Non-existence of Equilibrium for the Two-dimensional Three-firms Location Problem [1]

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Given a distribution of customers and a location of κ firms in the plane, assume that each customer buys one unit from the nearest firm and that the revenue of a firm is proportional to the number of customers it has. Underlying these assumptions is the assumption that the customer bears the transport costs and that these are determined by a monotonic function of the distance.

A location of κ firms is an equilibrium if no firm can change its location so as to increase its revenue while the other $\kappa-1$ firms are kept fixed (Nash equilibrium).

B. C. Eaton and R. G. Lipsey [1] conjecture that no such equilibrium exists for $\kappa \geqq 3$. This note proves that the conjecture is true for $\kappa = 3$, for a continuous and connected customer distribution.

Assumptions

$f(x, y)$ or $g(\rho, \theta)$ will denote the distribution of customers.

(A.1) $f \not\equiv 0$ is continuous all over the plane or in a domain whose boundary does not contain any linear segments.

(A.2) Every two points for which $f \neq 0$ can be connected by a " thick " curve for whose points $f \neq 0$ (connectedness).

1. THE STRANGE NATURE OF EQUILIBRIUM

Consider an equilibrium location of the three firms. By the connectedness assumption, the three firms cannot be located on a line, nor can two of them be " paired " at one point because it will clearly pay a border firm or the non-" paired " one to move in the direction of the others.

Suppose, therefore, that the three firms are located at the points A, B, C. The plane is, then, divided into three regions by three lines m, n, p intersecting at 0— the centre of the circle Z circumscribing ABC (Figure 1).

Note that the angle $\varepsilon = \pi-\alpha$, between m, n does not change as A moves along Z between B and C.

Keep B, C fixed and allow A to move along Z such that the new separating lines will be $m'n'$ (Figure 1). By doing this A loses the revenue M of the area between m, m' and gains N. Assuming we started at an equilibrium: $N \leqq M$. Now, let A return to its original location and allow C to move along Z so that it loses the revenue N and gains P, and repeat the same experiment with B. Summarizing the results we get:

$$M \leqq P \leqq N \leqq M,$$

[1] *First version received June* 1973; *final version accepted February* 1974 (*Eds.*).

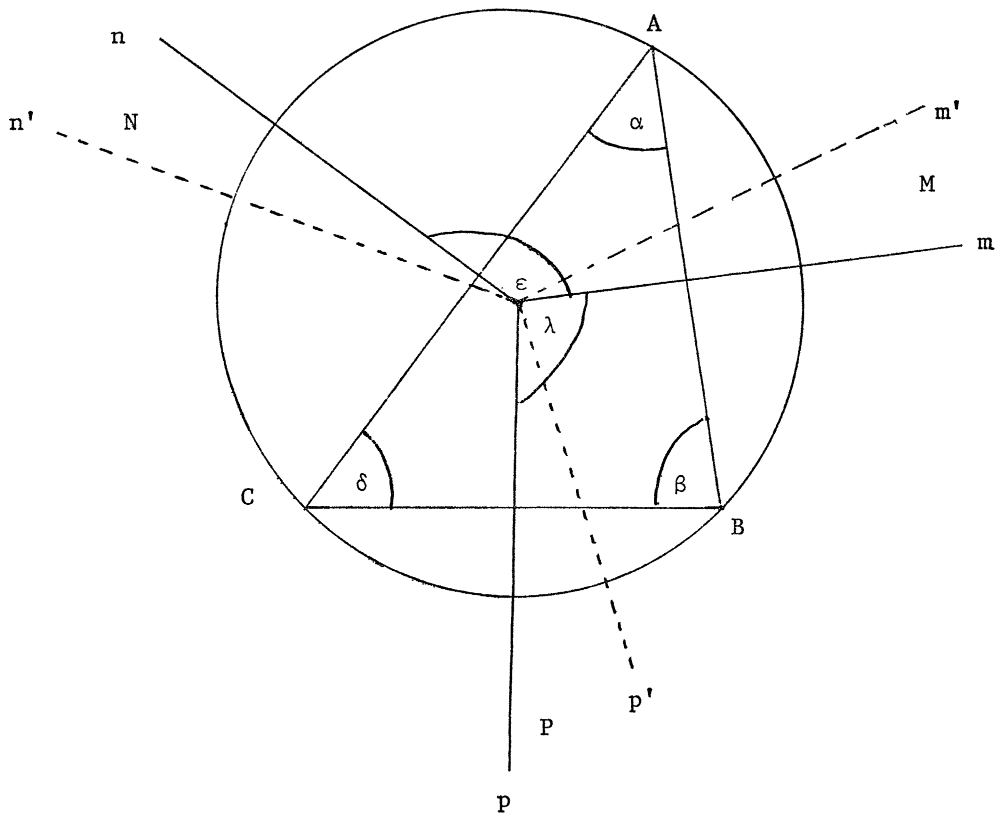

FIGURE 1

which implies:

$$M = P = N.$$

That is, A will maintain its maximal revenue level by moving along Z (keeping B, C fixed and not allowing A to cross over B or C). The same holds, of course, for B, C. (No continuity of the distribution function is needed.)

Note that A, B, C can all move a fixed distance in one direction along Z and thereby keep their revenues unchanged but the new location is not necessarily a Nash equilibrium.

Using the polar representation of the density function $g(\rho, \theta)$ with 0 as the origin and p as the pole, A's revenue is (taking the revenue to be the number of customers):

$$F(\lambda) = \int_{\lambda}^{\lambda+\varepsilon} \int_{0}^{\infty} \rho g(\rho, \theta) d\rho d\theta. \qquad \text{...(1)}$$

$$\frac{\alpha}{2} \leqq \lambda \leqq \pi + \frac{\alpha}{2},$$

where λ denotes the angle between m and p as m moves with A.

Since $F(\lambda)$ is a constant function for this range of λ:

$$F'(\lambda) = \int_{0}^{\infty} \rho g(\rho, \lambda+\varepsilon) d\rho - \int_{0}^{\infty} \rho g(\rho, \lambda) d\rho = 0. \qquad \text{...(2)}$$

This holds also for a customer distribution function defined and continuous on part of the plane only.

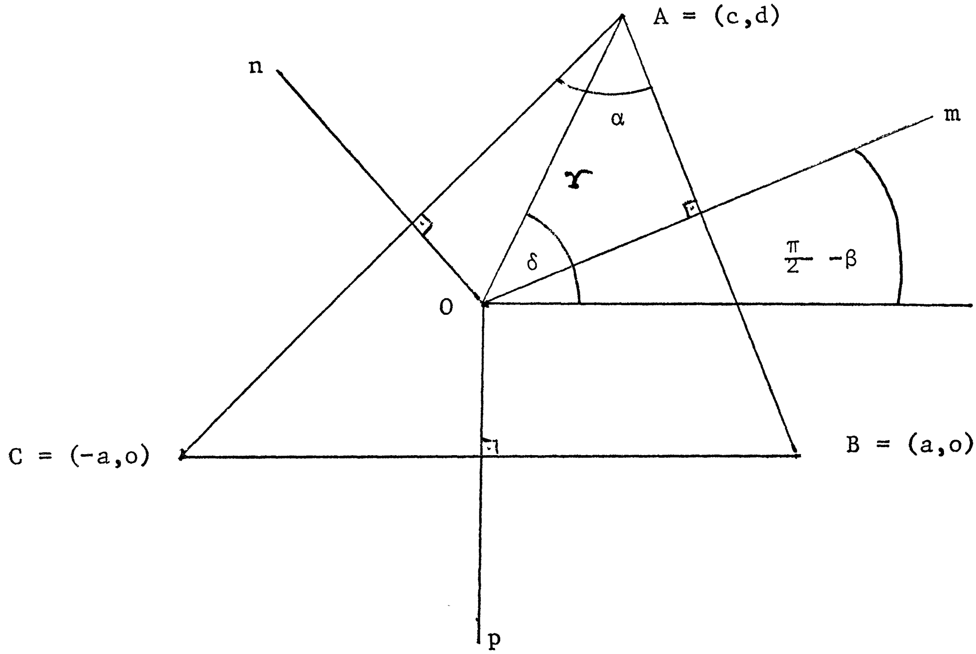

FIGURE 2

2. MOVING A OUTSIDE Z

Let BC be an x axis and p a y axis for the coordinate system in Figure 2.

Let $B = (a, 0)$, $C = (-a, 0)$ and $A = (c, d)$.

The line separating A and B is:

$$y(x, c, d) = y(x) = \frac{d}{2} + \frac{c^2 - a^2}{2d} + \frac{(a-c)}{d} x \qquad \text{...(3)}$$

and the line separating A and C is:

$$\bar{y}(x, c, d) = \bar{y}(x) = \frac{d}{2} + \frac{c^2 - a^2}{2d} - \frac{(a+c)}{d} x. \qquad \text{...(4)}$$

A's revenue is:

$$F(c, d) = \int_0^\infty \int_{y(x)}^\infty f(x, y) dy dx + \int_{-\infty}^0 \int_{\bar{y}(x)}^\infty f(x, y) dy dx \qquad \text{...(5)}$$

$$-F_c = \int_0^\infty f(x, y(x)) \frac{c-x}{d} dx + \int_{-\infty}^0 f(x, \bar{y}(x)) \frac{c-x}{d} dx. \qquad \text{...(6)}$$

Since, by the argument of the first section, A maintains its maximal level of revenues for all points of Z between B, C: $F_c = 0$ for all (c, d) on that part of Z. Transferring equation (6) to polar coordinates with origin 0 and a pole parallel to BC (Substituting $x = d\rho/AB$, $x = -d\rho/AC$ in the two integrals respectively):

$$\frac{c}{AB} \int_0^\infty g\left(\rho, \frac{\pi}{2} - \beta\right) d\rho - \frac{\sin \beta}{AB} \int_0^\infty \rho g\left(\rho, \frac{\pi}{2} - \beta\right) d\rho$$
$$+ \frac{c}{AC} \int_0^\infty g\left(\rho, \frac{\pi}{2} + \gamma\right) d\rho + \frac{\sin \gamma}{AC} \int_0^\infty \rho g\left(\rho, \frac{\pi}{2} + \gamma\right) d\rho = 0, \qquad \text{...(7)}$$

where AB is the length of the segment AB and $\pi/2+\gamma = (\pi/2-\beta)+\varepsilon$. ($\varepsilon = \pi-\alpha$, is the angle between m, n).

Denote:

$$r(\theta) = \frac{\int_0^\infty \rho g(\rho, \theta) d\rho}{\int_0^\infty g(\rho, \theta) d\rho} \qquad \text{...(8)}$$

using that,

$$AB = 2r \sin \gamma$$

$$AC = 2r \sin \beta$$

$$\frac{c}{r} = \cos \delta,\ \delta = \frac{\pi}{2} - (\beta-\gamma)$$

(r = radius of Z), and that by equation (2): $\int_0^\infty \rho g(\rho, \theta) d\rho$ is a periodic function of θ with period ε, we get:

$$\frac{\sin \alpha}{r} = \frac{\sin \beta}{r\left(\frac{\pi}{2} - \beta\right)} + \frac{\sin \gamma}{r\left(\frac{\pi}{2} + \gamma\right)}. \qquad \text{...(9)}$$

To get this we divided by $\int \rho g d\rho$ but this cannot be zero as this implies that A's portion of the plane is disconnected from B and C's portions by two lines with no customers on them. This violates (A.2) unless A's portion contains no customers at all in which case this location cannot be an equilibrium contrary to our assumption.

Since equation (9) holds for every position of A between B and C, reformulate it, taking as a parameter the angle θ between the right hand separating line and the axis through 0:

$$\left(\gamma = \frac{\pi}{2} - \alpha + \theta,\ \beta = \frac{\pi}{2} - \theta\right)$$

$$\frac{\sin \alpha}{r} = \frac{\cos \theta}{r(\theta)} - \frac{\cos(\pi-\alpha+\theta)}{r(\pi-\alpha+\theta)} \qquad \text{...(10)}$$

for

$$\alpha - \frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}.$$

As similar formulas can be written for every pair of separating lines, we have the following three equations:

$$\frac{\sin \alpha}{r} = \frac{\cos \theta}{r(\theta)} - \frac{\cos(\pi-\alpha+\theta)}{r(\pi-\alpha+\theta)}$$

$$\frac{\sin \gamma}{\theta} = \frac{\cos(\beta-\alpha+\theta)}{r(\pi-\alpha+\theta)} - \frac{\cos(\pi-\alpha-\gamma+\beta+\theta)}{r(2\pi-\alpha-\gamma+\theta)} \qquad \text{...(11)}$$

$$\frac{\sin \beta}{r} = \frac{\cos(\beta-\gamma+\theta)}{r(2\pi-\alpha-\gamma+\theta)} - \frac{\cos(\pi-\gamma+\theta)}{r(2\pi+\theta)}.$$

Note that the last two equations do not have the symmetry properties of the first because they are modified to measure θ starting from the axis parallel to BC.

Solving for $r(\theta)$ (for complete solution see appendix):

$$r(\theta) = \frac{r \sin(\theta+\beta)}{\cos \gamma} \qquad \ldots(12)$$

for

$$\alpha - \frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2},$$

solving $r(\theta)$ for the rays between A and C (by moving C and taking an axis paralel to AB):

$$r(\bar{\theta}) = \frac{r \sin(\bar{\theta}+\alpha)}{\cos \beta} \qquad \ldots(13)$$

(θ is measured from a line parallel to BC whereas $\bar{\theta}$ is measured from a line parallel to AB).

These two functions coincide for points between A and C.

Set:

$$\theta_0 = \delta+\beta = \left(\frac{\pi}{2} - \beta+\gamma\right) + \beta = \frac{\pi}{2} + \gamma,$$

$$\bar{\theta}_0 = \left(\gamma - \frac{\pi}{2}\right) + \beta = \gamma+\beta - \frac{\pi}{2},$$

these will describe a point half-way between A and C. Equality of the two functions implies:

$$\frac{\sin(\gamma+\beta)}{\cos \gamma} = \frac{\sin(\alpha+\beta+\gamma)}{\cos \beta}$$

but $\alpha+\beta+\gamma = \pi$, hence:

$$\gamma+\beta = 0 \quad \text{or} \quad \gamma+\beta = \pi.$$

In either case, A, B, C are on a line and hence not an equilibrium. This completes the proof for $\kappa = 3$.

3. COMMENTS

1. If the firms bear the transport costs and the revenue from a given customer is a decreasing function $R(d)$ of his distance from the firm, the proof is not valid. I suspect that the existence of equilibrium depends on the relation between the customer distribution and $R(d)$. If this last function is rapidly decreasing there could be an equilibrium with the firms well apart: take for example the extreme case where a firm has positive revenue from customers within a given radius only.

2. Though the proof presents some interesting properties of a possible equilibrium, its main failure is that it cannot be extended to $\kappa > 3$.

REFERENCE

[1] Eaton, B. C. and Lipsey, R. G. " The Principle of Minimum Differentiation Reconsidered: Some New Developments in Theory of Spatial Competition ", *Review of Economic Studies* (this issue).

APPENDIX

Substitute $\bar{\alpha} = \pi - \alpha$, $\bar{\beta} = \pi - \beta$, $\bar{\gamma} = \pi - \gamma$ in equation (13) and eliminate $r(\theta)$:

$$(a)\ \frac{1}{r}[\sin\bar{\alpha}\cos(\bar{\alpha}-\bar{\beta}+\theta)\cos(\bar{\gamma}-\bar{\beta}+\theta)+\sin\bar{\beta}\cos(\bar{\alpha}+\theta)\cos(\theta-2\bar{\beta})$$
$$+\sin\bar{\gamma}\cos(\bar{\alpha}+\theta)\cos(\bar{\gamma}-\bar{\beta}+\theta)]$$
$$=\frac{1}{r(\theta)}[\cos\theta\cos(\bar{\alpha}-\bar{\beta}+\theta)\cos(\bar{\gamma}+\bar{\beta}+\theta)-\cos(\bar{\gamma}+\theta)\cos(\bar{\alpha}+\theta)\cos(\theta-2\bar{\beta})].$$

All the next equations are obtained by using the formulas for

$$(\sin A+\sin B) \text{ or } (\cos A+\cos B)$$

and the fact that $\bar{\alpha}+\bar{\beta}+\bar{\gamma} = 2\pi$.

Denote the coefficient of $1/r(\theta)$ by Q:

$$(b)\ Q=\frac{\cos\theta}{2}[\cos(\bar{\alpha}+\bar{\gamma}-2\bar{\beta}+2\theta)+\cos(\bar{\alpha}-\bar{\gamma})]$$
$$-\frac{\cos(\theta-2\bar{\beta})}{2}[\cos(\bar{\alpha}+\bar{\gamma}+2\theta)+\cos(\bar{\alpha}-\bar{\gamma})]$$

or,

$$(c)\ Q=\frac{\cos(\bar{\alpha}-\bar{\gamma})}{2}[\cos\theta-\cos(\theta-2\bar{\beta})]+\tfrac{1}{4}[\cos(\bar{\alpha}+\bar{\gamma}-2\bar{\beta}+\theta)-\cos(\bar{\alpha}+\bar{\gamma}+\theta+2\bar{\beta})],$$

$$(d)\ Q=\sin\bar{\beta}\sin(\theta-\bar{\beta})[\cos\beta-\cos(\bar{\alpha}-\bar{\gamma})]$$
$$=-2\sin\bar{\alpha}\sin\bar{\beta}\sin\bar{\gamma}\sin(\theta-\bar{\beta})$$

Denote the coefficient of $1/r$ by R.

$$(e)\ R=\frac{\sin\bar{\alpha}}{2}[\cos(\bar{\alpha}+\bar{\gamma}-2\bar{\beta}+2\theta)+\cos(\bar{\alpha}-\bar{\gamma})]$$
$$+\frac{\sin\bar{\beta}}{2}[\cos(\bar{\alpha}-2\bar{\beta}+2\theta)+\cos(\bar{\alpha}+2\bar{\beta})]$$
$$+\frac{\sin\bar{\gamma}}{2}[\cos(\bar{\alpha}+\bar{\gamma}-\bar{\beta}+2\theta)+\cos(\bar{\alpha}+\bar{\beta}-\bar{\gamma})].$$

Let R' be the sum of the elements of R free of θ:

$$(f)\ R'=\tfrac{1}{4}[\sin(2\bar{\alpha}-\bar{\gamma})+\sin\bar{\gamma}+\sin(2\bar{\beta}-\bar{\gamma})+\sin\bar{\gamma}+\sin 3\bar{\gamma}-\sin\bar{\gamma}].$$

$$(g)\ R'=\tfrac{1}{2}[\sin 2\bar{\gamma}\cos\bar{\gamma}-\sin 2\bar{\gamma}\cos(\bar{\alpha}-\bar{\beta})]$$
$$=-\sin 2\bar{\gamma}\sin\bar{\alpha}\sin\bar{\beta}.$$

The rest of R: R'' — all the expressions with θ (to simplify denote $\mu = \theta-\bar{\beta}$):

$$(h)\ R''=\sin(2\mu+\bar{\alpha}-\bar{\beta})+\sin(\bar{\alpha}+\bar{\beta}-2\mu)+\sin(2\mu+\bar{\alpha}+\bar{\beta})+\sin(\bar{\beta}-2\mu-\bar{\alpha})$$
$$+\sin(2\mu+\bar{\gamma})+\sin(\bar{\gamma}-2\mu)\equiv 0.$$

Hence,

$$(i)\ \frac{-\sin 2\bar{\gamma}\sin\bar{\alpha}\sin\bar{\beta}}{r}=\frac{-2\sin\bar{\alpha}\sin\bar{\beta}\sin\bar{\gamma}\sin(\theta-\bar{\beta})}{r(\theta)}$$

$$(j)\ r(\theta)=\frac{r\sin(\theta-\bar{\beta})}{\cos\bar{\gamma}}=\frac{r\sin(\theta+\beta)}{\cos\gamma}.$$

QED