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Non-existence of Equilibrium for the Two-dimensional Three-firms Location Problem ¹

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Given a distribution of customers and a location of κ firms in the plane, assume that each customer buys one unit from the nearest firm and that the revenue of a firm is proportional to the number of customers it has. Underlying these assumptions is the assumption that the customer bears the transport costs and that these are determined by a monotonic function of the distance.

A location of κ firms is an equilibrium if no firm can change its location so as to increase its revenue while the other $\kappa - 1$ firms are kept fixed (Nash equilibrium).

B. C. Eaton and R. G. Lipsey [1] conjecture that no such equilibrium exists for $\kappa \geq 3$. This note proves that the conjecture is true for $\kappa = 3$, for a continuous and connected customer distribution.

Assumptions

$f(x, y)$ or $g(\rho, \theta)$ will denote the distribution of customers.

- (A.1) $f \neq 0$ is continuous all over the plane or in a domain whose boundary does not contain any linear segments.
- (A.2) Every two points for which $f \neq 0$ can be connected by a "thick" curve for whose points $f \neq 0$ (connectedness).

1. THE STRANGE NATURE OF EQUILIBRIUM

Consider an equilibrium location of the three firms. By the connectedness assumption, the three firms cannot be located on a line, nor can two of them be "paired" at one point because it will clearly pay a border firm or the non-"paired" one to move in the direction of the others.

Suppose, therefore, that the three firms are located at the points A, B, C . The plane is, then, divided into three regions by three lines m, n, p intersecting at O —the centre of the circle Z circumscribing ABC (Figure 1).

Note that the angle $\varepsilon = \pi - \alpha$, between m, n does not change as A moves along Z between B and C .

Keep B, C fixed and allow A to move along Z such that the new separating lines will be $m'n'$ (Figure 1). By doing this A loses the revenue M of the area between m, m' and gains N . Assuming we started at an equilibrium: $N \leq M$. Now, let A return to its original location and allow C to move along Z so that it loses the revenue N and gains P , and repeat the same experiment with B . Summarizing the results we get:

$$M \leq P \leq N \leq M,$$

¹ First version received June 1973; final version accepted February 1974 (Eds.).

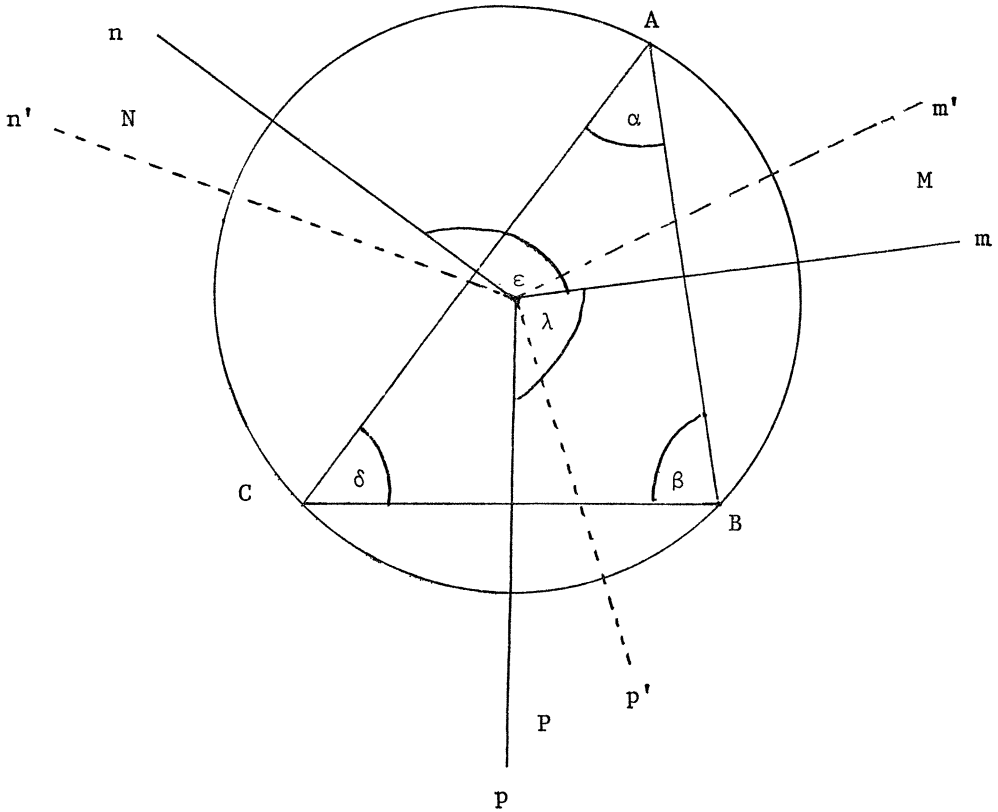


FIGURE 1

which implies:

$$M = P = N.$$

That is, *A* will maintain its maximal revenue level by moving along *Z* (keeping *B*, *C* fixed and not allowing *A* to cross over *B* or *C*). The same holds, of course, for *B*, *C*. (No continuity of the distribution function is needed.)

Note that *A*, *B*, *C* can all move a fixed distance in one direction along *Z* and thereby keep their revenues unchanged but the new location is not necessarily a Nash equilibrium.

Using the polar representation of the density function $g(\rho, \theta)$ with 0 as the origin and *p* as the pole, *A*'s revenue is (taking the revenue to be the number of customers):

$$F(\lambda) = \int_{\lambda}^{\lambda+\epsilon} \int_0^{\infty} \rho g(\rho, \theta) d\rho d\theta. \quad \dots(1)$$

$$\frac{\alpha}{2} \leq \lambda \leq \pi + \frac{\alpha}{2},$$

where λ denotes the angle between *m* and *p* as *m* moves with *A*.

Since $F(\lambda)$ is a constant function for this range of λ :

$$F'(\lambda) = \int_0^{\infty} \rho g(\rho, \lambda + \epsilon) d\rho - \int_0^{\infty} \rho g(\rho, \lambda) d\rho = 0. \quad \dots(2)$$

This holds also for a customer distribution function defined and continuous on part of the plane only.

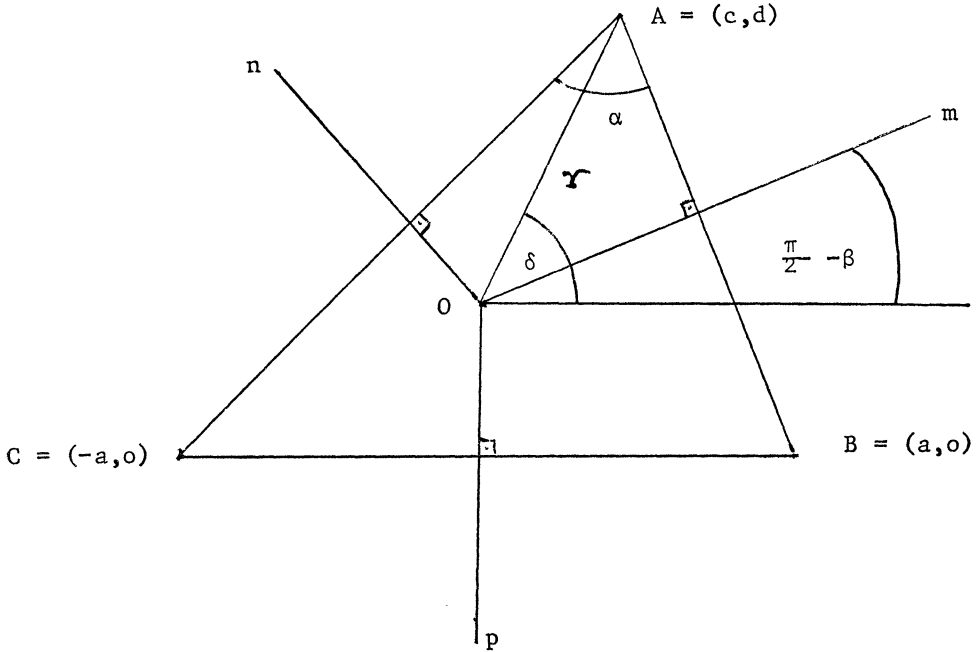


FIGURE 2

2. MOVING A OUTSIDE Z

Let BC be an x axis and p a y axis for the coordinate system in Figure 2.

Let B = (a, 0), C = (-a, 0) and A = (c, d).

The line separating A and B is:

$$y(x, c, d) = y(x) = \frac{d}{2} + \frac{c^2 - a^2}{2d} + \frac{(a - c)}{d} x \quad \dots(3)$$

and the line separating A and C is:

$$\bar{y}(x, c, d) = \bar{y}(x) = \frac{d}{2} + \frac{c^2 - a^2}{2d} - \frac{(a + c)}{d} x. \quad \dots(4)$$

A's revenue is:

$$F(c, d) = \int_0^\infty \int_{y(x)}^\infty f(x, y) dy dx + \int_{-\infty}^0 \int_{\bar{y}(x)}^\infty f(x, y) dy dx \quad \dots(5)$$

$$-F_c = \int_0^\infty f(x, y(x)) \frac{c-x}{d} dx + \int_{-\infty}^0 f(x, \bar{y}(x)) \frac{c-x}{d} dx. \quad \dots(6)$$

Since, by the argument of the first section, A maintains its maximal level of revenues for all points of Z between B, C: $F_c = 0$ for all (c, d) on that part of Z. Transferring equation (6) to polar coordinates with origin 0 and a pole parallel to BC (Substituting $x = d\rho/AB$, $x = -d\rho/AC$ in the two integrals respectively):

$$\begin{aligned} & \frac{c}{AB} \int_0^\infty g\left(\rho, \frac{\pi}{2} - \beta\right) d\rho - \frac{\sin \beta}{AB} \int_0^\infty \rho g\left(\rho, \frac{\pi}{2} - \beta\right) d\rho \\ & + \frac{c}{AC} \int_0^\infty g\left(\rho, \frac{\pi}{2} + \gamma\right) d\rho + \frac{\sin \gamma}{AC} \int_0^\infty \rho g\left(\rho, \frac{\pi}{2} + \gamma\right) d\rho = 0, \quad \dots(7) \end{aligned}$$

where AB is the length of the segment AB and $\pi/2 + \gamma = (\pi/2 - \beta) + \varepsilon$. ($\varepsilon = \pi - \alpha$, is the angle between m, n).

Denote:

$$r(\theta) = \frac{\int_0^\infty \rho g(\rho, \theta) d\rho}{\int_0^\infty g(\rho, \theta) d\rho} \quad \dots(8)$$

using that,

$$AB = 2r \sin \gamma$$

$$AC = 2r \sin \beta$$

$$\frac{c}{r} = \cos \delta, \quad \delta = \frac{\pi}{2} - (\beta - \gamma)$$

(r = radius of Z), and that by equation (2): $\int_0^\infty \rho g(\rho, \theta) d\rho$ is a periodic function of θ with period ε , we get:

$$\frac{\sin \alpha}{r} = \frac{\sin \beta}{r \left(\frac{\pi}{2} - \beta \right)} + \frac{\sin \gamma}{r \left(\frac{\pi}{2} + \gamma \right)}. \quad \dots(9)$$

To get this we divided by $\int \rho g d\rho$ but this cannot be zero as this implies that A 's portion of the plane is disconnected from B and C 's portions by two lines with no customers on them. This violates (A.2) unless A 's portion contains no customers at all in which case this location cannot be an equilibrium contrary to our assumption.

Since equation (9) holds for every position of A between B and C , reformulate it, taking as a parameter the angle θ between the right hand separating line and the axis through 0:

$$\left(\gamma = \frac{\pi}{2} - \alpha + \theta, \quad \beta = \frac{\pi}{2} - \theta \right)$$

$$\frac{\sin \alpha}{r} = \frac{\cos \theta}{r(\theta)} - \frac{\cos(\pi - \alpha + \theta)}{r(\pi - \alpha + \theta)} \quad \dots(10)$$

for

$$\alpha - \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

As similar formulas can be written for every pair of separating lines, we have the following three equations:

$$\frac{\sin \alpha}{r} = \frac{\cos \theta}{r(\theta)} - \frac{\cos(\pi - \alpha + \theta)}{r(\pi - \alpha + \theta)}$$

$$\frac{\sin \gamma}{\theta} = \frac{\cos(\beta - \alpha + \theta)}{r(\pi - \alpha + \theta)} - \frac{\cos(\pi - \alpha - \gamma + \beta + \theta)}{r(2\pi - \alpha - \gamma + \theta)} \quad \dots(11)$$

$$\frac{\sin \beta}{r} = \frac{\cos(\beta - \gamma + \theta)}{r(2\pi - \alpha - \gamma + \theta)} - \frac{\cos(\pi - \gamma + \theta)}{r(2\pi + \theta)}.$$

Note that the last two equations do not have the symmetry properties of the first because they are modified to measure θ starting from the axis parallel to BC .

Solving for $r(\theta)$ (for complete solution see appendix):

$$r(\theta) = \frac{r \sin(\theta + \beta)}{\cos \gamma} \quad \dots(12)$$

for

$$\alpha - \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

solving $r(\theta)$ for the rays between A and C (by moving C and taking an axis parallel to AB):

$$r(\bar{\theta}) = \frac{r \sin(\bar{\theta} + \alpha)}{\cos \beta} \quad \dots(13)$$

(θ is measured from a line parallel to BC whereas $\bar{\theta}$ is measured from a line parallel to AB).

These two functions coincide for points between A and C .

Set:

$$\theta_0 = \delta + \beta = \left(\frac{\pi}{2} - \beta + \gamma\right) + \beta = \frac{\pi}{2} + \gamma,$$

$$\bar{\theta}_0 = \left(\gamma - \frac{\pi}{2}\right) + \beta = \gamma + \beta - \frac{\pi}{2},$$

these will describe a point half-way between A and C . Equality of the two functions implies:

$$\frac{\sin(\gamma + \beta)}{\cos \gamma} = \frac{\sin(\alpha + \beta + \gamma)}{\cos \beta}$$

but $\alpha + \beta + \gamma = \pi$, hence:

$$\gamma + \beta = 0 \quad \text{or} \quad \gamma + \beta = \pi.$$

In either case, A , B , C are on a line and hence not an equilibrium. This completes the proof for $\kappa = 3$.

3. COMMENTS

1. If the firms bear the transport costs and the revenue from a given customer is a decreasing function $R(d)$ of his distance from the firm, the proof is not valid. I suspect that the existence of equilibrium depends on the relation between the customer distribution and $R(d)$. If this last function is rapidly decreasing there could be an equilibrium with the firms well apart: take for example the extreme case where a firm has positive revenue from customers within a given radius only.

2. Though the proof presents some interesting properties of a possible equilibrium, its main failure is that it cannot be extended to $\kappa > 3$.

REFERENCE

- [1] Eaton, B. C. and Lipsey, R. G. "The Principle of Minimum Differentiation Reconsidered: Some New Developments in Theory of Spatial Competition", *Review of Economic Studies* (this issue).

APPENDIX

Substitute $\bar{\alpha} = \pi - \alpha$, $\bar{\beta} = \pi - \beta$, $\bar{\gamma} = \pi - \gamma$ in equation (13) and eliminate $r(\theta)$:

$$(a) \frac{1}{r} [\sin \bar{\alpha} \cos (\bar{\alpha} - \bar{\beta} + \theta) \cos (\bar{\gamma} - \bar{\beta} + \theta) + \sin \bar{\beta} \cos (\bar{\alpha} + \theta) \cos (\theta - 2\bar{\beta}) \\ + \sin \bar{\gamma} \cos (\bar{\alpha} + \theta) \cos (\bar{\gamma} - \bar{\beta} + \theta)] \\ = \frac{1}{r(\theta)} [\cos \theta \cos (\bar{\alpha} - \bar{\beta} + \theta) \cos (\bar{\gamma} + \bar{\beta} + \theta) - \cos (\bar{\gamma} + \theta) \cos (\bar{\alpha} + \theta) \cos (\theta - 2\bar{\beta})].$$

All the next equations are obtained by using the formulas for

$$(\sin A + \sin B) \text{ or } (\cos A + \cos B)$$

and the fact that $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 2\pi$.

Denote the coefficient of $1/r(\theta)$ by Q :

$$(b) Q = \frac{\cos \theta}{2} [\cos (\bar{\alpha} + \bar{\gamma} - 2\bar{\beta} + 2\theta) + \cos (\bar{\alpha} - \bar{\gamma})] \\ - \frac{\cos (\theta - 2\bar{\beta})}{2} [\cos (\bar{\alpha} + \bar{\gamma} + 2\theta) + \cos (\bar{\alpha} - \bar{\gamma})]$$

or,

$$(c) Q = \frac{\cos (\bar{\alpha} - \bar{\gamma})}{2} [\cos \theta - \cos (\theta - 2\bar{\beta})] + \frac{1}{4} [\cos (\bar{\alpha} + \bar{\gamma} - 2\bar{\beta} + \theta) - \cos (\bar{\alpha} + \bar{\gamma} + \theta + 2\bar{\beta})],$$

$$(d) Q = \sin \bar{\beta} \sin (\theta - \bar{\beta}) [\cos \beta - \cos (\bar{\alpha} - \bar{\gamma})] \\ = -2 \sin \bar{\alpha} \sin \bar{\beta} \sin \bar{\gamma} \sin (\theta - \bar{\beta})$$

Denote the coefficient of $1/r$ by R .

$$(e) R = \frac{\sin \bar{\alpha}}{2} [\cos (\bar{\alpha} + \bar{\gamma} - 2\bar{\beta} + 2\theta) + \cos (\bar{\alpha} - \bar{\gamma})] \\ + \frac{\sin \bar{\beta}}{2} [\cos (\bar{\alpha} - 2\bar{\beta} + 2\theta) + \cos (\bar{\alpha} + 2\bar{\beta})] \\ + \frac{\sin \bar{\gamma}}{2} [\cos (\bar{\alpha} + \bar{\gamma} - \bar{\beta} + 2\theta) + \cos (\bar{\alpha} + \bar{\beta} - \bar{\gamma})].$$

Let R' be the sum of the elements of R free of θ :

$$(f) R' = \frac{1}{4} [\sin (2\bar{\alpha} - \bar{\gamma}) + \sin \bar{\gamma} + \sin (2\bar{\beta} - \bar{\gamma}) + \sin \bar{\gamma} + \sin 3\bar{\gamma} - \sin \bar{\gamma}].$$

$$(g) R' = \frac{1}{2} [\sin 2\bar{\gamma} \cos \bar{\gamma} - \sin 2\bar{\gamma} \cos (\bar{\alpha} - \bar{\beta})] \\ = -\sin 2\bar{\gamma} \sin \bar{\alpha} \sin \bar{\beta}.$$

The rest of R : R'' - all the expressions with θ (to simplify denote $\mu = \theta - \bar{\beta}$):

$$(h) R'' = \sin (2\mu + \bar{\alpha} - \bar{\beta}) + \sin (\bar{\alpha} + \bar{\beta} - 2\mu) + \sin (2\mu + \bar{\alpha} + \bar{\beta}) + \sin (\bar{\beta} - 2\mu - \bar{\alpha}) \\ + \sin (2\mu + \bar{\gamma}) + \sin (\bar{\gamma} - 2\mu) \equiv 0.$$

Hence,

$$(i) \frac{-\sin 2\bar{\gamma} \sin \bar{\alpha} \sin \bar{\beta}}{r} = \frac{-2 \sin \bar{\alpha} \sin \bar{\beta} \sin \bar{\gamma} \sin (\theta - \bar{\beta})}{r(\theta)}$$

$$(j) r(\theta) = \frac{r \sin (\theta - \bar{\beta})}{\cos \bar{\gamma}} = \frac{r \sin (\theta + \bar{\beta})}{\cos \bar{\gamma}}.$$

QED