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Non-Stationary Search and Assortative Matching

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Non-Stationary Search and Assortative Matching*

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Abstract

This paper studies assortative matching in a *non-stationary* search-and-matching model with non-transferable payoffs. Non-stationarity entails that the number and characteristics of agents searching evolve endogenously over time. Assortative matching can fail in non-stationary environments under conditions for which Morgan (1994) and Smith (2006) show that it occurs in the steady state. This is due to the risk of worsening match prospects inherent to non-stationary environments. The main contribution of this paper is to derive the weakest sufficient conditions on payoffs for which matching is assortative. In addition to known steady state conditions, more desirable individuals must be less risk-averse in the sense of Arrow-Pratt.

Keywords: non-stationary, assortative matching, random search, risk preferences, NTU

1 Introduction

Homer (Odyssey XVII, 218) claims that 'Gods join like things with like things.' This is one of the oldest mentions of *positive assortative matching* (PAM), where individuals with similar characteristics tend to match with one another. Interest in PAM is widespread, partly because it is so frequently observed.[1] To understand the determinants of PAM, it is imperative to study individual match decisions, as first recognized by Becker (1973). We follow his line of inquiry in a model with *time-varying* search frictions that render finding a potential partner haphazard and time-consuming.

The theory of assortative matching amid search frictions is extensive (Smith (2006); Morgan (1994); Shimer and Smith (2000); Atakan (2006)).[2] However, and in line with most of the literature

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[1] Examples include skilled workers employed by exporters (Davidson et al. (2014), Felbermayr et al. (2014)); marriages along wealth, education, or desired fertility (Mare (1991), Charles et al. (2012), Rasul (2008)); friends or study partners sharing altruism and risk attitudes (Jackson et al. (2023)).

[2] Chade et al. (2017) is a self-contained introduction to research on search and assortative matching.

on heterogeneous agent models (Achdou et al. (2014)), formal results are confined to the steady state where match prospects do not evolve, and individual expectations over the future remain unchanged as time goes on. The assumption of stationarity, commonly thought to be without loss, makes complex models more tractable.[3] But it also eclipses time-changing intertemporal trade-offs (e.g., due to seasonality or gradual market clearing) inherent in search.

This paper is the first to derive sufficient conditions for PAM in a non-stationary search-and-matching model. Following Shimer and Smith (2000), PAM means that, upon meeting, higher types match with sets of higher types. By deriving these conditions, we show that the steady state requirement is not always necessary for achieving tractability, nor is without loss: PAM fails in environments where it occurs in the steady state.

We consider a continuous-time, infinite-horizon matching model with two populations, in which pairs of vertically differentiated agents meet randomly at *time-varying* rates. Upon meeting, agents observe each other's type. We follow the NTU (non-transferable utility) paradigm where match payoffs solely depend on both partners' types.[4] If both agents agree, they permanently exit the search pool and enjoy their respective match payoffs. Otherwise, they continue waiting for a more suitable partner. Our model admits as a special case the classic pure search model without recall (McCall (1970), Mortensen (1970)) when one side of the population values all partners the same and is thus non-strategic.[5]

Much can be learned about PAM by studying the one-sided search problem. This is because PAM can be equivalently re-cast in terms of within-population sorting: PAM holds if, for any two agents from the same population, the higher type has a higher match acceptance threshold. Viewing PAM through this one-sided search lens naturally leads to a partial equilibrium analysis exclusively premised on optimal search and encompasses general equilibrium as a special case. To establish sufficient conditions for PAM, it is enough to compare the value-of-search across agents in one population, holding constant the meeting rates and match acceptance thresholds in the opposite population.

To date, the literature has derived equilibrium sorting conditions by drawing on an explicit characterization of the value-of-search in the steady state. Non-stationary analysis forecloses this avenue, as the time-varying value-of-search is a complicated object to handle.[6] We circumvent the ensuing tractability issues by using a revealed preference argument: superior types, being more desirable, can exploit their superior match opportunities and replicate the expected match outcomes of any inferior type. These deviations must be weakly dominated by the actual value-of-search—establishing lower bounds on superior types' value-of-search. The lower bounds serve as the keystone of all of our

[3]For instance, Smith (2011) writes that "Almost all successful research on equilibrium search and matching has assumed a steady-state model. For even the simplest of nonstationary environments can be notoriously intractable".

[4]The NTU paradigm applies, for instance, in environments characterized by the absence of bilateral bargaining (e.g., rent-controlled housing, collective bargaining agreements in the labor market, see Felbermayr et al. (2014), or national wage setting, see Hazell et al. (2022)) or those where bilateral bargaining does not precede match formation (e.g., the classical hold-up problem in household bargaining or team production, see Mazzocco (2007), Rasul (2008), Doepke and Kindermann (2019)).

[5]Smith (1999) studies a non-stationary pure search model without recall in which agents can quit employment and return to the search pool at will.

[6]The value-of-search is characterized by an integral over an infinite time horizon taking as its argument the population dynamics, which are themselves a solution to an infinite-dimensional system of integral equations.

equilibrium sorting results. In particular, we provide a concise proof that unifies several results that hold in stationary environments—two well-known (Theorems 1 and 1′) and two that are new (Propositions 2 and 2′): *if payoffs are log supermodular, then there is PAM when search is costly due to time discounting* as established by Smith (2006); *if payoffs are supermodular, then there is PAM when search entails an explicit time-invariant flow cost* as established by Morgan (1994). Moreover, we derive missing comparative static results in the pure search model under both discounting and explicit search costs: *under identical conditions, higher types pursue higher prizes.*

In a non-stationary environment, steady state sufficient conditions are insufficient to guarantee PAM. Here, unlike in the steady state, the lowest type accepted today need not be the worst possible match outcome for all future times. As the search pool evolves over time, agents may face a less favorable selection of types to match with in the future. And an agent who initially rejects a given type, may accept an inferior type at a later stage. In effect, the agent's decision problem involves weighing a sure match payoff today against both the upside risk of matching with a superior type and the downside risk of ending up with an inferior type in the future. Supermodularity and log supermodularity do not resolve this trade-off. Log supermodularity implies that higher types gain relatively more from being matched with higher types. But it also implies that higher types lose out more from being matched with a lower type. We provide an example of a gradually clearing search pool in which the latter effect dominates: lower, not higher types are choosier. PAM does not occur despite log supermodular payoffs.

The main contribution of this article (Theorems 2 and 2′) is to derive an intuitive condition that guarantees PAM in non-stationary environments. Proposition 3 and 3′ adapt this result to a pure search model. We establish that if the respective steady state sufficient condition holds and payoffs satisfy *log supermodularity in differences*, then there is positive assortative matching across all equilibria. By log supermodularity in differences we mean that, for all $y_1 < y_2 < y_3$ and $x_1 < x_2$, we have

$$\frac{\pi(y_3|x_2) - \pi(y_2|x_2)}{\pi(y_2|x_2) - \pi(y_1|x_2)} \geq \frac{\pi(y_3|x_1) - \pi(y_2|x_1)}{\pi(y_2|x_1) - \pi(y_1|x_1)},$$

where $\pi(y|x)$ represents agent type x's payoff if matched with an agent of type y. Assuming differentiability, this condition is equivalent to log supermodularity of $d_y\pi(y|x)$. Log supermodularity in differences emerges as the missing condition because it ensures that the upside of matching with a higher type vis-à-vis the downside of matching with a lower type is always greater for higher types. Observe that this result holds irrespective of how search cost is modeled. To ensure that PAM occurs in non-stationary environments, we require log supermodularity in differences under both discounting and explicit search cost.

We further prove that our conditions are the minimal ones under which matching away from the steady state is assortative: if one of the two is upset locally, then there exist environments for which PAM does not occur (Propositions 4 and 4′).

To interpret our result, it is instructive to link PAM to a ranking over risk preferences. In particular, when type x's payoff over partners y corresponds to a utility function, log supermodularity in differences defines a ranking over risk preferences in the sense of Arrow (1965)-Pratt (1964). Accordingly, if the respective steady state sufficient condition holds, our main contribution states that *the weakest sufficient conditions for positive assortative matching is that more desirable individuals are less risk-averse.*[7] In applied models, by contrast, the curvature of the payoff function may be unrelated to risk preferences.[8]

1.1 Related Work

Previous forays into non-stationary environments rely on two-type models or stylized payoffs.[9] Research shows that a sorting externality can give rise to endogenous cyclical equilibria (Burdett and Coles (1998)), render welfare-maximizing matching decisions non-stationary (Shimer and Smith (2001)), and sustain multiple equilibrium paths (Boldrin et al. (1993)). A notable exception is Wu (2015), who reports a limit result on the stability of equilibrium matches in a (non-stationary) gradually clearing search pool as search frictions vanish.

"When is matching assortative?" is the central question in the theory of decentralized matching. Becker (1973) famously studied it in an idealized frictionless marriage market. His analysis emphasizes the role of pre-match negotiation in sorting. Under "complete rigidity" in the division of output at the moment of match creation (the NTU paradigm), e.g., due to a hold-up problem, PAM occurs when match payoffs are increasing in the partner's type.[10] Under "complete negotiability" at the moment of match creation (the TU paradigm), PAM occurs when match output satisfies increasing differences.[11,12] Various authors have since extended Becker's initial analysis of frictionless matching markets.[13] Most related to ours is the strand of literature that takes into account search frictions, hitherto with an exclusive focus on the steady state.[14] A common finding is that Becker's conditions

[7]There is mounting empirical evidence that characteristics commonly attributed to desirability such as cognitive skills, education, health or income strongly correlate with risk preferences. See Dohmen et al. (2010) and Dohmen et al. (2011), as well as Guiso and Paiella (2004), Frederick (2005), Benjamin et al. (2013) and Noussair et al. (2013) for evidence. For instance, Dohmen et al. (2010) find that individuals with higher cognitive ability are both more willing to take financial risks and more patient. Moreover, Dohmen et al. (2011) find significant correlations between financial and non-financial measures of risk-aversion. This suggests that those individuals to which society attributes the greatest desirability are also the greatest risk-takers in matching markets.

[8]Online Appendix C illustrates this point by examining marriages between prospective partners who anticipate a hold-up problem over fertility decisions once matched. Match payoffs derive from a model due to Rasul (2008) wherein spouses Nash bargain over transfers after female fertility decisions have been made. The curvature of payoffs is unrelated to risk preferences and exclusively depends on the relevant threat point in the Nash bargaining problem over ex-post transfers.

[9]Recent applied papers, such as Baley et al. (2022) and Lise and Robin (2017), employ new modeling paradigms and numerical analysis to gain quantitative insights into non-stationary matching dynamics.

[10]More generally, Legros and Newman (2010) show that a co-ranking condition of types that requires local monotonicity of payoffs only is necessary and sufficient for PAM.

[11]This condition is commonly thought of as complementarity between assortative types. Increasing differences also plays a role for comparative statics: there is no less PAM with a more complementary production function Cambanis et al. (1976); more recently, Anderson and Smith (2024) impose additional structural assumptions under which they prove the stronger result that there is more PAM with a more complementary production function.

[12]Legros and Newman (2007) consider imperfect transfers that constitute a middle ground between the NTU and TU paradigm.

[13]The TU paradigm in particular has received great attention. Here the equilibrium matching coincides with the output-maximizing matching, allowing techniques from optimal transport to aid the analysis. See for instance Choo and Siow (2006), Chiappori et al. (2017) for the purpose of econometric analysis and Lindenlaub (2017) for studying PAM when agents' types are multidimensional.

[14]Following Postel–Vinay and Robin (2002), an applied literature incorporating search frictions in labor economics focuses on match-to-match transitions and simplifies the complexity of initial match creation by allowing firms to make

alone are insufficient to guarantee PAM, the exception being Atakan (2006). See Smith (2006) (time discounting) and Morgan (1994) (explicit search cost) for the NTU paradigm as well as Shimer and Smith (2000) (time discounting) and Atakan (2006) (explicit search cost) for the TU paradigm where payoffs are determined via Nash bargaining.[15] Smith (2011) reviews this literature.[16]

Log supermodularity in differences (LSD), often framed as a ranking of risk preferences (cf. Arrow (1965)-Pratt (1964) and Diamond and Stiglitz (1974)), plays a prominent role in the literature on monotone comparative statics.[17] It informs various sorting results in moral hazard, test design, mechanism design without transfers and menu pricing.[18] The search-and-matching literature, chiefly Shimer and Smith (2000) in the TU paradigm, has been an early adopter. Smith's (2011) review highlights that in their paper, the ranking of utility functions that sustains increasing choices under uncertainty is key to deriving conditions for PAM. While an as-if interpretation in the TU paradigm—marginal match output is re-cast as a utility function of an auxiliary decision maker—our paper shows that theirs is a prescient insight that applies literally to match payoffs in the NTU paradigm: away from the steady state, match payoffs satisfying LSD is the missing condition that guarantees PAM.

The link between risk preferences and assortative matching has also been made in frictionless contexts in which the purpose of matching is to share risk that materializes after[19] match creation (Serfes (2005), Chiappori and Reny (2016), Schulhofer-Wohl (2006) and Legros and Newman (2007)). These papers suggest that risk-loving individuals match with risk-averse ones to absorb the risk of the latter. Search frictions introduce risk that predates match creation.

2 The Model

There are two distinct populations, denoted X and Y, each containing a continuum of agents that seek to match with someone from the other population. Each agent is characterized by a type which belongs to the unit interval $[0, 1]$.[20] Throughout, we denote by x a type of an agent from population X, and y a type of an agent from population Y. Symmetric constructions apply throughout.

take–it-or–leave–it wage offers conditional on worker characteristics. Lindenlaub and Postel-Vinay (2024) build on this framework to identify the dimensions in which matching is assortative when agent characteristics are multi-dimensional.

[15] Eeckhout and Kircher (2010) depart from random search to derive sufficient conditions for PAM in a model with directed search. One key difference is that the sellers cannot discriminate their prices based on the buyer's type. This may be attributed to information frictions that are not present in the random search framework.

[16] In more recent work, Bonneton and Sandmann (2024), we expand the definition of positive assortative matching by allowing intermediate matching probabilities upon meeting as driven by unobserved heterogeneity. We show that in the TU paradigm, the literature's focus on binary match probabilities, zero or one, masks a shift away from assortative matching as search frictions rise. Since search frictions erode more the bargaining power of more productive agents, agents prioritize waiting for a more productive agents over matching with prospective partners of similar rank. On a technical level, this paper introduces a different inductive mimicking argument that we also rely on in Sandmann and Bonneton (2023).

[17] In the terminology pioneered by Karlin (1968), log supermodularity (LS) is referred to as total positivity of order 2 (STP2).

[18] See Chade and Swinkels (2019), Moreno de Barreda and Safonov (2024), Kattwinkel (2019) and Sandmann (2023)).

[19] Chade and Lindenlaub (2022) study how risk that precedes match creation affects risk-averse workers' skill investments. Atakan et al. (2024) study efficiency of skill investments in a search-and-matching model.

[20] Our focus on the continuum is without loss. Results on PAM extend naturally to the analogous model with finitely many types or agents.

2.1 Individual Problem

Agents engage in time-consuming and random searches for partners. When two agents meet, they observe each other's type. If both agree, they match and exit the search pool; otherwise, they continue searching for a more suitable partner. Each agent maximizes her expected present value of payoffs, discounted at rate $\rho > 0$.

Search. Meetings follow an (inhomogeneous) Poisson point process. Such a process is characterized by the time-variant (Poisson) meeting rate $\lambda_t = (\lambda_t^X, \lambda_t^Y)$ so that $\lambda_t^X(y|x)$ is the rate at which type x meets type y agents at time t. We assume that higher types are more likely to meet prospective partners:

Assumption 1 (hierarchical search)**.** *Higher types meet other agents at a weakly faster rate; that is, $\lambda_t^X(y|x_2) \geq \lambda_t^X(y|x_1)$ for $x_2 > x_1$ and all y and $\lambda_t^Y(x|y_2) \geq \lambda_t^Y(x|y_1)$ for $y_2 > y_1$ and all x.*

Assumption 1 encompasses the commonly studied case of anonymous search, where the meeting rate does not depend on one's type. However, it also allows for high-type-specific advantages in the search process.[21]

Match payoffs. Agents derive a time-independent one-time payoff if matched with another agent and zero if unmatched: denote $\pi^X(y|x) > 0$ the lump-sum payoff of agent type x from population X when matched with agent type y from population Y. Payoffs are bounded and continuous in the partner's type. We further assume that types are vertically differentiated.

Assumption 2 (vertical differentiation)**.** *Match payoffs $y \mapsto \pi^X(y|x)$ and $x \mapsto \pi^Y(x|y)$ are non-decreasing in the partner's type, i.e., $\pi^X(y_2|x) \geq \pi^X(y_1|x)$ for $y_2 > y_1$ and all x, and $\pi^Y(x_2|y) \geq \pi^Y(x_1|y)$ for $x_2 > x_1$ and all y.*

Assumptions 1 and 2 embed two advantages for higher-ranked agent types. First, they meet prospective partners at a weakly faster rate. Second, they are accepted by a greater number of prospective partners. Both assumptions are key to deriving a bound on the value-of-search under mimicking (Lemma 1).

Value-of-search. Upon meeting another unmatched agent, x weighs the immediate match payoff $\pi^X(y|x)$ against the value-of-search $V_t^X(x)$. Naturally, the (weakly dominant[22]) optimal matching decision is to accept to match with y whenever the payoff exceeds the option value-of-search:

$$\pi^X(y|x) \geq V_t^X(x). \tag{1}$$

The optimal stopping rule determines the match indicator function:

$$m_t(x,y) = \begin{cases} 1 & \text{if } \pi^X(y|x) \geq V_t^X(x) \text{ and } \pi^Y(x|y) \geq V_t^Y(y), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

[21]Note that homophily (as in Alger and Weibull (2013)), where agents of similar characteristics meet more frequently, is not encompassed by our analysis.

[22]By focusing on weakly dominant acceptance rules, we discard trivial equilibria in which agents mutually reject advantageous matches.

We denote $y_t(x)$ the infimum type with whom x is willing to match at time t so that $\pi^X(y|x) \geq V_t^X(x)$. As types are vertically differentiated, an agent type x is willing to match with any $y > y_t(x)$ at time t. A symmetric construction applies to $x_t(y)$.

The value-of-search is defined as the discounted expected future match payoff if currently unmatched:

$$V_t^X(x) = \int_t^{\infty} \int_0^1 e^{-\rho(\tau-t)} \pi^X(y|x)\, p_{t,\tau}^X(y|x) dy\, d\tau, \tag{3}$$

where $p_{t,\tau}^X(y|x)$ is the density of future matches with y at time τ conditional on x being unmatched at time t. This is a standard object and is characterized by the matching rate $\lambda_\tau^X(y|x) m_\tau(x, y)$.[23]

2.2 General Equilibrium

Our main result characterizes how match decisions differ across agents. In line with the literature, we consider a partial equilibrium approach —analyze the individual optimization problem when the meeting rate and match opportunities are exogenously given—to establish sufficient conditions under which more desirable individuals set higher search cut-offs. General equilibrium, described in the following, emerges as a special case of this analysis.

Endogenous meetings. Denote $\mu_t = (\mu_t^X, \mu_t^Y)$ the state so that for any $U \subseteq [0, 1]$ the mass of types $x \in U$ is $\int_U \mu_t^X(x)dx$. The initial time 0 distribution is given by μ_0.[24] Then agent type x's time t meeting rate $\lambda_t^X(y|x)$ is a function of the underlying state variable μ_t and time t. Coherence demands that the number of meetings of agent types x with agent types y must be equal to the number of meetings of agent types y with agent types x:[25]

$$\lambda_t^X(y|x)\mu_t^X(x) = \lambda_t^Y(x|y)\mu_t^Y(y).$$

Evolution of the search pool. Population dynamics are governed by entry and exit. The rate at which an individual agent type x matches and exits the market at time t—the hazard rate—is $\int_0^1 m_t(x, y)\lambda_t^X(y|x)dy$. Agent type x's time t entry rate $\eta_t^X(x)$ is a function of time t and the state μ_t. We have:

$$\mu_{t+h}^X(x) = \mu_t^X(x) + \int_t^{t+h} \left\{ -\mu_\tau^X(x) \int_0^1 \lambda_\tau^X(y|x) m_\tau(x, y) dy + \eta_\tau^X(x) \right\} d\tau. \tag{4}$$

The economy is non-stationary whenever the integrand is non-zero so that $\mu_{t+h} \neq \mu_t$.

[23]Formally, $p_{t,\tau}^X(y|x) = \lambda_\tau^X(y|x) m_\tau(x, y) \exp\{-\int_t^\tau \int_0^1 \lambda_r^X(z|x) m_r(x, z) dz dr\}$. Refer to Appendix A.1 in Sandmann and Bonneton (2023) for a formal derivation.

[24]Functions introduced are Lebesgue measurable throughout. This implies that the type distribution is atomless.

[25]To better understand the concepts of coherence and hierarchical search, write (without loss of generality) $\lambda_t^X(y|x) = \phi_t(x, y)\mu_t^Y(y)$ and $\lambda_t^Y(x|y) = \psi_t(x, y)\mu_t^X(x)$. Coherence then implies that $\psi_t(x, y) = \phi_t(x, y)$, while hierarchical search further implies that these functions are non-decreasing in both arguments. Moreover, if the populations are symmetric (and the equilibrium is symmetric), these functions are symmetric as well, i.e., $\psi_t(x, y) = \psi_t(y, x)$.

Equilibrium. An equilibrium of the search-and-matching economy of given initial search pool population μ_0 is a triple $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{m})$, solution to (2), (3) and (4). In a companion paper, Sandmann and Bonneton (2023), we show that a non-stationary search equilibrium exists under minimal regularity conditions.[26]

Note that our model relaxes common assumptions made in the literature, e.g. the economy is in the steady state, there are symmetric populations, search is anonymous and meeting and entry rates are given by specific functional forms. This level of generality helps identify the key assumptions to study equilibrium sorting: hierarchical search (Assumption 1) and vertically differentiated types (Assumption 2).[27]

3 Illustrative Example: One-block Block Segregation

To illustrate the richness of non-stationary matching patterns, we first characterize dynamic sorting in a highly-stylized non-stationary matching market: a closed market where future meetings become rarer over time, $t \mapsto \lambda_t^X(y|x), \lambda_t^Y(x|y)$ are decreasing, and agents have identical preferences over matched partners, $\pi^X(y|x) = y$ and $\pi^Y(x|y) = x$.[28] Our example shows that block segregation (see Smith (2006) and references therein) and perfect sorting can co-exist for different segments of the economy.[30]

A formal characterization of equilibrium matching (which is a corollary of our main result) is deferred to Proposition 5 in Appendix C. At the top, matching patterns bear resemblance to the steady state phenomenon of block segregation: at any moment in time, expanding sets of the most desirable types, $[\underline{y}_t, 1]$ and $[\underline{x}_t, 1]$, form a single matching class; agents within this class exclusively match with each other (see Figure 1b for an overlay of matching sets at different moments in time). Pooling emerges because individuals with the same preferences and identical match opportunities will make the same choices. Expansion of this set whereby $t \mapsto \underline{y}_t, \underline{x}_t$ is decreasing reflects declining meeting rates that make agents less selective over time. Observe that unlike at the top, intermediate types experience a discontinuous improvement in their match opportunities once the most desirable agents begin to accept them. This explains why match acceptance thresholds prior to joining the exclusive matching class rise in time (see Figure 1a). Matching decisions are ordered in the cross-section,

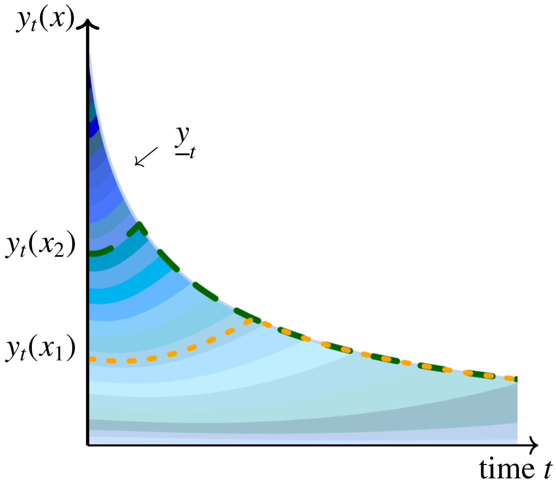
[26] Also see Shimer and Smith (2000), Smith (2006), Lauermann et al. (2020) in the context of a stationary equilibrium with a continuum of agent types.

[27] The meeting technology λ encompasses the most commonly studied meeting rates found in the literature: linear (e.g. Mortensen and Pissarides (1994), Burdett and Coles (1997)) and quadratic search technologies (e.g. Shimer and Smith (2000) and Smith (2006)). The entry rate η encompasses several natural entry rates such as no entry and constant flows of entry (as in Burdett and Coles (1997)). In addition, entry may be generated by exogenous match destruction (as in Shimer and Smith (2000) and Smith (2006)).

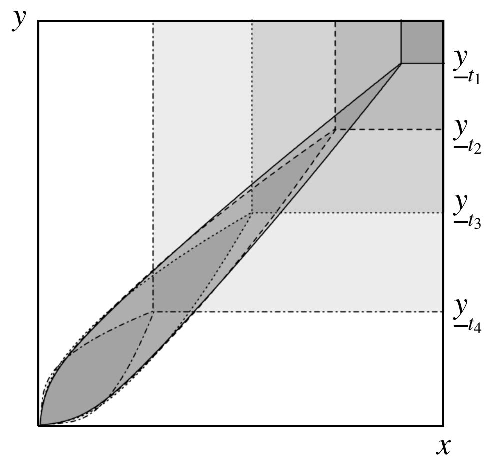
[28] As an example, consider the yearly junior academic job market where the historic norm of not reneging on acceptances resembles our framework with permanent matches.

[29] Matching patterns are derived in general equilibrium where the rate at which agents meet some type y is proportional to the number of unmatched agents $\mu_t^Y(y)$ in a gradually emptying search pool. Parameter values are $\lambda_t^X(y|x) = 5\mu_t^Y(y)$, $\eta_t^X(y|x) = 0$ and $\rho = 0.3$.

[30] Equilibrium behavior mirrors a result in Smith (1992) who considers a model with opportunistic match destruction: lesser types quit temporary matches when they become acceptable to the remaining top types. The key challenge in his paper is that match acceptance does not follow a threshold rule, so characterizing the non-stationary equilibrium becomes forbiddingly difficult.



(a) Match acceptance thresholds over time t



(b) Match acceptance sets at $t \in \{t_1, t_2, t_2, t_4\}$

Figure 1: Non-stationary sorting in a closed market with no new entrants

Populations are symmetric so that $\underline{x}_t = \underline{y}_t$. (a) The highest acceptance threshold $\underline{y}_t$ is depicted by the boundary of the color bands. Types x above $\underline{y}_t$ accept all other y types above $\underline{y}_t$. Color bands below depict the (perfectly ordered) acceptance thresholds $y_t(x)$ chosen by types x below $\underline{y}_t$. Darker colors correspond to higher types. We illustrate this by highlighting the acceptance thresholds for types x_1 and x_2: $y_t(x_2)$ is shown with a dashed line, and $y_t(x_1)$ with a dotted line. Once types are accepted by all agents, their acceptance thresholds coincide with $\underline{y}_t$. Observe that as agents anticipate joining the matching block at the top, they become choosier. (b) Initially (darker color), the exclusive matching class at the top is small. Over time, as the number of desirable agents shrinks, this highest matching class expands to include ever more agents (matching sets depicted using lighter colors).[29]

by contrast. At all times, more highly ranked individuals are weakly more selective in their match acceptances. The next section expands upon this observation.

4 Positive Assortative Matching

This section presents our main results. We derive the weakest sufficient conditions for positive assortative matching (PAM) in non-stationary environments.

4.1 Definition of PAM

PAM means that agents of similar characteristics or rank tend to match with one another. When finding a partner entails search, the flow number of created matches depends on both the number of meetings that take place and individual match decisions. We use the definition of PAM by Shimer and Smith (2000) that disentangles physical search frictions from individual matching decisions. They look at hypothetical matches that would be formed if a meeting took place. Formally, define $U_t \equiv \{(x, y) : m_t(x, y) = 1\}$ the set of pairs who are willing to form a match at time t. Matching is assortative if, when any two agreeable matches in U_t are severed, both the greater two and the lesser two types can be agreeably rematched.

Definition 1 (PAM, (Shimer and Smith (2000)))**.** *There is PAM at time t if $(x_1, y_2) \in U_t$ and $(x_2, y_1) \in U_t$ imply that $(x_1, y_1) \in U_t$ and $(x_2, y_2) \in U_t$ for all types $x_2 > x_1$ and $y_2 > y_1$.*

PAM can be recast in more intuitive terms: higher types match with sets of superior types; or, equivalently, higher types are relatively more selective about who they match with. The following proposition[31] develops this idea formally. Recall that $y_t(x)$ is the infimum type with whom x is willing to match at time t so that $\pi^X(y|x) \geq V_t^X(x)$.

Proposition 1. *(i) If $x \mapsto y_t(x)$ and $y \mapsto x_t(y)$ are non-decreasing then there is PAM at time t. (ii) If there is PAM at time t then $x \mapsto y_t(x)$ and $y \mapsto x_t(y)$ are non-decreasing for all types whose individual matching sets $U_t^X(x) \equiv \{y : m_t(x, y) = 1\}$ and $U_t^Y(y) \equiv \{x : m_t(x, y) = 1\}$ are non-empty.*

4.2 The Mimicking Argument

To derive equilibrium sorting properties, we need to compare the value-of-search across types. Such a comparison is challenging, as the law of motion is intractable in non-stationary environments, making it impossible to characterize the value-of-search in closed form. To circumvent this problem, we apply a revealed preference argument, which we refer to as the *mimicking argument*.[32]

We first note that the value-of-search, defined in Equation (3), admits an integral representation over payoffs that subsumes the time dimension:

$$V_t^X(x) = \int_0^1 \pi^X(y|x) Q_t^X(y|x) dy \quad \text{where} \quad Q_t^X(y|x) \equiv \int_t^\infty e^{-\rho(\tau - t)} \, p_{t,\tau}^X(y|x) d\tau. \tag{5}$$

Here $Q_t^X(y|x)$ corresponds to a density that does not integrate to one: $\int_U Q_t^X(y|x)dy$ represents type x's discounted probability of forming a match with some other agent type $y \in U \subseteq [0, 1]$ some time in the future.

Then observe that higher agent types have better match opportunities. The reasons are twofold. Since match payoffs are monotone (Assumption 2), an agent that is willing to match with a lower agent type x_1 is also willing to match with a higher agent type x_2. And since search is hierarchical (Assumption 1), x_2 meets other agents at a faster rate. Thus, agent type x_2 can in expectation match with all the agent types (and possibly even other, more attractive ones) that agent type x_1 is matching with. Both observations help establish the following lemma,[33] which is the keystone of our proofs for the sorting results in Theorems 1, 1′, 2 and 2′.

Lemma 1 (mimicking argument)**.** *The value-of-search admits the following lower bound:*

$$V_t^X(x_2) \geq \int_0^1 \pi^X(y|x_2) Q_t^X(y|x_1) dy \quad \textit{for all } x_2 > x_1 \in [0, 1]. \tag{6}$$

[31] Shimer and Smith (2000) prove this in the steady state with symmetric populations.

[32] Mimicking has a long tradition in economics, notably in the theory of incentives (cf. Laffont and Martimort (2002)). See especially Lauermann (2013) in the context of a stationary TU matching model and Kirkegaard (2009) in the context of asymmetric first-price auctions. Mimicking arguments also play a major role in our companion paper, Sandmann and Bonneton (2023), where we show that a non-stationary equilibrium exists under minimal regularity conditions.

[33] Lemma 5, and thereby all subsequent results on PAM, readily extends to an environment where higher types are more patient as expressed by their discount factor, i.e., $\rho(x_2) < \rho(x_1)$ for all $x_2 > x_1$.

To prove the lemma we define an auxiliary decision problem that allows more highly ranked agents x_2 to exactly replicate ("mimick") a lesser ranked agent x_1's matching rate. Such mimicking is feasible because higher types have better match opportunities. Then, by revealed preferences, mimicking leads to weakly smaller expected payoffs than following the optimal stopping rule (1).

Proof. Fix $x \in [0,1]$ and $t \in \mathbb{R}_+$. And let $\mathcal{Q}_t^X(x)$ be the space of discounted probabilities $y \mapsto Q_t(y) \in \mathbb{R}_+$ generated by some matching rate $(\tau, y) \mapsto \nu_\tau(y)$ that is feasible, i.e., $\nu_\tau(y) \le \lambda_\tau^X(y|x)$ and acceptable to y, i.e., $\nu_\tau(y) = 0$ if $\pi^Y(x|y) < V_\tau^Y(y)$. (Following standard arguments, the matching rate $(\tau, y) \mapsto \nu_\tau(y)$ defines the match density via $\tilde{p}_{t,\tau}^X(y|x) = \nu_\tau(y)\exp\{-\int_t^\tau \int_0^1 \nu_r(z)dzdr\}$, whence the discounted match probability via $\tilde{Q}_t(y) = \int_t^\infty e^{-\rho(\tau-t)}\tilde{p}_{t,\tau}^X(y)d\tau$.) By construction, $Q_t^X(\cdot|x) \in \mathcal{Q}_t^X(x)$ and

$$V_t^X(x) = \sup_{Q \in \mathcal{Q}_t(x)} \int_0^1 \pi^X(y|x)Q(y)dy.$$

Assumptions 1 and 2 imply that if $y \mapsto \nu_\tau(y)$ is feasible and acceptable for x_1, then it is feasible and acceptable for x_2. Hence, $\mathcal{Q}_t(x_1) \subseteq \mathcal{Q}_t(x_2)$ and

$$V_t^X(x_2) \ge \sup_{Q \in \mathcal{Q}_t(x_1)} \int_0^1 \pi^X(y|x_2)Q(y)dy.$$

The assertion of the lemma then follows because $Q_t^X(\cdot|x_1) \in \mathcal{Q}_t(x_1)$. □

4.3 Stationary Environment

We first use the mimicking argument to revisit the known steady state analysis. This allows us to make transparent how the assumption of stationarity facilitates PAM. A condition on payoffs, log supermodularity, is sufficient for PAM in stationary environments:

Definition 2 (Log supermodularity)**.** *Population X's payoffs are log supermodular if for all $y_2 > y_1$ and $x_2 > x_1$,*

$$\frac{\pi^X(y_2|x_2)}{\pi^X(y_1|x_2)} \ge \frac{\pi^X(y_2|x_1)}{\pi^X(y_1|x_1)}.$$

This condition means that higher types stand relatively more to gain from matching with higher types. If the inequality is reversed, payoffs are log submodular. We find it most instructive to view log supermodular payoffs as a property of time preferences: In a toy model with two agents that have the same discount factors, the higher type will be more inclined to choose a delayed, certain payoff over an immediate one if and only if payoffs are log supermodular.

The following result is due to Smith (2006).

Theorem 1 (stationary PAM, Smith (2006))**.** *Suppose that both populations' payoffs are log supermodular. Then there is positive assortative matching (PAM) in any stationary equilibrium.*

Smith's original proof, motivated by the analysis of block segregation, proceeds recursively from the highest type to the lowest type. Here, we present a shorter proof of a more granular result, Proposition 2, that is based on Lemma 1, which addresses the sorting patterns within a single population.

We deliver two new insights. First, our proof of Proposition 2 makes explicit why the sufficiency of log supermodular payoffs for PAM is specific to stationary environments: our proof uses the fact that, in the steady state, agents always match with a weakly better type than the most desirable type rejected previously. Second, our proof re-frames the across-population matching problem as a within-population sorting problem where match acceptance from the opposite population is held constant. This shows that equilibrium behavior on one side of the market is not a pre-condition for sorting on the other.

Proposition 2. *Suppose that population X's payoffs are log supermodular. Then, in any stationary environment, higher types x have a higher search cutoff,* $\underline{y}(x_2) \geq \underline{y}(x_1)$ *for all* $x_2 > x_1$.

Theorem 1 follows from here: PAM holds according to Proposition 1 (i) when higher types from both populations have higher search cutoffs.

Observe that PAM is but one implication of Proposition 2: when one side of the population acts non-strategically because of valuing all partners the same, $\pi^Y(x_1|y) = \pi^Y(x_2|y)$ for all x_1, x_2, our model simplifies to the classic pure search model without recall (McCall (1970), Mortensen (1970)). In effect, under log supermodular payoffs, Proposition 2 asserts that *under stationary search, higher types x pursue higher prizes (goods, assets, ideas...) y.*

Proof of Proposition 2. We prove the contrapositive: if some lower types have higher search cutoffs, it implies that payoffs are not log supermodular. Let $x_2 > x_1$ be such that $\underline{y}_t(x_2) < \underline{y}_t(x_1)$ (the environment being stationary, this applies to all moments in time). This means that for any type $\underline{y} \in (\underline{y}_t(x_2), \underline{y}_t(x_1))$, agent type x_2 accepts $\underline{y}$ and x_1 rejects $\underline{y}$; whence, due to (1), $\pi^X(\underline{y}|x_1) < V_t^X(x_1)$ and $\pi^X(\underline{y}|x_2) \geq V_t^X(x_2)$. Then recall the integral representation of the value-of-search (5) and apply the mimicking argument (Lemma 1):

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy > \pi^X(\underline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy \leq \pi^X(\underline{y}|x_2). \tag{7}$$

In the steady state, agents' matching decisions do not change over time. This implies that agents always match with a better type than any of the types that were rejected previously. Formally, $Q_t^X(y|x_1) = 0$ for all $y < \underline{y}_t(x_1)$ including $\underline{y}$, and we may adjust the bounds of integration in (7) accordingly. Finally, combining both inequalities yields

$$\int_{\underline{y}}^1 \frac{\pi^X(y|x_1)}{\pi^X(\underline{y}|x_1)}Q_t^X(y|x_1)dy > \int_{\underline{y}}^1 \frac{\pi^X(y|x_2)}{\pi^X(\underline{y}|x_2)}Q_t^X(y|x_1)dy, \tag{8}$$

which can only hold if match payoffs are not log supermodular. □

4.4 Non-Stationary Environments

In a non-stationary environment, log supermodularity is insufficient to guarantee PAM. Here, unlike in the steady state, the lowest type accepted today need not be the worst possible match outcome for all future times. As the search pool evolves over time, agents may face a less favorable selection of types to match with in the future; an agent who rejects a given type initially may accept an inferior type at a later stage. This requires an agent to weigh the current acceptance decision against both the upside risk of matching with a superior type and the downside risk of ending up with an inferior type in the future.

Log supermodularity does not resolve this trade-off. On the one hand, payoff log supermodularity implies that higher types relatively better like to be matched with higher types. On the other hand, it stipulates that higher types stand more to lose from matching with a lower type. Depending on which effect dominates, higher or lower types are choosier. In particular, the higher type's fear of the worst outcome may upset PAM, even though payoffs are log supermodular. To build intuition, we first develop a simple three-type example that illustrates this point (see Figure 3 for an example with a continuum of types).

Example (PAM does not occur in a gradually clearing matching market)**.** *We construct a three-type example in which PAM is upset despite log supermodular payoffs. Populations are symmetric. The market gradually clears with no entrants joining the search pool (*$\eta_t(x) = 0$*). Assuming quadratic search (*$\lambda_t(x'|x) = \mu_t(x')$*), meetings are less and less likely to occur over time. Then consider payoffs that are increasing and log supermodular. The intermediate* x_2 *and high type* x_3 *payoffs are given as follows where* $\epsilon > 0$ *is small:*

	x_3	x_2	x_1
$\pi(\cdot\|x_3)$	$10+\epsilon$	1	ϵ
$\pi(\cdot\|x_2)$	10	1	$1-\epsilon$

In effect, the high type x_3 *is highly averse to matching with the lowest type* x_1*. The intermediate type, by contrast, is almost indifferent between the lesser two types. Low type payoffs are not further specified —the lowest type accepts matching with everyone at all times whenever payoffs are log supermodular (Corollary 1 in Appendix A.2).*[34] *The example is solved numerically*[35] *and illustrated in Figure 2. Time is on the horizontal axis and the value-of-search on the vertical axis.*[36] *To facilitate the comparison of match acceptance thresholds across types, we use the payoff of matching with the medium type as a reference point on the horizontal axis. Hence, agents accept matching with the medium type whenever their value-of-search is above the horizontal axis. Owing to the gradually decreasing meeting rate, the high type's match opportunities deteriorate steadily. At the beginning of time she matches with high type agents* x_3 *only. But after time* t_1*, with only few agents left in the search pool, she also accepts to match with agents of intermediate type* x_2*. The intermediate type initially accepts fellow agents of type* x_2*. Yet, anticipating the possibility of matching with the highest type,* x_2 *experiences a surge in her value-of-search. This leaves her not only to reject the lowest, but also her own type between* t_0 *and* t_1*. (One could say that time interval* $[t_0, t_1]$ *is spent away from the search pool: Agents of type* x_2 *do not match with anyone!) Between time* t_1 *and* t_2 *type* x_2*, is the choosiest:*

[34] As an example, one can consider $\pi(x_3|x_1) = 10 - \epsilon,\ \pi(x_2|x_1) = 1,\ \pi(x_1|x_1) = 1 - \frac{\epsilon}{2}$.

[35] When the meeting rate is quadratic, solving the HJB differential equation characterizing the value-of-search in closed form is typically not possible. Closed-form solutions are reported in the examples on necessity (see Proposition 4).

[36] The equilibrium is constructed backward in time, starting with an almost empty search pool far into the future. We further consider $\epsilon = 0.01$ and $\rho = 1$.

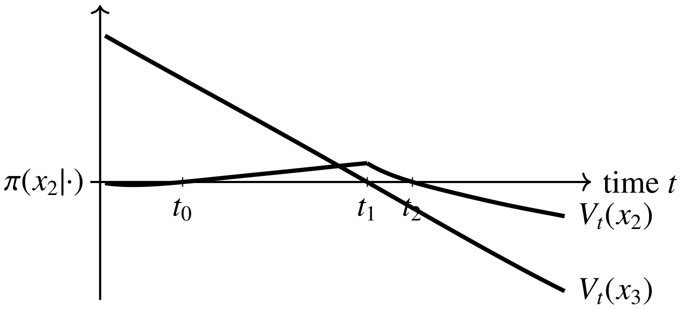


Figure 2: PAM is upset despite log supermodular payoffs —three type example

the highest type finds the intermediate type acceptable, whereas the intermediate type does not. This upsets PAM.

The main contribution of this paper is to establish sufficient conditions for which PAM occurs away from the steady state. First, a definition is in place.

Definition 3. *Population X's payoffs are log supermodular in differences if for all $y_3 > y_2 > y_1$ and $x_2 > x_1$,*

$$\frac{\pi^X(y_3|x_2) - \pi^X(y_2|x_2)}{\pi^X(y_2|x_2) - \pi^X(y_1|x_2)} \geq \frac{\pi^X(y_3|x_1) - \pi^X(y_2|x_1)}{\pi^X(y_2|x_1) - \pi^X(y_1|x_1)}.$$

If the inequality holds with the reverse sign, we say that payoffs satisfy log submodularity in differences. *Log supermodularity in differences*, a term that we introduce here, means that higher types stand relatively more to gain from matching with a high type than they stand to lose from matching with a low type. Log supermodularity in differences is equivalent to $d_y\pi^X(y|x)$ being log supermodular, insofar as such a derivative exists.[37,38] The payoffs in the preceding example do not satisfy this condition, for the downside loss from matching with x_1 instead of x_2 is much larger for higher types:

$$\frac{\pi(x_3|x_3) - \pi(x_2|x_3)}{\pi(x_2|x_3) - \pi(x_1|x_3)} = \frac{9+\epsilon}{1-\epsilon} < \frac{9}{\epsilon} = \frac{\pi(x_3|x_2) - \pi(x_2|x_2)}{\pi(x_2|x_2) - \pi(x_1|x_2)}.$$

We can interpret the payoff $\pi(\cdot|x) \equiv u_x(\cdot)$ as agent type x's utility function. This affords us an interpretation of log supermodularity in differences in terms of risk preferences. More specifically, Pratt (1964) shows that given arbitrary $x_2 > x_1$ the following statements are equivalent:

1. Agent type x_1 is weakly more risk-averse than agent type x_2; that is, x_1 does not accept a lottery that is rejected by x_2.[39]

2. For any $y_3 > y_2 > y_1$ we have

$$\frac{u_{x_2}(y_3) - u_{x_2}(y_2)}{u_{x_2}(y_2) - u_{x_2}(y_1)} \geq \frac{u_{x_1}(y_3) - u_{x_1}(y_2)}{u_{x_1}(y_2) - u_{x_1}(y_1)}.$$

[37] See Proposition 7 in the textbook by Gollier (2004) for a proof.

[38] Log supermodularity is a condition that affects both the level and the curvature of a function. By contrast, log supermodularity in differences governs the curvature of a function only and is invariant to its level. In particular, if $\pi^X(y|x)$ is log supermodular in differences, then so is $\pi^X(y|x) - \pi^X(0|x)$. Moreover, $\pi^X(y|x) - \pi^X(0|x)$ is also log supermodular, whereas $\pi^X(y|x)$ need not be.

[39] Formally, it holds that if $\int_0^1 u_{x_1}(y)dF(y) \geq (>)\, u_{x_1}(\bar{y})$, then also $\int_0^1 u_{x_2}(y)dF(y) \geq (>)\, u_{x_2}(\bar{y})$.

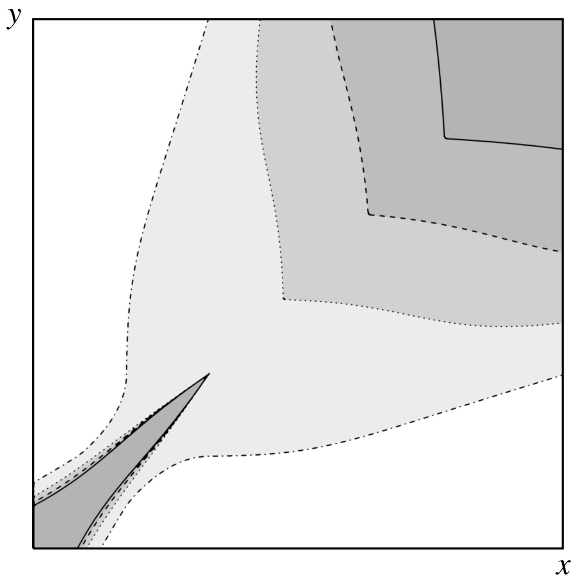


Figure 3: PAM is upset despite log supermodular payoffs

Note: Consider a rapidly clearing search pool with no entry. Symmetric payoffs are $\pi(y|x) = exp(1/16y - 2x^8(1-y)^8)$. These are log supermodular and log *sub*modular in differences. The figure depicts how match acceptance sets shrink over time: darker sets represent match acceptance sets at an earlier date. Initially, only the highest and the lowest types match. Intermediate types do not match up until they are accepted by the highest types. PAM fails initially because, prior to reaching an almost empty search pool, the most desirable agents are not the choosiest. Visually, at the top, the boundary of matching sets is decreasing.

The use of this result is twofold. First, it features prominently in the proof of Theorem 2. Second, it provides a simple interpretation of log supermodularity in differences: lesser ranked agent types are also more risk-averse. Here we are dealing with payoffs of course, not utilities. This is why we caution against viewing log supermodularity in differences solely in the guise of risk-aversion. The curvature of π is implied by the specific model in mind. It may consequently be derived from economic fundamentals rather than risk preferences.

Having established the terminology we can now state the main result:

Theorem 2 (non-stationary PAM). *Suppose that both populations' payoffs are log supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) at all times in any (non-stationary) equilibrium.*

The proof of this theorem directly follows from a more granular statement about within-population sorting, where higher types from one population are choosier about their matches. Similar to the steady state, increasing choosiness can also be interpreted through the lens of pure search theory (McCall (1970), Mortensen (1970)). Theorem 2 follows, as PAM holds when higher types in both populations have higher search cutoffs.

Proposition 3. *Suppose that population X's payoffs are log supermodular and log supermodular in differences. Then, higher types x have a higher search cutoff, $y_t(x_2) \geq y_t(x_1)$ for all $x_2 > x_1$.*

A clear division of labor emerges between the two sufficient conditions: One governs time, the other risk. Log supermodularity in differences ensures that higher types are more willing to bear the risk, while log supermodularity ensures that higher types are more willing to endure the delay associated with prolonged search.

Proof. We prove, as in the stationary case, the contrapositive. Let $x_2 > x_1$ be such that $y_t(x_2) < y_t(x_1)$ at some time t. This means that for any $\underline{y} \in (y_t(x_2), y_t(x_1))$, agent type x_2 accepts $\underline{y}$ and x_1 rejects $\underline{y}$. Using identical arguments as in the proof of Proposition 2, i.e., representation (5) and Lemma 1, yields

$$\int_0^1 \pi^X(y|x_1) Q_t^X(y|x_1) dy > \pi^X(\underline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2) Q_t^X(y|x_1) dy \leq \pi^X(\underline{y}|x_2). \tag{9}$$

Next, define $\bar{y}$ such that $\pi^X(\bar{y}|x_1) \int_0^1 Q_t^X(y|x_1)dy = \pi^X(\underline{y}|x_1)$. Since $Q_t^X(\cdot|x_1)$ integrates to less than one, $\bar{y} > \underline{y}$. (To see that such $\bar{y} \in [0,1]$ exists one must prove that $\pi^X(1|x_1) \int_0^1 Q_t^X(y|x_1)dy \geq \pi^X(\underline{y}|x_1) > \pi^X(\underline{y}|x_1) \int_0^1 Q_t^X(y|x_1)dy$ and apply the intermediate value theorem. The second inequality is trivially true. If the first inequality did not hold, then it must be that $\int_0^1 [\pi^X(y|x_1) - \pi^X(1|x_1)] Q_t^X(y|x_1)dy > 0$ due to (9) and in spite of non-decreasing match payoffs.) Log supermodularity implies that $1/\int_0^1 Q_t^X(y|x_1)dy = \frac{\pi^X(\bar{y}|x_1)}{\pi^X(\underline{y}|x_1)} \leq \frac{\pi^X(\bar{y}|x_2)}{\pi^X(\underline{y}|x_2)}$. Or, equivalently,

$$\pi^X(\underline{y}|x_2) \leq \pi^X(\bar{y}|x_2) \int_0^1 Q_t^X(y|x_1)dy. \tag{10}$$

Finally, normalize Q_t^X to recast the agents' decisions as a common choice in between a lottery F and the sure outcome $\bar{y}$. Formally, define $F(y) = \int_0^y Q_t^X(y'|x_1)dy' / \int_0^1 Q_t^X(y'|x_1)dy'$. It follows from (9) and

(10) that

$$\int_0^1 \pi^X(y|x_1)dF(y) > \pi^X(\overline{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)dF(y) \leq \pi^X(\overline{y}|x_2).$$

Or, type x_1 accepts the lottery that is rejected by type x_2. This runs counter to the characterization of log supermodularity in differences in terms of risk preferences and establishes a contradiction. □

To gain a visual understanding of the scope of Theorem 2, refer to Figure 4. In our simulations, we consider match acceptance thresholds with non-stationary cyclical entry, similar to the fluctuations in a dynamic seasonal housing market (cf. Ngai and Tenreyro (2014)). Despite the complex dynamics, when the conditions for PAM are met (as shown in Figure 4b), all acceptance thresholds remain in a specific order without any crossings. However, if these conditions are not satisfied, the sorting of thresholds may become intricate, leading to numerous crossings between agents' acceptance thresholds (as shown in Figure 4a). This is where PAM proves to be useful in imposing regularity on the dynamics of the matching problem.

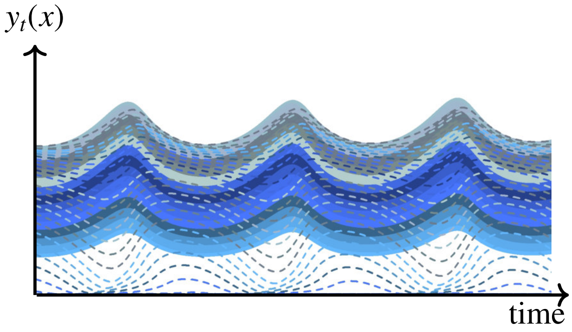
Discussion. It may come as a surprise that risk preferences do not play as prominent a role in the steady state. After all, the decision to reject a certain match payoff today is a revealed preference for a risky, random match payoff sometime in the future—regardless of whether the environment is stationary or not. Our analysis shows that the randomness of search translates into less risk in the steady state. Indeed, in a stationary world, the lowest type accepted initially constitutes a bound on the worst possible match outcome for all future dates; the prospect of future matches below one's current acceptance threshold does not arise. This renders downside risk a feature of non-stationary environments only. In consequence, sorting in the steady state solely relies on a preference ranking over upside risk. Non-stationarity in contrast requires a preference ranking over any kind of lottery, entailing both upside and downside risk.

4.5 Necessity

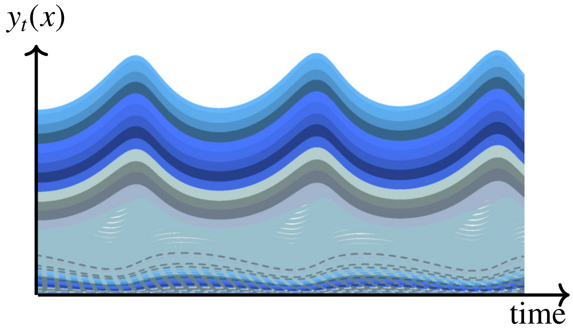
It is easy to provide examples in which PAM occurs, even when payoffs are neither log supermodular nor log supermodular in differences. As higher types are more likely to be accepted by others, higher types enjoy superior match opportunities and can therefore afford to be choosier, regardless of payoff curvature. Becker (1973) illustrates this point in a frictionless matching market. Adachi (2003) (cloning model), Lauermann and Nöldeke (2014) (steady state) and Wu (2015) (Markov equilibrium of the gradually emptying search pool) prove this to be the case more generally as search frictions vanish. This raises the question whether our conditions are needlessly strong.

In this section, we show that log supermodularity and log supermodularity in differences are the minimal conditions under which PAM occurs in non-stationary environments.[40] If either one condition reverses locally for some interval of types, then there exist primitives of the model under which PAM is upset. We show that this is particularly true when there is a gradually emptying search

[40]This exercise is similar in spirit to the frictionless result by Legros and Newman (2007).



(a) Not PAM —Payoffs are Lsub and LsubD



(b) PAM—Payoffs are LS and LSD

Figure 4: Illustration of Theorem 2 with cyclical entry

Note: Populations are symmetric with payoffs given by $\pi(y|x) = y^{\frac{1}{2}+\frac{1}{2}x}$ (b) and $\pi(y|x) = y^{1-\frac{1}{2}x}$ (a). The former is LS and LSD, i.e., the conditions from Theorem 2, and the latter is neither. Entry is cyclical: $\eta_t(x) = 10\sin(8t)\phi(x)(\mu_t(x))^4$ where $\phi(x)$ is the lognormal density with logmean and logvar equal to 0.5. Further parameters are $\lambda_t(y|x) = \mu_t(y)/(\int_0^1 \mu_t(z)dz)^{\frac{1}{2}}$ and $\rho = 10$. Each color band corresponds to the range of acceptance thresholds chosen by a small interval of types. To highlight the crossing of acceptance thresholds, acceptance thresholds of types $x \in [0.1, 1]$ are depicted in plain color and acceptance thresholds of types $x \in [0, 0.1]$ are dashed. In the example where PAM fails, it is not the most desirable agents, but rather agents of a lower-ranked type with $x = 0.1$, who exhibit the highest level of selectivity.

pool, arguably the simplest instance of a non-stationary environment. The additional requirement that there is zero entry and populations are symmetric merely disciplines the result.

Proposition 4 (weak sufficiency). *Consider an economy with symmetric populations and zero entry and suppose that payoffs satisfy either of the following:*

1. *payoffs restricted to $[\underline{x}, \overline{x}]^2 \subseteq [0,1]^2$ are strictly log submodular, or*

2. *payoffs restricted to $[\underline{x}, \overline{x}]^2 \subseteq [0,1]^2$ are strictly log submodular in differences;*

then there exist meeting rates λ and an initial search pool μ_0 such that PAM does not occur for some time preceding the (empty) steady state.

The proof of Proposition 4 is deferred to the appendix. To prove this statement, we show that the set of model primitives for which PAM fails is non-empty, which entails choosing an appropriate meeting rate and type distribution that foster negative sorting for the entire class of payoffs considered. The proof thus revolves around two counterexamples.[41]

Counterexample 1 derives from a ranking of time preferences across types that is implied by log submodular payoffs. This ranking states that lower ranked types will exhibit more patient behavior in the following choice problem, variations of which naturally arise in a non-stationary search pool: match instantaneously with a lower ranked type, or match with delay (possibly but not necessarily with probability less than 1) with a more attractive type in the future. In counterexample 2, we

[41]The proof of Proposition 4 relies on counterexamples involving finitely many types only. This is for analytical convenience only. Using bump functions, distributions over finitely many types can be approximated arbitrarily well by a continuous distribution over a continuum so that one can construct analogous counterexamples with a continuum of types for which PAM is equally upset.

emphasize the role of risk as opposed to time by letting the expected time spent in the search pool become exceedingly small, all the while maintaining the downside risk of matching with the lowest type.

5 Explicit Search Costs

So far, we have embedded search costs through time discounting (as espoused by Shimer and Smith (2000) and Smith (2006)). In this section, we re-establish sufficient conditions for PAM adopting the other prominent representation of search costs: explicit search costs (see Morgan (1994), Chade (2001) and Atakan (2006)).[42] Here, discounting plays no role ($\rho = 0$), and each agent in the search pool pays a flow cost c. Whereas time discounting captures the opportunity cost of time, explicit search costs elevate the act of search to be the critical cost. As was the case under discounting, this framework has been exclusively studied in the steady state (see Morgan (1994)). In what follows, we broaden the scope of the analysis to consider all equilibria. We show that log supermodularity in differences is as essential to PAM under explicit search costs as it is under discounting.

By adapting the proof of Proposition 2, Appendix B presents a short proof of the steady state result due to Morgan (1994) (see Theorem 1′): *Suppose that both populations' payoffs are supermodular. Then there is positive assortative matching (PAM) at all times in any stationary equilibrium.*[43] The corresponding search result (see Proposition 2′) is as follows: *Suppose that population X's payoffs are supermodular. Then, in any stationary environment, higher types x have a higher search cutoff,* $y(x_2) \geq y(x_1)$ *for all* $x_2 > x_1$.

Supermodularity is insufficient to guarantee positive assortative matching in non-stationary environments for the same reasons given in the analysis of search with discounting. Again, log supermodularity in differences turns out to be the missing sufficient condition that ensures PAM across all equilibria:

Theorem 2′ (non-stationary PAM with explicit search cost)**.** *Suppose that both populations' payoffs are supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) at all times in any (non-stationary) equilibrium.*

As with discounting, Theorem 2′ is due to a more granular result (proven in Appendix B):

Proposition 3′: *Suppose that population X's payoffs are supermodular and log supermodular in differences. Then, higher types x have a higher search cutoff,* $y_t(x_2) \geq y_t(x_1)$ *for all* $x_2 > x_1$.

Observe that unlike steady state sufficient conditions, which differ between environments with discounting and explicit search cost, the additional condition of log supermodularity in differences ensures monotone search cutoffs and thereby PAM in non-stationary equilibrium irrespective of how

[42] We are not aware of an existence result that applies under explicit search costs but conjecture that largely similar arguments as in Sandmann and Bonneton (2023) would establish the result.

[43] Population X's payoffs are supermodular if $\pi^X(y_2|x_2) + \pi^X(y_1|x_1) \geq \pi^X(y_1|x_2) + \pi^X(y_2|x_1)$ for all $y_1 < y_2$ and $x_1 < x_2$.

search cost is modeled.[44] We finally show that log supermodularity in differences is the weakest sufficient condition that guarantees PAM (see Online Appendix A for a proof along the lines of Proposition 4).

Proposition 4′—weak sufficiency with explicit search costs: *Consider an economy with explicit search cost, symmetric populations and zero entry and suppose that payoffs restricted to* $[\underline{x}, \overline{x}]^2 \subseteq [0,1]^2$ *are strictly log submodular in differences. Then there exist meeting rates* λ *and an initial search pool* μ_0 *such that PAM does not occur for some time preceding the (empty) steady state.*

The following table summarizes the conditions on payoffs that ensure PAM for various environments in the NTU paradigm.

Frictionless	$\pi_2 > 0$ Becker (1973)	
Search frictions	**Stationary**	**Non-Stationary**
i) discounting	$\pi_2 > 0$, $(\log\ \pi)_{12} > 0$ Smith (2006)	$\pi_2 > 0$, $(\log\ \pi)_{12} > 0$, $(\log\ \pi_2)_{12} > 0$ This paper
ii) explicit search costs	$\pi_2 > 0$, $\pi_{12} > 0$ Morgan (1994)	$\pi_2 > 0$, $\pi_{12} > 0$, $(\log\ \pi_2)_{12} > 0$ This paper

Table 1: Sufficient conditions for PAM—subscript 2 stands for the partial derivative in the partner's type and subscript 1 stands for the partial derivative in one's own type (assuming that these exist).

6 Model Variations

In this section, we discuss three natural alternative specifications of the model. Each of these highlights the scope of our main sorting result.

6.1 Aggregate Uncertainty

Note that Theorem 2 straightforwardly extends to environments where aggregate fluctuations are stochastic and not deterministic.[45] Algebraically, aggregate uncertainty merely compounds the individual idiosyncratic risk. Irrespective of the source of randomness—idiosyncratic or aggregate—future match prospects can be summarized by the discounted match probability $Q^X_t(y|x)$. Hence the integral representation of the value-of-search and the subsequent proofs of our main sorting results continue to apply without modification.

[44] In Online Appendix A, we consider the alternative explicit search costs model where agents can quit the search pool and exit unmatched. Quits prevent future expected search costs from accumulating to the point where the value-of-search becomes negative. Log supermodularity in differences also plays a critical role in this model.

[45] Our focus on deterministic aggregate dynamics owes to the literature's initial focus on the steady state. In Bonneton and Sandmann (2024), we explore a model with aggregate uncertainty, where uncertainty is driven by random entry into the search pool.

Our insights, therefore, carry over to environments where aggregate fluctuations in the state are uncertain, such as unemployment rising following a (random) economic crisis or sex imbalances being inflicted due to a low-probability event such as a war. Log supermodularity in differences plays a critical role whenever there is a positive probability that one's current match prospects deteriorate in the future.

6.2 Non-Stationary Types

It is also worthwhile to note that Theorem 2 extends to environments where time-variant match opportunities arise due to a change in individual characteristics rather than a change in the composition of the search pool.[46] To ensure PAM in this context, we require log supermodularity in differences, even in the steady state.

Formally, consider two-dimensional agent types (x, α_t) and (y, β_t). α_t and β_t capture, depending on the application, time spent in the search pool or age. Then $\alpha_{t''} - \alpha_{t'} = t'' - t'$ and $\beta_{t''} - \beta_{t'} = t'' - t'$. We assume that age types α_t and β_t affect the agents' attractiveness to others, but not their preferences. Then y's match payoff when matching with an agent of type (x, α_t) is $\Pi^Y(x, \alpha_t|y)$.[47]

The following Theorem (proven in Online Appendix B) states, as in our baseline model, that higher types of similar or more desirable age match with more desirable agents under identical conditions on payoffs as before.

Theorem 3 (PAM with non-stationary types)**.** *Suppose that both populations' payoffs are log supermodular and log supermodular in differences in x and y. Then for all α_t and $x_2 \geq x_1$, (x_1, α_t) accepts every (y, β_t) that (x_2, α_t) accepts. If moreover $\beta_t \mapsto \Pi^X(y, \beta_t|x)$ and $\alpha_t \mapsto \Pi^Y(x, \alpha_t|y)$ are non-increasing, then for all ages $\alpha''_t \geq \alpha'_t$ and types $x_2 \geq x_1$, (x_1, α''_t) accepts every (y, β_t) that (x_2, α'_t) accepts.*

This result extends our previous insight: whenever there is downside risk, log supermodularity in differences is necessary to sustain sorting. For downside risk to arise, the economy need not be out of steady state. With non-stationary types, downside risk also emerges when individual agents experience a decline in their value to others.

6.3 Strategic Match Destruction

In our model, agents do not return to the initial search pool once a match is formed. This provides a natural setting if (i) break-up costs are prohibitive (e.g., non-compete clauses as studied by Shi (2023)), (ii) the purpose of the match serves a one-time goal, or (iii) if agents enter a different search pool upon match destruction (e.g., as divorcees). The literature, by contrast, has largely considered

[46] For instance, Pissarides (1992) suggests that time spent unemployed in the search pool decreases future match payoffs.

[47] To illustrate, consider non-stationary flow payoffs $f^Y_{\alpha_t}(x|y)$ that depend on the partner's age type α_t. For instance, flow payoffs may be given by $f^Y_{\alpha_t}(x|y) = e^{-\alpha_t} x$. Then the match payoff of matching with an agent of type (x, α_t) is given by

$$\Pi^Y(x, \alpha_t|y) = \int_{\alpha_t}^{\infty} e^{-\rho(\alpha_\tau - \alpha_t)} f^Y_{\alpha_\tau}(x|y) d\tau.$$

environments in which agents repeatedly enter and exit the search pool and derive flow payoffs while matched. In the steady state, both modeling specifications are indistinguishable because there is no reason for agents to match temporarily. In non-stationary environments, however, agents could be tempted to break their matches strategically once their match opportunities have improved.

Whether PAM occurs in these environments depends on whether our mimicking argument holds, i.e., whether higher types enjoy better match opportunities. Intuitively, if agents *cannot commit* to staying in a match and leave whenever beneficial (as in Smith (1992)), then the mimicking argument, hence PAM, will not hold. The reason is simple: individuals may choose not to accept a match with a high-type agent because they anticipate being dumped in the future. If, however, agents *can commit* to staying in a match, they continue enjoying better match opportunities, so the mimicking argument, hence PAM, should hold.

7 Conclusion

This article studies sorting of heterogeneous agents in a general non-stationary matching model, showing that the study of sorting need not confine itself to particular examples or stationary environments. We hope that it will inspire future ventures into the study of non-stationary dynamics in related frameworks.

Our analysis reveals a close link between the time-variant nature of search frictions and risk preferences. We find that the weakest sufficient conditions for positive assortative matching entail that more desirable individuals are less risk-averse in the sense of Arrow-Pratt. This result, taken together with the empirical evidence, provides a theoretical foundation as to why positive assortative matching arises in decentralized matching markets where there is no bilateral bargaining that precedes match formation.

A Positive Assortative Matching

A.1 Definition of PAM

Proof of Proposition 1. (i) Fix $x_1 < x_2$ and $y_1 < y_2$ so that $(x_1, y_2), (x_2, y_1)$ belong to the set of pairs that match upon meeting, U_t. Then $y_t(x_2) \leq y_1$ and $x_t(y_2) \leq x_1$, whence also $y_t(x_2) \leq y_2$ and $x_t(y_2) \leq x_2$ due to Assumption 2. It follows that $(x_2, y_2) \in U_t$. As to (x_1, y_1), note that since $y_t(x)$ and $x_t(y)$ are non-decreasing, it holds that $y_t(x_1) \leq y_t(x_2) \leq y_1$ and $x_t(y_1) \leq x_t(y_2) \leq x_1$, whence $(x_1, y_1) \in U_t$.
(ii) Suppose by contradiction that there is PAM, yet $y_t^X(x_2) < y_t^X(x_1)$ for some types $x_2 > x_1$ whose time t matching sets are non-empty.
Case 1: Suppose that there exists $\underline{y} \in [y_t(x_2), y_t(x_1)) \cap U_t^X(x_2)$. Then pick arbitrary $y_2 \in U_t^X(x_1)$. Clearly, $y_2 \geq y_t(x_1) > y_1$. And due to the lattice property, $(x_2, y_1), (x_1, y_2) \in U_t$ implies that $(x_1, y_1) \in U_t$. This contradicts the assertion that $y_1 < y_t(x_1)$.
Case 2: Suppose that $[y_t(x_2), y_t(x_1)) \cap U_t^X(x_2)$ is empty. Then pick arbitrary $y_2 \in U_t^X(x_2)$ and $y_1 \in [y_t(x_2), y_t(x_1))$. Clearly, $y_2 > y_1$ and $x_t(y_1) > x_2$. Whence, for any $x_3 \in U_t^Y(y_1)$ it must be that $x_3 > x_2$.

In particular, $(x_2, y_2), (x_3, y_1) \in U_t$ implies $(x_2, y_1) \in U_t$ due to the lattice property. This contradicts the assertion that $x_t(y_1) > x_2$. □

A.2 Lowest-type self-acceptance

Corollary 1. *Suppose that payoffs are log supermodular and populations are symmetric. Then the lowest type will accept everyone,* $0 \in U_t(0)$ *for every* t.

Proof. We prove the contrapositive, i.e., if self-acceptance fails at some point in time, then payoffs cannot be log supermodular. Let (t_0, t_1) denote the maximal time interval during which $0 \notin U_t(0)$ for all $t \in (t_0, t_1)$. If $U_t(0)$ were empty throughout (t_0, t_1), $V_{t_0}(0) = e^{-(t_1-t_0)\rho} V_{t_1}(x) < \pi(0|0)$, yet $V_{t_0}(0) = \pi(0|0)$ which is absurd. Thus, there exists $t \in (t_0, t_1)$ and some non-zero type $x_2 \in U_t(0)$. Yet, due to identical arguments as in the proof of Theorem 1,

$$\int_0^1 \frac{\pi(x'|0)}{\pi(0|0)} Q_t(x'|0) dx' > \int_0^1 \frac{\pi(x'|x_2)}{\pi(0|x_2)} Q_t(x'|0) dx'.$$

As in the proof of Theorem 1, this can only hold if match payoffs are not log supermodular. □

A.3 Necessity

Proof of Proposition 4. Counterexample 1. There are two types, $x_2 > x_1$, payoffs are strictly log submodular, $\frac{\pi(x_2|x_2)}{\pi(x_1|x_2)} < \frac{\pi(x_2|x_1)}{\pi(x_1|x_1)}$, search is quadratic, $\lambda(t, \mu_t) = \mu_t$ and there is no entry.

As match prospects are bleakening over time, there exists a time t^* beyond which the high type will always accept the low type and $V_{t^*}(x_2) = \pi(x_1|x_2)$. Drawing on the integral representation of the value-of-search we can write $V_{t^*}(x_2) = \sum_{j\in\{1,2\}} \pi(x_j|x_2) Q_{t^*}(x_j)$ where $Q_{t^*}(x_j)$ is the probability of type x_2 matching with x_j—discounted by the time at which such event materializes. Now observe that if the low type found it desirable, she could always exactly replicate discounted match probabilities of the high type, that is $V_{t^*}(x_1) \geq \sum_{j\in\{1,2\}} \pi(x_j|x_1) Q_{t^*}(x_j)$. Then $V_{t^*}(x_1) > \pi(x_1|x_1)$ and the low type rejects other low types at time t^*. For otherwise the integral representation of the value-of-search combined with the inequalities implies that

$$\sum_{j\in\{1,2\}} \frac{\pi(x_j|x_2)}{\pi(x_1|x_2)} Q_{t^*}(x_j) \geq \sum_{j\in\{1,2\}} \frac{\pi(x_j|x_1)}{\pi(x_1|x_1)} Q_{t^*}(x_j) \quad \Leftrightarrow \quad \frac{\pi(x_2|x_2)}{\pi(x_1|x_2)} \geq \frac{\pi(x_2|x_1)}{\pi(x_1|x_1)}$$

in spite of strict log submodularity.

Counterexample 2. Consider symmetric populations consisting of three types $x_1 < x_2 < x_3$. Omit superscripts. Suppose that $\frac{\pi(x_3|x_3)-\pi(x_2|x_3)}{\pi(x_2|x_3)-\pi(x_1|x_3)} < \frac{\pi(x_3|x_2)-\pi(x_2|x_2)}{\pi(x_2|x_2)-\pi(x_1|x_2)}$. Then x_3 is strictly more risk-averse than x_2.

We construct a sequence of equilibra indexed by n in which, for n sufficiently large, there exists a moment in time such that x_3 accepts x_2 whereas x_2 rejects a fellow x_2. Specifically, consider two distinct moments in time, t_0^n and 0 where t_0^n precedes 0: at time t_0^n the high type x_3 begins accepting the intermediate type x_2 and at time 0 the high type begins accepting the low type x_1; PAM will be upset because type x_2 will reject another type x_2 at time t_0^n.

The construction makes apparent that the failure of PAM at time t_0^n arises due to a reversal of risk preferences. As n grows large both (i) $t_0^n \to 0$ and (ii) the probability of matching after time 0 will go to zero. As a consequence agent type x_3's future match outcomes at time t_0^n converge towards a lottery assigning positive probability to both the event that x_3 match with another x_3 and to the event that x_3 match with an agent type x_1. Crucially, at time t_0^n agent types x_2 are accepted by agent types x_3. They thus face identical match opportunities. Like agent types x_3, they may either choose to play the lottery—or accept x_2. Note that since agent type x_3 is indifferent between playing the lottery, i.e., waiting, or accepting x_2, by virtue of being less risk-averse agent type x_2 must strictly prefer the lottery and therefore reject another type x_2.

To construct the failure of PAM analytically, we consider the simplest non-stationary matching environment conceivable. There is zero entry. Agent type x_2 is present in zero proportion and solely of hypothetical interest. Due to log supermodularity agent type x_1 will accept any agent he meets. Proceed then to define the (anonymous) meeting rate: it becomes stationary eventually and is piecewise constant over time. We set

$$\lambda_t(x_1) = n(1 - h(n)) \quad \text{if } t \geq 0 \qquad \text{and} \qquad \lambda_t(x_3) = \begin{cases} nh(n) & \text{if } t \geq 0 \\ n & \text{if } t < 0. \end{cases}$$

$h(n)$ is determined as to ensure indifference of agent type x_3 between accepting and rejecting agent types x_1 for all $t \geq 0$. Then at time $t = 0$

$$\rho V_0^n(x_3) = n\big[h(n)\pi(x_3|x_3) + (1 - h(n))\pi(x_1|x_3) - V_0^n(x_3)\big] \quad \text{and} \quad V_0^n(x_3) = \pi(x_1|x_3).$$

Here the equation on the left is the stationary HJB equation and the equation on the right is the indifference condition. The latter holds if $h(n) = \frac{\rho}{n}\frac{\pi(x_1|x_3)}{\pi(x_3|x_3)-\pi(x_1|x_3)}$.

We assume that at time 0 agent types x_2 likewise accept agent types x_1 (log supermodular payoffs imply this). If they did not, PAM would be upset as we desire to show.

Finally, choose as time 0 'starting values' $(\mu_0(x_1), \mu_0(x_2), \mu_0(x_3))$ such that $\mu_0(x_2) = 0$ and $\frac{\mu_0(x_3)}{\mu_0(x_1)} = \frac{\lambda_0(x_3)}{\lambda_0(x_1)}$

Preceding time $t = 0$ the high type x_3's value-of-search is decreasing. Time $t_0^n < 0$, the moment in time at which agent type x_3 is indifferent between accepting and rejecting agent type x_2, likewise admits a closed-form representation: Recall that $V_0^n(x_3) = \pi(x_1|x_3)$ so that prior to time 0 the high type x_3 exclusively matches with other high types. Then an explicit characterization of x_3's value-of-search as defined in Equation (3) gives

$$V_{t_0^n}^n(x_3) = \int_{t_0^n}^{0} e^{-\rho(\tau - t_0^n)}\pi(x_3|x_3)n\, e^{-n(\tau - t_0^n)}d\tau + e^{\rho t_0^n}e^{n t_0^n}\pi(x_1|x_3).$$

And the indifference condition that characterizes t_0^n is $V_{t_0^n}^n(x_3) = \pi(x_2|x_3)$. The solution is given by $t_0^n = \frac{1}{\rho+n}\ln\frac{\frac{n}{\rho+n}\pi(x_3|x_3)-\pi(x_2|x_3)}{\frac{n}{\rho+n}\pi(x_3|x_3)-\pi(x_1|x_3)}$. Clearly, $t_0^n < 0$ due to Assumption 2 and $t_0^n \to 0$ as n goes to infinity.

Agent type x_3's discounted match probabilities of matching with agent types x_1 and x_3, as defined in Equation (5), are denoted by $Q_{t_0^n}^n(x_1) = \frac{\pi(x_3|x_3)-\pi(x_2|x_3)}{\pi(x_3|x_3)-\pi(x_1|x_3)} + o(1) \equiv q + o(1)$ and $Q_{t_0^n}^n(x_3) = 1 - q + o(1)$

respectively.[48] Here $o(1)$ denotes the Landau notation: $\lim_{n\to\infty} o(1) = 0$. In particular, note that $Q^n_{t^n_0}(x_1) + Q^n_{t^n_0}(x_3) = 1 + o(1)$, meaning that the x_3's probability of matching instantaneously approaches 1 as n tends to infinity.

Now observe that, beginning from time t^n_0, agent type x_2 is accepted by agent type x_3, and thus faces identical match opportunities as an agent type x_3. Accordingly, x_2 can mimic the higher type x_3's match probabilities (see Lemma 1) so that

$$V^n_{t^n_0}(x_2) \geq \pi(x_1|x_2)q + \pi(x_3|x_2)(1-q) + o(1).$$

(Recall by construction that $\pi(x_2|x_3) = V^n_{t^n_0}(x_3) = \pi(x_1|x_3)q + \pi(x_3|x_3)(1-q) + o(1)$.) We then claim that $V^n_{t^n_0}(x_2) > \pi(x_2|x_2)$ for n sufficiently large, so that PAM does not occur at time t^n_0: the intermediate type x_2 rejects a fellow intermediate type x_2 that is accepted by high type agents x_3. Indeed, this follows from the characterization of risk preferences. Suppose by contradiction that $V^n_{t^n_0}(x_2) \leq \pi(x_2|x_2)$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ gives

$$\pi(x_2|x_2) \geq \pi(x_1|x_2)q + \pi(x_3|x_2)(1-q) \quad \text{and} \quad \pi(x_2|x_3) = \pi(x_1|x_3)q + \pi(x_3|x_3)(1-q).$$

This means that (i) agent type x_3 is indifferent between the lottery assigning probability q to x_1 and $1-q$ to x_2 and the sure outcome x_2, whereas (ii) agent type x_2 weakly prefers the sure outcome x_2. This contradicts the assertion that agent type x_2 is strictly less risk-averse than agent type x_3. □

B Explicit search cost

We begin by re-stating an adapted version of the mimicking argument that incorporates explicit search cost. As under time-discounting, the value-of-search admits an integral representation over payoffs:

$$V^X_t(x) = \int_0^1 \pi^X(y|x) Q^X_t(y|x)dy - C^X_t(x) \quad \text{where} \quad Q^X_t(y|x) = \int_t^\infty p^X_{t,\tau}(y|x)d\tau. \tag{11}$$

Here $C^X_t(x)$ is the expected time that agent type x spends in the search pool from time t onward, multiplied by the explicit search cost c:

$$C^X_t(x) = c\int_t^\infty \int_0^1 (\tau - t)\, p^X_{t,\tau}(y|x)dyd\tau.$$

Higher types have better match opportunities, and so can mimick lesser ranked agents' matching rates. Then an identical reasoning as in the proof of Lemma 1 establishes the following lower bound on the

[48] Formally, following the above value-of-search, discounted probabilities are

$$Q^n_{t^n_0}(x_1) = e^{t^n_0(\rho+n)} \int_0^\infty e^{-\rho\tau} n(1-h(n))e^{-n\tau}d\tau = e^{t^n_0(\rho+n)} \frac{n(1-h(n))}{\rho+n} = \frac{\frac{n}{\rho+n}\pi(x_3|x_3) - \pi(x_2|x_3)}{\frac{n}{\rho+n}\pi(x_3|x_3) - \pi(x_1|x_3)} \frac{n(1-h(n))}{\rho+n} \equiv q + o(1)$$

$$Q^n_{t^n_0}(x_3) = \int_{t^n_0}^0 e^{-\rho(\tau-t^n_0)} n e^{-n(\tau-t^n_0)}d\tau + e^{t^n_0(\rho+n)} \int_0^\infty e^{-\rho\tau} nh(n)e^{-n\tau}d\tau \frac{n}{\rho+n} - \frac{n(1-h(n))}{\rho+n} \frac{\frac{n}{\rho+n}\pi(x_3|x_3) - \pi(x_2|x_3)}{\frac{n}{\rho+n}\pi(x_3|x_3) - \pi(x_1|x_3)} = (1-q) + o(1).$$

value-of-search:

$$V_t^X(x_2) \geq \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - C_t^X(x_1) \quad \text{for all } x_2 > x_1 \in [0,1]. \tag{12}$$

Theorem 1′ (stationary PAM with explicit search cost, Morgan (1994))**.** *Suppose that both populations' payoffs are supermodular. Then there is positive assortative matching (PAM) at all times in any stationary equilibrium.*

As under discounting, this holds if search cutoffs are monotone:

Proposition 2′: *Suppose that population X's payoffs are supermodular. Then, in any stationary environment, higher types x have a higher search cutoff,* $y(x_2) \geq y(x_1)$ *for all* $x_2 > x_1$.

Proof. We prove the contrapositive. Let $x_2 > x_1$ be such that $y_t(x_2) < y_t(x_1)$ (the environment being stationary, this applies to all moments in time). Then for any type $\underline{y} \in (y_t(x_2), y_t(x_1))$ the optimal matching decision implies that $\pi^X(\underline{y}|x_1) < V_t^X(x_1)$, yet $\pi^X(\underline{y}|x_2) \geq V_t^X(x_2)$. Then apply the integral representation of the value-of-search and apply the mimicking argument:

$$\pi^X(\underline{y}|x_1) < \int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy - C_t^X(x_1) \ \text{ and } \ \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - C_t^X(x_1) \leq \pi^X(\underline{y}|x_2).$$

In the steady state agents always match with a weakly better type than the one rejected initially. Formally, $Q_t^X(y|x_1) = 0$ for all $y < y_t(x_1)$ including $\underline{y}$, and we may adjust the bounds of integration accordingly. Isolating $C_t^X(x_1)$, it follows that

$$\int_{\underline{y}}^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy - \pi^X(\underline{y}|x_1) > \int_{\underline{y}}^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - \pi^X(\underline{y}|x_2).$$

Since $y_t(x_1) > 0$, agent type x_1's value-of-search exceeds the match payoff from matching with type 0. In effect, type x_1 must almost surely eventually exit the search pool so that $Q_t^X(\cdot|x_1)$ integrates to one. If not it must be that $V_t^X(x_1) = -\infty$, because there is a non-zero probability of incurring an infinite amount of search cost. The preceding inequality thus simplifies to

$$\int_{\underline{y}}^1 [\pi^X(y|x_1) + \pi^X(\underline{y}|x_2) - \pi^X(\underline{y}|x_1) - \pi^X(y|x_2)]Q_t^X(y|x_1)dy > 0,$$

which can impossibly hold if payoffs are not supermodular. □

Proof of Proposition 3′. Suppose that there exist $x_2 > x_1$ such that $y_t(x_2) < y_t(x_1)$ at some time t. Then for any type $\underline{y} \in (y_t(x_2), y_t(x_1))$ the optimal matching decision implies that $\pi^X(\underline{y}|x_1) < V_t^X(x_1)$, yet $\pi^X(\underline{y}|x_2) \geq V_t^X(x_2)$. As before, an application of the mimicking argument implies that

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy - C_t^X(x_1) > \pi^X(\underline{y}|x_1) \ \text{ and } \ \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy - C_t^X(x_1) \leq \pi^X(\underline{y}|x_2). \tag{13}$$

Next, define $\overline{y} > \underline{y}$ such that $\pi^X(\overline{y}|x_1) = \pi^X(\underline{y}|x_1) + C_t^X(x_1)$. Such $\overline{y} \in [0,1]$ does exist (for $\pi^X(\underline{y}|x_1) + C_t^X(x_1) \leq V_t^X(x_1) + C_t^X(x_1) \leq \pi^X(1|x_1)$; then conclude using the intermediate value theorem). Due to supermodularity,

$$\pi^X(\bar{y}|x_2) + \pi^X(\underline{y}|x_1) \geq \pi^X(\underline{y}|x_2) + \pi^X(\bar{y}|x_1) \quad \Leftrightarrow \quad \pi^X(\bar{y}|x_2) \geq \pi^X(\underline{y}|x_2) + C_t^X(x_1).$$

It follows that

$$\int_0^1 \pi^X(y|x_1)Q_t^X(y|x_1)dy > \pi^X(\bar{y}|x_1) \quad \text{and} \quad \int_0^1 \pi^X(y|x_2)Q_t^X(y|x_1)dy \leq \pi^X(\bar{y}|x_2). \tag{14}$$

It remains to observe that, as in the steady state, $Q_t^X(\cdot|x_1)$ is a density and integrates to one. Then type x_1 accepts a lottery that is rejected by type x_2. This runs counter to the characterization of log supermodularity in differences in terms of risk preferences and establishes a contradiction. □

C One-block Block Segregation

Proposition 5 (one-block block segregation)**.** *Suppose that payoffs are multiplicatively separable and continuous in the partner's type*[49]*, and meeting rates $t \mapsto \lambda_t^X(y|x)$ and $t \mapsto \lambda_t^Y(x|y)$ are anonymous, decreasing and (for item 2.) tend to zero. Then there exist thresholds $t \mapsto \underline{x}_t \in [0,1)$ and $t \mapsto \underline{y}_t \in [0,1)$, decreasing if non-zero, so that:*

1. *agents with the most advantageous match opportunities match with the same set of agents: $y_t(x) = \underline{y}_t$ for all $x \geq \underline{x}_t$, and $x_t(y) = \underline{x}_t$ for all $y \geq \underline{y}_t$;*
2. *among agents with inferior match opportunities higher types are more selective: $y_t(x_1) < y_t(x_2) < \underline{y}_t$ for all $x_1 < x_2 < \underline{x}_t$, and $x_t(y_1) < x_t(y_2) < \underline{x}_t$ for all $y_1 < y_2 < \underline{y}_t$.*

To prove Proposition 5, we use our main result, Theorem 2.

Proof of Proposition 5. Denote $\underline{y}_t = y_t(1)$. Step 1: We first show, as is to be expected, that $t \mapsto \underline{y}_t$ is decreasing. To see this, note from (3) and the fact that all other agents accept the highest agent type, that

$$V_t^X(1) = \sup_{(\hat{y}_\tau)_{\tau\geq t}} \int_t^\infty \int_0^1 e^{-\rho(\tau-t)}\pi^X(y|1)\lambda_\tau^X(y)1\{y \geq \hat{y}_\tau\} \exp\{-\int_t^\tau \int_0^1 \lambda_r^X(z)1\{z \geq \hat{y}_r\}dzdr\}dydr.$$

Then fix arbitrary times $t_1 > t_0$. And consider the strategy where, from time t_1 onward, at any time t type 1 accepts type y agents as if it were time $t + t_0 - t_1$. This gives a lower bound for $V_{t_1}^X(1)$. In effect, $V_{t_1}^X(1) - V_{t_0}^X(1)$ is weakly greater than

$$V_{t_1}^X(1) - V_{t_0}^X(1) \geq \int_{t_0}^\infty \int_0^1 \pi^X(y|1)\Big\{\Big(\lambda_{\tau+t_1-t_0}^X(y) - \lambda_\tau^X(y)\Big)1\{y \geq \underline{y}_\tau\} \cdot$$

$$\exp\{-\int_t^\tau \int_0^1 \rho + \lambda_{r+t_1-t_0}^X(z)1\{z \geq \underline{y}_r\}\,dzdr\} \ + \ \lambda_\tau^X(y)1\{y \geq \underline{y}_\tau\} \cdot$$

[49]Formally, payoffs are such that $\pi^X(y|x) = \gamma_1^X(x)\gamma_2^X(y)$ with γ_1^X strictly positive and γ_2^Y a continuous, increasing function.

$$\Big(\exp\{-\int_t^\tau\int_0^1 \rho+\lambda^X_{r+t_1-t_0}(z)1\{z\geq \underline{y}_r\}\,dzdr\} - \exp\{-\int_t^\tau\int_0^1 \rho+\lambda^X_r(z)1\{z\geq \underline{y}_r\}\,dzdr\}\Big)\Big\}dyd\tau.$$

The difference is strictly positive due to the fact that, having assumed decreasing meeting rates, both terms in round parentheses are strictly positive. This proves that $t\mapsto V^X_t(1)$ is decreasing in time, and since $y\mapsto \pi^X(y|1)$ is continuous, it follows that also $\underline{y}_t = \inf\{y : \pi^X(y|1) - V_t(1)\geq 0\}$ is decreasing.

Step 2: We prove item 1., i.e., that all agents $x\in[\underline{x}_t, 1]$ match with the same set of agents. To begin with, admit (as a corollary of Theorem 2) that $\underline{x}_\tau \geq x_\tau(y)$ for all $y\in[0,1]$ and $\tau\geq t$. Since $\tau\mapsto \underline{x}_\tau$ is decreasing, we deduce that all agents $x\in[\underline{y}_t, 1]$ have identical future match opportunities. In effect,

$$V^X_t(x) = \sup_{(\hat{y}_\tau)_{\tau\geq t}} \int_t^\infty\int_0^1 e^{-\rho(\tau-t)}\pi^X(y|x)\lambda^X_\tau(y)1\{y\geq\hat{y}_\tau\}\,\exp\{-\int_t^\tau\int_0^1 \lambda^X_r(z)1\{z\geq\hat{y}_r\}dzdr\}dydr$$

for all $x\in[\underline{x}_t, 1]$. Then recall that $\pi^X(y|x) = \gamma^X_1(x)\gamma^X_2(y)$ and compare with $V^X_t(1)$ as characterized above. It follows that $V^X_t(x) = \frac{\gamma^X_1(x)}{\gamma^X_1(1)}V^X_t(1)$ and

$$y_t(x) = \inf\{y : \gamma^X_2(y) - \frac{V_t(x)}{\gamma^X_1(x)}\geq 0\} = \inf\{y : \gamma^X_2(y) - \frac{V_t(1)}{\gamma^X_1(1)}\geq 0\} = y_t(1).$$

Step 3: We prove item 2., i.e., that among agents with inferior match opportunities, $x_1 < x_2 < \underline{x}_t$, higher agent types are more selective. That $y_t(x_1)\leq y_t(x_2)$ is an implication of Theorem 2. We now show that this inequality is strict when meeting rates tend to zero. First, observe that following standard arguments $t\mapsto V^X_t(x)$ is continuous (see Proposition 6 (i) in Sandmann and Bonneton (2023)) and tends to zero (because meeting rates tend to zero), and so the earliest times at which two agents with inferior match opportunities match with the most desirable agents are finite and favor the more desirable type: for any two $x_1 < x_2 < \underline{x}_t$, it holds that $\inf\{t : \underline{x}_t = x_1\} > \inf\{t : \underline{x}_t = x_2\}$. Then an identical construction as in step 1 implies that $\frac{V^X_t(x_2)}{\gamma^X_1(x_2)} > \frac{V^X_t(x_1)}{\gamma^X_1(x_1)}$. And since $y\mapsto\gamma^X_2(y)$ is continuous, it follows that

$$y_t(x_2) = \inf\{y : \gamma^X_2(y) - \frac{V_t(x_2)}{\gamma^X_1(x_2)}\geq 0\} > \inf\{y : \gamma^X_2(y) - \frac{V_t(x_1)}{\gamma^X_1(x_1)}\geq 0\} = y_t(x_1)$$

as was to be shown. □

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Online Appendix

The online appendix contains the missing proofs from the sections on explicit search costs and model variations. It further develops Rasul's 2008 model on fertility decisions within couples to demonstrate how sufficient conditions for PAM can shed light on applied problems.

A Explicit Search Costs: Missing Proofs

A.1 Necessity

The proof of Proposition 4′ is analogous to the proof of Proposition 4:

Proof of Proposition 4′. We follow the same steps as in Counterexample 2 in the proof of Proposition 4. For an identical set-up, type x_3's stationary HJB equation writes as $c = n[h(n)\pi(x_3|x_3) + (1 - h(n))\pi(x_1|x_3) - V_0^n(x_3)]$. The indifference condition continues unchanged as $V_0^n(x_3) = \pi(x_1|x_3)$. One then deduces algebraically that $h(n)$ is well-defined (as given by $h(n)n = c/(\pi(x_3|x_3) - \pi(x_1|x_3))$).

Next, consider the explicit characterization of the value-of-search preceding time 0 and succeeding the time t_0^n at which agent type x_3 is indifferent between accepting and rejecting agent type x_2:

$$\begin{aligned} V_{t_0^n}^n(x_3) = & \int_{t_0^n}^{0} ne^{-n(\tau - t_0^n)} d\tau \, \pi(x_3|x_3) + \Big(1 - \int_{t_0^n}^{0} ne^{-n(\tau - t_0^n)} d\tau\Big)\pi^X(x_1|x_3) \\ & - c\Big\{ \int_{t_0^n}^{0} (\tau - t_0^n) ne^{-n(\tau - t_0^n)} d\tau + (-t_0^n)\Big(1 - \int_{t_0^n}^{0} ne^{-n(\tau - t_0^n)} d\tau\Big)\Big\} \\ = & \underbrace{(1 - e^{nt_0^n})}_{\equiv Q_{t_0^n}^n(x_3)} \pi(x_3|x_3) + \underbrace{e^{nt_0^n}}_{\equiv Q_{t_0^n}^n(x_1)} \pi(x_1|x_3) - c\frac{1 - e^{nt_0^n}}{n}. \end{aligned}$$

Here, as before, we used that by construction $V_0^n(x_3) = \pi(x_1|x_3)$ and that during time interval $(t_0^n, 0)$ the high type only matches with fellow high type agents. The indifference condition is $V_{t_0^n}^n(x_3) = \pi(x_2|x_3)$ which implies that for all $n \in \mathbb{N}$

$$\pi(x_2|x_3) = Q_{t_0^n}^n(x_3)\pi(x_3|x_3) + Q_{t_0^n}^n(x_1)\pi(x_1|x_3) + o(1).$$

Beginning from time t_0^n, agent type x_2 is accepted by agent type x_3, and thus faces identical match opportunities as an agent type x_3. Accordingly, x_2 can mimic the higher type x_3's match probabilities so that

$$V_{t_0^n}^n(x_2) \geq Q_{t_0^n}^n(x_3)\pi(x_3|x_2) + Q_{t_0^n}^n(x_1)\pi(x_1|x_2) + o(1).$$

We then show that PAM does not occur at time t_0^n for n sufficiently large. Or, we show that $V_{t_0^n}^n(x_2) > \pi(x_2|x_2)$. If not, it must hold that

$$\pi(x_2|x_2) \geq Q_{t_0^n}^n(x_3)\pi(x_3|x_2) + Q_{t_0^n}^n(x_1)\pi(x_1|x_2) + o(1).$$

Since $Q^n_{t^n_0}(\cdot)$ is a probability measure, i.e. $Q^n_{t^n_0}(x_1) + Q^n_{t^n_0}(x_3) = 1$, this runs counter Arrow-Pratt's characterization of risk preferences whereby strict LsubD implies that x_3 is strictly more risk averse than x_2: $\pi(x_2|x_3) \leq Q^n_{t^n_0}(x_3)\pi(x_3|x_3) + Q^n_{t^n_0}(x_1)\pi(x_1|x_3)$ implies $\pi(x_2|x_2) < Q^n_{t^n_0}(x_3)\pi(x_3|x_2) + Q^n_{t^n_0}(x_1)\pi(x_1|x_2)$ Then taking the limit $n \to \infty$ establishes the desired contradiction. □

A.2 Explicit search costs with quits

We here study a further model variation within the explicit search costs case: agents can voluntarily exit the search pool unmatched. This variation is motivated by the possibility that, absent the option to exit, expected search costs may accumulate and outweigh the expected benefit of matching; staying unmatched forever is infinitely costly. If agents lacking significant future match opportunities could quit the search pool, they would. In keeping with the discounting paradigm, we set the value of rejecting all match opportunities or, analogously, quitting the search altogether to be zero.

The option to exit the search pool is irrelevant in the steady state—those who are currently searching would never have entered if they then wanted to quit. However, it invalidates the conclusion of Theorem 2′ for non-stationary environments. Coercing unmatched agents to keep searching ensures that agents who are selective about who they match with must eventually match with someone. If, instead agents exit the search pool after an unsuccessful search, the probability that they match must be bounded away from one. In effect, inequalities (14) no longer amount to a comparison between a lottery and a certain outside option. Nonetheless, Theorem 2″) shows that PAM can be recovered when in addition to the conditions from Theorem 2′, payoffs are *log supermodular*. Crucially, risk preferences play the same role as before. Adding log supermodularity to the sufficiency conditions is unsurprising in light of the analysis under time discounting. It allows us to normalize future match probabilities like in the proof of Theorem 2.

Theorem 2″ (non-stationary PAM with explicit search costs and endogenous quits)**.** *Suppose that both populations' payoffs are supermodular, log supermodular and log supermodular in differences. Then there is positive assortative matching (PAM) at all times in any (non-stationary) equilibrium.*

Proof. When there are outside options, there are two stopping rules: match if $\pi^X(y|x) \geq V^X_t(x)$, exit if $0 \geq V^X_t(x)$. Now suppose that matching is not assortative. As before an application of the mimicking argument guarantees that there exist $x_2 > x_1$ and $\underline{y}$ such that (13) holds with the exception that $y \mapsto Q^X_t(y|x_1)$ need not integrate to one. Then consider two normalizations: let, as in the proof of Theorem 2, $\hat{y}$ be such that $\pi^X(\hat{y}|x_1)\int_0^1 Q^X_t(y|x_1)dy = \pi^X(\underline{y}|x_1)$. Clearly $\hat{y} > \underline{y}$. Then $\pi^X(\hat{y}|x_2)\int_0^1 Q^X_t(y|x_1)dy \geq \pi^X(\underline{y}|x_2)$ because payoffs are log supermodular. Next, let, as in the proof of Theorem 2′, $\overline{y}$ be such that $\pi^X(\overline{y}|x_1)\int_0^1 Q^X_t(y|x_1)dy = \pi^X(\hat{y})\int_0^1 Q^X_t(y|x_1)dy + C^X_t(x_1)$. Clearly $\overline{y} > \hat{y}$ because search cost are non-negative. Then $\pi^X(\overline{y}|x_2)\int_0^1 Q^X_t(y|x_1)dy \geq \pi^X(\hat{y}|x_2)\int_0^1 Q^X_t(y|x_1)dy + C^X_t(x_1)$ because payoffs are supermodular. Given both normalizations, inequalities (14) continue to hold which (as before) upsets the posited ranking of risk preferences. □

B Model Variations

B.1 Non-Stationary Types

Proof of Theorem 3. The proof of this theorem is readily adapted from Theorem 2, since, just as in the baseline model, the value-of-search admits an integral representation over payoffs that subsumes the time dimension. For an analogously defined value-of-search that accounts for age, it holds that

$$V_t^X(x,\alpha_t) = \int_t^\infty \int_0^1 \Pi^X(y,\beta_\tau|x) Q_t^X(y,\beta_\tau|x,\alpha_t) dy d\tau.$$

Then the proof of Lemma 1 implies that

$$V_t^X(x_2,\alpha_t') \geq \int_t^\infty \int_0^1 \Pi^X(y,\beta_\tau|x_2) Q_t^X(y,\beta_\tau|x_1,\alpha_t'') dy d\tau.$$

for $x_2 > x_1$ and $\alpha_t' = \alpha_t''$ in general, and $\alpha_t' \leq \alpha_t''$ if match payoffs are non-increasing in age. If agents cease to be attractive to others as α_t grows, agents face downside risk even in the steady state. We therefore cannot simplify thie problem as in the steady state proof of the baseline model. Instead, we need to proceed with the proof of Theorem 2 requiring log supermodularity in differences. □

C Application

Our theory focused on ex-ante match creation and did not address the origin of payoffs. In contrast, many applied models in household bargaining or team production provide a detailed description of the strategic interactions that occur ex-post, once agents are already matched (see Chiappori and Mazzocco (2017) and references therein). Here, we integrate ex-ante and ex-post perspectives into a unified marriage model, where anticipation of having children informs match payoffs.

C.1 Marriage and Fertility Choice

We build on the work of Rasul (2008), who studies fertility among married ethnic Chinese and Malay couples in Malaysia. In marriage, spouses must resolve differences in fertility preferences. Rasul (2008) proposes a game-theoretic analysis based on differing threat points in bargaining, whose predictions align with the observed fertility outcomes among both ethnic groups. Here, we extend Rasul's analysis of married couples' decisions to the context of match formation. Using Rasul's equilibrium utilities as primitive match payoffs, we identify when these payoffs satisfy our sufficient conditions for assortative matching. When they do, our theory predicts that marriages form between individuals with similar fertility preferences.

Rasul (2008)'s analysis is particularly well-suited to our purposes. One of his central contributions is to uncover a hold-up problem in couple bargaining: husbands cannot commit ex-ante to compensate

Marriage threat point:	Divorce (Malay case)	No divorce (Chinese case)
Payoff properties:		
Increasing	no	yes
Log supermodular	no	yes
Log supermodular in diff.	yes	yes
Empirical observation:	Less PAM	More PAM

Table 2: Do equilibrium payoffs from Rasul (2008) satisfy our condition for PAM? By "yes", we mean that the condition is satisified for all parameters values, and "no" means that it is not always satisfied.

their wives for bearing the couple's children.[50] From a matching perspective, the absence of ex-ante agreed transfers suggests that marriage formation follows the NTU paradigm, where a lack of commitment prevents ex-ante redistribution within the couple.

C.2 Formal Analysis

Rasul (2008)'s model: The married couple comprises a husband (y) and a wife (x). Types $x, y \in [0, 1]$ encode preferences for greater fertility. Realized fertility q is at the sole discretion of the wife and subsumes both the quality and quantity of children born. Individual spouses wish to match their desired fertility, but can be compensated via transfers.

Excluding transfers and sunk cost, utility over realized fertility q is

$$u^X(q|x) = v^X - \frac{1}{2}(q-x)^2,$$
$$u^Y(q|y) = v^Y - \frac{1}{2}(q-y)^2,$$

where v^X and v^Y are private gains from marriage.

The timing of the game is as follow. In stage 1, the wife chooses fertility q and incurs sunk cost $cq^2/2$. In stage 2, the husband makes a transfer to his wife, determined via Nash bargaining with positive bargaining weights $\alpha^X, \alpha^Y : \alpha^X + \alpha^Y = 1$.[51] Bargaining outcomes hinge on the spouses' threat points.

Divorce Regime. If the relevant threat point is divorce (attributed to Malay couples), spouses lose the private benefits of marriage v^X, v^Y, and pursue their fertility goals with future marriage partners. Denote the regime $\mathscr{R} = D$. Threat point utility is $\overline{u}^X_D(q; x) = \overline{u}^Y_D(q; y) = 0$.

Non-cooperative Regime. If the relevant threat point is a distressed, non-cooperative marriage (at-

[50] In Malay marriages, both spouses' fertility preferences have an equal, positive, and significant impact on fertility outcomes. In ethnic Chinese couples, male fertility preferences have no statistical power to explain realized fertility levels. This finding is inconsistent with bargaining at the moment of match creation. If spouses could commit to transfers before marriage, the observed fertility outcomes should reflect a compromise between both spouses' preferences.

[51] About notation: in Rasul's paper the husband's bargaining power is θ, not α^X; α^Y becomes $1-\theta$. Fertility costs are non-parametric as given by $c(q)$. Finally, Rasul allows for a common fertility benefit, $\phi(q)$, that we normalize to be zero.

tributed to ethnic Chinese couples), the mismatch in desired and realized fertility levels is irrevocable. Denote the regime $\mathscr{R} = NC$. Threat point utility is $\overline{u}^X_{NC}(q;x) = -\frac{1}{2}(q-x)^2$ and $\overline{u}^Y_{NC}(q;y) = -\frac{1}{2}(q-y)^2$.

Resolution of the model: Rasul (2008) solves the game via backward induction.[52]

Second stage: once fertility decisions have been made, the couple determines ex-post transfers via Nash bargaining. Transfers $\boldsymbol{t}_{\mathscr{R}}(q;x,y)$ in both regimes $\mathscr{R} \in \{NC, D\}$ maximize the product of surplus utilities:

$$\boldsymbol{t}_{\mathscr{R}}(q;x,y) \in \arg\max_{t} \left(u^X(q;x) - \overline{u}^X_{\mathscr{R}}(q;x) + \boldsymbol{t}\right)^{\alpha^X} \left(u^Y(q;y) - \overline{u}^Y_{\mathscr{R}}(q;y) - \boldsymbol{t}\right)^{\alpha^Y}. \tag{15}$$

First stage: the female spouse anticipates second-stage transfers for given fertility decision q. The female fertility decision thus solves

$$q_{\mathscr{R}}(x,y) = \arg\max_{q} u^X(q;x) + \boldsymbol{t}_{\mathscr{R}}(q;x,y) - cq^2/2. \tag{16}$$

Plugging equilibrium transfers and fertility decisions into utilities gives the expected utilities of the bargaining game.[53] These correspond to the match payoffs of the search-and-matching model.[54]

Check conditions for PAM: We then check whether Rasul's match payoffs satisfy our conditions for PAM. Results are summarized in Table 2. The theory, in line with empirical observations, predicts PAM only in the non-cooperative regime where divorce is inadmissible.

Non-cooperative Regime It emerges that transfers (solution to (15)) and female fertility decisions (solution to (16)) do not depend on the husband's fertility preferences y: $\boldsymbol{t}_{NC}(q;x,y) = \alpha^X v^Y - \alpha^Y v^X$ and $q_{NC}(x,y) = \frac{x}{1+c}$. And so neither do female payoffs:

$$\pi^X_{NC}(y|x) = \alpha^X(v^Y + v^X) - \frac{1}{2}\frac{c}{1+c}x^2,$$

As a result, a single woman's optimal match acceptance strategy is to not discriminate among potential husbands and prioritize quick matching. Males, by contrast, can only realize their desired fertility by marrying a woman whose preferences closely align with their own. Male match payoffs are

$$\pi^Y_{NC}(x|y) = \alpha^Y(v^X + v^Y) - \frac{1}{2}\left(\frac{x}{1+c} - y\right)^2.$$

These are increasing in the partner's type x for all men with a sufficiently high fertility preference, more precisely for all $y \geq \frac{\max x}{1+c}$ where $\max x$ is the highest supported type x.[55]

[52] Transfers and fertility levels corresponds to Equations (14) and (15) in Rasul (2008).

[53] In greater detail, $\pi^X_{\mathscr{R}}(y|x) = u^X(q_{\mathscr{R}}(x,y);x) + \boldsymbol{t}_{\mathscr{R}}(q_{\mathscr{R}}(x,y);x,y) - cq_{\mathscr{R}}(x,y)^2/2$ and $\pi^Y_{\mathscr{R}}(x|y) = u^Y(q_{\mathscr{R}}(x,y);y) - \boldsymbol{t}_{\mathscr{R}}(q_{\mathscr{R}}(x,y);x,y)$.

[54] Using our notations, the transfers and fertility levels are

$$\boldsymbol{t}_{NC}(q;x;y) = \alpha^X v^Y - \alpha^Y v^X \quad \text{and} \quad \boldsymbol{t}_D(q;x,y) = \alpha^X(v^Y - \frac{1}{2}(q-y)^2) - \alpha^Y(v^X - \frac{1}{2}(q-x)^2),$$

and,

$$q_{NC}(x,y) = \frac{x}{1+c} \qquad \text{and} \qquad q_D(x,y) = \frac{\alpha^X(x+y)}{c+2\alpha^X}.$$

[55] Fortunately, this restriction is of little empirical relevance. In the data, low fertility preferences among men are rare,

The sufficient condition for assortative matching identified in this article is log supermodularity in differences. This holds because

$$d_{xy}^2 \log d_x \pi_{NC}^Y(x|y) = \frac{d_x q_{NC}(x,y)}{(q_{NC}(x,y)-y)^2} = \frac{1}{1+c} \frac{1}{(x/(1+c)-y)^2} > 0.$$

One can further check that payoffs are also log supermodular.

Divorce Regime. In the regime where divorce is prevalent, attributed to Malays, both spouses' preferences carry equal weight in determining fertility outcomes. As a result, individual payoffs are single-peaked in the partner's type. The asymmetric distributions over fertility preferences between men and women then imply that men with the highest fertility preference will be the least desirable husbands. Conversely, women with the lowest fertility preference are the least desirable wives. These individuals then face the worst match opportunities in equilibrium, giving rise to negative assortative matching between the two groups.

Overall, Theorem 2 predicts PAM along fertility preferences only when divorce is not an admissible threat point. This prediction is consistent with empirical observations. Fertility preferences play a negligible role in explaining marriage patterns among Malays: 44% of couples have fertility preferences that differ by at least two children; 10% differ by more than four (see Figure 3 in Rasul (2008)). The extent of these differences is much smaller among Chinese couples and not attributable to differences in the distribution of individual fertility preferences. This suggests that, remarkably, payoffs derived from a within-household decision model have predictive power for aggregate sorting in the marriage market.

as men typically desire more children than women. Payoff monotonicity arises because women unilaterally bear the cost of child birth, yet are not compensated for it.