^l)$, which is not blocked by the grand coalition [i.e., there is no allocation $c(\cdot)$ s.t. all individuals do better in T, $c(\cdot)$ than in $R^1,\dots,R^l,\boldsymbol{b}^1(\cdot),\dots,\boldsymbol{b}^l(\cdot)$].

We are now in a position to prove the theorem:

Proof of Theorem

Like in Scarf's proof we define a finite corner game and prove first that it has a non-empty core (a stable partition-allocation). We then increase the number of the corners, taking the limit we show that there is a stable partition-allocation for the limit game.

Let $\lambda(S)\subseteq\{1,\dots,m\}$ be the set of types that participate in the coalition S [i.e., $\alpha_i(S)>0$].

Denote by $v(S)$ the set of utility allocation that the types of S can achieve by forming S. $\boldsymbol{v}\in v(S)\subseteq R_+^{\lambda(S)}$ iff there exists an (equal treatment) S-allocation $\boldsymbol{a}^i$, $i\in\lambda(S)$, s.t.

$$v_i=u_i(\alpha(S),a^i).$$

Clearly if S and Q have the same composition of types $[\boldsymbol{\alpha}(S)=\boldsymbol{\alpha}(Q)]$ then $v(S)=v(Q)$. So that we can define v as a function of $\boldsymbol{\alpha}$, and $v(\boldsymbol{\alpha})$ is a compact set.

Individual rationality means that coalitions that can be blocked by a single type will not form. Clearly types who prefer to form their own coalitions will do so. We are then justified in deleting such coalitions from the m-simplex Δ^m. Denote the set of the remaining coalitions by Γ^m. This set is compact and not empty since it contains all the vertices of Δ^m.

Choose a dense sequence $\{\boldsymbol{\alpha}^j\}$ in Γ^m s.t. all the vertices of Δ^m appear in the sequence. For each $\boldsymbol{\alpha}^j$ in this sequence choose a dense sequence in $v(\boldsymbol{\alpha}^j):\{\boldsymbol{v}^{j,h}\}$.

Let r be an integer, consider the $r(r+1)$ vectors $\boldsymbol{\alpha}^j,\boldsymbol{v}^{j,1},\dots,\boldsymbol{v}^{j,r}$, $1\leqq j\leqq r$. Define G^r as a cooperative finite corner game in which the grand coalition T can partition in any way, but blocking is done only by coalitions whose composition is one of $\boldsymbol{\alpha}^1,\dots,\boldsymbol{\alpha}^r$ with the utility allocations $\boldsymbol{v}^{j,h}$, $1\leqq j,h\leqq r$. To prove that there is a stable partition-allocation in G^r choose a sufficiently large $K:K>\max_{i,j,h}v_i^{j,h}$. Complete each $\boldsymbol{v}^{j,h}$ to an m-vector by adding K as the missing coordinates;

$$\begin{aligned}\beta_i^{j,h}&=v_i^{j,h}\quad\text{if}\quad \alpha_i^j>0,\\&=K\quad\text{otherwise.}\end{aligned}$$

Define two matrices by their columns (following Scarf's notation):

$$A=(\boldsymbol{\alpha}^{j}), \qquad 1\leqq j\leqq r,$$

$$C=(\boldsymbol{\beta}^{j,h}), \qquad 1\leqq j,h\leqq r.$$

A's columns are the distributions of the permissible coalitions in G^r, and C's are the utility payoff vectors of the corresponding coalitions (completed to m-vectors by an arbitrary large K).

The triple $A, C, \boldsymbol{\varepsilon}$ satisfies Scarf's conditions (1967, p. 59) when r is sufficiently large so that all the vertices of Δ^m are listed in $\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^r$, since in Γ^m a type can only gain by mixing with other types. It is therefore possible to find $k\leqq m$ columns in $A:\boldsymbol{\alpha}^{j_1},\ldots,\boldsymbol{\alpha}^{j_k}$ representing coalitions, and corresponding columns in $C:\boldsymbol{\beta}^{j_s,h_s}$, $1\leqq j\leqq k$, representing the payoffs for these coalitions, s.t.

(1) $\boldsymbol{\alpha}^{j_1},\ldots,\boldsymbol{\alpha}^{j_k}$ is a balanced set of distributions, i.e., there exist non-negative δ_j s.t. $\sum_{s\leqq k}\delta_s\boldsymbol{\alpha}^{j_s}=\boldsymbol{\varepsilon}$.
(2) For each column in $C:\beta^{j,h}$, there exists a type i, for which

$$\beta_i^{j,h}\leqq \min_{1\leqq s\leqq k}\beta_i^{j_s,h_s}.$$

(type i does not prefer $\beta^{j,h}$ to any of the allocations β^{j_s,h_s}; note that because $\boldsymbol{\alpha}^{j_1},\ldots,\boldsymbol{\alpha}^{j_k}$ is balanced the minimum of the right-hand side is never K).

By Lemma 1, there is a partition of $T:R^1,R^2,\ldots,R^k$ with $\boldsymbol{\alpha}(R^j)=\boldsymbol{\alpha}^{j_s}$, and an R^s-allocation realizing the utility allocation β^{j_s,h_s}. This partition-allocation is not blocked by any of the permissible coalition-allocations in G^r other than (perhaps) the grand coalition. By Lemma 2 there exists a partition-allocation $S^1,\ldots,S^l,\boldsymbol{b}^1,\ldots,\boldsymbol{b}^l$, with $l\leqq m$, which dominates $R^1,\ldots,R^k$ and is not blocked by any coalition in G^r. $S^1,\ldots,S^l,\boldsymbol{b}^1,\ldots,\boldsymbol{b}^l$ is a stable partition-allocation for G^r.

It remains to prove that the existence of a stable partition-allocation for the limit game, where all coalitions are permissible.

Consider the stable partition-allocation for $G^r:S^{r,1},\ldots,S^{r,l_r},\boldsymbol{b}^{r,1},\ldots,\boldsymbol{b}^{r,l_r}$, $l_r\leqq m$. First we observe that without loss of generality we may have that for all $r:l_r\equiv l$, then by diagonalizing and taking a subsequence we may assume that $\{\boldsymbol{\alpha}(S^{r,h})\}_r$ converges for all $h\leqq l_r\equiv l\leqq m$. Let $\boldsymbol{\alpha}^h=\lim_r\boldsymbol{\alpha}(S^{r,h})$, since $\{S^{r,h}\}_h$ is a partition it follows that $\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^l$ represents a partition $S^1,\ldots,S^l$, with $\boldsymbol{\alpha}(S^h)=\boldsymbol{\alpha}^h$, since $\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^l$ is a balanced set. By further diagonalizing we ensure that $\{\boldsymbol{b}^{r,h}\}_r$ converges for all $h\leqq l$, let $\boldsymbol{b}^h=\lim_r\boldsymbol{b}^{r,h}$. $\boldsymbol{b}^h$ is an allocation for a coalition whose composition is $\boldsymbol{\alpha}^h$. $S^1,\ldots,S^l,\boldsymbol{b}^1,\ldots,\boldsymbol{b}^l$ is a stable partition-allocation. For, if it is blocked by $S,\boldsymbol{b}$ then for a sufficiently large $r:S^{r,1},\ldots,S^{r,l_r},\boldsymbol{b}^{r,1},\ldots,b^{r,l_r}$ will be blocked by $S, \boldsymbol{b}$ since $S^1, \ldots, S^l, \boldsymbol{b}^1, \ldots, \boldsymbol{b}^l$ is the limit of the partition-allocations for G^r. $S, \boldsymbol{b}$ may not be permissible in G^r, but for a sufficiently large r we can have permissible partition-allocations arbitrarily close to $S, \boldsymbol{b}$. Hence for a sufficiently large $r: S^{r,1}, \ldots, S^{r,l_r}, \boldsymbol{b}^{r,1}, \ldots, \boldsymbol{b}^{r,l_r}$ is not a stable partition-allocation for G^r, contrary to our construction. It therefore follows that there exists a stable partition-allocation for the general game. Q.E.D.

3. An example[2]

One of the fields which this model may be applied to is urban economics, where the question of neighbourhoods and their composition arises naturally.

We construct here a simple example, from which it is possible to derive the structure and change of neighbourhoods during industrial growth. This example is of course far too simple to describe a realistic situation. It indicates, however, how this model can be used. At the end of this section some further refinements are suggested.

Let there be three types of individuals: rich, poor and industrialists, the last category represents both industries and small businesses. Coalitions are represented on the triangle *pri* (fig. 1). The point p represents a coalition consisting of poor only (similarly for r,i). Individuals have no private goods and they derive their utility only from the coalition they are in. All types have indifference curves which are circles around their bliss point. The rich and the poor have identical preferences with a bliss point at $R=P$, and the industrialists have a bliss point at I. Thus industries prefer to be located in a poor area where they can find workers, whereas the rich and the poor prefer a neighbourhood with a large proportion of rich but with some poor in it as well as some shops.

The point i_r represents the coalition most preferred by industrialists among the coalitions without the poor (similarly for r_i, p_i and $r_p = p_r$). A and B are points on the interval RI s.t. industrialists are indifferent between A and i_r and the rich (poor) are indifferent between B and r_p. C is the intersection of Br and Rr_p.

The triangle is now divided into two categories of triangles:

(1) pr_pI, p_ir_pB, r_pBC, r_prC, p_iiR, ir_iR, i_rrA.

(The last one overlaps two of this category and one of the other.)

(2) p_ir_pI, p_iRB, rAB.

If the grand coalition is one of the triangles of the first category then a stable partition is a partition into the vertices of this triangle. The relative sizes of

[2] I am very grateful to John Sutton for suggesting this example.

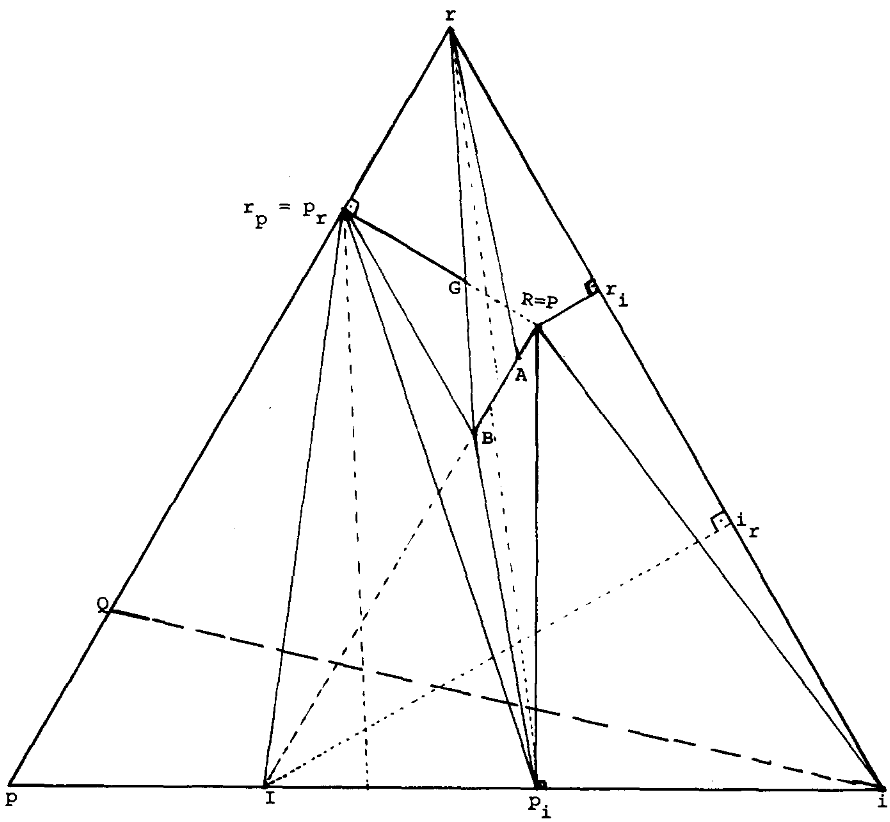

Fig. 1

the coalitions are determined by the exact position of the grand coalition in the triangle.

If the grand coalition is in one of the triangles of the second category a stable partition consists of two coalitions indicated by the lines in fig. 1. In $p_i r_p I$, for example, the partition consists of p_i and an appropriate coalition on the edge $\bar{I}p_i$. The proof that these partitions are indeed stable is somewhat cumbersome (but simple) and therefore omitted here.

Imagine now changing the structure of society (the grand coalition) as we move along the interval $\bar{Q}i$ toward i. This represents industrial growth where the ratio of rich to the poor remains fixed. When there is little industrialization we find three neighbourhoods: one with poor only, the other a mixture of the poor and all of the industry, with the poor being the majority in this coalition, and a coalition of the rich and poor. With industrial growth the poor evacuate the slums and move into industrial areas (triangle $p_i p_p I$: two coalitions). Further growth moves some of the businesses to areas with rich individuals and the poor industrial areas become more industrialized. As we move into p_iRB the neighbourhood of rich and poor with no businesses disappears and instead we get a more favourable (for the rich and poor) mixture of the three. In the last stage a new purely industrial area has developed and the proportion of industry in the other two areas (P_i and R) is optimal for the rich and poor.

In this example we have not used the other feature of the model: the private good. Allowing one good (money) to enter, enables individuals to bribe others in order to form a coalition with them. Thus in the triangle p_iiR the three vertices do not necessarily represent a stable partition anymore. If the industrialists have enough money they could bribe some of the poor to move towards I which is more favoured by the industrialists. An example which allows for that may attempt to describe the formation of certain neighbourhoods as well as explain various phenomena like the housing supplied by the industry for their workers, and the high prices of basic products and domestic services in rich areas.

4. Proofs of the lemmas

Lemma 1. Take a partition of T^i (the set of individuals of type i in T) $R^{i,1},\dots,R^{i,k}$ s.t. $\mu(R^{i,h})=\delta_h\alpha_i^h$; since $\sum\delta_h\alpha_i^h=\varepsilon_i=\mu(\mathrm{T}^i)$ this can be done.

Define $R^h=\bigcup_{i\leqq m}R^{i,h}$, clearly $\boldsymbol{\alpha}(R^h)=\boldsymbol{\alpha}^h$. Q.E.D.

Lemma 2. The proof consists of two parts, the first proves the existence of a dominating partition-allocation whose length is bounded by m, and the second shows the existence of a partition-allocation which is not dominated by the grand coalition.

We show first that, if $k>m$, then the partition-allocation can be shortened to a partition-allocation with a higher $\boldsymbol{u}$.

$\boldsymbol{\alpha}(\bigcup_{h\leqq k}R^h)=\boldsymbol{\alpha}(T)=\boldsymbol{\varepsilon}$ is a convex combination of $\boldsymbol{\alpha}(R^1),\dots,\boldsymbol{\alpha}(R^k)$. By Caratheodory's theorem, if $k>m$, $\boldsymbol{\varepsilon}$ is a convex combination of at most m vectors chosen from $\boldsymbol{\alpha}(R^1),\dots,\boldsymbol{\alpha}(R^k)$. It is therefore possible to choose a partition of T to at most m coalitions, each of which has a composition of types identical to one of $\boldsymbol{\alpha}(R^1),\dots,\boldsymbol{\alpha}(R^k)$ with the corresponding allocation. Thus we have shortened the partition while the corresponding $\boldsymbol{u}$ did not decrease. Let this partition-allocation be $S^1,\dots,S^l,\boldsymbol{b}^1,\dots,\boldsymbol{b}^l$ with $l\leqq m$. If it is blocked by some $T,\boldsymbol{a}$ then all we need to show is that there exists a maximal grand coalition-allocation which dominates $T,\boldsymbol{a}$. This is trivially satisfied, since a grand coalition-allocation is an $m\times n$ vector chosen from a compact set. This maximal grand coalition-allocation dominates $R^1,\dots,R^k,\boldsymbol{a}^1,\dots,\boldsymbol{a}^k$ and is not blocked by the grand coalition. Q.E.D.

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