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Existence of a Non-Stationary Equilibrium in Search-And-Matching Models: TU and NTU

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Existence of a Non-Stationary Equilibrium in Search-and-Matching Models: TU and NTU*

Christopher Sandmann† and Nicolas Bonneton‡

Abstract

This paper proves the existence of a non-stationary equilibrium in the canonical search-and-matching model with heterogeneous agents. Non-stationarity entails that the number and characteristics of unmatched agents evolve endogenously over time. An equilibrium exists under minimal regularity conditions and for both paradigms considered in the literature: transferable and non-transferable utility. To address potential discontinuities in match opportunities across types, our analysis introduces a generalized Schauder fixed-point theorem suitable for models with discontinuous value functions.

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1 Introduction

This paper builds tools for proving the existence of a non-stationary equilibrium in dynamic heterogeneous agent models, where the aggregate state evolves deterministically over time. Our focus is the canonical search-and-matching model. This model has been widely used to study productive and social interactions.[1] As in the pioneering work by Shimer and Smith (2000), a continuum of heterogeneous agents engage in a time-consuming and haphazard search for one another and exit the search pool upon forming a match. Following the two dominant paradigms in the literature, match payoffs can be transferable (TU), i.e., there is Nash bargaining over match surplus, or non-transferable (NTU), i.e., match payoffs are exogenously given. In this model, fluctuations arise naturally, e.g., due to a seasonal thick market externality, gradual market clearing or the business cycle. Known equilibrium existence results, with the exception of Manea (2017a), apply however only in the stylized stationary environment where entry and exit into the search pool are balanced at all moments in time (Burdett and Coles (1997), Shimer and Smith (2000), Smith (2006), Lauermann et al. (2020)).[2,3]

A non-stationary equilibrium resolves a complex feedback loop between a time-moving aggregate state and individual decisions. In the search-and-matching economy, the endogenous variables are: the distribution of agents' characteristics in the search pool, agents' value-of-search, and thereby determined matching decisions and transfers. Aggregate population dynamics and the individual decision problem are coupled; when the search pool evolves and therefore, future match prospects evolve, so do optimal matching decisions, and hence the rate at which agents exit the search pool. The interplay between aggregate dynamics and the individual decision problem is shared with virtually all dynamic general equilibrium models under rational expectations. Lasry and Lions (2007) refer to this class of models as mean field games.

We prove equilibrium existence in three steps. As in general equilibrium theory, existence will depend on the application of a topological fixed point theorem. In Section 2, we establish a non-trivial adaptation of the Schauder (1930) fixed point theorem, which imposes few constraints on the model. This theorem translates abstract concepts, notably compactness in function spaces, into premises that can be interpreted economically. In Section 5.1, we construct a value-of-search operator whose fixed points correspond to a non-stationary equilibrium. In

[1] Notable applications include sorting in the labor and the marriage market (see Chade et al. (2017) for a review) and foundations of Walrasian equilibrium (Rubinstein and Wolinsky (1985), Gale (1987), Lauermann (2013)).

[2] Relatedly, Lauermann and Nöldeke (2015) and Manea (2017b) prove the existence of a stationary equilibrium when there are finitely many types.

[3] Manea (2017a) proves existence in the non-stationary TU (but not NTU) search-and-matching model when there are finitely many types and time is discrete. The present paper deals with the continuum (in both the TU and NTU paradigm).

Sections 5.2 and 5.3, we prove that the operator satisfies the premises of our fixed point theorem. To that end, we construct bounds on the value-of-search across individuals that are derived from two revealed preferences arguments.

We first establish a fixed point theorem (Theorem 1). Due to its potential appeal to other models, we present it in a self-contained section. The domain of this fixed point theorem is the space of tuples $(F^1, ..., F^N) \in \mathscr{F}^N$ of measurable mappings $F^n : [0,1] \times \mathbb{R}_+$ endowed with a semimetric. In search-and-matching models, $N = 2$ is the number of populations, e.g., workers and firms, and $F^n(x,t)$ is the value-of-search of agent type x from population n at time t. We prove that *an operator $H : \mathscr{F}^N \to \mathscr{F}^N$ admits a fixed point if it is (i) continuous with respect to the seminorm, and (ii) maps into a function space whose (two-dimensional) total variation norm is uniformly bounded.* Premise (i) is the familiar continuity premise from Schauder. Both premises are sufficiently general to allow the value function to fluctuate endogenously over time and to be discontinuous with respect to time and type.[4]

The key step in proving our fixed point theorem is the construction of a sequence of approximating fixed point operators. By mapping any given value function profile into a smaller function space, the space of k-Lipschitz functions, the approximating fixed point operator is guaranteed to be compact-valued. Since the operator is also continuous due to Premise (i), Schauder's theorem guarantees the existence of a fixed point. This fixed point corresponds to an approximate equilibrium with vanishing approximating error as we increase the constant k. We then prove that the sequence of approximating equilibria converges. This is the consequence of a generalized multidimensional (time and type) Helly selection theorem which establishes that Premise (ii) implies sequential compactness of $H(\mathscr{F}^N)$.[5]

Second, we construct a value-of-search operator whose fixed points correspond to a non-stationary equilibrium. Under this operator, agents take others' value-of-search, hence matching decisions, as given to compute their own discounted expected future match payoff. The operator can be interpreted as the out-of-equilibrium value-of-search in that the value-of-search ascribed to other agents of equal type need not coincide with their own.

Third, we prove that the operator satisfies the premises of our fixed point theorem: continuity and uniformly bounded variation. This holds true for general primitives of the economy.

[4]In NTU search-and-matching models, it is known that block segregation (see McNamara and Collins (1990), Smith (2006) and references therein) prohibits continuity of the value-of-search: agents can cluster according to classes so that any two agents match upon meeting if and only if they belong to the same class. Similarly, discontinuities in the value function across types arise naturally, absent prices, under informational asymmetries. For example, Bardhi et al. (2023) show that, when employers learn about workers' skills by observing "bad news", workers who are almost equal ex-ante have very different expected career paths.

[5]Relatedly, Smith (2006) makes use of the Helly selection theorem in dimension one (type) to establish sequential compactness of the value function space.

The central assumptions are Lipschitz continuity and linear boundedness of entry and meeting rates. In particular, we allow both rates to depend generally on time and the current size and composition of the search pool, which relaxes assumptions considered in the literature.

We circumvent the tractability issues that come with non-stationary dynamics by constructing tight bounds on the difference in the value-of-search between two agents. Those bounds follow from two revealed preference arguments (NTU and TU) coined *mimicking arguments* whose underlying idea is to let one agent replicate someone else's matching decisions. In the TU paradigm, we establish bounds in terms of time-invariant output rather than time-varying payoffs by employing an inductive reasoning over the mimicking argument. These bounds are also key to studying sorting in non-stationary equilibrium (see Bonneton and Sandmann (2023) (NTU), and Bonneton and Sandmann (2021) (TU)).

Related Work This paper contributes to the theoretical literature on search and matching, see Chade et al. (2017) for an excellent review.

With the exception of Manea (2017a), all equilibrium existence results derive conditions on the primitives of the model for which a *stationary* equilibrium exists (Burdett and Coles (1997), Shimer and Smith (2000), Smith (2006), Lauermann and Nöldeke (2015), Manea (2017b) and Lauermann et al. (2020)). Many economic phenomena, however, are inherently non-stationary,[6] including time-variant entry as in a seasonal housing market (see Ngai and Tenreyro (2014)), and a gradually clearing job market (by which, e.g., academic economists have organized the junior job market for Ph.D. hires).

Manea (2017a) proves equilibrium existence in the TU non-stationary search-and-matching model when there are finitely many types and time is discrete. In line with the literature on assortative matching, we consider continuous time and a continuum of types. Continuum-type models ease the analysis of sorting (Chade et al. (2017)) and continuous-time models avoid pathological coordination failures across periods with instantaneous first-round exit by all agents (as reported in Damiano et al. (2005)). One reason to focus on finite-type models is technical simplicity; Tychonoff guarantees the compactness of the (countably finite) equilibrium domain. Our proof, notably the herein developed mimicking argument, reveals that the TU paradigm with a continuum of types poses no additional conceptual difficulties (cf. Remark 2).

[6] One critical insight is that aggregate fluctuations can amplify idiosyncratic risk. In a non-stationary Aiyagari (1994) model (see Achdou et al. (2022)), a looming rise in interest rates makes consecutive negative income shocks more costly, contributing to greater precautionary savings. In a growth model, the anticipation of future industry consolidation can dampen investment in long-run quality in favor of greater short-term intangible investment (see De Ridder (2024)). In a companion paper (see Bonneton and Sandmann (2023)), we show how the downside risk of future acceptance of an undesirable match in a depleted search pool can impede contemporaneous positive assortative matching.

In the NTU paradigm, by contrast, continuous model primitives alone do not guarantee that the equilibrium value-of-search is continuous in types. Herein introduced proof techniques allow us to establish equilibrium existence regardless and encompass discontinuous model primitives.

Questioning what happens outside the steady state is at the heart of burgeoning literature at the intersection of continuous-time macroeconomics and mean field games. Yet for many models of interest no one even knows whether an equilibrium exists when the economy is not assumed to be in the steady state (see Achdou et al. (2014)). Difficulties include the fact that it is usually impossible to characterize the value-of-search in closed form. Smith (2011) quipped that "the simplest non-stationary models can be notoriously intractable."

Our fixed point theorem relates to Jovanovic and Rosenthal (1988) who also propose a topological approach[7] to prove the existence of non-stationary (and stationary) equilibria in a general class of models coined anonymous games. These can be viewed as mean field games in discrete time.[8] Observe however that their critical assumption on the continuity of individual expected utilities do not hold in search-and-matching models; in two-sided search-and-matching models pooling of match acceptance decisions in one population can give rise to discontinuities in the other population's match prospects.[9,10] This has been extensively explored in the context of NTU block segregation where the unique steady state equilibrium exhibits a discontinuous value-of-search profile (refer to McNamara and Collins (1990), Smith (2006) and references therein). More generally, discontinuities arise when a non-negligible set of agents adopts identical match acceptance decisions, leading to a sharp drop in match prospects for the marginally rejected agents. A scenario of agents endowed with identical preferences and match opportunities provides the simplest example thereof.

Recent work by Balbus et al. (2022) (on supermodular anonymous games) and Pröhl (2023) (on the non-stationary Aiyagari (1994) model with aggregate uncertainty) has established non-topological existence results that rely on monotonicity conditions. Where such monotonicity exists, equilibria can be ranked or are unique. The interactive nature of search-and-matching

[7]We say that an equilibrium existence proof is topological if it uses a fixed point theorem that endows its domain with a topology, such as Brouwer's fixed point theorem or its generalizations. An example of a non-topological fixed point theorem is Tarski.

[8]Bergin and Bernhardt (1995) investigate the complementary case where there is aggregate uncertainty.

[9]Identify unmatched agents' utilities (labeled reward function in Jovanovic and Rosenthal (1988)) with the expected flow payoff of mutually acceptable matches weighted by the meeting rate, and equate action profiles with either match indicators or value-of-search profiles. Then utilities can impossibly be continuous at action profiles where a non-negligible mass of agents is indifferent between accepting and rejecting the same type.

[10]Continuous value function profiles naturally arise in Bewley-style economies where individual agents face uninsurable income risk. In particular, Miao (2006) Lemma 1 (cf. Cheridito and Sagredo (2016) and furhter qualifying assumptions in Cao (2020)) proves that individual saving and consumption decisions are continuous in asset holdings. The prevalence of prices aggregating agent heterogeneity is central for this result.

models rule out their structural assumptions.[11,12]

The mean field game literature has made strides as of late by allowing for aggregate uncertainty under the probabilistic approach (see Carmona and Delarue (2018) and Bilal (2023)). Mathematically, aggregate noise is a convenient tool, for it smoothes the value function across states, allowing the researcher to leverage PDE techniques. Conceptually, our approach is different since, like in the steady state, the aggregate dynamics we consider are, by construction, deterministic.

2 A Fixed Point Theorem for Non-Stationary Models

We first develop a fixed point theorem that will help us prove the existence of a non-stationary equilibrium of the search-and-matching economy. It is a non-trivial adaptation of the well-known Schauder-Tychonoff fixed point theorem (Schauder (1930) - Tychonoff (1935)). Due to its potential appeal for proving existence in other models, this section is self-contained.

Our theorem applies to continuous-time, infinite-horizon models in which a group of heterogeneous agents, as described by a type $x \in [0,1]$, take actions that affect others through the aggregate only. This class of models is sometimes referred to as anonymous or mean field games. Within this class of models, our fixed point theorem is sufficiently general to allow the value function to fluctuate endogenously over time and to be discontinuous with respect to time and type. It is therefore relevant in models that dispense with the steady state assumption or admit pooling behavior. Note that existing applications of fixed point theorems in economic models (see for instance Stokey and Lucas (1989)) often rely on the Arzelà-Ascoli theorem, which explicitly rules out discontinuities in the value function.

2.1 Preliminaries and Statement of the Theorem

We would like to establish the existence of a fixed point of the operator $H = (H^1, ..., H^N) : \mathscr{F}^N \to \mathscr{F}^N$ where $\mathscr{F}$ and $\mathscr{F}^N$ are the spaces of measurable mappings $[0,1] \times \mathbb{R}_+ \to [0,1]$ and $[0,1] \times \mathbb{R}_+ \to [0,1]^N$ respectively. In more detail, a fixed point is a mapping $\overline{F} = (\overline{F}^1, ..., \overline{F}^N) \in \mathscr{F}^N$ such that $H[\overline{F}] = \overline{F}$.

To state the theorem we introduce two notions of distance. First, the continuity premise

[11] Supermodularity posits incremental and monotone effect of others' actions on expected utility. Supermodularity is not satisfied in our context where there are threshold acceptance strategies in the same way that Bertrand competition cannot be modelled as a supermodular game.

[12] The Bewley-style Aiyagari (1994) model builds on a sufficient statistic approach whereby individual decisions aggregate into a single variable such as the interest rate. Match acceptance decisions of heterogeneous agents do not admit such aggregation.

of our fixed point theorem requires the following operator to measure the "distance" between functions.

Definition 1 (seminorm)**.** *Define, for all functions* $F = (F^1, ..., F^N) \in \mathscr{F}^N$*,*

$$||F|| = \max_{n\in\{1,...,N\}} \int_0^\infty \int_0^1 e^{-t} \big|F^n(x,t)\big| dx\, dt.$$

The mapping $(F, \bar{F}) \mapsto ||F - \bar{F}||$ is called a semimetric because it is induced by a seminorm. Following this terminology, we call $|| \cdot ||$ a seminorm.

Second, we introduce the total variation norm for mappings in $\mathscr{F}$. As we shall see, if a set of functions is uniformly bounded in the total variation norm, then it is sequentially compact. Our focus on two-dimensional functions is a special case of the general definition provided by Idczak and Walczak (1994) and Leonov (1996).[13]

Definition 2 (total variation norm)**.** *The total variation norm for functions* $F^n \in \mathscr{F}$ *and arbitrary bounded time interval* $[\underline{t}, \overline{t}]$ *is given by*

$$T\mathscr{V}\big(F^n, [0,1] \times [\underline{t}, \overline{t}]\big) = \mathscr{V}_0^1\big(F^n(\cdot, \underline{t})\big) + \mathscr{V}_{\underline{t}}^{\overline{t}}\big(F^n(0, \cdot)\big) + \mathscr{V}_2\big(F^n, [0,1] \times [\underline{t}, \overline{t}]\big)$$

with

$$\begin{aligned}
\mathscr{V}_0^1\big(F^n(\cdot, t_0)\big) &= \sup_{\mathscr{P}} \sum_{i=1}^{m} \big|F^n(x_i, t_0) - F^n(x_{i-1}, t_0)\big| \\
&\qquad \textit{where } \mathscr{P} \textit{ is a partition of } [0,1], \ \textit{i.e.}\ 0 = x_0 < x_1 < ... < x_m = 1, \\
\mathscr{V}_{\underline{t}}^{\overline{t}}\big(F^n(0, \cdot)\big) &= \sup_{\mathscr{P}} \sum_{i=1}^{m} \big|F^n(0, t_i) - F^n(0, t_{i-1})\big| \\
&\qquad \textit{where } \mathscr{P} \textit{ is a partition of } [\underline{t}, \overline{t}], \ \textit{i.e.}\ \underline{t} = t_0 < t_1 < ... < t_m = \overline{t}, \\
\mathscr{V}_2\big(F^n, [0,1] \times [\underline{t}, \overline{t}]\big) &= \sup_{\mathscr{P}} \sum_{i=1}^{m} \big|F^n(x_i, t_i) - F^n(x_i, t_{i-1}) - F^n(x_{i-1}, t_i) + F^n(x_{i-1}, t_{i-1})\big| \\
&\qquad \textit{where } \mathscr{P} \textit{ is a discrete path in } [0,1] \times [\underline{t}, \overline{t}], \ \textit{s.t.}\ \underline{t} = t_0 < t_1 < ... < t_m = \overline{t} \\
&\qquad \textit{and } 0 = x_0 < x_1 < ... < x_m = 1.
\end{aligned}$$

The total variation norm in dimension two is the sum of the variations, i.e., "up and down" movements, of a function F^n along three paths within the square $[0,1] \times [\underline{t}, \overline{t}]$. These paths

[13] Subsequent work by Chistyakov and Tretyachenko (2010) extends the total variation norm to more abstract spaces.

start from the origin point $(0, \underline{t})$ and move towards the three remaining extremal points of the square, capturing variations along the boundary and the interior.[14]

We can now state our fixed point theorem:

Theorem 1. *Suppose that $H : \mathscr{F}^N \to \mathscr{F}^N$ satisfies*

(i) *for all $\overline{F} = (\overline{F}^1, ..., \overline{F}^N) \in \mathscr{F}^N$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\big|\big|H[\overline{F}] - H[F]\big|\big| < \epsilon$ for all $F = (F^1, ..., F^N) \in \mathscr{F}^N$ such that $\big|\big|\overline{F} - F\big|\big| < \delta$;*

(ii) $\forall T > 0\ \exists C > 0$ *such that* $\forall n \in \{1, ..., N\} : T\mathscr{V}\big(H^n[F], [0,1] \times [0,T]\big) < C$ *for all* $F \in \mathscr{F}^N$.

Then H admits a fixed point.

We will refer to condition (i) as continuity and (ii) as uniformly bounded variation.[15]

2.2 Proof of the Fixed Point Theorem

Outline of the proof:

we construct a sequence of operators that approximate the fixed point operator H (Step 1). Each approximate fixed point operator will satisfy all the assumptions of the Schauder fixed point theorem (Step 0, Step 2) and hence admits a fixed point (Step 2). We then show that the sequence of approximate fixed point admits a convergent subsequence (Step 3). To conclude, we prove that H maps the convergent subsequence's limit point into a fixed point of H (Step 4).

To begin with, endow $\mathscr{F}$ and $\mathscr{F}^N$ with the discounted supremum metric. Discounting is what helps us deal with an infinite horizon.

Definition 3 (discounted sup metric)**.** *The discounted sup metric for functions $F = (F^1, ..., F^N) \in \mathscr{F}^N$ and $\overline{F} = (\overline{F}^1, ..., \overline{F}^N) \in \mathscr{F}^N$ is given by*

$$\mathbf{d}^N(F, \overline{F}) = \max_{n \in \{1,...,N\}} \mathbf{d}(F^n, \overline{F}^n) = \max_{n \in \{1,...,N\}} \sup_{x,t} e^{-t}\big|F^n(x,t) - \overline{F}^n(x,t)\big|.$$

[14] To establish a Helly-type selection theorem whereby a sequence of functions $(F_{(k)})_{k\in\mathbb{N}}$ defined on $[0,1] \times [\underline{t}, \overline{t}]$ of uniformly bounded variation admits a pointwise convergent subsequence, the inclusion of boundary variations is crucial. To see this, consider $F_{(k)}(x,t) = \begin{cases} \sin(kx) & \text{if } t = \underline{t} \\ 0 & \text{otherwise.} \end{cases}$ Thus defined sequence $(F_{(k)})_{k\in\mathbb{N}}$ does not admit a pointwise convergent subsequence, despite $\mathscr{V}_2(F_{(k)}, [0,1] \times [\underline{t}, \overline{t}]) = 1$. This shows that uniform bounded variation on the interior only does not guarantee the existence of a pointwise convergent subsequence.

[15] Also note that the domain $\mathscr{F}^N$ is convex.

Step 0 (Preliminary): A Compact Set of Functions

To apply Schauder's fixed point theorem, we require that the fixed point operator maps into a compact set of functions. As a preliminary step, we show that the set of k-Lipschitz functions is compact. (Note however that functions in the image of H need not be k-Lipschitz, let alone continuous.)

A function $F^m : [0,1] \times \mathbb{R}_+ \to [0,1]$ is k-Lipschitz if for any $(x,t),(y,r) \in [0,1] \times \mathbb{R}_+$

$$\left|F^m(x,t) - F^m(y,r)\right| \leq k \cdot \max\left\{|x-y|, |t-r|\right\}.$$

Denote $\mathscr{F}_{(k)} \subset \mathscr{F}$ the (convex) subset of k-Lipschitz functions.

Proposition 1. $(\mathscr{F}_{(k)}, \mathbf{d})$ *is compact.*

The proof of this Proposition mirrors that of the Arzelà-Ascoli theorem and is deferred to Appendix A.1.[16]

Step 1: Construction of the Approximate Fixed Point Operator

We construct an approximate fixed point operator that is continuous and maps into the set of k-Lipschitz functions. We achieve this via convolution with approximate identity functions. To handle the integration at the boundary points $x \in \{0,1\}$ and $t = 0$, where the convolution operation naturally extends beyond the domain $[0,1] \times [0,\infty)$, we extend the support of functions in $\mathscr{F}$ to $[-1,2] \times [-1,\infty)$. This extension ensures that the convolution is well-defined across the entire original domain.

Denote the approximate operator $H^m_{(k)} : \mathscr{F}^N \to \mathscr{F}$, and define, for any $(x_0,t_0) \in [0,1] \times \mathbb{R}_+$,

$$H^m_{(k)}[F](x_0,t_0) = \int_{-1}^{2} \int_{-1}^{\infty} \hat{H}^m[F](x,t)\delta_{(k)}(x_0 - x, t_0 - t)dt\, dx$$

where, first, $\hat{H}^m[F]$ is the extension of $H^m[F] \in \mathscr{F}$ to a mapping $[-1,2] \times [-1,\infty) \to [0,1]$,

$$\hat{H}^m[F](x,t) = \begin{cases} H^m[F](|x|,|t|) & \text{if } -1 \leq x < 0 \\ H^m[F](x,|t|) & \text{if } 0 \leq x \leq 1 \\ H^m[F](2-x,|t|) & \text{if } 1 < x \leq 2, \end{cases}$$

[16]The classical version of the Arzelà-Ascoli theorem (see Munkres (2015), Theorem 45.4) does not apply here directly because the domain of the functions in $\mathscr{F}_{(k)}$ is unbounded.

and, secondly, for $b_{(k)} = 4/k$ and $k \geq 4$ we define

$$\delta_{(k)}(x,t) = \frac{1}{(b_{(k)})^2} \quad \text{if } (x,t) \in B_{(k)}(0) \equiv \big\{(x',t') \in \mathbb{R}^2 : \max\{|x'|,|t'|\} \leq \frac{b_{(k)}}{2}\big\}, \text{ zero otherwise.}$$

Intuitively, convolution with a sufficiently dispersed approximate identity function smooths out a bounded function by averaging its values over a larger neighborhood. Our approximate identity function assigns equal weights to every point within its support. Consequently, the difference in the operator's value between two points is bounded by the size of the difference in the integration regions. Lipschitz continuity follows because this difference is at most proportional to the distance between the two points, as illustrated in Figure 2. At this point, it is immaterial whether the original function $H^m[F] \in \mathscr{F}$ is of uniformly bounded variation or not.

Step 2: Properties of the Approximate Fixed Point Operator

We now show that the approximate fixed point operator satisfies all the necessary properties that allow us to apply the Schauder fixed point theorem: compactness of its image and continuity.

Lemma 1. $H^m_{(k)}[\mathscr{F}^N] \subseteq \mathscr{F}_{(k)}$

Lemma 2. $H^m_{(k)} : (\mathscr{F}^N, \mathbf{d}^N) \to (\mathscr{F}, \mathbf{d})$ *is continuous.*

The proof of both Lemmata is deferred to Appendix A.2.

Proposition 2. $H_{(k)} = (H^1_{(k)}, ..., H^N_{(k)}) : \mathscr{F}^N \to \mathscr{F}^N$ *has a fixed point* $F^*_{(k)}$.

This Proposition is an application of the Schauder fixed point theorem.

Proof. First observe that $(\mathscr{F}^N, \mathbf{d}^N)$ is a complete metric space (see Theorem 43.5. in Munkres (2015)) and that $(\mathscr{F}^N_{(k)}, \mathbf{d}^N)$ is a subset of this space. Moreover, observe that the metric $\mathbf{d}^N$ satisfies the three axioms posited by Schauder (1930).[17] Second, since $(\mathscr{F}^N_{(k)}, \mathbf{d}^N)$ is the finite-dimensional product of compact sets (Proposition 1), it is compact. It is also closed (since $\mathscr{F}^N_{(k)}$ is compact) and convex. Finally, continuity of the component operator $H^n_{(k)}$ (Lemma 2) on the larger space $\mathscr{F}^N$ establishes continuity of $H_{(k)} = (H^1_{(k)}, ..., H^N_{(k)}) : (\mathscr{F}^N_{(k)}, \mathbf{d}^N) \to (\mathscr{F}^N_{(k)}, \mathbf{d}^N)$. Then the Schauder fixed point theorem (see Schauder (1930), Satz I) asserts that *if the continuous operator $H_{(k)}$ maps the convex, closed and compact set $\mathscr{F}^N_{(k)}$ into itself, then there exists a fixed point $F^*_{(k)}$, i.e., $H_{(k)}[F^*_{(k)}] = F^*_{(k)}$.* □

[17] Those axioms are 1° $\mathbf{d}^N(F,\bar{F}) = \mathbf{d}^N(F-\bar{F},0)$, 2° $\lim_{n\to\infty} \mathbf{d}^N(F_{(n)},\bar{F}) = \lim_{n\to\infty} \mathbf{d}^N(G_{(n)},\bar{G}) = 0$ implies $\lim_{n\to\infty} \mathbf{d}^N(F_{(n)}+G_{(n)},\bar{F}+\bar{G}) = 0$ and 3° for $\{\lambda_n\}$ a sequence of real numbers and $\{F_{(n)}\}$ a sequence in $\mathscr{F}^N$, $\lim_{n\to\infty}\lambda_n = \lambda$, $\lim_{n\to\infty} \mathbf{d}^N(F_{(n)},\bar{F}) = 0$ implies $\lim_{n\to\infty} \mathbf{d}^N(\lambda_n F_{(n)}, \lambda\bar{F}) = 0$. Those axioms are naturally satisfied if the metric is induced by a norm (which is prohibited by discounting in our case).

Step 3: Existence of a Convergent Subsequence of Approximate Fixed Points

By considering all $k \in \mathbb{N}$, Proposition 2 establishes that there exists a sequence of approximate fixed points $(F^*_{(k)})_{k\in\mathbb{N}}$. We now show that this sequence admits a convergent subsequence.

Proposition 3. *The fixed points $(F^*_{(k)})_{k\in\mathbb{N}}$ admit an accumulation point F^* in $(\mathscr{F}^N, \mathbf{d}^N)$.*

This follows from a higher-dimensional Helly-type selection theorem. It is here that we utilize our assumption—immaterial to the preceding results—that the total variation of functions $F^n \in \mathscr{F}$ admits a uniform bound.

Proof. Fix arbitrary $n \in \{1, ...N\}$. Idczak and Walczak (1994) and Leonov (1996) (Theorem 4) prove that *if all the elements (indexed by $k \in \mathbb{N}$) of a sequence of functions $(F^n_{(k)})_{k\in\mathbb{N}}$: $F^n_{(k)} \in \mathscr{F}$ satisfy $\mathcal{TV}\big(F^n_{(k)}, [0,1]\times[0,T]\big) < C$ for some constant C, then the sequence admits a subsequence of functions that converges pointwise on $[0,1]\times[0,T]$ to a function with the same property.* Assumption (ii) of the theorem asserts the existence of such a uniform bound C for all $H^n[F] \in \mathscr{F}$ where $n \in \{1, ..., N\}$ and $F \in \mathscr{F}^N$. Since the uniform bound on the total variation will be preserved under convolution, such bound also obtains for all $H^n_{(k)}[F] \in \mathscr{F}$ where $k \in \mathbb{N}$. In particular, this applies for the sequence $(H^n_{(k)}[F^*_{(k)}])_{k\in\mathbb{N}} = (F^{*,n}_{(k)})_{k\in\mathbb{N}}$. Then, for all $T > 0$ and $n = 1$, the Helly-type selection theorem ensures the existence of a pointwise convergent subsequence $(F^{*,1}_{(k_\ell)})_{k\in\mathbb{N}}$ on $[0,1]\times[0,T]$ where each $F^{*,1}_{(k_\ell)} \in \mathscr{F}$. Then, iterating over n ensures the existence of a pointwise convergent subsequence $(F^*_{(k_\ell)})_{k\in\mathbb{N}}$ in $[0,1]\times[0,T]$ where $F^*_{(k_\ell)} \in \mathscr{F}^N$. Following an identical reasoning, we can find a subsequence of the subsequence which converges pointwise in $[0,1]\times[0,T+1]$. Proceeding by induction then establishes pointwise convergence in $[0,1]\times[0,\infty)$; we denote $F^* \in \mathscr{F}^N$ the limit point. □

Step 4: Conclusion

The preceding steps established the existence of a convergent subsequence of approximate fixed points. Proposition 4 asserts that the image of this limit is a fixed point of H, which concludes the proof of Theorem 1.

Proposition 4. *$H[F^*]$ is a fixed point of $H : \mathscr{F}^N \to \mathscr{F}^N$.*

The proof of this Proposition is deferred to Appendix A.3.

Remark:

[18] Theorem 1 can be generalized to restrict attention to closed, convex subsets of measurable functions. Formally, consider the restriction to a subset of functions $\mathscr{C} \subseteq \mathscr{F}^N$ such that H maps $\mathscr{C}$ into itself, i.e., $H[\mathscr{C}] \subseteq \mathscr{C}$. And maintain that H satisfies the theorem's Premises (i) and (ii) where $\mathscr{F}^N$ is now replaced by $\mathscr{C}$. Then introduce the additional assumption that $\mathscr{C} \subseteq \mathscr{F}^N$ is convex and closed under $\mathbf{d}^N$. Then the theorem's conclusion can be strengthened: H admits a fixed point in $\mathscr{C}$. The proof is as follows:

Proof. Define $\mathscr{C}_{(k)} \subseteq \mathscr{F}^N_{(k)}$ the set of functions derived via convolution of functions $F \in \mathscr{C}$ with the approximate identity function $\delta_{(k)}$ as defined in Step 1. To simplify the notation, we write $F \star \delta_{(k)}$. Then $\mathscr{C}_{(k)} = \{F_{(k)} \in \mathscr{F}^N | F_{(k)} = F \star \delta_{(k)}$ for some $F \in \mathscr{C}\}$. We note that $\mathscr{C}_{(k)}$ inherits from $\mathscr{C}$ convexity and closedness. It follows that $(\mathscr{C}_{(k)}, \mathbf{d}^N)$ is compact (because the closed subset of a compact space, here $(\mathscr{F}^N_{(k)}, \mathbf{d}^N)$ as shown in Proposition 1, is compact). Then our application of Schauder's fixed point theorem in Proposition 2 applies to the *convex, closed and compact set* $\mathscr{C}_{(k)}$. Hence, as established in Step 2, there exists a sequence of approximate fixed points $(F^*_{(k)})_{k\in\mathbb{N}}$ where $F^*_{(k)} \in \mathscr{C}_{(k)}$. And according to Step 3, Proposition 3, this sequence admits an accumulation point, denoted F^*. Therefore, also the sequence $(\tilde{F}^*_{(k)})_{k\in\mathbb{N}}$ in $\mathscr{C}$ where $F^*_{(k)} = \tilde{F}^*_{(k)} \star \delta_{(k)}$ admits F^* as an accumulation point under $\mathbf{d}^N$. Since $(\mathscr{C}, \mathbf{d}^N)$ is closed, it follows that $F^* \in \mathscr{C}$. Then Proposition 4 allows us to conclude. □

3 The Search-and-Matching Economy

This section presents the continuous-time, infinite-horizon search-and-matching model. We first describe the set-up. We then lay out the assumptions that we rely on to prove the existence of a non-stationary equilibrium. The formal definition of equilibrium and the proof of its existence are deferred to later sections.

3.1 Set-up

Agents engage in time-consuming and random search for potential matches. When two agents meet, they observe each other's type. If both agents give their consent, they permanently exit the search pool and consume their respective match payoffs. Otherwise they continue waiting for a more suitable partner. Each agent maximizes their expected present value of payoffs, discounted at rate $\rho > 0$.

[18] We thank the editor for suggesting this generalization.

Agents

There are two distinct populations denoted X and Y, each containing a continuum of agents that seek to match with someone from the other population. Each agent is characterized by a type which belongs to the unit interval $[0, 1]$. We usually denote by x a type of an agent from population X, and y a type of an agent from population Y. The distribution of types in the search pool at time t is characterized by a pair of functions $\mu_t = (\mu_t^X, \mu_t^Y)$, such that for any $U \subseteq [0, 1]$, the mass of types $x \in U$ in the search pool is $\int_U \mu_t^X(x)dx$. The initial distribution at time 0 is given by some uniformly bounded μ_0.

Note: We typically construct the value-of-search and related concepts from the perspective of population X; symmetric constructions apply to agent types y from population Y. Furthermore, we impose that all functions introduced are Lebesgue measurable.

Search

Over time agents randomly meet each other. Meetings follow an (inhomogeneous) Poisson point process. Such a process is characterized by the time-variant (Poisson) meeting rate $\lambda = (\lambda^X, \lambda^Y)$ where $\lambda_t^X(y|x)$ is agent type x's time-t meeting rate with an agent type y. In the simplest case, the meeting rate is proportional to the search pool population so that $\lambda_t^X(y|x) = \mu_t^X(x)$, as in Shimer and Smith (2000) and Smith (2006). More generally, we take λ to be a function of the underlying state variable μ_t and time t. Then the subindex t is short-hand for dependence on both the prevailing time t and state μ_t, i.e., $\lambda_t^X(y|x) \equiv \lambda^X(t, \mu_t)(y|x)$.

The meeting rates λ_t^X and λ_t^Y are not arbitrary but intricately linked. Coherence of the model demands that the number of meetings of agent types x with agent types y must be equal to the number of meetings of agent types y with agent types x:

$$\lambda_t^X(y|x)\mu_t^X(x) = \lambda_t^Y(x|y)\mu_t^Y(y).$$

Population Dynamics

Population dynamics are governed by entry and exit. Any two agents x and y of opposite populations that meet and mutually consent to form a match exit the search pool. The rate at which an individual agent type x matches and exits the market at time t—the hazard rate—is

$$\int_0^1 m_t(x, y)\lambda_t^X(y|x)dy;$$

$m_t(x, y) \in \{0, 1\}$, determined in equilibrium, denotes the time-t match indicator. This is equal to one if agent types x and y match upon meeting and zero otherwise. Entry is characterized by a time-variant rate $\eta = (\eta^X, \eta^Y)$. We take η to be a function of the underlying state variable μ_t and time t. Then $\eta_t^X(x) \equiv \eta^X(t, \mu_t)(x)$ is agent type x's time-t entry rate.

The economy can be non-stationary in that entry and exit need not be equal, leading to a time-variant state $\mu_t = (\mu_t^X, \mu_t^Y)$.[19] The population dynamics are given by

$$\mu_{t+h}^X(x) = \mu_t^X(x) + \int_t^{t+h} \left\{ -\mu_\tau^X(x) \int_0^1 \lambda_\tau^X(y|x) m_\tau(x, y) dy + \eta_\tau^X(x) \right\} d\tau. \tag{1}$$

Value-of-Search

Any given agent's experience in the search pool is characterized by random encounters with other agents. Presented with a match opportunity, an agent must weigh the immediate match payoff against the option value-of-search, the discounted expected future match payoff were one to continue one's search. Denote agent type x's time-t value-of-search $V_t^X(x)$ and $\pi_t^X(y|x)$ the one-time match payoff when matching with y. Naturally, the optimal matching decision is to accept to match with another agent whenever the payoff exceeds the option value-of-search:

$$\pi_t^X(y|x) \geq V_t^X(x). \tag{OS}$$

Knowledge of the value-of-search uniquely determines the match indicator:

$$m_t(x, y) = \begin{cases} 1 & \text{if } \pi_t^X(y|x) \geq V_t^X(x) \text{ and } \pi_t^Y(x|y) \geq V_t^Y(y), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Our definition of the value-of-search is recursive: agents form beliefs about future match probabilities and payoffs. Future match probabilities depend jointly on the Poisson rate λ and match outcomes upon meeting m—which depends on the value-of-search. Reflecting optimality of individual strategies, we define the value-of-search to be the solution to

$$V_t^X(x) = \int_t^\infty e^{-\rho(\tau - t)} \int_0^1 \pi_\tau^X(y|x)\, p_{t,\tau}^X(y|x) dy\, d\tau, \tag{3}$$

[19]Our formulation is that of a system of integral equations rather than differential equations, because the left- and right time derivative of $\mu_t^X(x)$ do not always coincide as will be the case if $z \mapsto \int_0^z m_t(x, y) dy$ is discontinuous.

where $p^X_{t,\tau}(y|x)$ is the density of future matches with y at time τ conditional on x being unmatched at time t. This is a standard object and is characterized by the matching rate (see Appendix B.1):

$$p^X_{t,\tau}(y|x) = \Lambda^X_\tau(y|x)\ \exp\big\{-\int\limits_t^\tau \int\limits_0^1 \Lambda^X_r(z|x)dzdr\big\}\ \text{ where }\ \Lambda^X_\tau(y|x) = \lambda^X(\tau,\mu_\tau)(y|x)m_\tau(x,y).$$

We consider the two most studied paradigms for defining match payoffs: *non-transferable utility (NTU)* and *transferable utility (TU)*.

Payoffs: NTU

In the NTU paradigm, match payoffs are exogenously given and time-invariant. We denote $\pi^X(y|x) = \pi^X_t(y|x)$ and normalize payoffs, i.e., $\pi^X(y|x) \in [0,1]$. This paradigm precludes individualized price-setting and bargaining.

Payoffs: TU

Alternatively, the TU paradigm takes as its primitive the match output $f(x,y) \in [0,1]$, generated when agent types x and y match with one another. Any division of output is conceivable. As in the Diamond-Mortensen-Pissarides model, we use Nash bargaining as a solution concept for the bargaining problem in which agents can claim their value-of-search $V^X_t(x)$ as a threat point. Surplus $f(x,y) - V^X_t(x) - V^Y_t(y)$ is shared according to bargaining weights α^X and α^Y (where $\alpha^X + \alpha^Y = 1$). Formally,

$$\pi^X_t(y|x) = V^X_t(x) + \alpha^X[f(x,y) - V^X_t(x) - V^Y_t(y)]. \tag{4}$$

It follows that match decisions (2) are intratemporally efficient: $m_t(x,y) = 1$ if and only if $f(x,y) - V^X_t(x) - V^Y_t(y) \geq 0$.

3.2 Assumptions

Our assumptions on search and entry rates make use of the L^1 seminorm:

$$N(\mu'_t,\mu''_t) \equiv \max\big\{\int\limits_0^1 \big|{\mu'}^X_t(x) - {\mu''}^X_t(x)\big|dx, \int\limits_0^1 \big|{\mu'}^Y_t(y) - {\mu''}^Y_t(y)\big|dy\big\}.$$

Search

We first assume that higher types meet other agents at a weakly faster rate.[20]

Assumption 1 (hierarchical search)**.** *Higher types meet other agents at a weakly faster rate; that is,* $\lambda_t^X(y|x_2) \geq \lambda_t^X(y|x_1)$ *for* $x_2 > x_1$ *and* $\lambda_t^Y(x|y_2) \geq \lambda_t^Y(x|y_1)$ *for* $y_2 > y_1$*.*[21]

We further require that the meeting rate is linearly bounded and Lipschitz continuous in the following sense.[22]

Assumption 2 (regularity of meetings)**.** *There exists* $L^\lambda > 0$ *such that for all* x, y *and* z

(i) $\lambda_t^X(y|x) \leq L^\lambda\big(1 + \mu_t^Y(y)\big)$ *and* $\lambda_t^Y(x|y) \leq L^\lambda\big(1 + \mu_t^X(x)\big)$*;*

(ii) $N(\lambda(t, \mu_t')(\cdot|z), \lambda(t, \mu_t'')(\cdot|z)) \leq L^\lambda N(\mu_t', \mu_t'')$*.*

Entry

Entry rates satisfy analogous conditions as meeting rates:

Assumption 3 (regularity of entry)**.** *There exists* $L^\eta > 0$ *such that for all* x, y *and* t

(i) $\eta_t^X(x) \leq L^\eta$ *and* $\eta_t^Y(y) \leq L^\eta$*;*

(ii) $N(\eta(t, \mu'_t), \eta(t, \mu''_t)) \leq L^\eta N(\mu_t', \mu_t'')$*.*

Examples The entry rate η encompasses several natural entry rates, including no entry, to study, for instance, a gradually clearing job market (by which, e.g., academic economists have organized the junior job market for Ph.D. hires) and constant flows of entry (as in Burdett and Coles (1997)). Moreover, the entry rate can be time-dependent to account for seasonal fluctuations, e.g., in the housing market (see Ngai and Tenreyro (2014)) or the business cycle (see Beaudry et al. (2020)).

[20] Hierarchical search encompasses, as a special case, anonymous meeting rates whereby the meeting rate does not depend on the agent's type and yet, critically, preserves later established bounds on the value-of-search across types under mimicking (Lemmata 3 and 4).

[21] To better understand the concepts of coherence and hierarchical search, write (without loss of generality) $\lambda_t^X(y|x) = \phi_t(x,y)\mu_t^Y(y)$ and $\lambda_t^Y(x|y) = \psi_t(x,y)\mu_t^X(x)$. Coherence then implies that $\psi_t(x,y) = \phi_t(x,y)$, while hierarchical search further implies that these functions are non-decreasing in both arguments. Moreover, if the populations are symmetric, i.e., $\mu_t^X(x) = \mu_t^Y(x)$ (and the equilibrium is symmetric), these functions are symmetric as well, i.e., $\psi_t(x,y) = \psi_t(y,x)$.

[22] Assumptions 1 and 2 relax a proportionality assumption in Lauermann et al. (2020). In order to prove the fundamental matching lemma in the steady state (see their Condition 32) they assume that $\lambda^X(t, \mu_t)(y|x)$ is proportional to $\mu_t^Y(y)$. Our non-stationary analysis does not require this.

Population Dynamics

To ensure that the population dynamics are well-defined, we adapt the proof of the well-known Cauchy-Lipschitz-Picard-Lindelöf theorem, which typically establishes the local existence of a unique solution for a system of finite-dimensional ODEs, to our infinite-dimensional context (see Appendix B.2).[23] To ensure that the unique solution exists globally for all t, we rely on Assumption 3, whereby in the absence of exit the search pool population grows at most linearly in time. We denote this bound $\overline{\overline{\mu}}_t$ and reference it in the proof.[24]

Proposition 5. *System (1) admits a unique solution for any $(\mu_0, \lambda, \eta, m)$ satisfying Assumptions 2 and 3.*

Payoffs: NTU

In line with the literature, we consider vertically differentiated types.[25] To encompass payoffs that are not strictly increasing for some types, e.g., $\pi^X(y|x) = y^x$, we impose the slightly weaker assumption that the set of types for whom the gains of matching with a superior type are small has vanishingly small mass as captured by Hölder continuity.

Assumption 4 (NTU-increasing match payoffs)**.** *Match payoffs are non-decreasing in the partner's type. Moreover, there exist positive constants C and α such that for all $\Delta > 0$, there is a measurable subset $\mathscr{Z}^\Delta$ of pairs (x, y) satisfying the following:*

(i) *payoffs for pairs in $\mathscr{Z}^\Delta$ are at least Δ-differentiated, i.e.,*

$$\pi^X(y'|x) - \pi^X(y|x) > \Delta \quad \text{and} \quad \pi^Y(x'|y) - \pi^X(x|y) > \Delta$$

for all pairs $(x, y), (x, y'), (x', y)$ (where $x' > x$ and $y' > y$) in $\mathscr{Z}^\Delta$;

(ii) *pairs not in $\mathscr{Z}^\Delta$ have Hölder-vanishing mass, i.e.,*

$$\int_0^1 \int_0^1 1\{(x, y) \notin \mathscr{Z}^\Delta\} dxdy < C\Delta^\alpha.$$

[23]There is one difference from the classical result: owing to our focus on a continuum of types, the system is infinite-dimensional. We draw on the more general treatment by Dieudonné (2013) (see Chapter 10.4) to deal with the dimensionality of our problem. What is key when passing from the finite to the infinite is the mean field property embedded in Assumptions 2 (ii) and 3 (ii), whereby changes in the pool of unmatched agents, driven by individual types, have a negligible impact on entry and meeting rates.

[24] It holds that $\mu^X_{t+h}(x) - \mu^X_t(x) \le L^\eta h$, and so $\mu^X_t(x) \le \overline{\overline{\mu}}_0 + tL^\eta \equiv \overline{\overline{\mu}}_t$ where the initial upper bound $\overline{\overline{\mu}}_0$ is given by the supremum over μ^X_0 and μ^Y_0.

[25]Vertical differentiation guarantees that higher types face superior match opportunities, a property we exploit when developing the NTU mimicking argument.

We moreover require a regularity condition regarding match payoffs:

Assumption 5 (NTU)**.** *$x \mapsto \pi^X(y|x)$ and $y \mapsto \pi^Y(x|y)$ admit a uniform bound L^π on total variation.*

Payoffs: TU

To ensure that small changes in individual match prospects do not result in large changes in matching patterns, Shimer and Smith (2000) impose super- or submodular output. Here, we relax their assumption and allow for output functions that are supermodular for some types and submodular for others, e.g., $f(x, y) = x^y + y^x$.[26] It suffices that the set of types for whom complementarity gains are small has vanishingly small mass, as captured by Hölder continuity.[27]

Assumption 6 (TU-differentiated marginal output)**.** *There exist positive constants C and α such that for all $\Delta > 0$, there are measurable subsets $\mathscr{Z}^{>\Delta}$ and $\mathscr{Z}^{<-\Delta}$ of pairs (x, y) satisfying the following:*

(i) *marginal output is at least Δ-differentiated for pairs in $\mathscr{Z}^{>\Delta}$ and at most $-\Delta$-differentiated for pairs in $\mathscr{Z}^{<-\Delta}$, i.e.,*

$$\begin{aligned} & f(x,y) - f(x,y') - f(x',y_2) + f(x',y') > \Delta(x'-x)(y'-y) \\ \textit{and} \quad & f(x,y) - f(x,y') - f(x',y_2) + f(x',y') < -\Delta(x'-x)(y'-y) \end{aligned}$$

for all pairs $(x, y), (x, y'), (x', y), (x', y')$ (where $x' > x$ and $y' > y$) in $\mathscr{Z}^{>\Delta}$ and $\mathscr{Z}^{<-\Delta}$ respectively;

(ii) *pairs not in $\mathscr{Z}^{>\Delta} \cup \mathscr{Z}^{<-\Delta}$ have Hölder-vanishing mass, i.e.,*

$$\int_0^1 \int_0^1 1\big\{(x, y) \notin \mathscr{Z}^{>\Delta} \cup \mathscr{Z}^{<-\Delta}\big\} dx dy < C\Delta^\alpha.$$

As in the NTU paradigm, we require a further regularity condition for output:

[26] Anderson and Smith (2024) study sorting in a frictionless matching market where output functions are neither supermodular nor submodular.

[27] If the cross-partial derivative $D^2_{xy} f(x, y)$ is well-defined, Assumption 6 says that *there exist positive constants C and α so that for all $\Delta > 0$*

$$\int_0^1 \int_0^1 1\big\{(x, y) : |D^2_{xy} f(x, y)| < \Delta\big\} dx dy < C\Delta^\alpha.$$

Assumption 7 (TU)**.** *$x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ admit a uniform bound L^f on total variation.*

These assumptions are weaker than requiring that $\pi^X(y|x)$, $\pi^Y(x|y)$ and $f(x,y)$ are continuously differentiable (as in Smith (2006) and Shimer and Smith (2000)).[28] We will use these assumptions to prove bounded variation of the value-of-search (Proposition 6).

Remark:

Where Assumptions 4 and 6 hold, our focus on pure strategies is without loss; As Lemmata 8 and 9 make clear, at any moment in time only a negligible mass of agents can lie on the indifference threshold of a non-negligible mass of agents. Further note that Assumptions 4 and 6 rule out embedding discrete types in our continuum type space.

4 Equilibrium

An equilibrium jointly determines the evolution of the endogenous variables of the search-and-matching economy: the distribution of agents' characteristics in the search pool, agents' continuation values of search, matching decisions and transfers (under bargaining in the TU paradigm). None of those can be determined in isolation. Agents compute their value-of-search given their beliefs about the economy at large. In equilibrium, each individual correctly anticipates future match opportunities and payoffs. This generates a feedback loop between the population dynamics and the value-of-search.

Definition 4. *An equilibrium of the search-and-matching economy of given initial search pool population μ_0 is a triple $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{m})$, solution to (1),(2) and (3), where (NTU) payoffs are exogenously given, or (TU) determined via Nash bargaining (4).*

The interplay between aggregate dynamics and the individual decision problem is a feature shared with virtually all dynamic general equilibrium models under rational expectations.

The main result of this paper is to show that an equilibrium exists, both in the NTU and TU paradigm.

Theorem 2. *Posit Assumptions 1, 2, 3 for both paradigms, 4, 5 for NTU, and 6, 7 for TU. Then there exists an equilibrium of the search-and-matching economy.*

[28] Discontinuities in payoffs can arise naturally, e.g., when agents differ in discrete attributes such as workers' professional degrees, location or export focus of a firm, or number of bedrooms in the rental market (see Glaeser and Luttmer (2003)).

The proof of both results will be developed jointly in Sections 2 and 5. These sections develop tools to deal with discontinuous value functions

5 Proving Equilibrium Existence

In this section, we use Theorem 1 to prove that the search-and-matching economy admits an equilibrium. This proof relies on the construction of a fixed point operator $\boldsymbol{V} : \mathscr{F}^2 \to \mathscr{F}^2$ that maps a value-of-search profile $v = (v^X, v^Y) \in \mathscr{F}^2$ into a new value-of-search profile. As in the preceding section, $\mathscr{F}^2$ is the space of jointly measurable mappings $v : [0,1] \times \mathbb{R}_+ \to [0,1]^2$.

Our focus on value function space is common in the literature (see Shimer and Smith (2000), Smith (2006)). Even though an equilibrium is a triple $(\mathbf{V}, \mu, \mathbf{m})$, the value-of-search $\mathbf{V}$ encodes all the information needed to recover the match indicator function (through Equation (2)), whence the state μ (through Equation (1) as shown in Proposition 5).

To apply Theorem 1, we construct an operator that satisfies the two conditions that guarantee the existence of a fixed point: (i) continuity and (ii) uniformly bounded variation. The proof is extensive. Figure 1 provides a schematic overview.

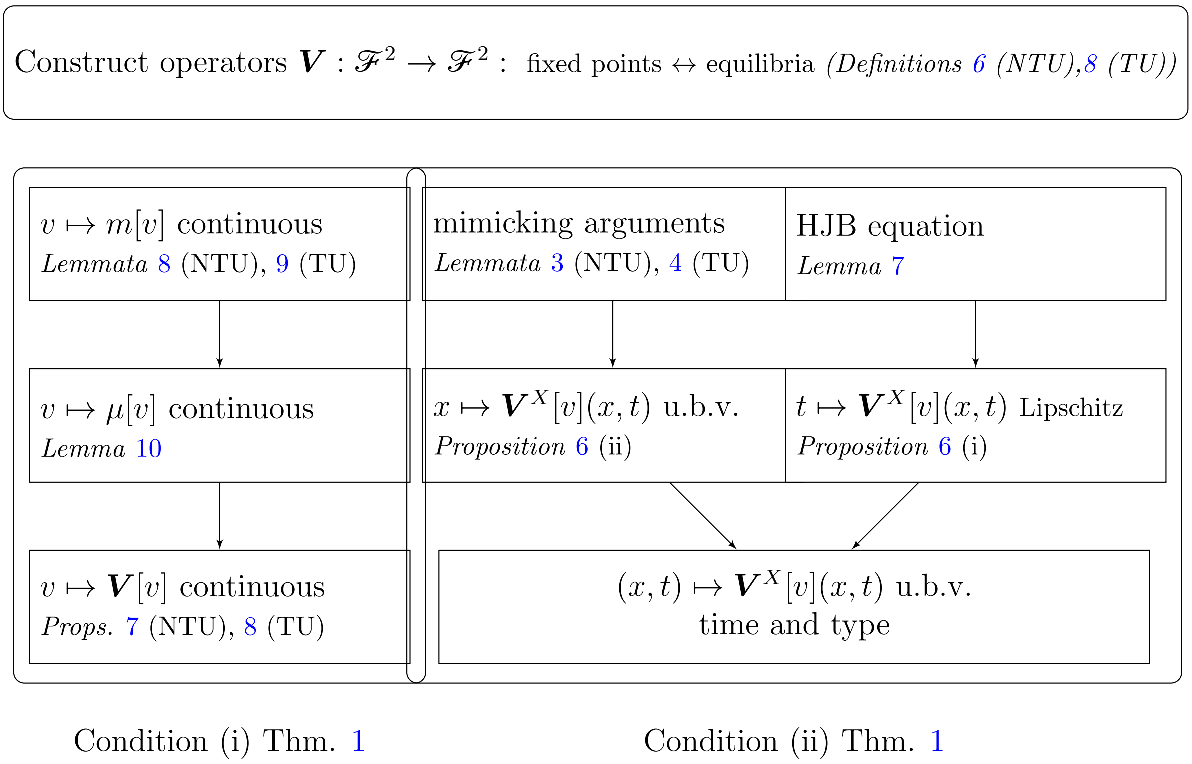

Figure 1: A schematic overview of the proof of Theorem 2.

5.1 Construction of the Fixed Point Operators

We construct two separate fixed point operators, denoted $\overset{NTU}{\boldsymbol{V}}$ and $\overset{TU}{\boldsymbol{V}}$. In the interest of brevity, we detail only the NTU construction in the main text and relegate the TU construction to the appendix.

Non-Transferable Utility

To compute his value-of-search, an agent must hold a belief over the likelihood of future meetings. This is a function of the underlying state variable μ_t and time. We begin by defining the aggregate population dynamics under the point belief that other agents' value-of-search is v.

Definition 5. $\overset{NTU}{\boldsymbol{\mu}}_t[v]$ *is the unique solution to* (1) *for given* $(\mu_0, \lambda, \eta, \overset{NTU}{m}[v])$*, where*

$$\overset{NTU}{m}_t[v](x,y) = \begin{cases} 1 & \textit{if } \pi^X(y|x) \geq v_t^X(x) \textit{ and } \pi^Y(x|y) \geq v_t^Y(x) \\ 0 & \textit{otherwise} \end{cases}$$

is the aggregate probability of matching upon meeting under v*.*

In contrast, agent type x accepts any match whose payoff exceeds his expected discounted match payoff under v, not the value-of-search $v_t^X(x)$ he ascribes to other agents of identical type x. As in the set-up, this match acceptance rule gives rise to an implicit definition of the value-of-search.

Definition 6. *The out-of-equilibrium value-of-search given* v[29] *is the solution to*

$$\overset{NTU}{\boldsymbol{V}}{}_t^X[v](x) = \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \pi^X(y|x) \overset{NTU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x) dy \, d\tau, \tag{5}$$

where x's match acceptance decisions are individually rational,

$$\overset{NTU}{\boldsymbol{m}}{}_t[v](x,y) = \begin{cases} 1 & \textit{if } \pi^X(y|x) \geq \overset{NTU}{\boldsymbol{V}}{}_t^X[v](x) \textit{ and } \pi^Y(x|y) \geq v_t^Y(y) \\ 0 & \textit{otherwise,} \end{cases}$$

and the probability of meetings is pinned down by aggregate match decisions,

$$\overset{NTU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x) = \lambda^X(\tau, \overset{NTU}{\boldsymbol{\mu}}_\tau[v])(y|x) \overset{NTU}{\boldsymbol{m}}{}_\tau[v](x,y) \exp\Big\{-\int_t^\tau \int_0^1 \lambda^X(r, \overset{NTU}{\boldsymbol{\mu}}_r[v])(y'|x) \overset{NTU}{\boldsymbol{m}}{}_r[v](x,y') dy' dr\Big\}.$$

[29] We remark that the operator $\overset{NTU}{\boldsymbol{V}}{}_t^X[v](x)$ in Definition 6 is well-defined. The unique existence of a solution $\overset{NTU}{\boldsymbol{V}}{}_t^X[v](x)$ to equation (5) follows, because the recursively defined value-of-search is the supremum of the right-hand side of equation (5) over the set of match indicators $\boldsymbol{m}_t(x,y)$ satisfying $\boldsymbol{m}_t(x,y) = 0$ if $\pi^Y(x|y) < v_t^Y(y)$.

Critically, the fixed points of our operator coincide with the set of equilibria.

Remark 1. *For given μ_0, there exists an equilibrium of the NTU search-and-matching economy if and only if $\overset{NTU}{\boldsymbol{V}} : \mathscr{F}^2 \to \mathscr{F}^2$ admits a fixed point.*

To conceptualize this construction, think of $v \in \mathscr{F}^2$ as a point belief about other agents' value-of-search. Under this interpretation $\boldsymbol{V}_t^X[v](x)$ becomes agent type x's time-t out-of-equilibrium value-of-search when expecting other agents to match according to v, yet computing his own value-of-search under the rule that he accepts a match whenever it is optimal for him to do so: accept if the offered match payoff exceeds the discounted expected future match payoff.[30] Observe that this is an interpretation only. Our objective here is to preserve desirable in-equilibrium properties of the value-of-search, not decide what is the most "reasonable" out-of-equilibrium behavior.

5.2 The Mimicking Arguments and Uniformly Bounded Variation

To satisfy Condition (ii) of Theorem 1, we establish the following result.

Proposition 6 (bounded variation of the value-of-search)**.** *In both paradigms:*

(i) *Posit Assumptions 2 and 3. Then the value-of-search is Lipschitz continuous in time; i.e., for all moments in time T : $0 < T < \infty$ there exists $C > 0$ such that for all $0 \leq t_1 < t_2 \leq T$ and $x \in [0,1]$*

$$\left|\boldsymbol{V}_{t_2}^X[v](x) - \boldsymbol{V}_{t_1}^X[v](x)\right| \leq C\left|t_2 - t_1\right| \quad \text{for all } v \in \mathscr{F}^2;$$

(ii) *Posit Assumptions 1, 2, 3 and 4, 5 (NTU) and 7 (TU). Then the value-of-search is of uniformly bounded variation in type; i.e., for all time indices $t \geq 0$, there exists $C > 0$ such that for all partitions of the type interval $[0,1]$*

$$\sum_{i=0}^{m}\left|\boldsymbol{V}_t^X[v](x_{i+1}) - \boldsymbol{V}_t^X[v](x_i)\right| \leq C \quad \text{for all } v \in \mathscr{F}^2.$$

Lipschitz continuity in time of any agent's value-of-search (Condition (i)) is due to an immediate application of the Hamilton-Jacobi-Bellman equation. Uniformly bounded variation in types (Condition (ii)) relies on what we refer to as the mimicking arguments and discuss below. The relevance of these conditions is readily apparent:

[30] For comparison, payoffs in the TU paradigm are computed under x's belief that her threat point will be $\boldsymbol{V}_t^X[v](x)$ whereas her potential partner's threat point is $v_t^Y(y)$.

Corollary 1. *Posit Assumptions 1, 2, 3 (both paradigms) and 4, 5 (NTU), 7 (TU). Then the operators* $\overset{NTU}{\boldsymbol{V}}$ *and* $\overset{TU}{\boldsymbol{V}}$ *satisfy Condition (ii) of Theorem 1.*

Proof. Both $\mathscr{V}_0^1\big(x \mapsto \boldsymbol{V}_t^X[v](x)\big)$ and $\mathscr{V}_{\underline{t}}^{\bar{t}}\big(t \mapsto \boldsymbol{V}_t^X[v](x)\big)$ are uniformly bounded in both paradigms due to Proposition 6 (i) and (ii). Moreover, due to (i), there exists C such that for all $x \in [0,1]$: $\left|\boldsymbol{V}_{t_2}^X[v](x) - \boldsymbol{V}_{t_2}^X[v](x)\right| \leq C\,|t_2 - t_1|$. Hence,

$$\mathscr{V}_2\big((x,t) \mapsto \boldsymbol{V}_t[v](x), [0,1] \times [0,T]\big) \leq \sup_{\mathscr{P}} \sum_{i=1}^{m} \sup_{x\in[0,1]} \left|\boldsymbol{V}_{t_i}^X[v](x) - \boldsymbol{V}_{t_{i-1}}^X[v](x)\right| \leq 2\,C\,T,$$

where $\mathscr{P}$ is any partition of the time interval $[0,T]$. □

The Mimicking Arguments

The remainder of this subsection discusses the proof of Proposition 6 (ii) by means of the mimicking arguments: two lemmata first developed in Bonneton and Sandmann (2021) and Bonneton and Sandmann (2023) to establish sorting results in equilibrium, provided an equilibrium exists. They are equally indispensable in the pursuit of proving equilibrium existence. The reason is that bounded variation is a property of the difference in values-of-search across types. However, non-stationary dynamics typically preclude a closed-form characterization of the value-of-search. Instead, we employ a revealed preferences argument whereby one type replicates ('mimicks') another type's probability of matching with other agents. Since mimicking is not the revealed preference, this bounds the difference in values-of-search by the difference in expected payoffs or outputs with such expectation formed under a discounted measure of the agents' future match prospects.

More specifically, Bonneton and Sandmann (2023) prove the following: *In the NTU paradigm, posit Assumptions 1, 2, 3 and 4. Then for all $x_2 > x_1$ the equilibrium value-of-search satisfies*

$$\overset{NTU}{V}{}_t^X(x_2) - \overset{NTU}{V}{}_t^X(x_1) \geq \int_0^1 \big(\pi^X(y|x_2) - \pi^X(y|x_1)\big) Q_t^X(y|x_1)dy$$

for some discounted density $Q_t^X(y|x_1)$.[31] This result relies on payoff monotonicity (Assumption 4): superior types, being more desirable, can exploit their superior match offerings and replicate match outcomes of any inferior type.

Bonneton and Sandmann (2021) prove an analogous mimicking argument in the TU paradigm. This relies on the preliminary (and well-known) observation that the infratemporal efficiency of

[31] More precisely, $Q_t^X(y|x_1)$ is the discounted future match density of agent type x_1 matching with agent type y.

matching decisions under Nash bargaining bounds the value-of-search: for all $x_2 > x_1$ it holds that

$$\overset{TU}{V}{}_t^X(x_2) \geq \int_t^{\infty} e^{-\rho(\tau-t)} \int_0^1 \pi_t^X(y|x_2) p_{t,\tau}^X(y|x_1) dy \, d\tau.$$

From a mathematical viewpoint, alas, TU payoffs (as defined in (4)) depend on the non-stationary value-of-search: $\pi_t^X(y|x) = V_t^X(x) + \alpha^X[f(x,y) - V_t^X(x) - V_t^Y(y)]$. To establish time-invariant bounds, we must eliminate the value-of-search on the right-hand-side of the integral above. We proceed via iteration and apply the infratemporal efficiency bound to (the weighted discounted future average of) $V_\tau(x_2) - V_\tau(x_1)$. This gives two terms: The first term is, as desired, a weighted sum over the difference in match output. The second term is (a weighted discounted future average over averages of) the expected difference in values-of-search. Recursively, k iterations give $k+1$ terms: The first k terms converge to a (weighted) difference in match output. The $k+1$th term converges to zero as k grows large. In sum, this proves the following: *In the TU paradigm, posit Assumption 1, 2 and 3. Then for all $x_2 > x_1$*[32] *the equilibrium value-of-search satisfies*

$$\overset{TU}{V}{}_t^X(x_2) - \overset{TU}{V}{}_t^X(x_1) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big) Q_t^X(y|x_1) dy$$

for some discounted density $Q_t^X(y|x_1)$. These arguments are fully developed in the appendix (cf. Lemmata 3,5,4). Unlike in the aforementioned papers, we prove here the stronger property that the mimicking arguments hold not just in equilibrium but are satisfied by our operators for any value-of-search profile.

5.3 Continuity of the Fixed Point Operators

We then turn to Item (i) from Theorem 1: continuity of the operator $v \mapsto \boldsymbol{V}[v]$.

Proposition 7 (NTU)**.** *In the NTU paradigm, posit Assumptions 2, 3 and 4. Then for all $\overline{v} \in \mathscr{F}^2$, $t \in [0,\infty)$: for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_0^1 \Big| \overset{NTU}{\boldsymbol{V}}{}_t^X[v](x) - \overset{NTU}{\boldsymbol{V}}{}_t^X[\overline{v}](x) \Big| dx < \epsilon \quad \textit{for all } v : ||v - \overline{v}|| < \delta.$$

[32] Assuming that the meeting rate is not just hierarchical but in fact identical for all types, it is easy to see that this holds for any x_2, x_1 irrespective of ordering. If so, one implication of the bound below proves that the value-of-search is continuous in types.

A (needlessly) stronger result obtains in the TU paradigm.

Proposition 8 (TU)**.** *In the TU paradigm, posit Assumptions 2, 3 and 6. For all* $\overline{v} \in \mathscr{F}^2$*,* $t \in [0, \infty)$*,* $x \in [0,1]$*: for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left| \overset{TU}{\boldsymbol{V}}{}_t^X[v](x) - \overset{TU}{\boldsymbol{V}}{}_t^X[\overline{v}](x) \right| < \epsilon \quad \textit{for all } v : ||v - \overline{v}|| < \delta.$$

To establish these Propositions we rely on two intuitive preliminary results. In Appendix F we show that match indicator functions $v \mapsto m_t[v]$ are continuous in both paradigms (see Lemmata 8 and 9).[33] Following arguments from differential calculus, we then use Grönwall's lemma to track the non-stationary evolution of the state and deduce the continuity of $v \mapsto \boldsymbol{\mu}_t[v]$ (see Lemma 10). Unlike the forward-looking population dynamics, the value-of-search is backward-looking in time. The proof of Propositions 7 and 8 (see Appendices F.5 and F.6) encompasses the infinite time-horizon by considering the auxiliary function $v_t^X(x) = e^{\rho t} V_t^X(x)$ (see Corollary 2) that admits a more tractable HJB equation.

The juxtaposition of both Propositions makes apparent a key difference between the NTU and the TU paradigm. In the TU paradigm, the operator $v \mapsto \boldsymbol{V}_t^X[v](x)$ is continuous. In the NTU paradigm, it need not be. To see this, it is instructive to decompose any type x's time-t match opportunities into marginal and inframarginal prospective partner types. Marginal types y are indifferent between accepting and rejecting x, inframarginal types y strictly prefer entering the match. An increase in other agents' time-t value-of-search has two effects. First, in the TU paradigm x matches with inframarginal consumers at reduced payoffs. Continuity is preserved because the decrease in x's payoff is proportional to the increase in y's value-of-search. Second, in both paradigms x ceases to match with marginal types. The loss of marginal types hurts x in the NTU paradigm because marginal types can be strictly profitable to match with. This gives rise to a discontinuity in the value-of-search operator if the set of marginal types is non-negligible. In the TU paradigm, the loss of marginal types is inconsequential due to the intratemporal efficiency of Nash bargaining: if y is indifferent in between matching and not matching with x, then so is x with regard to y.[34]

[33] Lemma 8 is analogous to Smith (2006) Lemma 8 a) in the steady state. Lemma 9 relaxes Shimer and Smith (2000) Lemma 3 who impose global super- or submodularity. Those are special cases of the Hölder continuity assumption 6.

[34] Note that we did not solve the model by passing to the mean field limit, i.e., by gradually decreasing the scope of individual agents to influence the future evolution of the search pool. Lemma 10 and Propositions 7 and 8 suggest that doing so would not lead to the selection of a different set of equilibria. Suppose that one agent could control the behavior of an interval of agents and thereby exert some non-negligible influence on the evolution of the state. Our results show that as this interval shrinks, such control has an exceedingly vanishing effect on other agents' matching decisions.

6 Discussion

This section motivates the generality of our proof and provides some indications as to how limitations of our model can be addressed.

6.1 Discontinuous Value-of-Search Profiles

Our existence proof, notably Theorem 1, is sufficiently general to accommodate discontinuities in the value-of-search profile across types. We here discuss how discontinuities can arise—even in the absence of discontinuous primitives such as meeting rates, payoffs, or output.

In the non-transferable utility paradigm, the most well-known example of such discontinuity is the steady-state phenomenon of "block segregation" (cf. Smith (2006) and reference therein) whereby identical Neumann-Morgenstern preferences over partners, $\pi^X(y|x) = \gamma(x)\phi(y)$ with ϕ increasing, give rise to distinct matching classes in *all equilibria*. The intuition for this phenomenon, prominently conveyed by Burdett and Coles (1997) for $\pi^X(y|x) = y$, is straightforward: every agent agrees to match with a set that includes the highest types because search frictions make any sufficiently high type desirable to all. Since these high types are universally accepted and, by assumption, have identical preferences, they all share the same acceptance threshold, denoted $\hat{x}$. As a result, the value-of-search jumps at $\hat{x}$.

Beyond the von Neumann-Morgenstern case, two key factors are essential to observe block segregation and thereby discontinuous values-of-search. First, utility must be non-transferable (NTU). Otherwise, in the TU case, the type below $\hat{x}$ could compensate the higher type for accepting a partner below the threshold by making a small transfer.[35,36]

Remark 2 (TU continuity)**.** *Suppose that an equilibrium exists in the TU paradigm. If the output $x \mapsto f(x, y)$ is continuous for population X and meetings are anonymous (i.e., $\lambda_t^X(y|x_1) = \lambda_t^X(y|x_2)$ for all x_1, x_2), then population X's value-of-search $x \mapsto V_t^X(x)$ is continuous.*

Second, when utility is non-transferable (NTU), discontinuities typically require identical acceptance thresholds across types. Identical acceptance thresholds, however, signify a violation of strict positive assortative matching, whereby higher-ranked, more desirable agents are

[35] Transferability guarantees more broadly that if agent heterogeneity aggregates via the price mechanism, e.g., Bewley-style economies, the value function is continuous in types.

[36] We require anonymous meeting rates for this result to hold to ensure that the mimicking argument applies symmetrically, i.e., both a higher type mimicking a lower type and a lower type mimicking a higher type provide bounds on the value-of-search.

choosier. Complementarity conditions on payoffs prevent this.[37]

Remark 3 (NTU continuity)**.** *Suppose that an equilibrium exists in the NTU paradigm. If population Y match acceptance thresholds $\underline{x}_t(y) \equiv \inf\{x : \pi^Y(x|y) \geq V_t^Y(y)\}$ are increasing and population X payoffs $x \mapsto \pi^X(y|x)$ and meeting rates $x \mapsto \lambda_t^X(y|x)$ are continuous in their own type, then population X's value-of-search $x \mapsto V_t^X(x)$ is continuous.*

In the absence of complementarity conditions that ensure strict assortative matching in the NTU paradigm, indifference regions cannot be ruled out.[38]

6.2 Possible Extensions

A vanilla matching model typically assumes quadratic search, constant entry, and supermodular output. Our main result ensures that a non-stationary equilibrium exists in this special case. Going beyond these stylized assumptions, our model also allows to address phenomena of matching markets that are rarely modeled in the literature (including in the steady state). Our assumption of hierarchical search allows to model greater visibility of highly ranked individuals (e.g., embedding skewed attention in professional social media or the added benefit of elite versus state university alumni networks). Time dependence in meeting and entry rates can capture seasonality (e.g., daily demand and traffic jam peaks affecting ride hailing services or different market thickness in hot & cold housing markets). Discontinuities in exogenous payoffs also allow to capture qualitative differences between agents (e.g., houses versus apartments, degree- versus non-degree workers). Finally, our approach invites directly contrasting equilibrium predictions of NTU and TU models (e.g., sticky versus flexible wages; marriages with or without dowry).

Other common model specifications are ruled out by our framework. We here discuss how those could be addressed in extensions of our model.

Free entry

Our model does not allow the entry rate to depend on the value-of-search, as would be the case under free entry. In practice, researchers assume that entry is finitely elastic (e.g., Moll (2020) discussing the original firm-size model by Hopenhayn (1992)). It would be feasible to

[37] Theorem 2 from Bonneton and Sandmann (2023) entails that if $\pi^Y(x|y)$ is log submodular and log submodular in differences, with at least one of these conditions holding strictly, then $\underline{x}_t(y)$ is increasing.

[38] Higher types generally have better matching opportunities, making them more selective. However, without such complementarity, some higher types are less selective about matching with lower types. Depending on the distribution of agents and the meeting rate, an interval of types may be indifferent between accepting or rejecting the same threshold type, $\hat{x}$. As with identical von Neumann-Morgenstern preferences, the consequence is that match opportunities and, therefore, the value-of-search is discontinuous at $\hat{x}$.

accommodate dependency on the value-of-search in our context as well. The only adjustment required would be in Lemma 10, where we propose that the search pool population μ is continuous in the value-of-search V. Upon closer inspection of the proof, the key arguments continue to hold assuming that η^X is Lipschitz continuous in both the state μ and the value-of-search V.

Match destruction

In Shimer and Smith (2000) and Smith (2006), exogenous match destruction allows to maintain a steady state population of unmatched agents. This involves a time-invariant distribution of agents $\ell(x)dx$, with matches destroyed at an exogenous rate δ. Unlike in our framework, agents anticipate re-entry into the search pool. Since core proof concepts, notably the mimicking argument, are unaffected by exogenous re-entry, this extension is straightforward. Endogenous match destruction, where agents may opportunistically destroy matches to re-enter the search pool, has received less attention (Smith (1992) is a notable exception). In the NTU paradigm, endogenous match destruction raises new challenges as the lack of commitment over match duration can make higher types less desirable, invalidating the mimicking argument (cf. Kreutzkamp et al. (2022), Bonneton and Sandmann (2023)). Conversely, in the TU paradigm, such decisions preserve intratemporal efficiency, allowing the proof to accommodate opportunistic match destruction or on-the-job search, as seen in labor economics (e.g., Cahuc et al. (2006)).

Homophily

(as in Alger and Weibull (2013)) occurs when agents of similar characteristics meet more frequently. If this affects all types, our analysis rules this out. Bottom types cannot meet other bottom types at a higher rate because the added heterogeneity at the meeting stage introduces additional variation in the value-of-search that cannot be accounted for by the mimicking arguments (Lemmata 3 and 4). "Homophily at the top" is ruled in, by contrast. Specifically, the assumption of hierarchical search allows higher types to be more likely to meet more desirable prospective partners.

Linear Search

Linear and Cobb-Douglas meeting rates violate Lipschitz continuity in the state and are thus ruled out. Either could be incorporated if the search pool population was bounded away from zero (e.g., due to lower bounds on entry rates).

Discrete types

With discrete types, an equilibrium in pure strategies may not exist. To accommodate mixing, the Kakutani–Glicksberg–Fan fixed point theorem, as utilized by Jovanovic and Rosenthal (1988) and Manea (2017a), provides a path forward.

7 Conclusion

Although many economic questions in the search-and-matching literature concern non-stationary dynamics (see for instance Lise and Robin (2017)), the theoretical literature has confined itself, with few exceptions,[39] to the steady state. This paper proves the existence of a non-stationary equilibrium for a general class of search-and-matching models, encompassing model specifications in Shimer and Smith (2000), Smith (2006) and Lauermann et al. (2020).

The tools we develop here have scope, however, that goes beyond search-and-matching. Our fixed point theorem, coupled with the economic insight born out by the mimicking arguments, is applicable in many related dynamic general equilibrium models with heterogeneous agents (see Achdou et al. (2014)) where the aggregate state evolves deterministically over time.

An interesting open question remains: how large is the set of non-stationary equilibria? Could it even be unique? Our paper does not speak to this question.[40,41] Existing examples of multiplicity (see Burdett and Coles (1998), Manea (2017a), Eeckhout and Lindenlaub (2019)) rely on explicit equilibrium constructions with finitely many (typically two) types. In the continuum model, it is conceivable that the inability of a single agent type to coordinate on different equilibria (e.g., high types accepting or rejecting low types) implies that uniqueness can be restored via an iterated dominance argument.

A Theorem 1: Omitted Proofs

A.1 Proof of Proposition 1

Proof of Proposition 1. We show that $(\mathscr{F}_{(k)}, \mathbf{d})$ is complete and totally bounded. This establishes compactness (see for instance Munkres (2015), Theorem 45.1, p. 274).

We focus on completeness first. By abuse of notation, omit superscripts and let $(F_n)_{n\in\mathbb{N}}$ a

[39] See for instance Boldrin et al. (1993), Burdett and Coles (1998), Shimer and Smith (2001).

[40] We failed in both paradigms at our attempt to construct a contraction mapping, but felt that we came closer in the TU paradigm.

[41] Whether or not an equilibrium is unique is less critical if one is interested in properties that occur in any equilibrium, as is the case for the literature on assortative matching.

Cauchy sequence in $(\mathscr{F}_{(k)}, \mathbf{d})$. Then for each $(x,t) \in [0,1] \times [0,\infty)$ the sequence $\big(F_n(x,t)\big)_{n\in\mathbb{N}}$ converges as $n \to \infty$. Denote $F(x,t)$ its pointwise limit and F the thereby obtained function in $\mathscr{F}$. We first show that $F \in \mathscr{F}_{(k)}$. Fix arbitrary $\epsilon > 0$ and $(x,t),(y,r) \in [0,1] \times [0,\infty)$. Due to pointwise convergence there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\max\Big\{\big|F_n(x,t) - F(x,t)\big|, \big|F_n(y,r) - F(y,r)\big|\Big\} < \frac{\epsilon}{2}.$$

It follows from the triangle inequality and k-Lipschitz continuity of F_N that

$$\begin{aligned}\big|F(x,t) - F(y,r)\big| &\leq \big|F(x,t) - F_N(x,t)\big| + \big|F_N(x,t) - F_N(y,r)\big| + \big|F_N(y,r) - F(y,r)\big| \\ &< \epsilon + k \cdot \max\big\{|x-y|, |t-r|\big\}.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this establishes that $F \in \mathscr{F}_{(k)}$. We then show that $F_n \to F$ in the $\mathbf{d}$-metric. Again fix arbitrary $\epsilon > 0$. If for any given $n \in \mathbb{N}$ the sup is attained for some $t > T$ where $e^{-T} < \epsilon$, clearly $\mathbf{d}(F_n, F) < \epsilon$. Let us then focus our attention on the case $(x,t) \in [0,1] \times [0,T]$. Define $B^{\epsilon}_{(k)}(x,t) = \big\{(y,r) :\, \max\big\{|x-y|, |t-r|\big\} < \frac{\epsilon}{4k}\big\}$ and let $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|F_n(x,t) - F(x,t)| < \frac{\epsilon}{2}$. Then for any $(y,r) \in B^{\epsilon}_{(k)}(x,t)$

$$\begin{aligned}\big|F_n(y,r) - F(y,r)\big| &\leq \big|F_n(y,r) - F_n(x,t)\big| + \big|F_n(x,t) - F(x,t)\big| + \big|F(x,t) - F(y,r)\big| \\ &< 2k \,\max\big\{|x-y|, |t-r|\big\} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Finally observe that the set $\big\{B^{\epsilon}_{(k)}(x,t) :\, (x,t) \in [0,1] \times [0,T]\big\}$ forms an open covering of the compact set $[0,1] \times [0,T]$. Whence there exists a finite subcovering of that set, $\big\{B^{\epsilon}_{(k)}(x_j,t_j) :\, j \in \{1,...,M\}\big\}$. For any $j \in \{1,...,M\}$ let N_j such that for all $n \geq N_j$ we have $\big|F_n(x_j,t_j) - F(x_j,t_j)\big| < \frac{\epsilon}{2}$. Then it follows from the preceding arguments that for all $n \geq N \equiv \max\big\{N_j :\, j \in \{1,...,M\}\big\}$ we have $\mathbf{d}(F_n, F) < \epsilon$. This establishes completeness.

Let's now focus attention on total boundedness. That is, for every $\epsilon > 0$ there exists a finite number M of functions $F_j \in \mathscr{F}$ such that for all $F \in \mathscr{F}_{(k)}$ we have $\mathbf{d}(F_j, F) < \epsilon$ for some $j \in \{1,...,M\}$. We achieve this by choosing a grid $\mathscr{R}^{\epsilon}$ on $[0,1]$ as well as a grid $\mathscr{P}^{\epsilon}$ on $[0,1] \times [0,T]$ for some $T > 0$ such that $e^{-T} < \epsilon$. In particular, let $\mathscr{R}^{\epsilon} = \big\{0, \epsilon, ..., l^{\epsilon}\epsilon\big\}$ where $l^{\epsilon}\epsilon \leq 1 < (l^{\epsilon}+1)\epsilon$ and $\mathscr{P}^{\epsilon} = \big\{\big(\frac{m}{k}\frac{\epsilon}{2}, \frac{n}{k}\frac{\epsilon}{2}\big) :\, m,n \in \{0,...,m^{\epsilon}\} \times \{0,...n^{\epsilon}\}\big\}$ where $\frac{m^{\epsilon}}{k}\frac{\epsilon}{2} \leq 1 < \frac{m^{\epsilon}+1}{k}\frac{\epsilon}{2}$ and $\frac{n^{\epsilon}}{k}\frac{\epsilon}{2} \leq T < \frac{n^{\epsilon}+1}{k}\frac{\epsilon}{2}$. We then consider the (finite) set of grid functions $\mathscr{G}^{\epsilon} = \big\{g : \mathscr{P}^{\epsilon} \to \mathscr{R}^{\epsilon}\big\}$. Let g an element in this set. The corresponding function F_g is defined pointwise where $F_g(x,t) = g\big(\frac{m}{k}\frac{\epsilon}{2}, \frac{n}{k}\frac{\epsilon}{2}\big)$ for $(x,t) \in [\frac{m}{k}\frac{\epsilon}{2}, \frac{m+1}{k}\frac{\epsilon}{2}) \times [\frac{n}{k}\frac{\epsilon}{2}, \frac{n+1}{k}\frac{\epsilon}{2})$. Denote $\mathscr{F}^{\epsilon}_{(k)} \equiv \big\{F_g \in \mathscr{F} :\, g \in \mathscr{G}^{\epsilon}\big\}$ the desired finite set of functions.

ϵ-proximity of $\mathscr{F}_{(k)}$ to $\mathscr{F}^{\epsilon}_{(k)}$ then follows immediately: for arbitrary $F \in \mathscr{F}_{(k)}$ there exists $g \in \mathscr{G}^{\epsilon}$ such that for all $(y,\tau) \in \mathscr{P}^{\epsilon}$ we have $\left|F(y,\tau) - g(y,\tau)\right| \leq \frac{\epsilon}{2}$. Then consider any $(x,t) \in [0,1] \times [0,T]$. Let $(x^{\epsilon}, t^{\epsilon})$ the greatest element in $\mathscr{P}^{\epsilon}$ such that $x^{\epsilon} \leq x$ and $t^{\epsilon} \leq t$. Then by construction $F_g(x^{\epsilon}, t^{\epsilon}) = F_g(x,t)$ and $\max\left\{|x - x^{\epsilon}|, |t - t^{\epsilon}|\right\} \leq \frac{1}{k}\frac{\epsilon}{2}$. Using the triangle inequality and the fact that F is k-Lipschitz continuous, we obtain

$$\left|F(x,t) - F_g(x,t)\right| \leq \left|F(x,t) - F(x^{\epsilon}, t^{\epsilon})\right| + \left|F(x^{\epsilon}, t^{\epsilon}) - \underbrace{F_g(x,t)}_{=F_g(x^{\epsilon},t^{\epsilon})}\right| \leq k \max\left\{|x - x^{\epsilon}|, |t - t^{\epsilon}|\right\} + \frac{\epsilon}{2} \leq \epsilon.$$

As (x,t) was arbitrary, this bound holds uniformly across $[0,1] \times [0,T]$. Meanwhile, for $t > T$ ϵ-closeness is satisfied vacuously, whence the result. □

A.2 Proof of Lemmata 1 and 2

Proof of Lemma 1. $H^m_{(k)}[\mathscr{F}^N] \subseteq \mathscr{F}_{(k)}$. Pick arbitrary $F \in \mathscr{F}^N$. Pick arbitrary $(x_1,t_1),(x_0,t_0) \in [0,1] \times [0,\infty)$. We show that

$$\left|H^m_{(k)}[F](x_1,t_1) - H^m_{(k)}[F](x_0,t_0)\right| \leq k \, \max\left\{|x_1 - x_0|, |t_1 - t_0|\right\} \equiv k\,C.$$

Or, this is vacuously the case if $k\,C > 1$. Thus suppose otherwise that $k\,C \leq 1$. In particular this implies that $C \leq \frac{1}{k} < \frac{2}{k} = b_{(k)}/2$. Then, as Figure 2 illustrates,

$$\begin{aligned}\left|H^m_{(k)}[F](x_1,t_1) - H^m_{(k)}[F](x_0,t_0)\right| &\leq \frac{1}{(b_{(k)})^2} \int\limits_{B_{(k)}(x_1,t_1) \triangle B_{(k)}(x_0,t_0)} d(x,t) \\ &\leq \frac{1}{(b_{(k)})^2} \int\limits_{B_{(k)}(x_1,t_1) \triangle B_{(k)}(x_1+C,t_1+C)} d(x,t) \leq 2\frac{C\,b_{(k)} + (b_{(k)} - C)C}{(b_{(k)})^2} \leq k\,C,\end{aligned}$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference. □

Proof of Lemma 2. Fix $\overline{F} \in \mathscr{F}^N$. Fix $\epsilon > 0$. Let $T \geq 1 : \, e^{-T} < \epsilon$. Define

$$A_n = \left\{t \in [0,T] : \int\limits_0^1 \left|H^m[F](x,t) - H^m[\overline{F}](x,t)\right| dx \leq \frac{\epsilon}{18}\frac{1}{T} \quad \forall F \in \mathscr{F}^N : \mathbf{d}^N(F,\overline{F}) < \frac{1}{n}\right\}.$$

By the continuity Assumption (i) of the theorem there exists $N \in \mathbb{N}$ so that for all $n \geq N$ and $F : \mathbf{d}^N(F,\overline{F}) < \frac{1}{n}$ the Lebesgue measure of $[0,T] \setminus A_n$ is less than $\frac{\epsilon}{18}(b_{(k)})^2$.[42] Then for all

[42]The peculiar number $18 = 2 \cdot 9$ is the pertinent bound here because for any point $(x,t) \in [0,1] \times [0,T]$ there could exist a ball containing nine distinct points $(x',t') \in [-1,2] \times [-1,\infty)$ so that the extension $\hat{H}$ interprets

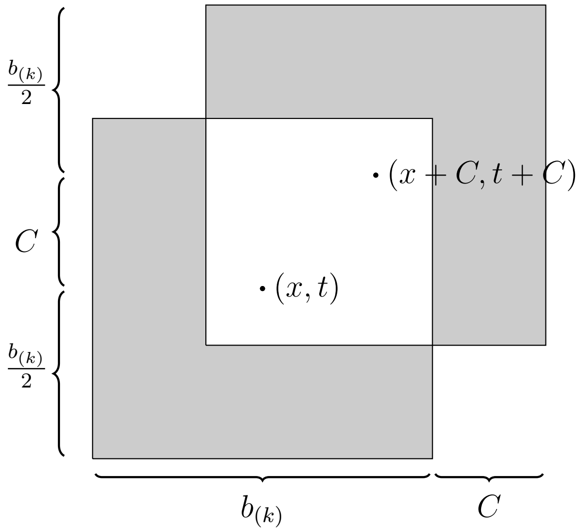

Figure 2: The shaded area corresponds to the measure of $B_{(k)}(x,t)\triangle B_{(k)}(x+C,t+C)$.

$(x_0,t_0)\in[0,1]\times\mathbb{R}_+$ and $\frac{b_{(k)}}{2}\leq 1$

$$e^{-t}\Big|H^m_{(k)}[F](x_0,t_0)-H^m_{(k)}[\overline{F}](x_0,t_0)\Big|\leq\int\limits_{B_{(k)}(x_0,t_0)}\frac{\big|\hat{H}^m[F](x',t')-\hat{H}^m[\overline{F}](x',t')\big|}{(b_{(k)})^2}d(x',t')$$
$$\leq\int\limits_{-1}^{T+1}\int\limits_{-1}^{2}\big|\hat{H}^m[F](x,t)-\hat{H}^m[\overline{F}](x,t)\big|dx\,dt\leq 9\int\limits_{0}^{T}\int\limits_{0}^{1}\big|H^m[F](x,t)-H^m[\overline{F}](x,t)\big|dx\,dt$$
$$\leq 9\int\limits_{A_n}\int\limits_{0}^{1}\big|H^m[F](x,t)-H^m[\overline{F}](x,t)\big|\,dx\,dt+\frac{\epsilon}{2}\leq 9\,T\frac{\epsilon}{18}\frac{1}{T}+\frac{\epsilon}{2}=\epsilon.$$

Since (x_0,t_0) was arbitrary, this bound is uniform, i.e., $\mathbf{d}(H^m_{(k)}[F],H^m_{(k)}[\overline{F}])<\epsilon$ for all F : $\mathbf{d}^N(F,\overline{F})<\delta$ where $\delta\leq\frac{1}{N}$. □

A.3 Proof of Proposition 4

Proof of Proposition 4. By abuse of notation denote $(F^*_{(k)})_{k\in\mathbb{N}}$ the pointwise convergent subsequence with limit point $F^*\in\mathscr{F}^N$. This sequence exists due to Propositions 2 and 3. Then, due to the triangle inequality,

$$\big|\big|F^*-H[F^*]\big|\big|\leq\big|\big|F^*-F^*_{(k)}\big|\big|+\big|\big|H_{(k)}[F^*_{(k)}]-H_{(k)}[F^*]\big|\big|+\big|\big|H_{(k)}[F^*]-H[F^*]\big|\big|,$$

(x',t') as if it were (x,t): $\hat{H}[F](x',t')=H[F](x,t)$ for all $F\in\mathscr{F}^N$.

where we have made use of the fact that $F^*_{(k)}$ is a fixed point, i.e., $H_{(k)}[F^*_{(k)}] = F^*_{(k)}$. By construction, the first term converges as $k \to \infty$; the third term converges because the convolution with an approximate delta function converges in the seminorm defined on compact sets $[0,1] \times [0,T]$ to the function itself (see for instance Königsberger (2004) 10.1 II). This property extends to $[0,1] \times \mathbb{R}_+$ under the discounted semimetric.

With regard to the second term, fix arbitrary $\epsilon > 0$. Then there exists $T > 0$ so that $e^{-T} < \epsilon/2$. Therefore, for arbitrary $\frac{b_{(k)}}{2} \leq 1$, $||H_{(k)}[F^*_{(k)}] - H_{(k)}[F^*]||$ is bounded from above by

$$\max_{n \in \{1,\ldots,N\}} \int_0^T \int_0^1 \Big| \frac{1}{(b_{(k)})^2} \int_{B_{(k)}(x,t)} \big(\hat{H}^n[F^*_{(k)}](x',t') - \hat{H}^n[F^*](x',t')\big) d(x',t') \Big| dx\, dt + \frac{\epsilon}{2}.$$

And the first term is bounded by

$$\begin{aligned}
&\max_{n \in \{1,\ldots,N\}} \int_0^T \int_0^1 \frac{1}{(b_{(k)})^2} \int_{B_{(k)}(x,t)} \Big| \hat{H}^n[F^*_{(k)}](x',t') - \hat{H}^n[F^*](x',t') \Big| d(x',t')\, dx\, dt \\
&\leq \max_{n \in \{1,\ldots,N\}} \int_{-1}^{T+1} \int_{-1}^{2} \Big| \hat{H}^n[F^*_{(k)}](x,t) - \hat{H}^n[F^*](x,t) \Big| dx\, dt \\
&\leq \max_{n \in \{1,\ldots,N\}} 9 \int_0^T \int_0^1 \Big| \hat{H}^n[F^*_{(k)}](x,t) - \hat{H}^n[F^*](x,t) \Big| dx\, dt \\
&= \max_{n \in \{1,\ldots,N\}} 9 \int_0^T \int_0^1 \Big| H^n[F^*_{(k)}](x,t) - H^n[F^*](x,t) \Big| dx\, dt.
\end{aligned}$$

Then recall that Proposition 3 establishes that $F^*_{(k)}$ converges pointwise to F^*. Whence due to the continuity Assumption (i) of the Theorem the expression goes to zero as $k \to \infty$. □

B Set-up: Omitted Proofs

B.1 Derivation of the Match Density

The probability of agent type x not matching during $[t,\tau]$ is $\exp\big\{ - \int_t^\tau \int_0^1 \Lambda^X_r(z|x) dz dr \big\}$ (as defined by the inhomogenous Poisson process). By definition of the density of future matches this expression is equal to $1 - \int_t^\tau \int_0^1 p^X_{t,r}(z|x) dz dr$. Then differentiating with respect to time τ implies that $\int_0^1 p^X_{t,\tau}(z|x) dz = \int_0^1 \Lambda^X_\tau(z|x) dz \exp\big\{ - \int_t^\tau \int_0^1 \Lambda^X_r(z|x) dz dr \big\}$. Since this equation must hold for every time-τ match indicator function $m_\tau(x,y)$, the claimed functional form of $p^X_{t,\tau}(y|x)$ follows from here.

B.2 Proof of Proposition 5

Step 1: We equip the set of possible evolutions of the state μ over a finite time interval with a norm.

Denote $I_\delta(t_0)$ the time interval $[t_0, t_0+\delta)$. Let M_+ be the set of measurable, bounded and non-negative functions $h:[0,1]\to\mathbb{R}_+$. Denote M the identical set without the requirement that functions must be non-negative. Equip M with the seminorm, denoted $||\cdot||_1$, i.e., $||h||_1 = \int_0^1 |h(x)|dx$, and, by abuse of notation, identify M and M_+ with the set of equivalence classes where any two functions that agree almost everywhere belong to the same class. It is well-known that $(M, ||\cdot||_1)$ is a Banach space and $(M_+, ||\cdot||_1)$ is complete. Then define $\mathscr{M}_\delta(t_0)$ the set of continuous mappings $\mu : I_\delta(t_0) \to M_+^2$ where $\mu^X_t(x) \le \overline{\overline{\mu}}_t$ and $\mu^Y_t(y) \le \overline{\overline{\mu}}_t$. We equip $\mathscr{M}_\delta(t_0)$ with the norm

$$||\mu||_{\mathscr{M}_\delta(t_0)} = \sup_{t\in I_\delta(t_0)} \max\big\{||\mu^X_t||_1, ||\mu^Y_t||_1\big\}.$$

Following standard arguments (see Munkres (2015) Theorem 43.6), $\mathscr{M}_\delta(t_0)$ is complete.

Step 2: Fix a time-and type-dependent match probability $m_t(x,y)$ and initial condition $\mu_{t_0}\in M_+^2$. We define a mapping $T:\mathscr{M}_\delta(t_0)\to\mathscr{M}_\delta(t_0)$ whose fixed points $\mu\in\mathscr{M}_\delta(t_0)$ correspond to the solutions of (1) within time interval $I_\delta(t_0)$:

$$(T^X\mu)_t(x) = \min\Big\{\max\Big\{\mu^X_{t_0} + \int_{t_0}^{t} h^X(\tau,\mu_\tau)d\tau; 0\Big\}, \overline{\overline{\mu}}_t\Big\}$$

where $h=(h^X,h^Y): I_\delta(t_0)\times M_+^2 \to M^2$ is

$$h^X(t,\mu_t)(x) = -\mu^X_t(x)\int_0^1 \lambda^X(t,\mu_t)(y|x)m_t(x,y)dy + \eta^X(t,\mu_t)(x).$$

Step 3: We show that T is a contraction mapping for δ sufficiently small. Whence by the contraction mapping theorem it admits a unique fixed point. To begin with, consider arbitrary $\mu',\mu''\in\mathscr{M}_\delta(t_0)$. Then

$$\sup_{t\in I_\delta(t_0)} ||(T^X\mu')_t - (T^X\mu'')_t||_1 \le \delta \sup_{t\in I_\delta(t_0)} ||h^X(t,\mu'_t) - h^X(t,\mu''_t)||_1.$$

Expanding gives, for all $x \in [0,1]$ and $t \in I_\delta(t_0)$,

$$|h^X(t,\mu'_t)(x) - h^X(t,\mu''_t)(x)| \leq |\mu'^X_t(x) - \mu''^X_t(x)| \int_0^1 \lambda^X(t,\mu''_t)(y|x) m_t(x,y) dy$$
$$+ \mu'^X_t(x) \int_0^1 |\lambda^X(t,\mu''_t)(y|x) - \lambda^X(t,\mu'_t)(y|x)| m_t(x,y) dy + |\eta(t,\mu'_t)(x) - \eta(t,\mu''_t)(x)|.$$

We then make use of Assumptions 2 and 3:

$$||h^X(t,\mu'_t)(x) - h^X(t,\mu''_t)(x)||_1 \leq ||\mu'^X_t - \mu''^X_t||_1 \, L^\lambda\big(1+\overline{\overline{\mu}}_t\big) + \big(\overline{\overline{\mu}}_t L^\lambda + L^\eta\big) N(\mu'_t, \mu''_t),$$

whence $||T\mu' - T\mu''||_{\mathscr{M}_\delta(t_0)} \leq \delta\big(L^\eta + L^\lambda + 2L^\lambda \overline{\overline{\mu}}_t\big) ||\mu' - \mu''||_{\mathscr{M}_\delta(t_0)}$. Hence T is a contraction mapping for δ sufficiently small.

Step 4: We establish existence of a unique solution on successive time intervals $[\sum_{\ell=0}^{k} \delta_\ell, \sum_{\ell=0}^{k+1} \delta_\ell)$ beginning at initial time $t = 0$.

To ensure that $T : \mathscr{M}_{\delta_{k+1}}(\sum_{\ell=1}^{k} \delta_k) \to \mathscr{M}_{\delta_{k+1}}(\sum_{\ell=1}^{k} \delta_k)$ is a contraction mapping for each k, we construct the sequence $(\delta_k)_{k \geq 1}$ as the solution to

$$\delta_{k+1}\big(L^\eta + L^\lambda + 2L^\lambda\big(\overline{\overline{\mu}}_0 + \sum_{\ell=1}^{k} \delta_\ell L^\lambda + \delta_{k+1} L^\lambda\big) = \frac{1}{2}.$$

Here we used that $\mu^X_t(x)$ is uniformly bounded by $\overline{\overline{\mu}}_t = \overline{\overline{\mu}}_0 + tL^\lambda$ (cf. Footnote 24).

With T a contraction mapping, the Banach fixed point theorem guarantees the existence of a unique fixed point of T in $\mathscr{M}_{\delta_{k+1}}(\sum_{\ell=1}^{k} \delta_k)$, solution to (1) in $I_{\delta_{k+1}}(\sum_{\ell=1}^{k} \delta_k)$.[43]

Step 5: It remains to show that $\sum_{k=1}^{\infty} \delta_k = \infty$, as this guarantees that the solution to the population dynamics is globally defined for all $t \geq 0$.

Or, solving for δ_{k+1} yields

$$\delta_{k+1} = -\frac{\overline{\overline{\mu}}_0 + \frac{L^\eta + L^\lambda}{2(L^\lambda)^2} + \sum_{\ell=1}^{k} \delta_\ell}{2} + \left[\left(\frac{\overline{\overline{\mu}}_0 + \frac{L^\eta + L^\lambda}{2(L^\lambda)^2} + \sum_{\ell=1}^{k} \delta_\ell}{2}\right)^2 + \frac{1}{4(L^\lambda)^2}\right]^{\frac{1}{2}}.$$

Then suppose by contradiction that $\sum_{k=1}^{\infty} \delta_k$ is finite. If so, per the formula above, δ_{k+1} is uniformly bounded away from 0. And so the sum $\sum_{k=1}^{\infty} \delta_k$ must be infinite. Absurd.

[43]Strictly speaking, the proof of Proposition 5 identifies a unique solution μ to the system (1) *within* the equivalence class of states $\mathscr{M}_{\sum_{k=1}^{\infty} \delta_k}(0)$. Existence of a unique solution to the system (1) *for a fixed type* x is then established as follows: solve (1) for type x only while maintaining that $\mu^X_t(x')$ for $x' \neq x$ and $\mu^Y_t(y)$ are given by the solution $\mu \in \mathscr{M}_{\sum_{k=1}^{\infty} \delta_k}(0)$.

C Construction of the TU Fixed Point Operator

As in the NTU construction, we begin by defining the aggregate population dynamics under the belief v.

Definition 7. $\overset{TU}{\boldsymbol{\mu}}_t[v]$ *is the unique solution to* (1) *for given* $(\mu_0, \lambda, \eta, \overset{TU}{m}[v])$*, where*

$$\overset{TU}{m}_t[v](x,y) = \begin{cases} 1 & \text{if } f(x,y) - v_t^X(x) - v_t^Y(y) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is the aggregate probability of matching upon meeting under the value-of-search profile v*.*

To define the individual value-of-search in the TU paradigm we must also specify future match payoffs. Those are defined implicitly by the Nash bargaining solution. The individual agent believes that her threat point is her actual value-of-search whereas her potential partner's threat point is $v_t^Y(y)$.

Definition 8. *The out-of-equilibrium value-of-search given belief* v *is the solution to*

$$\overset{TU}{\boldsymbol{V}}{}_t^X[v](x) = \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}_t^X[v](y|x) \overset{TU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x) dy \, d\tau,$$

where x*'s subjective match payoffs are*

$$\overset{TU}{\boldsymbol{\pi}}{}_t^X[v](y|x) = \overset{TU}{\boldsymbol{V}}{}_t^X[v](x) + \alpha^X\big(f(x,y) - \overset{TU}{\boldsymbol{V}}{}_t^X[v](x) - v_t^Y(y)\big),$$

x*'s match acceptance decisions are individually rational,*

$$\overset{TU}{\boldsymbol{m}}_t[v](x,y) = \begin{cases} 1 & \text{if } f(x,y) - \overset{TU}{\boldsymbol{V}}{}_t^X[v](x) - v_t^Y(y) \geq 0 \;\;\Leftrightarrow\;\; \overset{TU}{\boldsymbol{\pi}}{}_t^X[v](y|x) \geq \overset{TU}{\boldsymbol{V}}{}_t^X[v](x) \\ 0 & \text{otherwise} \end{cases}$$

and the probability of meetings is pinned down by aggregate match decisions,

$$\overset{TU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x) = \lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x) \overset{TU}{\boldsymbol{m}}_\tau[v](x,y) \exp\Big\{ - \int_t^\tau \int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x) \overset{TU}{\boldsymbol{m}}_r[v](x,y') dy' dr \Big\}.$$

Definition 8 is well-posed for identical reasons as in the NTU paradigm.

D Mimicking Argument

We prove the following results:

Lemma 3 (NTU mimicking argument)**.** *In the NTU paradigm, posit Assumptions 1, 2, 3 and 4. Then, for all $x_2 > x_1$ there exists a non-negative operator $Q^X_t[v](y|x_1)$, with $\int_0^1 Q^X_t[v](y|x_1)dy < 1$, such that*

$$\overset{NTU}{\boldsymbol{V}}{}^X_t[v](x_2) - \overset{NTU}{\boldsymbol{V}}{}^X_t[v](x_1) \geq \int_0^1 \big(\pi^X(y|x_2) - \pi^X(y|x_1)\big)Q^X_t[v](y|x_1)dy.$$

Lemma 4 (TU mimicking argument)**.** *In the TU paradigm, posit Assumption 1, 2 and 3. Then for all $x_2 > x_1$ there exists a non-negative operator $Q^X_t[v](y|x_1)$, with $\int_0^1 Q^X_t[v](y|x_1)dy < 1$, such that*

$$\overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x_1) \geq \int_0^1 \big(f(x_2, y) - f(x_1, y)\big)Q^X_t[v](y|x_1)dy.$$

D.1 Proof of Lemma 4

We introduce a preliminary Lemma.

Lemma 5 (TU intratemporal efficiency)**.** *Under hierarchical search 1 and non-decreasing match payoffs 4: for all $x_2 > x_1$*

$$\overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) \geq \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x_2)\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1)dy\, d\tau.$$

Proof. Define $u(t) = e^{-\rho t}\left\{\overset{TU}{\boldsymbol{V}}{}^X_t[v](x_2) - \int_t^\infty e^{-\rho(\tau-t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}^X_t[v](y|x_2)\overset{TU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1)dy\, d\tau\right\}$.

An identical construction as in Corollary 2 guarantees that for all $T > t$

$$u(T) - u(t) = -\int_t^T e^{-\rho\tau} \int_0^1 \big(\overset{TU}{\boldsymbol{\pi}}{}^X_\tau[v](y|x_2) - \overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x_2)\big)$$
$$\big(\lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x_2)\overset{TU}{\boldsymbol{m}}_\tau[v](x_2, y) - \lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x_1)\overset{TU}{\boldsymbol{m}}_\tau[v](x_1, y)\big)\, dy\, d\tau.$$

Since $\overset{TU}{\boldsymbol{m}}_\tau[v](x_2, y)$ is intratemporally efficient for given payoffs and search is hierarchical, i.e., Assumption 1 holds, it follows that $u(T) - u(t) \leq 0$. Then noting that $u(T) \leq e^{-\rho T}$ and taking the limit establishes that $u(t) \geq 0$. □

Proof of Lemma 4. Define $\bar{\boldsymbol{p}}^{X}_{t_0,t_1}[v](x) = \int_0^1 \overset{TU}{\boldsymbol{p}}{}^{X}_{t_0,t_1}[v](y|x)dy$.

Define for $k = 1, 2, ...$

$$\overset{[k]}{M}{}^{X}_{t}[v](y|x_1) = \int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k - t)}\alpha^X \overset{TU}{\boldsymbol{p}}{}^{X}_{\tau_{k-1},\tau_k}[v](y|x_1)d\tau_k\,(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}^{X}_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell$$

$$\overset{[k]}{R}{}^{X}_{t}[v](x_1,x_2) = \int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k - t)}\big(\overset{TU}{\boldsymbol{V}}{}^{X}_{\tau_k}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^{X}_{\tau_k}[v](x_1)\big)\,(1-\alpha^X)^{k}\prod_{\ell=k}^{1}\bar{\boldsymbol{p}}^{X}_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell.$$

(Note that, due to the order of integration, the product counts downwards from $\ell = k-1$ or $\ell = k$ respectively to 1.) We then prove by induction that

$$\overset{TU}{\boldsymbol{V}}{}^{X}_{t}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^{X}_{t}[v](x_1) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big)\sum_{\ell=1}^{k}\overset{[\ell]}{M}{}^{X}_{t}[v](y|x_1)dy + \overset{[k]}{R}{}^{X}_{t}[v](x_1,x_2).$$

Base case: due to the preceding Lemma 5

$$\overset{TU}{\boldsymbol{V}}{}^{X}_{t}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^{X}_{t}[v](x_1) \geq \int_t^T e^{-\rho(\tau-t)}\int_0^1 \big(\overset{TU}{\boldsymbol{\pi}}{}^{X}_{t}[v](y|x_2) - \overset{TU}{\boldsymbol{\pi}}{}^{X}_{t}[v](y|x_1)\big)\overset{TU}{\boldsymbol{p}}{}^{X}_{t,\tau}[v](y|x_1)dyd\tau$$

$$= \int_0^1\big(f(x_2,y) - f(x_1,y)\big)\int_t^\infty e^{-\rho(\tau-t)}\overset{TU}{\boldsymbol{p}}{}^{X}_{t,\tau}[v](y|x_1)d\tau dy$$

$$+ \int_t^\infty e^{-\rho(\tau-t)}\big(\overset{TU}{\boldsymbol{V}}{}^{X}_{\tau}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^{X}_{\tau}[v](x_1)\big)(1-\alpha^X)\bar{\boldsymbol{p}}^{X}_{t,\tau}[v](x_1)d\tau$$

$$= \int_0^1 \big(f(x_2,y) - f(x_1,y)\big)\overset{[1]}{M}{}^{X}_{t}[v](y|x_1)dy + \overset{[1]}{R}{}^{X}_{t}[v](x_1,x_2)$$

Induction step: Suppose that

$$\overset{TU}{\boldsymbol{V}}{}^{X}_{t}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^{X}_{t}[v](x_1) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big)\sum_{\ell=1}^{k-1}\overset{[\ell]}{M}{}^{X}_{t}[v](y|x_1)dy + \overset{[k-1]}{R}{}^{X}_{t}[v](x_1,x_2).$$

We show that $\overset{[k-1]}{R}{}^{X}_{t}[v](x_1,x_2) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big)\overset{[k]}{M}{}^{X}_{t}[v](y|x_1)dy + \overset{[k]}{R}{}^{X}_{t}[v](x_1,x_2)$ from which the claim follows.

To see this, it suffices to note that once more due to the preceding Lemma we have

$$\overset{[k-1]}{R}{}_t^X[v](x_1,x_2) = \int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-2}}^{\infty} e^{-\rho(\tau_{k-1}-t)}\big(\overset{TU}{\boldsymbol{V}}{}^X_{\tau_{k-1}}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_{\tau_{k-1}}[v](x_1)\big)$$
$$(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}^X_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell$$

$$\geq \int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-2}}^{\infty} e^{-\rho(\tau_{k-1}-t)}\left[\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k-\tau_{k-1})}\int_0^1 \big(\overset{TU}{\boldsymbol{\pi}}{}^X_{\tau_k}[v](y|x_2) - \overset{TU}{\boldsymbol{\pi}}{}^X_{\tau_k}[v](y|x_1)\big)\overset{TU}{\boldsymbol{p}}{}^X_{\tau_{k-1},\tau_k}[v](y|x_1)d\tau_k\right]$$
$$(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}^X_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell$$
$$= \int_0^1 \big(f(x_2,y) - f(x_1,y)\big)\left[\int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k-t)}\alpha^X\overset{TU}{\boldsymbol{p}}{}^X_{\tau_{k-1},\tau_k}[v](y|x_1)\,d\tau_k\right.$$
$$\left.(1-\alpha^X)^{k-1}\prod_{\ell=k-1}^{1}\bar{\boldsymbol{p}}^X_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell\right]dy$$
$$+\int_{\tau_0=t}^{\infty}\int_{\tau_1}^{\infty}\cdots\int_{\tau_{k-1}}^{\infty} e^{-\rho(\tau_k-t)}\big(\overset{TU}{\boldsymbol{V}}{}^X_{\tau_k}[v](x_2) - \overset{TU}{\boldsymbol{V}}{}^X_{\tau_k}[v](x_1)\big)(1-\alpha^X)^k\prod_{\ell=k}^{1}\bar{\boldsymbol{p}}^X_{\tau_{\ell-1},\tau_\ell}[v](x_1)d\tau_\ell.$$

Then define $Q_t^X[v](y|x_1) = \sum_{\ell=1}^{k}\overset{[\ell]}{M}{}_t^X[v](y|x_1)$. $Q_t^X[v](y|x_1)$ which is non-negative. It remains to verify that its integral over y is less than one. To see this, it suffices to note that $\int_0^1 \overset{[k]}{M}{}_t^X[v](y|x_1)dy \leq \alpha^X(1-\alpha^X)^{k-1}$, whence $\int_0^1 Q_t^X[v](y|x_1)dy = \sum_{\ell=1}^{k}\int_0^1 \overset{[k]}{M}{}_t^X[v](y|x_1)dy \leq \sum_{\ell=1}^{k}\alpha^X(1-\alpha^X)^{k-1} = 1.$ □

D.2 Proof of Lemma 3

We prove a slightly stronger result than Lemma 3.[44]

Lemma 6 (NTU mimicking argument)**.** *Under hierarchical search 1: for all $x_2 > x_1$*

$$\overset{NTU}{\boldsymbol{V}}{}_t^X[v](x_2) \geq \int_t^{\infty} e^{-\rho(\tau-t)}\int_0^1 \pi^X(y|x_2)\overset{NTU}{\boldsymbol{p}}{}^X_{t,\tau}[v](y|x_1)dy\,d\tau.$$

Proof. Define $u(t)$ as in the proof of Lemma 5, but now consider exogenous payoffs and the

[44]Our companion paper Bonneton and Sandmann (2023) contains another proof of this result.

NTU value-of-search. Then for all $T > t$

$$u(T)-u(t) = -\int_t^T e^{-\rho\tau} \int_0^1 \big(\pi^X(y|x_2) - \overset{NTU}{\boldsymbol{V}}{}^X_\tau[v](x_2)\big)$$
$$\big(\lambda^X(\tau, \overset{NTU}{\boldsymbol{\mu}}_\tau[v])(y|x_2) \underbrace{\overset{NTU}{\boldsymbol{m}}_\tau[v](x_2,y)}_{1\{\pi^Y(x_2|y)\geq v_t^Y(y)\}1\{\pi^X(y|x_2)\geq v_t^X(x_2)\}} - \lambda^X(\tau, \overset{NTU}{\boldsymbol{\mu}}_\tau[v])(y|x_1)\overset{TU}{\boldsymbol{m}}_\tau[v](x_1,y)\big)\, dy\, d\tau$$

Note that x_2, being of a superior type, is accepted by a greater number of agents. Formally, $1\{\pi^Y(x_1|y) \geq v_t^Y(y)\} = 1 \Rightarrow 1\{\pi^Y(x_2|y) \geq v_t^Y(y)\} = 1$. It follows that the preceding is weakly smaller than

$$-\int_t^T e^{-\rho\tau} \int_0^1 \big(\pi^X(y|x_2) - \overset{NTU}{\boldsymbol{V}}{}^X_\tau[v](x_2)\big)$$
$$1\big\{\pi^Y(x_1|y) \geq v_t^Y(y)\big\}\Big(\lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x_2)1\big\{\pi^X(y|x_2) \geq \overset{NTU}{\boldsymbol{V}}{}^X_\tau[v](x_2)\big\}$$
$$-\lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x_1)1\big\{\pi^X(y|x_1) \geq \overset{NTU}{\boldsymbol{V}}{}^X_\tau[v](x_1)\big\}\Big)dy\, d\tau.$$

This expression is less than zero: First, x_2's acceptance threshold is weakly more desirable for x_2 than whichever threshold is instituted by x_1. Secondly, search is hierarchical. We conclude that $u(t) \geq 0$ by letting T converge to infinity. □

E Bounded Variation: Proof of Proposition 6

Proof of Proposition 6 (ii). Consider a generic function $h(x, y)$ short for $\pi^X(y|x)$ or $f(x, y)$ and L^h short for L^π or L^f. Consider an arbitrary partition of the unit interval $[0, 1]$: $0 = x_0 < x_1 < ... < x_m = 1$. Recall the mimicking argument, namely Lemmata 3 and 4. Those assert that in both the NTU and TU paradigm the difference in values of search can be bounded as follows:

$$\boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1}) \geq \int_0^1 \big(h(x_i, y) - h(x_{i-1}, y)\big)Q_t^X(y|x_i)dy.$$

Further recall that match payoff or output is normalized, i.e., $h(x, y) \in [0, 1]$. Then

$$\sum_{i=1}^m \big|\boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1})\big|$$
$$= -2\sum_{i=1}^m \min\big\{\boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1}), 0\big\} + \sum_{i=1}^m \big(\boldsymbol{V}_t^X[v](x_i) - \boldsymbol{V}_t^X[v](x_{i-1})\big)$$

$$\leq -2\sum_{i=1}^{m} \min\Big\{ \int_0^1 \big(h(x_i,y)-h(x_{i-1},y)\big) Q_t^X(y|x_i)dy,\, 0\Big\} + 1$$

$$\leq 2\int_0^1 \sum_{i=1}^{m} \big|(h(x_i,y)-h(x_{i-1},y))\big| Q_t^X(y|x_i)dy + 1 \leq 2\sup_y \sum_{i=1}^{m} \big|(h(x_i,y)-h(x_{i-1},y)\big| + 1 \leq 2L^h + 1.$$

The last inequality is due to Assumptions 5 and 7 which posit that match payoff and output is of uniformly bounded total variation. □

F Continuity

The proofs of Propositions 7 and 8 make use of the following results:

F.1 Preliminary Results

Lemma 7 (dynamic programming)**.** *In both paradigms:*

$$\frac{\boldsymbol{V}_{t+h}^X[v](x) - \boldsymbol{V}_t^X[v](x)}{h} = \rho \boldsymbol{V}_{t+h}^X[v](x)$$
$$- \frac{1}{h}\int_t^{t+h} e^{-\rho(\tau-t)} \int_0^1 \big(\pi_\tau^X[v](y|x) - \boldsymbol{V}_\tau^X[v](x)\big)\lambda^X(\tau,\boldsymbol{\mu}_\tau[v])(y|x)\boldsymbol{m}_\tau[v](x,y)\,dy\,d\tau + o(1)$$

where $\pi_\tau^X[v](y|x) = \overset{TU}{\boldsymbol{\pi}}{}_\tau^X[v](y|x)$ *in the TU and* $= \pi^X(y|x)$ *in the NTU paradigm.*

The proof is in Appendix F.2.

Corollary 2. *In both paradigms: for* $\boldsymbol{v}_t^X[v](x) = e^{-\rho t}\boldsymbol{V}_t^X[v](x)$

$$\frac{\boldsymbol{v}_{t+h}^X[v](x) - \boldsymbol{v}_t^X[v](x)}{h}$$
$$= -\frac{1}{h}\int_t^{t+h} e^{-\rho\tau} \int_0^1 \big(\pi_\tau^X[v](y|x) - \boldsymbol{V}_\tau^X[v](x)\big)\lambda^X(\tau,\boldsymbol{\mu}_\tau[v])(y|x)\boldsymbol{m}_\tau[v](x,y)\,dy\,d\tau + o(1)$$

where $\pi_\tau^X[v](y|x) = \overset{TU}{\boldsymbol{\pi}}{}_\tau^X[v](y|x)$ *in the TU and* $= \pi^X(y|x)$ *in the NTU paradigm.*

Proof. Observe that

$$\frac{\boldsymbol{v}_{t+h}^X[v](x) - \boldsymbol{v}_t^X[v](x)}{h} = \frac{\boldsymbol{V}_{t+h}^X[v](x) - \boldsymbol{V}_t^X[v](x)}{h}e^{-\rho t} + e^{-\rho t}\frac{e^{-\rho h}-1}{h}\boldsymbol{V}_{t+h}^X[v](x).$$

Then use Lemma 7 to conclude. □

Next, we require, as in the main text (see Definition 1) a notion of distance between arbitrary match indicator functions $m_t(x,y)$. Define $||m|| = \int_0^\infty \int_0^1 \int_0^1 e^{-t}|m_t(x,y)|dx\,dy\,dt$.

Lemma 8 (NTU)**.** *In the NTU paradigm, posit Assumption 4. Then for all* $\overline{v}$ *and* $t \in [0,\infty)$*: for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $||\overset{NTU}{m}[v] - \overset{NTU}{m}[\overline{v}]|| < \epsilon$ *for all* $||v - \bar{v}|| < \delta$.

Lemma 9 (TU)**.** *In the TU paradigm, posit Assumption 6. Then for all* $\overline{v}$ *and* $t \in [0,\infty)$*: for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $||\overset{TU}{m}[v] - \overset{TU}{m}[\overline{v}]|| < \epsilon$ *for all* $||v - \bar{v}|| < \delta$.

Continuity of the match indicator functions implies continuity of the state at all times.

Lemma 10. *In both the NTU and TU paradigm, posit Assumptions 2, 3, 4 and 6. Fix* $\overline{v}$*. Then for all* $t \in [0,\infty)$*: for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_0^1 \left|\boldsymbol{\mu}_t^X[v](x) - \boldsymbol{\mu}_t^X[\overline{v}](x)\right|dx < \epsilon \quad \textit{for all} \quad v : ||v - \bar{v}|| < \delta.$$

F.2 Proof of Lemma 7

We only prove this in the TU paradigm, NTU follows from identical arguments.

Proof of Lemma 7: TU. We make use of the dynamic programming principle:

$$\overset{TU}{\boldsymbol{V}}{}_t^X[v](x) = \int_t^{t+h} e^{-\rho(\tau - t)} \int_0^1 \overset{TU}{\boldsymbol{\pi}}{}_\tau^X[v](y|x)\overset{TU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x)dy\,d\tau$$
$$+ e^{-\rho h}\exp\Big\{-\int_t^{t+h}\int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\}\overset{TU}{\boldsymbol{V}}{}_{t+h}^X(x)[v](y|x)$$

which used that $\overset{TU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x) = \exp\Big\{-\int_t^{t+h}\int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\}\overset{TU}{\boldsymbol{p}}{}_{t+h,\tau}^X[v](y|x)$.

Equivalently, we can write

$$\frac{\overset{TU}{\boldsymbol{V}}{}_{t+h}^X[v](x) - \overset{TU}{\boldsymbol{V}}{}_t^X[v](x)}{h} = -\Bigg\{\frac{1}{h}\int_t^{t+h} e^{-\rho(\tau-t)}\int_0^1 \overset{TU}{\boldsymbol{\pi}}{}_\tau^X[v](y|x)\overset{TU}{\boldsymbol{p}}{}_{t,\tau}^X[v](y|x)dy\,d\tau + \frac{e^{-\rho h}-1}{h}\overset{TU}{\boldsymbol{V}}{}_{t+h}^X[v](x)$$
$$+ e^{-\rho h}\frac{\exp\Big\{-\int_t^{t+h}\int_0^1 \lambda^X(r, \overset{TU}{\boldsymbol{\mu}}_r[v])(y'|x)\overset{TU}{\boldsymbol{m}}_r[v](x,y')dy'dr\Big\} - 1}{h}\overset{TU}{\boldsymbol{V}}{}_{t+h}^X[v](x)\Bigg\}.$$

The term in the curled brackets is finite, whence $\overset{TU}{\boldsymbol{V}}{}_{t+h}^X[v](x) = \overset{TU}{\boldsymbol{V}}{}_t^X[v](x) + o(1)$.[45] This proves

[45]The little-o refers to the Landau notation; $o(1)$ means that $\lim_{h\to 0} o(1) = 0$

Proposition 6 (i). Further note that $\frac{e^{-\rho h}-1}{h} = -\rho + o(1)$ and

$$\overset{TU}{\boldsymbol{p}}{}^{X}_{t,\tau}[v](y|x) = \lambda^X(\tau, \overset{TU}{\boldsymbol{\mu}}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y) + o(1)$$
$$\frac{\exp\{\cdot\}-1}{h} = -\int_t^{t+h}\int_0^1 \lambda^X(\tau, \overset{TU}{\mu}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y)dy\,d\tau + o(1).$$

It follows that

$$\frac{\overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x) - \overset{TU}{\boldsymbol{V}}{}^X_t[v](x)}{h} = \rho \overset{TU}{\boldsymbol{V}}{}^X_{t+h}[v](x)$$
$$-\frac{1}{h}\int_t^{t+h} e^{-\rho(\tau-t)}\int_0^1 \Big(\underbrace{\overset{TU}{\boldsymbol{\pi}}{}^X_\tau[v](y|x) - \overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x)}_{\alpha^X\left(f(x,y)-\overset{TU}{\boldsymbol{V}}{}^X_\tau[v](x)-Z^Y_\tau(y)\right)}\Big)\lambda^X(\tau, \overset{TU}{\mu}_\tau[v])(y|x)\overset{TU}{\boldsymbol{m}}_\tau[v](x,y)\,dy\,d\tau + o(1).$$

□

F.3 Proof of Lemma 8

Proof of Lemma 8. Observe that $\left|\overset{NTU}{m}_t[v](x,y) - \overset{NTU}{m}_t[\overline{v}](x,y)\right|$ is smaller than

$$\left|1\{\pi(y|x) \geq v^X_t(x)\} - 1\{\pi(y|x) \geq \overline{v}^X_t(x)\}\right| + \left|1\{\pi(x|y) \geq v^Y_t(y)\} - 1\{\pi(x|y) \geq \overline{v}^Y_t(y)\}\right|.$$

We bound the double integral of the first term: For any x observe that

$$\left|1\{\pi^X(y|x) \geq v^X_t(x)\} - 1\{\pi^X(y|x) \geq \overline{v}^X_t(x)\}\right|$$
$$= 1\Big\{y : \min\{v^X_t(x), \overline{v}^X_t(x)\} \leq \pi^X(y|x) < \max\{v^X_t(x), \overline{v}^X_t(x)\}\Big\}.$$

Fix some y such that $\pi^X(y|x) \in \big[\min\{v^X_t(x), \overline{v}^X_t(x)\}, \max\{v^X_t(x), \overline{v}^X_t(x)\}\big]$.

Case 1: $y : \pi^X(y|x) \leq \overline{v}^X_t(x)$. Denote $\overline{y}$ the greatest y such that $\lim_{y\uparrow\overline{y}} \pi(y|x) \leq \overline{v}^X_t(x)$. Then

$$\pi^X(\overline{y}|x) - \pi^X(y|x) \leq \left|v^X_t(x) - \overline{v}^X_t(x)\right|.$$

Case 2: $y : \pi^X(y|x) \geq \overline{v}^X_t(x)$. Denote $\overline{y}$ the smallest y such that $\lim_{y\downarrow\overline{y}} \pi(y|x) \geq \overline{v}^X_t(x)$.

$$\pi^X(y|x) - \pi^X(\overline{y}|x) \leq \left|v^X_t(x) - \overline{v}^X_t(x)\right|.$$

It follows that

$$\begin{aligned}
&\int_0^1\int_0^1 1\Big\{y:\min\{v_t^X(x),\overline{v}_t^X(x)\}\le \pi^X(y|x)\le \max\{v_t^X(x),\overline{v}_t^X(x)\}\Big\}dy\,dx\\
&\le \int_{\mathscr{X}^\Delta} 1\Big\{y:\min\{v_t^X(x),\overline{v}_t^X(x)\}\le \pi^X(y|x)\le \max\{v_t^X(x),\overline{v}_t^X(x)\}\Big\}d(x,y)+C\Delta^\alpha\\
&\le \int_0^1 \frac{1}{\Delta}\big|v_t^X(x)-\overline{v}_t^X(x)\big|dx+C\Delta^\alpha,
\end{aligned}$$

and we can similarly bound

$$\begin{aligned}
&\int_0^1\int_0^1 1\Big\{x:\min\{v_t^Y(y),\overline{v}_t^Y(y)\}\le \pi^Y(x|y)\le \max\{v_t^Y(y),\overline{v}_t^Y(y)\}\Big\}dy\,dx\\
&\le \int_0^1 \frac{1}{\Delta}\big|v_t^Y(y)-\overline{v}_t^Y(y)\big|dy+C\Delta^\alpha.
\end{aligned}$$

In effect, we can conclude that

$$\begin{aligned}
\big|\big|\overset{NTU}{m}[v]-\overset{NTU}{m}[\overline{v}]\big|\big| &\le \frac{1}{\Delta}\int_0^\infty e^{-t}\Bigg\{\int_0^1 \big|v_t^X(x)-\overline{v}_t^X(x)\big|dx+\int_0^1 \big|v_t^Y(y)-\overline{v}_t^Y(y)\big|dy+2C\Delta^\alpha\Bigg\}dt\\
&\le \frac{2}{\Delta}\big|\big|v-\overline{v}\big|\big|+2C\Delta^\alpha.
\end{aligned}$$

Then for any $\epsilon>0$ let $\Delta:\ 2C\Delta^\alpha<\frac{\epsilon}{2}$. And let $\delta:\ \frac{2}{\Delta}\delta<\frac{\epsilon}{2}$. □

F.4 Proof of Lemma 9

Proof of Lemma 9. Step 1: Observe that for all v

$$\begin{aligned}
&\big|\overset{TU}{m}_t[v](x,y)-\overset{TU}{m}_t[\overline{v}](x,y)\big| = \big|1\{f(x,y)\ge v_t^X(x)+v_t^Y(y)\}-1\{f(x,y)\ge \overline{v}_t^X(x)+\overline{v}^Y(y)\}\big|\\
&\quad = 1\Big\{y:\min\big\{v_t^X(x)+v_t^Y(y),\overline{v}_t^X(x)+\overline{v}_t^Y(y)\big\}\le f(x,y)<\max\big\{v_t^X(x)+v_t^Y(y),\overline{v}_t^X(x)+\overline{v}_t^Y(y)\big\}\Big\}\\
&\quad < 1\Big\{(x,y):\big|f(x,y)-\overline{v}_t^X(x)-\overline{v}_t^Y(y)\big|<2\max\big\{\big|v_t^X(x)-\overline{v}_t^X(x)\big|,\big|v_t^Y(y)-\overline{v}_t^Y(y)\big|\big\}\Big\}.
\end{aligned}$$

Step 2: Denote $\overline{S}_t(x,y)=f(x,y)-\overline{v}_t^X(x)-\overline{v}_t^Y(y)$ and define

$$\begin{aligned}
\mathscr{D}_t^{>\Delta,\delta}(x) &= \big\{y:|\overline{S}_t(x,y)|<\delta\wedge(x,y)\in\mathscr{X}^{>\Delta}\big\}\\
\mathscr{D}_t^{<-\Delta,\delta}(x) &= \big\{y:|\overline{S}_t(x,y)|<\delta\wedge(x,y)\in\mathscr{X}^{<-\Delta}\big\}.
\end{aligned}$$

Then consider pairs $(x,y),(x',y),(x,y'),(x',y')$ in $\mathscr{Z}^{>\Delta}$ (an analogous construction applies for $\mathscr{Z}^{>\Delta}$). The triangular inequality implies that

$$|\overline{S}_t(x',y')| \geq |\overline{S}_t(x,y)-\overline{S}_t(x',y)-\overline{S}_t(x,y')+\overline{S}_t(x',y')|-|\overline{S}_t(x,y)-\overline{S}_t(x',y)-\overline{S}_t(x,y')| \geq \Delta-3\delta > \delta$$

if $\Delta > 4\delta$. In effect, for such chosen δ, Δ the set $\mathscr{D}_t^{>\Delta,\delta}(x) \cap \mathscr{D}_t^{>\Delta,\delta}(x')$ contains at most one point y for all x, x'.

Step 3: We claim that $\int_{\mathscr{Z}^{>\Delta}} 1\{|\overline{S}_t(x,y)| < \delta\} d(x,y) = \int_0^1 \int_{\mathscr{D}_t^{>\Delta,\delta}} dy dx \underset{\delta \to 0}{\longrightarrow} 0$.[46] An analogous construction applies for $\mathscr{Z}^{<-\Delta}$. If not, for some k there are infinitely many $\{x_n\}$ with $\int_{\mathscr{D}_t^{>\Delta,\delta}(x)} dy > \frac{1}{k}$, whereupon $\sum_{n=1}^{\infty}\big(\int_{\mathscr{D}_t^{>\Delta,\delta}(x_n)} dy\big) = \infty$. Since, for $\delta < \Delta/4$, $\mathscr{D}_t^{>\Delta,\delta}(x_i) \cap \mathscr{D}_t^{>\Delta,\delta}(x_j) \equiv y_{ij}$ contains at most one point, $N_t^{>\Delta,\delta} = \cup_{i,j=1}^{\infty} y_{i,j}$ is countable and so $\int_{N_t^{>\Delta,\delta}} dy = 0$. Also, $\mathscr{D}_t^{>\Delta,\delta}(x_i) \setminus N_t^{>\Delta,\delta}$ and $\mathscr{D}_t^{>\Delta,\delta} \setminus N_t^{>\Delta,\delta}$ are disjoint for all $i \neq j$. This gives the absurd assertion that

$$1 \geq \int_{\cup_{n=1}^{\infty} \mathscr{D}_t^{>\Delta,\delta}(x_n) \setminus N_t^{>\Delta,\delta}} dy = \sum_{n=1}^{\infty} \int_{\mathscr{D}_t^{>\Delta,\delta}(x_n) \setminus N_t^{>\Delta,\delta}} dy = \sum_{n=1}^{\infty} \int_{\mathscr{D}_t^{>\Delta,\delta}(x_n)} dy = \infty.$$

Step 4: Pick arbitrary $\epsilon > 0$. Let $T > 0$ so that $(1-e^{-T}) < \frac{\epsilon}{4}$. And let $\Delta > 0 : C\Delta^{\alpha} T \leq \frac{\epsilon}{4}$. Then due to Step 1 and Assumption 6 it holds that

$$\begin{aligned}
&\big|\big|\overset{TU}{m}[v] - \overset{TU}{m}[\overline{v}]\big|\big| \\
&< \int_0^T \int_{\mathscr{Z}^{>\Delta}} 1\Big\{(x,y) : \big|\overline{S}_t(x,y)\big| < 2\max\big\{\big|v_t^X(x) - \overline{v}_t^X(x)\big|, \big|v_t^Y(y) - \overline{v}_t^Y(y)\big|\big\}\Big\} d(x,y) dt \\
&\quad + \int_0^T \int_{\mathscr{Z}^{<-\Delta}} 1\Big\{(x,y) : \big|\overline{S}_t(x,y)\big| < 2\max\big\{\big|v_t^X(x) - \overline{v}_t^X(x)\big|, \big|v_t^Y(y) - \overline{v}_t^Y(y)\big|\big\}\Big\} d(x,y) dt + \frac{\epsilon}{2}.
\end{aligned}$$

We then twice apply Egorov's theorem. First, due to Step 3, there exists $\delta_1 > 0$ with $4\delta_1 < \Delta$ so that for all $\delta' < \delta_1$ the set of $t \in [0,T]$ for which

$$\int_{\mathscr{Z}^{>\Delta}} 1\{|\overline{S}_t(x,y)| < \delta'\} d(x,y) + \int_{\mathscr{Z}^{<-\Delta}} 1\{|\overline{S}_t(x,y)| < \delta'\} d(x,y) \geq \frac{1}{T}\frac{\epsilon}{4}$$

has mass at most $\frac{\epsilon}{8} \geq 0$. Second, we choose $\delta_2 > 0$ so that for all $\delta' < \delta_2$ the set of $(t,x,y) \in$

[46] We follow Shimer and Smith (2000), Lemma 3, Step 1, second paragraph.

$[0, T] \times [0, 1] \times [0, 1]$ for which

$$\left|v_t^X(x) - \overline{v}_t^X(x)\right|, \left|v_t^Y(y) - \overline{v}_t^Y(y)\right| > \delta'$$

has mass at most $\frac{\epsilon}{8}$. In effect, for $\delta^\star = \min\{\delta_1, \delta_2\}$ it holds that $\left|\left|\overset{TU}{m}[v] - \overset{TU}{m}[\overline{v}]\right|\right| < \epsilon$ for all $v : \left|\left|v - \overline{v}\right|\right| < \delta^*$ as claimed. □

F.5 Proof of Lemma 10

Proof of Lemma 10. Step 1: Manipulating (1) gives

$$\boldsymbol{\mu}_t^X[v](x) - \boldsymbol{\mu}_t^X[\overline{v}](x) = \int_0^t \Bigg\{ \boldsymbol{\mu}_\tau^X[\overline{v}](x) \int_0^1 \lambda^X(\tau, \boldsymbol{\mu}_\tau[\overline{v}])(y|x) m_\tau[\overline{v}](x, y) dy$$

$$- \boldsymbol{\mu}_\tau^X[v](x) \int_0^1 \lambda^X(\tau, \boldsymbol{\mu}_\tau[v])(y|x) m_\tau[v](x, y) dy + \eta^X(\tau, \boldsymbol{\mu}_\tau[v])(x) - \eta^X(\tau, \boldsymbol{\mu}_\tau[\overline{v}])(x) \Bigg\} d\tau$$

$$= \int_0^t \Bigg\{ \big(\boldsymbol{\mu}_\tau^X[\overline{v}](x) - \boldsymbol{\mu}_\tau^X[v](x)\big) \int_0^1 \lambda^X(\tau, \boldsymbol{\mu}_\tau[\overline{v}])(y|x) m_\tau[\overline{v}](x, y) dy$$

$$+ \boldsymbol{\mu}_\tau^X[v](x) \int_0^1 \big(\lambda^X(\tau, \boldsymbol{\mu}_\tau[\overline{v}])(y|x) - \lambda^X(\tau, \boldsymbol{\mu}_\tau[v])(y|x)\big) m_\tau[\overline{v}](x, y) dy$$

$$+ \boldsymbol{\mu}_\tau^X[v](x) \int_0^1 \lambda^X(\tau, \boldsymbol{\mu}_\tau[v])(y|x) \big(m_\tau[\overline{v}](x, y) - m_\tau[v](x, y)\big) dy + \eta^X(\tau, \boldsymbol{\mu}_\tau[v])(x) - \eta^X(\tau, \boldsymbol{\mu}_\tau[\overline{v}])(x) \Bigg\} d\tau.$$

Using Assumptions 2 and 3 we obtain the following upper bound:

$$\int_0^1 \left|\boldsymbol{\mu}_t^X[v](x) - \boldsymbol{\mu}_t^X[\overline{v}](x)\right| dx \leq \big(1 + \overline{\overline{\mu}}_t\big) L^\lambda \int_0^t \int_0^1 \left|\boldsymbol{\mu}_\tau^X[\overline{v}](x) - \boldsymbol{\mu}_\tau^X[v](x)\right| dx\, d\tau$$

$$+ \overline{\overline{\mu}}_t L^\lambda \int_0^t N(\boldsymbol{\mu}_\tau[v], \boldsymbol{\mu}_\tau[\overline{v}]) d\tau + \overline{\overline{\mu}}_t (1 + \overline{\overline{\mu}}_t) L^\lambda \int_0^t \int_0^1 \int_0^1 \left|m_\tau[\overline{v}](x, y) - m_\tau[v](x, y)\right| dy\, dx\, d\tau$$

$$+ L^\eta \int_0^t N(\boldsymbol{\mu}_\tau[v], \boldsymbol{\mu}_\tau[\overline{v}]) d\tau. \qquad (\star)$$

Step 2: The preceding Lemmata 8 and 9 imply that in both paradigms for all $\xi > 0$ (to be determined) there exists $\delta > 0$ such that for all $v : ||v - \overline{v}|| < \delta$:

$$\int_0^t \int_0^1 \int_0^1 \left| m_\tau[\overline{v}](x, y) - m_\tau[v](x, y) \right| dy\, dx\, d\tau < \xi. \qquad (\star\star)$$

Step 3: We show that $\forall \epsilon > 0\ \exists \delta > 0$ such that $N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) < \epsilon \quad \forall\, v : ||v - \overline{v}|| < \delta$.

Indeed, inequalities ($\star$) and ($\star\star$) jointly imply that

$$N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) \leq \underbrace{\left(\left(1 + \overline{\overline{\mu}}_t\right) L^\lambda + \overline{\overline{\mu}}_t L^\lambda + L^\eta\right)}_{\equiv K_1} \int_0^t N(\boldsymbol{\mu}_\tau[v], \boldsymbol{\mu}_\tau[\overline{v}]) d\tau + \underbrace{\overline{\overline{\mu}}_t \left(1 + \overline{\overline{\mu}}_t\right) L^\lambda}_{\equiv K_2} \xi$$

for all $v : ||v - \overline{v}|| < \delta$. And an application of Grönwall's inequality gives $N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) \leq K_1 \xi e^{K_2 t}$.

Then to satisfy the $\epsilon - \delta$ argument, choose $\xi \equiv \frac{\epsilon}{K_1} e^{-K_2 t}$. □

Proof of Proposition 7

Proof of Proposition 7. Pick h such that $1/h \in \mathbb{N}$. Then, due to Corollary 2, it holds that $\left|\boldsymbol{V}_{t_0}^X[v](x) - \boldsymbol{V}_{t_0}^X[\overline{v}](x)\right|$

$$\begin{aligned}
&\leq e^{\rho t_0} \left\{ e^{-\rho t_0} \left| \boldsymbol{V}_{t_0}^X[v](x) - \boldsymbol{V}_{t_0}^X[\overline{v}](x) \right| - e^{-\rho t_1} \left| \boldsymbol{V}_{t_1}^X[v](x) - \boldsymbol{V}_{t_1}^X[\overline{v}](x) \right| \right\} + e^{-\rho(t_1 - t_0)} \\
&= e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left\{ e^{-\rho(t_0 + nh)} \frac{\left| \boldsymbol{V}_{t_0+nh}^X[v](x) - \boldsymbol{V}_{t_0+nh}^X[\overline{v}](x) \right|}{h} \right. \\
&\qquad \left. - e^{-\rho(t_0 + (n+1)h)} \frac{\left| \boldsymbol{V}_{t_0+(n+1)h}^X[v](x) - \boldsymbol{V}_{t_0+(n+1)h}^X[\overline{v}](x) \right|}{h} \right\} + e^{-\rho(t_1 - t_0)} \\
&= e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left\{ \frac{\left| \boldsymbol{v}_{t_0+nh}^X[v](x) - \boldsymbol{v}_{t_0+nh}^X[\overline{v}](x) \right|}{h} - \frac{\left| \boldsymbol{v}_{t_0+(n+1)h}^X[v](x) - \boldsymbol{v}_{t_0+(n+1)h}^X[\overline{v}](x) \right|}{h} \right\} + e^{-\rho(t_1 - t_0)} \\
&\leq e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left| \frac{\boldsymbol{v}_{t_0+(n+1)h}^X[v](x) - \boldsymbol{v}_{t_0+nh}^X[v](x)}{h} - \frac{\boldsymbol{v}_{t_0+(n+1)h}^X[\overline{v}](x) - \boldsymbol{v}_{t_0+nh}^X[\overline{v}](x)}{h} \right| + e^{-\rho(t_1 - t_0)} \\
&\leq e^{\rho t_0} h \sum_{n=0}^{\frac{1}{h}-1} \left| \frac{1}{h} \int_{t_0+nh}^{t_0+(n+1)h} e^{-\rho t} \int_0^1 \left\{ \left(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x) \right) \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \boldsymbol{m}_t[\overline{v}](x, y) \right. \right.
\end{aligned}$$

$$- \big(\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\big)\lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\boldsymbol{m}_t[v](x,y)\Bigg\} dy\, dt\Bigg| + o(1) + e^{-\rho(t_1-t_0)}$$

$$= e^{\rho t_0}\Bigg| \int_{t_0}^{t_1} e^{-\rho t} \int_0^1 \Bigg\{ \big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big)\lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x)\boldsymbol{m}_t[\overline{v}](x,y)$$

$$- \big(\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\big)\lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\boldsymbol{m}_t[v](x,y)\Bigg\} dy\, dt\Bigg| + e^{-\rho(t_1-t_0)}.$$

Next, recall the definition of $\boldsymbol{m}_t[v](x,y)$. In the NTU paradigm the preceding term is

$$= e^{\rho t_0}\Bigg| \int_{t_0}^{t_1} e^{-\rho t} \int_0^1 \Bigg\{ \big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big)\lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x)$$

$$\Big(1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}\Big)1\big\{\pi^X(y|x) \geq \overline{v}_t^X(x)\big\}$$

$$+ \Bigg[\big(\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\big)\lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x)1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}1\big\{\pi^X(y|x) \geq \overline{v}_t^X(x)\big\}$$

$$- \big(\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\big)\lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}1\big\{\pi^X(y|x) \geq v_t^X(x)\big\}\Bigg]\Bigg\} dy\, dt\Bigg|$$

$$+ e^{-\rho(t_1-t_0)}$$

$$\leq e^{-\rho(t_1-t_0)} + \int_{t_0}^{t_1} e^{-\rho(t-t_0)} \int_0^1 \Bigg\{ (1 + \overline{\overline{\mu}}_t)L^\lambda\Big|1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}\Big|$$

$$+ \Bigg|\Big[\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) - \Big[\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\Big]_+ \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\Bigg|\Bigg\} dy\, dt$$

$$\leq e^{-\rho(t_1-t_0)} + (1 + \overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \Big|1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}\Big| dy\, dt$$

$$+ (1 + \overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \Bigg|\Big[\pi^X(y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ - \Big[\pi^X(y|x) - \boldsymbol{V}_t^X[v](x)\Big]_+\Bigg| dy\, dt$$

$$+ \int_{t_0}^{t_1}\int_0^1 \Big|\lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) - \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x)\Big| dy\, dt$$

$$\leq (1 + \overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \Big|1\big\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\big\} - 1\big\{\pi^Y(x|y) \geq v_t^Y(y)\big\}\Big| dy\, dt + e^{-\rho(t_1-t_0)}$$

$$+ (1 + \overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1} \big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| dt + L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}]) dt$$

where we have made use of Assumption 2. By integrating over all $x \in [0,1]$, it follows that $\int_0^1 \left| \boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x) \right| dx$ is bounded by

$$(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1\int_0^1 \left| 1\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\} - 1\{\pi^Y(x|y) \geq v_t^Y(y)\} \right| dx\, dy\, dt + e^{-\rho(t_1-t_0)}$$

$$+ (1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \left| \boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x) \right| dx\, dt + L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}])dt.$$

To conclude, fix some ξ (yet to be determined). Let t_1 be the smallest time such that $e^{-\rho(t_1-t_0)} < \xi$. The proof of Lemma 8 implies that there exists $\delta_1 > 0$ such that for all $v: ||v - \overline{v}|| < \delta_1$

$$\int_{t_0}^{t_1}\int_0^1\int_0^1 \left| 1\{\pi^Y(x|y) \geq \overline{v}_t^Y(y)\} - 1\{\pi^Y(x|y) \geq v_t^Y(y)\} \right| dx\, dy\, dt < \xi.$$

And Lemma 10 implies that there exists $\delta_2 > 0$ such that $\int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v], \boldsymbol{\mu}_t[\overline{v}])dt < \xi$ for all $v: ||v - \overline{v}|| < \delta_2$ Then set $\delta = \min\{\delta_1, \delta_2\}$. It follows that for all $v: ||v - \overline{v}|| < \delta$

$$\int_0^1 \left| \boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x) \right| dx \leq \underbrace{(1+\overline{\overline{\mu}}_{t_1})L^\lambda}_{\equiv K_1} \int_{t_0}^{t_1}\int_0^1 \left| \boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x) \right| dx\, dt + \underbrace{\left((2+\overline{\overline{\mu}}_{t_1})L^\lambda + 1\right)}_{\equiv K_2} \xi.$$

And an application of Grönwall's inequality gives $\int_0^1 \left| \boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x) \right| dx \leq K_1 \xi e^{K_2(t_1-t_0)}$. Then to satisfy the $\epsilon - \delta$ argument, choose $\xi \equiv \frac{\epsilon}{K_1} e^{-K_2(t_1-t_0)}$. □

F.6 Proof of Proposition 8

Proof of Proposition 8. Pick h such that $1/h \in \mathbb{N}$. Then (details for the first inequality that is not specific to the TU paradigm are given in the proof of Proposition 7)

$$\left| \boldsymbol{V}_{t_0}^X[v](x) - \boldsymbol{V}_{t_0}^X[\overline{v}](x) \right| \leq e^{\rho t_0} \left| \int_{t_0}^{t_1} e^{-\rho t} \int_0^1 \left\{ \left(\pi_t^X[\overline{v}](y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\right) \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \boldsymbol{m}_t[\overline{v}](x,y) \right.\right.$$

$$\left.\left. - \left(\pi_t^X[v](y|x) - \boldsymbol{V}_t^X[v](x)\right) \lambda^X(t, \boldsymbol{\mu}_t[v])(y|x) \boldsymbol{m}_t[v](x,y) \right\} dy\, dt \right| + e^{-\rho(t_1-t_0)}$$

$$\leq \int_{t_0}^{t_1}\int_0^1 \left| \left[\pi_t^X[v](y|x) - \boldsymbol{V}_t^X[\overline{v}](x)\right]_+ \lambda^X(t, \boldsymbol{\mu}_t[\overline{v}])(y|x) \right.$$

$$
- \Big[\pi_t^X[v](y|x) - \boldsymbol{V}_t^X[v](x)\Big]_+ \lambda^X(t,\boldsymbol{\mu}_t[v])(y|x)\Bigg| dy\,dt + e^{-\rho(t_1-t_0)}
$$

$$
= \alpha^X \int_{t_0}^{t_1}\int_0^1 \Bigg\{\bigg|\Big[f(x,y) - \overline{v}_t^Y(y) - \boldsymbol{V}_t^X[\overline{v}](x)\Big]_+ - \Big[f(x,y) - v_t^Y(y) - \boldsymbol{V}_t^X[v](x)\Big]_+\bigg|\lambda^X(t,\boldsymbol{\mu}_t[\overline{v}])(y|x)
$$

$$
+ \Big|\lambda^X(t,\boldsymbol{\mu}_t[\overline{v}])(y|x) - \lambda^X(t,\boldsymbol{\mu}_t[v])(y|x)\Big|\Bigg\} dy\,dt + e^{-\rho(t_1-t_0)}
$$

$$
\leq \alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \Big|\big(\overline{v}_t^Y(y) - \boldsymbol{V}_t^X[\overline{v}](x)\big) - \big(v_t^Y(y) - \boldsymbol{V}_t^X[v](x)\big)\Big| dy\,dt
$$

$$
+ \alpha^X L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v],\boldsymbol{\mu}_t[\overline{v}])dt + e^{-\rho(t_1-t_0)}
$$

$$
\leq \alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1}\int_0^1 \big|\overline{v}_t^Y(y) - v_t^Y(y)\big| dy\,dt + \alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda \int_{t_0}^{t_1} \big|\boldsymbol{V}_t^X[\overline{v}](x) - \boldsymbol{V}_t^X[v](x)\big| dt
$$

$$
+ \alpha^X L^\lambda \int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v],\boldsymbol{\mu}_t[\overline{v}])dt + e^{-\rho(t_1-t_0)}.
$$

To conclude, fix some ξ (yet to be determined). Let t_1 be the smallest time such that $e^{-\rho(t_1-t_0)} < \xi$. To bound the first term, set $\delta_1 = e^{-t_1}\xi$. Lemma 10 implies that there exists $\delta_2 > 0$ such for all $v : ||v - \overline{v}|| < \delta_2$: $\int_{t_0}^{t_1} N(\boldsymbol{\mu}_t[v],\boldsymbol{\mu}_t[\overline{v}])dt < \xi$.

Then set $\delta = \min\{\delta_1,\delta_2\}$. It follows that for all $v : ||v - \overline{v}|| < \delta$

$$
\big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| \leq \underbrace{\alpha^X(1+\overline{\overline{\mu}}_{t_1})L^\lambda}_{\equiv K_1} \int_{t_0}^{t_1} \big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| dt + \underbrace{\alpha^X\big((2+\overline{\overline{\mu}}_{t_1})L^\lambda + 1\big)}_{\equiv K_2}\xi.
$$

And an application of Grönwall's inequality gives $\big|\boldsymbol{V}_t^X[v](x) - \boldsymbol{V}_t^X[\overline{v}](x)\big| \leq K_1\xi e^{K_2(t_1-t_0)}$.

Then to satisfy the $\epsilon - \delta$ argument, choose $\xi \equiv \frac{\epsilon}{K_1}e^{-K_2(t_1-t_0)}$.

□

G Discussion: Omitted Proofs

Proof of Remark 2. Careful inspection of the proof of Lemma 4 reveals that under anonymous meeting rates, the following inequalities hold irrespective of the order of types for any two types

x_1, x_2:

$$\int_0^1 \big(f(x_2,y) - f(x_1,y)\big) Q_t^X[v](y|x_2)dy \geq \overset{TU}{\boldsymbol{V}}{}_t^X[v](x_2) - \overset{TU}{\boldsymbol{V}}{}_t^X[v](x_1) \geq \int_0^1 \big(f(x_2,y) - f(x_1,y)\big) Q_t^X[v](y|x_1)dy.$$

In light of continuity of $x \mapsto f(x, y)$, the difference in values-of-search then tends to zero as $x_2 \to x_1$. □

Proof of Remark 3. Fix types x and $x - \delta$ with $\delta > 0$. We show that:

$$V_t^X(x - \delta) \to V_t^X(x) \quad \text{as} \quad \delta \downarrow 0.$$

(The mapping $x \mapsto V_t^X(x)$ is right-continuous because, by construction, agents accept one another if indifferent. Given our continuity assumptions, this readily implies $\lim_{\delta \downarrow 0} V_t^X(x + \delta) = V_t^X(x)$.)

Step 1: We construct an individual type $x - \delta$ matching rate whereby, at all times, one individual of type $x - \delta$ accepts a match with type y if a type x individual accepts a match with type y, whereas all other agents of type $x - \delta$ continue to best-respond to their match opportunities. Since individual deviations from match acceptance rules do not change the evolution of the search pool, meeting rates remain unaffected. Individual type $x - \delta$ matches with type y thus occur at rate $\lambda_t^X(y|x - \delta)1\{\pi^Y(x - \delta|y) \geq V_t^Y(y)\}1\{\pi^X(y|x) \geq V_t^X(x)\}$. Considering hierarchical search, whereby $\lambda_t^X(y|x - \delta) \leq \lambda_t^X(y|x)$, and vertically differentiated types, whereby $1\{\pi^Y(x - \delta|y) \geq V_t^Y(y)\} \leq 1\{\pi^Y(x|y) \geq V_t^Y(y)\}$, this matching rate is weakly lower than that of x. We then introduce the object $\mathscr{P}_t^X(y|x - \delta)$, which captures, analogous to the construction in the proof of Lemma 3, type $x - \delta$'s resulting discounted probability of matching. In particular, the discounted probability of matching with some type less than $\bar{y}$ is $\int_0^{\bar{y}} \mathscr{P}_t^X(y|x - \delta)dy$.

Step 2: By revealed preferences, the value-of-search for type $x - \delta$ for $\delta \geq 0$ must weakly exceed the expected discounted match payoff when following the constructed matching rate. Thus:

$$\begin{aligned} V_t^X(x - \delta) &\geq \int_0^1 \pi^X(y|x - \delta)\mathscr{P}_t^X(y|x - \delta)dy \\ &= \int_0^1 \pi^X(y|x - \delta)Q_t^X(y|x)dy + \int_0^1 \pi^X(y|x - \delta)(\mathscr{P}_t^X(y|x - \delta) - Q_t^X(y|x))dy. \end{aligned}$$

Step 3: We show that $\mathscr{P}_t^X(y|x-\delta) \to Q_t^X(y|x)$ as $\delta \to 0$. First, note that by assumption, meeting rates satisfy $\lambda_t^X(y|x-\delta) \to \lambda_t^X(y|x)$ as $\delta \to 0$. Additionally, since $y \mapsto x_t(y)$ is assumed to be increasing, the match opportunities for type $x-\delta$, denoted by $y : \pi^Y(x-\delta|y) \geq V_t(y)$, almost surely coincide with the interval $[0, \inf y : x_t(y) \geq x-\delta]$, and $\inf\{y : x_t(y) \geq x-\delta\} \to \inf\{y : x_t(y) \geq x\}$ as δ tends to zero. Consequently, the individual type $x-\delta$ matching rate converges to type x's matching rate, thereby implying the convergence of discounted match probabilities.

Step 4: We deduce from the preceding that $V_t^X(x) - V_t^X(x-\delta)$ is bounded from above by

$$\int_0^1 \big(\pi^X(y|x) - \pi^X(y|x-\delta)\big) Q_t^X(y|x) dy + \int_0^1 \pi^X(y|x-\delta)(\mathscr{P}_t^X(y|x-\delta) - Q_t^X(y|x)) dy.$$

Moreover, due to the mimicking argument (Lemma 3), $V_t^X(x) - V_t^X(x-\delta)$ is bounded from below by

$$\int_0^1 \big(\pi^X(y|x) - \pi^X(y|x-\delta)\big) Q_t^X(y|x-\delta) dy.$$

Then continuity of $x \mapsto \pi^X(y|x)$ and convergence of $\mathscr{P}_t^X(y|x-\delta) \to Q_t^X(y|x)$ as $\delta \to 0$ ensure that both bounds tend to zero as $\delta \to 0$. □

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