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Concentration Indices, Welfare Distortions, and Misallocation in Oligopoly

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Concentration Indices, Welfare Distortions, and Misallocation in Oligopoly*

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Abstract

We study welfare distortions in a multiproduct-firm pricing game with constant elasticity of substitution (CES) or multinomial logit (MNL) demand. Using approximations both around small market shares and around monopolistic competition conduct, we identify sufficient statistics to gauge the extent of inefficiencies caused by oligopolistic market power. We find that, at a low order, the oligopoly distortions are proportional to the Herfindahl index of industry concentration. At a higher order, distortions also depend on the cubic Hannah-Kay concentration index. Additionally, we show that the welfare loss from resource misallocation is approximately proportional to the difference between the cubic Hannah-Kay index and the square of the Herfindahl index.

Keywords: Market power, oligopoly pricing, misallocation, industry concentration, Herfindahl index, Hannah-Kay indices, sufficient statistics.
JEL Codes: L13, D43, E20

1 Introduction

There is a growing empirical literature that reports worrying trends in the evolution of market power and industry concentration. For instance, in an influential paper, De Loecker, Eeckhout, and Unger (2020) estimate that the sales-weighted average markups in the U.S. rose from 21% in 1980 to 61% in 2016.[1] De Loecker and Eeckhout (2018) report comparable figures for the evolution of markups in the global economy. In a similar vein, Barkai (2020)

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[1]Consistent with the findings on rising markups, the labor share of GDP has declined in the U.S. and other countries since the early 1980s (see, e.g., Autor, Dorn, Katz, Patterson, and Reenen, 2020).

reports that the average four-firm concentration ratio in U.S. NAICS 6-digit industries increased by around 5 percentage points between 1997 and 2012. Koltay, Lorincz, and Valletti (2023) find that the share of high-concentration industries (defined as ISIC four-digit industries where the four-firm concentration ratio exceeds 50%) grew by more than 60% from 1998 to 2018.[2]

The aim of this paper is to identify sufficient statistics to gauge the extent of welfare distortions caused by oligopolistic market power. These sufficient statistics take the form of (combinations of) concentration indices, notably the Herfindahl index and the cubic Hannah-Kay index (see Hannah and Kay, 1977). To this end, we consider a multiproduct-firm oligopoly model with constant elasticity of substitution (CES) and multinomial logit (MNL) demand featuring arbitrary firm and product heterogeneity. As shown in Nocke and Schutz (2018, forthcoming), this framework has several appealing properties. First, the pricing game is fully aggregative, in that each firm's profit and incentives depend on rivals' behavior solely through a uni-dimensional sufficient statistic summarizing the intensity of competition. Second, that statistic is also a sufficient statistic for consumer surplus. Third, the type aggregation property holds, in that each firm's product portfolio can be summarized by a uni-dimensional sufficient statistic—the firm's type.

Given that products are horizontally differentiated, we take the outcome under monopolistic competition, where firms do not internalize the effects of their actions on the industry aggregator, as the relevant competitive benchmark. The reason is that, under monopolistic competition, the equilibrium markups are identical across products and invariant to which firm offers which products. To gauge the extent of welfare distortions due to oligopolistic market power, we thus compare our welfare measures in oligopoly to their values under monopolistic competition. Our results are based on Taylor approximations. Our first set of results relies on approximations around small market shares, and the second on approximations around monopolistic competition conduct. We find that, with both sets of approximations, the oligopoly distortions to consumer surplus and aggregate surplus are proportional to the Herfindahl index at a low order. At a higher order, the oligopoly distortions also depend on the cubic Hannah-Kay concentration index.

Next, we propose a decomposition of the oligopoly distortions in the spirit of the macroeconomics literature on misallocation (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). The decomposition features two terms. The first term, which is driven by the overall level of the average markup, measures the welfare loss generated by the fact that an oligopolistic industry does not utilize the efficient amount of resources for production. The second term, which is driven by markup heterogeneity, captures the extent to which the resources used in the industry are not efficiently allocated to the production of the various products. It thus corresponds to the welfare loss from within-industry misallocation.

Beginning with the first term of the decomposition, we show that oligopolistic market power results in too few resources being utilized under CES demand. For the case of MNL

[2] For critical takes on this literature, see Miller (2025) and Shapiro and Yurukoglu (forthcoming).

demand, however, we show that both excessive and insufficient resource employment can arise. Turning to the second term, we show that for *any* well-behaved quasi-linear demand system, within-industry resource allocation is efficient if and only if percentage markups are equalized across products. We then approximate the aggregate surplus loss from within-industry misallocation under the assumption that firms have small market shares or in the neighborhood of monopolistic competition conduct. For reasons that are explained in Section 4.2, we focus on the case of CES demand. With both sets of approximations, we find that the misallocation term is negligible at a low order, implying that the oligopoly distortion is entirely driven by the industry under-utilizing resources. At a higher order, the misallocation is proportional to the difference between the cubic Hannah-Kay concentration index and the square of the Herfindahl index. As that difference is neither concave nor convex in the vector of market shares, it is generally not true that greater industry concentration results in larger within-industry misallocation.

Related literature. There is a small literature studying the relationship between industry concentration and market performance in the special case of the homogeneous-goods Cournot model. Cowling and Waterson (1976) show that the Herfindahl index provides a measure of an industry's average markup and profitability (see Belleflamme and Peitz, 2010, for a textbook treatment). Dansby and Willig (1979) show that the industry performance gradient index, which measures the rate of potential improvement in aggregate surplus from a small variation in the output vector, is proportional to the square root of the Herfindahl index. Assuming that demand has constant curvature, Corchon (2008) shows that the percentage deadweight loss in Cournot oligopoly depends solely on the Herfindahl index, demand curvature, and the market share of the top firm (see also Ritz, 2014).[3] Spiegel (2021) shows that the ratio of producer surplus to consumer surplus in Cournot oligopoly is equal to the product of the Herfindahl index and the elasticity of consumer suplus with respect to aggregate output.

We are aware of only few results linking the Herfindahl index to industry performance measures in models of differentiated-products industries. In an oligopoly model with CES preferences and price or quantity competition, Grassi (2017) relates the industry average markup to the Herfindahl index and higher-order Hannah-Kay indices. In Feenstra and Weinstein (2017)'s model with translog preferences, the representative consumer's indirect utility depends on the Herfindahl index both directly, due to translog preferences, and indirectly, due to endogenous markups. In earlier work, Nocke and Schutz (2023, forthcoming), we use the same approximation techniques as in the present paper to show that the welfare effects of a merger without synergies are approximately proportional to the naively-computed, merger-induced variation in the Herfindahl index.[4] In a similar spirit, Nocke and Whinston (2022) show both theoretically and empirically that the level of synergies required for a merger not

[3]Demand has constant curvature if and only if it is ρ-linear, i.e., inverse demand takes the form $P(Q) = a - bQ^\delta$ (Bulow and Pfleiderer, 1983).

[4]Breinlich, Fadinger, Nocke, and Schutz (2020) use an approximation around monopolistic competition conduct to derive an industry-level gravity equation for trade flows in oligopoly.

to harm consumers is positively related to the induced change in the Herfindahl index, but not to its level. To the best of our knowledge, our paper is the first to link explicitly the oligopoly distortions to consumer surplus and aggregate surplus to concentration indices, including the Herfindahl index.

Our paper is also related to the macroeconomics literature on misallocation, pioneered by Restuccia and Rogerson (2008) and Hsieh and Klenow (2009), and surveyed by Restuccia and Rogerson (2017). That literature is concerned with welfare or aggregate productivity losses due to frictions at the firm or plant level that imply that the marginal revenue product of inputs is not equalized across firms, both within and across industries. A source of friction that has recently received much attention is markup heterogeneity across firms. In a macroeconomic model with flexible input-output linkages, Baqaee and Farhi (2020) take markups as exogenous and approximate the aggregate productivity loss in the neighborhood of small markups. In a quantitative macroeconomic model, in which markups vary due to either monopolistic competition with Kimball (1995) preferences or single-product-firm oligopoly with (nested) CES preferences, Edmond, Midrigan, and Xu (2023) evaluate the welfare loss from markups. They quantitatively break down the loss into three components: one arising from the level of the average markup, one from the misallocation of factors of production due to markup heterogeneity, and one from inefficient entry. In a Cournot model with differentiated products and linear demand, Pellegrino (forthcoming) estimates the deadweight loss from oligopolistic competition and the component stemming from misallocation. We contribute to this literature by providing formulas that link both the overall welfare loss from markups and the induced misallocation to concentration measures.

Road map. The remainder of the paper is organized as follows. In Section 2, we describe the multiproduct-firm oligopoly model and characterize its equilibrium using aggregative games techniques. In Section 3, we approximate the oligopoly distortions to consumer and aggregate surplus. In Section 4, we relate the within-industry misallocation to concentration indices. We conclude in Section 5.

2 The Multiproduct-Firm Oligopoly Framework

In this section, we present the multiproduct-firm pricing game with CES and MNL demand of Nocke and Schutz (forthcoming). We describe the model in Section 2.1 and characterize equilibrium behavior using aggregative games techniques in Section 2.2.

2.1 The Model

Consider an industry with a set $\mathcal{N}$ of horizontally differentiated products. Demand is of either the MNL type or the CES type. The representative consumer's quasi-linear preferences are described by the indirect utility function

$$y + V(\boldsymbol{p}) = y + V_0 \log \left[H^0 + \sum_{j \in \mathcal{N}} h_j(p_j) \right], \tag{1}$$

where $y > 0$ is the consumer's income, $V_0 > 0$ is a market size parameter, and $H^0 \geq 0$ is a baseline-utility parameter. In the case of MNL demand, the functions h_j take the exponential form $h_j(p_j) = \exp\left(\frac{a_j - p_j}{\lambda}\right)$; in the case of CES demand, they are power functions, $h_j(p_j) = a_j p_j^{1-\sigma}$. The parameter $a_j > 0$ represents the quality of product j; $\lambda > 0$ and $\sigma > 1$ measure the substitutability of products. Defining the industry aggregator as

$$H(\boldsymbol{p}) \equiv H^0 + \sum_{j \in \mathcal{N}} h_j(p_j),$$

indirect utility can be rewritten as $V(\boldsymbol{p}) = V_0 \log H(\boldsymbol{p})$.

Applying Roy's identity, we obtain the demand for product i:

$$D_i(\boldsymbol{p}) = V_0 \frac{-h_i'(p_i)}{H(\boldsymbol{p})}. \tag{2}$$

This demand system can also be derived from a discrete/continuous choice model, in which each consumer first chooses a product (or the outside option), and then decides how many units of that product to consume (see Anderson, de Palma, and Thisse, 1992; Nocke and Schutz, 2018). Under this micro-foundation, any given consumer ends up choosing product i with probability h_i/H; conditional on having chosen product i, a consumer purchases $-h_i'/h_i$ units of that product. Moreover, V_0 is the total mass of consumers, and $\log H^0$ is the mean utility of the outside option. In the remainder of the paper, we normalize V_0 to 1.

The set of firms, $\mathcal{F}$, is a partition of $\mathcal{N}$. That is, each product is sold by one and only one firm. The profit of firm $f \in \mathcal{F}$ is given by

$$\Pi^f(\boldsymbol{p}) = \sum_{j \in f} (p_j - c_j) \frac{-h_j'(p_j)}{H(\boldsymbol{p})}, \tag{3}$$

where $c_j > 0$ denotes product j's constant marginal cost of production, incurred in units of the Hicksian composite commodity. Firms compete by simultaneously setting prices. The solution concept is Nash equilibrium.

The equilibrium characterization below and our results on concentration indices as measures of oligopoly distortions rely on firm-level market shares. We define the market share of firm f as

$$s^f = \sum_{j \in f} \frac{h_j}{H}.$$

In the discrete/continuous choice micro-foundation, this corresponds to the probability that any given consumer chooses one of firm f's products. Moreover, s^f corresponds to firm f's output share under MNL demand and to firm f's revenue share under CES demand. In both cases, the firms' market shares add up to $1 - H^0/H$, where H^0/H is the market share of the outside option.

2.2 Equilibrium Behavior

In this subsection, we summarize the key properties of the equilibrium. We refer the reader to Nocke and Schutz (2018, forthcoming) for details and proofs.

The equilibrium exists and is unique. Each firm f's pricing behavior can be summarized by a uni-dimensional sufficient statistic, $\mu^f > 1$, referred to as firm f's ι-markup. Specifically, given the ι-markup μ^f, firm f prices each of its products $i \in f$ according to the following inverse elasticity rule:

$$\frac{p_i - c_i}{p_i} = \frac{\mu^f}{p_i h_i''(p_i)/(-h_i'(p_i))}.$$

Under MNL demand, the elasticity term $p_i h_i''/(-h_i')$ is equal to p_i/λ, and so firm f charges the same absolute markup, $p_i - c_i = \lambda \mu^f$, on every product in its portfolio. Under CES demand, the elasticity term is constant and equal to σ, implying that firm f sets the same Lerner index, $(p_i - c_i)/p_i = \mu^f/\sigma$, on every product $i \in f$. With such a common ι-markup, the profit of firm f is simply given by

$$\Pi^f = \alpha \mu^f s^f. \tag{4}$$

The equilibrium ι-markup and market share of firm f satisfy the following condition:

$$\mu^f = \frac{1}{1 - \alpha s^f}, \tag{5}$$

where $\alpha \equiv 1$ under MNL demand and $\alpha \equiv (\sigma - 1)/\sigma$ under CES demand. Equation (5) can be understood as an upward-sloping firm-level supply relation: a firm that has a high market share has significant market power, and therefore sets a high ι-markup.

In equilibrium, μ^f and s^f must also be consistent with the following firm-level demand relation:

$$s^f = \begin{cases} \frac{T^f}{H} \left(1 - (1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{under CES demand,} \\ \frac{T^f}{H} \mathrm{e}^{-\mu^f} & \text{under MNL demand,} \end{cases} \tag{6}$$

where $T^f \equiv \sum_{j \in f} h_j(c_j)$ is firm f's type. According to equation (6), firm f's market share is a decreasing function of firm f's markup, with T^f/H acting as a demand shifter. The sufficient statistic T^f summarizes all the relevant information about firm f's product portfolio. As it is higher if firm f sells more products, with lower marginal costs or with higher qualities, it can be viewed as a measure of how good the firm is.

The system of equations (5)–(6) has a unique solution in (μ^f, s^f), which pins down $m(T^f/H)$ and $S(T^f/H)$, firm f's *markup fitting-in function* and *market-share fitting-in func-*

tion, respectively. Both functions are strictly increasing: a firm that has a higher type or operates in a less competitive environment (lower H) sets a higher ι-markup and receives a larger market share. It can also be shown that firm f's equilibrium profit satisfies $\Pi^f = \mu^f - 1$. We thus obtain the *profit fitting-in function* of firm f:

$$\pi\left(\frac{T^f}{H}\right) \equiv m\left(\frac{T^f}{H}\right) - 1 = \frac{\alpha S(H/T^f)}{1 - \alpha S(H/T^f)}, \tag{7}$$

where we have used equation (5) to obtain the second equality.

Finally, the equilibrium level of the aggregator, H^*, is uniquely pinned down by the condition that market shares add up to one:

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right) = 1. \tag{8}$$

Summing up: the aggregator level, H^*, is pinned down by equilibrium condition (8). Firm-level equilibrium behavior is then determined by the corresponding fitting-in functions: in equilibrium, firm f sets an ι-markup of $\mu^{f*} = m(T^f/H^*)$, commands a market share of $s^{f*} = S(T^f/H^*)$, and earns a profit of $\Pi^{f*} = \pi(T^f/H^*)$. Equilibrium prices can be backed out from the inverse elasticity rule: for every $i \in f$, p_i^* is equal to $c_i + \lambda\mu^{f*}$ under MNL demand, and to $c_i\sigma/(\sigma - \mu^{f*})$ under CES demand.

Conduct parameter. For some of our approximation results, it is useful to adopt a flexible approach to industry conduct. Specifically, suppose that, when firm f increases the price of its product $i \in f$ by dp_i, it believes that the impact on the aggregator is $\theta(\partial H/\partial p_i)dp_i$, where $\theta \in [0, 1]$ is the industry conduct parameter.[5] The baseline model studied above, in which firms have correct conjectures, corresponds to the case of $\theta = 1$. If instead $\theta < 1$, then firms do not fully internalize the impact of their behavior on the industry aggregator.

For a given conduct parameter, θ, the analysis proceeds as above. The firm-level supply relation (equation (5)) must be adapted as follows:

$$\mu^f = \frac{1}{1 - \alpha\theta s^f}. \tag{9}$$

The markup and market-share fitting-in functions, $m(T^f/H, \theta)$ and $S(T^f/H, \theta)$, jointly solve equations (6) and (9). The expression for the profit fitting-in function becomes:

$$\pi\left(\frac{T^f}{H}, \theta\right) \equiv \frac{m(T^f/H, \theta) - 1}{\theta} = \frac{\alpha S(H/T^f, \theta)}{1 - \alpha\theta S(H/T^f, \theta)}.$$

The equilibrium aggregator level, $H^*(\theta)$, continues to be pinned down by market shares

[5] Our treatment of industry conduct under product differentiation is the natural counterpart to the classic approach for homogeneous-products industries, as surveyed in Bresnahan (1989).

having to add up to one:

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}, \theta\right) = 1. \tag{10}$$

The analysis in the remainder of the paper should be understood as being carried out under the assumption that $\theta = 1$, unless explicitly stated otherwise.

Monopolistic competition. The special case of monopolistic competition conduct arises when $\theta = 0$, i.e., when firms believe that they have no impact on the industry aggregator. From equation (9), we see that each firm sets its ι-markup to the lowest possible value, $\mu^f = 1$, regardless of the firm's market share. We thus refer to $\mu^f = 1$ as the monopolistic competition ι-markup.

Under fully fledged oligopoly conduct ($\theta = 1$), the outcome in which each firm sets the monopolistic competition ι-markup arises in the limit as firms' market shares tend to zero, i.e., when firms become atomless. Such a limiting outcome can be obtained by infinitely replicating the population of firms, or by making the mean utility of the outside option, $\log H^0$, go to infinity.

Nested demand systems. The CES and MNL demand systems have the independence of irrelevant alternatives (IIA) property, as the ratio of demands for any two products is independent of the price of any third product. This restriction on substitution patterns can be relaxed by considering *nested* CES and *nested* MNL demand systems, in which the set of products is partitioned into a set of nests, with product being closer substitutes within nests than across nests. In Nocke and Schutz (forthcoming), we show that, provided each firm owns one or several entire nests of products, the pricing game remains aggregative with consumer surplus being equal to the log of the aggregator, and the common ι-markup and type aggregation properties continue to hold. Moreover, fitting-in functions are still defined by equations (5), (6), and (7), and the aggregator remains being pinned down by equation (8). As the analysis in the present paper relies only on these properties of the pricing game, this implies that all the results stated below continue to hold under nested CES or MNL demand, provided each firm owns one or several entire nests of products.

3 Concentration Indices and Oligopoly Distortions

In this section, we relate the loss in consumer surplus and aggregate surplus from oligopolistic market power to concentration indices using Taylor approximations. We provide two sets of approximation results: when firms have small market shares (Section 3.1), and when industry conduct is close to monopolistic competition (Section 3.2).

As a competitive benchmark for the case without market power, we use the equilibrium outcome under monopolistic competition rather than perfect competition. If goods were homogeneous, then under both price and quantity competition, the equilibrium outcome would converge to perfect competition as the population of firms is infinitely replicated, so that each firm's limiting size is negligible relative to the size of the market. By contrast, in our differentiated-goods framework, such an infinite replication results in the monopolistic competition outcome, as explained in Section 2.2.

In our model, if consumers have access to an outside option ($H^0 > 0$), then the sum of the choice probabilities for the firms' products, $\Sigma \equiv \sum_{f \in \mathcal{F}} s^f$, is strictly less than one. By contrast, in antitrust practice, the outside option is not accounted for, implying that firms' market shares always add up to one. In the following, we therefore renormalize firms' market shares when defining concentration indices. Specifically, the Herfindahl index (HHI) is defined as

$$\text{HHI} \equiv \sum_{f \in \mathcal{F}} \left(\frac{s^f}{\Sigma} \right)^2 .$$

As we will see below, our higher-order approximations also give rise to the cubic Hannah-Kay concentration index (CHK), defined as

$$\text{CHK} \equiv \sum_{f \in \mathcal{F}} \left(\frac{s^f}{\Sigma} \right)^3 .$$

3.1 Approximation Results for Small Firms

For the approximations around small market shares, we assume that consumers have access to an outside option ($H^0 > 0$), so that $\Sigma < 1$. We begin by expressing consumer surplus and aggregate surplus as functions of the equilibrium profile of firm-level market shares, $\boldsymbol{s} \equiv (s^f)_{f \in \mathcal{F}}$. From equation (8), the equilibrium aggregator level H^* is equal to $H^0/(1 - \Sigma(\boldsymbol{s}))$. It follows that consumer surplus can be written as:

$$\text{CS}(\boldsymbol{s}) = \log H^0 - \log(1 - \Sigma(\boldsymbol{s})) . \tag{11}$$

Note that consumer surplus depends on the firms' market shares only through their sum. This is a feature of demand systems satisfying the independence of irrelevant alternatives (IIA) property, as all the products are equally good substitutes to the outside option.

Adding the firms' profits (see equation (7)) to consumer surplus, we obtain an expression for aggregate surplus:

$$\text{AS}(\boldsymbol{s}) = \log H^0 - \log(1 - \Sigma(\boldsymbol{s})) + \sum_{f \in \mathcal{F}} \frac{\alpha s^f}{1 - \alpha s^f} .$$

Note that aggregate surplus is strictly increasing and strictly convex in the vector of market shares. In particular, a mean-preserving spread of market shares, which necessarily leaves consumer surplus unchanged, raises industry profit and thus aggregate surplus by reallocating market shares towards larger firms charging higher markups.

To approximate the oligopoly distortion to consumer surplus and aggregate surplus when firms have small market shares, we proceed as follows. We first fix a vector of market shares $\boldsymbol{s}$, and compute the welfare measures $\mathrm{CS}(\boldsymbol{s})$ and $\mathrm{AS}(\boldsymbol{s})$. Using $\boldsymbol{s}$, we then recover the type vector $\boldsymbol{T}(\boldsymbol{s}) = (T^f(\boldsymbol{s}))_{f\in\mathcal{F}}$ that must have given rise to this profile of market shares under oligopoly. Next, using $\boldsymbol{T}(\boldsymbol{s})$, we compute our welfare measures under monopolistic competition as functions of firms' market shares under oligopoly, $\mathrm{CS}^m(\boldsymbol{s})$ and $\mathrm{AS}^m(\boldsymbol{s})$. In Appendix A (see Lemma A.6), we show that

$$\begin{aligned} \mathrm{CS}^m(\boldsymbol{s}) &= \log\left(H^0 + \mathcal{T}(\boldsymbol{s})S'(0)\right), \qquad (12)\\ \text{and } \mathrm{AS}(\boldsymbol{s}) &= \log\left(H^0 + \mathcal{T}(\boldsymbol{s})S'(0)\right) + \alpha\frac{\mathcal{T}(\boldsymbol{s})S'(0)}{H^0 + \mathcal{T}(\boldsymbol{s})S'(0)}, \end{aligned}$$

where $\mathcal{T}(\boldsymbol{s}) \equiv \sum_{f\in\mathcal{F}} T^f(\boldsymbol{s})$. Finally, we apply Taylor's Theorem to approximate the oligopoly distortions $\mathrm{CS}(\boldsymbol{s}) - \mathrm{CS}^m(\boldsymbol{s})$ and $\mathrm{AS}(\boldsymbol{s}) - \mathrm{AS}^m(\boldsymbol{s})$:

Proposition 1. *In the neighborhood of $\boldsymbol{s} = \boldsymbol{0}$, the oligopoly distortions are:*[6]

$$\mathrm{CS}(\boldsymbol{s}) - \mathrm{CS}^m(\boldsymbol{s}) = -\alpha\Sigma(\boldsymbol{s})^2\left(\mathrm{HHI}(\boldsymbol{s}) + \frac{1}{2}(1+2\alpha)\Sigma(\boldsymbol{s})\,\mathrm{CHK}(\boldsymbol{s})\right) + o\left(\|\boldsymbol{s}\|^3\right),$$

$$\mathrm{AS}(\boldsymbol{s}) - \mathrm{AS}^m(\boldsymbol{s}) = -\alpha\Sigma(\boldsymbol{s})^2\left(\mathrm{HHI}(\boldsymbol{s})(1-\alpha\Sigma(\boldsymbol{s})) + \frac{1}{2}(1+3\alpha)\Sigma(\boldsymbol{s})\,\mathrm{CHK}(\boldsymbol{s})\right) + o\left(\|\boldsymbol{s}\|^3\right).$$

Proof. See Appendix A. □

To see why the distortion to consumer surplus increases with the Herfindahl index and the cubic Hannah-Key index, consider a mean-preserving spread of the market share vector $\boldsymbol{s}$ under oligopoly. This raises both concentration indices (as these are convex functions of $\boldsymbol{s}$) while leaving consumer surplus unchanged, as $\mathrm{CS}(\boldsymbol{s})$ depends only on $\Sigma(\boldsymbol{s})$ (see equation (11)). The concavity of the market-share fitting-in function $S(\cdot)$, which comes from the fact that a firm with a higher type tends to charge a higher markup, implies that the mean-preserving spread of the market share vector must have been caused by a sum-increasing change in the vector of firm types.[7] As consumer surplus under monopolistic competition depends only on the sum of those types (see equation (12) above), this change increases $\mathrm{CS}^m(\boldsymbol{s})$.

Note that the oligopoly distortions to consumer surplus and aggregate surplus coincide at the second order, as $\Sigma(\boldsymbol{s})^3$ is of third order. Put differently, at the second order, there is no difference between industry profit under oligopoly and monopolistic competition. At the third order, the difference is given by

$$\alpha^2\Sigma(\boldsymbol{s})^3\left[\mathrm{HHI}(\boldsymbol{s}) - \frac{1}{2}\,\mathrm{CHK}(\boldsymbol{s})\right],$$

[6] $o(\cdot)$ is Landau's little-o notation: $f(x) = o(g(x))$ in the neighborhood of $x = x^0$ if $f(x)/g(x) \underset{x\to x^0}{\longrightarrow} 0$.

[7] The concavity of S is stated and proven in Lemma XXVI in the Online Appendix to Nocke and Schutz (2018); see also equation (21) in the Appendix.

with the term inside brackets being strictly positive and strictly convex in $\boldsymbol{s}$. Thus, the gap between industry profit under oligopoly and monopolistic competition widens as the industry becomes more concentrated.

3.2 Approximation Results around Monopolistic Competition Conduct

We now provide an alternative approximation of the oligopoly distortions, namely one involving only small departures from monopolistic competition conduct, but without restricting the size of firms or imposing that there is a positive outside option.

Fix a conduct parameter $\theta \in [0, 1]$ and a type vector $(T^f)_{f \in \mathcal{F}}$. The equilibrium aggregator level is denoted $H^*(\theta)$, and firm f's market share is $s^f(\theta) = S\left(T^f / H^*(\theta), \theta\right)$. Equilibrium consumer surplus and aggregate surplus are given by

$$\text{CS}(\theta) = \log H^*(\theta) \quad \text{and} \quad \text{AS}(\theta) = \left[\log H^*(\theta) + \sum_{f \in \mathcal{F}} \frac{\alpha s^f(\theta)}{1 - \alpha\theta s^f(\theta)}\right].$$

We can now provide a first-order Taylor approximation of the oligopoly distortions in the neighborhood of $\theta = 0$, i.e., close to monopolistic competition conduct:

Proposition 2. *In the neighborhood of $\theta = 0$, the oligopoly distortions are*

$$\text{CS}(\theta) - \text{CS}(0) = -\alpha\Sigma(\theta)^2\,\text{HHI}(\theta)\theta - \frac{\alpha}{2}\Sigma(\theta)^3\left[(1 + 2\alpha)\,\text{CHK}(\theta) - \alpha\Sigma(\theta)\,\text{HHI}(\theta)^2\right]\theta^2 + o(\theta^2)$$

and

$$\begin{aligned}\text{AS}(\theta) - \text{AS}(0) = {} & -\alpha\Sigma(\theta)^2\,\text{HHI}(\theta)\left(1 - \alpha\Sigma(\theta)\right)\theta - \frac{1}{2}\alpha\Sigma(\theta)^3 \times \\ & \left[\left(1 + 3\alpha - (\alpha + 2\alpha^2)\Sigma(\theta)\right)\text{CHK}(\theta) - \left(\alpha + 2\alpha^2(1 - \Sigma(\theta))\right)\Sigma(\theta)\,\text{HHI}(\theta)^2\right] + o(\theta^2).\end{aligned}$$

Proof. See Appendix B. □

As in the approximation with small market shares, at the first order in θ, the oligopoly distortions to consumer surplus and aggregate surplus are proportional to the Herfindahl index. At the second order, the distortions also depend on the cubic Hannah-Kay index.

An interesting special case arises under MNL demand ($\alpha = 1$) without an outside option ($\Sigma = 1$). The oligopoly distortion to aggregate surplus simplifies to

$$\text{AS}(\theta) - \text{AS}(0) = -\frac{1}{2}\theta^2\left(\text{CHK} - \text{HHI}^2\right) + o(\theta^2).$$

Thus, at the first order in θ, there is no distortion. The intuition is that, in the absence of an outside option, the level of the average markup is irrelevant for aggregate surplus, as it is merely a transfer from consumers to firms. Hence, any inefficiencies must be due to markup heterogeneity, which are however of second order. At that order, the oligopoly distortion to aggregate surplus is proportional to the difference between the cubic Hannah-Kay index and the square of the Herfindahl index. The Cauchy-Schwarz inequality implies that that difference is strictly positive if firms are asymmetric and equal to zero otherwise (see footnote 12 below). Note that, as $\text{CHK} - \text{HHI}^2$ is not convex in the market share vector, it is generally not true that greater industry concentration results in a larger oligopoly distortion. For example, it is easily seen that $\text{CHK} - \text{HHI}^2$ vanishes not only when firms become perfectly symmetric, but also in the limit as one of the firms' market share tends to 1 (while those of the others tend to zero).

4 Concentration Indices and Resource (Mis-)Allocation

In this section, we propose a resource-based decomposition of the oligopoly distortion to aggregate surplus in the spirit of the macroeconomics literature on misallocation (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). The decomposition involves two terms. The first term, which is driven by the overall level of the average markup, measures the extent to which the industry does not utilize the efficient amount of resources for production. The second term, which is driven by markup heterogeneity, captures the misallocation of resources within the industry. In Section 4.1, we provide results on the first term of the decomposition. We show that oligopolistic market power results in resources being under-utilized under CES demand, but not necessarily under MNL demand.

Turning our attention to the second term, we show in Section 4.2 that, regardless of the details of the demand system, within-industry resource allocation is efficient if and only if relative markups are equalized across products. As, with MNL demand, this condition is typically not satisfied under monopolistic competition (our competitive benchmark), not even across products within the same firm, this leads us to focus on CES demand for the remainder of the section. We approximate the loss in aggregate surplus from within-industry resource misallocation around small market shares in Section 4.3, and around monopolistic competition conduct in Section 4.4.

4.1 Over- or Under-Utilization of Resources

In this subsection, we focus on the first term of our decomposition. In general, it is unclear whether the industry uses *too few* or *too many* resources relative to the competitive benchmark. On the one hand, equilibrium prices are higher than in the competitive benchmark, implying a lower aggregate demand, and thus fewer resources being used for production. On the other hand, as larger, more efficient firms charge higher equilibrium markups, consumption is distorted towards less efficient firms, which may result in more resources being used.

We now show that, under the assumption of CES demand, oligopolistic market power always results in *too few* resources being used. The amount of resources employed in the equilibrium with conduct parameter θ is equal to

$$\begin{aligned} C(\theta) &= \sum_{j\in\mathcal{N}} c_j D_j(\boldsymbol{p}) = \sum_{j\in\mathcal{N}} p_j D_j(\boldsymbol{p}) - \sum_{j\in\mathcal{N}} (p_j - c_j) D_j(\boldsymbol{p}) \\ &= \sum_{j\in\mathcal{N}} \frac{-p_j h_j'(p_j)}{H(\boldsymbol{p})} - \sum_{f\in\mathcal{F}} \Pi^f(\boldsymbol{p}) = (\sigma - 1) \sum_{j\in\mathcal{N}} \frac{h_j(p_j)}{H(\boldsymbol{p})} - \sum_{f\in\mathcal{F}} \Pi^f(\boldsymbol{p}) \\ &= \frac{\alpha}{1-\alpha}\Sigma(\theta) - \Pi(\theta), \end{aligned} \tag{13}$$

where

$$\Pi(\theta) \equiv \alpha \sum_{f\in\mathcal{F}} \frac{s^f(\theta)}{1 - \alpha\theta s^f(\theta)}$$

is equilibrium industry profit.

As θ increases from 0 (monopolistic competition conduct) to 1 (fully fledged oligopoly conduct), the industry becomes less competitive and all prices increase. This implies that Σ, the firms' aggregate market share, weakly decreases; the decrease is strict if consumers have access to an outside option. Moreover, the equilibrium profit of every firm f increases for two reasons: first, firm f's rivals raise their prices; second, firm f best-responds to that price increase. It follows that $C(0) > C(1)$. The following proposition generalizes these insights by showing that $C(\theta)$ is strictly decreasing in θ:

Proposition 3. *Under CES demand, an increase in the conduct parameter θ strictly reduces the amount of resources used for production, $C(\theta)$. In particular, under fully fledged oligopoly ($\theta = 1$), the industry uses strictly fewer resources than under monopolistic competition ($\theta = 0$). That is, an oligopolistic industry under-utilizes resources relative to the competitive benchmark.*

Proof. See Appendix C. □

Although the idea that an oligopoly tends to under-utilize resources relative to the competitive benchmark may seem intuitive (because oligopolies tend to underproduce), it is easy to construct counterexamples. Indeed, we now show that, under MNL demand, both over- and under-utilization can arise. The examples below feature MNL demand with two products, 1 and 2, sold by firms f_1 and f_2, respectively. With this specification, the total amount of resources used when the conduct parameter is θ is $C(\theta) = c_1 s^{f_1}(\theta) + c_2 s^{f_2}(\theta)$.[8] We denote by $\mu^f(\theta)$ the equilibrium ι-markup of firm f when the conduct parameter is θ.

Example 1 (over-utilization of resources under MNL demand): Suppose that there is no outside option, and marginal costs and qualities satisfy $c_1 < c_2$ and $a_1 = a_2$, respectively. Under monopolistic competition, the two firms charge the same absolute markup, $\mu^{f_1}(0) =$

[8] Without loss of generality, we set the price sensitivity parameter λ equal to 1 to ease notation.

$\mu^{f_2}(0) = 1$. Instead, when $\theta = 1$, we have that

$$\mu^{f_1}(1) = m\left(\frac{\mathrm{e}^{a-c_1}}{H^*(1)}, 1\right) > m\left(\frac{\mathrm{e}^{a-c_2}}{H^*(1)}, 1\right) = \mu^{f_2}(1),$$

as the markup fitting-in function is strictly increasing in its first argument. It follows that $s^{f_1}(1) < s^{f_1}(0)$ and $s^{f_2}(1) > s^{f_2}(0)$, so that $C(1) > C(0)$. Thus, this duopoly over-utilizes resources relative to the competitive benchmark.

Example 2 (under-utilization of resources under MNL demand): Suppose that there is an outside option ($H^0 > 0$), and marginal costs and qualities satisfy $c_1 = c_2 = c$ and $a_1 = a_2 = a$. As firms are symmetric, they charge the same absolute markup in equilibrium, $\mu^{f_1}(\theta) = \mu^{f_2}(\theta)$. As that common markup is strictly higher under oligopoly than under monopolistic competition, $\mu^f(1) > 1 = \mu^f(0)$, we have that $s^f(1) < s^f(0)$, which implies that $C(1) < C(0)$. Thus, this oligopoly under-utilizes resources relative to the competitive benchmark.

4.2 Efficient Within-Industry Resource Allocation

Turning to the second term of our decomposition, we now fully characterize the efficient within-industry resource allocation under minimal assumptions on the demand system. Suppose that the representative consumer has quasi-linear preferences that can be represented by the decreasing, concave, and $\mathcal{C}^1$ sub-utility function $U(x)$, where $x = (x_j)_{j\in\mathcal{N}}$ is the output vector. Let $p \equiv \nabla U$ denote the inverse demand function. Each product j has a constant unit cost of production c_j, incurred in units of the Hicksian composite commodity. Consider the problem of maximizing aggregate surplus subject to the constraint of using C units of the resource for production:

$$\max_{x\in\mathbb{R}_+^{\mathcal{N}}} U(x) - \sum_{j\in\mathcal{N}} c_j x_j \quad \text{s.t.} \quad \sum_{j\in\mathcal{N}} c_j x_j = C. \tag{14}$$

The following proposition has appeared in various forms in the literature on misallocation:[9]

Proposition 4. *Any interior solution x^* to maximization problem* (14) *must feature a common relative markup on all of the products. That is, $p_i(x^*)/c_i = p_j(x^*)/c_j$ for every $i, j \in \mathcal{N}$.*

Proof. Suppose indeed that x^* is an interior solution to maximization problem (14). Then, for some Lagrange multiplier ζ, x^* must maximize the Lagrangian

$$U(x) - \sum_{j\in\mathcal{N}} c_j x_j + \zeta\left(C - \sum_{j\in\mathcal{N}} c_j x_j\right).$$

[9] For instance, Pellegrino (forthcoming) obtains the same result for the special case in which the sub-utility function U is quadratic, giving rise to linear demand (see his Proposition 5).

Hence, for each product i, the first-order condition $\partial U/\partial x_i - (1+\zeta)c_i = 0$ must hold at x^*. As $\partial U/\partial x_i = p_i(x)$, this implies that $p_i(x^*)/c_i = 1+\zeta$ for every i. □

Thus, there is misallocation whenever relative markups are not equalized across products. Recall from Section 2.2 that, under MNL demand, a firm charges the same *absolute* markup over all of its products. This means that, unless all of the firm's products have the same unit cost, relative markups vary within the firm's product portfolio, implying that resources are misallocated within the firm. For the same reason, the monopolistic competition outcome, which we view as the benchmark outcome undistorted by oligopolistic market power, features misallocation whenever there is cost heterogeneity. We therefore focus on the case of CES demand in the remainder of this section.

Suppose indeed that demand is of the CES type. It is easily shown that maximization problem (14) has a unique solution, which must be interior. By Proposition 4, that solution features a common relative markup over all of the products, which, given the assumption of CES demand, is equivalent to there being a common ι-markup on all the products. This property clearly holds in the monopolistic competition outcome, where all firms set their ι-markups equal to 1. Hence, that benchmark outcome does not suffer from misallocation. Under oligopoly, each firm f charges the same ι-markup μ^f over all the products in its portfolio, implying that there is no resource misallocation within a given firm. However, unless equilibrium ι-markups are equalized across firms, which happens in equilibrium only if all firms have the same type, there is misallocation *across* firms.

4.3 Misallocation: Approximation for Small Firms

In this subsection, we approximate the loss in aggregate surplus from resource misallocation in the neighborhood of small smarket shares for the case of CES demand. As in Section 3.1, we assume that consumers have access to an outside option, $H^0 > 0$, so that the equilibrium aggregator level is given by $H^*(\boldsymbol{s}) = H^0/(1-\Sigma(\boldsymbol{s}))$. Consider the version of maximization problem (14) in which prices are set to maximize aggregate surplus subject to the constraint of using as much resources as in the oligopolistic equilibrium. From equation (13) above, we see that the amount of resources used in the oligopolistic equilibrium is given by

$$C(\boldsymbol{s}) = \frac{\alpha}{1-\alpha}\Sigma(\boldsymbol{s}) - \Pi(\boldsymbol{s}), \tag{15}$$

where

$$\Pi(\boldsymbol{s}) \equiv \alpha \sum_{f\in\mathcal{F}} \frac{s^f}{1-\alpha s^f}$$

is equilibrium industry profit.

Moreover, by Proposition 4, the solution to the maximization problem features a common relative markup, and thus, under CES demand, a common ι-markup $\tilde{\mu}$, over all of the products (regardless of ownership). With such a common ι-markup, the amount of resources used is given by[10]

$$\tilde{C} = \frac{\alpha}{1-\alpha}\tilde{\Sigma} - \alpha\tilde{\mu}\tilde{\Sigma}, \tag{16}$$

where

$$\tilde{\Sigma} \equiv \frac{(1-(1-\alpha)\tilde{\mu})^{\frac{\alpha}{1-\alpha}}\mathcal{T}}{H^0 + (1-(1-\alpha)\tilde{\mu})^{\frac{\alpha}{1-\alpha}}\mathcal{T}}$$

is the market share of the inside goods given the industry-wide ι-markup $\tilde{\mu}$. (Recall from above that $\mathcal{T}$ is the sum of the firms' types.)

For what follows, it is more convenient to work directly with the market share $\tilde{\Sigma}$. We thus invert the relationship between $\tilde{\mu}$ and $\tilde{\Sigma}$ to obtain:

$$\tilde{\mu} = \frac{1}{1-\alpha}\left[1 - \left(\frac{H^0}{\mathcal{T}}\frac{\tilde{\Sigma}}{1-\tilde{\Sigma}}\right)^{\frac{1-\alpha}{\alpha}}\right].$$

The amount of resources used can then be rewritten as a function of $\tilde{\Sigma}$ only:

$$\tilde{C}(\tilde{\Sigma}) = \frac{\alpha}{1-\alpha}\left(\frac{H^0}{\mathcal{T}}\frac{\tilde{\Sigma}}{1-\tilde{\Sigma}}\right)^{\frac{1-\alpha}{\alpha}}\tilde{\Sigma}.$$

The resource constraint therefore becomes: $\tilde{C}(\tilde{\Sigma}) = C(\boldsymbol{s})$, which uniquely pins down $\tilde{\Sigma}(\boldsymbol{s})$. Industry profit in this allocation is given by

$$\tilde{\Pi}(\boldsymbol{s}) = \alpha\tilde{\mu}\tilde{\Sigma} = \frac{\alpha}{1-\alpha}\left[1 - \left(\frac{H^0}{\mathcal{T}}\frac{\tilde{\Sigma}(\boldsymbol{s})}{1-\tilde{\Sigma}(\boldsymbol{s})}\right)^{\frac{1-\alpha}{\alpha}}\right]\tilde{\Sigma}(\boldsymbol{s}).$$

Using the resource constraint, this simplifies to

$$\tilde{\Pi}(\boldsymbol{s}) = \frac{\alpha}{1-\alpha}(\tilde{\Sigma}(\boldsymbol{s}) - \Sigma(\boldsymbol{s})) + \Pi(\boldsymbol{s})$$

[10]To see this, note that

$$\begin{aligned}\tilde{C} &= \sum_{j\in\mathcal{N}} p_j D_j(\boldsymbol{p}) - \sum_{j\in\mathcal{N}} (p_j - c_j) D_j(\boldsymbol{p}) = \sum_{j\in\mathcal{N}} \frac{-p_j h_j'(p_j)}{H(\boldsymbol{p})} - \sum_{j\in\mathcal{N}} \frac{p_j - c_j}{p_j}\frac{-p_j h_j'(p_j)}{H(\boldsymbol{p})} \\ &= (\sigma - 1)\left(\sum_{j\in\mathcal{N}} \frac{h_j(p_j)}{H(\boldsymbol{p})} - \frac{\tilde{\mu}}{\sigma}\sum_{j\in\mathcal{N}} \frac{h_j(p_j)}{H(\boldsymbol{p})}\right) = \frac{\alpha}{1-\alpha}\tilde{\Sigma} - \alpha\tilde{\mu}\tilde{\Sigma}.\end{aligned}$$

The expression for $\tilde{\Sigma}$ follows immediately, as

$$\sum_{j\in\mathcal{N}} h_j(p_j) = \sum_{j\in\mathcal{N}} h_j(c_j)\left(1-(1-\alpha)\tilde{\mu}\right)^{\frac{\alpha}{1-\alpha}} = \mathcal{T}\left(1-(1-\alpha)\tilde{\mu}\right)^{\frac{\alpha}{1-\alpha}}.$$

As the industry aggregator in the resource-efficient allocation is $\tilde{H}(\boldsymbol{s}) = H^0/(1-\tilde{\Sigma}(\boldsymbol{s}))$, consumer surplus is given by $\widetilde{\mathrm{CS}}(\boldsymbol{s}) = \log H^0 - \log(1-\tilde{\Sigma}(\boldsymbol{s}))$. Adding up, we obtain aggregate surplus in the resource-efficient allocation:

$$\widetilde{\mathrm{AS}}(\boldsymbol{s}) = \log H^0 - \log(1-\tilde{\Sigma}(\boldsymbol{s})) + \frac{\alpha}{1-\alpha}(\tilde{\Sigma}(\boldsymbol{s}) - \Sigma(\boldsymbol{s})) + \Pi(\boldsymbol{s}).$$

Subtracting this from equilibrium aggregate surplus, we obtain the welfare loss from resource misallocation:

$$M(\boldsymbol{s}) = \log(1-\tilde{\Sigma}(\boldsymbol{s})) - \log(1-\Sigma(\boldsymbol{s})) + \frac{\alpha}{1-\alpha}(\Sigma(\boldsymbol{s}) - \tilde{\Sigma}(\boldsymbol{s})). \tag{17}$$

Our goal is to approximate $M(\boldsymbol{s})$ around small market shares. We proceed as in Section 3.1. We fix an equilibrium vector of market shares $\boldsymbol{s}$ and recover the implied vector of types $\boldsymbol{T}$. Using this vector of types, we obtain $\tilde{\Sigma}(\boldsymbol{s})$ from the resource constraint. We can then compute $M(\boldsymbol{s})$ and approximate it. We obtain:

Proposition 5. *Under CES demand, in the neighborhood of $\boldsymbol{s} = \mathbf{0}$, the impact of resource misallocation on aggregate surplus is:*

$$M(\boldsymbol{s}) = -\frac{1}{2}\alpha\Sigma(\boldsymbol{s})^3\left(\mathrm{CHK}(\boldsymbol{s}) - \mathrm{HHI}(\boldsymbol{s})^2\right) + o\left(\|\boldsymbol{s}\|^3\right).$$

Proof. See Appendix D. □

A first observation is that $M(\boldsymbol{s})$ is negligible at the second order. This holds because $\Sigma(\boldsymbol{s})^3$ is of order 3, while $\mathrm{HHI}(\boldsymbol{s})^2$ and $\mathrm{CHK}(\boldsymbol{s})$ are both of order 0, as

$$\frac{1}{|\mathcal{F}|^2} \le \mathrm{HHI}(\boldsymbol{s})^2 \le 1 \quad \text{and} \quad \frac{1}{|\mathcal{F}|^2} \le \mathrm{CHK}(\boldsymbol{s}) \le 1.$$

Thus, there is no resource misallocation at the second order. This means that, at that order, the oligopoly distortion to aggregate surplus, approximated in Proposition 1, is entirely driven by resources being underutilized in equilibrium (rather than by resources flowing to the wrong firms). Misallocation becomes relevant at the third order, where it is proportional to $\mathrm{CHK}(\boldsymbol{s}) - \mathrm{HHI}^2(\boldsymbol{s})$.[11] As mentioned in Section 3.2, we have that $\mathrm{CHK}(\boldsymbol{s}) - \mathrm{HHI}^2(\boldsymbol{s}) \ge 0$, with the inequality being strict if and only if firms are asymmetric.[12] Moreover, as $\mathrm{CHK}(\boldsymbol{s}) - \mathrm{HHI}^2(\boldsymbol{s})$ is not convex in the market share vector (see again Section 3.2), it is generally not true that greater industry concentration results in larger misallocation.

[11] Note that the function on the right-hand side of the Taylor approximation of Proposition 5 is not a polynomial in $\boldsymbol{s}$. In fact, that function is not thrice differentiable at $\boldsymbol{s} = \mathbf{0}$, implying that it cannot be approximated by a polynomial at the third order. For this reason, the proof of the proposition is somewhat non-standard, and involves iterating over the resource constraint, with each iteration generating a higher-order approximation of $\tilde{\Sigma}(\boldsymbol{s})$ (see the proof of Lemma D.2 in Appendix D).

[12] To see this, define the vectors $\boldsymbol{x} \equiv ((s^f)^{3/2})_{f\in\mathcal{F}}$ and $\boldsymbol{y} \equiv ((s^f)^{1/2})_{f\in\mathcal{F}}$. By the Cauchy-Schwarz inequality, we have that $\boldsymbol{x}\cdot\boldsymbol{y} \le \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2$, with equality if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are collinear. Taking squares on both

4.4 Misallocation: Approximation around Monopolistic Competition Conduct

In this subsection, we approximate the loss in aggregate surplus from resource misallocation in the neighborhood of monopolistic competition conduct for the case of CES demand. To simplify the analysis, we assume that consumers do not have access to an outside option, $H^0 = 0$. As in Section 4.3, we consider the version of maximization problem (14) in which prices are set to maximize aggregate surplus subject to the constraint of using as much resources as in equilibrium. From equation (13) above, we see that the amount of resources used in equilibrium is given by

$$C(\theta) = \frac{\alpha}{1-\alpha} - \Pi(\theta), \tag{18}$$

where we have used the fact that $\Sigma(\theta) = 1$ in the absence of an outside option.

The resource-efficient allocation features a common ι-markup, $\tilde{\mu}(\theta)$, over all of the products. The amount of resources used given $\tilde{\mu}(\theta)$ continues to be

$$\tilde{C} = \frac{\alpha}{1-\alpha}\tilde{\Sigma} - \alpha\tilde{\mu}(\theta)\tilde{\Sigma} = \frac{\alpha}{1-\alpha} - \alpha\tilde{\mu}(\theta),$$

where we have used the fact that $\tilde{\Sigma} = 1$ in the absence of an outside option. The resource constraint, $C(\theta) = \tilde{C}$, therefore boils down to $\alpha\tilde{\mu}(\theta) = \Pi(\theta)$.

Note that, under the common ι-markup $\tilde{\mu}(\theta)$, industry profit equals $\tilde{\Pi}(\theta) = \alpha\tilde{\mu}(\theta)\tilde{\Sigma} = \alpha\tilde{\mu}(\theta)$ in the resource-efficient allocation. It follows that $\Pi(\theta) = \tilde{\Pi}(\theta)$, i.e., industry profit in the equilibrium allocation is equal to that in the resource-efficient allocation. The impact of resource misallocation on aggregate surplus is therefore equal to

$$M(\theta) \equiv \log H^*(\theta) - \log \tilde{H}(\theta),$$

where

$$\tilde{H}(\theta) \equiv (1 - (1-\alpha)\tilde{\mu}(\theta))^{\frac{\alpha}{1-\alpha}}\mathcal{T} \tag{19}$$

is the value of the industry aggregator in the resource-efficient allocation. The following proposition approximates $M(\theta)$ at the second order:

Proposition 6. *Under CES demand without an outside option, in the neighborhood of* $\theta = 0$*,*

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sides, this is equivalent to

$$\left(\sum_{f\in\mathcal{F}} (s^f)^2\right)^2 \leq \left(\sum_{f\in\mathcal{F}} (s^f)^3\right)\left(\sum_{f\in\mathcal{F}} s^f\right),$$

with equality if and only if all of the components of $\boldsymbol{s}$ are identical. Dividing both sides by $\left(\sum_{f\in\mathcal{F}} s^f\right)^4$, the inequality simplifies to $\text{CHK}(\boldsymbol{s}) \geq \text{HHI}(\boldsymbol{s})^2$.

the impact of resource misallocation on aggregate surplus is

$$M(\theta) = -\frac{1}{2}\alpha \left(\mathrm{CHK}(\theta) - \mathrm{HHI}(\theta)^2\right)\theta^2 + o(\theta^2).$$

Proof. See Appendix E. □

Thus, as in the approximation around small firms, we find that there is no misallocation at a low order. At a higher order, the welfare loss from resource misallocation is again proportional to the difference between the cubic Hannah-Kay concentration index and the squared Herfindahl index.

5 Conclusion

We study welfare distortions in the Nocke and Schutz (forthcoming) multiproduct-firm pricing game with CES or MNL demand.[13] The model allows for arbitrary product heterogeneity in terms of marginal costs and qualities, and allows firms to differ in their product portfolios. The pricing game is fully aggregative, with the industry aggregator being a sufficient statistic for consumer surplus. Moreover, the model permits type aggregation, in the sense that all relevant information about a firm's product portfolio can be summarized by a uni-dimensional sufficient statistic. Importantly, consumer surplus and aggregate surplus can be expressed as functions of firm-level equilibrium market shares.

To gauge the welfare distortions from oligopolistic market power (relative to monopolistic competition, our competitive benchmark), we derive Taylor approximations both around small market shares and around monopolistic competition conduct. We find that, at a low order, the oligopoly distortions to consumer surplus and aggregate surplus are proportional to the Herfindahl index. At a higher order, these distortions also depend on the cubic Hannah-Kay concentration index.

We also connect to the macroeconomics literature on resource misallocation by decomposing, for the case of CES demand, the welfare loss from oligopoly into two components. The first component arises from markups being above the monopolistically competitive level, implying that the industry does not utilize the efficient amount of resources. As we show, oligopolistic market power results in too few resources being utilized under CES demand, whereas, under MNL demand, both excessive and insufficient resource employment may arise.

The second component of the decomposition is driven by markup heterogeneity and captures the extent to which resources are misallocated across firms. We show that for any well-behaved quasi-linear demand system, efficient within-industry resource allocation requires equalization of relative markups across products. Under MNL demand, such markup equalization typically does not arise under monopolistic competition, not even within the

[13] As mentioned in Section 2.2, all of our results continue to hold under nested CES or MNL demand, provided each firm owns one or several entire nests of products.

same firm. Focusing therefore on the case of CES demand, we approximate the aggregate surplus loss from within-industry misallocation under the assumption that firms have small market shares or in the neighborhood of monopolistic competition conduct. We find that, at a low order, the misallocation component is negligible, implying that the oligopoly distortion is primarily driven by the industry under-utilizing resources. At a higher order, the misallocation component is proportional to the difference between the cubic Hannah-Kay index and the square of the Herfindahl index. Thus, it is generally not true that greater industry concentration leads to larger within-industry misallocation.

In earlier work (Nocke and Schutz, 2018), we proposed an alternative, demand-side decomposition of the welfare distortion from oligopolistic market power. The first component arises from the industry aggregator being below its efficient level. The second component arises from some firms contributing too little and others too much to the aggregator, due to markup heterogeneity.[14] Relating these components to concentration indices is left for future research.

Appendix

Throughout this appendix, we use the unnormalized versions of the Herfindahl and Hannah-Kay concentration indices to ease notation:

$$\widehat{\text{HHI}} \equiv \sum_{f\in\mathcal{F}} (s^f)^2 \quad \text{and} \quad \widehat{\text{CHK}} \equiv \sum_{f\in\mathcal{F}} (s^f)^3.$$

A Proof of Proposition 1

We prove a series of lemmas that jointly imply Proposition 1.[15]

The begin by establishing some key properties of the market-share fitting-in function:

Lemma A.1. *The continuous extension of S to $\mathbb{R}_+$ is $\mathcal{C}^3$. Moreover, $S(0) = 0$,*

$$S'(0) = \begin{cases} \alpha^{\frac{\alpha}{1-\alpha}} & \textit{under CES demand,} \\ \mathrm{e}^{-1} & \textit{under MNL demand,} \end{cases}$$

$S''(0) = -2\alpha S'(0)^2$, and $S'''(0) = -3\alpha(1-2\alpha)S'(0)^3$.

The inverse function $\Theta \equiv S^{-1}$ is $\mathcal{C}^3$ on $[0,1)$. Moreover, $\Theta(0) = 0$, $\Theta'(0) = 1/S'(0)$, $\Theta''(0) = 2\alpha/S'(0)$, and $\Theta'''(0) = 3\alpha(1+2\alpha)/S'(0)$.

[14] Nocke and Schutz (2018) show that, holding fixed a firm's contribution to the industry aggregator, its pricing structure is efficient, in that it maximizes aggregate surplus.

[15] Lemmas A.1, A.2, A.3, and A.4 below are third-order extensions to Lemmas 6, 9, 10, and 11 in the Online Appendix to Nocke and Schutz (forthcoming).

Proof. We begin by computing S'. Applying the implicit function theorem to equations (5)–(6) yields:

$$S'(x) = \frac{S(x)}{x}\frac{(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}. \tag{20}$$

Differentiating equation (20), we obtain

$$S''(x) = -\left(\frac{S(x)}{x}\right)^2 \frac{\alpha(2-S(x))(1-S(x))(1-\alpha S(x))}{\left(1-S(x)+\alpha S(x)^2\right)^3}. \tag{21}$$

Differentiating once more gives

$$\begin{aligned} S'''(x) = -\left(\frac{S(x)}{x}\right)^3 \frac{\alpha(1-S(x))(1-\alpha S(x))}{\left(1-S(x)+\alpha S(x)^2\right)^5}\Big(&3(1-2\alpha) - 4(1+\alpha)S(x) \\ &+ (1+13\alpha+6\alpha^2)S(x)^2 - 2\alpha(2+5\alpha)S(x)^3 + 3\alpha^2 S(x)^4\Big). \end{aligned} \tag{22}$$

We thus require the value of $\lim_{x\downarrow 0}\frac{S(x)}{x}$. In the MNL case,

$$\frac{S(x)}{x} = \mathrm{e}^{-m(x)} = \exp\left(\frac{-1}{1-S(x)}\right) \underset{x\downarrow 0}{\longrightarrow} \mathrm{e}^{-1}.$$

In the CES case,

$$\frac{S(x)}{x} = (1-(1-\alpha)m(x))^{\frac{\alpha}{1-\alpha}} = \left(1 - \frac{1-\alpha}{1-\alpha S(x)}\right)^{\frac{\alpha}{1-\alpha}} \underset{x\downarrow 0}{\longrightarrow} \alpha^{\frac{\alpha}{1-\alpha}}.$$

Taking limits in equations (20), (21), and (22) gives us the values of $S'(0)$, $S''(0)$, and $S'''(0)$.

As S is $\mathcal{C}^3$ with strictly positive derivative on $\mathbb{R}_+$, it establishes a $\mathcal{C}^3$-diffeomorphism from $\mathbb{R}_+$ to

$$\Big[S(0), \lim_{x\to\infty} S(x)\Big) = [0,1).$$

It follows that Θ is $\mathcal{C}^3$. Moreover,

$$\begin{aligned} \Theta'(s) &= \frac{1}{S'\circ S^{-1}(s)}, \\ \Theta''(s) &= -\frac{S''\circ S^{-1}(s)}{\left(S'\circ S^{-1}(s)\right)^3}, \\ \Theta'''(s) &= -\frac{\frac{S'''\circ S^{-1}(s)}{S'\circ S^{-1}(s)}\left(S'\circ S^{-1}(s)\right)^3 - S''\circ S^{-1}(s)\times 3\left(S'\circ S^{-1}(s)\right)^2\frac{S''\circ S^{-1}(s)}{S'\circ S^{-1}(s)}}{\left(S'\circ S^{-1}(s)\right)^6} \\ &= \frac{1}{S'\circ S^{-1}(s)}\left(-\frac{S'''\circ S^{-1}(s)}{\left(S'\circ S^{-1}(s)\right)^3} + 3\left(\frac{S''\circ S^{-1}(s)}{\left(S'\circ S^{-1}(s)\right)^2}\right)^2\right). \end{aligned}$$

Hence,

$$\Theta'(0) = \frac{1}{S'(0)},$$

$$\Theta''(0) = -\frac{1}{S'(0)}\frac{S''(0)}{S'(0)^2} = \frac{2\alpha}{S'(0)},$$

$$\Theta'''(0) = \frac{1}{S'(0)}\left(-\frac{S'''(0)}{S'(0)^3} + 3\left(\frac{S''(0)}{S'(0)^2}\right)^2\right) = \frac{3\alpha(1+2\alpha)}{S'(0)}. \qquad \square$$

Next, we approximate consumer surplus under oligopoly. We have:

Lemma A.2. $H^*(\boldsymbol{s}) = H^0/(1-\Sigma(\boldsymbol{s}))$. *Moreover, in the neighborhood of* $\boldsymbol{s} = \boldsymbol{0}$,

$$\mathrm{CS}(\boldsymbol{s}) = \log H^0 + \Sigma(\boldsymbol{s}) + \frac{1}{2}\Sigma(\boldsymbol{s})^2 + \frac{1}{3}\Sigma(\boldsymbol{s})^3 + o\left(\|\boldsymbol{s}\|^3\right).$$

Proof. The first part of the lemma follows immediately from equilibrium condition (8). The second part follows from the fact that, in the neighborhood of $x = 0$,

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3). \qquad \square$$

Next, we compute the first, second, and third (cross-)partial derivatives of the type vector $\boldsymbol{T}(\boldsymbol{s})$:

Lemma A.3. *For every* $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{\boldsymbol{s}=\boldsymbol{0}} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{otherwise.} \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\left.\frac{\partial^2 T^f}{\partial s^{f'}\partial s^{f''}}\right|_{\boldsymbol{s}=\boldsymbol{0}} = \begin{cases} \frac{H^0}{S'(0)}2(1+\alpha) & \text{if } f = f' = f'', \\ 0 & \text{if } f' \neq f \text{ and } f'' \neq f, \\ \frac{H^0}{S'(0)} & \text{otherwise.} \end{cases}$$

Finally, for every $(f, f', f'', f''') \in \mathcal{F}^4$,

$$\left.\frac{\partial^3 T^f}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}}\right|_{\boldsymbol{s}=\boldsymbol{0}} = \frac{H^0}{S'(0)}\begin{cases} 6 + 9\alpha + 6\alpha^2 & \text{if } f = f' = f'' = f''', \\ 2\alpha + 4 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{P}^1(f) = \left\{(f^1, f^2, f^3) \in \mathcal{F}^3 : \ f = f^i \neq f^j, f^k, \text{ for some permutation } (i,j,k) \text{ of } (1,2,3)\right\},$$

and

$$\mathcal{P}^2(f) = \left\{(f^1, f^2, f^3) \in \mathcal{F}^3 : \ f = f^i = f^j \neq f^k, \text{ for some permutation } (i,j,k) \text{ of } (1,2,3)\right\}.$$

Proof. Let $f \in \mathcal{F}$. As $s^f = S(T^f(\boldsymbol{s})/H^*(\boldsymbol{s}))$, we have that

$$T^f(\boldsymbol{s}) = H^*(\boldsymbol{s})S^{-1}(s^f) = H^0 \frac{\Theta(s^f)}{1-\Sigma(\boldsymbol{s})} \equiv H^0\Theta(s^f)\Psi(\boldsymbol{s}),$$

where the inverse function Θ was defined in Lemma A.1.

Note that, for every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned} \Psi(0) = \left.\frac{\partial \Psi}{\partial s^f}\right|_{\boldsymbol{s}=\boldsymbol{0}} &= 1, \\ \left.\frac{\partial^2 \Psi}{\partial s^f \partial s^{f'}}\right|_{\boldsymbol{s}=\boldsymbol{0}} &= 2, \\ \left.\frac{\partial^3 \Psi}{\partial s^f \partial s^{f'} \partial s^{f''}}\right|_{\boldsymbol{s}=\boldsymbol{0}} &= 6. \end{aligned}$$

Therefore, for every $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{\boldsymbol{s}=\boldsymbol{0}} = H^0 \left(\frac{\partial \Theta(s^f)}{\partial s^{f'}}\Psi(\boldsymbol{s}) + \Theta(s^f)\frac{\partial \Psi}{\partial s^{f'}}\right)\Bigg|_{\boldsymbol{s}=\boldsymbol{0}} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{if } f \neq f'. \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned} \left.\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}}\right|_{\boldsymbol{s}=\boldsymbol{0}} &= H^0 \left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}}\Psi(\boldsymbol{s}) + \Theta(s^f)\frac{\partial^2 \Psi}{\partial s^{f'} \partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}}\frac{\partial \Psi}{\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\frac{\partial \Psi}{\partial s^{f'}}\right)\Bigg|_{\boldsymbol{s}=\boldsymbol{0}} \\ &= H^0 \left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\right)\Bigg|_{\boldsymbol{s}=\boldsymbol{0}} \\ &= H^0 \times \begin{cases} \Theta''(0) + 2\Theta'(0) & \text{if } f = f' = f'' \\ 0 & \text{if } f', f'' \neq f \\ \Theta'(0) & \text{otherwise} \end{cases} \\ &= \frac{H^0}{S'(0)} \times \begin{cases} 2(\alpha+1) & \text{if } f = f' = f'' \\ 0 & \text{if } f', f'' \neq f \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, for every $(f, f', f'', f''') \in \mathcal{F}^3$,

$$
\begin{aligned}
\left.\frac{\partial^3 T^f}{\partial s^{f'} \partial s^{f''} \partial s^{f'''}}\right|_{\boldsymbol{s}=\mathbf{0}} &= H^0 \left(\frac{\partial^3 \Theta(s^f)}{\partial s^{f'} \partial s^{f''} \partial s^{f'''}} \Psi(\boldsymbol{s}) + \Theta(s^f) \frac{\partial^3 \Psi}{\partial s^{f'} \partial s^{f''} \partial s^{f'''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}} \frac{\partial \Psi}{\partial s^{f'''}} \right. \\
&\quad + \frac{\partial \Theta(s^f)}{\partial s^{f'''}} \frac{\partial^2 \Psi}{\partial s^{f'} \partial s^{f''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f'''}} \frac{\partial \Psi}{\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}} \frac{\partial^2 \Psi}{\partial s^{f''} \partial s^{f'''}} \\
&\quad \left.\left. + \frac{\partial^2 \Theta(s^f)}{\partial s^{f''} \partial s^{f'''}} \frac{\partial \Psi}{\partial s^{f'}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}} \frac{\partial^2 \Psi}{\partial s^{f'} \partial s^{f'''}} \right) \right|_{\boldsymbol{s}=\mathbf{0}} \\
&= H^0 \left(\frac{\partial^3 \Theta(s^f)}{\partial s^{f'} \partial s^{f''} \partial s^{f'''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}} + 2 \frac{\partial \Theta(s^f)}{\partial s^{f'''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f'''}} \right. \\
&\quad \left.\left. + 2 \frac{\partial \Theta(s^f)}{\partial s^{f'}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f''} \partial s^{f'''}} + 2 \frac{\partial \Theta(s^f)}{\partial s^{f''}} \right) \right|_{\boldsymbol{s}=\mathbf{0}} \\
&= \frac{H^0}{S'(0)} \begin{cases} 3\alpha(1+2\alpha) + 3(2\alpha+2) & \text{if } f = f' = f'' = f''' \\ 2\alpha + 2 + 2 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f) \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f) \\ 0 & \text{otherwise} \end{cases} \\
&= \frac{H^0}{S'(0)} \begin{cases} 6 + 9\alpha + 6\alpha^2 & \text{if } f = f' = f'' = f''' \\ 2\alpha + 4 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f) \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

□

We now use Lemma A.3 to obtain a third-order Taylor approximation of $T^f(\boldsymbol{s})$ in the neighborhood of $\boldsymbol{s} = \mathbf{0}$:

Lemma A.4. *In the neighborhood of* $\boldsymbol{s} = \mathbf{0}$,

$$
T^f(\boldsymbol{s}) = \frac{H^0}{S'(0)} \left(s^f + \left(\alpha (s^f)^2 + s^f \Sigma(\boldsymbol{s}) \right) + \left(\frac{\alpha(1+2\alpha)}{2} (s^f)^3 + \alpha (s^f)^2 \Sigma(\boldsymbol{s}) + s^f \Sigma(\boldsymbol{s})^2 \right) \right) + o(\|\boldsymbol{s}\|^3).
$$

Proof. By Lemma A.3, first-order terms are simply given by $\frac{H^0}{S'(0)} s^f$. Second-order terms are given by

$$
\frac{H^0}{S'(0)} \frac{1}{2} \left(2(1+\alpha)(s^f)^2 + 2 s^f \sum_{g \neq f} s^g \right) = \frac{H^0}{S'(0)} \left(\alpha (s^f)^2 + s^f \Sigma \right).
$$

Finally, third-order terms are:

$$
\begin{aligned}
&\frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + (2\alpha + 4) \sum_{(f', f'', f''') \in \mathcal{P}^2(f)} s^{f'} s^{f''} s^{f'''} + 2 \sum_{(f', f'', f''') \in \mathcal{P}^1(f)} s^{f'} s^{f''} s^{f'''} \right) \\
&= \frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2 \sum_{g \neq f} s^g + 6 s^f \sum_{g, g' \neq f} s^g s^{g'} \right)
\end{aligned}
$$

$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2(\Sigma - s^f) + 6s^f(\Sigma - s^f)^2\right)$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((-6 + 3\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2\Sigma + 6s^f(\Sigma^2 - 2\Sigma s^f + (s^f)^2)\right)$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((3\alpha + 6\alpha^2)(s^f)^3 + 6\alpha(s^f)^2\Sigma + 6s^f\Sigma^2\right).$$

The lemma follows by Taylor's theorem. □

Adding up over all the firms, we obtain a third-order Taylor approximation of $\mathcal{T}(\boldsymbol{s})$, the sum of the firms' types:

Lemma A.5. *In the neighborhood of $\boldsymbol{s} = \boldsymbol{0}$,*

$$\frac{S'(0)}{H^0}\mathcal{T}(\boldsymbol{s}) = \Sigma(\boldsymbol{s}) + \left(\alpha\widehat{\mathrm{HHI}}(\boldsymbol{s}) + \Sigma(\boldsymbol{s})^2\right) + \left(\frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}}(\boldsymbol{s}) + \alpha\widehat{\mathrm{HHI}}(\boldsymbol{s})\Sigma(\boldsymbol{s}) + \Sigma(\boldsymbol{s})^3\right) + o(\|\boldsymbol{s}\|^3).$$

Proof. This follows immediately from Lemma A.4. □

Next, we provide expressions for consumer surplus, industry profit, and aggregate surplus under monopolistic competition, $\mathrm{CS}^m(\boldsymbol{s})$, $\Pi^m(\boldsymbol{s})$, and $\mathrm{AS}^m(\boldsymbol{s})$, as functions of the market shares vector in the oligopolistic equilibrium.

Lemma A.6. *The following holds:*

$$\mathrm{CS}^m(\boldsymbol{s}) = \log H^0 + \log\left(1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}\right),$$
$$\Pi^m(\boldsymbol{s}) = \alpha\left(1 - \frac{1}{1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}}\right),$$
$$\text{and } \mathrm{AS}(\boldsymbol{s}) = \log H^0 + \log\left(1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}\right) + \alpha\left(1 - \frac{1}{1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}}\right).$$

Proof. Under monopolistic competition, all the firms set their ι-markups equal to 1. Hence, in the case of MNL demand,

$$\mathrm{CS}^m(\boldsymbol{s}) = \log\left(H^0 + \sum_{f\in\mathcal{F}} T^f(\boldsymbol{s})\,\mathrm{e}^{-1}\right) = \log H^0 + \log\left(1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}\right)$$

and

$$\Pi^m(\boldsymbol{s}) = \sum_{f\in\mathcal{F}} \frac{T^f(\boldsymbol{s})\,\mathrm{e}^{-1}}{H^0 + \sum_{g\in\mathcal{F}} T^g(\boldsymbol{s})\,\mathrm{e}^{-1}} = 1 - \frac{1}{1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}},$$

where we have used the fact that $S'(0) = \mathrm{e}^{-1}$ (see Lemma A.1).

In the case of CES demand,

$$\mathrm{CS}^m(\boldsymbol{s}) = \log\left(H^0 + \sum_{f\in\mathcal{F}} T^f(\boldsymbol{s})\alpha^{\frac{\alpha}{1-\alpha}}\right) = \log H^0 + \log\left(1 + \mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}\right)$$

and

$$\Pi^m(\boldsymbol{s}) = \sum_{f\in\mathcal{F}} \alpha \frac{T^f(\boldsymbol{s})\alpha^{\frac{\alpha}{1-\alpha}}}{H^0 + \sum_{g\in\mathcal{F}} T^g(\boldsymbol{s})\alpha^{\frac{\alpha}{1-\alpha}}} = \alpha\left(1 - \frac{1}{1+\mathcal{T}(\boldsymbol{s})\frac{S'(0)}{H^0}}\right). \qquad \square$$

We now provide a third-order Taylor expansion of $\mathrm{CS}^m(\boldsymbol{s})$:

Lemma A.7. *In the neighborhood of $\boldsymbol{s} = \boldsymbol{0}$,*

$$\mathrm{CS}^m(\boldsymbol{s}) = \log H^0 + \Sigma(\boldsymbol{s}) + \frac{1}{2}\Sigma(\boldsymbol{s})^2 + \frac{1}{3}\Sigma(\boldsymbol{s})^3 + \alpha\widehat{\mathrm{HHI}}(\boldsymbol{s}) + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}}(\boldsymbol{s}) + o(\|\boldsymbol{s}\|^3).$$

Proof. At the third order in the neighborhood of $x = 0$, we have that

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3).$$

Combining this with Lemma A.5 and eliminating higher-order terms, we obtain

$$\begin{aligned}
\mathrm{CS}^m(\boldsymbol{s}) = &\ \log H^0 + \Sigma + \alpha\widehat{\mathrm{HHI}} + \Sigma^2 + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} + \alpha\widehat{\mathrm{HHI}}\Sigma + \Sigma^3 \\
&- \frac{1}{2}\left(\Sigma + \alpha\widehat{\mathrm{HHI}} + \Sigma^2\right)^2 + \frac{1}{3}\Sigma^3 + o(\|\boldsymbol{s}\|^3) \\
= &\ \log H^0 + \Sigma + \alpha\widehat{\mathrm{HHI}} + \Sigma^2 + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} + \alpha\widehat{\mathrm{HHI}}\Sigma + \Sigma^3 \\
&- \frac{1}{2}\left(\Sigma^2 + 2\alpha\widehat{\mathrm{HHI}}\Sigma + 2\Sigma^3\right) + \frac{1}{3}\Sigma^3 + o(\|\boldsymbol{s}\|^3) \\
= &\ \log H^0 + \Sigma + \frac{1}{2}\Sigma^2 + \frac{1}{3}\Sigma^3 + \alpha\widehat{\mathrm{HHI}} + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} + o(\|\boldsymbol{s}\|^3). \qquad \square
\end{aligned}$$

Combining Lemmas A.2 and A.7, we obtain the approximation of the distortion to consumer surplus:

Lemma A.8. *In the neighborhood of $\boldsymbol{s} = \boldsymbol{0}$,*

$$\mathrm{CS}(\boldsymbol{s}) - \mathrm{CS}^m(\boldsymbol{s}) = -\alpha\left[\widehat{\mathrm{HHI}}(\boldsymbol{s}) + \frac{1+2\alpha}{2}\widehat{\mathrm{CHK}}(\boldsymbol{s})\right] + o\left(\|\boldsymbol{s}\|^3\right).$$

Next, we turn our attention to industry profit,

$$\Pi(\boldsymbol{s}) \equiv \sum_{f\in\mathcal{F}}\left(\frac{1}{1-\alpha s^f} - 1\right).$$

Lemma A.9. *In the neighborhood of* $\boldsymbol{s} = \boldsymbol{0}$,

$$\Pi(\boldsymbol{s}) = \alpha\Sigma(\boldsymbol{s}) + \alpha^2\widehat{\mathrm{HHI}}(\boldsymbol{s}) + \alpha^3\widehat{\mathrm{CHK}}(\boldsymbol{s}) + o\left(\|\boldsymbol{s}\|^3\right).$$

Proof. This follows immediately from the fact that, in the neighborhood of $x = 0$,

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + o(x^3). \qquad \square$$

The next involves approximating industry profit under monopolistic competition.

Lemma A.10. *In the neighborhood of* $\boldsymbol{s} = \boldsymbol{0}$,

$$\Pi^m(\boldsymbol{s}) = \alpha\left(\Sigma(\boldsymbol{s}) + \alpha\widehat{\mathrm{HHI}}(\boldsymbol{s}) + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}}(\boldsymbol{s}) - \alpha\widehat{\mathrm{HHI}}(\boldsymbol{s})\Sigma(\boldsymbol{s})\right) + o(\|\boldsymbol{s}\|^3).$$

Proof. Note that, at the third order in the neighborhood of $x = 0$,

$$1 - \frac{1}{1+x} = x - x^2 + x^3 + o(x^3).$$

Combining this with the formula for $\Pi^m(\boldsymbol{s})$ in Lemma A.6 and the approximation of $\mathcal{T}(\boldsymbol{s})$ in Lemma A.5, and eliminating higher-order terms, we obtain:

$$\begin{aligned}
\Pi^m(\boldsymbol{s}) &= \alpha\left(\Sigma + \alpha\widehat{\mathrm{HHI}} + \Sigma^2 + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} + \alpha\widehat{\mathrm{HHI}}\Sigma + \Sigma^3\right. \\
&\quad \left. - \left(\Sigma + \alpha\widehat{\mathrm{HHI}} + \Sigma^2\right)^2 + \Sigma^3\right) + o(\|\boldsymbol{s}\|^3) \\
&= \alpha\left(\Sigma + \alpha\widehat{\mathrm{HHI}} + \Sigma^2 + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} + \alpha\widehat{\mathrm{HHI}}\Sigma + \Sigma^3\right. \\
&\quad \left. - \left(\Sigma^2 + 2\alpha\widehat{\mathrm{HHI}}\Sigma + 2\Sigma^3\right) + \Sigma^3\right) + o(\|\boldsymbol{s}\|^3) \\
&= \alpha\left(\Sigma + \alpha\widehat{\mathrm{HHI}} + \frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} - \alpha\widehat{\mathrm{HHI}}\Sigma\right) + o(\|\boldsymbol{s}\|^3). \qquad \square
\end{aligned}$$

Combining Lemmas A.8, A.9, and A.10, we obtain the approximation of the aggregate surplus distortion:

Lemma A.11. *In the neighborhood of* $\boldsymbol{s} = \boldsymbol{0}$,

$$\mathrm{AS}(\boldsymbol{s}) - \mathrm{AS}^m(\boldsymbol{s}) = -\alpha\left(\widehat{\mathrm{HHI}}(\boldsymbol{s})(1 - \alpha\Sigma(\boldsymbol{s})) + \frac{1}{2}(1+3\alpha)\widehat{\mathrm{CHK}}(\boldsymbol{s})\right) + o\left(\|\boldsymbol{s}\|^3\right).$$

B Proof of Proposition 2

We prove a series of lemmas that jointly imply Proposition 2. These lemmas correspond to higher-order versions of Lemmas 16–19 in the Online Appendix to Nocke and Schutz (forthcoming).

Recall from Section 2.2 that the markup and market-share fitting-in functions, $m(x,\theta)$ and $S(x,\theta)$, jointly solve the system

$$\mu = \frac{1}{1-\theta\alpha s},$$
$$s = \begin{cases} x\,(1-(1-\alpha)\mu)^{\frac{\alpha}{1-\alpha}} & \text{under CES demand,} \\ x\,\mathrm{e}^{-\mu} & \text{under MNL demand.} \end{cases}$$

We compute the first and second partial derivatives of $S(x,\theta)$ at $\theta = 0$:

Lemma B.1. *For every $\alpha \in (0,1]$ and $x > 0$, the partial derivatives of S at $\theta = 0$ are:*

$$\left.\frac{\partial S}{\partial x}\right|_{(x,0)} = \frac{S(x,0)}{x} \quad \textit{and} \quad \left.\frac{\partial S}{\partial \theta}\right|_{(x,0)} = -\alpha S(x,0)^2.$$

The second partial derivatives of S at $\theta = 0$ are:

$$\left.\frac{\partial^2 S}{\partial x^2}\right|_{(x,0)} = 0, \quad \left.\frac{\partial^2 S}{\partial \theta^2}\right|_{(x,0)} = (2\alpha-1)\alpha S(x,0)^3, \quad \textit{and} \quad \left.\frac{\partial^2 S}{\partial x \partial \theta}\right|_{(x,0)} = -2\alpha\frac{S(x,0)^2}{x}.$$

Proof. Under CES demand,

$$S = x\,(1-(1-\alpha)m)^{\frac{\alpha}{1-\alpha}} = x\left(1-\frac{1-\alpha}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}} = x\alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{1-\theta S}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}}.$$

Taking logs and differentiating yields

$$\frac{dS}{S} = \frac{dx}{x} + \frac{\alpha}{1-\alpha}\left[\frac{dS}{S}+\frac{d\theta}{\theta}\right]\left(\frac{\alpha\theta S}{1-\alpha\theta S} - \frac{\theta S}{1-\theta S}\right).$$

Rearranging terms, we obtain

$$\frac{dS}{S}\frac{1-\theta S+\alpha(\theta S)^2}{(1-\theta S)(1-\alpha\theta S)} = \frac{dx}{x} - \frac{\alpha S}{(1-\theta S)(1-\alpha\theta S)}d\theta.$$

The partial derivatives of S are therefore given by

$$\frac{\partial S}{\partial x} = \frac{S}{x}\frac{(1-\theta S)(1-\alpha\theta S)}{1-\theta S+\alpha(\theta S)^2} \quad \text{and} \quad \frac{\partial S}{\partial \theta} = \frac{-\alpha S^2}{1-\theta S+\alpha(\theta S)^2}. \tag{23}$$

Under MNL demand,

$$S = x\,\mathrm{e}^{-m} = x\exp\left(-\frac{1}{1-\theta S}\right).$$

Proceeding as above, we obtain

$$\frac{dS}{S}\frac{1-\theta S+(\theta S)^2}{(1-\theta S)^2}=\frac{dx}{x}-\frac{Sd\theta}{(1-\theta S)^2}$$

Hence,

$$\frac{\partial S}{\partial x}=\frac{S}{x}\frac{(1-\theta S)^2}{1-\theta S+(\theta S)^2} \quad \text{and} \quad \frac{\partial S}{\partial \theta}=\frac{-S^2}{1-\theta S+(\theta S)^2},$$

which boils down to equation (23) with $\alpha = 1$. Plugging $\theta = 0$ into equation (23), we obtain the values of the first partial derivatives announced in the statement of the lemma.

Next, we differentiate one more time at $\theta = 0$. We obtain:

$$\begin{aligned}
\left.\frac{\partial^2 S}{\partial x^2}\right|_{(x,0)} &= -\frac{S(x,0)}{x^2}\times 1+\left.\frac{\partial S}{\partial x}\right|_{(x,0)}\frac{1}{x}\times 1+\left.\frac{\partial S}{\partial x}\right|_{(x,0)}\left.\frac{d}{ds}\frac{(1-\theta s)(1-\alpha\theta s)}{1-\theta s+\alpha(\theta s)^2}\right|_{\theta=0,s=S(x,0)}=0,\\
\left.\frac{\partial^2 S}{\partial x\partial\theta}\right|_{(x,0)} &= -2\alpha S(x,0)\left.\frac{\partial S}{\partial x}\right|_{(x,0)}-\alpha S(x,0)^2\left.\frac{\partial S}{\partial x}\right|_{(x,0)}\left.\frac{d}{ds}\frac{1}{1-\theta s+\alpha(\theta s)^2}\right|_{\theta=0,s=S(x,0)}\\
&= -2\alpha\frac{S(x,0)^2}{x}+0,\\
\left.\frac{\partial^2 S}{\partial\theta^2}\right|_{(x,0)} &= -2\alpha S(x,0)\left.\frac{\partial S}{\partial \theta}\right|_{(x,0)}-\alpha S(x,0)^2\left.\frac{\partial S}{\partial \theta}\right|_{(x,0)}\left.\frac{d}{ds}\frac{1}{1-\theta s+\alpha(\theta s)^2}\right|_{\theta=0,s=S(x,0)}\\
&\quad -\alpha S(x,0)^2\left.\frac{d}{d\theta}\frac{1}{1-\theta s+\alpha(\theta s)^2}\right|_{\theta=0,s=S(x,0)}\\
&= 2\alpha^2 S(x,0)^3+0-\alpha S(x,0)^3.
\end{aligned}$$

□

Next, we compute the first and second derivatives of $H^*(\theta)$:

Lemma B.2. *The following holds:*

$$\left.\frac{d\log H^*}{d\theta}\right|_{\theta=0}=-\alpha\widehat{\mathrm{HHI}}(0) \quad \textit{and} \quad \left.\frac{d^2\log H^*}{d\theta^2}\right|_{\theta=0}=(2\alpha-1)\alpha\widehat{\mathrm{CHK}}(0)-3\alpha^2\widehat{\mathrm{HHI}}(0)^2.$$

Proof. Totally differentiating equilibrium condition (10), we obtain:

$$-\frac{dH^*}{H^*(\theta)}\left(\frac{H^0}{H^*(\theta)}+\sum_{f\in\mathcal{F}}\frac{T^f}{H^*(\theta)}\frac{\partial S}{\partial x}\left(\frac{T^f}{H^*(\theta)},\theta\right)\right)+d\theta\sum_{f\in\mathcal{F}}\frac{\partial S}{\partial\theta}\left(\frac{T^f}{H^*(\theta)},\theta\right)=0.$$

Hence,

$$\frac{d\log H^*}{d\theta}=\frac{\sum_{f\in\mathcal{F}}\frac{\partial S}{\partial\theta}\left(\frac{T^f}{H^*(\theta)},\theta\right)}{\frac{H^0}{H^*(\theta)}+\sum_{f\in\mathcal{F}}\frac{T^f}{H^*(\theta)}\frac{\partial S}{\partial x}\left(\frac{T^f}{H^*(\theta)},\theta\right)}. \tag{24}$$

Using Lemma B.1 and the equilibrium condition, we see that, at $\theta = 0$, the denominator of the above expression is equal to 1, whereas the numerator is equal to $-\alpha\widehat{\mathrm{HHI}}(0)$, as stated.

Next, we differentiate the right-hand side of equation (24) with respect to θ at $\theta = 0$, keeping in mind that the denominator is equal to 1 at that point and that $\partial^2 S(x,0)/\partial x^2 = 0$:

$$\begin{aligned}
\left.\frac{d^2 \log H^*}{d\theta^2}\right|_{\theta=0} &= \sum_{f\in\mathcal{F}} \frac{\partial^2 S(T^f/H^*,0)}{\partial\theta^2} - \frac{d\log H^*}{d\theta}\sum_{f\in\mathcal{F}}\frac{T^f}{H^*}\frac{\partial^2 S(T^f/H^*,0)}{\partial\theta\partial x} - \sum_{f\in\mathcal{F}}\frac{\partial S(T^f/H^*,0)}{\partial\theta} \\
&\quad\times\left[-\frac{d\log H^*}{d\theta}\left(\frac{H^0}{H^*}+\sum_{f\in\mathcal{F}}\frac{T^f}{H^*}\frac{\partial S(T^f/H^*,0)}{\partial x}\right)+\sum_{f\in\mathcal{F}}\frac{T^f}{H^*}\frac{\partial^2 S(T^f/H^*,0)}{\partial x\partial\theta}\right] \\
&= (2\alpha-1)\alpha\widehat{\mathrm{CHK}}(0) - 2\alpha^2\widehat{\mathrm{HHI}}(0)^2 + \alpha\widehat{\mathrm{HHI}}(0)\left[\alpha\widehat{\mathrm{HHI}}(0) - 2\alpha\widehat{\mathrm{HHI}}(0)\right] \\
&= (2\alpha-1)\alpha\widehat{\mathrm{CHK}}(0) - 3\alpha^2\widehat{\mathrm{HHI}}(0)^2,
\end{aligned}$$

where we have used Lemma B.1 and the fact that, at $\theta = 0$, $d\log H^*/d\theta = -\alpha\widehat{\mathrm{HHI}}$ to obtain the second equality. □

Next, we obtain the Taylor approximations of $\Sigma(\theta)$ and $\mathrm{HHI}(\theta)$:

Lemma B.3. *In the neighborhood of $\theta = 0$,*

$$\begin{aligned}
\Sigma(\theta) &= \Sigma(0) - \alpha\widehat{\mathrm{HHI}}(\theta)\left(1-\Sigma(\theta)\right)\theta \\
&\quad - \frac{1}{2}\alpha\left(1-\Sigma(\theta)\right)\left[(1+2\alpha)\widehat{\mathrm{CHK}}(\theta) - 2\alpha\widehat{\mathrm{HHI}}(\theta)^2\right]\theta^2 + o(\theta^2), \\
\widehat{\mathrm{HHI}}(0) &= \widehat{\mathrm{HHI}}(\theta) + 2\alpha\left[\widehat{\mathrm{CHK}}(\theta) - \widehat{\mathrm{HHI}}(\theta)^2\right]\theta + o(\theta)
\end{aligned}$$

Proof. We begin by computing $s^{f\prime}(0)$, $s^{f\prime\prime}(0)$, $\Sigma'(0)$, and $\Sigma''(0)$. As $s^f(\theta) = S(T^f/H^*(\theta),\theta)$, we have that

$$\frac{ds^f}{d\theta} = -\frac{d\log H^*}{d\theta}\frac{T^f}{H^*}\frac{\partial S(T^f/H^*,\theta)}{\partial x} + \frac{\partial S(T^f/H^*,\theta)}{\partial\theta}.$$

Applying Lemma B.1, we obtain

$$s^{f\prime}(0) = \alpha\left(\widehat{\mathrm{HHI}}(0)s^f(0) - s^f(0)^2\right),$$

and thus

$$\Sigma'(0) = -\alpha\widehat{\mathrm{HHI}}(0)(1-\Sigma(0)).$$

Differentiating one more time at $\theta = 0$ and applying Lemmas B.1 and B.2 yields

$$\begin{aligned}
s^{f\prime\prime}(0) &= -\frac{d^2\log H^*}{d\theta^2}s^f(0) + \alpha\widehat{\mathrm{HHI}} \times \left(-\frac{d\log H^*}{d\theta}\right)\frac{T^f}{H^*}\frac{\partial S(T^f/H^*,\theta)}{\partial x} \\
&\quad + \alpha\widehat{\mathrm{HHI}}\frac{T^f}{H^*}\frac{\partial^2 S(T^f/H^*,\theta)}{\partial x\partial\theta} - \frac{d\log H^*}{d\theta}\frac{T^f}{H^*}\frac{\partial^2 S(T^f/H^*,\theta)}{\partial x\partial\theta} + \frac{\partial^2 S(T^f/H^*,\theta)}{\partial\theta^2} \\
&= \left(-(2\alpha-1)\alpha\widehat{\mathrm{CHK}}(0) + 3\alpha^2\widehat{\mathrm{HHI}}(0)^2\right)s^f(0) + \alpha^2\widehat{\mathrm{HHI}}(0)^2 s^f(0) - 2\alpha^2\widehat{\mathrm{HHI}}(0)s^f(0)^2 \\
&\quad - 2\alpha^2\widehat{\mathrm{HHI}}(0)s^f(0)^2 + (2\alpha-1)\alpha s^f(0)^3
\end{aligned}$$

$$= (2\alpha - 1)\alpha \left(s^f(0)^3 - \widehat{\text{CHK}}(0) s^f(0) \right) + 4\alpha^2 \left(\widehat{\text{HHI}}(0)^2 s^f(0) - \widehat{\text{HHI}}(0) s^f(0)^2 \right),$$

so that

$$\Sigma''(0) = (1 - \Sigma(0)) \left((2\alpha - 1)\alpha \widehat{\text{CHK}}(0) - 4\alpha^2 \widehat{\text{HHI}}(0)^2 \right).$$

By Taylor's theorem, we have that, in the neighborhood of $\theta = 0$,

$$\Sigma(\theta) = \Sigma(0) - \alpha \widehat{\text{HHI}}(0) \left(1 - \Sigma(0)\right) \theta + o(\theta) = \Sigma(0) - \alpha \widehat{\text{HHI}}(\theta) \left(1 - \Sigma(\theta)\right) \theta + o(\theta),$$

where the second equality follows, as $\widehat{\text{HHI}}(\theta) = \widehat{\text{HHI}}(0) + o(1)$ and $\Sigma(\theta) = \Sigma(0) + o(1)$ by continuity of $\widehat{\text{HHI}}(\cdot)$ and $\Sigma(\cdot)$. Moreover,

$$\begin{aligned}
\widehat{\text{HHI}}(\theta) &= \sum_{f \in \mathcal{F}} \left(s^f(0) + s^{f\prime}(0)\theta \right)^2 + o(\theta) \\
&= \sum_{f \in \mathcal{F}} \left(s^f(0)^2 + 2 s^f(0) \alpha \left[(\widehat{\text{HHI}}(0) s^f(0) - s^f(0)^2 \right] \theta \right) + o(\theta) \\
&= \widehat{\text{HHI}}(0) + 2\alpha \left[\widehat{\text{HHI}}(0)^2 - \widehat{\text{CHK}}(0) \right] \theta + o(\theta) \\
&= \widehat{\text{HHI}}(0) + 2\alpha \left[\widehat{\text{HHI}}(\theta)^2 - \widehat{\text{CHK}}(\theta) \right] \theta + o(\theta).
\end{aligned}$$

Combining these approximations, we obtain an approximation of $\widehat{\text{HHI}}(\cdot)(1 - \Sigma(\cdot))$:

$$\begin{aligned}
\widehat{\text{HHI}}(0) \left(1 - \Sigma(0)\right) = &\left(\widehat{\text{HHI}}(\theta) + 2\alpha \left[\widehat{\text{CHK}}(\theta) - \widehat{\text{HHI}}(\theta)^2 \right] \theta \right) \\
&\times \left(1 - \Sigma(\theta) - \alpha \widehat{\text{HHI}}(\theta) \left(1 - \Sigma(\theta)\right) \theta \right) + o(\theta) \\
= &\widehat{\text{HHI}}(\theta) \left(1 - \Sigma(\theta)\right) + \alpha \left(1 - \Sigma(\theta)\right) \left[2\widehat{\text{CHK}}(\theta) - 3\widehat{\text{HHI}}(\theta)^2 \right] \theta + o(\theta).
\end{aligned}$$

Finally, we apply again Taylor's theorem to obtain the second-order approximation of $\Sigma(\cdot)$:

$$\begin{aligned}
\Sigma(\theta) &= \Sigma(0) - \alpha \widehat{\text{HHI}}(0)(1 - \Sigma(0))\theta \\
&\quad + \frac{1}{2}\alpha \left(1 - \Sigma(0)\right) \left[((2\alpha - 1)\widehat{\text{CHK}}(0) - 4\alpha \widehat{\text{HHI}}(0)^2 \right] \theta^2 + o(\theta^2) \\
&= \Sigma(0) - \alpha \widehat{\text{HHI}}(\theta)(1 - \Sigma(\theta))\theta - \alpha^2 \left(1 - \Sigma(\theta)\right) \left[2\widehat{\text{CHK}}(\theta) - 3\widehat{\text{HHI}}(\theta)^2 \right] \theta^2 \\
&\quad + \frac{1}{2}\alpha \left(1 - \Sigma(\theta)\right) \left[((2\alpha - 1)\widehat{\text{CHK}}(\theta) - 4\alpha \widehat{\text{HHI}}(\theta)^2 \right] \theta^2 + o(\theta^2) \\
&= \Sigma(0) - \alpha \widehat{\text{HHI}}(\theta)(1 - \Sigma(\theta))\theta \\
&\quad + \frac{1}{2}\alpha \left(1 - \Sigma(\theta)\right) \left[2\widehat{\text{HHI}}(\theta)^2 - (2\alpha + 1)\widehat{\text{CHK}}(\theta) \right] \theta^2 + o(\theta^2).
\end{aligned}$$

□

Combining Lemmas B.2 and B.3, we obtain the approximation of the distortion to consumer surplus:

Lemma B.4. *In the neighborhood of $\theta = 0$,*

$$\mathrm{CS}(\theta) - \mathrm{CS}(0) = -\alpha\widehat{\mathrm{HHI}}(\theta)\theta - \frac{1}{2}\alpha\left[(1+2\alpha)\widehat{\mathrm{CHK}}(\theta) - \alpha\widehat{\mathrm{HHI}}(\theta)^2\right]\theta + o(\theta^2).$$

Proof. Combining Taylor's theorem with Lemma B.2, the distortion to consumer surplus is

$$\begin{aligned}
\mathrm{CS}(\theta) - \mathrm{CS}(0) = & -\alpha\widehat{\mathrm{HHI}}(0)\theta + \frac{1}{2}\alpha\left[(2\alpha-1)\widehat{\mathrm{CHK}}(0) - 3\alpha\widehat{\mathrm{HHI}}(0)^2\right]\theta^2 + o(\theta^2) \\
= & -\alpha\widehat{\mathrm{HHI}}(\theta)\theta + 2\alpha^2\left[\widehat{\mathrm{HHI}}(\theta)^2 - \widehat{\mathrm{CHK}}(\theta)\right]\theta^2 \\
& + \frac{1}{2}\alpha\left[(2\alpha-1)\widehat{\mathrm{CHK}}(\theta) - 3\alpha\widehat{\mathrm{HHI}}(\theta)^2\right]\theta^2 + o(\theta^2) \\
= & -\alpha\widehat{\mathrm{HHI}}(\theta)\theta + \frac{1}{2}\alpha\left[\alpha\widehat{\mathrm{HHI}}(\theta)^2 - (2\alpha+1)\widehat{\mathrm{CHK}}(\theta)\right]\theta^2 + o(\theta^2),
\end{aligned}$$

where the second line follows by Lemma B.3. □

Let $\Pi(\theta)$ denote aggregate equilibrium profits when the conduct parameter is θ. The following lemma approximates Π at the second order:

Lemma B.5. *In the neighborhood of $\theta = 0$,*

$$\begin{aligned}
\Pi(\theta) = & \Pi(0) + \alpha^2\widehat{\mathrm{HHI}}(\theta)\Sigma(\theta)\theta \\
& + \frac{1}{2}\alpha^2\left[2\alpha\widehat{\mathrm{CHK}}(\theta) - (1-\Sigma(\theta))\left((1+2\alpha)\widehat{\mathrm{CHK}}(\theta) - 2\alpha\widehat{\mathrm{HHI}}(\theta)^2\right)\right]\theta^2 + o(\theta^2).
\end{aligned}$$

Proof. Let $\pi^f(\theta) = \alpha s^f(\theta)/(1-\alpha\theta s^f(\theta))$ denote firm f's equilibrium profit. Using the Taylor approximation $1/(1-x) = 1 + x + x^2 + o(x^2)$ in the neighborhood of $x = 0$, we obtain

$$\pi^f(\theta) = \alpha s^f(\theta)\left(1 + \alpha\theta s^f(\theta) + \alpha^2\theta^2 s^f(\theta)^2\right) + o(\theta^2).$$

Adding up over all the firms yields

$$\begin{aligned}
\Pi(\theta) &= \alpha\Sigma(\theta) + \alpha^2\widehat{\mathrm{HHI}}(\theta)\theta + \alpha^3\widehat{\mathrm{CHK}}(\theta)\theta^2 + o(\theta^2) \\
&= \Pi(0) + \alpha^2\widehat{\mathrm{HHI}}(\theta)\Sigma(\theta)\theta \\
&\quad + \frac{1}{2}\alpha^2\left[2\alpha\widehat{\mathrm{CHK}}(\theta) + (1-\Sigma(\theta))\left(2\alpha\widehat{\mathrm{HHI}}(\theta)^2 - (1+2\alpha)\widehat{\mathrm{CHK}}(\theta)\right)\right]\theta^2 + o(\theta^2),
\end{aligned}$$

where the second line follows by Lemma B.3. □

Combining Lemmas B.4 and B.5, we obtain the second part of Proposition 2:

Lemma B.6. *In the neighborhood of $\theta = 0$,*

$$\mathrm{AS}(\theta) - \mathrm{AS}(0) = -\alpha\widehat{\mathrm{HHI}}(\theta)\left(1 - \alpha\Sigma(\theta)\right)\theta$$

$$-\frac{1}{2}\alpha\left[(1+3\alpha-\alpha(1+2\alpha)\Sigma(\theta))\,\widehat{\text{CHK}}(\theta)-\alpha\left(1+2\alpha(1-\Sigma(\theta))\right)\widehat{\text{HHI}}(\theta)^2\right]\theta^2+o(\theta^2).$$

C Proof of Proposition 3

Proof. Let us show that $\Sigma(\theta)$ is weakly decreasing in θ while $\Pi(\theta)$ is strictly increasing. The former follows immediately, as

$$\Sigma(\theta)=1-\frac{H^0}{H^*(\theta)}$$

and $H^*(\theta)$ is strictly decreasing (see equations (23) and (24) above). To prove the latter, let us show that each firm f's equilibrium profit, $\pi^f(\theta)$, is strictly increasing.

As a first step, we show that the markup fitting-in function has strictly positive derivatives. Differentiating equation (9) with respect to θ (and dropping the firm superscript to ease notation), we obtain

$$\begin{aligned}\frac{\partial m}{\partial\theta}&=\frac{\alpha}{(1-\alpha\theta S(x,\theta))^2}\left(S(x,\theta)+\theta\frac{\partial S}{\partial\theta}\right)\\&=\frac{\alpha S(x,\theta)(1-\theta S(x,\theta))}{(1-\alpha\theta S(x,\theta))(1-\theta S(x,\theta)+\alpha(\theta S(x,\theta))^2)}>0,\end{aligned}$$

where we have used equation (23) to obtain the second line. Moreover,

$$\frac{\partial m}{\partial x}=\frac{\alpha\theta}{(1-\alpha\theta S(x,\theta))^2}\frac{\partial S}{\partial x}$$

is also strictly positive (see again equation (23)). It follows that, for every $g\in\mathcal{F}$, the equilibrium ι-markup of firm g, $\mu^g(\theta)=m(T^g/(H^*(\theta)),\theta)$, has a strictly positive derivative.

Let

$$\widehat{\Pi}^f\left((\mu^g)_{g\in\mathcal{F}}\right)\equiv\alpha\mu^f\frac{T^f\left(1-(1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}}}{H^0+\sum_{g\in\mathcal{F}}T^g\left(1-(1-\alpha)\mu^g\right)^{\frac{\alpha}{1-\alpha}}}$$

be firm f's profit when firms set the profile of ι-markups $(\mu^g)_{g\in\mathcal{F}}$. Clearly, $\widehat{\Pi}^f$ has strictly positive partial derivatives with respect to the ι-markups of rival firms and $\pi^f(\theta)=\widehat{\Pi}^f\left((\mu^g(\theta))_{g\in\mathcal{F}}\right)$. (Recall from equation (4) that, under the ι-markup μ^f, firm f makes a profit of $\alpha\mu^f s^f$.) Moreover, when the profile of ι-markups is equal to $(\mu^g(\theta))_{g\in\mathcal{F}}$, the fraction in the definition of $\widehat{\Pi}^f$ is equal to $s^f(\theta)$ and its denominator is equal to $H^*(\theta)$ (see equations (6) and (10)). We have:

$$\begin{aligned}\left.\frac{\partial\widehat{\Pi}^f}{\partial\mu^f}\right|_{(\mu^g(\theta))_{g\in\mathcal{F}}}&=\alpha T^f\left(1-(1-\alpha)\mu^f(\theta)\right)^{\frac{\alpha}{1-\alpha}-1}\\&\quad\times\frac{(1-\mu^f(\theta))H^*(\theta)+\alpha\mu^f(\theta)T^f\left(1-(1-\alpha)\mu^f(\theta)\right)^{\frac{\alpha}{1-\alpha}}}{H^*(\theta)^2}\end{aligned}$$

$$= \frac{\alpha T^f \left(1-(1-\alpha)\mu^f(\theta)\right)^{\frac{\alpha}{1-\alpha}-1}}{H^*(\theta)} \left[1-\mu^f(\theta)(1-\alpha s^f(\theta))\right]$$
$$= \frac{\alpha T^f \left(1-(1-\alpha)\mu^f(\theta)\right)^{\frac{\alpha}{1-\alpha}-1}}{H^*(\theta)} \frac{\alpha s^f(\theta)(1-\theta)}{1-\alpha\theta s^f(\theta)},$$

where we have used the fact that $\mu^f(\theta)$ and $s^f(\theta)$ satisfy condition (9) to obtain the last line. Note that the final expression is strictly positive unless $\theta = 1$. We can conclude: for every $\theta \in [0,1)$,

$$\frac{d\pi^f}{d\theta} = \sum_{g\in\mathcal{F}} \left.\frac{\partial\widehat{\Pi}^f}{\partial\mu^g}\right|_{(\mu^g(\theta))_{g\in\mathcal{F}}} \frac{d\mu^g}{d\theta} > 0. \qquad \square$$

D Proof of Proposition 5

We require a third-order Taylor approximation of $\tilde{\Sigma}(\boldsymbol{s})$. To derive such an approximation, we use the resource constraint, $C(\boldsymbol{s}) = \tilde{C}(\tilde{\Sigma}(\boldsymbol{s}))$, which can be rewritten as:

$$\frac{\tilde{\Sigma}(\boldsymbol{s})}{\mathcal{T}(\boldsymbol{s})/H^0} = (1-\tilde{\Sigma}(\boldsymbol{s}))^{1-\alpha} \left[\frac{\sum_{f\in\mathcal{F}} s^f - (1-\alpha)\frac{s^f}{1-\alpha s^f}}{\mathcal{T}(\boldsymbol{s})/H^0}\right]^{\alpha}. \tag{25}$$

It is useful to obtain first a second-order Taylor approximation of the bracketed term on the right-hand side of equation (25).

Lemma D.1. *In the neighborhood of* $\boldsymbol{s} = \boldsymbol{0}$,

$$\left[\frac{\sum_{f\in\mathcal{F}} s^f - (1-\alpha)\frac{s^f}{1-\alpha s^f}}{\alpha S'(0)\mathcal{T}(\boldsymbol{s})/H^0}\right]^{\alpha} = 1 - \alpha\left(\frac{\widehat{\mathrm{HHI}}(\boldsymbol{s})}{\Sigma(\boldsymbol{s})} + \Sigma(\boldsymbol{s})\right)$$
$$+ \alpha\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\mathrm{HHI}}(\boldsymbol{s})}{\Sigma(\boldsymbol{s})}\right)^2 + \alpha\widehat{\mathrm{HHI}}(\boldsymbol{s}) - \frac{3}{2}\alpha\frac{\widehat{\mathrm{CHK}}(\boldsymbol{s})}{\Sigma(\boldsymbol{s})} - \frac{1}{2}(1-\alpha)\Sigma(\boldsymbol{s})^2\right) + o(\|\boldsymbol{s}\|^2).$$

Proof. We begin by approximating the term inside square brackets. Approximating the numerator at the third order and using Lemma A.5 to approximate the denominator yields

$$\frac{\sum_{f\in\mathcal{F}} s^f - (1-\alpha)\frac{s^f}{1-\alpha s^f}}{\alpha S'(0)\mathcal{T}(\boldsymbol{s})/H^0} = \frac{\Sigma - (1-\alpha)\widehat{\mathrm{HHI}} - \alpha(1-\alpha)\widehat{\mathrm{CHK}} + o(\|\boldsymbol{s}\|^3)}{\Sigma + \left(\alpha\widehat{\mathrm{HHI}} + \Sigma^2\right) + \left(\frac{\alpha(1+2\alpha)}{2}\widehat{\mathrm{CHK}} + \alpha\widehat{\mathrm{HHI}}\Sigma + \Sigma^3\right) + o(\|\boldsymbol{s}\|^3)}$$
$$= \frac{1 - (1-\alpha)\frac{\widehat{\mathrm{HHI}}}{\Sigma} - \alpha(1-\alpha)\frac{\widehat{\mathrm{CHK}}}{\Sigma} + o(\|\boldsymbol{s}\|^2)}{1 + \left(\alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right) + \left(\frac{\alpha(1+2\alpha)}{2}\frac{\widehat{\mathrm{CHK}}}{\Sigma} + \alpha\widehat{\mathrm{HHI}} + \Sigma^2\right) + o(\|\boldsymbol{s}\|^2)},$$

where we have used the fact that $o(\|\boldsymbol{s}\|^3)/\Sigma(\boldsymbol{s})$ is a little-o of $\|\boldsymbol{s}\|^2$ in the neighborhood of $\boldsymbol{s} = \boldsymbol{0}$, as $\|\boldsymbol{s}\|/\Sigma(\boldsymbol{s}) \leq 1$. For what follows, it is useful to keep in mind that $\widehat{\mathrm{HHI}}(\boldsymbol{s})/\Sigma(\boldsymbol{s})$ is of order 1 and $\widehat{\mathrm{CHK}}(\boldsymbol{s})/\Sigma(\boldsymbol{s})$ is of order 2 in the neighborhood of $\boldsymbol{s} = \mathbf{0}$, as

$$\frac{1}{|\mathcal{F}|} \leq \frac{\widehat{\mathrm{HHI}}(\boldsymbol{s})}{\Sigma(\boldsymbol{s})^2} \leq 1 \quad \text{and} \quad \frac{1}{|\mathcal{F}|} \leq \frac{\widehat{\mathrm{CHK}}(\boldsymbol{s})}{\Sigma(\boldsymbol{s})^3} \leq 1$$

and $\Sigma(\boldsymbol{s})$ is of order 1.

Using the Taylor approximation $1/(1+x) = 1 - x + x^2 + o(x^2)$ in the neighborhood of $x = 0$, we obtain

$$\begin{aligned}
\frac{\sum_{f\in\mathcal{F}} s^f - (1-\alpha)\frac{s^f}{1-\alpha s^f}}{\alpha S'(0)\mathcal{T}(\boldsymbol{s})/H^0} &= \left[1 - (1-\alpha)\frac{\widehat{\mathrm{HHI}}}{\Sigma} - \alpha(1-\alpha)\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right]\left[1 - \left(\alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right)\right. \\
&\quad \left. - \left(\frac{\alpha(1+2\alpha)}{2}\frac{\widehat{\mathrm{CHK}}}{\Sigma} + \alpha\widehat{\mathrm{HHI}} + \Sigma^2\right) + \left(\alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right)^2\right] + o(\|\boldsymbol{s}\|^2) \\
&= \left[1 - (1-\alpha)\frac{\widehat{\mathrm{HHI}}}{\Sigma} - \alpha(1-\alpha)\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right]\left[1 - \left(\alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right)\right. \\
&\quad \left. + \left(\alpha^2\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)^2 + \alpha\widehat{\mathrm{HHI}} - \frac{\alpha(1+2\alpha)}{2}\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right)\right] + o(\|\boldsymbol{s}\|^2) \\
&= 1 - (\alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma) + \left(\alpha^2\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)^2 + \alpha\widehat{\mathrm{HHI}} - \frac{\alpha(1+2\alpha)}{2}\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right) \\
&\quad - (1-\alpha)\frac{\widehat{\mathrm{HHI}}}{\Sigma}\left(1 - (\alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma)\right) - \alpha(1-\alpha)\frac{\widehat{\mathrm{CHK}}}{\Sigma} + o(\|\boldsymbol{s}\|^2) \\
&= 1 - \left(\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right) + \left(\alpha\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)^2 + \widehat{\mathrm{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right) + o(\|\boldsymbol{s}\|^2).
\end{aligned}$$

Combining this approximation with the fact that $(1+x)^\alpha = 1 + \alpha x - \alpha(1-\alpha)x^2/2 + o(x^2)$ in the neighborhood of $x = 0$, we obtain

$$\begin{aligned}
\left[\frac{\sum_{f\in\mathcal{F}} s^f - (1-\alpha)\frac{s^f}{1-\alpha s^f}}{\alpha S'(0)\mathcal{T}(\boldsymbol{s})/H^0}\right]^\alpha &= \left(1 - \left(\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right)\right. \\
&\quad \left. + \left(\alpha\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)^2 + \widehat{\mathrm{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right)\right)^\alpha + o(\|\boldsymbol{s}\|^2) \\
&= 1 + \alpha\left[-\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right) + \left(\alpha\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)^2 + \widehat{\mathrm{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\mathrm{CHK}}}{\Sigma}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{2}\alpha(1-\alpha)\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right)^2 + o(\|\boldsymbol{s}\|^2) \\
&= 1 - \alpha\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right) + \alpha\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)^2 + \alpha\widehat{\mathrm{HHI}}\right. \\
& \left. - \frac{3}{2}\alpha\frac{\widehat{\mathrm{CHK}}}{\Sigma} - \frac{1}{2}(1-\alpha)\Sigma^2\right) + o(\|\boldsymbol{s}\|^2). \qquad \square
\end{aligned}
$$

Combining Lemma D.1 with equation (25), we obtain an approximation of $\tilde{\Sigma}$.

Lemma D.2. *In the neighborhood of* $\boldsymbol{s} = \boldsymbol{0}$,

$$
\tilde{\Sigma}(\boldsymbol{s}) = \Sigma(\boldsymbol{s}) + \frac{1}{2}\alpha(1-\alpha)\left(\widehat{\mathrm{CHK}}(\boldsymbol{s}) - \frac{\widehat{\mathrm{HHI}}(\boldsymbol{s})^2}{\Sigma(\boldsymbol{s})}\right) + o(\|\boldsymbol{s}\|^3).
$$

Proof. Using Lemma D.1 and the fact that $\lim_{\boldsymbol{s}\to\boldsymbol{0}} \tilde{\Sigma}(\boldsymbol{s}) = 0$, we can take limits in equation (25) to obtain

$$
\lim_{\boldsymbol{s}\to\boldsymbol{0}} \frac{\tilde{\Sigma}(\boldsymbol{s})}{\mathcal{T}(\boldsymbol{s})/H^0} = (\alpha S'(0))^\alpha = S'(0),
$$

where the second equality follows by Lemma A.1. It follows that

$$
\tilde{\Sigma}(\boldsymbol{s}) = S'(0)\frac{\mathcal{T}(\boldsymbol{s})}{H^0} + o(\|\mathcal{T}(\boldsymbol{s})\|) = \Sigma(\boldsymbol{s}) + o(\|\boldsymbol{s}\|),
$$

where we have used Lemma A.5 to obtain the second equality. Inserting this approximation into equation (25), making use of Lemma D.1, and using the Taylor approximation $(1-x)^{1-\alpha} = 1-(1-\alpha)x + o(x)$ for $x \simeq 0$, we obtain

$$
\begin{aligned}
\frac{\tilde{\Sigma}(\boldsymbol{s})}{\mathcal{T}(\boldsymbol{s})/H^0} &= S'(0)\left(1-(1-\alpha)\Sigma\right)\left[1 - \alpha\left(\frac{\widehat{\mathrm{HHI}}}{\Sigma} + \Sigma\right)\right] + o(\|\boldsymbol{s}\|) \\
&= S'(0)\left[1 - \Sigma - \alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right] + o(\|\boldsymbol{s}\|).
\end{aligned}
$$

(Recall from the proof of Lemma (D.1) that $\widehat{\mathrm{HHI}}(\boldsymbol{s})/\Sigma(\boldsymbol{s})$ is of first order.) Multiplying both sides of the approximation by $\mathcal{T}(\boldsymbol{s})/H^0$ yields a second-order approximation of $\tilde{\Sigma}(\boldsymbol{s})$:

$$
\begin{aligned}
\tilde{\Sigma}(\boldsymbol{s}) &= \frac{S'(0)}{H^0}\left[1 - \left(\Sigma + \alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)\right]\mathcal{T} + o(\|\boldsymbol{s}\|^2) \\
&= \left[1 - \left(\Sigma + \alpha\frac{\widehat{\mathrm{HHI}}}{\Sigma}\right)\right]\left[\Sigma + \left(\alpha\widehat{\mathrm{HHI}} + \Sigma^2\right)\right] + o(\|\boldsymbol{s}\|^2)
\end{aligned}
$$

$$= \Sigma(\boldsymbol{s}) + o(\|\boldsymbol{s}\|^2),$$

where the second line follows by Lemma A.5.

Inserting this new approximation of $\tilde{\Sigma}(\boldsymbol{s})$ and the Taylor approximation $(1-x)^{1-\alpha} = 1-(1-\alpha)x - \alpha(1-\alpha)x^2/2 + o(x^2)$ for $x \simeq 0$ into equation (25), and making use of Lemma D.1, we obtain

$$\begin{aligned}
\frac{\tilde{\Sigma}(\boldsymbol{s})}{\mathcal{T}(\boldsymbol{s})/H^0} &= S'(0)\left[1-(1-\alpha)\Sigma - \frac{1}{2}\alpha(1-\alpha)\Sigma^2\right]\left[1-\alpha\left(\frac{\widehat{\text{HHI}}}{\Sigma}+\Sigma\right)\right. \\
&\quad \left. +\alpha\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\text{HHI}}}{\Sigma}\right)^2 + \alpha\widehat{\text{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\text{CHK}}}{\Sigma} - \frac{1}{2}(1-\alpha)\Sigma^2\right)\right] + o(\|\boldsymbol{s}\|^2) \\
&= S'(0)\left[1-\alpha\left(\frac{\widehat{\text{HHI}}}{\Sigma}+\Sigma\right) + \alpha\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\text{HHI}}}{\Sigma}\right)^2 + \alpha\widehat{\text{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\text{CHK}}}{\Sigma}\right.\right. \\
&\quad \left.\left. -\frac{1}{2}(1-\alpha)\Sigma^2\right) - (1-\alpha)\Sigma\left(1-\alpha\left(\frac{\widehat{\text{HHI}}}{\Sigma}+\Sigma\right)\right) - \frac{1}{2}\alpha(1-\alpha)\Sigma^2\right] + o(\|\boldsymbol{s}\|^2) \\
&= S'(0)\left[1-\left(\Sigma+\alpha\frac{\widehat{\text{HHI}}}{\Sigma}\right) + \alpha\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\text{HHI}}}{\Sigma}\right)^2 + \widehat{\text{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\text{CHK}}}{\Sigma}\right)\right] \\
&\quad + o(\|\boldsymbol{s}\|^2).
\end{aligned}$$

Multiplying both sides by $\mathcal{T}(\boldsymbol{s})/H^0$ and applying Lemma A.5 yields

$$\begin{aligned}
\tilde{\Sigma}(\boldsymbol{s}) = &\left[\Sigma + \left(\alpha\widehat{\text{HHI}}+\Sigma^2\right) + \left(\frac{\alpha(1+2\alpha)}{2}\widehat{\text{CHK}} + \alpha\widehat{\text{HHI}}\Sigma + \Sigma^3\right)\right]\left[1-\left(\Sigma+\alpha\frac{\widehat{\text{HHI}}}{\Sigma}\right)\right. \\
&\quad \left. +\alpha\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\text{HHI}}}{\Sigma}\right)^2 + \widehat{\text{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\text{CHK}}}{\Sigma}\right)\right] + o(\|\boldsymbol{s}\|^3) \\
&= \Sigma + \left(\alpha\widehat{\text{HHI}}+\Sigma^2\right) + \left(\frac{\alpha(1+2\alpha)}{2}\widehat{\text{CHK}} + \alpha\widehat{\text{HHI}}\Sigma + \Sigma^3\right) \\
&\quad - \left(\Sigma+\alpha\frac{\widehat{\text{HHI}}}{\Sigma}\right)\left(\Sigma + \left(\alpha\widehat{\text{HHI}}+\Sigma^2\right)\right) \\
&\quad + \alpha\Sigma\left(\frac{1}{2}(3\alpha-1)\left(\frac{\widehat{\text{HHI}}}{\Sigma}\right)^2 + \widehat{\text{HHI}} - \frac{3}{2}\alpha\frac{\widehat{\text{CHK}}}{\Sigma}\right) + o(\|\boldsymbol{s}\|^3) \\
&= \Sigma + \frac{1}{2}\alpha(1-\alpha)\left(\widehat{\text{CHK}} - \frac{\widehat{\text{HHI}}^2}{\Sigma}\right) + o(\|\boldsymbol{s}\|^3). \qquad \square
\end{aligned}$$

We are now in a position to prove Proposition 5:

Proof. Using Lemma D.2 and the fact that $\log(1-x) = -x - x^2/2 - x^3/3 + o(x^3)$ in the neighborhood of $x = 0$, equation (17) can be approximated as follows:

$$\begin{aligned} M(\boldsymbol{s}) &= \left(\Sigma + \frac{1}{2}\Sigma^2 + \frac{1}{2}\Sigma^3\right) - \left(\tilde{\Sigma} + \frac{1}{2}\tilde{\Sigma}^2 + \frac{1}{2}\tilde{\Sigma}^3\right) + \frac{\alpha}{1-\alpha}(\Sigma - \tilde{\Sigma}) + o(\|\boldsymbol{s}\|^3) \\ &= -\frac{1}{1-\alpha}(\tilde{\Sigma} - \Sigma) + o(\|\boldsymbol{s}\|^3) \\ &= -\frac{1}{2}\alpha\left(\widehat{\text{CHK}} - \frac{\widehat{\text{HHI}}^2}{\Sigma}\right) + o(\|\boldsymbol{s}\|^3). \end{aligned}$$

□

E Proof of Proposition 6

Proof. Plugging the resource constraint, $\alpha\tilde{\mu} = \Pi(\theta)$, into equation (19), we obtain

$$\tilde{H}(\theta) \equiv \left(1 - \frac{1-\alpha}{\alpha}\Pi(\theta)\right)^{\frac{\alpha}{1-\alpha}} \mathcal{T}.$$

Hence,

$$\begin{aligned} \log\tilde{H}(\theta) - \log\tilde{H}(0) &= \frac{\alpha}{1-\alpha}\left[\log\left(1 - \frac{1-\alpha}{\alpha}\Pi(\theta)\right) - \log\alpha\right] \\ &= \frac{\alpha}{1-\alpha}\log\left(1 - \frac{1-\alpha}{\alpha^2}(\Pi(\theta) - \Pi(0))\right), \end{aligned}$$

where we have used the fact that $\Pi(0) = \alpha\Sigma(0) = \alpha$. Using the Taylor approximation $\log(1-x) = -x - x^2/2 + o(x^2)$ for $x \simeq 0$ and inserting the approximation of $\Pi(\theta)$ from Lemma B.5 (with $\Sigma(\theta) = 1$), we obtain

$$\begin{aligned} \log\frac{\tilde{H}(\theta)}{\tilde{H}(0)} &= -\frac{1}{\alpha}(\Pi(\theta) - \Pi(0)) - \frac{1}{2}\frac{1-\alpha}{\alpha^3}(\Pi(\theta) - \Pi(0))^2 + o(\theta^2) \\ &= -\alpha\widehat{\text{HHI}}(\theta)\theta - \frac{1}{2}(1-\alpha)\alpha\widehat{\text{HHI}}(\theta)^2\theta^2 - \alpha^2\widehat{\text{CHK}}(\theta)\theta^2 + o(\theta^2). \end{aligned}$$

Subtracting this from the Taylor approximation of $\log H^*(\theta)$ (see Lemma B.4) and simplifying proves the proposition. □

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