Discussion Paper Series – CRC TR 224

Discussion Paper No. 678
Project B 03, B 04

Dual Pricing in a Model of Sales

Nicolas Schutz[1]
Anton Sobolev[2]

June 2025
(First Version : March 2025)

[1]University of Mannheim, MaCCI and CEPR, Email: schutz@uni-mannheim.de
[2]University of Mannheim and MaCCI, Email: anton.sobolev@uni-mannheim.de

Support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

Collaborative Research Center Transregio 224 - www.crctr224.de
Rheinische Friedrich-Wilhelms-Universität Bonn - Universität Mannheim

Dual Pricing in a Model of Sales*

Nicolas Schutz[†] Anton Sobolev[‡]

June 22, 2025

Abstract

We study the competitive effects of dual pricing, a vertical restraint that involves charging a distributor different prices for units intended to be resold online versus offline. We develop a model in which a manufacturer contracts with hybrid retailers selling both in-store and online. We find that, by eliminating wasteful price dispersion, dual pricing allows the manufacturer to induce the industry monopoly outcome, whereas uniform pricing does not. Despite this, a ban on dual pricing has negative welfare effects if the online market is small, if the offline consumers' search costs are high, and if the monopoly pass-through is high.

Keywords: Dual pricing, price dispersion, consumer search, vertical restraints

Journal of Economics Literature Classification: L13, L42, D43, D83

1 Introduction

Internet sales are becoming increasingly important in retailing. Many retailers have adopted a so-called hybrid business model, which involves both owning and operating brick-and-mortar stores, and selling online—such online sales can take place on the retailer's online

*We thank Florian Baumann, Gregorio Curello, Maarten Janssen, Thomas Jungbauer, Leonardo Madio, Jeanine Miklós-Thal, Massimo Motta, Martin Peitz, Patrick Rey, Nikita Roketskiy, Johannes Schneider, as well as seminar participants at Humboldt University, Madrid (CEMFI & Carlos III), Monash University, Toulouse School of Economics, University of Freiburg, University of Illinois Urbana-Champaign, and University Pompeu Fabra, and conference participants at APIOC, CEPR VIOS, CLEEN Workshop, the Consumer Search Digital Seminar, the CRC TR 224 Spring Retreat, the annual conference of the theory committee of the German Economic Association, IIOC, MaCCI Annual Meeting, the MaCCI/EPoS Workshop on Digital Markets, MaCCI IO Day, Mannheim ICT Conference, and the Workshop on Search and Platform at Kyoto University for helpful comments and suggestions. We gratefully acknowledge financial support from the Deutsche Forschungsgemeinschaft (DFG) through CRC TR 224 (Projects B03 and B04).

[†]Department of Economics and MaCCI, University of Mannheim; also affiliated with CEPR. Email: schutz@uni-mannheim.de.

[‡]Department of Economics and MaCCI, University of Mannheim. Email: anton.sobolev@uni-mannheim.de.

store or on a platform. This shift towards online sales has been a cause for concern for manufacturers. To the extent that searching and comparing prices is easier online than offline, a greater prevalence of online sales may intensify intra-brand competition, which may ultimately be detrimental to manufacturers' profits.[1] This has led manufacturers to seek greater control over their distribution network, using vertical restraints that make it harder or more costly for retailers to sell online. A prominent example is the *Pierre Fabre* case, in which cosmetic and body-hygiene products manufacturer Pierre Fabre required from its retailers that a pharmacist be present to assist customers. The European Court of Justice (case C-439/09) viewed that practice as an explicit ban on online sales, and thus as an infringement of Article 101 of the Treaty on the Functioning of the European Union. More generally, the fact that retailers face numerous contractual restrictions when selling online is well documented in the European Commission's 2017 e-commerce sector inquiry.[2]

In this paper, we study *dual pricing*, a vertical restraint that involves charging the same distributor a different price for units intended to be resold online than for units intended to be resold offline. In the European Union, dual pricing was a hardcore restriction under the Vertical Block Exemption Regulation (VBER) until June 2022. It remains a hardcore restriction "if its goal is to prevent online sales." In recent years, a number of manufacturers in Germany have been led to discontinue their use of dual pricing after the Bundeskartellamt, Germany's competition authority, initiated proceedings. For example, in the market for household appliances in 2013, Bosch-Siemens was found to have offered price discounts to retailers based on their share of online versus offline sales. Similar concerns were raised in 2013 and 2016, respectively, about Gardena and Lego's trade discounts.[3] Lego's price discount system also ran afoul of the French Competition Authority in 2021 (case 21-D-02).

We explore the incentives to use dual pricing and the associated welfare implications in online and brick-and-mortar markets. We develop a clearinghouse model with vertical relations, based on Varian (1980), Stahl (1989), and Janssen and Shelegia (2015). A manufacturer M distributes its product through two hybrid retailers, operating on- and offline. There are two types of consumers: some search online for free; others use the offline channel, where search is costly. The manufacturer first offers contracts to both retailers. Next, retailers compete in prices and consumers make search and purchase decisions. Throughout

[1] Another concern, which we will not address in this paper, is that manufacturers may find it harder to maintain high-quality sales and post-sales services online. See Section II in Miklós-Thal and Shaffer (2022) for a discussion of such service theories of dual pricing.

[2] Such restrictions include limitations to sell on marketplaces, limitations to sell on own website, limitations to use price comparison tools, and limitations to advertise online. For cases involving restrictions on online sales, see the ASICS case in 2015 (Bundeskartellamt, B2-98/11), the Coty case in 2017 (European Court of Justice, C-230/16), and the Guess case in 2018 (European Commission, AT.40428).

[3] For the Bosch-Siemens and Gardena cases, see Bundeskartellamt B7-11/13 and B5-144/13. The case number for the Lego case does not seem to be publicly available, but press releases can be found on the website of the Bundeskartellamt.

the paper, we maintain the assumption that retailers do not price discriminate between the online and the offline channel—we discuss this assumption in detail below. We compare two regimes: under *uniform pricing*, the manufacturer offers a standard two-part tariff; under *dual pricing*, we again have a two-part tariff, but with potentially distinct variable parts for online and offline sales.

We find that dual pricing is strictly optimal for the manufacturer. Specifically, we show that the manufacturer is unable to induce the industry profit-maximizing outcome with a uniform pricing contract. The reason is that, regardless of what uniform-pricing contract is used, retailers face conflicting incentives. On the one hand, retailers want to undercut each other to capture the online market; on the other, they also have an incentive to set high prices to exploit their offline customer base. This tension gives rise to a mixed-strategy equilibrium. The resulting price dispersion prevents the retail market from being supplied at the industry monopoly price.

By contrast, a well-chosen dual pricing contract can induce the industry monopoly outcome. The optimal contract involves discriminating against the online market by setting a high variable part on online sales. This suppresses the retailers' incentives to undercut each other to corner the online market. Thus, by eliminating wasteful price dispersion, dual pricing allows the manufacturer to fully exploit its monopoly power, whereas uniform pricing does not. We also show that these results are robust to various alterations of the oligopoly and search model, to allowing for uniform-pricing contracts that go beyond two-part tariffs, and to upstream contracts being private rather than public.

Even though the industry monopoly outcome arises under dual pricing, the welfare effects of banning this practice are *a priori* unclear. The reason is that, under uniform pricing retailers set prices both above and below the monopoly level with positive probability. As a result, a ban on dual pricing does not systematically reduce prices, and the question is whether such a ban raises consumer surplus and/or aggregate surplus *in expectation*. Using approximation techniques, we show that the answer is negative in a wide range of scenarios. Specifically, a ban on dual pricing reduces both consumer and aggregate surplus when the online market is relatively small, when offline consumers face low search costs and monopoly pass-through is high, or when offline consumers have high search costs and demand is ρ-linear (Bulow and Pfleiderer, 1983; Anderson and Renault, 2003).[4] By contrast, banning dual pricing has positive welfare effects when offline search costs are small and monopoly pass-through is low.

The interests of online and offline consumers regarding dual pricing are often misaligned. This divergence arises because online consumers receive two draws from the equilibrium price

[4]The demand function D is said to be ρ-linear if $D(p)^\rho$ is linear. This is equivalent to the inverse demand function having constant curvature, and thus to monopoly pass-through being constant.

distribution, while offline consumers receive only one. As a result, online consumers are more likely to benefit from a ban on dual pricing. Indeed, we find that when search costs are low or the online market is large, online consumers favor a ban on dual pricing, whereas offline consumers prefer it to remain legal. This suggests that competition policy measures aimed at protecting online sales, such as the prohibition of dual pricing, may ultimately backfire by harming offline consumers.

The above results have implications beyond the welfare effects of dual pricing. In the Stahl (1989) model without vertical relations, it is well known that an increase in the share of shoppers or a decrease in the non-shoppers' search cost results in a first-order stochastic dominance shift toward lower prices, thus increasing consumer and aggregate surplus. Our analysis implies that these intuitive results can be easily overturned once the manufacturer's optimal response to these changes, which typically involves raising the wholesale price to soften retail competition, is taken into account. For instance, our Proposition 9 implies that under uniform pricing, starting with no shoppers, a small increase in the share of shoppers *lowers* consumer and aggregate surplus for any well-behaved demand function. In the case where demand is ρ-linear and offline consumers' search costs are high, our Proposition 7 implies that consumer and aggregate surplus are in fact highest in the absence of shoppers. This highlights the importance of taking vertical relations into account when assessing the welfare effects of policies aimed at reducing search frictions.

On the assumption that retailers do not discriminate. One of the paper's central assumptions is that retailers charge the same prices on- and offline. This assumption is broadly in line with the evidence reported in Cavallo (2017). Cavallo collected data on the online and offline prices charged by 56 large hybrid retailers in 10 countries and found that on- and offline prices are identical 72% of the time. Moreover, there is substantial variation in that key statistic both across countries (from 42% in Brazil to 91% in Canada) and across sectors (from 25% for office products to 83% and 92% for electronics and clothing, respectively). Thus, the assumption of no discrimination has empirical relevance, at least for some countries and sectors.[5]

There are many good reasons why a retailer might choose not to price discriminate. First, the firm may worry about losing customer goodwill, as its offline consumers would feel deceived if they found out that the product they just purchased was offered online at a lower price by the same retailer. Second, an individual (or collective) commitment not to price discriminate may provide a strategic advantage. Third, if the online channel is a marketplace, the platform may impose a price parity clause, thus making discrimination

[5]For another well-known real-world instance in which retailers do not discriminate as much as one would expect them to, see the paper by DellaVigna and Gentzkow (2019), which shows that most U.S. food, drugstore, and mass-merchandise chains set nearly uniform prices across their stores.

infeasible. From this, we conclude that both the discrimination case and the no-discrimination case are relevant and worth studying. This paper focuses on the no-discrimination case. We refer the reader to Miklós-Thal and Shaffer (2021) for a thorough study of dual pricing in a setting in which retailers do discriminate across markets.

Related literature. The literature on clearinghouse models was pioneered by Varian (1980), Rosenthal (1980), and Stahl (1989, 1996). More recent contributions include Baye and Morgan (2001), Montez and Schutz (2021), Shelegia and Wilson (2021), and Armstrong and Vickers (2022). Janssen and Shelegia (2015) embed Stahl (1989)'s clearinghouse setup into a vertical-relations model with an upstream monopolist and study the consequences of consumers not observing vertical contracts. Garcia and Janssen (2018) study the upstream manufacturer's incentives to price-discriminate between retailers in Janssen and Shelegia (2015)'s framework; Janssen and Reshidi (2022, 2023) address a similar question but in a search model without shoppers. We contribute to this literature by studying the effects on industry performance of non-linear upstream contracts that can condition on the sales channel. We also perform novel comparative statics in the vertical-relations version of the Stahl (1989) model, showing that parameters such as search costs or the share of informed consumers have radically different welfare effects once upstream contracts are endogenized.

Our work is also related to the literature on vertical restraints in e-commerce, where consumer search plays a central role.[6] Retail-price recommendations (RPRs) are among the most widely used vertical restraints in e-commerce and were shown empirically to influence prices and consumer search (De los Santos et al., 2018). Lubensky (2017) shows that a manufacturer can use RPRs as a cheap-talk signal about its marginal cost to influence the search behavior of consumers. In Buehler and Gärtner (2013), a manufacturer uses RPRs to convey information on cost or demand conditions to retailers. Janssen and Reshidi (2022) find that regulations requiring that at least some sales be made at RPRs can harm retailers and consumers. Asker and Bar-Isaac (2020) study the effects of minimum-advertised-price policies on search frictions, retail competition, and upstream profits. In a model in which consumers have context-dependent preferences, Helfrich and Herweg (2020) show that a manufacturer can benefit from banning online sales, as low online prices can negatively affect the perceived quality of the manufacturer's product.

The closest paper to ours is Miklós-Thal and Shaffer (2021), who also study input price discrimination across resale markets. Our paper differs from theirs in several dimensions. First, as discussed above, Miklós-Thal and Shaffer (2021) allow retailers to price discriminate, whereas we do not. Second, they consider an exogenously given system of market demand functions, whereas we explicitly model the underlying asymmetry in search frictions online

[6]For classic references on vertical restraints, see Mathewson and Winter (1984), Rey and Tirole (1986), Hart and Tirole (1990), and Winter (1993).

and offline, which generates different demand functions across the retail markets. Miklós-Thal and Shaffer (2021) show that, everything else equal, the manufacturer has an incentive to discriminate against those resale markets in which competition is more intense. In the context of online versus offline sales, this means setting a higher wholesale price for online units so as to bring online and offline retail prices closer together. This mechanism is absent in our model, as retailers are required to price uniformly across markets. Instead, the reason why discriminating against online sales is profitable for the manufacturer is that it eliminates the price dispersion that is inherent to clearinghouse models.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we characterize equilibria under uniform and dual pricing. In our baseline result in Section 3.1, we show that the manufacturer strictly benefits from using dual pricing. In Section 3.2, we discuss several extensions of the model, including a richer class of uniform-pricing contracts, the possibility of discriminating between retailers, and secret contracts. In Section 4, we examine the welfare effects of banning dual pricing. We conclude in Section 5.

2 The Model

A manufacturer, M, sells its product through two identical retailers, R_1 and R_2, who compete in prices. The manufacturer has a constant unit cost of production, $c > 0$. Retailing costs are linear and normalized to zero. The retailers have a hybrid business model, in that they operate both in an online market (O) and in a brick-and-mortar market (B). We assume throughout that each retailer R_i sets the same price p_i on- and offline.

The online market is populated by a mass $\lambda \in (0, 1)$ of buyers. These consumers, sometimes referred to as shoppers, purchase from the retailer offering the lowest price (and flip a coin in case of indifference). The remaining mass of consumers, $1 - \lambda$, is composed of offline buyers, sometimes referred to as captive consumers. These consumers search sequentially with perfect recall, as in Stahl (1989). Specifically, each offline consumer initially observes the price of one of the retailers, drawn at random with probability of 50/50. Then, the consumer can either purchase from this retailer, or pay a search cost of s, discover the price charged by the other retailer, and purchase from the retailer charging the lowest price. We assume throughout that retailers play an advertising role in the following sense: if the product is not available at retailer R_i (because that retailer has not signed a contract with the manufacturer), then the consumers that are captive to R_i are not aware of the product's existence and will therefore not attempt to purchase it from R_j. This assumption rules out the trivial outcome in which the manufacturer can induce the industry monopoly outcome by only dealing with one retailer.

Per-consumer demand is given by the continuous function $D(\cdot)$, assumed to be strictly decreasing and twice continuously differentiable up to its (potentially infinite) choke price $\check{p}$. Moreover, D satisfies Marshall's second law of demand (i.e., the absolute value of the price elasticity of demand is non-decreasing in price), and $\lim_{p\to\infty} r(p) = 0$, where $r(p) \equiv (p-c)D(p)$ is industry profit at price p. These assumptions imply that, for every unit cost $w < \check{p}$, the function $p \in (w, \check{p}) \mapsto \pi(p,w) = (p-w)D(p)$ is strictly quasi-concave and has a unique maximizer, which we denote $p^m(w)$. For what follows, it is useful to define $p_0 \equiv p^m(c)$, the price that maximizes industry profit, and $r_0 \equiv r(p_0)$, the industry monopoly profit.

The manufacturer offers public, non-discriminatory contracts to the retailers. A contract is a triple (w_o, w_b, T), where T denotes the lump-sum transfer made to the manufacturer, $w_o > 0$ the per-unit wholesale price for units sold online, and $w_b > 0$ the per-unit wholesale price for units sold offline. We shall distinguish two cases. Under uniform pricing, on- and offline wholesale prices must coincide, $w_o = w_b = w$, and the contract thus boils down to a standard two-part tariff, (w, T). Under dual pricing, there is no such restriction.

The game unfolds as follows. In stage 1, the manufacturer publicly announces its contract. In stage 2, retailers simultaneously decide whether to accept the contract and their decisions become common knowledge. In stage 3, retailers that have accepted the contract choose their retail prices. In stage 4, consumers make search and purchase decisions as described above.[7] The equilibrium concept is perfect Bayesian equilibrium.[8] Moreover, we confine attention to equilibria in which the manufacturer does not mix,[9] retailers do not randomize their acceptance decisions,[10] and retailers behave symmetrically in stage 3, both on and off the equilibrium path.

Some of the results in Section 4 require stronger assumptions on the shape of demand. We say that the demand function D is ρ-linear if it takes the form[11]

$$D(p) = M\left[1 + \frac{1-\alpha}{\alpha}(a - bp)\right]^{\frac{\alpha}{1-\alpha}} \tag{1}$$

[7]Note that we are assuming that consumers observe the upstream contract to simplify the analysis. The case in which the upstream contract is unobservable to consumers gives rise to major technical complications. Janssen and Shelegia (2015) show that an equilibrium in which the manufacturer's choice of upstream contract is pure and the consumers' search strategy is a cutoff rule fails to exist for a wide range of parameters. It is unclear whether an equilibrium exists at all in this case, and, if so, how to characterize it.

[8]The reason for not using subgame-perfect equilibrium is that offline consumers' information sets at the beginning of stage 4 are not singletons. Note that, since consumers observe the manufacturer's contract and by virtue of the no-signaling-what-you-don't-know condition (Fudenberg and Tirole, 1991), a non-shopper that observes R_i's price must hold passive beliefs about R_j's price.

[9]Given our assumption that consumers observe the upstream contract, this assumption is generically without loss of generality, as the manufacturer's maximization problem will typically have a unique solution.

[10]It is clear that, for any contract, the acceptance subgame has at least one pure-strategy equilibrium. If T is so high that retailers would make losses if they both accepted the contract, but sufficiently low so that a retailer that accepts the contract makes positive profits conditional on the other retailer rejecting, then the acceptance subgame has the same structure as Chicken. That subgame therefore has two pure-strategy equilibria and one non-degenerate mixed-strategy equilibrium. We select one of the pure-strategy equilibria.

[11]The case of $\alpha = 1$ arises from taking the limit in equation (1), which gives $D(p) = M\,e^{a-bp}$.

for some parameters $\alpha > 0$, a, $b > 0$, and $M > 0$.[12] Note that linear demand and iso-elastic demand are nested as special cases. This family of demand functions was first introduced by Bulow and Pfleiderer (1983), who showed that the ρ-linearity of demand is equivalent to the monopoly cost pass-through being constant. Routine calculations show that, under this demand specification,

$$p^m(w) = \frac{\alpha + (1-\alpha)a}{b} + \alpha w,$$

so that monopoly pass-through, dp^m/dw, is indeed constant and equal to α.

3 Uniform vs. Dual Pricing

In this section, we show that dual pricing allows the manufacturer to fully exploit its monopoly power and induce the monopoly outcome, whereas uniform pricing does not. The main result is in Section 3.1. We provide extensions and discuss the robustness of this result in Section 3.2.

3.1 Baseline Result

Uniform pricing. We proceed by backward induction. Consider the subgame in which the manufacturer has offered uniform-pricing contract (w, T), which both retailers have accepted. Equilibrium behavior in the retail pricing game was studied in Varian (1980) and Stahl (1989). It is well known that, in equilibrium, retailers draw their prices from a continuous cumulative distribution function (CDF), F, with support $[\underline{p}, \overline{p}]$, and offline consumers never search on the equilibrium path. The retailers' indifference condition

$$\left[\lambda(1 - F(p)) + \frac{1-\lambda}{2}\right]\pi(p, w) = \frac{1-\lambda}{2}\pi(\overline{p}, w)$$

pins down the CDF as

$$F(p, \overline{p}, w) = 1 - \frac{1-\lambda}{2\lambda}\left(\frac{\pi(\overline{p}, w)}{\pi(p, w)} - 1\right) \tag{2}$$

for every $p \in [\underline{p}, \overline{p}]$. The lower bound of the support, $\underline{p}(\overline{p}, w)$, is the unique solution to $F(\underline{p}, \overline{p}, w) = 0$, which can be rewritten as:

$$(1-\lambda)\pi(\overline{p}, w) = (1+\lambda)\pi(\underline{p}, w). \tag{3}$$

[12]This parametrization of ρ-linear demand is due to Anderson and Renault (2003). Genesove and Mullin (1998) use a variant of this specification in their empirical investigation of oligopoly conduct in a homogeneous-products industry.

To determine the upper bound of the support, let

$$H(\overline{p}, w) \equiv \int_{\underline{p}(\overline{p},w)}^{\overline{p}} D(p)F(p, \overline{p}, w)dp \tag{4}$$

denote the expected gains from searching (gross of the search cost) when receiving price quote $\overline{p}$, and expecting the other firm to draw its price from $F(\cdot, \overline{p}, w)$. To ensure that offline consumers do not search on path and deter retailers from pricing above $\overline{p}$, it must be that either (i) $\overline{p} = p^m(w)$ and $H(\overline{p}, w) \leq s$, or (ii) $\overline{p} < p^m(w)$ and $H(\overline{p}, w) = s$. It is easily checked that $\lim_{\overline{p}\downarrow w} H(\overline{p}, w) = 0$ and $H(\cdot, w)$ is continuous on $(w, \overline{p}]$. Moreover, by Lemma A.1.1 in Appendix A.1, Marshall's second law of demand implies that $H(\cdot, w)$ is strictly increasing on the interval $(w, p^m(w))$. It follows that there exists a unique $\overline{p}$ such that condition (i) or (ii) holds. Hence, the retail equilibrium is unique; we denote the CDF of prices by $F(\cdot, w)$ and its support by $[\underline{p}(w), \overline{p}(w)]$.

Moving backward, it is clear that in any equilibrium in which both retailers are active, the fixed part of the tariff must fully extract retailers' profits, i.e., $T = T(w) \equiv \frac{1-\lambda}{2}\pi(\overline{p}, w)$. The manufacturer then earns an expected profit of

$$\Pi(w) = \int_{\underline{p}(w)}^{\overline{p}(w)} r(p)dG(p, w), \tag{5}$$

where

$$G(p, w) \equiv (1 - \lambda)F(p, w) + \lambda \left[1 - (1 - F(p, w))^2\right]$$

is the CDF of prices paid by consumers. To understand the formula for G, note that a share $1 - \lambda$ of consumers receive one draw from F, whereas the remaining share receives two draws and picks the lowest.

Let us now show that the maximization problem

$$\max_{w>0} \Pi(w) \tag{6}$$

has a solution. Standard arguments imply that the objective function is continuous. We show in Appendix A.1 that $\Pi(\cdot)$ is strictly increasing on $(0, c]$ and strictly decreasing on $[p_0, \check{p})$ (see Lemma A.1.3). It follows that maximization problem (6) is equivalent to maximizing the continuous function $\Pi(\cdot)$ over the compact set $[c, p_0]$. Hence, a solution exists.

Alternatively, the manufacturer could choose to offer a contract with a fixed fee so high that only one retailer would be willing to accept it. In that case, the manufacturer has no incentives to introduce double marginalization: it sets its variable part equal to c, and the active retailer prices at p_0 and supplies the online consumers as well as its captive consumers. The manufacturer can then adjust its fixed fee to extract retail profits, which results in the optimal tariff $\left(c, \frac{1+\lambda}{2}r_0\right)$ and a profit of $\frac{1+\lambda}{2}r_0$ for the manufacturer. Note that even though the manufacturer induces a downstream price of p_0, it does not earn the industry monopoly profit, as the consumers that are captive to the excluded retailer are not served.

Taking stock, we have that, if $\max_w \Pi(w) > \frac{1+\lambda}{2}r_0$, then in equilibrium the manufacturer offers $(w^*, T(w^*))$, where w^* is any solution to maximization problem (6), both retailers accept, and retailers mix according to $F(\cdot, w^*)$. If the inequality is reversed, then in equilibrium the manufacturer offers $\left(c, \frac{1+\lambda}{2}r_0\right)$, and only one retailer accepts and prices at p_0. Finally, in the knife-edged case where $\max_w \Pi(w) = \frac{1+\lambda}{2}r_0$, there exist equilibria with one active retailer and equilibria with two active retailers, as characterized above. This concludes the equilibrium characterization.

Observe that, in any equilibrium, either both retailers accept and retail prices are drawn from a continuous distribution, or only one retailer accepts and a share $(1-\lambda)/2$ of consumers are not served. This means that the industry monopoly outcome, in which *all* consumers purchase at p_0 with probability one, never arises, regardless of which equilibrium is selected.

We summarize these insights in the following proposition:

Proposition 1. *An equilibrium exists under uniform pricing. Moreover, there exists no equilibrium in which the industry monopoly outcome arises.*

In standard models of vertical relations with smooth product differentiation and no information frictions on the consumer side, the retail equilibrium is always pure. This implies that a simple two-part tariff is always sufficient to soften downstream competition and induce the industry monopoly outcome (see, e.g., Mathewson and Winter, 1984). By contrast, in our model two-part tariff contracts always induce wasteful mixing (unless one of the retailers is excluded), and so the monopoly outcome cannot be achieved.

Dual pricing. Consider now the dual-pricing contract $(w_o, w_b, T) = \left(p_0, c, \frac{1-\lambda}{2}r_0\right)$. Let us argue that there exists an equilibrium in the continuation subgame in which both retailers accept this contract and price at p_0. Retailer R_i does not have an incentive to deviate upward, as that firm would then lose the online consumers and charge a sub-optimal price on its offline consumers. Similarly, a downward deviation would involve charging a sub-optimal price on offline consumers and receiving a negative margin on online consumers. Hence, conditional on both retailers having accepted the contract, pricing at p_0 is indeed a Nash equilibrium. As the fixed fee was set to fully extract retail profits, it follows that this dual pricing contract generates the industry monopoly outcome and allows the manufacturer to obtain industry monopoly profits. We go one step further and show that the industry monopoly outcome with full surplus extraction arises in any equilibrium under dual pricing:

Proposition 2. *There is an equilibrium under dual pricing. In any equilibrium, the manufacturer sets $w_o = p_0$ and $w_b \leq c$, both retailers are active and price at p_0 with probability 1, and the manufacturer makes a profit of r_0. This outcome can be induced by the dual-pricing contract $\left(p_0, c, \frac{1-\lambda}{2} r_0\right)$.*[13]

Proof. The proof involves constructing a continuation equilibrium in the subgame following any dual pricing contract. Interestingly, we find subgames in which firms mix in equilibrium and offline consumers occasionally search on the equilibrium path, which cannot arise in the standard Stahl (1989) model. Moreover, conditional on both retailers accepting contract $\left(p_0, c, \frac{1-\lambda}{2} r_0\right)$, we show that the unique equilibrium of the retail competition subgame involves both retailers pricing at p_0. See Appendix C for details. □

Combining Propositions 1 and 2, we see that dual pricing is an essential tool, as it allows the manufacturer to fully exploit its monopoly power and induce the monopoly outcome. The manufacturer does so by discriminating against online sales. This eliminates retailers' incentives to undercut to corner the online market, thus giving rise to a pure-strategy equilibrium.

Another less subtle (but, from the point of view of competition law, certainly riskier) way of inducing the monopoly outcome would be to offer a two-part tariff with a variable part equal to c and prohibit one of the retailers from selling online. An even less subtle approach would be an outright ban on online sales—the manufacturer can then obtain the industry monopoly profit, under the assumption that online consumers would be willing to purchase from their local stores. What all these approaches have in common is that they involve explicitly discriminating against online sales.

3.2 Extensions and Robustness

We begin this subsection by discussing the robustness of our results to various alterations of the oligopoly and search model. Next, we undertake two more involved extensions. In the first extension, we explore whether a richer class of uniform-pricing contracts could allow the manufacturer to induce the industry monopoly outcome. In the second, we study a version of the model in which retailers do not observe each other's contracts.

Some robustness considerations. It is straightforward to show that Propositions 1 and 2 are robust to various alterations of the retail oligopoly and search model, such as having N retailers instead of 2, endogenizing the share of online consumers (as in Varian, 1980; Baye and Morgan, 2001), or allowing offline consumers to have heterogeneous search costs (as

[13] In Appendix C, we show that a dual pricing contract (w_o, w_b, T) induces this outcome if and only if $w_o = p_0$, $w_b \in \left[\frac{(1+\lambda)c - 2\lambda p_0}{1-\lambda}, c\right]$, and $T = \frac{1-\lambda}{2}\pi(p_0, w_b)$.

in Stahl, 1996). In all these alterations, the monopoly outcome remains out of reach under uniform pricing, as retailers continue to randomize their prices (or the manufacturer excludes some of them). By contrast, a well-chosen dual-pricing contract, with $w_o = p_0$ and $w_b = c$, still gives rise to the monopoly outcome. Similar insights obtain if we relax the assumption that consumers observe the manufacturer's contract: if an equilibrium exists under uniform pricing, then it must give rise to price dispersion (or exclusion of one of the retailers), whereas the monopoly outcome arises under dual pricing.[14] Our results also extend to the case with asymmetric shares μ_i of offline consumers, if the manufacturer can charge different fixed fees to the retailers.[15] Again, either wasteful price dispersion or exclusion arises under uniform pricing, whereas the manufacturer obtains the industry monopoly profit with the dual-pricing contracts $(p_o, c, \mu_i r_0)$ for $i = 1, 2$.

Our results also continue to hold if online and offline consumers have different demand functions, D_o and D_b, under some conditions. Suppose for instance that D_o and D_b satisfy the assumptions made above and that online consumers are less price-elastic than offline consumers, i.e., $D_o'(p)/D_o(p) \geq D_b'(p)/D_b(p)$ for every p. Then, Corollary 1 in Armstrong and Vickers (2023) implies that a vertically integrated monopolist controlling the prices of both retailers would find it optimal to set uniform prices, i.e., $p_1 = p_2$. Let p^* be such an industry-profit-maximizing uniform price. The manufacturer can induce the outcome in which both retailers price at p^* by offering a dual-pricing contract with $w_o = p^*$ and w_b such that $p^* \in \arg\max_p (p - w_b) D_b(p)$. This outcome cannot be achieved with a uniform-pricing contract for the same reasons as in the baseline model.

Proposition 2 does, however, rely on the assumption that search frictions are absent online. If, for instance, a fraction of the online consumers faced strictly positive search costs, then dual pricing would no longer give rise to a pure-strategy equilibrium, as a retailer would always have an incentive to either cut its price to attract the online "shoppers" or raise its price to exploit the online "captives". That being said, the manufacturer would still find it profitable to set a higher variable part on online units so as to mitigate these incentives and reduce wasteful price dispersion. In this sense, the main message of Propositions 1 and 2 is robust. The fact that the industry monopoly outcome arises under dual pricing in our baseline model will be particularly useful for our welfare analysis in Section 4, as this gives us a clean outcome against which to compare the equilibrium under uniform pricing.

Richer class of uniform-pricing contracts. According to Proposition 1, no uniform two-part tariff contract can induce the industry monopoly outcome. This raises the question of whether the industry monopoly outcome could be induced by a more flexible uniform-

[14]See Janssen and Shelegia (2015) for conditions under which an equilibrium exists under uniform pricing when consumers do not observe the wholesale contract.

[15]We study discriminatory contracts more thoroughly below.

pricing contract, i.e., by some arbitrary mapping $\mathcal{T}(\cdot)$, which, to any quantity q ordered by the retailer, associates the payment $\mathcal{T}(q)$ to be made to the manufacturer. In the baseline model of Section 2, it is easy to see that the answer is positive. Consider indeed the quantity-forcing contract

$$\mathcal{T}(q) = \begin{cases} \frac{1}{2}p_0 D(p_0) & \text{if } q = \frac{1}{2}D(p_0), \\ \overline{T} & \text{otherwise,} \end{cases}$$

where $\overline{T}$ is a sufficiently large number. It is clear that, following such a contract, there exists an equilibrium in which both retailers accept and price at p_0, so that the monopoly outcome arises.

One may contend that this contract is somewhat special, as it is neither monotonic nor continuous. Yet, consider the following continuous and monotonic contract:

$$\mathcal{T}(q) = \begin{cases} \frac{1}{2}p_0 D(p_0) & \text{if } q \leq \frac{1}{2}D(p_0), \\ p_0 q & \text{if } q > \frac{1}{2}D(p_0), \end{cases}$$

which resembles a two-part tariff with a minimum purchase requirement. Starting from a situation in which both retailers accept the contract and price at p_0, a retailer deviating upward would lower its revenue without affecting the payment to be made to the manufacturer, whereas a retailer deviating downward would end up pricing below its average cost. Hence, this continuous and monotonic contract also induces the monopoly outcome.

Thus, the ability to discriminate against the online market brought about by dual pricing no longer seems that essential if the manufacturer can use sufficiently rich uniform-pricing contracts. Let us now show that dual pricing becomes essential again in a slightly richer version of the model of Section 2 with stochastic demand. Specifically, let us assume that demand per consumer at price p is given by $M \times D(p)$, where M is a non-contractible random variable, which is drawn and becomes common knowledge at the beginning of stage 3, before retailers set their prices. We assume that M is supported on the interval $[\underline{m}, \overline{m}]$ with $0 \leq \underline{m} < \overline{m} \leq \infty$. The rest of the model is as in Section 2, except that we allow the manufacturer to offer any continuous uniform-pricing contract.

Proposition 3. *Consider the model with demand uncertainty, and suppose that the support of M satisfies*

$$\overline{m} \geq \frac{1+\lambda}{1-\lambda}\underline{m}.$$

Then, there exists no continuous uniform-pricing contract that induces the industry monopoly outcome in almost every state.

By contrast, regardless of the support of M, the dual-pricing contract $\left(p_0, c, \frac{1-\lambda}{2}\mathbb{E}(M)r_0\right)$ induces the monopoly outcome in every state and allows the manufacturer to earn the expected *industry monopoly profit.*

Proof. See Appendix A.3. □

The proposition states that the manufacturer cannot induce the industry monopoly outcome with a flexible uniform-pricing contract if retail demand is sufficiently uncertain. Intuitively, for a flexible uniform-pricing contract to induce the monopoly outcome, it would have to satisfy the following properties. First, in high demand states, a retailer should not have an incentive to raise its price above p_0, thus losing the shoppers and ordering fewer units from the manufacturer. This implies that payments made to the manufacturer when ordering few units should be high enough. Second, retailers should make non-negative profits when pricing at p_0 in low demand states, implying that payments when ordering few units cannot be too high. We show in the proof that these conditions cannot hold simultaneously when demand uncertainty is sufficiently high. By contrast, the monopoly outcome does arise with a well chosen dual-pricing contract, as wholesale discrimination against the online market continues to suppress retailers' incentives to undercut.

Wholesale discrimination and secret contracts. We assumed in our baseline model that the manufacturer does not discriminate between retailers, which automatically implies that each retailer knows the terms at which the other retailer is purchasing. One justification for this assumption is that, in many jurisdictions, there are laws (such as the Robinson-Patman Act in the U.S. and Article 102 of the Treaty on the Functioning of the European Union) restricting a manufacturer's ability to price discriminate. Nevertheless, it seems important to study whether our results continue to hold if the manufacturer can discriminate.

Under the assumption of publicly observable contracts, it is clear that the manufacturer will remain unable to induce the monopoly outcome using discriminatory uniform-pricing two-part tariffs, as such contracts will induce either mixing by both retailers, or pure pricing with at least one retailer not setting p_0, or the exclusion of one retailer. By contrast, the non-discriminatory dual-pricing contract $\left(p_0, c, \frac{1-\lambda}{2} r_0\right)$ continues to induce the monopoly outcome.

Note however that, if the manufacturer is able to discriminate between its retailers, then there is no longer a compelling reason to assume that contracts are observable to rivals. Bilateral vertical contracts are typically private information to the contracting parties; and even though the manufacturer could in principle show its contract with retailer R_1 to retailer R_2, nothing would prevent it from secretly renegotiating that contract thereafter. In the following, we therefore solve a version of the model with secret contracts. That is, we now assume that retailer R_i never observes retailer R_j's contract; nor does it observe R_j's acceptance decision in stage 3. Let $C_i = (w_{i,o}, w_{i,b}, T_i)$ denote the contract offered to R_i under dual pricing (resp., $C_i = (w_i, T_i)$ under uniform pricing). For simplicity (and to avoid taking a stance on whether consumers observe wholesale contracts), we assume that the search cost parameter s is so high that offline consumers would never consider searching in stage 4.

As perfect Bayesian equilibria are notoriously hard to solve for when downstream competition is in price (Rey and Vergé, 2004), we use contract equilibrium in passive beliefs as our solution concept (Crémer and Riordan, 1987; Horn and Wolinsky, 1988; O'Brien and Shaffer, 1992; Rey and Vergé, 2020). A pair of contracts (C_1, C_2) and a strategy profile for the two retailers form a contract equilibrium in passive beliefs if: (i) contract C_i maximizes the manufacturer's profit holding fixed C_j ($i \neq j$ in $\{1, 2\}$); (ii) retailer R_i believes that R_j's contract is C_j, regardless of what contract M offers to R_i; (iii) the retailers' strategies are sequentially rational given their beliefs.[16] Moreover, for simplicity, we confine attention to symmetric equilibria, i.e., $C_1 = C_2$.

Let us first solve the model under uniform pricing. Consider an equilibrium candidate in which the manufacturer offers the two-part tariff (w, T) to both retailers, with $w > c$. By sequential rationality, on the candidate equilibrium path both retailers mix according to the CDF $F(\cdot)$ of equation (2) (where we have dropped argument w to ease notation). As the manufacturer's profits must be maximized, it must be that $T = \frac{1-\lambda}{2}\pi(\overline{p}, w)$. The manufacturer earns an expected profit of

$$\Pi^* = 2T + (1-\lambda)(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p) + \lambda(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF_{\min}(p), \qquad (7)$$

where $F_{\min}$ denotes the CDF of the minimum of (p_1, p_2).

Suppose now that the manufacturer deviates by offering $(w - \varepsilon, T)$ to retailer R_1, for some small $\varepsilon > 0$. Then, R_1 accepts this new contract and (correctly) believes that R_2 will continue to draw its price from F. Before the deviation, firm R_1 was indifferent between all the prices in $[\underline{p}, \overline{p}]$. After the deviation, R_1's marginal cost is strictly lower, and R_1's profit is therefore strictly decreasing on $[\underline{p}, \overline{p}]$.[17] It follows that, after the deviation, R_1 prices at $\underline{p}$ with probability 1. Taking ε to zero, the manufacturer's expected deviation profits can be

[16] Any perfect Bayesian equilibrium in passive beliefs must be a contract equilibrium in passive beliefs, but the converse is not necessarily true, as condition (i) in the definition does not consider deviations in which the manufacturer changes *both* contracts. Note that, to fix ideas, we are using the version of contract equilibrium in which the manufacturer has all the bargaining power.

[17] To see this, let $\tilde{D}(p)$ be the expected demand that R_1 faces when R_2 is mixing according to F:

$$\tilde{D}(p) = \left(\frac{1-\lambda}{2} + \lambda(1 - F(p))\right) D(p).$$

By definition of F, the function $p \mapsto (p - w)\tilde{D}(p)$ is strictly increasing up to $\underline{p}$, constant on $[\underline{p}, \overline{p}]$, and strictly decreasing on $[\overline{p}, \check{p})$. This implies that $(p - w + \varepsilon)\tilde{D}(p) = (p - w)\tilde{D}(p) + \varepsilon\tilde{D}(p)$ is strictly increasing up to $\underline{p}$ (for ε small enough) and strictly decreasing on $[\underline{p}, \check{p})$.

made arbitrarily close to:

$$2T + \underbrace{\frac{1-\lambda}{2}(w-c)D(\underline{p}) + \lambda(w-c)D(\underline{p})}_{\text{variable profits on } R_1} + \underbrace{\frac{1-\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p)}_{\text{variable profits on } R_2}.$$

The change in the manufacturer's profit is given by

$$\frac{1-\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} (D(\underline{p}) - D(p))dF(p) + \lambda(w-c)\int_{\underline{p}}^{\overline{p}} (D(\underline{p}) - D(p))dF_{\min}(p),$$

which is strictly positive since D is strictly decreasing. Hence, there is no equilibrium in which the variable part of the tariff is strictly above cost.

Next, consider an equilibrium candidate in which the manufacturer offers the two-part tariff (c, T) to the retailers. Again, we must have that $T = \frac{1-\lambda}{2}\pi(\overline{p}, c) = \frac{1-\lambda}{2}r_0$. On the candidate equilibrium path, the manufacturer makes an expected profit of $2T$. Suppose the manufacturer deviates and offers (w', T') to retailer R_i. If $w' < c$, then the argument used in the previous paragraph implies that R_i responds by pricing at $p' = \min(\underline{p}(c), p^m(w'))$ with probability 1. The retailer therefore supplies the online market and makes an operating profit of $\frac{1+\lambda}{2}\pi(p', w')$, which the manufacturer extracts with its fixed fee. Hence, the manufacturer's expected deviation profit is

$$T + \frac{1+\lambda}{2}r(p') \leq T + \frac{1+\lambda}{2}r(\underline{p}(c)) = T + \frac{1-\lambda}{2}r_0 = 2T,$$

where we have used equation (3). The deviation is therefore not profitable. If instead $w' > c$, then R_i optimally sets $p' = p^m(w') > p_0$ and makes an operating profit of $\frac{1-\lambda}{2}\pi(p', w')$, which the manufacturer extracts. The manufacturer earns

$$T + \frac{1-\lambda}{2}r(p') < T + \frac{1-\lambda}{2}r_0 = 2T,$$

and so the deviation is not profitable.

Summing up:

Proposition 4. *Consider the model with secret contracts. Under uniform pricing, there is a unique symmetric equilibrium. The manufacturer offers the contract* $\left(c, \frac{1-\lambda}{2}r_0\right)$ *to both retailers. Both retailers accept and draw their prices from* $F(\cdot, c)$.

Proof. See Appendix A.3. □

Thus, under secret contracts and uniform pricing, the manufacturer loses its ability to soften downstream competition by choosing a high variable part. As the proposition shows, the only variable part that can be sustained in equilibrium is $w = c$. Hence, as far as the retail competition outcome is concerned, it is as if the upstream market were perfectly competitive. The intuition is the same as in Hart and Tirole (1990) and the literature that followed: starting from a contract (w, T) with $w > c$, manufacturer M and retailer R_i have a joint incentive to free-ride on retailer R_j's margin by setting a lower p_i.

Next, we turn our attention to dual pricing. Consider an equilibrium candidate in which the manufacturer offers the dual-pricing contract $\left(p_0, c, \frac{1-\lambda}{2} r_0\right)$, i.e., the same contract as in Proposition 2. On the equilibrium path, it is sequentially rational for both retailers to accept this contract and price at p_0, as argued in Section 3.1. The manufacturer then earns r_0.

Now, suppose that the manufacturer deviates and offers some alternative contract $C' = (w'_o, w'_b, T')$ to retailer R_i. As R_j does not observe this deviation, it continues to price at p_0 with probability 1. This implies that, regardless of the deviation contract C' and regardless of how R_i behaves after having been offered that contract, R_j earns zero profit. The reason is that R_j makes zero profit on shoppers and receives the monopoly profit on its captives, which is transferred to the manufacturer. Hence, for any sequentially rational decision made by R_i, industry profit must be equal to the sum of the manufacturer's and R_i's profits. Since R_i's profit must be non-negative by sequential rationality, it follows that the manufacturer's deviation profit is weakly less than industry profit, which is bounded above by r_0. The deviation is therefore unprofitable.

We thus have:

Proposition 5. *Consider the model with secret contracts. Under dual pricing, there exists an equilibrium in which the manufacturer offers contract* $\left(p_0, c, \frac{1-\lambda}{2} r_0\right)$ *and both retailers accept and price at* p_0*.*

As mentioned above, the reason why a uniform-pricing equilibrium with a positive variable part can not be sustained is that the manufacturer and one of the retailers would have an incentive to agree on more favorable terms so as to free-ride on the other retailer's margin. Under the dual-pricing contract of Proposition 5, such incentives are absent since there is no margin that can be free-ridden on in the competitive (online) segment.

Comparing Propositions 1–2 and Propositions 4–5, there is a sense in which being able to use dual pricing becomes even more crucial for the manufacturer when contracts are secret. That is, in addition to eliminating wasteful mixing as in the case of public contracts, under secret contracts dual pricing also solves the supplier opportunism problem.

4 The Welfare Effects of Dual Pricing

In this section, we study how a ban on dual pricing affects expected consumer surplus and expected aggregate surplus. We are thus interested in

$$\Delta CS = CS^{UP} - CS^{DP} \quad \text{and} \quad \Delta AS = AS^{UP} - AS^{DP},$$

where the superscript UP (resp. DP) stands for uniform pricing (resp. dual pricing). Some of our results rely on approximating equilibrium behavior under uniform pricing in the neighborhood of $s = 0$, $\lambda = 0$, or $\lambda = 1$.[18] As we shall see below, monopoly pass-through and its behavior will play a key role in these approximations. We thus introduce the following notation:

$$\alpha = \left.\frac{dp^m(w)}{dw}\right|_{w=c} \quad \text{and} \quad \beta = \left.\frac{d^2p^m(w)}{dw^2}\right|_{w=c}.$$

That is, α is the monopoly pass-through (of a cost increase, evaluated at the industry marginal cost) and β is the derivative of pass-through with respect to cost. Recall from our discussion at the end of Section 2 that pass-through is constant under ρ-linear demand, i.e., $\beta = 0$.

We begin by putting on record some key properties of the equilibrium under uniform pricing:

Proposition 6. *In any equilibrium under uniform pricing, the manufacturer deals with both retailers if (at least) one of the following conditions holds:*

(i) $\lambda \leq 1/3$*;*

(ii) λ *is sufficiently close to 1 and* $s \neq \int_{p_0}^{p^m(p_0)} D(p)dp$*;*[19]

(iii) s *is sufficiently close to 0;*

(iv) s *is sufficiently high and demand is* ρ*-linear.*

Moreover, whenever an equilibrium under uniform pricing involves dealing with both retailers, the wholesale price w^* *satisfies* $w^* \in (c, p_0)$ *and the support of the equilibrium distribution of retail prices,* $[\underline{p}, \overline{p}]$*, satisfies* $\underline{p} < p_0 < \overline{p}$*.*

Proof. For the first part of the proposition, see Lemma B.2 in Appendix B for condition (i), Lemma D.5.13 in Appendix D.5.6 for condition (ii), Lemma D.2.1 in Appendix D.2.1 for

[18] This will require strengthening the differentiability properties of the demand function D on $(0, \check{p})$. Specifically, we need D to be $\mathcal{C}^4$ for Propositions 6, 8, 10, and 11, and $\mathcal{C}^5$ for Proposition 9. We refrain from mentioning these technical assumptions in the propositions to avoid overloading their statements.

[19] The reason why we need to exclude the knife-edged case $s = \int_{p_0}^{p^m(p_0)} D(p)dp$ is discussed in footnote 23 below.

condition (iii), and Lemma B.1 in Appendix B for condition (iv). For the second part, see Lemma A.1.3 in Appendix A.1. □

That conditions (i) and (iii) are sufficient for the manufacturer to prefer dealing with both retailers is quite intuitive. If λ is small, excluding one of the retailers means giving up on a large chunk of the retail market, which cannot be optimal. If s is small, then the retail equilibrium is close to the Bertrand outcome, implying that the manufacturer can make a profit close to r_0 by setting w equal to p_0.

The sufficiency of condition (ii) is far less obvious: as λ approaches 1, double marginalization and price dispersion vanish, and so the cost associated with dealing with both retailers disappears; at the same time, the benefit from using both retailers also becomes negligible, as offline consumers cease to exist. We prove the result by obtaining a Taylor approximation of equilibrium behavior under the optimal w in the neighborhood of $\lambda = 1$. The sufficiency of condition (iv) is also non-trivial. We establish it by exploiting the mean-value theorem to obtain a lower bound on the profit from dealing with both retailers under ρ-linear demand and high search costs, and showing that that lower bound is greater than $(1 + \lambda)/2r_0$, the profit from excluding one retailer.

To see the intuition for the second part of the proposition, suppose to the contrary that $\underline{p} \geq p_0$ in equilibrium. Then, the retailers are systematically pricing above the industry monopoly level. The manufacturer can then gain by lowering its variable part to induce a first-order stochastic dominance shift towards lower prices, thus mitigating double marginalization. If instead $\overline{p} \leq p_0$, then the retailers are systematically pricing below the industry monopoly level, and the manufacturer should increase w to induce a first-order stochastic dominance shift towards higher prices, thus softening downstream competition. The optimal w solves the trade-off between double marginalization and excessive downstream competition, which results in $c < w^* < p_0$ and $\underline{p} < p_0 < \overline{p}$.

An implication of the second part of Proposition 6 is that the welfare effects of a ban on dual pricing are generally ambiguous whenever the manufacturer chooses to deal with both retailers under uniform pricing. Under dual pricing, we know from Proposition 2 that the monopoly outcome arises in equilibrium. This means that, under dual pricing retailers price at p_0 with probability 1, whereas under uniform pricing retailers randomize between pricing above and below p_0. Hence, a ban on dual pricing, despite preventing the manufacturer from implementing the industry monopoly outcome, may or may not raise consumer surplus and aggregate surplus. In the following, we show how the welfare effects of a ban on dual pricing depend on the shape of demand, the relative size of the online segment, and the search cost faced by offline consumers.

The case of high search costs. This first set of results does not rely on approximations but requires assuming that demand is ρ-linear. Suppose that the search cost is sufficiently high, so that the offline consumers' threat of searching does not constrain the retailers' pricing behavior, i.e., $\overline{p}(w) = p^m(w)$ for every w. That is, suppose that $s \geq \overline{s} \equiv \max_{w \in [c, p_0]} H(p^m(w), w)$. (Recall from Proposition 6 that any optimal wholesale price must lie in $[c, p_0]$.)

We now argue that a ban on dual pricing strictly lowers consumer surplus and aggregate surplus. Under dual pricing, the monopoly outcome arises. It is well known that, in the monopoly outcome under ρ-linear demand, the ratio of consumer surplus to producer surplus is equal to α, the monopoly pass-through (see, e.g., Anderson and Renault, 2003). That is, $CS^{DP} = \alpha \Pi^{DP}$, where Π^{DP} is industry profit under dual pricing. Recall from Proposition 6 that the manufacturer deals with both retailers under uniform pricing. In Appendix A.2, we show that at any optimal variable part under uniform pricing, the ratio of consumer surplus to producer surplus is also equal to α: $CS^{UP} = \alpha \Pi^{UP}$. It follows that

$$CS^{UP} = \alpha \Pi^{UP} < \alpha \Pi^{DP} = CS^{DP},$$

where the inequality follows by Propositions 1 and 2. Hence, ΔCS and ΔAS are both strictly negative.

Summing up:

Proposition 7. *Assume that demand is ρ-linear and $s \geq \overline{s}$. Then, a ban on dual pricing strictly reduces consumer surplus, industry profit, and aggregate surplus.*

Proof. See Appendix A.2. □

The case of low search costs. When the search cost s is equal to zero, the retail pricing game reduces to a simple homogeneous-goods Bertrand model. Hence, for any uniform tariff $(w, 0)$, both retailers price at w with probability 1, i.e., $\underline{p} = \overline{p} = w$. The manufacturer therefore finds it optimal to set $w = p_0$, thereby inducing the industry monopoly outcome. Suppose now that s is strictly positive, but small. By Proposition 6, the manufacturer finds it profitable to deal with both retailers under uniform pricing. For every such small s, let $w(s)$ be an equilibrium variable part; denote also by $\underline{p}(s)$ and $\overline{p}(s)$ the lower and upper bounds of the support of the retailers' equilibrium mixed strategy given s and $w(s)$. We show in Appendix D.2.1 that, as s tends to zero, $w(s)$, $\underline{p}(s)$, and $\overline{p}(s)$ all tend to p_0, i.e., equilibrium behavior does converge to the equilibrium of the limiting game without search costs (see Lemma D.2.1).

The following proposition approximates the welfare effects of a ban on dual pricing in the neighborhood of $s = 0$:

Proposition 8. *In the neighborhood of $s = 0$, we have:*[20]

$$\Delta CS(s) = K\left[\alpha(2-\alpha) - \beta(p_0 - c)\right]s^2 + o(s^2),$$
$$\Delta AS(s) = K\left[\alpha(1-\alpha) - \beta(p_0 - c)\right]s^2 + o(s^2),$$

where
$$K = \frac{(1-\lambda)(\lambda-\psi)}{2\alpha^2 r_0 \psi^2} > 0 \quad \textit{and} \quad \psi = 1 - \frac{1-\lambda}{2\lambda}\log\frac{1+\lambda}{1-\lambda}.$$

Therefore, when the search cost is small, a ban on dual pricing raises consumer surplus (resp. aggregate surplus) if $\alpha(2-\alpha) - \beta(p_0 - c) > 0$ (resp. $\alpha(1-\alpha) - \beta(p_0 - c) > 0$) and reduces it if the inequality is reversed.

Proof. The proof is lengthy and non-trivial. We provide here a brief sketch and refer the reader to Appendix D.2 for details.

Integrating by parts in the definition of ΔCS and ΔAS, we obtain

$$\Delta CS = \int_{\underline{p}}^{\overline{p}} D(p)G(p,\overline{p},w)dp - \int_{p_0}^{\overline{p}} D(p)dp \tag{8}$$

$$\text{and } \Delta AS = \Delta CS + r(\overline{p}) - r(p_0) - \int_{\underline{p}}^{\overline{p}} r'(p)G(p,\overline{p},w)dp, \tag{9}$$

which we wish to approximate. Recall that G is the CDF of prices paid by consumers, i.e.,

$$G(p,\overline{p},w) = (1-\lambda)F(p,\overline{p},w) + \lambda\left[1 - (1-F(p,\overline{p},w))^2\right],$$

where F is given by
$$F(p,\overline{p},w) = 1 - \frac{1-\lambda}{2\lambda}\left(\frac{\pi(\overline{p},w)}{\pi(p,w)} - 1\right).$$

The variables $\underline{p}$, $\overline{p}$, and w are jointly pinned down by the fact that a retailer should be indifferent between pricing at $\underline{p}$ and pricing at $\overline{p}$, the fact that a non-shopper should be indifferent between searching and not searching when receiving a price quote of $\overline{p}$, and the manufacturer's first-order condition:

$$(1-\lambda)\pi(\overline{p},w) = (1+\lambda)\pi(\underline{p},w), \tag{10}$$

$$\int_{\underline{p}}^{\overline{p}} D(p)F(p,\overline{p},w)dp = s, \tag{11}$$

$$\text{and } \int_{\underline{p}}^{\overline{p}} r'(p)\left[\frac{\partial G(p,\overline{p},w)}{\partial w} + \frac{\partial \overline{p}}{\partial w}\frac{\partial G(p,\overline{p},w)}{\partial \overline{p}}\right]dp = 0, \tag{12}$$

[20] $o(\cdot)$ is Landau's little-o notation: $f(x) = o(g(x))$ in the neighborhood of $x = x_0$ if $f(x)/g(x) \underset{x \to x_0}{\longrightarrow} 0$.

where, on the last line, $\partial\overline{p}/\partial w$ corresponds to the partial derivative of the function $\overline{p}$ implicitly defined by equations (10)–(11).[21]

We begin by exploiting equation (10) to show that $\underline{p} - w \sim \frac{1-\lambda}{1+\lambda}(\overline{p} - w)$ when s is close to zero. Next, we apply the implicit function theorem to condition (11) and take limits to obtain that $\partial\overline{p}/\partial w \underset{s\to 0}{\longrightarrow} 1$. Combining this with a first-order Taylor approximation around p_0 of the integrand in condition (12), we then show that $p_0 - w \sim (1-\lambda)(\overline{p} - w)$ when s is close to zero. It follows that $\overline{p} - w$, $\underline{p} - w$, and $p_0 - w$ are all of the same order in the neighborhood of $s = 0$. This allows us to further apply the Taylor theorem to conditions (10), (11), and (12) to obtain approximations of $\underline{p} - w$ and $p_0 - w$ at the second order in $\overline{p} - w$:

$$\underline{p} - w = \frac{1-\lambda}{1+\lambda}\left[\overline{p} - w - \frac{2\lambda}{1+\lambda}\frac{1}{p_0 - c}(\overline{p} - w)^2\right] + o((\overline{p} - w)^2)$$

$$p_0 - w = (1-\lambda)\left[\overline{p} - w - \left((\lambda - \psi)\left(\frac{\beta}{2\alpha^2} + \frac{2\alpha - 1}{\alpha(p_0 - c)}\right) + \frac{\lambda}{p_0 - c}\right)(\overline{p} - w)^2\right] + o((\overline{p} - w)^2).$$

Further exploiting condition (11), we also show that s and $\overline{p} - w$ are of the same order when s is close to zero and derive the approximation

$$\overline{p} - w = \frac{1}{\psi D_0}s + o(s). \tag{13}$$

The final step involves approximating the integrands in equations (8) and (9) in the neighborhood of $p = p_0$ and inserting the approximations of $\underline{p} - w$ and $p_0 - w$ to obtain approximations of ΔCS and ΔAS at the second order in $\overline{p} - w$. We can then use equation (13) to obtain approximations of these welfare measures with s as the right-hand side variable. □

According to the proposition, when offline consumers have sufficiently low search costs, the consumer surplus effect of a ban on dual pricing has the same sign as $\alpha(2-\alpha) - \beta(p_0 - c)$. Thus, a sufficient condition for the ban to benefit consumers is that monopoly pass-through is less than 2 and non-increasing in cost.[22] The condition under which such a ban raises aggregate surplus is naturally more stringent, as we know from Propositions 1 and 2 that industry profit is unambiguously higher under dual pricing. Specifically, a ban on dual pricing raises aggregate surplus provided monopoly pass-through is less than 1 and non-increasing in costs.

[21] More precisely, equations (10)–(11) jointly define two functions, $\overline{p}(w, s)$ and $\underline{p}(w, s)$.

[22] Although there is a large empirical literature analyzing the pass-through of costs, estimates of the *monopoly* pass-through are harder to find. Genakos and Pagliero (2022) study the price effects of excise duties in isolated gasoline markets and find a pass-through of 0.43 in monopoly markets. In their study of the portland cement industry, Miller et al. (2017) use an empirical specification that allows pass-through to depend on the intensity of competition. Their estimate of the monopoly pass-through (which corresponds to coefficient α_0 in their specification) ranges from 2.05 to 2.76. Ganapati et al. (2020) estimate pass-through rates for six imperfectly competitive U.S. industries and find rates ranging from 0.27 to 1.84.

In the special case of ρ-linear demand, monopoly pass-through is constant and we thus have $\beta = 0$. It follows that, if search costs are small and demand is ρ-linear, then a ban on dual pricing raises consumer surplus if monopoly pass-through is less than 2 and lowers it if monopoly pass-through is greater than 2. The pass-through cutoff for aggregate surplus is 1.

The case of a small online market. When the share of online consumers, λ, is equal to zero, retailers are no longer in competition with each other. The manufacturer therefore finds it optimal to set $w = c$ to eliminate double marginalization. The retailers respond by pricing at p_0, i.e., $\underline{p} = \overline{p} = p_0$. Suppose now that λ is strictly positive but small. By Proposition 6, the manufacturer optimally chooses to deal with both retailers in equilibrium. For every such small λ, let $w(\lambda)$ be an equilibrium variable part; denote also by $\overline{p}(\lambda)$ and $\underline{p}(\lambda)$ the upper and lower bounds of the support of the retailers' equilibrium mixed strategy given λ and $w(\lambda)$. We show in the appendix that, as λ tends to zero, $w(\lambda)$ tends to c, whereas $\underline{p}(\lambda)$ and $\overline{p}(\lambda)$ tend to p_0, i.e., equilibrium behavior converges to the equilibrium of the limiting game in which the online market does not exist.

The following proposition approximates the welfare effects of a ban on dual pricing in the neighborhood of $\lambda = 0$:

Proposition 9. *In the neighborhood of $\lambda = 0$, we have:*

$$\Delta CS(\lambda) = -\frac{1}{9} r_0 \alpha \lambda + o(\lambda).$$

Thus, when the share of online consumers is small, a ban on dual pricing lowers consumer surplus and aggregate surplus.

Proof. We follow similar steps as in the proof of Proposition 8. See Appendix D.3 for details. □

In contrast to Proposition 8, the sign of the welfare effect of a ban on dual pricing when the online market is small does not depend on how high or low the monopoly pass-through is. Such a ban lowers consumer surplus (and thus aggregate surplus) for any well-behaved demand function when the online market is small.

The case of a large online market. When the share of online consumers, λ, is equal to 1, retailers compete *à la* Bertrand for consumers. As there is no double marginalization, the manufacturer optimally sets $w = p_0$ and the retailers respond by pricing at p_0. By Proposition 6, when λ is close to but strictly less than 1, the manufacturer finds it optimal to deal with both retailers. We show in Appendix D.4 that the equilibrium variable part, $w(\lambda)$, and lower bound of the support, $\underline{p}(\lambda)$, both tend to p_0 as λ goes to 1, while the equilibrium

CDF of prices converges weakly to a unit mass on p_0. The upper bound of the support, $\overline{p}(\lambda)$ converges to $\tilde{p}(s) \equiv \min(p^m(p_0), \hat{p}(s))$, where $\hat{p}(s)$ is the solution of equation $\int_{p_0}^{\hat{p}} D(p)dp = s$.

Define $\hat{s} \equiv \int_{p_0}^{p^m(p_0)} D(p)dp$ and

$$\tilde{\mu}(s) \equiv \frac{(D'(\tilde{p}(s))(\tilde{p}(s) - p_0) + D(\tilde{p}(s)))D(p_0)}{D(\tilde{p}(s))\pi(\tilde{p}(s), p_0)} - \frac{1}{\tilde{p}(s) - p_0}.$$

The following proposition approximates the welfare effects of a ban on dual pricing in the neighborhood of $\lambda = 1$:

Proposition 10. *In the neighborhood of $\lambda = 1$, if $s \neq \hat{s}$, we have:*[23]

$$\begin{aligned}
\Delta CS(\lambda) = & \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2} \left[\alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)\right] (1-\lambda)^2 \left|\log(1-\lambda)\right| \\
& + o\left((1-\lambda)^2 \log(1-\lambda)\right), \\
\Delta AS(\lambda) = & \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2} \left[\alpha(1-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)\right] (1-\lambda)^2 \left|\log(1-\lambda)\right| \\
& + o((1-\lambda)^2 \log(1-\lambda)).
\end{aligned}$$

Therefore, when the share of online consumers is large, a ban on dual pricing raises consumer surplus (resp. aggregate surplus) if $\alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s) > 0$ (resp. $\alpha(1-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s) > 0$) and lowers it if the inequality is reversed.

The function $\tilde{\mu}$ is continuous and strictly negative. If monopoly pass-through is non-increasing in cost (i.e., $p^{m\prime\prime} \leq 0$), then $\tilde{\mu}$ is strictly decreasing on $(0, \hat{s})$ and constant thereafter.

Proof. We follow similar steps as in the proof of Proposition 8. See Appendices D.4 and D.5 for details. ☐

According to the proposition, when λ is large the consumer-surplus effect of a ban on dual pricing has the same sign as $\alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)$; and the aggregate-surplus effect of a ban has the same sign as $\alpha(1-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)$. If s is close to zero, then $\tilde{p}(s) \simeq p_0$, implying that $\tilde{\mu}(s) \simeq 0$. The sign of ΔCS is then determined by $\alpha(2-\alpha) - \beta(p_0 - c)$ while the sign of ΔAS is determined by $\alpha(1-\alpha) - \beta(p_0 - c)$, as in Proposition 8.

Under the (perhaps natural) assumption that pass-through is non-increasing in cost, $\tilde{\mu}$ is strictly decreasing on $(0, \hat{s})$ and constant thereafter. This implies that if a ban on dual

[23] The reason why we exclude the case $s = \hat{s}$ is that, when $s = \hat{s}$ and λ is close to 1, it is unclear whether the manufacturer chooses a w such that $\overline{p} < p^m(w)$ or $\overline{p} = p^m(w)$. This gives rise to major complications when approximating $\overline{p}$ in the neighborhood of $\lambda = 1$. Note that $\tilde{\mu}(\cdot)$ is continuous on the strictly positive domain, so we have every reason to expect our approximations to remain valid when $s = \hat{s}$.

pricing raises (resp. lowers) consumer surplus for some s_1, then such a ban raises (resp. lowers) consumer surplus for any $s_2 < s_1$ (resp. $s_2 > s_1$).[24] Likewise, if a ban on dual pricing raises (resp. lowers) aggregate surplus for some s_1, then such a ban raises (resp. lowers) aggregate surplus for any $s_2 < s_1$ (resp. $s_2 > s_1$). In this sense, if λ is close to 1, then a ban on dual pricing is "more likely" to be detrimental to consumer surplus and aggregate surplus if s is large.

In the special case where demand is ρ-linear, we have that $\beta = 0$, so that the expression to be signed for the consumer-surplus effect reduces to $\alpha(2-\alpha) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)$. Moreover, as $p^{m\prime}(w)$ is constant and thus non-increasing, the proposition implies that $\tilde{\mu}(s)$ is decreasing in s. If s is high, then $\tilde{\mu}(s) = -\frac{1}{p^m(p_0)-p_0} = -\frac{1}{(p_0-c)\alpha}$, so that $\Delta CS(\lambda) < 0$, consistent with Proposition 7. This implies that if λ is large and the pass-through rate is sufficiently high, i.e., $\alpha \geq 2$, then a ban on dual pricing lowers consumer surplus for any $s > 0$. Otherwise, if $\alpha < 2$, then there exists a cutoff $\check{s}$ such that a ban on dual pricing raises consumer surplus if $s < \check{s}$, and lowers it if the inequality is reversed. Similar results obtain for aggregate surplus, with the pass-through cutoff being 1 instead of 2.

Numerical simulations. Taking stock, Propositions 7–10 characterize the welfare effects of banning dual pricing when s is large (assuming demand is ρ-linear), when s is low, when λ is low, and when λ is large. To explore those welfare effects when both s and λ are intermediate, we run numerical simulations under ρ-linear demand (with $a = b = M = c = 1$).[25] The results are reported in Figure 1. We experiment with low (0.5), intermediate (1.5), and high (2.5) values of the pass-through parameter α, as well as low (0.2), intermediate (0.5), and high (0.8) values of the share of online consumers λ; the search-cost parameter s varies continuously.

When the pass-through parameter is high, we know from Propositions 7 and 8 that a ban on dual pricing negatively affects consumer and aggregate surplus both when s is high and when s is low. Our simulations suggest that the welfare effects remain negative for intermediate s, regardless of λ. When the pass-through parameter is low, Propositions 7 and 8 imply that a ban on dual pricing has a positive effect on consumer and aggregate surplus when s is low, but a negative one when s is high. Our simulations suggest the existence of cutoffs $\sigma^{CS}(\lambda)$ and $\sigma^{AS}(\lambda)$ such that banning dual pricing raises consumer surplus (resp. aggregate surplus) if and only if $s < \sigma^{CS}(\lambda)$ (resp. $s < \sigma^{AS}(\lambda)$). Moreover, these cutoffs appear to be increasing in λ. This is consistent with the results of Propositions 9 and 10, according to which a ban on dual pricing always lowers consumer and aggregate surplus when

[24]Excluding the knife-edged case where the search cost is equal to $\hat{s}$.

[25]This is a normalization, as the game with arbitrary values of the parameters a, b, c, and M corresponds to a version of the game in which all these parameters are equal to 1 and the players' payoffs and the search cost parameter s are scaled up by $\frac{M}{b}\left(1 + \frac{1-\alpha}{\alpha}(a - bc)\right)^{\frac{1}{1-\alpha}}$. This can be shown by defining the new strategic variable $p' = 1 + \frac{b\alpha}{\alpha+(1-\alpha)(a-bc)}(p - c)$ and by applying a similar transformation to the wholesale price.

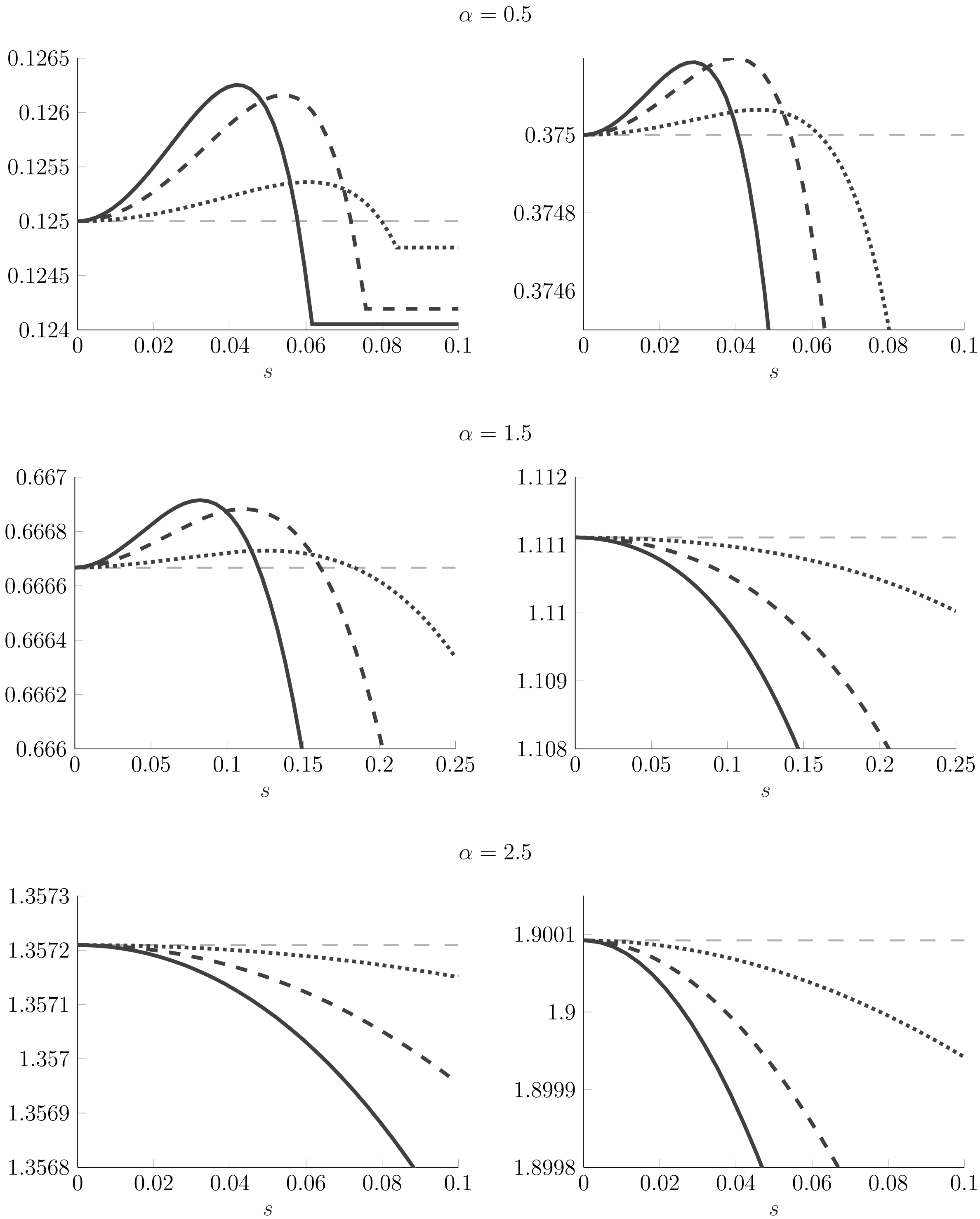

Figure 1: Consumer Surplus (left panel) and Aggregate Surplus (right panel)

Notes: Figure is constructed for ρ-linear demand ($a = b = M = 1$), $c = 1$, and various values of λ: $\lambda = 0.2$ (solid), $\lambda = 0.5$ (dashed) and $\lambda = 0.8$ (dotted).

λ is small, but raises them when λ is high (provided s is low and α is not too high). Finally, the case where the pass-through parameter is intermediate is similar to the low pass-through case for consumer surplus, and to the high pass-through case for aggregate surplus.

The general picture that emerges from our propositions and simulations is that a ban on dual pricing is "more likely" to have positive welfare effects if the monopoly pass-through is low, the offline consumers' search cost is low, and the online market is large. Interestingly, the European Commission has adopted a friendlier approach towards dual pricing in recent years, with the 2022 revision of the Vertical Block Exemption Regulation labeling it as a hardcore restriction only if its goal is to prevent online sales. As the share of consumers shopping online has in recent years increased, not decreased, our analysis does not endorse this less aggressive stance.

The distributional effects of dual pricing. We now separately examine the effects of a ban on dual pricing on offline and online consumers. Under dual pricing, the online and offline consumers end up with the same utility level, as both types obtain the good at p_0 with probability 1. By contrast, under uniform pricing the online consumers systematically receive a higher utility than the offline ones because the former receive two draws from the equilibrium price distribution, whereas the latter only receive one draw. This implies that online consumers are more likely to benefit from a ban on dual pricing in the following sense: if online consumers suffer from such a ban, then so do the offline ones; similarly, if offline consumers benefit from such a ban, then so do the online ones. The following proposition provides conditions under which online and offline consumers disagree on whether dual pricing should be banned:[26]

Proposition 11. *A ban on dual pricing makes offline consumers worse off and online consumers better off if at least one of the following conditions holds:*

(i) s is small;

(ii) λ is high, $s \neq \hat{s}$, and monopoly pass-through is non-increasing in cost.

Proof. See Online Appendices D.2.4, D.4.4, and D.5.5. □

The fact that the interests of online and offline consumers with regard to dual pricing are not aligned extends beyond the cases considered in Proposition 11. Figure 2 plots the consumer surplus effects in online and offline markets of a ban on dual pricing under ρ-linear using the same parameter values as in Figure 1. Our numerical simulations indicate

[26]It is easily seen that offline consumers suffer from a ban on dual pricing when (i) λ is small, or (ii) s is high and demand is ρ-linear. The reason is that, if offline consumers were not harmed, then the above reasoning would imply that the overall consumer surplus effect would be positive, which would contradict Propositions 7 and 9.

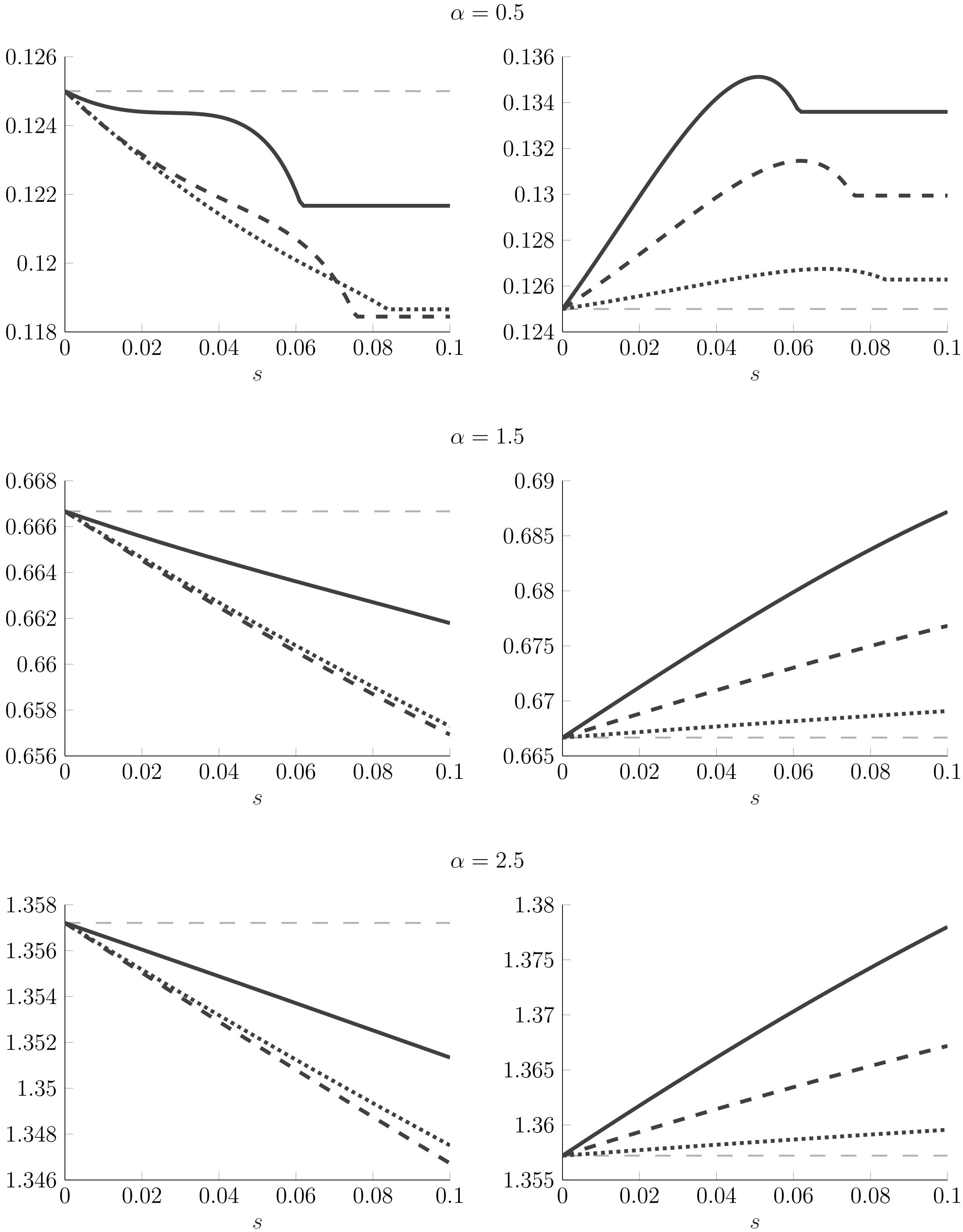

Figure 2: Offline (left panel) and Online (right panel) Consumer Surplus per Consumer

Notes: Figure is constructed for ρ-linear demand ($a = b = M = 1$), $c = 1$ and various values of λ: $\lambda = 0.2$ (solid), $\lambda = 0.5$ (dashed) and $\lambda = 0.8$ (dotted).

that, typically, online consumers benefit from such a ban, whereas offline consumers suffer from it.[27] Overall, our analysis suggests that policy measures aimed at protecting the online market may well backfire by harming consumers in the offline market.

An alternative interpretation of Propositions 7–10. Propositions 7–10 can also be interpreted as providing comparative statics in the vertical-relations version of the Stahl (1989) model. Without vertical relations (i.e., when w is exogenously given), it is well known that an increase in λ or a decrease in s gives rise to a first-order stochastic dominance shift towards lower prices, thus resulting in higher consumer surplus and aggregate surplus (see Propositions 6 and 7 in Stahl, 1989).

Consider now the version of the model with vertical relations, i.e., suppose w is optimally chosen by an upstream monopolist to maximize industry profit. According to Proposition 8, if the monopoly pass-through is sufficiently low ($\alpha < 1$) and does not vary too much with cost ($\beta \simeq 0$), then, starting from 0 search cost, a small increase in s *raises* both consumer surplus and aggregate surplus. The intuition is that the manufacturer responds to the increase in s by decreasing w to mitigate double marginalization. This effect counteracts the upward pressure on retail prices that arises from the increase in search costs.

Similarly, Proposition 9 shows that with vertical relations, starting from $\lambda = 0$, a small increase in λ *always reduces* consumer surplus and aggregate surplus. The intuition is again that the manufacturer responds to the increase in λ by raising w, which ends up outweighing the downward pressure on prices brought about by the intensification of retail competition.[28] In fact, assuming that demand is ρ-linear and s is high, Proposition 7 implies that consumer surplus and aggregate surplus are *highest* when $\lambda = 0$, i.e., when the retailers have no overlap in their customer bases. These results highlight the importance of accounting for vertical relations when evaluating the welfare effects of, e.g., policy changes that make markets more transparent for consumers.

5 Conclusion

We examine dual pricing, a vertical restraint that enables manufacturers to condition contract terms offered to hybrid retailers based on whether a product is sold online or offline. The new EU Vertical Guidelines adopted in June 2022 have taken a less aggressive stance towards this practice, suggesting that online sales no longer require protection relative to offline sales.

[27]More precisely, we have found a small set of parameters for which offline consumers benefit from a ban, and no parameters for which online consumers suffer from it.

[28]Similarly, suppose that the terms inside square brackets in Proposition 10 are strictly positive, and that λ is initially high. Then, increasing λ to 1, i.e., making everybody a shopper, also reduces consumer surplus and aggregate surplus.

Under this new approach, dual pricing benefits from the exemption provided by Article 2(1) if it encourages retailer investments, but it continues to be considered a hardcore restriction if it is primarily used to limit online sales. This raises the question of why a manufacturer would benefit from discriminating against online sales.

We propose a new rationale for using dual pricing that is based on search cost heterogeneity across online and offline markets. In our model, online consumers can search and compare prices for free, whereas search is costly in the offline market. Due to these search cost asymmetries, a uniform-pricing contract, which does not distinguish between online and offline sales, necessarily gives rise to wasteful price dispersion. By offering more favorable terms for offline sales, the manufacturer can weaken the retailers' incentives to cut prices to corner the online market, thereby mitigating price dispersion. This restores the manufacturer's control over retail prices. Indeed, a well-chosen dual-pricing contract allows the manufacturer to implement the industry monopoly outcome.

This insight applies to a broad class of non-linear uniform-pricing contracts. Moreover, if upstream contracts are secret, dual pricing not only eliminates wasteful price dispersion but also solves the classic supplier opportunism problem. Our results also extend to a more general search model with more than two retailers, asymmetric shares of offline consumers among retailers, heterogeneous search costs for offline consumers, an endogenous share of online consumers, and heterogeneous demands for online and offline consumers. As discussed in Section 3.2, the industry monopoly outcome would no longer arise under dual pricing if there were search cost heterogeneity in the online market. However, as long as search frictions are greater offline than online, the manufacturer would still find it profitable to discriminate against online sales to mitigate price dispersion.

Our second set of results suggests that dual pricing, despite inducing the industry monopoly outcome, is not necessarily detrimental to consumer surplus or aggregate surplus. Under dual pricing, the retail market is always supplied at the industry monopoly price, whereas under uniform pricing retailers price above and below the industry monopoly level with positive probability. We find that a ban on dual pricing tends to harm consumers and society at large when the online market is small, offline search frictions are large, and the monopoly pass-through is high. Interestingly, our analysis indicates that the interests of online and offline consumers are often misaligned: typically, while online consumers benefit from a ban on dual pricing, offline consumers suffer from it. Thus, policy measures aimed at protecting online sales may well come at the cost of harming offline consumers.

References

ANDERSON, S. AND R. RENAULT (2003): "Efficiency and surplus bounds in Cournot competition," *Journal of Economic Theory*, 113, 253–264.

ARMSTRONG, M. AND J. VICKERS (2022): "Patterns of Competitive Interaction," *Econometrica*, 90, 153–191.

——— (2023): "Multibrand Price Dispersion," Economics Series Working Papers 1029, University of Oxford, Department of Economics.

ASKER, J. AND H. BAR-ISAAC (2020): "Vertical Information Restraints: Pro-and Anticompetitive Impacts of Minimum-Advertised-Price Restrictions," *The Journal of Law and Economics*, 63, 111–148.

BAYE, M. R. AND J. MORGAN (2001): "Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets," *American Economic Review*, 91, 454–474.

BUEHLER, S. AND D. L. GÄRTNER (2013): "Making sense of nonbinding retail-price recommendations," *American Economic Review*, 103, 335–359.

BULOW, J. I. AND P. PFLEIDERER (1983): "A Note on the Effect of Cost Changes on Prices," *Journal of Political Economy*, 91, 182–85.

CAVALLO, A. (2017): "Are Online and Offline Prices Similar? Evidence from Large Multichannel Retailers," *American Economic Review*, 107, 283–303.

CRÉMER, J. AND M. RIORDAN (1987): "On Governing Multilateral Transactions with Bilateral Contracts," *RAND Journal of Economics*, 18, 436–451.

DE LOS SANTOS, B., I. K. KIM, AND D. LUBENSKY (2018): "Do MSRPs decrease prices?" *International Journal of Industrial Organization*, 59, 429–457.

DELLAVIGNA, S. AND M. GENTZKOW (2019): "Uniform Pricing in U.S. Retail Chains," *The Quarterly Journal of Economics*, 134, 2011–2084.

FUDENBERG, D. AND J. TIROLE (1991): *Game Theory*, MIT Press, Cambridge, USA.

GANAPATI, S., J. S. SHAPIRO, AND R. WALKER (2020): "Energy Cost Pass-Through in US Manufacturing: Estimates and Implications for Carbon Taxes," *American Economic Journal: Applied Economics*, 12, 303–342.

Garcia, D. and M. Janssen (2018): "Retail channel management in consumer search markets," *International Journal of Industrial Organization*, 58, 162–182.

Genakos, C. and M. Pagliero (2022): "Competition and Pass-Through: Evidence from Isolated Markets," *American Economic Journal: Applied Economics*, 14, 35–57.

Genesove, D. and W. P. Mullin (1998): "Testing Static Oligopoly Models: Conduct and Cost in the Sugar Industry, 1890-1914," *RAND Journal of Economics*, 29, 355–377.

Hart, O. and J. Tirole (1990): "Vertical Integration and Market Foreclosure," *Brookings Papers on Economic Activity*, 21, 205–286.

Helfrich, M. and F. Herweg (2020): "Context-dependent preferences and retailing: Vertical restraints on internet sales," *Journal of Behavioral and Experimental Economics*, 87, 101556.

Horn, H. and A. Wolinsky (1988): "Bilateral Monopolies and Incentives for Merger," *RAND Journal of Economics*, 19, 408–419.

Janssen, M. and E. Reshidi (2022): "Regulating recommended retail prices," *International Journal of Industrial Organization*, 85, 102872.

——— (2023): "Discriminatory Trade Promotions in Consumer Search Markets," *Marketing Science*, 42, 401–422.

Janssen, M. and S. Shelegia (2015): "Consumer Search and Double Marginalization," *American Economic Review*, 105, 1683–1710.

Lubensky, D. (2017): "A model of recommended retail prices," *The RAND Journal of Economics*, 48, 358–386.

Mathewson, G. and R. Winter (1984): "An Economic Theory of Vertical Restraints," *RAND Journal of Economics*, 15, 27–38.

Miklós-Thal, J. and G. Shaffer (2021): "Input price discrimination by resale market," *RAND Journal of Economics*, 52, 727–757.

——— (2022): "The Economics of Dual Pricing in Vertical Agreements," *Concurrences*, 2-2022.

Miller, N. H., M. Osborne, and G. Sheu (2017): "Pass-through in a concentrated industry: empirical evidence and regulatory implications," *RAND Journal of Economics*, 48, 69–93.

Montez, J. and N. Schutz (2021): "All-pay oligopolies: Price competition with unobservable inventory choices," *The Review of Economic Studies*, 88, 2407–2438.

O'Brien, D. P. and G. Shaffer (1992): "Vertical Control with Bilateral Contracts," *RAND Journal of Economics*, 23, 299–308.

Rey, P. and J. Tirole (1986): "The logic of vertical restraints," *American Economic Review*, 921–939.

Rey, P. and T. Vergé (2004): "Bilateral Control with Vertical Contracts," *RAND Journal of Economics*, 35, 728–746.

——— (2020): "Secret contracting in multilateral relations," TSE Working Papers 16-744, Toulouse School of Economics (TSE).

Rosenthal, R. W. (1980): "A Model in Which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica*, 48, 1575–1579.

Shelegia, S. and C. M. Wilson (2021): "A generalized model of advertised sales," *American Economic Journal: Microeconomics*, 13, 195–223.

Stahl, D. O. (1989): "Oligopolistic Pricing with Sequential Consumer Search," *American Economic Review*, 79, 700–712.

——— (1996): "Oligopolistic pricing with heterogeneous consumer search," *International Journal of Industrial Organization*, 14, 243–268.

Varian, H. (1980): "A Model of Sales," *American Economic Review*, 70, 651–59.

Winter, R. A. (1993): "Vertical control and price versus nonprice competition," *The Quarterly Journal of Economics*, 108, 61–76.

Appendix

A Technical Details and Omitted Proofs

A.1 Technical Details for the Equilibrium Analysis under Uniform Pricing

This appendix gathers some omitted technical details for the equilibrium analysis under uniform pricing.

As discussed in the main text, for a given w, the retailers draw their prices from the CDF $F(\cdot, \overline{p}, w)$ defined in equation (2). The support of F is $[\underline{p}, \overline{p}]$, where, for a given $\overline{p}$, $\underline{p}$ uniquely solves equation (3). The following lemma implies that the upper endpoint of the support is uniquely pinned down (recall that the function H was defined in equation (4)):

Lemma A.1.1. *Given Marshall's second law of demand, $H(\cdot, w)$ has a strictly positive derivative on the interval $(w, p^m(w))$.*

Proof. We show that Marshall's second law of demand implies Assumption C in Stahl (1989). Once this is established, the lemma follows from Lemma 3 in Stahl (1989). In our framework, Stahl's Assumption C can be expressed as follows: for every w, the function

$$\chi : p \in (w, p^m(w)) \mapsto \frac{(p-w)\frac{\partial \pi(p,w)}{\partial p}}{\pi(p,w)^2}$$

is strictly decreasing. Observe that

$$\chi(p) = \frac{(p-w)D'(p) + D(p)}{(p-w)D(p)^2} = \frac{1}{\pi(p,w)}\left[1 - \frac{p-w}{p}|\varepsilon(p)|\right],$$

where $\varepsilon(\cdot)$ is the price elasticity of demand. By Marshall's second law of demand, the term inside square brackets is strictly positive and non-increasing on $(w, p^m(w))$. Moreover, $\pi(\cdot, w)$ is strictly increasing on that interval. The result follows. □

Thus, the upper endpoint of the support is equal to the monopoly price, $p^m(w)$, if $H(p^m(w), w) \leq s$, and otherwise to the unique solution of equation $H(\overline{p}, w) = s$. In the following, we use the notation $\overline{p}(w)$, $\underline{p}(w)$, and $F(\cdot, w)$ to describe the retail equilibrium. For what follows, it is useful to study the differentiability properties of $\overline{p}(\cdot)$:

Lemma A.1.2. *At every w, $\overline{p}(w)$ has strictly positive left and right derivatives. If $H(p^m(w), w) \neq s$, then $\overline{p}(w)$ is differentiable. If $H(p^m(w), w) > s$, then the derivative is equal to*

$$\overline{p}'(w) = \frac{\frac{1-\lambda}{2\lambda}\left(-1 + \frac{\overline{p}-w}{\underline{p}-w} - \log\left(\frac{\overline{p}-w}{\underline{p}-w}\right)\right)}{1 - \frac{1-\lambda}{2\lambda}\left(\frac{D'(\overline{p})}{D(\overline{p})}(\overline{p}-w) + 1\right)\log\left(\frac{\overline{p}-w}{\underline{p}-w}\right)}, \tag{14}$$

where $\overline{p} = \overline{p}(w)$ and $\underline{p} = \underline{p}(w)$. If instead $H(p^m(w), w) < s$, then $\overline{p}'(w) = p^{m\prime}(w)$.

Proof. Let $H^m(w) \equiv H(p^m(w), w)$. Suppose first that $H^m(w) > s$. Then, for every w' close enough to w, $\overline{p}(w')$ is strictly less than $p^m(w')$ and given by the unique solution of equation $H(\overline{p}, w') = s$. Applying the implicit function theorem to $H(\overline{p}, w') = s$ at $w' = w$

and $\overline{p} = \overline{p}(w)$, we obtain $\overline{p}'(w) = -(\partial H/\partial w)/(\partial H/\partial \overline{p})$. As

$$\frac{\partial F}{\partial \overline{p}} = -\frac{(1-\lambda)}{2\lambda}\frac{D(\overline{p}) + (\overline{p}-w)D'(\overline{p})}{\pi(p,w)}, \tag{15}$$

$$\frac{\partial F}{\partial w} = -\frac{(1-\lambda)}{2\lambda}\frac{-D(\overline{p})\pi(p,w) + D(p)\pi(\overline{p},w)}{\pi^2(p,w)} = -\frac{(1-\lambda)}{2\lambda}\frac{D(\overline{p})}{\pi(p,w)}\frac{\overline{p}-p}{p-w}, \tag{16}$$

we have that

$$\begin{aligned}
\frac{\partial H}{\partial \overline{p}} &= D(\overline{p}) + \int_{\underline{p}}^{\overline{p}} D(p)\frac{\partial F(p,\overline{p},w)}{\partial \overline{p}}dp \\
&= D(\overline{p}) - \frac{1-\lambda}{2\lambda}(D'(\overline{p})(\overline{p}-w) + D(\overline{p}))\log\left(\frac{\overline{p}-w}{\underline{p}-w}\right), \qquad (17)\\
\frac{\partial H}{\partial w} &= \int_{\underline{p}(w)}^{\overline{p}(w)} D(p)\frac{\partial F(p,\overline{p}(w),w)}{\partial w}dp \\
&= -\frac{1-\lambda}{2\lambda}D(\overline{p})\left(-1 + \frac{\overline{p}-w}{\underline{p}-w} - \log\left(\frac{\overline{p}-w}{\underline{p}-w}\right)\right). \qquad (18)
\end{aligned}$$

Inserting these expressions into the formula for $\overline{p}'(w)$, we obtain equation (14). Note that this derivative is indeed strictly positive by Lemma A.1.1 and since $\log(1+x) < x$ for every $x > 0$.

Next, suppose that $H^m(w) < s$, so that, for every w' close enough to w, $\overline{p}(w')$ is equal to $p^m(w')$. Then, $\overline{p}'(w) = p^{m\prime}(w)$, as stated.

Finally, suppose that $H^m(w) = s$. Observe that

$$\left.\frac{\partial H}{\partial \overline{p}}\right|_{(p^m(w),w)} = D(p^m(w)),$$

which is strictly positive. It follows that, for some sufficiently small $\eta > 0$, $H(p^m(w)+\eta, w) > s$. By continuity of H, this implies that $H(p^m(w)+\eta, w') > s$ for w' close enough to w. Hence, for every such w', the equation $H(p, w') = s$ has a solution by the intermediate value theorem, and we let $\rho(w')$ denote the smallest such solution (which exists by continuity of H). Below, w' should always be understood as being part of the neighborhood of w such that $\rho(w')$ is well defined. Since $\partial H/\partial \overline{p}$ and $\partial H/\partial w$ at $(p^m(w), w)$ are respectively strictly positive and strictly negative (see above), the implicit function theorem implies that ρ is continuously differentiable in a neighborhood of w and $\rho'(w)$ is strictly positive.

We distinguish three cases. Assume first that $H^{m\prime}(w) > 0$. Then, $\overline{p}(w')$ is equal to $\rho(w')$ if $w' \geq w$, and to $p^m(w')$ otherwise. It follows that the right derivative of $\overline{p}$ at w is equal to $\rho'(w)$, while the left derivative is equal to $p^{m\prime}(w)$. If instead $H^{m\prime}(w) < 0$, then the right derivative of $\overline{p}$ at w is equal to $p^{m\prime}(w)$, while the left derivative is equal to $\rho'(w)$.

Finally suppose that $H^{m\prime}(w) = 0$, which implies that

$$\frac{\partial H}{\partial w} + \frac{\partial H}{\partial \overline{p}} p^{m\prime}(w) = 0,$$

i.e., $p^{m\prime}(w) = -(\partial H/\partial w)/(\partial H/\partial \overline{p}) = \rho'(w)$. For w' in the neighborhood of w, we have that $\overline{p}(w') = \min(\rho(w'), p^m(w'))$. It follows that

$$\begin{aligned}
\left|\frac{\overline{p}(w') - \overline{p}(w)}{w' - w} - \rho'(w)\right| &\leq \max\left(\left|\frac{p^m(w') - p^m(w)}{w' - w} - \rho'(w)\right|, \left|\frac{\rho(w') - \rho(w)}{w' - w} - \rho'(w)\right|\right) \\
&= \max\left(\left|\frac{p^m(w') - p^m(w)}{w' - w} - p^{m\prime}(w)\right|, \left|\frac{\rho(w') - \rho(w)}{w' - w} - \rho'(w)\right|\right) \\
&\underset{w' \to w}{\longrightarrow} 0.
\end{aligned}$$

Hence, $\overline{p}$ is differentiable at w with strictly positive derivative. □

The manufacturer's profit is given by

$$\Pi(w) = \int_{\underline{p}(w)}^{\overline{p}(w)} r(p) dG(p, w) = r(\overline{p}(w)) - \int_{\underline{p}(w)}^{\overline{p}(w)} r'(p) G(p, w) dp,$$

where the second equality follows by integrating by parts and $G(\cdot, w)$ was defined in the main text as the CDF of prices paid by consumers. Using equation (2), G simplifies to:

$$G(p, w) = \frac{1}{4\lambda}\left[(1 + \lambda)^2 - (1 - \lambda)^2 \left(\frac{\pi(\overline{p}(w), w)}{\pi(p, w)}\right)^2\right]. \tag{19}$$

Differentiating $\Pi(w)$ yields (if $\overline{p}$ is kinked at w (see Lemma A.1.2), the derivatives below are one-sided derivatives):

$$\begin{aligned}
\Pi'(w) &= -\int_{\underline{p}}^{\overline{p}} r'(p) \frac{\partial G(p, w)}{\partial w} dp \\
&= \frac{(1 - \lambda)^2}{2\lambda} \pi(\overline{p}, w) \int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D^2(p)} \left(\frac{\pi_1'(\overline{p}, w)\overline{p}'(w) - D(\overline{p})}{(p - w)^2} + \frac{\pi(\overline{p}, w)}{(p - w)^3}\right) dp,
\end{aligned}$$

where $\pi_1'(p, w)$ denotes the partial derivative of $\pi(p, w)$ with respect to its first argument. Simplifying, we obtain the following first-order condition, which is necessary for optimality whenever $\overline{p}(w)$ is differentiable:

$$\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D^2(p)} \left(\frac{\pi_1'(\overline{p}, w)\overline{p}'(w) - D(\overline{p})}{(p - w)^2} + \frac{\pi(\overline{p}, w)}{(p - w)^3}\right) dp = 0. \tag{20}$$

In the following lemma, we establish some basic properties of the manufacturer's maximization problem and the resulting retail price distribution:

Lemma A.1.3. *For every $\lambda \in (0,1)$ and $s > 0$, the manufacturer's maximization problem has a solution. Moreover, any solution w must satisfy $w \in (c, p^0)$ and $\underline{p} < p_0 < \overline{p}$, where $\overline{p}$ and $\underline{p}$ are the associated upper and lower bounds of the support of the retail distribution.*

Proof. We begin by showing that $\Pi(\cdot)$ has a strictly positive right-hand derivative at every $w \leq c$ and a strictly negative right-hand derivative at every $w \in [p_0, \check{p})$. That right-hand derivative is given by:

$$\Pi'^{+}(w) = \frac{(1-\lambda)^2}{2\lambda}\pi(\overline{p}, w) \int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D^2(p)} \left(\frac{\pi_1'(\overline{p}, w)\overline{p}'^{+}(w) - D(\overline{p})}{(p-w)^2} + \frac{\pi(\overline{p}, w)}{(p-w)^3} \right) dp, \tag{21}$$

where $\overline{p} = \overline{p}(w)$, $\underline{p} = \underline{p}(w)$, and $\overline{p}'^{+}(w)$ is the right-hand derivative of the upper bound with respect to the wholesale price, which, by Lemma A.1.2, exists and is strictly positive.

The expression inside parentheses in the integrand in equation (21),

$$\frac{\pi_1'(\overline{p}, w)\overline{p}'^{+}(w) - D(\overline{p})}{(p-w)^2} + \frac{\pi(\overline{p}, w)}{(p-w)^3} = \frac{\pi_1'(\overline{p}, w)\overline{p}'^{+}(w)}{(p-w)^2} + \frac{D(\overline{p})(\overline{p}-p)}{(p-w)^3}, \tag{22}$$

is strictly positive for every $p \in [\underline{p}, \overline{p})$ since $\overline{p}(w) \leq p^m(w)$ and $\overline{p}'^{+}(w) > 0$.

If $w \leq c$, then $\overline{p} \leq p^m(w) \leq p^m(c) = p_0$, so that $r'(p) > 0$ for $p < \overline{p}$. It follows that the integrand in equation (21) is strictly positive on $p \in [\underline{p}, \overline{p})$, implying that $\Pi'^{+}(w) > 0$. If instead $w \geq p_0$, then $\underline{p} > p_0$, so that $r'(p) < 0$ for $p \in [\underline{p}, \overline{p}]$. It follows that the integrand in equation (21) is strictly negative, implying that $\Pi'^{+}(w) < 0$. Hence, $\Pi(\cdot)$ is strictly increasing on $(0, c]$ and strictly decreasing on $[p_0, \check{p})$. The manufacturer's problem is therefore equivalent to $\max_{w \in [c, p_0]} \Pi(w)$, which has a solution by the Weierstrass theorem.

Next, let w be a solution to the maximization problem, with associated upper and lower endpoints of the support, $\overline{p}$ and $\underline{p}$, respectively. Then, $w \geq c$. Moreover, as $\Pi'^{+}(c) > 0$, we have that $w > c$. Let us prove that $\overline{p} > p_0$. Assume for a contradiction that $\overline{p} \leq p_0$. The right-hand derivative of the manufacturer's profit function at the optimal wholesale price is given by equation (21). As $\overline{p} \leq p_0$, we have that $r'(p) > 0$ for every $p \in [\underline{p}, \overline{p})$, implying again that the integrand in that equation is strictly positive. It follows that $\Pi'^{+}(w) > 0$ at the optimal wholesale price, which is a contradiction. Hence, $\overline{p} > p_0$. Following the exact same approach, we also obtain that $\underline{p} < p_0$. Since $\underline{p} > w$, it follows that $w < p_0$. □

A.2 Proof of Proposition 7

Suppose demand is ρ-linear and $s \geq \overline{s} = \max_{w \in [c, p_0]} H(p^m(w), w)$. We need to show that, at any optimal w, the ratio of consumer surplus to industry profit is equal to α. This proof and the proof of Lemma B.1 below will rely on the following formulas for D' and $\partial\pi/\partial p$, which are easily verified under ρ-linear demand:

$$D'(p) = -\frac{\alpha D(p)}{p_0 - \alpha c - (1-\alpha)p} \tag{23}$$

$$\text{and } \frac{\partial \pi(p,w)}{\partial p} = \frac{D(p)}{p_0 - \alpha c - (1-\alpha)p}(p^m(w) - p). \tag{24}$$

We begin by deriving expressions for consumer surplus and industry profit:

Lemma A.2.1. *Under ρ-linear demand, for $s \geq \overline{s}$, expected industry profit and expected consumer surplus are given by*

$$\Pi(w) = (1-\lambda)\pi^m(w) \int_{\underline{p}(w)}^{p^m(w)} \frac{p-c}{p-w} dF(p,w) \tag{25}$$

$$= r(p^m(w)) - \int_{\underline{p}(w)}^{p^m(w)} r'(p)G(p,w)dp \tag{26}$$

$$\textit{and } CS(w) = \alpha\Pi(w) + (1-\lambda)\pi^m(w) \int_{\underline{p}(w)}^{p^m(w)} \frac{p_0 - p}{p-w} dF(p,w). \tag{27}$$

Proof. Equation (25) follows immediately by inserting

$$\begin{aligned} dG(p,w) &= d\left[(1-\lambda)F(p,w) + \lambda\left(1-(1-F(p,w))^2\right)\right] \\ &= (1-\lambda)\frac{\pi^m(w)}{\pi(p,w)} dF(p,w). \end{aligned}$$

into the definition of $\Pi(w)$. Integrating by parts in that definition and using the fact that $G(p^m(w),w) = 1$ and $G(\underline{p}(w),w) = 0$, we obtain equation (26).

Expected consumer surplus is given by

$$\begin{aligned} CS(w) &= \int_{\underline{p}(w)}^{p^m(w)} \int_p^\infty D(x)dx \; dG(p,w) \\ &= \frac{1-\lambda}{b}\pi^m(w) \int_{\underline{p}(w)}^{p^m(w)} (\alpha + (1-\alpha)(a-bp)) D(p) \frac{dF(p,w)}{\pi(p,w)} \\ &= (1-\lambda)\pi^m(w) \int_{\underline{p}(w)}^{p^m(w)} (p_0 - \alpha c - (1-\alpha)p) \frac{dF(p,w)}{p-w}. \end{aligned}$$

Combining this with equation (25), we obtain equation (27). □

Next, we obtain the optimality conditions for the wholesale price w:

Lemma A.2.2. *Under ρ-linear demand with $s \geq \overline{s}$, for any optimal wholesale price w, we have*

$$\int_{\underline{p}(w)}^{p^m(w)} \frac{r'(p)}{\pi^2(p,w)} \frac{p^m(w)-p}{p-w} dp = 0, \tag{28}$$

or, equivalently,

$$\int_{\underline{p}(w)}^{p^m(w)} \frac{p_0 - p}{p-w} dF(p,w) = 0. \tag{29}$$

Proof. Suppose that w maximizes $\Pi(\cdot)$. Then, $\Pi'(w) = 0$. Applying the Leibniz integral rule to equation (26) and using the fact that $G(p^m(w), w) = 1$ and $G(\underline{p}(w), w) = 0$, we obtain

$$\begin{aligned}
0 &= -\int_{\underline{p}(w)}^{p^m(w)} r'(p) \frac{\partial G}{\partial w} dp \\
&= \int_{\underline{p}(w)}^{p^m(w)} r'(p)(1-\lambda) \frac{\pi^m(w)}{\pi(p,w)} \frac{1-\lambda}{2\lambda} \frac{\partial}{\partial w} \frac{\pi^m(w)}{\pi(p,w)} dp \\
&= \frac{(1-\lambda)^2}{2\lambda} \pi^m(w) D(p^m(w)) \int_{\underline{p}(w)}^{p^m(w)} \frac{r'(p)}{\pi^3(p,w)} \left[p^m(w) - p\right] D(p) dp,
\end{aligned}$$

which yields condition (28).

As

$$dF(p,w) = \frac{1-\lambda}{2\lambda} \frac{\pi^m(w)}{\pi(p,w)^2} \frac{\partial \pi(p,w)}{\partial p} dp,$$

equation (28) can be rewritten as

$$\int_{\underline{p}(w)}^{p^m(w)} \frac{r'(p)}{\partial \pi(p,w)/\partial p} \frac{p^m(w)-p}{p-w} dF(p,w) = 0.$$

Plugging equation (24) into the above condition to eliminate $\partial\pi/\partial p$ and $r'(p)$, using the fact that $p^m(c) = p_0$, and simplifying, we obtain condition (29). □

Combining Lemmas A.2.1 and A.2.2, we obtain that $CS(w) = \alpha \Pi(w)$.

A.3 Proofs for Section 3.2

Proof of Proposition 3. The second part of the proposition is obvious given the analysis in Section 3.1. To prove the first part, we take the contrapositive. Suppose indeed that the continuous uniform-pricing tariff $\mathcal{T}$ implements the monopoly outcome in almost every demand state.

We begin by showing that, for every $x \in \left[\underline{m}\frac{D(p_0)}{2}, \overline{m}\frac{D(p_0)}{2}\right]$ and $y \in (0, (1-\lambda)x)$,

$$p_0 x - \mathcal{T}(x) > p_0 y - \mathcal{T}(y). \tag{30}$$

To see this, recall that, in almost every state m and for every price $p > p_0$, retailers should have no incentive to deviate from p_0 to p:

$$\frac{m}{2}p_0 D(p_0) - \mathcal{T}\left(\frac{m}{2}D(p_0)\right) \geq \frac{1-\lambda}{2}mpD(p) - \mathcal{T}\left(\frac{1-\lambda}{2}mD(p)\right).$$

By continuity of D and $\mathcal{T}$, the above inequality must in fact hold for every $m \in [\underline{m}, \overline{m}]$ and $p > p_0$. It follows that, for every $m \in [\underline{m}, \overline{m}]$ and $p \in (p_0, \check{p})$,

$$\frac{m}{2}p_0 D(p_0) - \mathcal{T}\left(\frac{m}{2}D(p_0)\right) > \frac{1-\lambda}{2}mp_0 D(p) - \mathcal{T}\left(\frac{1-\lambda}{2}mD(p)\right).$$

Letting $x \equiv mD(p_0)/2$ and noting that, as p varies between p_0 and $\check{p}$, $y \equiv \frac{1-\lambda}{2}mD(p)$ takes all the values in the interval $(0, (1-\lambda)x)$, establishes inequality (30).

Next, let $m_0 = \underline{m} + \varepsilon$, where $\varepsilon > 0$ is small, and for every n, $m_{n+1} = (1+\lambda)m_n$. Let N be the highest n such that $m_n \leq \overline{m}$ (or ∞ if $\overline{m} = \infty$). For every n, let $x_n = m_n D(p_0)/2$. In state m_n, retailers should not have an incentive to price "just below" p_0:[29]

$$p_0 \frac{m_n}{2} D(p_0) - \mathcal{T}\left(\frac{m_n}{2}D(p_0)\right) \geq p_0 \frac{1+\lambda}{2} m_n D(p_0) - \mathcal{T}\left(\frac{1+\lambda}{2}m_n D(p_0)\right).$$

Rewriting, this means that

$$p_0 x_n - \mathcal{T}(x_n) \geq p_0 x_{n+1} - \mathcal{T}(x_{n+1}).$$

Hence,

$$p_0 x_0 - \mathcal{T}(x_0) \geq p_0 x_n - \mathcal{T}(x_n) \tag{31}$$

for every $n \leq N$.

Suppose that there is an integer $n \leq N$ such that $x_n > \frac{x_0}{1-\lambda}$. Then, as $x_n \in [\underline{m}\frac{D(p_0)}{2}, \overline{m}\frac{D(p_0)}{2}]$ and $x_0 < (1-\lambda)x_n$, inequality (30) must hold for $x = x_n$ and $y = x_0$, i.e.,

$$p_0 x_n - \mathcal{T}(x_n) > p_0 x_0 - \mathcal{T}(x_0),$$

contradicting condition (31). Hence, it must be that $x_n \leq x_0/(1-\lambda)$ whenever $n \leq N$. The following condition must therefore hold for every small $\varepsilon > 0$:

$$\forall n > 0,\ (1+\lambda)^n \leq \frac{\overline{m}}{\underline{m}+\varepsilon} \quad \Longrightarrow \quad (1+\lambda)^n \leq \frac{1}{1-\lambda}. \tag{32}$$

[29]The continuity of D and $\mathcal{T}$ implies that this inequality must hold in every state, rather than just in almost every state.

As this condition is clearly violated for $\varepsilon > 0$ small enough when $\underline{m} = 0$, let us assume that $\underline{m} > 0$. Then, ε can be made arbitrarily small in condition (32) to obtain the simpler condition

$$\forall n > 0,\ (1+\lambda)^n \leq \frac{\overline{m}}{\underline{m}} \quad \Longrightarrow \quad (1+\lambda)^n \leq \frac{1}{1-\lambda},$$

or, taking the contrapositive,

$$\forall n > 0,\ (1+\lambda)^n > \frac{1}{1-\lambda} \quad \Longrightarrow \quad (1+\lambda)^n > \frac{\overline{m}}{\underline{m}}.$$

The premise is equivalent to

$$n \geq 1 + \left[\frac{-\log(1-\lambda)}{\log(1+\lambda)}\right],$$

where $[\cdot]$ is the integer-part function. Hence, the condition reduces to

$$\overline{m} < \underline{m}(1+\lambda)^{1+\left[\frac{-\log(1-\lambda)}{\log(1+\lambda)}\right]}.$$

As $[x] \leq x$, this implies that

$$\overline{m} < \underline{m}(1+\lambda)^{1-\frac{\log(1-\lambda)}{\log(1+\lambda)}},$$

which simplifies to the negation of the condition in the statement of the proposition. □

Proof of Proposition 4. All that is left to do is check that there is no equilibrium in which the variable part w is strictly below cost. Assume for a contradiction that such an equilibrium exists. As usual, the fixed part is given by $T = \frac{1-\lambda}{2}\pi(\overline{p}, w)$. The manufacturer's equilibrium expected profit is still given by equation (7). Suppose that the manufacturer deviates by offering (c, T') to retailer R_1. If R_1 accepts this new contract, it reacts by pricing at $p^m(c) = p_0$ with probability 1, thus never supplying the online market. The manufacturer extracts its profit by setting $T' = \frac{1-\lambda}{2}r_0$ and thus makes a deviation profit of

$$\begin{aligned}
\Pi' &= T + \frac{1+\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p) + \frac{1-\lambda}{2}(p_0 - c)D(p_0) \\
&> T + \frac{1-\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p) + \lambda(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF_{\min}(p) + \frac{1-\lambda}{2}(\overline{p}-c)D(\overline{p}) \\
&= T + \frac{1-\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p) + \lambda(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF_{\min}(p) \\
&\quad + \frac{1-\lambda}{2}\left(\pi(\overline{p}, w) + (w-c)D(\overline{p})\right) \\
&> 2T + \frac{1-\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p) + \lambda(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF_{\min}(p)
\end{aligned}$$

$$+\frac{1-\lambda}{2}(w-c)\int_{\underline{p}}^{\overline{p}} D(p)dF(p)$$
$$= \Pi^*.$$

The deviation is therefore profitable. $\square$

B On the Optimality of Dealing with Both Retailers

Lemma B.1. *Suppose that $D(p)$ is ρ-linear and $s \geq \overline{s} = \max_{w\in[c,p_0]} H(p^m(w), w)$. Then, in any equilibrium under uniform pricing, the manufacturer deals with both retailers.*

Proof. As $s \geq \overline{s}$, we have that for any $w \in [c, p_0]$, the upper bound of the retail price distribution is given by the monopoly price, $p^m(w) = p_0 + \alpha(w-c)$. By Lemma A.2.1, the profit of the manufacturer can be represented as

$$\Pi(w) = (1-\lambda)\pi^m(w)\int_{\underline{p}(w)}^{p^m(w)} \frac{p-c}{p-w}dF(p,w)$$
$$= (1-\lambda)\pi^m(w)\left(1+(w-c)\int_{\underline{p}(w)}^{p^m(w)}\frac{dF(p,w)}{p-w}\right).$$

Let w be a global maximizer of $\Pi(\cdot)$. By Lemma A.2.2, the following condition must hold:

$$0 = \int_{\underline{p}(w)}^{p^m(w)} \frac{p_0-p}{p-w}dF(p,w) = (p_0-w)\int_{\underline{p}(w)}^{p^m(w)}\frac{dF(p,w)}{p-w} - 1.$$

Inserting this into the above expression for $\Pi(w)$, we obtain that

$$\Pi(w) = (1-\lambda)\pi^m(w)\frac{p_0-c}{p_0-w} = 2\frac{1-\lambda}{1+\lambda}\frac{\pi^m(w)}{\pi(p_0,w)} \times \frac{1+\lambda}{2}r_0$$
$$= 2\frac{\pi(\underline{p}(w),w)}{\pi(p_0,w)} \times \frac{1+\lambda}{2}r_0,$$

where the second line follows by equation (3).

Next, we show that $(p_0-w)/(\underline{p}(w)-w) < 2$. By Lemma A.2.2, w must satisfy condition (28). Using equation (24) to replace $r'(p)$ in that condition, we obtain:

$$0 = \int_{\underline{p}(w)}^{p^m(w)} \psi(p)\frac{p_0-p}{(p-w)^3}dp, \tag{33}$$

where

$$\psi(p) \equiv \frac{1}{D(p)} \frac{p^m(w) - p}{p_0 - \alpha c - (1-\alpha)p}.$$

The derivative of $\psi(p)$ is given by

$$\begin{aligned}
\psi'(p) &= -\frac{D'(p)}{D^2(p)} \frac{p^m(w) - p}{p_0 - \alpha c - (1-\alpha)p} + \frac{1}{D(p)} \frac{-(p_0 - \alpha c - (1-\alpha)p) + (1-\alpha)(p^m(w) - p)}{(p_0 - \alpha c - (1-\alpha)p)^2} \\
&= \frac{1}{D(p)} \frac{\alpha(p^m(w) - p) + (1-\alpha)p^m(w) - (p_0 - \alpha c)}{(p_0 - \alpha c - (1-\alpha)p)^2} = -\frac{\alpha}{D(p)} \frac{p - w}{(p_0 - \alpha c - (1-\alpha)p)^2},
\end{aligned}$$

where we have used equation (23) to obtain the second equality and the fact that $p^m(w) = p_0 + \alpha(w - c)$ to obtain the third. Note that $\psi'(\cdot) < 0$ for every $p > w$.

By the mean-value theorem, for every $p \in [\underline{p}(w), p^m(w)]$ there exists $\xi = \xi(p)$ between p and p_0 such that

$$\psi(p) = \psi(p_0) - \psi'(\xi(p))(p_0 - p).$$

Therefore, condition (33) can be rewritten as

$$\psi(p_0) \int_{\underline{p}(w)}^{p^m(w)} \frac{p_0 - p}{(p - w)^3} dp = \int_{\underline{p}(w)}^{p^m(w)} \psi'(\xi(p)) \frac{(p_0 - p)^2}{(p - w)^3} dp.$$

As $\psi(p_0) > 0$ and $\psi'(p) < 0$ for every $p > w$, we have that the integral on the left-side is strictly negative for any optimal w. Thus,

$$\begin{aligned}
\int_{\underline{p}(w)}^{p^m(w)} \frac{p_0 - p}{(p - w)^3} dp &= \int_{\underline{p}(w)}^{p^m(w)} \left(\frac{p_0 - w}{(p - w)^3} - \frac{1}{(p - w)^2} \right) dp \\
&= \frac{p_0 - w}{2} \left(\frac{1}{(\underline{p}(w) - w)^2} - \frac{1}{(p^m(w) - w)^2} \right) - \left(\frac{1}{\underline{p}(w) - w} - \frac{1}{p^m(w) - w} \right) < 0.
\end{aligned}$$

Multiplying this inequality by $\underline{p}(w) - w$, we obtain

$$\frac{1}{2} \frac{p_0 - w}{\underline{p}(w) - w} \left(1 - \left(\frac{\underline{p}(w) - w}{p^m(w) - w} \right)^2 \right) < \left(1 - \frac{\underline{p}(w) - w}{p^m(w) - w} \right).$$

This implies that

$$\frac{p_0 - w}{\underline{p}(w) - w} < \frac{2}{1 + \frac{\underline{p}(w) - w}{p^m(w) - w}} < 2.$$

Hence, $\underline{p}(w) > \frac{1}{2}w + \frac{1}{2}p_0$. Moreover, as $\pi(\cdot, w)$ is strictly concave and strictly increasing on

$(w, p^m(w))$ by Marshall's second law of demand, we have that

$$\pi(\underline{p}(w), w) > \pi\left(\frac{1}{2}w + \frac{1}{2}p_0, w\right) > \frac{1}{2}\pi(w, w) + \frac{1}{2}\pi(p_0, w) = \frac{1}{2}\pi(p_0, w).$$

This establishes the final result that $\Pi(w) > \frac{1+\lambda}{2}r_0$. □

Lemma B.2. *Regardless of $D(\cdot)$, the manufacturer deals with both retailers in any equilibrium under uniform pricing provided $\lambda \leq \frac{1}{3}$.*

Proof. We begin by showing that the manufacturer can secure a profit strictly greater than $(1-\lambda)r_0$ by dealing with both retailers. Suppose first that $s \geq H(p_0, c)$, where $H(\cdot,\cdot)$ was defined in equation (4). If the manufacturer offers the tariff $\left(c, \frac{1-\lambda}{2}r_0\right)$, then both retailers accept and draw their prices from the CDF F defined in equation (2), with support $[\underline{p}(c), p_0]$. This results in a profit of $(1-\lambda)r_0$ for the manufacturer. By Lemma A.1.3, $w = c$ is not an optimal wholesale price for the manufacturer. Hence, for some w, $\Pi(w) > (1-\lambda)r_0$.

Next, suppose instead that $s < H(p_0, c)$. It is easily checked that $\lim_{w \uparrow p_0} H(p_0, w) = 0$ and $H(p_0, \cdot)$ is continuous on $[c, p_0)$. The intermediate value theorem implies the existence of a $w' \in (c, p_0)$ such that $H(p_0, w') = s$. If the manufacturer offers the tariff $\left(w', \frac{1-\lambda}{2}\pi(p_0, w')\right)$, then both retailers accept and the support of the equilibrium CDF of retail prices is $[\underline{p}(w'), p_0]$. The manufacturer earns a profit of

$$\begin{aligned}
\Pi(w') &= (1-\lambda)(p_0 - w')D(p_0) + (w' - c)\int_{\underline{p}(w')}^{p_0} D(p)dG(p, w') \\
&= (1-\lambda)r_0 - (1-\lambda)(w' - c)D(p_0) + (w' - c)\int_{\underline{p}(w')}^{p_0} D(p)dG(p, w') \\
&> (1-\lambda)r_0.
\end{aligned}$$

Hence, the manufacturer can secure a profit strictly greater than $(1-\lambda)r_0$ by dealing with both retailers. This exceeds $(1+\lambda)r_0/2$, the maximum profit from dealing with one retailer, provided $\lambda \leq 1/3$. □

C Equilibrium Analysis under Dual Pricing

Fix a profile of wholesale prices (w_o, w_b), where $w_o > 0$ and $w_b > 0$. In Sections C.1–C.3 below, we establish the existence of a symmetric equilibrium in each retail pricing subgame; we also fully characterize the set of symmetric pure-strategy equilibria. Building on this, we prove Proposition 2 in Section C.4.

A symmetric equilibrium is a common CDF of prices F (which may be degenerate) and a search rule for the offline consumers such that: (i) for every firm i, drawing prices from

F is optimal, conditional on the offline consumers' search rule and on the other firm mixing according to F; and (ii) the offline consumers' search rule is sequentially rational.

Let $\overline{p}$ and $\underline{p}$ be the maximum and minimum of the support of F. It is easily shown that any sequentially rational search rule must involve a cutoff strategy. That is, there exists $\rho \in [\underline{p}, \infty]$ such that a non-shopper that samples a price of p always searches if $p > \rho$, and never searches if $p < \rho$. A non-shopper that samples $p = \rho$ is indifferent between searching and not searching; the search rule should specify the non-shopper's behavior in that case. We let $\nu \in [0, 1]$ denote the probability that a non-shopper searches when it samples $p = \rho$. Note that $\rho > \underline{p}$, as the gains from search (gross of the search cost) vanish as the sampled price approaches $\underline{p}$.

To sum up, a symmetric equilibrium is fully described by a CDF F (with $\overline{p}$ and $\underline{p}$ as the maximum and minimum of the support) and a search rule $(\rho, \nu) \in (\underline{p}, \infty] \times [0, 1]$. We show equilibrium existence in the retail competition subgame separately for the following cases: 1. $w_o < p^m(w_b)$; 2. $w_o > p^m(w_b)$; and 3. $w_o = p^m(w_b)$. We do so in Sections C.1, C.2, and C.3, respectively. Define $\pi_b(p) \equiv \pi(p, w_b)$, $\pi_o(p) \equiv \pi(p, w_o)$, and $\pi_b^m \equiv \pi(p_b^m(w_b), w_b)$.

C.1 Case 1: Dual Pricing Subgames when $w_o < p^m(w_b)$

We begin by ruling out pure-strategy equilibria:

Lemma C.1.1. *For any (w_o, w_b) such that $w_o < p^m(w_b)$, there is no symmetric pure-strategy equilibrium in the retail competition subgame.*

Proof. Assume for a contradiction that there exists a symmetric pure-strategy equilibrium, in which retailers price at p. If $p > w_o$, then retailer i can profitably deviate to pricing just below p, a contradiction. Suppose instead that $p \leq w_o$. If equilibrium profits are strictly negative, then firm i can deviate to $p^m(w_b)$ and make non-negative profits, a contradiction. Suppose instead that equilibrium profits are non-negative, which implies that $w_o \geq p > w_b$. Then, retailer i can deviate to $p + \varepsilon$. This deviation is profitable, as firm i no longer serves the shoppers (on which it was making losses), continues to serve its captives, and makes more profits on its captives (as $p + \varepsilon < p^m(w_b)$). Hence, there is no pure-strategy equilibrium. □

We construct an equilibrium in which retailers mix symmetrically and continuously and the offline consumers never search on path. Define the function

$$k(p, x; w_o, w_b) = 1 - \frac{1-\lambda}{2\lambda} \frac{\pi_b(x) - \pi_b(p)}{\pi_o(p)}, \tag{34}$$

for every $x \in (w_o, p^m(w_b)]$ and $p \in (w_o, x]$. Define also $\underline{p}$ as the unique solution to $k(\underline{p}, x; w_o, w_b) =$

0, which can be rewritten as

$$2\lambda\pi_o(\underline{p}) + (1-\lambda)\pi_b(\underline{p}) = (1-\lambda)\pi_b(x). \tag{35}$$

We establish the existence and uniqueness of $\underline{p}$ in the proof of the proposition below.

The following proposition establishes the existence of a symmetric equilibrium for the case $w_o < p^m(w_b)$.

Lemma C.1.2. *Suppose that $w_o < p^m(w_b)$. There exists a $\overline{p} \in (w_o, p^m(w_b)]$ such that the continuous CDF $F(\cdot, \overline{p}) \equiv k(\cdot, \overline{p}; w_o, w_b)$ with support $[\underline{p}, \overline{p}]$ and the search rule $(\rho, \nu) = (\overline{p}, 1)$ form a symmetric equilibrium in the retail competition subgame.*

Proof. We begin by showing that, for any $x \in (w_o, p^m(w_b)]$, the function $k(\cdot, x)$ (where we have dropped the arguments (w_o, w_b) to ease notation) is strictly increasing on (w_o, x). We have:

$$\frac{\partial k(p,x)}{\partial p} = \frac{1-\lambda}{2\lambda}\frac{1}{\pi_o^2(p)}\left[\pi_b'(p)\pi_o(p) + (\pi_b(x) - \pi_b(p))\pi_o'(p)\right]. \tag{36}$$

If $w_o > w_b$, then $p^m(w_o) > p^m(w_b)$, and we immediately obtain that $\partial k/\partial p > 0$, as the two terms inside the square brackets in equation (36) are strictly positive. Suppose instead that $w_o \leq w_b$. We have:

$$\begin{aligned}
\frac{\partial k(p,x)}{\partial p} &= \frac{1-\lambda}{2\lambda}\frac{1}{\pi_o^2(p)}\left[(p-w_b)D'(p) + D(p)\right](p-w_o)D(p) \\
&\quad + \frac{1-\lambda}{2\lambda}\frac{1}{\pi_o^2(p)}\left[\pi_b(x) - (p-w_b)D(p)\right]\left[(p-w_o)D'(p) + D(p)\right] \\
&= \frac{1-\lambda}{2\lambda}\frac{1}{\pi_o^2(p)}\left[(w_b-w_o)D^2(p) + \pi_b(x)\left((p-w_o)D'(p) + D(p)\right)\right] \\
&= \frac{1-\lambda}{2\lambda}\frac{D(p)}{\pi_o^2(p)}\left[(w_b-w_o)D(p) + \pi_b(x)\left(1 - \frac{p-w_o}{p}|\varepsilon(p)|\right)\right],
\end{aligned} \tag{37}$$

where ε is the price elasticity of demand. As $x \leq p^m(w_b)$, we see from equation (36) that $\frac{\partial k(x,x)}{\partial p} = \frac{1-\lambda}{2\lambda}\frac{\pi_b'(x)}{\pi_o(x)} \geq 0$. Moreover, given Marshall's second law of demand and the fact that $w_o \leq w_b$, the term inside the square brackets on the right-hand side of equation (37) is strictly decreasing in p. It follows that $\partial k/\partial p > 0$ for every $p \in (w_o, x)$.

Note that $k(p,x) < 1$, for every $p < x$ and $k(x,x) = 1$. Moreover, k is continuous on the set of pairs (p,x) such that $x \in (w_o, p^m(w_b)]$ and $p \in (w_o, x]$. As $x > w_o$, we have that $\lim_{p \downarrow w_o} k(p,x) = -\infty$. The continuity and monotonicity of k uniquely pin down a $\underline{p} = \underline{p}(x) \in (w_o, x)$ such that $k(\underline{p}(x), x) = 0$. The properties of k imply that F is continuous in (p,x) and non-decreasing in p. Therefore, for every $x \in (w_o, p^m(b)]$, the function $F(\cdot, x)$ is the CDF of a probability measure with support $[\underline{p}(x), x]$.

Next, we show that, for some x, there exists an equilibrium in which retailers mix symmetrically according to the CDF $F(\cdot, x)$ with support $[\underline{p}(x), x]$, and offline consumers do not search on the equilibrium path. Define $H(x)$ as the expected gain from searching (gross of the search cost) when receiving a price of x and expecting the new price to be drawn from $F(\cdot, x)$:

$$H(x) = \int_{\underline{p}}^{x} \left(\int_{p}^{\infty} D(t)dt - \int_{x}^{\infty} D(t)dt \right) dF(p, x) = \int_{\underline{p}}^{x} D(p)F(p, x)dp,$$

where the second equality was obtained by integrating by parts.

Suppose first that $s \geq H(p^m(w_b))$, and let us show that there is an equilibrium in which firms mix according to $F(\cdot, x)$ with $x = p^m(w_b)$ and offline consumers do not search on path. The latter property follows as the net gains from searching when receiving price $p^m(w_b)$ are non-positive, and the gains from searching when receiving a lower price are even lower. Next, we show that the retailers have no incentives to deviate. The expected profit of a retailer setting price $p \in [\underline{p}(x), x]$ is given by

$$\frac{1-\lambda}{2}\pi_b(p) + \lambda(1 - F(p, x))\pi_o(p) = \frac{1-\lambda}{2}\pi_b(x),$$

implying that the firm is indifferent between any prices in $[\underline{p}(x), x]$. Deviating to a price above $p^m(w_b)$ is not profitable, as the deviating firm does not serve the shoppers and makes sub-optimal profits on the offline consumers. Deviating to any price $p \in [w_o, \underline{p})$ is not profitable, as the expected profit from the deviation would satisfy

$$\frac{1-\lambda}{2}\pi_b(p) + \lambda\pi_o(p) < \frac{1-\lambda}{2}\pi_b(p) + \lambda(1 - k(p, x))\pi_o(p) = \frac{1-\lambda}{2}\pi_b(x).$$

Finally, deviating to a price $p < w_o$ involves making a loss on shoppers and charging a sub-optimal price on the offline consumers, which is not profitable. Therefore, the proposed strategy profile is a symmetric equilibrium.

Next, suppose that $s < H(p^m(w_b))$. Let us show that, $H(x) = s$ for some $x \in (w_o, p^m(w_b))$. Note that $\lim_{x \downarrow w_o} H(x) = 0$, as $H(x) \leq (x - w_o)D(w_o) \underset{x \downarrow w_o}{\longrightarrow} 0$. Moreover, H is continuous, as the integrand is continuous in (p, x) and bounded above by the integrable function $D(p)$. By the intermediate value theorem, there therefore exists an $x \in (w_o, p^m(w_b))$ such that $H(x) = s$. We can again construct a symmetric equilibrium in which firms mix according to $F(\cdot, x)$ and offline consumers do not search on path. As above, firms are indifferent between all the prices in $[\underline{p}(x), x]$ and have no incentives to price below $\underline{p}(x)$. Moreover, deviating to a price above x would result in zero profit, as the firm's offline consumers would search and find a lower price with probability 1. $\square$

C.2 Case 2: Dual Pricing Subgames when $w_o > p^m(w_b)$

We begin by characterizing the set of symmetric pure-strategy equilibria. For every $\theta \in [0,1]$, define

$$\tilde{w}(\theta) \equiv \frac{\frac{1-\lambda}{2}(1+\theta)w_b + \lambda w_o}{\frac{1-\lambda}{2}(1+\theta) + \lambda} \tag{38}$$

and $\tilde{\pi}^m(\theta) \equiv \pi(p^m(\tilde{w}(\theta)), w(\theta))$.

Lemma C.2.1. *Suppose that $w_o > p^m(w_b)$. If $w_o \leq p^m(\tilde{w}(0))$, then the retail competition subgame has a unique symmetric pure-strategy equilibrium, in which both firms price at w_o. If instead $w_o > p^m(\tilde{w}(0))$, then no pure-strategy equilibrium exists.*

Proof. Suppose that there exists a symmetric pure-strategy equilibrium, in which retailers price at p. Assume for a contradiction that $p < w_o$. If equilibrium profits are strictly negative, then firm i can obtain non-negative profits by deviating to w_o, a contradiction. Suppose instead that equilibrium profits are non-negative, which implies that $p > w_b$. Then, firm i can raise its profits by deviating to $p + \varepsilon$ to stop serving the shoppers (on which it was making losses) without inducing offline consumers to search. Hence, $p \geq w_o$. Assume for a contradiction that $p > w_o$; then, firm i can profitably deviate to $p - \varepsilon$, a contradiction. It follows that $p = w_o$.

We show that $p = w_o$ is a pure-strategy equilibrium if and only if $w_o \leq p^m(\tilde{w}(0))$. Clearly, starting from this equilibrium candidate, deviating upwards is not profitable, as $w_o \geq p^m(w_b)$. If firm i deviates downward, it obtains

$$\frac{1-\lambda}{2}(p_i - w_b)D(p_i) + \lambda(p_i - w_o)D(p_i) = \frac{1+\lambda}{2}\left(p_i - \tilde{w}(0)\right)D(p_i),$$

which tends to $\frac{1-\lambda}{2}(w_o - w_b)D(w_o)$ (the candidate equilibrium profits) as p_i tends to w_o. If $w_o \leq p^m(\tilde{w}(0))$, then $p_i \mapsto (p_i - \tilde{w}(0))D(p_i)$ is strictly increasing on (w_b, w_o), and so the downward deviation is not profitable. If instead $w_o > p^m(\tilde{w}(0))$, then that function is locally strictly decreasing around $p_i = w_o$, and there is a profitable downward deviation. □

For the case where $w_o > p^m(\tilde{w}(0))$, we construct a symmetric mixed-strategy equilibrium. Consider the function

$$k(p, \theta; w_o, w_b) = \left[\frac{1-\lambda}{2\lambda}(1+\theta) + 1\right] \frac{\tilde{\pi}^m(\theta) - \pi(p, \tilde{w}(\theta))}{-\pi_o(p)}, \tag{39}$$

defined for every $p \in [p^m(\tilde{w}(\theta)), w_o)$ and $\theta \in [0,1]$. We have:

Lemma C.2.2. *Suppose that $w_o > p^m(\tilde{w}(0))$. There exist $\theta \in [0,1]$, $\nu \in [0,1)$ and $\overline{p} \in (p^m(\tilde{w}), w_o)$ such that the following strategy profile is a symmetric equilibrium of the retail* *competition subgame: firms draw their prices from the CDF*

$$F(p) = \begin{cases} k(p, \theta; w_o, w_b) & \text{if } p \in [p^m(\tilde{w}(\theta)), \overline{p}) \\ 1 & \text{otherwise} \end{cases}$$

and the search rule is $(\overline{p}, \nu)$.

Proof. For every $\theta \in [0, 1]$, we have that $k(p^m(\tilde{w}(\theta)), \theta) = 0$, and k is continuous and strictly positive on $(p^m(\tilde{w}(\theta)), w_o)$, where we have dropped the arguments (w_o, w_b) to ease notation. Moreover, $\lim_{p \uparrow w_o} k(p, \theta) = \infty$, and k is strictly increasing on $(p^m(\tilde{w}(\theta)), w_o)$, as

$$\frac{\partial k(p, \theta)}{\partial p} = \left[\frac{1-\lambda}{2\lambda}(1+\theta) + \lambda\right] \left[\frac{\partial \pi(p, \tilde{w}(\theta))}{\partial p} \pi_o(p) + (\tilde{\pi}^m(\theta) - \pi(p, \tilde{w}(\theta)))\, \pi_o'(p)\right] > 0.$$

Therefore, there exists a unique $\hat{p}(\theta) \in (p^m(\tilde{w}(\theta)), w_o)$ such that $k(\hat{p}(\theta), \theta) = 1$.

For every $x \in (p^m(\tilde{w}(\theta)), \hat{p}(\theta)]$, define

$$F(p, x, \theta) = \begin{cases} k(p, \theta) & \text{if } p \in [p^m(\tilde{w}(\theta)), x) \\ 1 & \text{otherwise.} \end{cases}$$

Then, $F(\cdot, x, \theta)$ is the CDF of a probability measure. Define also $H(x, \theta)$ as the expected gain from searching (gross of the search cost) when receiving a price of x and expecting the new price to be drawn from $F(\cdot, x, \theta)$:

$$H(x, \theta) \equiv \int_{p^m(\tilde{w}(\theta))}^{x} \left(\int_p^\infty D(t)dt - \int_x^\infty D(t)dt\right) dF(p, x, \theta) = \int_{p^m(\tilde{w}(\theta))}^{x} D(p)F(p, x, \theta)dp,$$

where we have integrated by parts to obtain the second equality.

Suppose first that $H(\hat{p}(0), 0) \leq s$. We show that there is an equilibrium in which firms mix according to $F(\cdot, \hat{p}(0), 0)$ and offline consumers never search on path. Clearly, this search behavior is sequentially rational for the offline consumers. The expected profit of a firm pricing at any $p \in [p^m(\tilde{w}(0)), \hat{p}(0)]$ is

$$\begin{aligned} &\frac{1-\lambda}{2}\pi_b(p) + \lambda(1 - F(p, \hat{p}(0), 0))\pi_o(p) \\ &\qquad = \left(\frac{1-\lambda}{2} + \lambda\right)\pi(p, \tilde{w}(0)) - \lambda k(p, 0)\pi_o(p) = \frac{1+\lambda}{2}\tilde{\pi}^m(0), \end{aligned}$$

implying that firms are indifferent between all the prices in the support of $F(\cdot, \hat{p}(0), 0)$. In particular, a firm pricing at $\hat{p}(0)$ receives a profit of $\frac{1-\lambda}{2}\pi_b(\hat{p}(0)) = \frac{1+\lambda}{2}\tilde{\pi}^m(0)$. A deviation to a price $p > \hat{p}(0)$ results in a profit of at most $\frac{1-\lambda}{2}\pi_b(p)$, which is lower than $\frac{1-\lambda}{2}\pi_b(\hat{p}(0))$, as

$\hat{p}(0) > p^m(\tilde{w}(0)) > p^m(w_b)$. Similarly, a deviation to a price $p < p^m(\tilde{w}(0))$ results in a profit of $\frac{1+\lambda}{2}\pi(p, \tilde{w}(0)) < \frac{1+\lambda}{2}\tilde{\pi}^m(0)$.

Next, suppose that $H(\hat{p}(0), 0) > s$. Consider the following equation:

$$\Psi(x, \theta) \equiv [1 - k(x, \theta)]^2 \frac{\lambda}{1-\lambda}\frac{w_o - x}{x - w_b} - \theta = 0. \tag{40}$$

Let us show that there exists a $\bar{\theta} > 0$, such that for every $\theta \in [0, \bar{\theta}]$, equation (40) has a unique solution in x on the interval $[p^m(\tilde{w}(\theta)), \hat{p}(\theta)]$, which we define as $\tilde{p}(\theta)$. Note that $\partial\Psi/\partial x < 0$, implying that, $\Psi(\cdot, \theta)$ is maximized at $x = p^m(\tilde{w}(\theta))$. Thus, define

$$\Phi(\theta) \equiv \Psi(p^m(\tilde{w}(\theta)), \theta) = \frac{\lambda}{1-\lambda}\frac{w_o - p^m(\tilde{w}(\theta))}{p^m(\tilde{w}(\theta)) - w_b} - \theta.$$

As $w_o > p^m(\tilde{w}(0))$, we have that $\Phi(0) > 0$. Moreover, as $p^m(\tilde{w}(1)) > \tilde{w}(1) = (1-\lambda)w_b + \lambda w_o$, we have that $\lambda(w_o - p^m(\tilde{w}(1))) < (1-\lambda)(p^m(\tilde{w}(1)) - w_b)$, implying that $\Phi(1) < 0$. By the intermediate value theorem, there exists a solution to the equation $\Phi(\theta) = 0$. Let $\bar{\theta}$ be the smallest θ that solves $\Phi(\theta) = 0$ (which exists by continuity of Φ). Note that $\bar{\theta} > 0$, as $\Phi(0) > 0$. Thus, by the monotonicity of $\Psi(\cdot, \theta)$, we have that for every $\theta \in [0, \bar{\theta}]$, there exists a unique $\tilde{p}(\theta) \in [p^m(\tilde{w}(\theta)), \hat{p}(\theta)]$ that solves equation (40). The properties of Ψ imply that $\tilde{p}(\cdot)$ is continuous and satisfies $\tilde{p}(0) = \hat{p}(0)$, and $\tilde{p}(\bar{\theta}) = p^m(\tilde{w}(\bar{\theta}))$.

Next, let

$$\tilde{H}(\theta) \equiv H(\tilde{p}(\theta), \theta) = \int_{p^m(\tilde{w}(\theta))}^{\tilde{p}(\theta)} D(p)F(p, \tilde{p}(\theta), \theta)dp$$

for every $\theta \in [0, \bar{\theta}]$. Let us show that equation $\tilde{H}(\theta) = s$ has a solution. When $\theta = 0$, we have that $\tilde{H}(0) > s$ by assumption. Note that $\lim_{\theta\uparrow\bar{\theta}} \tilde{H}(\theta) = 0$, as $\tilde{H}(\theta) \leq (\tilde{p}(\theta) - p^m(\tilde{w}(\theta)))D(\tilde{w}(1)) \underset{\theta\uparrow\bar{\theta}}{\longrightarrow} 0$. Moreover, $\tilde{H}(\cdot)$ is continuous in θ, as the integrand is continuous in (p, θ) and bounded above by the integrable function $D(p)$. Hence, by the intermediate value theorem, there exists a $\theta^* \in [0, \bar{\theta}]$ such that $\tilde{H}(\theta^*) = s$.

Consider the strategy profile in which: firms draw their prices from the CDF $F(\cdot, \tilde{p}(\theta^*), \theta^*)$ with support $[p^m(\tilde{w}(\theta^*)), \tilde{p}(\theta^*)]$; offline consumers never search if they receive a price strictly below $\tilde{p}(\theta^*)$; offline consumers search with probability

$$\nu = \frac{\theta^*}{1 - k(\tilde{p}(\theta^*), \theta^*)} \tag{41}$$

if they sample a price of $\tilde{p}(\theta^*)$. Let us verify that this strategy profile constitutes a symmetric equilibrium. Note first that the offline consumers' search rule is sequentially rational by construction. Following the same steps as above, it is easily shown that firms are indifferent between all the prices in $[p^m(\tilde{w}(\theta^*)), \tilde{p}(\theta^*))$, which all result in a profit of

$\left(\frac{1-\lambda}{2}(1+\theta^*)+\lambda\right)\tilde{\pi}^m(\theta^*)$. Deviating to $p < p^m(\tilde{w}(\theta^*))$ is unprofitable, as the resulting profit would be $\left(\frac{1-\lambda}{2}(1+\theta^*)+\lambda\right)\pi(p,\tilde{w}(\theta^*))$. Deviating to $p > \tilde{p}(\theta^*)$ would result in zero profit. All that is left to do is show that the profit from pricing at $\tilde{p}(\theta^*)$ is equal to $\left(\frac{1-\lambda}{2}(1+\theta^*)+\lambda\right)\tilde{\pi}^m(\theta^*)$. This holds, as the expected profit at $\tilde{p}(\theta^*)$ is

$$
\begin{aligned}
&\frac{1-\lambda}{2}(1-\nu)\pi_b(\tilde{p}(\theta^*)) + (1-k(\tilde{p}(\theta^*),\theta^*))\left(\frac{\lambda}{2}\pi_o(\tilde{p}(\theta^*)) + \frac{1-\lambda}{2}\nu\pi_b(\tilde{p}(\theta^*))\right)\\
&= \left(\frac{1-\lambda}{2}(1+\theta^*)+\lambda\right)\tilde{\pi}^m(\theta^*) - \frac{1-\lambda}{2}\frac{\theta^*}{1-k(\tilde{p}(\theta^*),\theta^*)}\pi_b(\tilde{p}(\theta^*))\\
&\quad - \frac{\lambda}{2}\left(1-k(\tilde{p}(\theta^*),\theta^*)\right)\pi_o(\tilde{p}(\theta^*))\\
&= \left(\frac{1-\lambda}{2}(1+\theta^*)+\lambda\right)\tilde{\pi}^m(\theta^*) - \frac{1-\lambda}{2}\frac{\pi_b(\tilde{p}(\theta^*))}{1-k(\tilde{p}(\theta^*),\theta^*)}\Psi(\tilde{p}(\theta^*),\theta^*)\\
&= \left(\frac{1-\lambda}{2}(1+\theta^*)+\lambda\right)\tilde{\pi}^m(\theta^*).
\end{aligned}
$$

□

C.3 Case 3: Dual Pricing Subgames when $w_o = p^m(w_b)$

Lemma C.3.1. *Suppose that $w_o = p^m(w_b)$. Then, the retail competition subgame has a unique equilibrium, in which both firms price at w_o.*

Proof. Consider a symmetric equilibrium characterized by a CDF F, where $\overline{p}$ and $\underline{p}$ are the maximum and minimum of the support, and a search rule (ρ,ν). Assume for a contradiction that $\underline{p} < w_o$. Then, given that firm j draws its price from F, firm i's equilibrium expected profit is locally strictly increasing in p_i at $\underline{p}$. The reason is that a small increase in p_i (i) does not induce the offline consumers to start searching, (ii) raises profit per non-shopper, (iii) reduces the probability of selling to the shoppers (on which firm i makes losses), and (iv) raises profit per shopper. This contradicts the fact that $\underline{p}$ is the minimum of the support of F. It follows that $\underline{p} \geq w_o$, and that expected equilibrium profits are strictly positive.

Next, assume for a contradiction that F is non-degenerate, i.e., $\overline{p} > \underline{p}$. Suppose first that F puts strictly positive mass on $\overline{p}$. Then, firm i would be strictly better off pricing at $\overline{p}-\varepsilon$ than pricing at $\overline{p}$, a contradiction. Suppose instead that F puts no mass on $\overline{p}$. Then, as p_i approaches $\overline{p}$ from below, firm i sells with vanishingly small probability to the shoppers, and the price at which it sells to its captives is strictly sub-optimal. Firm i would therefore be strictly better off pricing at $p^m(w_b)$ instead, a contradiction. It follows that F is degenerate, i.e., puts full weight on some $p \geq w_o$.

If $p > w_o$, then firm i is strictly better off undercutting p, a contradiction. It follows that $p = w_o$. This is clearly an equilibrium. □

C.4 Proof of Proposition 2

Proof. Consider the dual pricing contract $(w_o, w_b, T) = \left(p_0, c, \frac{1-\lambda}{2} r_0 - \varepsilon\right)$ for some small $\varepsilon > 0$. By Lemma C.3.1, if both retailers accept this contract, there exists a unique equilibrium of the continuation subgame, in which both retailers price at p_0 and earn ε. Hence, both retailers accept, and the manufacturer makes a profit of $r_0 - 2\varepsilon$. Thus, the manufacturer can guarantee itself a profit that can be made arbitrarily close to r_0.

Assume for a contradiction that there is a subgame-perfect equilibrium in which $w_o \neq p_0$, or $w_o = p_0$ and $w_b \notin \left[\frac{(1+\lambda)c - 2\lambda p_0}{1-\lambda}, c\right]$. If only one retailer (resp., no retailer) accepts, then the manufacturer earns at most $\frac{1+\lambda}{2} r_0$ (resp., 0), which is less than r_0.

Suppose instead that both retailers accept. If $w_o < p^m(w_b)$ or $w_o > p^m(\tilde{w}(0))$ (using the notation of Section C.2), then by Lemmas C.1.1 and C.2.1, the retail pricing game does not have a pure-strategy equilibrium. In any mixed-strategy equilibrium (which exists by Lemmas C.1.2 and C.2.2), the retailers do not price at p_0 with probability 1, implying that industry profit, and thus the manufacturer's profit, is strictly less than r_0.

Next, suppose that $w_o \in [p^m(w_b), p^m(\tilde{w}(0))]$. Assume for a contradiction that $w_o = p_0$. Then, $p_0 \geq p^m(w_b)$ implies that $w_b \leq c$, while $p_0 \leq p^m(\tilde{w}(0))$ implies that $\tilde{w}(0) \geq c$, and thus $w_b \geq \frac{(1+\lambda)c - 2\lambda p_0}{1-\lambda}$. This contradicts our original assumption that $w_b \notin \left[\frac{(1+\lambda)c - 2\lambda p_0}{1-\lambda}, c\right]$.

It follows that $w_o \neq p_0$. Then, by Lemmas C.2.1 and C.3.1, there exists a unique pure-strategy equilibrium, in which both firms price at $w_o \neq p_0$, implying that industry profit, and thus the manufacturer's profit, is strictly less than r_0. If instead a non-degenerate mixed-strategy equilibrium is selected, then the manufacturer again earns strictly less than r_0.

Thus, in this subgame-perfect equilibrium candidate, the manufacturer earns strictly less than r_0. It follows that the manufacturer can profitably deviate to $(w_o, w_b, T) = \left(p_0, c, \frac{1-\lambda}{2} r_0 - \varepsilon\right)$ for $\varepsilon > 0$ sufficiently small, a contradiction.

Thus, in any subgame-perfect equilibrium, $w_o = p_0$ and $w_b \in \left[\frac{(1+\lambda)c - 2\lambda p_0}{1-\lambda}, c\right]$. Moreover, the above reasoning implies that in any such equilibrium, $T = \frac{1-\lambda}{2} \pi(p_0, w_b)$ and both retailers accept the contract and price at p_0 with probability 1 (for otherwise the manufacturer would earn strictly less than r_0 and could profitably deviate to $(w_o, w_b, T) = \left(p_0, c, \frac{1-\lambda}{2} r_0 - \varepsilon\right)$ for $\varepsilon > 0$ sufficiently small). The fact that this strategy profile forms a subgame-perfect equilibrium follows by Lemmas C.2.1 and C.3.1. □

D Proofs of Approximation Results

In this appendix, we provide the proofs of our approximation results (Propositions 8–10). The structure of this appendix is as follows. In Section D.1, we present some preliminaries, including expressions for consumer and aggregate surplus and a technical lemma that will be used repeatedly to approximate integrals. In Section D.2, we study the welfare effects of dual pricing for small s and prove Proposition 8. In Section D.3, we turn to the approximation results for small λ and prove Proposition 9. Finally, Sections D.4 and D.5 contain the welfare results for the case of high λ and the proof of Proposition 10.

D.1 Preliminaries for the Proofs of Propositions 8–10

D.1.1 Expressions for Consumer and Aggregate Surplus

In principle, the manufacturer's maximization problem may have multiple solutions. In the following, for every (λ, s), we let $w = w(\lambda, s)$ be such a solution. We also define $\overline{p}(\lambda, s)$ and $\underline{p}(\lambda, s)$ as the associated upper and lower endpoints of the support of the retail price distribution.

Let $CS(p) = \int_p^\infty D(t)dt$. The consumer surplus effect of a ban on dual pricing is given by

$$\Delta CS(\lambda, s) = \int_{\underline{p}(\lambda,s)}^{\overline{p}(\lambda,s)} CS(p)dG(p, w(\lambda, s), \lambda, s) - CS(p_0).$$

Integrating by parts and dropping the arguments (λ, s) to ease notation, we obtain:

$$\begin{aligned}\Delta CS &= CS(\overline{p}) + \int_{\underline{p}}^{\overline{p}} D(p)G(p, w)dp - CS(p_0) \\ &= -\int_{p_0}^{\overline{p}} D(p)dp + \int_{\underline{p}}^{\overline{p}} D(p)G(p, w)dp \\ &= -\int_{p_0}^{\overline{p}} D(p)dp + \frac{(1+\lambda)^2}{4\lambda}\int_{\underline{p}}^{\overline{p}} D(p)dp - \frac{(1-\lambda)^2}{4\lambda}\pi^2(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \frac{dp}{D(p)(p-w)^2}. \qquad (42)\end{aligned}$$

The change in producer surplus is given by

$$\begin{aligned}\Delta\Pi &= \int_{\underline{p}}^{\overline{p}} r(p)dG(p, w) - r_0 \\ &= r(\overline{p}) - r_0 - \int_{\underline{p}}^{\overline{p}} r'(p)G(p, w)dp \\ &= r(\overline{p}) - r_0 - \frac{(1+\lambda)^2}{4\lambda}(r(\overline{p}) - r(\underline{p})) + \frac{(1-\lambda)^2}{4\lambda}\pi^2(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2}. \qquad (43)\end{aligned}$$

The change in aggregate surplus is given by

$$\Delta AS = \Delta CS + \Delta\Pi. \qquad (44)$$

We show below that, as $s \to 0$, $\lambda \to 0$ or $\lambda \to 1$, the equilibrium of the model under uniform pricing converges to that under dual pricing. Thus, to determine the sign of ΔCS and ΔAS, it is sufficient to investigate whether consumer surplus and aggregate surplus under uniform pricing increases or decreases in the neighborhood of $s = 0$, $\lambda = 0$ and $\lambda = 1$. Our approximation results will rely on the (local) monopoly cost pass-through and its behavior:

$$\alpha \equiv \left.\frac{dp^m}{dw}\right|_{w=c} = \frac{D_0'}{r_0''} \tag{45}$$

$$\text{and } \beta \equiv \left.\frac{d^2p^m}{dw^2}\right|_{w=c} = \alpha^2 \left(\frac{2D_0''}{D_0'} - \frac{r_0'''}{r_0''}\right), \tag{46}$$

where $D_0^{(k)}$ and $r_0^{(k)}$ are the k-th derivative of $D(p)$ and $r(p)$ at $p = p_0$, respectively, and the expressions were derived using the implicit function theorem.

D.1.2 Taylor Approximation under the Integral Sign

The following lemma will allow us to derive Taylor approximations of expressions involving integrals:

Lemma D.1.1. *Consider the integral*

$$I(x) = \int_{a(x)}^{b(x)} \chi(p)\xi(p,x)dp.$$

Let p_0 such that $p_0 \in [a(x), b(x)]$ for every x. Assume $\lim_{x\to x_0} a(x) = a_0 \leq b_0 = \lim_{x\to x_0} b(x)$ and χ is $\mathcal{C}^{N+1}$ on an open interval J such that $[a_0, b_0] \subseteq J$. Then, in the neighborhood of $x = x_0$,

$$I(x) = \sum_{k=0}^{N} \frac{\chi^{(k)}(p_0)}{k!} \int_{a(x)}^{b(x)} (p - p_0)^k \xi(p,x)dp + O\left(\int_{a(x)}^{b(x)} |p - p_0|^{N+1} |\xi(p,x)| dp\right).$$

Proof. By the Taylor-Lagrange theorem, for every $p \in J$, there exists a $\psi(p)$ between p and p_0 such that

$$\chi(p) = \sum_{k=0}^{N} \chi^{(k)}(p_0)\frac{(p-p_0)^k}{k!} + \chi^{(N+1)}(\psi(p))(p-p_0)^{N+1}.$$

Let $\varepsilon > 0$ be sufficiently small and

$$\kappa \equiv \sup_{p\in[a_0-\varepsilon, b_0+\varepsilon]} |\chi^{(N+1)}(\psi(p))| \leq \sup_{p\in[a_0-\varepsilon, b_0+\varepsilon]} |\chi^{(N+1)}(p)| < \infty.$$

Then, for x in the neighborhood of x_0,

$$I(x) = \sum_{k=0}^{N} \frac{\chi^{(k)}(p_0)}{k!} \int_{a(x)}^{b(x)} (p-p_0)^k \xi(p,x)dp + \underbrace{\int_{a(x)}^{b(x)} \chi^{(N+1)}(\psi(p))(p-p_0)^{N+1}\xi(p,x)dp}_{\equiv R(x)}.$$

Note that

$$|R(x)| \leq \kappa \int_{a(x)}^{b(x)} |p-p_0|^{N+1} |\xi(p,x)|dp$$

for x sufficiently close to x_0, which implies that

$$R(x) = O\left(\int_{a(x)}^{b(x)} |p-p_0|^{N+1} |\xi(p,x)|dp\right)$$

in the neighborhood of $x = x_0$. □

D.2 Proofs of Welfare Results when s is Small

In this appendix, we study the welfare effects of a ban on dual pricing when s is small and provide the proof of Proposition 8. Specifically, we derive the second-order Taylor approximations of ΔCS and ΔAS with respect to s in the neighborhood of $s = 0$.

We proceed as follows. In Section D.2.1, we discuss the properties of the retail price distribution for $s \simeq 0$. Next, in Section D.2.2, we derive auxiliary Taylor approximations for the wholesale price and the lower bound of the support of the retail price distribution. In Section D.2.3, we derive the second-order Taylor approximations of ΔCS and ΔAS, which concludes the proof of Proposition 8. In section D.2.4, we study the distributional effects of banning dual pricing.

D.2.1 Basic Properties of the Equilibrium for Low s

By Lemma A.1.3, without loss of generality, we can restrict attention to wholesale prices in $[c, p_0]$. Let us show that in the neighborhood of $s = 0$, the upper endpoint of the support, $\overline{p}(w)$, satisfies $H(\overline{p}, w) = s$. Define $\tilde{s} \equiv \min_{w \in [c,p_0]} H(p^m(w), w)$. Note that $H(p^m(w), w)$ does not depend on s. Moreover, the minimum exists, as we are minimizing a continuous function over a compact set, and it must be strictly positive, as retailers play mixed strategies at any minimizing w. Therefore, for all $s < \tilde{s}$ and all $w \in [c, p_0]$ we have that $H(p^m(w), w) \geq \tilde{s} > s$. This implies that for s sufficiently close to zero, $\overline{p}(w)$ is strictly less than $p^m(w)$ and must therefore satisfy $H(\overline{p}, w) = s$.

We now show that, as $s \to 0$, the equilibrium of the model under uniform pricing converges to that under dual pricing. In particular, we show that, when s goes to 0, $\underline{p}$, $\overline{p}$, and w converge to the industry monopoly price p_0; the change in consumer and aggregate surplus, ΔCS and ΔAS respectively, both converge to 0.

Lemma D.2.1. *Under uniform pricing, the limiting equilibrium behavior as s goes to zero is as follows:*

$$\lim_{s\to 0} w(\lambda, s) = \lim_{s\to 0} \underline{p}(\lambda, s) = \lim_{s\to 0} \overline{p}(s, \lambda) = p_0.$$

This implies that

$$\lim_{s\to 0} \Delta CS(\lambda, s) = \lim_{s\to 0} \Delta AS(\lambda, s) = 0.$$

Proof. Below, we drop argument λ to ease notation. First, we show that $\overline{p}(s) - \underline{p}(s)$ converges to 0 as s tends to 0. We begin by putting on record that, for any w, the equilibrium price CDF is strictly concave. From equation (2), the density function of the retail price distribution is

$$f(p, w, s) = \frac{1-\lambda}{2\lambda} \pi(\overline{p}(w, s), w) \frac{\partial \pi(p, w)/\partial p}{\pi^2(p, w)}.$$

In Lemma A.1.1, we established that Marshall's second law of demand implies Assumption C in Stahl (1989), i.e., $(p-w)\frac{\partial \pi(p,w)}{\partial p}/\pi^2(p, w)$ is strictly decreasing on $(w, p^m(w))$. It follows that $\frac{\partial \pi(p,w)}{\partial p}/\pi^2(p, w)$ is also strictly decreasing on $(w, p^m(w))$, implying that $F(\cdot, w, s)$ is strictly concave over its support. As a concave function lies above its secant lines, we have that for every $p \in [\underline{p}(w, s), \overline{p}(w, s)]$, $F(p, w, s) \geq (p - \underline{p}(w, s))/(\overline{p}(w, s) - \underline{p}(w, s))$.

As, in the neighborhood of $s = 0$, $\overline{p}(s)$ satisfies $H(\overline{p}(s), w(s), s) = s$, we have that

$$s = \int_{\underline{p}(s)}^{\overline{p}(s)} D(p)F(p, w(s), s)dp \geq \int_{\underline{p}(s)}^{\overline{p}(s)} D(p)\frac{p - \underline{p}(s)}{\overline{p}(s) - \underline{p}(s)}dp \geq \frac{D(p^m(p_0))}{2}(\overline{p}(s) - \underline{p}(s)),$$

where the second inequality follows as $w(s) \leq p_0$ by Lemma A.1.3. Taking the limit as s tends to zero and applying the squeeze theorem, we obtain that $\overline{p}(s) - \underline{p}(s) \underset{s\to 0}{\longrightarrow} 0$. As $\underline{p}(s) < p_0 < \overline{p}(s)$ by Lemma A.1.3, this implies that $\overline{p}(s) \to p_0$ and $\underline{p}(s) \to p_0$ as s goes to 0.

Next, we study the limit of $w(s)$ as s goes to 0. Solving out for $w(s)$ in equation (3) yields:

$$w(s) = \frac{(1+\lambda)\underline{p}(s)D(\underline{p}(s)) - (1-\lambda)\overline{p}(s)D(\overline{p}(s))}{(1+\lambda)D(\underline{p}(s)) - (1-\lambda)D(\overline{p}(s))},$$

which tends to p_0 as s goes to 0.

Finally, to show that $\Delta CS(s)$ and $\Delta AS(s)$ converge to 0 as s goes to 0, let us show that $(F(\cdot, w(s_n), s_n))_{n\geq 0}$ converges weakly to a unit mass at p_0 for any sequence $(s_n)_{n\geq 0}$ that converges to 0. For every $p > p_0$, we have that for n large enough, $\overline{p}(s_n) < p$, and so $F(p, w(s_n), s_n) = 1$. Similarly, for every $p < p_0$, $\underline{p}(s_n) > p$ for n high enough, and so $F(p, w(s_n), s_n) = 0$. This establishes the weak convergence of $(F(\cdot, w(s_n), s_n))_{n\geq 0}$ to a unit mass at p_0. Hence, for any sequence $(s_n)_{n\geq 0}$ that converges to 0, we have that $\lim_{n\to\infty} \Delta CS(s_n) = \lim_{n\to\infty} \Delta AS(s_n) = 0$. It follows that $\Delta CS(s)$ and $\Delta AS(s)$ converge to 0 as s goes to 0. □

D.2.2 Taylor Approximation of Equilibrium Behavior

In this section, we derive the Taylor approximations of $\underline{p}(\lambda, s) - w(\lambda, s)$ and $p_0 - w(\lambda, s)$ with respect to $\overline{p}(\lambda, s) - w(\lambda, s)$ when $s \simeq 0$. We drop arguments (λ, s) to ease notation. For what follows, it is useful to define

$$\psi \equiv 1 - \frac{1-\lambda}{2\lambda} \log \frac{1+\lambda}{1-\lambda}. \tag{47}$$

First-order Taylor approximation of $\underline{p} - w$.

Lemma D.2.2. *In the neighborhood of $s = 0$, we have*

$$\underline{p} - w = \frac{1-\lambda}{1+\lambda}(\overline{p} - w) + o(\overline{p} - w). \tag{48}$$

Proof. Rearranging equation (3), we have that

$$\underline{p} - w = \frac{1-\lambda}{1+\lambda}\frac{D(\overline{p})}{D(\underline{p})}(\overline{p} - w) = \frac{1-\lambda}{1+\lambda}\left(1 + o(1)\right)(\overline{p} - w) = \frac{1-\lambda}{1+\lambda}(\overline{p} - w) + o(\overline{p} - w),$$

where the second equality follows by Lemma D.2.1. □

Lemma D.2.2 implies that, in the neighborhood of $s = 0$, $o((\overline{p} - w)^k) = o((\underline{p} - w)^k)$, $o((\underline{p} - w)^k) = o((\overline{p} - w)^k)$ for all $k \geq 0$, and

$$z \equiv \frac{\overline{p} - w}{\underline{p} - w} = \frac{1+\lambda}{1-\lambda} + o(1). \tag{49}$$

First-order Taylor approximation of $p_0 - w$. By Lemma A.1.2, as $H(p^m(w), w) > s$ in the neighborhood of $s = 0$, $\overline{p}'(w)$ exists and is given by equation (14). Therefore, the manufacturer's first-order condition, given by equation (20), must hold. We start by stating an auxiliary lemma that establishes the limiting behavior of $\overline{p}'(w)$ when $s \to 0$.

Lemma D.2.3. *The derivative of $\overline{p}$ with respect to w at $w = w(\lambda, s)$ converges to 1 as s goes to 0: $\overline{p}'(w) \underset{s\to 0}{\longrightarrow} 1$.*

Proof. By Lemma A.1.2, we have that

$$\overline{p}'(w) = \frac{\frac{1-\lambda}{2\lambda}\left(-1+z-\log z\right)}{1-\frac{1-\lambda}{2\lambda}\left(\frac{D'(\overline{p})}{D(\overline{p})}(\overline{p}-w)+1\right)\log z},$$

where z was defined in equation (49). As $\overline{p}-w$ tends to 0 and $D'(\overline{p})/D(\overline{p})$ tends to D_0'/D_0 as $s \to 0$, we have that

$$\lim_{s\to 0}\overline{p}'(w) = \frac{\frac{1-\lambda}{2\lambda}\left(-1+\frac{1+\lambda}{1-\lambda}-\log\frac{1+\lambda}{1-\lambda}\right)}{1-\frac{1-\lambda}{2\lambda}\log\frac{1+\lambda}{1-\lambda}} = \frac{\psi}{\psi} = 1. \qquad \square$$

Next, we define the following functions, which we will use throughout the proofs:

$$\phi(p) \equiv \frac{r'(p)}{D^2(p)}, \tag{50}$$

$$\iota(\overline{p}, w) \equiv \frac{\pi_1'(\overline{p}, w)\overline{p}'(w) - D(\overline{p})}{D(\overline{p})}. \tag{51}$$

Then, equation (20) can be rewritten as

$$(\overline{p}-w)\int_{\underline{p}}^{\overline{p}}\phi(p)\frac{1}{(p-w)^3}dp + \iota(\overline{p},w)\int_{\underline{p}}^{\overline{p}}\phi(p)\frac{1}{(p-w)^2}dp = 0. \tag{52}$$

In the following lemma, we use equation (52) to derive the Taylor approximation of $p_0 - w$ with respect to $\overline{p}-w$ for $s \simeq 0$:

Lemma D.2.4. *In the neighborhood of $s = 0$, we have*

$$p_0 - w = (1-\lambda)(\overline{p}-w) + o(\overline{p}-w).$$

Proof. Applying Lemma D.1.1 to the two integrals in equation (52), we obtain the existence of bounded functions $M(s)$ and $N(s)$ such that

$$0 = (\overline{p}-w)\phi_0'\int_{\underline{p}}^{\overline{p}}\frac{p-p_0}{(p-w)^3}dp + \overbrace{\iota(\overline{p},w)\phi_0'\int_{\underline{p}}^{\overline{p}}\frac{p-p_0}{(p-w)^2}dp}^{\equiv R_1(s)}$$
$$+ \underbrace{M(s)(\overline{p}-w)\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^2}{(p-w)^3}dp}_{\equiv R_2(s)} + \underbrace{N(s)\iota(\overline{p},w)\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^2}{(p-w)^2}dp}_{\equiv R_3(s)},$$

where

$$\phi_0' \equiv \left.\frac{d\phi}{dp}\right|_{p=p_0} = \frac{r_0''}{D_0^2}$$

is strictly negative. Simplifying further, we have that

$$\begin{aligned}
0 &= (\overline{p}-w)\phi_0'\left(\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2} - (p_0-w)\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^3}\right) + \sum_{i=1}^{3} R_i(s) \\
&= \phi_0'\left(\left(\frac{\overline{p}-w}{\underline{p}-w}-1\right) - \frac{1}{2}\frac{p_0-w}{\overline{p}-w}\left(\left(\frac{\overline{p}-w}{\underline{p}-w}\right)^2-1\right)\right) + R(s),
\end{aligned}$$

where $R(s) \equiv \sum_{i=1}^{3} R_i(s)$. Next, we obtain upper bounds for each remainder term R_i:

$$\begin{aligned}
|R_1| &\leq |\iota(\overline{p},w)||\phi_0'|\int_{\underline{p}}^{\overline{p}} \frac{|p-p_0|}{(p-w)^2}dp \leq |\iota(\overline{p},w)||\phi_0'|(\overline{p}-w)\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2} \\
&= |\iota(\overline{p},w)||\phi_0'|\left(\frac{\overline{p}-w}{\underline{p}-w}-1\right), \\
|R_2| &\leq |M(s)|(\overline{p}-w)\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^3}dp \leq |M(s)|(\overline{p}-w)^3\int_{\underline{p}}^{\overline{p}} \frac{1}{(p-w)^3}dp, \\
&= |M(s)|(\overline{p}-w)^3\frac{1}{2}\left(\frac{1}{(\underline{p}-w)^2}-\frac{1}{(\overline{p}-w)^2}\right) = (\overline{p}-w)\frac{|M(s)|}{2}\left(\left(\frac{\overline{p}-w}{\underline{p}-w}\right)^2-1\right) \\
|R_3| &\leq |N(s)||\iota(\overline{p},w)|\int_{\underline{p}}^{\overline{p}} \left(\frac{p-p_0}{p-w}\right)^2 dp \leq |N(s)||\iota(\overline{p},w)|(\overline{p}-w)^2\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2} \\
&= |N(s)||\iota(\overline{p},w)|(\overline{p}-w)\left(\frac{\overline{p}-w}{\underline{p}-w}-1\right).
\end{aligned}$$

By Lemma D.2.3, $\overline{p}'(w) \underset{s\to 0}{\longrightarrow} 1$, and so $\iota(\overline{p},w)$ tends to 0 as $s \to 0$. Moreover, $z = \frac{\overline{p}-w}{\underline{p}-w}$ is bounded in the neighborhood of $s=0$ by Lemma D.2.2. It follows that $\lim_{s\to 0}|R_i(s)| = 0$ for $i=1,2,3$. Rearranging the first-order condition of the manufacturer and taking absolute values yields

$$\left|\frac{2}{z+1} - \frac{p_0-w}{\overline{p}-w}\right| = \frac{2}{|z^2-1||\phi_0'|}|R(s)| \leq \frac{2}{|z^2-1||\phi_0'|}\sum_{i=1}^{3}|R_i(s)|.$$

It follows that

$$\begin{aligned}
\left|1-\lambda-\frac{p_0-w}{\overline{p}-w}\right| &\leq \left|1-\lambda-\frac{2}{z+1}\right| + \left|\frac{2}{z+1}-\frac{p_0-w}{\overline{p}-w}\right| \\
&\leq \left|1-\lambda-\frac{2}{z+1}\right| + \frac{2}{|z^2-1||\phi_0'|}\sum_{i=1}^{3}|R_i(s)| \underset{s\to 0}{\longrightarrow} 0.
\end{aligned}$$

Hence,

$$\lim_{s\to 0} \frac{p_0 - w}{\overline{p} - w} = 1 - \lambda,$$

which implies that $p_0 - w = (1-\lambda)(\overline{p} - w) + o(\overline{p} - w)$. $\square$

Lemma D.2.2 and Lemma D.2.4 imply that, in the neighborhood of $s = 0$, for all $k \geq 0$

$$o((\overline{p} - w)^k) = o((\underline{p} - w)^k) = o((p_0 - w)^k) = o((\overline{p} - p_0)^k) = o((p_0 - \underline{p})^k)$$

and vice versa. Moreover, Lemma D.2.4 implies that for $s \simeq 0$, we have

$$\tau \equiv \frac{p_0 - w}{\overline{p} - w} = 1 - \lambda + o(1). \tag{53}$$

Second-order Taylor approximation of $\underline{p} - w$.

Lemma D.2.5. *In the neighborhood of $s = 0$, we have*

$$\underline{p} - w = \frac{1-\lambda}{1+\lambda}(\overline{p} - w) + \frac{2\lambda(1-\lambda)}{(1+\lambda)^2}\frac{D_0'}{D_0}(\overline{p} - w)^2 + o((\overline{p} - w)^2). \tag{54}$$

Proof. Rearranging equation (3) yields

$$\underline{p} - w = \frac{1-\lambda}{1+\lambda}\frac{D(\overline{p})}{D(\underline{p})}(\overline{p} - w).$$

Let us obtain the first-order approximations of $D(\overline{p})$ and $1/D(\underline{p})$ in the neighborhood of $s = 0$ using the first-order approximations of $\underline{p} - w$ and $p_0 - w$ derived in Lemmas D.2.2 and D.2.4. As $o(\overline{p} - p_0) = o(p_0 - \underline{p}) = o(\overline{p} - w)$, we have that

$$\begin{aligned} D(\overline{p}) &= D_0 + D_0'(\overline{p} - p_0) + o(\overline{p} - p_0) = D_0 + D_0'(\overline{p} - w - (p_0 - w)) + o(\overline{p} - w) \\ &= D_0 + \lambda D_0'(\overline{p} - w) + o(\overline{p} - w) \end{aligned} \tag{55}$$

and

$$\begin{aligned} \frac{1}{D(\underline{p})} &= \frac{1}{D_0} - \frac{D_0'}{D_0^2}(\underline{p} - p_0) + o(\underline{p} - p_0) = \frac{1}{D_0} - \frac{D_0'}{D_0^2}(\underline{p} - w - (p_0 - w)) + o(\overline{p} - w) \\ &= \frac{1}{D_0} - \frac{D_0'}{D_0^2}\left(\frac{1-\lambda}{1+\lambda} - (1-\lambda)\right)(\overline{p} - w) + o(\overline{p} - w) \\ &= \frac{1}{D_0} + \frac{\lambda(1-\lambda)}{(1+\lambda)}\frac{D_0'}{D_0^2}(\overline{p} - w) + o(\overline{p} - w). \end{aligned}$$

Plugging these approximations into equation (3), we find that for $s \simeq 0$,

$$\begin{aligned}
\underline{p} - w &= \frac{1-\lambda}{1+\lambda}\left(D_0 + \lambda D_0'(\overline{p}-w)\right)\left(\frac{1}{D_0} + \frac{\lambda(1-\lambda)}{(1+\lambda)}\frac{D_0'}{D_0^2}(\overline{p}-w)\right)(\overline{p}-w) + o((\overline{p}-w)^2) \\
&= \frac{1-\lambda}{1+\lambda}\left(1 + \frac{2\lambda}{(1+\lambda)^2}\frac{D_0'}{D_0}(\overline{p}-w)\right)(\overline{p}-w) + o((\overline{p}-w)^2). \qquad \square
\end{aligned}$$

Lemma D.2.5 implies:

Lemma D.2.6. *In the neighborhood of $s = 0$, we have*

$$\begin{aligned}
z &= \frac{1+\lambda}{1-\lambda} - \frac{2\lambda}{1-\lambda}\frac{D_0'}{D_0}(\overline{p}-w) + o(\overline{p}-w), \qquad (56) \\
\log z &= \log\frac{1+\lambda}{1-\lambda} - \frac{2\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w) + o(\overline{p}-w), \\
\frac{\overline{p}-\underline{p}}{\overline{p}-w} &= \frac{2\lambda}{1+\lambda}\left(1 - \frac{1-\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w)\right) + o(\overline{p}-w).
\end{aligned}$$

Proof. The first and third approximations follow immediately from Lemma D.2.5. To obtain the second approximation, note that $\log(1-x) = -x + o(x)$ in the neighborhood of $x = 0$, so that

$$\begin{aligned}
\log z &= \log\frac{1+\lambda}{1-\lambda} + \log\left(1 - \left(1 - \frac{1-\lambda}{1+\lambda}z\right)\right) \\
&= \log\frac{1+\lambda}{1-\lambda} - 1 + \frac{1-\lambda}{1+\lambda}z + o\left(1 - \frac{1-\lambda}{1+\lambda}z\right) \\
&= \log\frac{1+\lambda}{1-\lambda} - \frac{2\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w) + o(\overline{p}-w). \qquad \square
\end{aligned}$$

Second-order Taylor approximation of $p_0 - w$. We begin by deriving the first-order approximation of $\overline{p}'(w)$:

Lemma D.2.7. *In the neighborhood of $s = 0$, we have*

$$\overline{p}'(w) = 1 - \frac{D_0'}{D_0}(\overline{p}-w) + o(\overline{p}-w). \qquad (57)$$

Proof. We derive the first-order approximation of the partial derivatives of $H(\overline{p}, w)$ in the neighborhood of $s = 0$. Combining Lemmas D.2.5 and D.2.6 with equations (17) and (18), we obtain:

$$\begin{aligned}
\frac{\partial H}{\partial \overline{p}}\frac{1}{D(\overline{p})} &= \left[1 - \frac{1-\lambda}{2\lambda}\left(\frac{D_0'}{D_0}(\overline{p}-w) + 1\right)\left(\log\frac{1+\lambda}{1-\lambda} - \frac{2\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w)\right)\right] + o(\overline{p}-w) \\
&= \left[\psi + \frac{D_0'}{D_0}\left(\psi - \frac{2\lambda}{1+\lambda}\right)(\overline{p}-w)\right] + o(\overline{p}-w),
\end{aligned}$$

$$
\begin{aligned}
-\frac{\partial H}{\partial w}\frac{1}{D(\overline{p})} &= \frac{1-\lambda}{2\lambda}\left[-1+\frac{1+\lambda}{1-\lambda}-\frac{2\lambda}{1-\lambda}\frac{D_0'}{D_0}(\overline{p}-w)-\log\frac{1+\lambda}{1-\lambda}+\frac{2\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w)\right] \\
&\quad + o(\overline{p}-w) \\
&= \left[\psi-\frac{2\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w)\right]+o(\overline{p}-w).
\end{aligned}
$$

Using the implicit function theorem and the fact that $\frac{1}{a+bx}=\frac{1}{a}-\frac{b}{a^2}x+o(x)$ in the neighborhood of $x=0$, we obtain

$$
\begin{aligned}
\overline{p}'(w) &= \left[\psi-\frac{2\lambda}{1+\lambda}\frac{D_0'}{D_0}(\overline{p}-w)\right]\left[\frac{1}{\psi}-\frac{1}{\psi^2}\frac{D_0'}{D_0}\left(\psi-\frac{2\lambda}{1+\lambda}\right)(\overline{p}-w)\right]+o(\overline{p}-w) \\
&= 1-\frac{D_0'}{D_0}(\overline{p}-w)+o(\overline{p}-w).
\end{aligned}
$$

□

We continue to work with the function $\phi(p)$ defined in equation (50). The following lemma relies on ϕ_0' (computed above) and $\phi_0''=2\phi_0'\gamma$, where

$$
\gamma \equiv \frac{1}{2}\frac{r_0'''}{r_0''}-2\frac{D_0'}{D_0}.
$$

Using the definitions of α and β (equations (45) and (46)), we can rewrite

$$
\begin{aligned}
\gamma &= -\frac{\beta}{2\alpha^2}+\frac{D_0''}{D_0'}-2\frac{D_0'}{D_0} = -\frac{\beta}{2\alpha^2}+\left(\frac{1}{\alpha(p_0-c)}-\frac{2}{p_0-c}\right)+\frac{2}{p_0-c} \\
&= \frac{1}{2\alpha^2}\left(\frac{2\alpha}{p_0-c}-\beta\right),
\end{aligned}
$$

where we used the fact that $D_0'=-\frac{D_0}{p_0-c}$ and $D_0''=\frac{D_0'}{p_0-c}\frac{1-2\alpha}{\alpha}$.

We are ready to derive the second-order Taylor approximation of p_0-w with respect to $\overline{p}-w$ in the neighborhood of $s=0$:

Lemma D.2.8. *In the neighborhood of $s=0$, we have*

$$
p_0-w=(1-\lambda)(\overline{p}-w)+\chi(\overline{p}-w)^2+o((\overline{p}-w)^2) \tag{58}
$$

and

$$
\tau = 1-\lambda+\chi(\overline{p}-w)+o(\overline{p}-w), \tag{59}
$$

where

$$
\chi \equiv (1-\lambda)\left(\gamma(\lambda-\psi)+\lambda\frac{D_0'}{D_0}\right).
$$

Proof. Applying Lemma D.1.1 to equation (20), we have that

$$
\begin{aligned}
0 &= \int_{\underline{p}}^{\overline{p}} \phi(p) \left(\frac{\frac{d\pi(\overline{p},w)}{dw}}{(p-w)^2} + \frac{\pi(\overline{p},w)}{(p-w)^3} \right) dp \\
&= \phi_0' \underbrace{\int_{\underline{p}}^{\overline{p}} (p-p_0) \left(\frac{\frac{d\pi(\overline{p},w)}{dw}}{(p-w)^2} + \frac{\pi(\overline{p},w)}{(p-w)^3} \right) dp}_{\equiv B_1} + \phi_0' \gamma \underbrace{\int_{\underline{p}}^{\overline{p}} (p-p_0)^2 \left(\frac{\frac{d\pi(\overline{p},w)}{dw}}{(p-w)^2} + \frac{\pi(\overline{p},w)}{(p-w)^3} \right) dp}_{\equiv B_2} \\
&\quad + \underbrace{O\left(\int_{\underline{p}}^{\overline{p}} |p-p_0|^3 \left(\frac{\frac{d\pi(\overline{p},w)}{dw}}{(p-w)^2} + \frac{\pi(\overline{p},w)}{(p-w)^3} \right) dp \right)}_{\equiv B_3}.
\end{aligned}
$$

Next, we approximate B_1 and B_2 in the neighborhood of $s = 0$. We will then show that $B_3 = o(\overline{p} - w)$.

First, applying Lemma D.2.7, we have that

$$
\begin{aligned}
\frac{d\pi(\overline{p},w)}{dw} &= (D'(\overline{p})(\overline{p}-w) + D(\overline{p}))\,\overline{p}'(w) - D(\overline{p}) \\
&= D(\overline{p}) \left(\left(\frac{D_0'}{D_0}(\overline{p}-w) + 1 \right) \left(1 - \frac{D_0'}{D_0}(\overline{p}-w) \right) - 1 \right) + o(\overline{p}-w) = o(\overline{p}-w).
\end{aligned}
$$

Note that B_1 can be rewritten as

$$
\begin{aligned}
B_1 &= \int_{\underline{p}}^{\overline{p}} (p-p_0) \left(\frac{d\pi(\overline{p},w)/dw}{(p-w)^2} + \frac{\pi(\overline{p},w)}{(p-w)^3} \right) dp \\
&= \frac{d\pi(\overline{p},w)}{dw} \int_{\underline{p}}^{\overline{p}} \frac{p-p_0}{(p-w)^2} dp + \pi(\overline{p},w) \int_{\underline{p}}^{\overline{p}} \frac{p-p_0}{(p-w)^3} dp.
\end{aligned}
$$

Computing the integrals, we obtain

$$
\begin{aligned}
B_1 &= \frac{d\pi(\overline{p},w)}{dw} \left(\log \frac{\overline{p}-w}{\underline{p}-w} - \frac{p_0-w}{\overline{p}-w} \frac{\overline{p}-\underline{p}}{\underline{p}-w} \right) \\
&\quad + \pi(\overline{p},w) \left(\frac{\overline{p}-\underline{p}}{(\overline{p}-w)(\underline{p}-w)} + \frac{1}{2} \frac{p_0-w}{(\overline{p}-w)^2} - \frac{1}{2} \frac{p_0-w}{(\underline{p}-w)^2} \right) \\
&= \frac{d\pi(\overline{p},w)}{dw} (\log z - \tau(z-1)) + D(\overline{p}) \left((z-1) - \frac{1}{2} \tau (z^2-1) \right).
\end{aligned}
$$

By Lemma D.2.2, $\log z - \tau(z-1)$ is bounded in the neighborhood of $s = 0$, implying that the first term of B_1 is a little-o of $(\overline{p} - w)$. Next, we simplify the second term of B_1. From equation (55) above, we have :

$$D(\overline{p}) = D_0 + D_0'\lambda(\overline{p} - w) + o(\overline{p} - w).$$

As $(z-1) - \frac{1}{2}\tau(z^2-1)$ tends to 0 when s goes to 0, we have that

$$\lambda\left((z-1) - \frac{1}{2}\tau(z^2-1)\right)(\overline{p} - w) = o(\overline{p} - w)$$

in the neighborhood of $s = 0$. Therefore, the first-order Taylor approximation of B_1 is given by

$$\begin{aligned}
B_1 &= D_0\left((z-1) - \frac{1}{2}\tau(z^2-1)\right) + o(\overline{p} - w)\\
&= D_0\left(\frac{2\lambda}{1-\lambda} - \frac{2\lambda}{1-\lambda}\frac{D_0'}{D_0}(\overline{p} - w) - \frac{1}{2}\tau\left(\left(\frac{1+\lambda}{1-\lambda}\right)^2 - 1 - \frac{4\lambda(1+\lambda)}{(1-\lambda)^2}\frac{D_0'}{D_0}(\overline{p} - w)\right)\right)\\
&\quad + o(\overline{p} - w)\\
&= D_0\left(\frac{2\lambda}{1-\lambda} - \frac{2\lambda}{1-\lambda}\frac{D_0'}{D_0}(\overline{p} - w) - \tau\frac{2\lambda}{(1-\lambda)^2}\left(1 - (1+\lambda)\frac{D_0'}{D_0}(\overline{p} - w)\right)\right) + o(\overline{p} - w)\\
&= \frac{2\lambda}{1-\lambda}\left(1 - \frac{\tau}{1-\lambda}\right)D_0 + \frac{2\lambda^2}{1-\lambda}D_0'(\overline{p} - w) + o(\overline{p} - w),
\end{aligned}$$

where in the last equality we used the fact that $\tau = 1 - \lambda + o(1)$.

Next, we rewrite B_2 by splitting the integral into two parts:

$$B_2 = \frac{d\pi(\overline{p}, w)}{dw}\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2}dp + \pi(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^3}dp.$$

We start by showing that the first term in B_2 is a little-o of $\overline{p} - w$. Note that

$$\begin{aligned}
&\int_{\underline{p}}^{\overline{p}} \left(1 - 2\frac{p_0 - w}{p - w} + \frac{(p_0 - w)^2}{(p-w)^2}\right)dp\\
&\qquad = \left[\frac{\overline{p} - \underline{p}}{\overline{p} - w} - 2\frac{p_0 - w}{\overline{p} - w}\log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) + \left(\frac{p_0 - w}{\overline{p} - w}\right)^2\left(\frac{\overline{p} - w}{\underline{p} - w} - 1\right)\right](\overline{p} - w).
\end{aligned}$$

By Lemmas D.2.2 and D.2.4, the expression inside the square brackets is bounded for $s \simeq 0$. Since $\frac{d\pi(\overline{p},w)}{dw} = o(\overline{p} - w)$, we obtain that the first term in B_2 is a little-o of $(\overline{p} - w)^2$, and thus of $(\overline{p} - w)$.

The second term in B_2 can be rewritten as

$$B_2 = D(\overline{p})(\overline{p}-w)\left[\log\left(\frac{\overline{p}-w}{\underline{p}-w}\right) - 2\frac{p_0-w}{\overline{p}-w}\left(\frac{\overline{p}-w}{\underline{p}-w}-1\right) + \frac{1}{2}\left(\frac{p_0-w}{\overline{p}-w}\right)^2\left(\left(\frac{\overline{p}-w}{\underline{p}-w}\right)^2-1\right)\right]$$
$$= D_0\left[\log\frac{1+\lambda}{1-\lambda} - \frac{4\lambda}{1-\lambda}(1-\lambda) + \frac{1}{2}(1-\lambda)^2\frac{4\lambda}{(1-\lambda)^2}\right](\overline{p}-w) + o(\overline{p}-w)$$
$$= D_0\left[-2\lambda + \log\frac{1+\lambda}{1-\lambda}\right](\overline{p}-w) + o(\overline{p}-w).$$

Next, we show that $B_3 = o(\overline{p}-w)$. Note that

$$\int_{\underline{p}}^{\overline{p}} |p-p_0|^3\left(\frac{\frac{d\pi(\overline{p},w)}{dw}}{(p-w)^2} + \frac{\pi(\overline{p},w)}{(p-w)^3}\right)dp \leq \frac{d\pi(\overline{p},w)}{dw}\frac{(\overline{p}-\underline{p})^4}{(\underline{p}-w)^2} + D(\overline{p})\frac{(\overline{p}-w)(\overline{p}-\underline{p})^4}{(\underline{p}-w)^3}$$

Since $d\pi(\overline{p},w)/dw$ is bounded in the neighborhood of $s=0$, we have that $B_3 = O((\overline{p}-w)^2) = o(\overline{p}-w)$.

Putting together the approximations of B_1, B_2 and B_3 and dividing by $2\lambda\phi_0' D_0/(1-\lambda)$, we obtain

$$0 = 1 - \frac{\tau}{1-\lambda} + \left[\lambda\frac{D_0'}{D_0} + \gamma\left(-(1-\lambda) + \frac{1-\lambda}{2\lambda}\log\frac{1+\lambda}{1-\lambda}\right)\right](\overline{p}-w) + o(\overline{p}-w)$$
$$= 1 - \frac{\tau}{1-\lambda} + \left[\lambda\frac{D_0'}{D_0} + \gamma(\lambda-\psi)\right](\overline{p}-w) + o(\overline{p}-w).$$

Solving out for τ proves the lemma. □

First-order Taylor approximation of $\overline{p}-w$ in s.

Lemma D.2.9. *In the neighborhood of $s=0$, we have that*

$$\overline{p}-w = \frac{1}{\psi D_0}s + o(s). \tag{60}$$

Proof. The expected gain from search $H(\overline{p},w)$ given by equation (4) can be rewritten as

$$H(\overline{p},w) = \frac{1+\lambda}{2\lambda}\int_{\underline{p}}^{\overline{p}} D(p)dp - \frac{1-\lambda}{2\lambda}\pi(\overline{p},w)\log\frac{\overline{p}-w}{\underline{p}-w}.$$

By Lemma D.1.1, we have that

$$\int_{\underline{p}}^{\overline{p}} D(p)dp = \int_{\underline{p}}^{\overline{p}} D_0 dp + O\left(\int_{\underline{p}}^{\overline{p}} |p-p_0|dp\right)$$

$$= D_0 \left(1 - \frac{\underline{p} - w}{\overline{p} - w}\right) (\overline{p} - w) + O\left((\overline{p} - \underline{p})^2\right)$$
$$= \frac{2\lambda}{1+\lambda} D_0(\overline{p} - w) + o(\overline{p} - w).$$

This yields the approximation

$$H(\overline{p}, w) = \frac{1+\lambda}{2\lambda}\frac{2\lambda}{1+\lambda} D_0(\overline{p} - w) - \frac{1-\lambda}{2\lambda}\log\frac{1+\lambda}{1-\lambda} D_0(\overline{p} - w) + o(\overline{p} - w)$$
$$= \psi D_0(\overline{p} - w) + o(\overline{p} - w).$$

Since $H(\overline{p}, w) = s$ for $s \simeq 0$, we have that $s = \psi D_0(\overline{p} - w) + o(\overline{p} - w)$, implying that

$$\overline{p} - w = \frac{1}{\psi D_0} s + o(s). \qquad \square$$

The lemma implies that, in the neighborhood of $s = 0$, $o(s^k) = o((\overline{p} - w)^k)$ and vice versa for every $k \geq 0$.

D.2.3 Proof of Proposition 8

We are now in a position to derive the second-order Taylor approximations of ΔCS and ΔAS with respect to s for $s \simeq 0$. The expressions for ΔCS and ΔAS are given by equations (42) and (44), respectively.

Approximation of consumer surplus.

Lemma D.2.10. *In the neighborhood of $s = 0$, we have*

$$\Delta CS = K(\alpha(2-\alpha) - \beta(p_0 - c))s^2 + o(s^2), \tag{61}$$

where

$$K = \frac{(1-\lambda)^2}{2\alpha^2 r_0 \psi^2}\left(\frac{1}{2\lambda}\log\frac{1+\lambda}{1-\lambda} - 1\right) > 0. \tag{62}$$

Proof. Recall from equation (42) that

$$\Delta CS = -\underbrace{\int_{p_0}^{\overline{p}} D(p)dp}_{\equiv A_1} + \frac{(1+\lambda)^2}{4\lambda}\underbrace{\int_{\underline{p}}^{\overline{p}} D(p)dp}_{\equiv A_2} - \frac{(1-\lambda)^2}{4\lambda}\underbrace{\pi^2(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)}}_{\equiv A_3}.$$

We derive the second-order approximations of A_1, A_2 and A_3 with respect to $\overline{p} - w$. By

Lemma D.1.1, we have

$$
\begin{aligned}
A_1 &= \int_{p_0}^{\overline{p}} D(p)dp = \int_{p_0}^{\overline{p}} (D_0 + D_0'(p-p_0))dp + O\left(\int_{p_0}^{\overline{p}} (p-p_0)^2 dp\right) \\
&= \int_{p_0}^{\overline{p}} (D_0 + D_0'(p-p_0))dp + O\left((\overline{p}-p_0)^3\right) \\
&= D_0\left(1 - \frac{p_0 - w}{\overline{p}-w}\right)(\overline{p}-w) + \frac{1}{2}D_0'\left(1 - \frac{p_0-w}{\overline{p}-w}\right)^2 (\overline{p}-w)^2 + o((\overline{p}-w)^2) \\
&= D_0\lambda(\overline{p}-w) - \left(\chi - \frac{1}{2}\lambda^2\frac{D_0'}{D_0}\right) D_0(\overline{p}-w)^2 + o((\overline{p}-w)^2),
\end{aligned}
$$

where we have used Lemma D.2.8 and $\chi = (1-\lambda)(\gamma(\lambda-\psi) + \lambda D_0'/D_0)$. Next, we use Lemmas D.2.4 and D.2.5 to compute the approximation of A_2:

$$
\begin{aligned}
A_2 &= \int_{\underline{p}}^{\overline{p}} D(p)dp = \int_{\underline{p}}^{\overline{p}} (D_0 + D_0'(p-p_0))dp + O((\overline{p}-p_0)^3) \\
&= D_0\left(\frac{\overline{p}-\underline{p}}{\overline{p}-w}\right)(\overline{p}-w) + \frac{1}{2}D_0'\left(\left(1 - \frac{p_0-w}{\overline{p}-w}\right)^2 - \left(\frac{p_0-w}{\overline{p}-w} - \frac{\underline{p}-w}{\overline{p}-w}\right)^2\right)(\overline{p}-w)^2 \\
&\quad + o((\overline{p}-w)^2) \\
&= \frac{2\lambda}{1+\lambda}D_0(\overline{p}-w) - \left(\frac{2\lambda(1-\lambda)}{(1+\lambda)^2}\frac{D_0'}{D_0}D_0 - \frac{2\lambda^3}{(1+\lambda)^2}D_0'\right)(\overline{p}-w)^2 + o((\overline{p}-w)^2) \\
&= \frac{2\lambda}{1+\lambda}D_0(\overline{p}-w) + \frac{2\lambda(\lambda^2+\lambda-1)}{(1+\lambda)^2}D_0'(\overline{p}-w)^2 + o((\overline{p}-w)^2).
\end{aligned}
$$

To approximate A_3 we start by deriving the first-order approximation of the following integral

$$
I = \int_{\underline{p}}^{\overline{p}} \frac{1}{D(p)} \frac{\overline{p}-w}{(p-w)^2} dp.
$$

Applying Lemma D.1.1, we obtain

$$
I = (\overline{p}-w)\left(\frac{1}{D_0}\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2} - \frac{D_0'}{D_0^2}\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)}{(p-w)^2}dp\right) + O\left((\overline{p}-w)\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2}dp\right).
$$

As

$$
(\overline{p}-w)\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2}dp \leq (\overline{p}-w)\frac{(\overline{p}-\underline{p})^3}{(\underline{p}-w)^2},
$$

the remainder is a little-o of $(\overline{p}-w)$. Therefore, by Lemmas D.2.4 and D.2.5, we have that

$$\begin{aligned}
I &= \frac{1}{D_0}\frac{\overline{p}-\underline{p}}{\underline{p}-w} - \left(\log\frac{\overline{p}-w}{\underline{p}-w} - \frac{(p_0-w)(\overline{p}-\underline{p})}{(\overline{p}-w)(\underline{p}-w)}\right)\frac{D_0'}{D_0}(\overline{p}-w) + o(\overline{p}-w) \\
&= \left(\frac{2\lambda}{1-\lambda} - \frac{2\lambda}{1-\lambda}\frac{D_0'}{D_0}(\overline{p}-w)\right)\frac{1}{D_0} - \left(\log\frac{1+\lambda}{1-\lambda} - 2\lambda\right)\frac{D_0'}{D_0}(\overline{p}-w) + o(\overline{p}-w) \\
&= \frac{2\lambda}{1-\lambda}\frac{1}{D_0} + \frac{2\lambda}{1-\lambda}(\psi-(1-\lambda))\frac{D_0'}{D_0^2}(\overline{p}-w) + o(\overline{p}-w).
\end{aligned}$$

Combining this with the fact that $D^2(\overline{p})(\overline{p}-w) = D_0^2(\overline{p}-w) + o(\overline{p}-w)$, we obtain the first-order Taylor approximation of A_3 in the neighborhood of $s=0$,

$$A_3 = \frac{2\lambda}{1-\lambda}D_0(\overline{p}-w) + \frac{2\lambda}{1-\lambda}(\psi-(1-\lambda))D_0'(\overline{p}-w)^2 + o((\overline{p}-w)^2).$$

Finally, we can derive the second-order Taylor approximation of ΔCS

$$\begin{aligned}
\Delta CS = &-A_1 + \frac{(1+\lambda)^2}{4\lambda}A_2 - \frac{(1-\lambda)^2}{4\lambda}A_3 \\
= &\left(-\lambda + \frac{1+\lambda}{2} - \frac{1-\lambda}{2}\right)D_0(\overline{p}-w) \\
&+ \left(\chi - \left(\frac{\lambda^2}{2} - \frac{\lambda^2+\lambda-1}{2} + \frac{1-\lambda}{2}(\psi-(1-\lambda))\right)\frac{D_0'}{D_0}\right)D_0(\overline{p}-w)^2 + o((\overline{p}-w)^2) \\
= &\left(\chi + \frac{1-\lambda}{2}(\psi+\lambda)\frac{1}{p_0-c}\right)D_0(\overline{p}-w)^2 + o((\overline{p}-w)^2),
\end{aligned}$$

where we used $D_0 + D_0'(p_0-c) = 0$ to obtain the last expression. Inserting the formulas for χ and γ, we obtain

$$\begin{aligned}
\Delta CS &= (1-\lambda)\left(\gamma(\lambda-\psi) - \lambda\frac{1}{p_0-c} + \frac{1}{2}(\psi+\lambda)\frac{1}{p_0-c}\right)D_0(\overline{p}-w)^2 + o((\overline{p}-w)^2) \\
&= (1-\lambda)(\lambda-\psi)\left(\gamma + \frac{1}{2(p_0-c)}\right)(\overline{p}-w)^2 + o((\overline{p}-w)^2) \\
&= \frac{(1-\lambda)(\lambda-\psi)D_0}{2\alpha^2(p_0-c)}(\alpha(2-\alpha) - \beta(p_0-c))(\overline{p}-w)^2 + o((\overline{p}-w)^2).
\end{aligned}$$

Using Lemma D.2.9 to re-express the approximation in terms of s, we obtain equation (61). □

Approximation of aggregate surplus Next, we approximate industry profit:

Lemma D.2.11. *In the neighborhood of $s = 0$, we have that*

$$\Delta\Pi = -\alpha K s^2 + o(s^2), \tag{63}$$

where K is given in equation (62).

Proof. The change in industry profit is given by equation (43):

$$\Delta\Pi = r(s) - r_0 = \underbrace{r(\overline{p}) - r_0}_{\equiv C_1} - \frac{(1+\lambda)^2}{4\lambda}\underbrace{(r(\overline{p}) - r(\underline{p}))}_{\equiv C_2} + \frac{(1-\lambda)^2}{4\lambda}\underbrace{\pi^2(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D^2(p)}\frac{dp}{(p-w)^2}}_{\equiv C_3}.$$

We start with the second-order Taylor approximation of C_1. By Lemma D.2.4, we have that

$$C_1 = r(\overline{p}) - r_0 = \frac{1}{2}r_0''\left(\frac{\overline{p} - p_0}{\overline{p} - w}\right)^2(\overline{p} - w)^2 + o((\overline{p} - p_0)^2) = \frac{1}{2}\lambda^2 r_0''(\overline{p} - w)^2 + o((\overline{p} - w)^2).$$

As for C_2, we have that $r(\overline{p}) - r(\underline{p}) = (r(\overline{p}) - r_0) - (r(\underline{p}) - r_0)$. The first part of this equation coincides with C_1. The second part can be approximated by

$$r(\underline{p}) - r_0 = \frac{1}{2}r_0''\left(\frac{\underline{p} - p_0}{\overline{p} - w}\right)^2(\overline{p} - w)^2 + o((\underline{p} - p_0)^2) = \frac{1}{2}\frac{\lambda^2(1-\lambda)^2}{(1+\lambda)^2}r_0''(\overline{p} - w)^2 + o((\overline{p} - w)^2).$$

Adding up the approximations for $r(\overline{p}) - r_0$ and $-(r(\underline{p}) - r_0)$, we obtain

$$C_2 = r(\overline{p}) - r(\underline{p}) = \frac{2\lambda^3}{(1+\lambda)^2}r_0''(\overline{p} - w)^2 + o((\overline{p} - w)^2).$$

Next, we approximate C_3. Applying Lemmas D.1.1, D.2.2, and D.2.4, we obtain

$$\begin{aligned}
C_3 = \pi^2(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D^2(p)}\frac{dp}{(p-w)^2} &= \frac{r_0''}{D_0^2}D_0^2(\overline{p} - w)^2\int_{\underline{p}}^{\overline{p}} \frac{p - p_0}{(p-w)^2}dp + o((\overline{p} - w)^2) \\
&= r_0''\left(\log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) - \frac{p_0 - w}{\overline{p} - w}\left(\frac{\overline{p} - w}{\underline{p} - w} - 1\right)\right)(\overline{p} - w)^2 + o((\overline{p} - w)^2) \\
&= \left(\log\left(\frac{1+\lambda}{1-\lambda}\right) - 2\lambda\right)r_0''(\overline{p} - w)^2 + o((\overline{p} - w)^2) \\
&= \frac{2\lambda}{1-\lambda}(\lambda - \psi)r_0''(\overline{p} - w)^2 + o((\overline{p} - w)^2).
\end{aligned}$$

Finally, we can compute the approximation of $\Delta\Pi$. By the definition of α given in equation (45), we have that $r_0'' = \frac{D_0'}{\alpha} = -\frac{D_0}{\alpha(p_0-c)}$. Thus,

$$\begin{aligned}
\Delta\Pi &= C_1 - \frac{(1+\lambda)^2}{4\lambda}C_2 + \frac{(1-\lambda)^2}{4\lambda}C_3 \\
&= \left(\frac{\lambda^2}{2} - \frac{\lambda^2}{2} + \frac{1-\lambda}{2}(\lambda-\psi)\right) r_0''(\overline{p}-w)^2 + o((\overline{p}-w)^2) \\
&= -\frac{(1-\lambda)(\lambda-\psi)D_0}{\alpha(p_0-c)}(\overline{p}-w)^2 + o((\overline{p}-w)^2).
\end{aligned}$$

Using Lemma D.2.9, we can then rewrite the approximation with respect to s to obtain equation (63). □

Combining Lemmas D.2.10 and D.2.11, we obtain the approximation of aggregate surplus:

Lemma D.2.12. *In the neighborhood of $s = 0$, we have*

$$\Delta AS = K(\alpha(1-\alpha) - \beta(p_0-c))s^2 + o(s^2). \tag{64}$$

D.2.4 Distributional Effects

We now separately derive the approximation of consumer surplus for online and offline consumers. The change in the consumer surplus of offline and online consumers after the introduction of the ban on dual pricing can be written respectively as:

$$\begin{aligned}
\Delta CS_B &= (1-\lambda)\left[\int_{\overline{p}}^{\infty} D(p)dp + \int_{\underline{p}}^{\overline{p}} D(p)F(p,w)dp\right] - (1-\lambda)CS(p_0) \\
&= (1-\lambda)\left[-\int_{p_0}^{\overline{p}} D(p)dp + \int_{\underline{p}}^{\overline{p}} D(p)F(p,w)dp\right] \\
&= (1-\lambda)\left[-\int_{p_0}^{\overline{p}} D(p)dp + \frac{1+\lambda}{2\lambda}\int_{\underline{p}}^{\overline{p}} D(p)dp - \frac{1-\lambda}{2\lambda}\pi(\overline{p},w)\log\frac{\overline{p}-w}{\underline{p}-w}\right]
\end{aligned} \tag{65}$$

and

$$\Delta CS_O = \Delta CS - \Delta CS_B, \tag{66}$$

where ΔCS is given by equation (42).

Lemma D.2.13. *In the neighborhood of $s = 0$, we have that*

$$\Delta CS_B = -\frac{(1-\lambda)(\lambda-\psi)}{\psi}s + o(s), \tag{67}$$

$$\Delta CS_O = \frac{(1-\lambda)(\lambda-\psi)}{\psi}s + o(s). \tag{68}$$

Proof. In the neighborhood of $s = 0$, we have:

$$\begin{aligned}\Delta CS_B &= (1-\lambda)\left[-D_0(\overline{p}-p_0) + \frac{1+\lambda}{2\lambda}D_0(\overline{p}-\underline{p}) - \frac{1-\lambda}{2\lambda}\log\frac{1+\lambda}{1-\lambda}D_0(\overline{p}-w)\right] + o(\overline{p}-w)\\ &= (1-\lambda)\left[-\frac{\overline{p}-p_0}{\overline{p}-w} + \frac{1+\lambda}{2\lambda}\frac{\overline{p}-\underline{p}}{\overline{p}-w} - \frac{1-\lambda}{2\lambda}\log\frac{1+\lambda}{1-\lambda}\right] D_0(\overline{p}-w) + o(\overline{p}-w)\\ &= -(1-\lambda)(\lambda-\psi)D_0(\overline{p}-w) + o(\overline{p}-w)\\ &= -\frac{(1-\lambda)(\lambda-\psi)}{\psi}s + o(s),\end{aligned}$$

where the first line follows by Lemma D.1.1, the third line follows by Lemmas D.2.2 and D.2.4, and the fourth line follows by Lemma D.2.9. By Lemma D.2.10, $\Delta CS = o(s)$ in the neighborhood of $s = 0$. It follows that $\Delta CS_O = -\Delta CS_B + o(s)$. □

D.3 Proofs of Welfare Results when λ is Small

In this appendix, we study the welfare effects of banning dual pricing when λ is small and provide the proof of Proposition 9. Specifically, we derive the first-order Taylor approximation of ΔCS (equation (42)) with respect to λ when $\lambda \simeq 0$. As industry profit is lower under uniform pricing, and as we will show that $\Delta CS < 0$ in the neighborhood of $\lambda = 0$, it will not be necessary to approximate ΔAS to conclude that $\Delta AS < 0$ for small λ.

We proceed as follows. In Section D.3.1, we establish the limiting equilibrium behavior and describe the properties of the retail price distribution when λ is small. In Section D.3.2, we derive some auxiliary Taylor approximations required to approximate ΔCS. Section D.3.3 is devoted to the approximation of ΔCS.

D.3.1 Basic Properties of the Equilibrium for Low λ

In the following lemma, we show that for small λ, the upper endpoint of the retail price distribution is given by the monopoly price for all $w \in [c, p_0]$. (Recall from Proposition 6 that wholesale prices $w \notin [c, p_0]$ are suboptimal for the manufacturer.) Define $H^m(w, \lambda) \equiv H(p^m(w), w, \lambda)$ for every $w \in [c, p_0]$, where H is given by equation (4), and we have made its dependence on λ explicit. We have:

Lemma D.3.1. $\lim_{\lambda\to 0}\max_{w'\in[c,p_0]} H^m(w', \lambda) = 0$. *Therefore, there exists a neighborhood of* $\lambda = 0$ *such that* $\overline{p}(w) = p^m(w)$ *for every* $w \in [c, p_0]$.

Proof. In this proof, we include the argument λ in all functions to avoid confusion. Note that for any $w \in [c, p_0]$, $H^m(w, \lambda)$ is strictly increasing in λ, as

$$\begin{aligned}\frac{\partial H^m(w,\lambda)}{\partial \lambda} &= \int_{\underline{p}(p^m(w),w,\lambda)}^{p^m(w)} D(p)\frac{\partial F(p,p^m(w),w,\lambda)}{\partial \lambda}dp \\ &= \frac{1}{2\lambda^2}\int_{\underline{p}(p^m(w),w,\lambda)}^{p^m(w)} D(p)\left(\frac{\pi(p^m(w),w)}{\pi(p,w)}-1\right)dp > 0.\end{aligned}$$

Let $(\lambda_n)_{n\geq 0}$ be a strictly decreasing sequence over $(0,1)$, converging to 0 as $n \to \infty$. Let us show that the sequence of functions $(H^m(\cdot, \lambda_n))_{n\geq 0}$ converges uniformly to 0 on $[c, p_0]$. By Proposition 2 in Stahl (1989), holding fixed w the equilibrium lower endpoint of the support, $\underline{p}(w, \lambda_n)$, converges to $p^m(w)$ as $n \to 0$. Therefore, as

$$\underline{p}(w,\lambda_n) = \underline{p}(\overline{p}(w,\lambda_n), w, \lambda_n) \leq \underline{p}(p^m(w), w, \lambda_n),$$

we have that

$$H^m(w,\lambda_n) = \int_{\underline{p}(w,\lambda_n)}^{p^m(w)} D(p)F(p,p^m(w),w,\lambda_n)dp \leq D(c)(p^m(w) - \underline{p}(w,\lambda_n)) \underset{n\to\infty}{\longrightarrow} 0,$$

implying that for any $w \in [c, p_0]$, the sequence $(H^m(w, \lambda_n))_{n\geq 0}$ converges to 0 as $n \to \infty$.

Thus, for every $w \in [c, p_0]$, $(H^m(w, \lambda_n))_{n\geq 0}$ is decreasing in n and converges to 0 as n tends to ∞. By Dini's theorem, $(H^m(\cdot, \lambda_n))_{n\geq 0}$ converges uniformly to 0. Let $\eta > 0$. There exists $N > 0$ such that $H^m(w', \lambda_n) \leq \eta$ for all $n \geq N$ and $w' \in [c, p_0]$. By monotonicity of $H^m(w, \lambda)$ in λ, this implies that $\max_{w'\in[c,p_0]} H^m(w', \lambda) \leq \eta$ for every $\lambda \leq \lambda_N$. Hence, $\lim_{\lambda\to 0} \max_{w'\in[c,p_0]} H^m(w', \lambda) = 0$. It follows that, for some $\overline{\lambda} > 0$, for every $w \in [c, p_0]$, $H^m(w, \lambda) < s$ for every $\lambda < \overline{\lambda}$. Therefore, $\overline{p}(w, \lambda) = p^m(w)$ for every such λ and w. □

Next, we show that, as $\lambda \to 0$, the equilibrium of the model under uniform pricing converges to that under dual pricing. In particular, the following lemma establishes that, as $\lambda \to 0$, the upper and lower endpoints, $\underline{p}$, $\overline{p}$, converge to p_0 and the wholesale price w converges to c. The changes in consumer and aggregate surplus, ΔCS and ΔAS, converge to 0.

Lemma D.3.2. *Under uniform pricing, the limiting equilibrium behavior as λ goes to zero is as follows:*

$$\lim_{\lambda\to 0} w(\lambda, s) = c, \qquad \lim_{\lambda\to 0} \underline{p}(\lambda, s) = \lim_{\lambda\to 0} \overline{p}(\lambda, s) = p_0.$$

This implies that

$$\lim_{\lambda\to 0} \Delta CS(\lambda, s) = \lim_{\lambda\to 0} \Delta AS(\lambda, s) = 0.$$

Proof. We drop arguments s below to ease notation. We start by showing that $\overline{p}(\lambda)-\underline{p}(\lambda) \to 0$ as $\lambda \to 0$. In the proof of Lemma D.2.1, we showed that $F(p, w, \lambda)$ is strictly concave in p on $[\underline{p}(\lambda), \overline{p}(\lambda)]$ and, therefore, lies above its secant lines. This implies that

$$H(\overline{p}(\lambda), w(\lambda), \lambda) \geq \int_{\underline{p}(\lambda)}^{\overline{p}(\lambda)} D(p)\frac{p-\underline{p}(\lambda)}{\overline{p}(\lambda)-\underline{p}(\lambda)}dp \geq \frac{D(p^m(p_0))}{2}(\overline{p}(\lambda)-\underline{p}(\lambda)).$$

Since $H(\cdot, w, \lambda)$ is increasing on $(w, p^m(w))$ by Lemma A.1.1, we also have that

$$H(\overline{p}(\lambda), w(\lambda), \lambda) \leq H(p^m(w(\lambda)), w(\lambda), \lambda) \leq \max_{w' \in [c, p_0]} H(p^m(w'), w', \lambda).$$

Using Lemma D.3.1, it follows that

$$0 \leq \frac{D(p^m(p_0))}{2}(\overline{p}(\lambda)-\underline{p}(\lambda)) \leq \max_{w' \in [c, p_0]} H(p^m(w'), w', \lambda) \underset{\lambda \to 0}{\longrightarrow} 0,$$

so that $\overline{p}(\lambda) - \underline{p}(\lambda) \underset{\lambda \to 0}{\longrightarrow} 0$ by the squeeze theorem. As $\underline{p}(\lambda) < p_0 < \overline{p}(\lambda)$ by Lemma A.1.3, this implies that $\overline{p}(\lambda) \to p_0$ and $\underline{p}(\lambda) \to p_0$ as λ goes to 0.

By Lemma D.3.1, we have that in the neighborhood of $\lambda = 0$, $\overline{p}(\lambda) = p^m(w(\lambda))$. Using the first-order condition for that monopoly price yields:

$$w(\lambda) = c + \frac{r'(\overline{p}(\lambda))}{D'(\overline{p}(\lambda))},$$

which tends to c as λ goes to 0.

Finally, using the same argument as in the proof of Lemma D.2.1, it is straightforward to show that, for every sequence $(\lambda_n)_{n\geq 0}$ over $(0,1)$ that converges to 0, $(F(\cdot, w(\lambda_n), \lambda_n))_{n\geq 0}$ converges weakly to a unit mass at p_0. This implies that $\lim_{n\to\infty} \Delta CS(\lambda_n) = \lim_{n\to\infty} \Delta AS(\lambda_n) = 0$ for any such sequence, so that $\Delta CS(\lambda)$ and $\Delta AS(\lambda)$ both tend to 0 as $\lambda \to 0$. □

D.3.2 Taylor Approximation of Equilibrium Behavior

The goal of this section is to obtain, in the neighborhood of $\lambda = 0$, the Taylor approximations of $\underline{p}(\lambda, s)$, $\overline{p}(\lambda, s)$, and λ with respect to $w(\lambda, s) - c$. These approximations are required to calculate the first-order Taylor approximations of ΔCS with respect to λ in Section D.3.3. To ease notation, we define $\tilde{w}(\lambda, s) \equiv w(\lambda, s) - c$, and drop arguments (λ, s) in the following.

Third-order Taylor approximation of $\overline{p}$.

Lemma D.3.3. *In the neighborhood of $\lambda = 0$, we have*

$$\overline{p} = p_0 + \alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3 + o(\tilde{w}^3), \tag{69}$$

where α and β were defined in equations (45) and (46), and δ is a constant.

Proof. By Lemma D.3.1, in the neighborhood of $\lambda = 0$, $\overline{p}$ is equal to $p^m(w)$ and solves $\pi_1'(\overline{p}, w) = 0$. Manipulating this equation yields

$$\begin{aligned}\tilde{w} &= \frac{r'(\overline{p})}{D'(\overline{p})} \\ &= \underbrace{\frac{r_0''}{D_0'}}_{\tilde{\alpha}}(\overline{p} - p_0) + \underbrace{\left(\frac{1}{2}\frac{r_0'''}{D_0'} - r_0''\frac{D_0''}{(D_0')^2}\right)}_{\tilde{\beta}}(\overline{p} - p_0)^2 + \underbrace{\frac{1}{6}\left(\frac{r'}{D'}\right)'''\bigg|_{p_0}}_{\tilde{\delta}}(\overline{p} - p_0)^3 + o((\overline{p} - p_0)^3).\end{aligned}$$

This implies in particular that, for every $k \geq 0$, $o(\tilde{w}^k) = o((\overline{p} - p_0)^k)$ (and vice versa), and that

$$\begin{aligned}\tilde{w}^2 &= \tilde{\alpha}^2(\overline{p} - p_0)^2 + 2\tilde{\alpha}\tilde{\beta}(\overline{p} - p_0)^3 + o(\tilde{w}^3), \\ \tilde{w}^3 &= \tilde{\alpha}^3(\overline{p} - p_0)^3 + o(\tilde{w}^3).\end{aligned}$$

Hence,

$$\begin{aligned}\overline{p} &= p_0 + \frac{1}{\tilde{\alpha}}\tilde{w} - \frac{\tilde{\beta}}{\tilde{\alpha}}(\overline{p} - p_0)^2 - \frac{\tilde{\delta}}{\tilde{\alpha}}(\overline{p} - p_0)^3 + o((\overline{p} - p_0)^3) \\ &= p_0 + \frac{1}{\tilde{\alpha}}\tilde{w} - \frac{\tilde{\beta}}{\tilde{\alpha}}\left[\frac{\tilde{w}^2}{\tilde{\alpha}^2} - 2\frac{\tilde{\beta}}{\tilde{\alpha}^4}\tilde{w}^3\right] - \frac{\tilde{\delta}}{\tilde{\alpha}^4}\tilde{w}^3 + o(\tilde{w}^3) \\ &= p_0 + \frac{1}{\tilde{\alpha}}\tilde{w} - \frac{\tilde{\beta}}{\tilde{\alpha}^3}\tilde{w}^2 + \left[\frac{\tilde{\beta}^2}{\tilde{\alpha}^5} - \frac{\tilde{\delta}}{\tilde{\alpha}^4}\right]\tilde{w}^3 + o(\tilde{w}^3).\end{aligned}$$

It is easily verified that the coefficient on $\tilde{w}$ is α and the coefficient on $\tilde{w}^2$ is $\beta/2$. □

First-order Taylor approximation of $\underline{p}$. By Lemma D.3.1, in the neighborhood of $\lambda = 0$, $\overline{p} = p^m(w)$, implying that equation (20) reduces to

$$\int_{\underline{p}}^{\overline{p}} \varphi(p, w)(\overline{p} - p)dp = 0, \tag{70}$$

where

$$\varphi(p, w) \equiv \frac{r'(p)}{D^2(p)}\frac{1}{(p - w)^3}. \tag{71}$$

In the following lemma, we use equation (70) to derive the first-order Taylor approximation of $\underline{p}$ with respect to $\tilde{w}$ for $\lambda \simeq 0$.

Lemma D.3.4. *In the neighborhood of $\lambda = 0$, we have*

$$\underline{p} = p_0 - \frac{1}{2}\alpha\tilde{w} + o(\tilde{w}). \tag{72}$$

Proof. Applying Lemma D.1.1 to equation (70), we obtain the existence of a bounded function $M(\lambda)$ such that

$$0 = \varphi_0'(w)\int_{\underline{p}}^{\overline{p}}(\overline{p}-p)(p-p_0)dp + \underbrace{M(\lambda)\int_{\underline{p}}^{\overline{p}}(\overline{p}-p)(p-p_0)^2dp}_{\equiv R(\lambda)},$$

where

$$\varphi_0'(w) \equiv \left.\frac{\partial\varphi}{\partial p}\right|_{(p_0,w)} = \frac{r_0''}{D_0^2}\frac{1}{(p_0-w)^3}.$$

It follows that:

$$\begin{aligned}
0 &= \varphi_0'(w)\left[\frac{1}{2}(\overline{p}-p_0)(p-p_0)^2 - \frac{1}{3}(p-p_0)^3\right]_{\underline{p}}^{\overline{p}} + R(\lambda)\\
&= (\overline{p}-p_0)^3\left\{\varphi_0'(w)\left[\frac{1}{2}\left(1-\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^2\right) - \frac{1}{3}\left(1+\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^3\right)\right] + \frac{R(\lambda)}{(\overline{p}-p_0)^3}\right\}\\
&= \varphi_0'(w)\left[\frac{1}{2}\left(1-\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^2\right) - \frac{1}{3}\left(1+\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^3\right)\right] + \frac{R(\lambda)}{(\overline{p}-p_0)^3}.
\end{aligned} \tag{73}$$

Next, we derive an upper bound for the absolute value of the remainder term:

$$\begin{aligned}
\left|\frac{R(\lambda)}{(\overline{p}-p_0)^3}\right| &\le \frac{|M(\lambda)|}{(\overline{p}-p_0)^3}\int_{\underline{p}}^{\overline{p}}(\overline{p}-p)(p-p_0)^2dp\\
&= \frac{|M(\lambda)|}{(\overline{p}-p_0)^3}\left[\frac{1}{3}(\overline{p}-p_0)(p-p_0)^3 - \frac{1}{4}(p-p_0)^4\right]_{\underline{p}}^{\overline{p}}\\
&= |M(\lambda)|\left[\frac{1}{3}(\overline{p}-p_0)\left(1+\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^3\right) - \frac{1}{4}\left((\overline{p}-p_0)-(p_0-\underline{p})\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^3\right)\right]\\
&= |M(\lambda)|\left[\frac{1}{12}(\overline{p}-p_0) + \left(\frac{\overline{p}-p_0}{3}+\frac{p_0-\underline{p}}{4}\right)\left(\frac{p_0-\underline{p}}{\overline{p}-p_0}\right)^3\right].
\end{aligned}$$

We wish to show that

$$\varsigma \equiv \frac{p_0-\underline{p}}{\overline{p}-p_0} \xrightarrow[\lambda\to 0]{} \frac{1}{2}.$$

Assume for a contradiction that this is not the case. Then, there exists $\varepsilon_0 > 0$, a sequence $(\lambda^n)_{n\geq 0}$ that converges to 0, and an associated sequence of equilibrium upper and lower endpoints of the support and wholesale price $(\overline{p}^n, \underline{p}^n, w^n)_{n\geq 0}$ that converges to (p_0, p_0, c), such that

$$\left| \underbrace{\frac{p_0 - \underline{p}^n}{\overline{p}^n - p_0}}_{\equiv \varsigma^n} - \frac{1}{2} \right| > \varepsilon_0$$

for every n.

Suppose first that $(\varsigma^n)_{n\geq 0}$ is bounded. Then, we can extract a subsequence that converges to some $\varsigma^* \neq 1/2$. Moreover, the boundedness of $(\varsigma^n)_{n\geq 0}$ and the above upper bound on the remainder term imply that

$$\frac{R(\lambda^n)}{(\overline{p}^n - p_0)^3} \underset{n\to\infty}{\longrightarrow} 0.$$

Taking limits along the convergent subsequence in equation (73), this implies that

$$\frac{1}{2}(1 - \varsigma^{*2}) - \frac{1}{3}(1 + \varsigma^{*3}) = 0.$$

The above polynomial has exactly two roots: -1 and $1/2$. As $\varsigma^n > 0$ for every n, this implies that $\varsigma^* = 1/2$, a contradiction.

Next, suppose that $(\varsigma^n)_{n\geq 0}$ is not bounded, and extract a subsequence that diverges to $+\infty$. Along the divergent subsequence, for n sufficiently high, we have

$$\begin{aligned} |\varphi_0'(w^n)| \left[\frac{1}{2} \left((\varsigma^n)^2 - 1 \right) + \frac{1}{3}((\varsigma^n)^3 + 1) \right] &= \left| \frac{R(\lambda^n)}{(\overline{p}^n - p_0)^3} \right| \\ &\leq |M(\lambda^n)| \left[\frac{1}{12}(\overline{p}^n - p_0) + \left(\frac{\overline{p}^n - p_0}{3} + \frac{p_0 - \underline{p}^n}{4} \right) (\varsigma^n)^3 \right]. \end{aligned}$$

Therefore,

$$\frac{1}{2} \left((\varsigma^n)^2 - 1 \right) + \frac{1}{3} + \left(\frac{1}{3} - \frac{|M(\lambda^n)|}{|\varphi_0'(w^n)|} \left(\frac{\overline{p}^n - p_0}{3} + \frac{p_0 - \underline{p}^n}{4} \right) \right) (\varsigma^n)^3 \leq \frac{|M(\lambda^n)|}{|\varphi_0'(w^n)|} \frac{1}{12} (\overline{p}^n - p_0).$$

As n tends to ∞, the left-hand side of the above inequality tends to $+\infty$ whereas the right-hand side tends to zero, which is again a contradiction.

It follows that $\varsigma \underset{\lambda\to 0}{\longrightarrow} 1/2$. Hence, in the neighborhood of $\lambda = 0$,

$$\underline{p} - p_0 = -\frac{1}{2} (\overline{p} - p_0) + o(\overline{p} - p_0),$$

and so, by Lemma D.3.3

$$\underline{p} = p_0 - \frac{1}{2}\alpha\tilde{w} + o(\tilde{w}).$$ □

Lemma D.3.4 implies that $o((\underline{p} - p_0)^k) = o(\tilde{w}^k)$ (and vice versa) for every $k \geq 0$.

Second-order Taylor approximation of $\underline{p}$.

Lemma D.3.5. *In the neighborhood of $\lambda = 0$, we have*

$$\underline{p} = p_0 - \frac{1}{2}\alpha\tilde{w} + \gamma\tilde{w}^2 + \varepsilon\tilde{w}^3 + o(\tilde{w}^3), \tag{74}$$

where ε is a constant and

$$\gamma \equiv -\frac{5}{32}\beta + \frac{3}{16}\frac{\alpha}{p_0 - c}(3\alpha - 1). \tag{75}$$

Proof. Applying Lemma D.1.1 to equation (70), we have that

$$0 = \varphi_0'(w)\int_{\underline{p}}^{\overline{p}}(\overline{p} - p)(p - p_0)dp + \frac{1}{2}\varphi_0''(w)\int_{\underline{p}}^{\overline{p}}(\overline{p} - p)(p - p_0)^2 dp$$
$$+ \frac{1}{6}\varphi_0'''(w)\int_{\underline{p}}^{\overline{p}}(\overline{p} - p)(p - p_0)^3 dp + O\left(\int_{\underline{p}}^{\overline{p}}(\overline{p} - p)(p - p_0)^4 dp\right),$$

where $\varphi_0'(w)$ was defined in the proof of Lemma D.3.4,

$$\varphi_0''(w) \equiv \left.\frac{\partial^2\varphi}{\partial p^2}\right|_{(p_0,w)} = \varphi_0'(w)\left(\frac{r_0'''}{r_0''} - \frac{6D_0 + 4(p_0 - w)D_0'}{(p_0 - w)D_0}\right),$$

and $\varphi_0'''(w) \equiv \partial^3\varphi/\partial p^3|_{(p_0,w)}$. Note that $\varphi_0'''(w)$ has a finite limit as λ tends to 0, as the equilibrium w is bounded away from p_0.

Define again $\varsigma \equiv (p_0 - \underline{p})/(\overline{p} - p_0)$. Then, the first integral can be computed as

$$\int_{\underline{p}}^{\overline{p}}(\overline{p} - p)(p - p_0)dp = (\overline{p} - p_0)^3\left(\frac{1}{2}(1 - \varsigma^2) - \frac{1}{3}(1 + \varsigma^3)\right) = -(\overline{p} - p_0)^3\frac{1}{3}(\varsigma + 1)^2\left(\varsigma - \frac{1}{2}\right).$$

The second integral can be computed as

$$\int_{\underline{p}}^{\overline{p}}(\overline{p} - p)(p - p_0)^2 dp = \left[\frac{1}{3}(\overline{p} - p)(p - p_0)^3\right]_{\underline{p}}^{\overline{p}} + \frac{1}{3}\int_{\underline{p}}^{\overline{p}}(p - p_0)^3 dp$$
$$= \frac{1}{3}(\overline{p} - \underline{p})(p_0 - \underline{p})^3 + \frac{1}{12}\left((\overline{p} - p_0)^4 - (p_0 - \underline{p})^4\right)$$
$$= (\overline{p} - p_0)^3\left[\left(\frac{\overline{p} - \underline{p}}{3} - \frac{p_0 - \underline{p}}{12}\right)\varsigma^3 + \frac{\overline{p} - p_0}{12}\right],$$

where we have obtained the first line by integrating by parts. Finally, the third integral is equal to

$$\begin{aligned}\int_{\underline{p}}^{\overline{p}}(\overline{p}-p)(p-p_0)^3dp &= \left[\frac{1}{4}(\overline{p}-p)(p-p_0)^4\right]_{\underline{p}}^{\overline{p}} + \frac{1}{4}\int_{\underline{p}}^{\overline{p}}(p-p_0)^4dp \\ &= -\frac{1}{4}(\overline{p}-\underline{p})(p_0-\underline{p})^4 + \frac{1}{20}\left((\overline{p}-p_0)^5+(p_0-\underline{p})^5\right) \\ &= (\overline{p}-p_0)^4\left[\left(\frac{p_0-\underline{p}}{20}-\frac{\overline{p}-\underline{p}}{4}\right)\varsigma^4+\frac{\overline{p}-p_0}{20}\right].\end{aligned}$$

Note that

$$\int_{\underline{p}}^{\overline{p}}(p-p_0)^4(\overline{p}-p)dp \leq (\overline{p}-\underline{p})^6,$$

implying that the remainder term is $O\left((\overline{p}-\underline{p})^6\right)$ and thus, by Lemmas D.3.3 and D.3.4, a little-o of $\tilde{w}^5$.

Combining the four terms, dividing through by $(\overline{p}-p_0)^3$, and rearranging terms yields:

$$\begin{aligned}\varsigma-\frac{1}{2} &= \frac{1}{2}\frac{\varphi_0''(w)}{\varphi_0'(w)}\frac{3}{(\varsigma+1)^2}\left[\left(\frac{\overline{p}-\underline{p}}{3}-\frac{p_0-\underline{p}}{12}\right)\varsigma^3+\frac{\overline{p}-p_0}{12}\right] \\ &\quad+\frac{1}{2}\frac{\varphi_0'''(w)}{\varphi_0'(w)}\frac{1}{(\varsigma+1)^2}(\overline{p}-p_0)\left[\left(\frac{p_0-\underline{p}}{20}-\frac{\overline{p}-\underline{p}}{4}\right)\varsigma^4+\frac{\overline{p}-p_0}{20}\right]+o(\tilde{w}^2). \quad (76)\end{aligned}$$

We begin by using the above expression to obtain an approximation of ς at the first order in $\tilde{w}$. By Lemmas D.3.3 and D.3.4, the term inside square brackets on the first line is at most first order in $\tilde{w}$, while the term on the second line is a little-o of $\tilde{w}$. Thus, this expression simplifies to

$$\begin{aligned}\varsigma &= \frac{1}{2}+\frac{1}{2}\frac{\varphi_0''(c)}{\varphi_0'(c)}\frac{3}{(\frac{1}{2}+1)^2}\left[\left(\frac{\overline{p}-\underline{p}}{3}-\frac{p_0-\underline{p}}{12}\right)\left(\frac{1}{2}\right)^3+\frac{\overline{p}-p_0}{12}\right]+o(\tilde{w}) \\ &= \frac{1}{2}+\frac{2}{3}\left(\frac{r_0'''}{r_0''}-\frac{2}{p_0-c}\right)\left[\left(\frac{\overline{p}-\underline{p}}{3}-\frac{p_0-\underline{p}}{12}\right)\frac{1}{8}+\frac{\overline{p}-p_0}{12}\right]+o(\tilde{w}) \\ &= \frac{1}{2}+\frac{2}{3}\left(\frac{r_0'''}{r_0''}-\frac{2}{p_0-c}\right)\alpha\left[\left(\frac{3}{2}\times\frac{1}{3}-\frac{1}{2}\times\frac{1}{12}\right)\frac{1}{8}+\frac{1}{12}\right]\tilde{w}+o(\tilde{w}) \\ &= \frac{1}{2}+\underbrace{\frac{3}{32}\left(\frac{r_0'''}{r_0''}-\frac{2}{p_0-c}\right)\alpha}_{\equiv\kappa}\,\tilde{w}+o(\tilde{w}).\end{aligned}$$

Next, we use equation (76) again to obtain an approximation of ς at the second order in

$\tilde{w}$:

$$\begin{aligned}
\varsigma &= \frac{1}{2} + \frac{1}{2}\left[\frac{\varphi_0''(c)}{\varphi_0'(c)} + \left(\frac{\varphi_0''(w)}{\varphi_0'(w)}\right)'\bigg|_{w=c}\tilde{w}\right]\left(\frac{4}{3} - \frac{16}{9}\kappa\tilde{w}\right)\frac{9}{64}\alpha\tilde{w} + \frac{1}{160}\frac{\varphi_0'''(c)}{\varphi_0'(c)}\alpha^2\tilde{w}^2 + o(\tilde{w}^2) \\
&= \frac{1}{2} + \kappa\tilde{w} + \underbrace{\alpha\left[\frac{3}{32}\left(\frac{\varphi_0''(w)}{\varphi_0'(w)}\right)'\bigg|_{w=c} - \frac{1}{8}\frac{\varphi_0''(c)}{\varphi_0'(c)}\kappa + \frac{1}{160}\frac{\varphi_0'''(c)}{\varphi_0'(c)}\alpha\right]}_{\equiv\tilde{\varepsilon}}\tilde{w}^2 + o(\tilde{w}^2).
\end{aligned}$$

Therefore, applying Lemma D.3.3, we have that

$$\begin{aligned}
p_0 - \underline{p} &= \left(\frac{1}{2} + \kappa\tilde{w} + \tilde{\varepsilon}\tilde{w}^2\right)(\overline{p} - p_0) + o(\tilde{w}^3) \\
&= \left(\frac{1}{2} + \kappa\tilde{w} + \tilde{\varepsilon}\tilde{w}^2\right)\left(\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3\right) + o(\tilde{w}^3) \\
&= \frac{1}{2}\alpha\tilde{w} + \left[\frac{1}{4}\beta + \alpha\kappa\right]\tilde{w}^2 + \underbrace{\left[\frac{1}{2}\delta + \frac{1}{2}\beta\kappa + \alpha\tilde{\varepsilon}\right]}_{\equiv-\varepsilon}\tilde{w}^3 + o(\tilde{w}^3).
\end{aligned}$$

It remains to show that the expression in the square brackets is equal to γ defined in equation (75). Note that

$$\begin{aligned}
-\frac{1}{4}\beta - \alpha\kappa &= -\frac{1}{4}\beta - \frac{3}{32}\left(\frac{r_0'''}{r_0''} - \frac{2}{p_0 - c}\right)\alpha^2 \\
&= -\frac{1}{4}\beta - \frac{3}{16}\alpha^2\left[\frac{1}{2}\frac{r_0'''}{r_0''} - \frac{D_0''}{D_0'} + \frac{D_0''}{D_0'} - \frac{1}{p_0 - c}\right] \\
&= -\frac{5}{32}\beta - \frac{3}{16}\alpha^2\frac{1}{(p_0 - c)D_0'}\left[(p_0 - c)D_0'' - D_0'\right] \\
&= -\frac{5}{32}\beta - \frac{3}{16}\alpha^2\frac{1}{(p_0 - c)D_0'}\left[r_0'' - 3D_0'\right] \\
&= -\frac{5}{32}\beta + \frac{3}{16}\frac{\alpha}{p_0 - c}(3\alpha - 1) \\
&= \gamma.
\end{aligned}$$

□

Relating the derivatives of demand to α and β.

Lemma D.3.6. *We have:*

$$\begin{aligned}
D_0' &= -\frac{D_0}{p_0 - c}, \\
D_0'' &= \frac{D_0}{(p_0 - c)^2}\frac{2\alpha - 1}{\alpha}, \\
D_0''' &= \frac{D_0}{(p_0 - c)^3}\left[\frac{\beta(p_0 - c)}{\alpha^3} + \frac{(2 - 3\alpha)(2\alpha - 1)}{\alpha^2}\right].
\end{aligned}$$

Proof. The expression for D_0' follows immediately from the monopolist's first-order condition. Moreover,

$$D_0'' = \frac{1}{p_0 - c}\left((p_0 - c)D_0'' + 2D_0' - 2D_0'\right) = \frac{D_0'}{p_0 - c}\left(\frac{r_0''}{D_0'} - 2\right) = \frac{D_0}{(p_0 - c)^2}\frac{2\alpha - 1}{\alpha}.$$

Finally,

$$\begin{aligned}
D_0''' &= \frac{1}{p_0 - c}\left(r_0''' - 3D_0''\right) \\
&= \frac{1}{p_0 - c}\left(\frac{1}{2}\frac{r_0'''}{r_0''}2r_0'' - 3D_0''\right) \\
&= \frac{1}{p_0 - c}\left(\left(\frac{1}{2}\frac{r_0'''}{r_0''} - \frac{D_0''}{D_0'}\right)2r_0'' + 2\frac{D_0''}{D_0'}r_0'' - 3D_0''\right) \\
&= \frac{1}{p_0 - c}\left(-\beta\frac{r_0''}{\alpha^2} + 2\frac{D_0''}{D_0'}r_0'' - 3D_0''\right) \\
&= \frac{1}{p_0 - c}\left[-\frac{\beta}{\alpha^3}D_0' + \left(\frac{2}{\alpha} - 3\right)D_0''\right] \\
&= \frac{D_0}{(p_0 - c)^3}\left[\frac{\beta(p_0 - c)}{\alpha^3} + \frac{(2 - 3\alpha)(2\alpha - 1)}{\alpha^2}\right].
\end{aligned}$$

□

Third-order Taylor approximations of λ and $\pi(\overline{p}, w)$.

Lemma D.3.7. *In the neighborhood of $\lambda = 0$, we have:*

$$\lambda = \frac{9}{16}\alpha\frac{\tilde{w}^2}{(p_0 - c)^2} + \frac{9}{64}\left(5\alpha - 3\alpha^2 + \frac{3}{2}\beta(p_0 - c)\right)\frac{\tilde{w}^3}{(p_0 - c)^3} + o(\tilde{w}^3),$$

$$\pi(\overline{p}, w) = r_0 - D_0\tilde{w} + \frac{D_0}{p_0 - c}\frac{\alpha}{2}\tilde{w}^2 + \frac{D_0}{(p_0 - c)^2}\left[-\frac{1}{6}(2\alpha - 1)\alpha + \frac{1}{6}\beta(p_0 - c)\right]\tilde{w}^3 + o(\tilde{w}^3).$$

Proof. Solving out for λ in equation (3) yields:

$$\lambda = \frac{\pi(\overline{p}, w) - \pi(\underline{p}, w)}{\pi(\overline{p}, w) + \pi(\underline{p}, w)}.$$

We seek third-order Taylor approximations of $\overline{\pi}(\tilde{w}) \equiv \pi(\overline{p}, c + \tilde{w})$ and $\underline{\pi}(\tilde{w}) \equiv \pi(\underline{p}, c + \tilde{w})$. To ease notation, we define $\tilde{p}_0 \equiv p_0 - c$.

Applying Lemma D.3.3, we have:

$$\begin{aligned}
\overline{\pi}(\tilde{w}) &= \left(\tilde{p}_0 + (\alpha - 1)\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3\right)D\left(p_0 + \alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3\right) + o(\tilde{w}^3) \\
&= \left(\tilde{p}_0 + (\alpha - 1)\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(D_0 + D_0' \left[\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3\right] + \frac{D_0''}{2}\left[\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2\right]^2 + \frac{D_0'''}{6}\alpha^3\tilde{w}^3\right) + o(\tilde{w}^3) \\
&= \tilde{p}_0 D_0 - D_0\tilde{w} + \tilde{p}_0\left[\frac{D_0''}{2}\left[\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2\right]^2 + \frac{D_0'''}{6}\alpha^3\tilde{w}^3\right] \\
&\quad + D_0'\left((\alpha-1)\tilde{w} + \frac{1}{2}\beta\tilde{w}^2\right)\left(\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2\right) + \frac{D_0''}{2}(\alpha-1)\alpha^2\tilde{w}^3 + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} + \left[\tilde{p}_0\frac{D_0''}{2}\alpha^2 + D_0'\alpha(\alpha-1)\right]\tilde{w}^2 \\
&\quad + \left[\tilde{p}_0\frac{D_0''}{2}\alpha\beta + \tilde{p}_0\frac{D_0'''}{6}\alpha^3 - D_0'\left(-\alpha\beta + \frac{1}{2}\beta\right) + \frac{D_0''}{2}(\alpha-1)\alpha^2\right]\tilde{w}^3 + o(\tilde{w}^3),
\end{aligned}
$$

which, using Lemma D.3.6, can be further simplified to

$$
\begin{aligned}
\overline{\pi}(\tilde{w}) &= r_0 - D_0\tilde{w} + \frac{D_0}{\tilde{p}_0}\left[\frac{1}{2}(2\alpha-1)\alpha - \alpha(\alpha-1)\right]\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[(2\alpha-1)\left(\frac{1}{2}(\alpha-1)\alpha + \frac{1}{2}\beta\tilde{p}_0\right)\right. \\
&\quad \left. -\frac{1}{2}\beta\tilde{p}_0(2\alpha-1) + \frac{1}{6}\beta\tilde{p}_0 + \frac{1}{6}\alpha(2-3\alpha)(2\alpha-1)\right]\tilde{w}^3 + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} + \frac{D_0}{\tilde{p}_0}\frac{\alpha}{2}\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[\frac{2\alpha-1}{6}\left(3\alpha(\alpha-1) + \alpha(2-3\alpha)\right) + \frac{1}{6}\beta\tilde{p}_0\right]\tilde{w}^3 + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} + \frac{D_0}{\tilde{p}_0}\frac{\alpha}{2}\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[-\frac{1}{6}(2\alpha-1)\alpha + \frac{1}{6}\beta\tilde{p}_0\right]\tilde{w}^3 + o(\tilde{w}^3).
\end{aligned}
$$

We also have:

$$
\begin{aligned}
\underline{\pi}(\tilde{w}) &= \left(\tilde{p}_0 - \left(\frac{\alpha}{2}+1\right)\tilde{w} + \gamma\tilde{w}^2 + \epsilon\tilde{w}^3\right) D\left(p_0 - \frac{\alpha}{2}\tilde{w} + \gamma\tilde{w}^2 + \epsilon\tilde{w}^3\right) + o(\tilde{w}^3) \\
&= \left(\tilde{p}_0 - \left(\frac{\alpha}{2}+1\right)\tilde{w} + \gamma\tilde{w}^2 + \epsilon\tilde{w}^3\right) \\
&\quad \times \left(D_0 + D_0'\left[-\frac{\alpha}{2}\tilde{w} + \gamma\tilde{w}^2 + \epsilon\tilde{w}^3\right] + \frac{D_0''}{2}\left[-\frac{\alpha}{2}\tilde{w} + \gamma\tilde{w}^2\right]^2 - \frac{D_0'''}{48}\alpha^3\tilde{w}^3\right) + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} + \tilde{p}_0\left[\frac{D_0''}{2}\left[-\frac{\alpha}{2}\tilde{w} + \gamma\tilde{w}^2\right]^2 - \frac{D_0'''}{48}\alpha^3\tilde{w}^3\right] \\
&\quad + D_0'\left(-\left(1+\frac{\alpha}{2}\right)\tilde{w} + \gamma\tilde{w}^2\right)\left(-\frac{\alpha}{2}\tilde{w} + \gamma\tilde{w}^2\right) - \frac{D_0''}{2}\left(1+\frac{\alpha}{2}\right)\frac{\alpha^2}{4}\tilde{w}^3 + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} + \left[\tilde{p}_0\frac{D_0''}{8}\alpha^2 + D_0'\frac{\alpha}{2}\left(1+\frac{\alpha}{2}\right)\right]\tilde{w}^2 \\
&\quad + \left[-\tilde{p}_0\frac{D_0''}{2}\alpha\gamma - \tilde{p}_0\frac{D_0'''}{48}\alpha^3 - D_0'\left(1+\alpha\right)\gamma - \frac{D_0''}{16}\alpha^2(2+\alpha)\right]\tilde{w}^3 + o(\tilde{w}^3),
\end{aligned}
$$

which, using again Lemma D.3.6, simplifies to

$$
\begin{aligned}
\underline{\pi}(\tilde{w}) &= r_0 - D_0\tilde{w} + \frac{D_0}{\tilde{p}_0}\left[\frac{1}{8}\alpha(2\alpha-1) - \frac{\alpha}{2} - \frac{\alpha^2}{4}\right]\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[-\frac{(2\alpha-1)}{2}\gamma\tilde{p}_0\right. \\
&\quad \left. -\frac{1}{48}\left(\beta\tilde{p}_0 + \alpha(2-3\alpha)(2\alpha-1)\right) + (1+\alpha)\gamma\tilde{p}_0 - \frac{2\alpha-1}{16}\alpha(2+\alpha)\right]\tilde{w}^3 + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} - \frac{D_0}{\tilde{p}_0}\frac{5}{8}\alpha\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[\frac{3}{2}\gamma\tilde{p}_0 - \frac{1}{48}\beta\tilde{p}_0 - \frac{1}{6}\alpha(2\alpha-1)\right]\tilde{w}^3 + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} - \frac{D_0}{\tilde{p}_0}\frac{5}{8}\alpha\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[-\frac{15}{64}\beta\tilde{p}_0 + \frac{9}{32}\alpha(3\alpha-1) - \frac{1}{48}\beta\tilde{p}_0 - \frac{1}{6}\alpha(2\alpha-1)\right]\tilde{w}^3 \\
&\quad + o(\tilde{w}^3) \\
&= r_0 - D_0\tilde{w} - \frac{D_0}{\tilde{p}_0}\frac{5}{8}\alpha\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\left[-\frac{49}{192}\beta\tilde{p}_0 + \frac{\alpha}{96}(49\alpha-11)\right]\tilde{w}^3 + o(\tilde{w}^3).
\end{aligned}
$$

Hence,

$$
\overline{\pi}(\tilde{w}) - \underline{\pi}(\tilde{w}) = \frac{D_0}{\tilde{p}_0}\frac{9}{8}\alpha\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\frac{9}{32}\left(\alpha - 3\alpha^2 + \frac{3}{2}\beta\tilde{p}_0\right)\tilde{w}^3 + o(\tilde{w}^3)
$$

and $\overline{\pi}(\tilde{w}) + \underline{\pi}(\tilde{w}) = 2r_0 - 2D_0\tilde{w} + o(\tilde{w})$. It follows that

$$
\begin{aligned}
\lambda &= \frac{\frac{9}{16}\alpha\frac{\tilde{w}^2}{\tilde{p}_0^2} + \frac{9}{64}\left(\alpha - 3\alpha^2 + \frac{3}{2}\beta\tilde{p}_0\right)\frac{\tilde{w}^3}{\tilde{p}_0^3}}{1 - \frac{\tilde{w}}{\tilde{p}_0}} + o(\tilde{w}^3) \\
&= \frac{9}{16}\alpha\frac{\tilde{w}^2}{\tilde{p}_0^2} + \frac{9}{64}\left(5\alpha - 3\alpha^2 + \frac{3}{2}\beta\tilde{p}_0\right)\frac{\tilde{w}^3}{\tilde{p}_0^3} + o(\tilde{w}^3). \qquad \square
\end{aligned}
$$

Lemma D.3.7 implies that $o(\lambda^k) = o(\tilde{w}^{2k})$ (and vice versa) for any $k \geq 0$, and that

$$
\tilde{w}^2 = \frac{16}{9}\frac{(p_0 - c)^2}{\alpha}\lambda + o(\lambda).
$$

D.3.3 Proof of Proposition 9

It follows from equation (42) that $\Delta CS = \Psi/(4\lambda)$, where

$$
\Psi \equiv -4\lambda\int_{p_0}^{\overline{p}} D(p)dp + (1+\lambda)^2\int_{\underline{p}}^{\overline{p}} D(p)dp - (1-\lambda)^2\overline{\pi}(\tilde{w})^2\int_{\underline{p}}^{\overline{p}}\frac{dp}{(p-w)^2D(p)},
$$

and $\overline{\pi}(\tilde{w}) \equiv \pi(\overline{p}, \tilde{w} + c)$. We seek a fourth-order Taylor approximation of Ψ with respect to $\tilde{w}$. As λ is second order in $\tilde{w}$, we have that

$$
(1+\lambda)^2 = 1 + 2\lambda + o(\tilde{w}^3) \quad \text{and} \quad (1-\lambda)^2 = 1 - 2\lambda + o(\tilde{w}^3),
$$

which implies that

$$\Psi = 2\lambda \overbrace{\left(\int_{\underline{p}}^{\overline{p}} D(p)dp + \overline{\pi}(\tilde{w})^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)} - 2\int_{p_0}^{\overline{p}} D(p)dp \right)}^{\equiv A} + \underbrace{\int_{\underline{p}}^{\overline{p}} D(p)dp - \overline{\pi}(\tilde{w})^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)}}_{\equiv B} + o(\tilde{w}^4).$$

As λ is second order, we require a second-order approximation of A and a fourth-order approximation of B.

Approximation of B. Put

$$B_1 = \int_{\underline{p}}^{\overline{p}} D(p)dp \quad \text{and} \quad B_2 = \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)},$$

and note that $B = B_1 - \overline{\pi}(\tilde{w})^2 B_2$.

We start by approximating B_1 at the fourth order:

$$\begin{aligned}
B_1 &= D_0(\overline{p} - \underline{p}) + \frac{D_0'}{2}\left((\overline{p}-p_0)^2 - (p_0 - \underline{p})^2\right) + \frac{D_0''}{6}\left((\overline{p}-p_0)^3 + (p_0-\underline{p})^3\right) \\
&\quad + \frac{D_0'''}{24}\left((\overline{p}-p_0)^4 - (p_0-\underline{p})^4\right) + o(\tilde{w}^4) \\
&= D_0(\overline{p}-\underline{p}) - \frac{D_0}{\tilde{p}_0}\frac{1}{2}\left(\left[\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2 + \delta\tilde{w}^3\right]^2 - \left[\frac{\alpha}{2}\tilde{w} - \gamma\tilde{w}^2 - \epsilon\tilde{w}^3\right]^2\right) \\
&\quad + \frac{D_0}{\tilde{p}_0^2}\frac{2\alpha-1}{6\alpha}\left(\left[\alpha\tilde{w} + \frac{1}{2}\beta\tilde{w}^2\right]^3 + \left[\frac{\alpha}{2}\tilde{w} - \gamma\tilde{w}^2\right]^3\right) \\
&\quad + \frac{D_0}{\tilde{p}_0^3}\frac{5}{128}\alpha\left[\alpha(2\alpha-1)(2-3\alpha) + \beta\tilde{p}_0\right]\tilde{w}^4 + o(\tilde{w}^4),
\end{aligned}$$

which further simplifies to

$$\begin{aligned}
B_1 &= D_0(\overline{p}-\underline{p}) - \frac{D_0}{\tilde{p}_0}\frac{1}{2}\left(\frac{3}{4}\alpha^2\tilde{w}^2 + \alpha(\gamma+\beta)\tilde{w}^3 + \left[\frac{1}{4}\beta^2 - \gamma^2 + 2\alpha\delta + \alpha\epsilon\right]\tilde{w}^4\right) \\
&\quad + \frac{D_0}{\tilde{p}_0^2}\frac{2\alpha-1}{6\alpha}\left(\frac{9}{8}\alpha^3\tilde{w}^3 - 3\alpha^2\left(-\frac{1}{2}\beta + \frac{\gamma}{4}\right)\tilde{w}^4\right) \\
&\quad + \frac{D_0}{\tilde{p}_0^3}\frac{5}{128}\alpha\left[\alpha(2\alpha-1)(2-3\alpha) + \beta\tilde{p}_0\right]\tilde{w}^4 + o(\tilde{w}^4) \\
&= D_0(\overline{p}-\underline{p}) - \frac{D_0}{\tilde{p}_0}\frac{3}{8}\alpha^2\tilde{w}^2 + \frac{D_0}{\tilde{p}_0^2}\alpha\left[\frac{3}{16}\alpha(2\alpha-1) - \frac{1}{2}\beta\tilde{p}_0 - \frac{\gamma}{2}\tilde{p}_0\right]\tilde{w}^3
\end{aligned}$$

$$+\frac{D_0}{\tilde{p}_0^3}\left[\frac{1}{2}\left(\gamma^2-\frac{1}{4}\beta^2-2\alpha\delta-\alpha\epsilon\right)\tilde{p}_0^2+\frac{5}{128}\alpha^2(2\alpha-1)(2-3\alpha)\right.$$
$$\left.+\frac{5}{128}\alpha\beta\tilde{p}_0-\frac{\alpha(2\alpha-1)}{2}\left(-\frac{1}{2}\beta+\frac{\gamma}{4}\right)\tilde{p}_0\right]\tilde{w}^4+o(\tilde{w}^4). \tag{77}$$

Applying Lemma D.1.1 to B_2, we obtain:

$$B_2=\frac{1}{D_0}\int_{\underline{p}}^{\overline{p}}\frac{dp}{(p-w)^2}-\frac{D_0'}{D_0^2}\int_{\underline{p}}^{\overline{p}}\frac{p-p_0}{(p-w)^2}dp+\left(\frac{(D_0')^2}{D_0^3}-\frac{1}{2}\frac{D_0''}{D_0^2}\right)\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^2}{(p-w)^2}dp$$
$$+\left(\frac{D_0'D_0''}{D_0^3}-\frac{(D_0')^3}{D_0^4}-\frac{1}{6}\frac{D_0'''}{D_0^2}\right)\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^3}{(p-w)^2}dp+o(\tilde{w}^4)$$
$$=\frac{1}{D_0}\overbrace{\int_{\underline{p}}^{\overline{p}}\frac{dp}{(p-w)^2}}^{=B_2^0}+\frac{1}{\tilde{p}_0D_0}\overbrace{\int_{\underline{p}}^{\overline{p}}\frac{p-p_0}{(p-w)^2}dp}^{=B_2^1}+\frac{1}{\tilde{p}_0^2D_0}\frac{1}{2\alpha}\overbrace{\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^2}{(p-w)^2}dp}^{=B_2^2}$$
$$+\frac{1}{\tilde{p}_0^3D_0}\frac{2\alpha-\alpha^2-\beta\tilde{p}_0}{6\alpha^3}\underbrace{\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^3}{(p-w)^2}dp}_{=B_2^3}+o(\tilde{w}^4).$$

We require a fourth-order approximation for each of the above integrals.[30] As $p-w=p-p_0-\tilde{w}+\tilde{p}_0$, we have:

$$B_2^0=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}\frac{1}{1+\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{1}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2}dp$$
$$=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}\left[1-\left(\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{1}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2\right)+\left(\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{1}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2\right)^2\right.$$
$$\left.-\left(\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{1}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2\right)^3\right]dp+o(\tilde{w}^4)\quad\text{(by Lemma D.1.1)}$$
$$=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}\left[1-\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{3}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2-\frac{4}{\tilde{p}_0^3}(p-p_0-\tilde{w})^3\right]dp+o(\tilde{w}^4)$$
$$=\frac{\overline{p}-\underline{p}}{\tilde{p}_0^2}-\frac{1}{\tilde{p}_0^3}\left[(\overline{p}-p_0-\tilde{w})^2-\left(\underline{p}-p_0-\tilde{w}\right)^2\right]+\frac{1}{\tilde{p}_0^4}\left[(\overline{p}-p_0-\tilde{w})^3-\left(\underline{p}-p_0-\tilde{w}\right)^3\right]$$
$$-\frac{1}{\tilde{p}_0^5}\left[(\overline{p}-p_0-\tilde{w})^4-\left(\underline{p}-p_0-\tilde{w}\right)^4\right]+o(\tilde{w}^4)$$
$$=\frac{\overline{p}-\underline{p}}{\tilde{p}_0^2}-\frac{1}{\tilde{p}_0^3}\left[\left((\alpha-1)\tilde{w}+\frac{1}{2}\beta\tilde{w}^2+\delta\tilde{w}^3\right)^2-\left(-\left(\frac{\alpha}{2}+1\right)\tilde{w}+\gamma\tilde{w}^2+\epsilon\tilde{w}^3\right)^2\right]$$

[30] Although those integrals can be computed in closed form, it is less cumbersome to first approximate the integrands at the third order.

$$+\frac{1}{\tilde{p}_0^4}\left[\left((\alpha-1)\tilde{w}+\frac{1}{2}\beta\tilde{w}^2\right)^3+\left(\left(\frac{\alpha}{2}+1\right)\tilde{w}-\gamma\tilde{w}^2\right)^3\right]$$
$$-\frac{1}{\tilde{p}_0^5}\left[(\alpha-1)^4-\left(\frac{\alpha}{2}+1\right)^4\right]\tilde{w}^4+o(\tilde{w}^4)\quad\text{(by Lemmas D.3.3 and D.3.5)}$$
$$=\frac{\overline{p}-\underline{p}}{\tilde{p}_0^2}+\left(3\alpha-\frac{3}{4}\alpha^2\right)\frac{\tilde{w}^2}{\tilde{p}_0^3}+\left[\frac{9}{2}\alpha-\frac{9}{4}\alpha^2+\frac{9}{8}\alpha^3-(2+\alpha)\gamma\tilde{p}_0+(1-\alpha)\beta\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^4}$$
$$+\left[\left[\gamma^2-\frac{1}{4}\beta^2-2(\alpha-1)\delta-(2+\alpha)\epsilon\right]\tilde{p}_0^2-3\left(-\frac{1}{2}(\alpha-1)^2\beta+\left(\frac{\alpha}{2}+1\right)^2\gamma\right)\tilde{p}_0\right.$$
$$\left.+6\alpha-\frac{9}{2}\alpha^2+\frac{9}{2}\alpha^3-\frac{15}{16}\alpha^4\right]\frac{\tilde{w}^4}{\tilde{p}_0^5}+o(\tilde{w}^4).$$

For the second integral, we have:

$$B_2^1=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}\frac{p-p_0}{1+\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{1}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2}dp$$
$$=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}(p-p_0)\left[1-\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{3}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2\right]dp$$
$$+o(\tilde{w}^4)\quad\text{(by Lemma D.1.1)}$$
$$=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}(p-p_0)\left[1+\frac{2\tilde{w}}{\tilde{p}_0}+\frac{3\tilde{w}^2}{\tilde{p}_0^2}-\frac{2}{\tilde{p}_0}\left(1+\frac{3\tilde{w}}{\tilde{p}_0}\right)(p-p_0)+\frac{3}{\tilde{p}_0^2}(p-p_0)^2\right]dp+o(\tilde{w}^4)$$
$$=\frac{1}{\tilde{p}_0^2}\left(1+\frac{2\tilde{w}}{\tilde{p}_0}+\frac{3\tilde{w}^2}{\tilde{p}_0^2}\right)\frac{1}{2}\left[(\overline{p}-p_0)^2-\left(\underline{p}-p_0\right)^2\right]$$
$$-\frac{1}{\tilde{p}_0^3}\left(1+\frac{3\tilde{w}}{\tilde{p}_0}\right)\frac{2}{3}\left[(\overline{p}-p_0)^3-\left(\underline{p}-p_0\right)^3\right]+\frac{1}{\tilde{p}_0^4}\frac{3}{4}\left[(\overline{p}-p_0)^4-\left(\underline{p}-p_0\right)^4\right]+o(\tilde{w}^4)$$
$$=\frac{1}{\tilde{p}_0^2}\left(1+\frac{2\tilde{w}}{\tilde{p}_0}+\frac{3\tilde{w}^2}{\tilde{p}_0^2}\right)\frac{1}{2}\left(\frac{3}{4}\alpha^2\tilde{w}^2+\alpha(\gamma+\beta)\tilde{w}^3+\left[\frac{1}{4}\beta^2-\gamma^2+2\alpha\delta+\alpha\epsilon\right]\tilde{w}^4\right)$$
$$-\frac{1}{\tilde{p}_0^3}\left(1+\frac{3\tilde{w}}{\tilde{p}_0}\right)\frac{2}{3}\left(\frac{9}{8}\alpha^3\tilde{w}^3-3\alpha^2\left(-\frac{1}{2}\beta+\frac{\gamma}{4}\right)\tilde{w}^4\right)$$
$$+\frac{45}{64}\alpha^4\frac{\tilde{w}^4}{\tilde{p}_0^4}+o(\tilde{w}^4)\quad\text{(by Lemmas D.3.3 and D.3.5)}$$
$$=\frac{3}{8}\alpha^2\frac{\tilde{w}^2}{\tilde{p}_0^2}+\left[\frac{3}{4}\alpha^2-\frac{3}{4}\alpha^3+\alpha\left(\frac{\gamma}{2}+\frac{\beta}{2}\right)\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^3}+\left[\frac{9}{8}\alpha^2-\frac{9}{4}\alpha^3+\frac{45}{64}\alpha^4\right.$$
$$\left.+\alpha(\gamma+\beta)\tilde{p}_0+\alpha^2\left(-\beta+\frac{\gamma}{2}\right)\tilde{p}_0+\frac{1}{2}\left[\frac{1}{4}\beta^2-\gamma^2+2\alpha\delta+\alpha\epsilon\right]\tilde{p}_0^2\right]\frac{\tilde{w}^4}{\tilde{p}_0^4}+o(\tilde{w}^4).$$

For the third integral, we have:

$$B_2^2=\frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}\frac{(p-p_0)^2}{1+\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})+\frac{1}{\tilde{p}_0^2}(p-p_0-\tilde{w})^2}dp$$

$$= \frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}(p-p_0)^2\left[1-\frac{2}{\tilde{p}_0}(p-p_0-\tilde{w})\right]dp+o(\tilde{w}^4) \quad \text{(by Lemma D.1.1)}$$

$$= \frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}(p-p_0)^2\left[1+\frac{2\tilde{w}}{\tilde{p}_0}-\frac{2}{\tilde{p}_0}(p-p_0)\right]dp+o(\tilde{w}^4)$$

$$= \frac{1}{\tilde{p}_0^2}\left(1+\frac{2\tilde{w}}{\tilde{p}_0}\right)\frac{1}{3}\left[(\overline{p}-p_0)^3-(\underline{p}-p_0)^3\right]-\frac{1}{\tilde{p}_0^3}\frac{1}{2}\left[(\overline{p}-p_0)^4-(\underline{p}-p_0)^4\right]+o(\tilde{w}^4)$$

$$= \frac{1}{\tilde{p}_0^2}\left(1+\frac{2\tilde{w}}{\tilde{p}_0}\right)\frac{1}{3}\left[\frac{9}{8}\alpha^3\tilde{w}^3-3\alpha^2\left(-\frac{\beta}{2}+\frac{\gamma}{4}\right)\tilde{w}^4\right]-\frac{1}{\tilde{p}_0^3}\frac{15}{32}\alpha^4\tilde{w}^4$$

$$+ o(\tilde{w}^4) \quad \text{(by Lemmas D.3.3 and D.3.5)}$$

$$= \frac{1}{3}\left[\frac{9}{8}\alpha^3\frac{\tilde{w}^3}{\tilde{p}_0^2}+\left(\frac{9}{4}\alpha^3-3\alpha^2\left(-\frac{\beta}{2}+\frac{\gamma}{4}\right)\tilde{p}_0\right)\frac{\tilde{w}^4}{\tilde{p}_0^3}\right]-\frac{15}{32}\alpha^4\frac{\tilde{w}^4}{\tilde{p}_0^3}+o(\tilde{w}^4)$$

$$= \frac{3}{8}\alpha^3\frac{\tilde{w}^3}{\tilde{p}_0^2}+\left[\frac{3}{4}\alpha^3-\frac{15}{32}\alpha^4-\alpha^2\left(-\frac{\beta}{2}+\frac{\gamma}{4}\right)\tilde{p}_0\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}+o(\tilde{w}^4).$$

Finally, applying again Lemmas D.1.1, D.3.3, and D.3.5, the fourth integral can be approximated as:

$$B_2^3 = \frac{1}{\tilde{p}_0^2}\int_{\underline{p}}^{\overline{p}}(p-p_0)^3dp+o(\tilde{w}^4) = \frac{1}{\tilde{p}_0^2}\frac{1}{4}\left[(\overline{p}-p_0)^4-(\underline{p}-p_0)^4\right]+o(\tilde{w}^4) = \frac{15}{64}\alpha^4\frac{\tilde{w}^4}{\tilde{p}_0^2}+o(\tilde{w}^4).$$

Combining those approximations allows us to approximate B_2 as:

$$B_2 = \frac{1}{r_0}\left(\frac{\overline{p}-\underline{p}}{\tilde{p}_0}+b_2^2\frac{\tilde{w}^2}{\tilde{p}_0^2}+b_2^3\frac{\tilde{w}^3}{\tilde{p}_0^3}+b_2^4\frac{\tilde{w}^4}{\tilde{p}_0^4}\right)+o(\tilde{w}^4), \tag{78}$$

where

$$b_2^2 = 3\alpha-\frac{3}{8}\alpha^2,$$

$$b_2^3 = \frac{9}{2}\alpha-\frac{21}{16}\alpha^2+\frac{3}{8}\alpha^3-2\left(\gamma-\frac{\beta}{2}\right)\tilde{p}_0-\alpha\frac{\beta}{2}\tilde{p}_0-\frac{1}{2}\alpha\gamma\tilde{p}_0,$$

$$b_2^4 = \left[\frac{1}{2}\left(\gamma^2-\frac{1}{4}\beta^2\right)+(2-\alpha)\delta-\left(2+\frac{\alpha}{2}\right)\epsilon\right]\tilde{p}_0^2-\left(-\alpha^2+\frac{229}{64}\alpha-3\right)\frac{\beta}{2}\tilde{p}_0$$

$$-\left(\frac{1}{4}\alpha^2+\frac{17}{8}\alpha+3\right)\gamma\tilde{p}_0+6\alpha-\frac{187}{64}\alpha^2+\frac{253}{128}\alpha^3-\frac{15}{64}\alpha^4.$$

Next, we use our approximation of B_2 to approximate $\overline{\pi}(\tilde{w})^2B_2$. By Lemma D.3.7, we have:

$$\overline{\pi}(\tilde{w})^2 = r_0^2\left(1-\frac{\tilde{w}}{\tilde{p}_0}+\frac{\alpha}{2}\frac{\tilde{w}^2}{\tilde{p}_0^2}-\left[\frac{1}{6}(2\alpha-1)\alpha-\frac{1}{6}\beta\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^3}\right)^2+o(\tilde{w}^3)$$

$$= r_0^2\left(1 - 2\frac{\tilde{w}}{\tilde{p}_0} + \alpha\frac{\tilde{w}^2}{\tilde{p}_0^2} - \left[\frac{1}{3}(2\alpha-1)\alpha - \frac{1}{3}\beta\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^3} + \left[-\frac{\tilde{w}}{\tilde{p}_0} + \frac{\alpha}{2}\frac{\tilde{w}^2}{\tilde{p}_0^2}\right]^2\right) + o(\tilde{w}^3)$$
$$= r_0^2\left(1 - 2\frac{\tilde{w}}{\tilde{p}_0} + (\alpha+1)\frac{\tilde{w}^2}{\tilde{p}_0^2} - \frac{2}{3}\left[\alpha(\alpha+1) - \frac{\beta}{2}\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^3}\right) + o(\tilde{w}^3).$$

The approximation of the first term in equation (78) multiplied by $\overline{\pi}(\tilde{w})^2$ is:

$$\begin{aligned}
\frac{\overline{p}-\underline{p}}{\tilde{p}_0^2 D_0}\overline{\pi}(\tilde{w})^2 &= D_0\left(\overline{p}-\underline{p}\right)\left(1 - 2\frac{\tilde{w}}{\tilde{p}_0} + (1+\alpha)\frac{\tilde{w}^2}{\tilde{p}_0^2} - \frac{2}{3}\left[\alpha(\alpha+1) - \frac{\beta}{2}\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^3}\right) + o(\tilde{w}^4)\\
&= D_0\left(\overline{p} - \underline{p} - 2\frac{\tilde{w}}{\tilde{p}_0}\left(\frac{3}{2}\alpha\tilde{w} - \left(-\frac{\beta}{2}+\gamma\right)\tilde{w}^2 + (\delta-\epsilon)\tilde{w}^3\right)\right.\\
&\quad\left. +(1+\alpha)\left[\frac{3}{2}\alpha\tilde{w} - \left(-\frac{\beta}{2}+\gamma\right)\tilde{w}^2\right]\frac{\tilde{w}^2}{\tilde{p}_0^2} - \alpha\left[\alpha(\alpha+1) - \frac{\beta}{2}\tilde{p}_0\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4)\\
&= D_0\left(\overline{p} - \underline{p} - 3\alpha\frac{\tilde{w}^2}{\tilde{p}_0} + \left[\frac{3}{2}\alpha(1+\alpha) + 2\left(-\frac{\beta}{2}+\gamma\right)\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^2}\right.\\
&\quad\left. + \left[2(\epsilon-\delta)\tilde{p}_0^2 - \left[-\frac{\beta}{2} + \gamma - \alpha\beta + \alpha\gamma\right]\tilde{p}_0 - \alpha^2(\alpha+1)\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4),
\end{aligned}$$

where we have used Lemmas D.3.3 and D.3.5 to obtain the second line. The second term is:

$$\begin{aligned}
\left[3\alpha - \frac{3}{8}\alpha^2\right]\frac{\tilde{w}^2}{\tilde{p}_0^3 D_0}\overline{\pi}(\tilde{w})^2 &= D_0\left[3\alpha - \frac{3}{8}\alpha^2\right]\frac{\tilde{w}^2}{\tilde{p}_0}\left(1 - 2\frac{\tilde{w}}{\tilde{p}_0} + (\alpha+1)\frac{\tilde{w}^2}{\tilde{p}_0^2}\right) + o(\tilde{w}^4)\\
&= D_0\left(\left[3\alpha - \frac{3}{8}\alpha^2\right]\frac{\tilde{w}^2}{\tilde{p}_0} - 2\left[3\alpha - \frac{3}{8}\alpha^2\right]\frac{\tilde{w}^3}{\tilde{p}_0^2} + (\alpha+1)\left[3\alpha - \frac{3}{8}\alpha^2\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4).
\end{aligned}$$

The third term is:

$$\begin{aligned}
b_2^3\frac{\tilde{w}^3}{\tilde{p}_0^4 D_0}\overline{\pi}(\tilde{w})^2 = D_0&\left(\left[\frac{9}{2}\alpha - \frac{21}{16}\alpha^2 + \frac{3}{8}\alpha^3 - (2\gamma-\beta)\tilde{p}_0 - \alpha\frac{\beta}{2}\tilde{p}_0 - \frac{1}{2}\alpha\gamma\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^4}\right.\\
&\left. + \left[-9\alpha + \frac{21}{8}\alpha^2 - \frac{3}{4}\alpha^3 + 2(2\gamma-\beta)\tilde{p}_0 + \alpha\beta\tilde{p}_0 + \alpha\gamma\tilde{p}_0\right]\frac{\tilde{w}^4}{\tilde{p}_0^5}\right) + o(\tilde{w}^4).
\end{aligned}$$

Combining these three terms yields:

$$B_2\overline{\pi}(\tilde{w})^2 = D_0\left(\overline{p} - \underline{p} + \tilde{b}_2^2\frac{\tilde{w}^2}{\tilde{p}_0} + \tilde{b}_2^3\frac{\tilde{w}^3}{\tilde{p}_0^2} + \tilde{b}_2^4\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4),$$

where

$$\begin{aligned}
\tilde{b}_2^2 &= -\frac{3}{8}\alpha^2\\
\tilde{b}_2^3 &= \frac{3}{8}\alpha^3 + \frac{15}{16}\alpha^2 - \frac{1}{2}\alpha\left(\beta+\gamma\right)\tilde{p}_0
\end{aligned}$$

$$\tilde{b}_2^4 = \left[\frac{1}{2}\left(\gamma^2 - \frac{1}{4}\beta^2\right) - \alpha\delta - \frac{\alpha}{2}\epsilon\right]\tilde{p}_0^2 + \left[\frac{\alpha^2}{2} + \frac{27}{128}\alpha\right]\beta\tilde{p}_0 - \left[\frac{1}{4}\alpha^2 + \frac{17}{8}\alpha\right]\gamma\tilde{p}_0$$
$$+ \frac{85}{64}\alpha^2 - \frac{19}{128}\alpha^3 - \frac{15}{64}\alpha^4.$$

Combining the approximations of B_1 and $\overline{\pi}(\tilde{w})^2 B_2$ yields:

$$B = D_0\left(-\frac{9}{8}\alpha^2\frac{\tilde{w}^3}{\tilde{p}_0^2} + \left[-\frac{45}{32}\alpha^2 + \frac{27}{64}\alpha^3 - \frac{27}{64}\alpha\beta\tilde{p}_0 + \frac{9}{4}\alpha\gamma\tilde{p}_0\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4).$$

Using the expression we derived above for γ from Lemma D.3.5, we finally obtain:

$$B = D_0\left(-\frac{9}{8}\alpha^2\frac{\tilde{w}^3}{\tilde{p}_0^2} + \frac{9}{64}\alpha\left[-13\alpha + 12\alpha^2 - \frac{11}{2}\beta\tilde{p}_0\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4).$$

Approximation of A. There are three integrals in the definition of A. We have shown above (see the approximation of B_1) that the first integral can be approximated as

$$\int_{\underline{p}}^{\overline{p}} D(p)dp = D_0\left(\overline{p} - \underline{p} - \frac{3}{8}\alpha^2\frac{\tilde{w}^2}{\tilde{p}_0}\right) + o(\tilde{w}^2).$$

In addition (see the approximation of $\overline{\pi}(\tilde{w})^2 B_2$), the second integral can be approximated as

$$\overline{\pi}(\tilde{w})^2\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)} = D_0\left(\overline{p} - \underline{p} - \frac{3}{8}\alpha^2\frac{\tilde{w}^2}{\tilde{p}_0}\right) + o(\tilde{w}^2),$$

i.e., the first and second integrals coincide at the second order. The third can be easily approximated as,

$$\int_{p_0}^{\overline{p}} D(p)dp = D_0\left(\overline{p} - p_0 - \frac{1}{2\tilde{p}_0}(\overline{p} - p_0)^2\right) + o(\tilde{w}^2) = D_0\left(\overline{p} - p_0 - \frac{1}{2}\alpha^2\frac{\tilde{w}^2}{\tilde{p}_0}\right) + o(\tilde{w}^2),$$

where we have used Lemma D.1.1 to obtain the first equality and Lemma D.3.3 to obtain the second one.

Combining these three approximations yields:

$$A = 2D_0\left(p_0 - \underline{p} + \frac{1}{8}\alpha^2\frac{\tilde{w}^2}{\tilde{p}_0}\right) + o(\tilde{w}^2)$$
$$= D_0\left(\alpha\tilde{w} + \left[\frac{3}{8}\alpha - \frac{7}{8}\alpha^2 + \frac{5}{16}\beta\tilde{p}_0\right]\frac{\tilde{w}^2}{\tilde{p}_0}\right) + o(\tilde{w}^2) \quad \text{(by Lemma D.3.5).}$$

Combining this with Lemma D.3.7, we obtain:

$$2\lambda A = D_0 \left(\frac{9}{8}\alpha \frac{\tilde{w}^2}{\tilde{p}_0^2} + \frac{9}{32}\left[5\alpha - 3\alpha^2 + \frac{3}{2}\beta\tilde{p}_0\right]\frac{\tilde{w}^3}{\tilde{p}_0^3}\right)\left(\alpha\tilde{w} + \left[\frac{3}{8}\alpha - \frac{7}{8}\alpha^2 + \frac{5}{16}\beta\tilde{p}_0\right]\frac{\tilde{w}^2}{\tilde{p}_0}\right) + o(\tilde{w}^4)$$
$$= D_0\left(\frac{9}{8}\alpha^2\frac{\tilde{w}^3}{\tilde{p}_0^2} + \frac{9}{64}\alpha\left[13\alpha - 13\alpha^2 + \frac{11}{2}\beta\tilde{p}_0\right]\frac{\tilde{w}^4}{\tilde{p}_0^3}\right) + o(\tilde{w}^4).$$

Approximation of ΔCS. Combining the Taylor approximations of $2\lambda A$ and B, we obtain a Taylor approximation of $\Psi = 2\lambda A + B$:

$$\Psi = -\frac{9}{64}D_0\alpha^3\frac{\tilde{w}^4}{\tilde{p}_0^3} + o(\tilde{w}^4).$$

The approximation of consumer surplus is therefore given by

$$\Delta CS = \frac{\Psi}{4\lambda} = -\frac{1}{16}D_0\alpha^2\frac{\tilde{w}^2}{\tilde{p}_0} + o(\tilde{w}^2),$$

where we have used Lemma D.3.7. Using again Lemma D.3.7, we obtain:

$$\Delta CS = -\frac{1}{9}r_0\alpha\lambda + o(\lambda).$$

D.4 Proofs of Welfare Results When λ is High and $s > \hat{s}$

In this appendix, we study the welfare effects of banning dual pricing when λ is close to 1 and $s > \hat{s}$, thus proving Proposition 10 for the case of high search costs. We derive the Taylor approximations of ΔCS and ΔAS with respect to $(1-\lambda)^2\log(1-\lambda)$ when $\lambda \simeq 1$. We proceed as follows. In Section D.4.1, we study the limiting equilibrium behavior when λ tends to 1. In Section D.4.2, we derive some auxiliary Taylor approximations required to approximate ΔCS and ΔAS. Section D.4.3 is devoted to the approximations of ΔCS and ΔAS. In Section D.4.4, we study the distributional effects of a ban on dual pricing.

D.4.1 Basic Properties of the Equilibrium for High λ

The following lemma applies regardless of whether s is high or low:

Lemma D.4.1. *Let $s \neq \hat{s}$. Under uniform pricing, the limiting equilibrium behavior as λ goes to 1 is as follows:*

$$\lim_{\lambda\to 1} w(\lambda, s) = \lim_{\lambda\to 1} \underline{p}(\lambda, s) = p_0, \quad \lim_{\lambda\to 1} \overline{p}(\lambda, s) = \tilde{p},$$

where $\tilde{p}(s)$ is equal to $p_1 \equiv p^m(p_0)$ if $\int_{p_0}^{p^m(p_0)} D(p)dp < s$, and solves $\int_{p_0}^{\tilde{p}} D(p)dp = s$ otherwise.

This implies that

$$\lim_{\lambda\to 1} \Delta CS(\lambda, s) = \lim_{\lambda\to 1} \Delta AS(\lambda, s) = 0.$$

In the neighborhood of $\lambda = 1$*, the derivative* $\overline{p}'(w(s,\lambda), \lambda, s)$ *exists; moreover, the upper endpoint of the support is equal to the monopoly price if* $s > \hat{s}$*, and solves* $H(\overline{p}, w(s,\lambda)) = s$ *if* $s < \hat{s}$*.*

Proof. We drop argument s to ease notation. Note first that $\underline{p}(\lambda) - w(\lambda)$ tends to 0 as $\lambda \to 1$, as

$$0 \le \underline{p}(\lambda) - w(\lambda) = \frac{1-\lambda}{1+\lambda} \frac{\pi(\overline{p}(\lambda), w(\lambda))}{D(\underline{p}(\lambda))} \le \frac{1-\lambda}{1+\lambda} \frac{r_0}{D(p^m(p_0))} \underset{\lambda\to 1}{\longrightarrow} 0,$$

where we have used equation (3) and the inequality follows as $w(\lambda) \in [c, p_0]$ by Proposition 6.

Assume for a contradiction that $w(\lambda)$ does not converge to p_0 as λ tends to 1. Then, there exists a sequence $(\lambda^n)_{n\ge 0}$ that tends to 1 and such that the associated sequence of equilibrium wholesale prices $(w^n)_{n\ge 0}$ remains bounded away from p_0. As the latter sequence lives in the compact set $[c, p_0]$, we can extract a subsequence that converges to some $\breve{w} \ne p_0$. In the following, all limits will be taken along that convergent subsequence. Note that the associated sequence of equilibrium lower endpoints $(\underline{p}^n)_{n\ge 0}$ tends to $\breve{w}$.

Let us show that the associated sequence of CDFs of prices, $(F^n)_{n\ge 0}$ converges weakly to a unit mass on $\breve{w}$. Let $p < \breve{w}$. Then, for n high enough, $\underline{p}^n > p$, and so $F^n(p) = 0$. Next, let $p > \breve{w}$. Then, for n high enough, $F^n(p) > 0$ and $\pi(p, w^n)$ is bounded away from 0, and so

$$F^n(p) = \min\left(1 - \frac{1-\lambda^n}{2\lambda^n}\left[\frac{\pi(\overline{p}^n, w^n)}{\pi(p, w^n)} - 1\right], 1\right) \underset{n\to\infty}{\longrightarrow} 1.$$

It follows from the weak convergence of $(F^n)_{n\ge 0}$ and the continuity of $r(\cdot)$ that the manufacturer's expected profit converges to $r(\breve{w}) < r_0$ as n goes to infinity.

Note however that, for every n, the manufacturer could set a wholesale price of p_0, which would result in an equilibrium CDF of prices denoted by $F_0^n(\cdot)$ and a profit of

$$\Pi_0^n = \int r(p) d\left[\lambda^n(1 - (1 - F_0^n(p))^2) + (1-\lambda^n)F_0^n(p))\right].$$

Thus, the manufacturer's equilibrium expected profit must be at least Π_0^n. By Proposition 2(b) in Stahl (1989), $(F_0^n)_{n\ge 0}$ converges weakly to a unit mass on p_0 as n goes to infinity, implying that $\Pi_0^n \underset{n\to\infty}{\longrightarrow} r_0$, which contradicts the fact that the manufacturer's equilibrium profit tends to $r(\breve{w}) < r_0$ as n tends to infinity.

Hence, $w(\lambda)$ converges to p_0 as λ tends to 1. Using the same argument as above, it follows that the equilibrium CDF of prices converges weakly to a unit mass on p_0 as λ tends to 1.

This implies that $\lim_{\lambda\to 1}\Delta CS(\lambda)=\lim_{\lambda\to 1}\Delta AS(\lambda)=0$.

Finally, we turn to the properties of the upper endpoint of the support, $\overline{p}$. We have:

$$\begin{aligned}
H(p^m(w(\lambda)),w(\lambda),\lambda) &= \int_{\underline{p}(p^m(w(\lambda)),w(\lambda),\lambda)}^{p^m(w(\lambda))} D(p)\left(1-\frac{1-\lambda}{2\lambda}\left(\frac{\pi(p^m(w(\lambda)),w(\lambda))}{\pi(p,w(\lambda))}-1\right)\right)dp \\
&= \frac{1+\lambda}{2\lambda}\int_{\underline{p}(p^m(w(\lambda)),w(\lambda),\lambda)}^{p^m(w(\lambda))} D(p)dp \\
&\quad -\frac{1-\lambda}{2\lambda}\pi(p^m(w(\lambda)),w(\lambda))\log\frac{p^m(w(\lambda))-w(\lambda)}{\underline{p}(p^m(w(\lambda)),w(\lambda),\lambda)-w(\lambda)} \\
&= \frac{1+\lambda}{2\lambda}\int_{\underline{p}(p^m(w(\lambda)),w(\lambda),\lambda)}^{p^m(w(\lambda))} D(p)dp-\frac{1-\lambda}{2\lambda}\pi(p^m(w(\lambda)),w(\lambda)) \\
&\quad \times\left(\log\frac{1+\lambda}{1-\lambda}+\log\frac{D(\underline{p}(p^m(w(\lambda)),w(\lambda),\lambda))}{D(p^m(w(\lambda)))}\right) \\
&\underset{\lambda\to 1}{\longrightarrow}\int_{p_0}^{p^m(p_0)} D(p)dp=\hat{s}.
\end{aligned}$$

where we have used equation (3) to obtain the third equality. By Lemma A.1.2, it follows that, for λ close enough to 1, $\overline{p}'(w(\lambda),\lambda)$ exists (since $H(p^m(w(\lambda)),w(\lambda),\lambda)\neq s$); moreover, $\overline{p}(\lambda)$ is equal to $p^m(w(\lambda))$ if $s>\hat{s}$, and solves $H(\overline{p}(\lambda),w(\lambda),\lambda)=s$ if $s<\hat{s}$. In the case where $s>\hat{s}$, we immediately obtain that $\overline{p}(\lambda)\underset{\lambda\to 1}{\longrightarrow}p^m(p_0)=\tilde{p}$.

Suppose instead that $s<\hat{s}$. Then, for λ high enough, we have

$$\int_{\underline{p}(\lambda)}^{\overline{p}(\lambda)} D(p)F(p,w(\lambda),\lambda)dp=s=\int_{p_0}^{\tilde{p}} D(p)dp.$$

Rearranging terms and taking absolute values yields:

$$\left|\int_{\tilde{p}}^{\overline{p}(\lambda)} D(p)F(p,w(\lambda),\lambda)dp\right|=\left|\int_{\underline{p}(\lambda)}^{p_0} D(p)F(p,w(\lambda),\lambda)dp+\int_{p_0}^{\tilde{p}} D(p)\left[F(p,w(\lambda),\lambda)-1\right]dp\right|.$$

As λ goes to 1, the first integral on the right-hand side tends to 0, as the integrand is bounded and $\underline{p}(\lambda)-p_0$ tends to 0. By Lebesgue's dominated convergence theorem, the second integral on the right-hand side also tends to 0, as the integrand is again bounded and converges pointwise to 0 on $(p_0,\tilde{p}]$. It follows that

$$|\overline{p}(\lambda)-\tilde{p}|F(\tilde{p},w(\lambda),\lambda)D(p^m(p_0))\leq\left|\int_{\tilde{p}}^{\overline{p}} D(p)F(p,w(\lambda),\lambda)dp\right|\underset{\lambda\to 1}{\longrightarrow}0.$$

As $\tilde{p}>p_0$, we have that $F(\tilde{p},w(\lambda),\lambda)\underset{\lambda\to 1}{\longrightarrow}1$, implying that $\overline{p}(\lambda)\underset{\lambda\to 1}{\longrightarrow}\tilde{p}$. □

D.4.2 Taylor Approximation of Equilibrium Behavior

In this section, we derive Taylor approximations of $p_0 - w(\lambda, s)$ and $\pi(\overline{p}(\lambda, s), w(\lambda, s))$ in the neighborhood of $\lambda = 1$ when $s > \hat{s}$. We drop arguments (λ, s) to ease notation. We introduce new notation: $D_1 \equiv D(p_1)$, $D_1' \equiv D'(p_1)$, $r_1'' \equiv \partial^2\pi(p, p_0)/\partial p^2$, $r_1''' \equiv \partial^3\pi(p, p_0)/\partial p^3$,

$$\alpha_1 = \left.\frac{dp^m(w)}{dw}\right|_{w=p_0} = \frac{D_1'}{r_1''},$$
$$\text{and } \beta_1 = \left.\frac{d^2p^m(w)}{dw^2}\right|_{w=p_0} = \alpha_1^2\left(\frac{2D_1''}{D_1'} - \frac{r_1'''}{r_1''}\right).$$

Taylor approximation of $p_0 - w$. As in Appendix D.2.2, we define $\phi(p) = r'(p)/D(p)^2$ to rewrite equation (20):

$$\int_{\underline{p}}^{\overline{p}} \phi(p)\frac{\overline{p} - p}{(p - w)^3}dp = 0. \tag{79}$$

Put $\phi_0' \equiv \phi'(p_0) = r_0''/D_0^2$ and

$$\phi_0'' \equiv \phi''(p_0) = \frac{r_0'''}{D_0^2} - 4\frac{D_0' r_0''}{D_0^3} = \phi_0'\left(\frac{r_0'''}{r_0''} - 4\frac{D_0'}{D_0}\right).$$

Lemma D.4.2. *For $s > \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$p_0 - w = 2(\underline{p} - w) + \left(\frac{2}{p_1 - p_0} - \frac{\phi_0''}{\phi_0'}\right)(\underline{p} - w)^2\log(\underline{p} - w) + o\left((\underline{p} - w)^2\log(\underline{p} - w)\right). \tag{80}$$

Proof. Let $\epsilon \equiv (p_0 - w)/(\underline{p} - w)$. We begin by showing that $\epsilon \underset{\lambda\to 1}{\longrightarrow} 2$. Applying Lemma D.1.1 to equation (79), we obtain the existence of a bounded function $M(\lambda)$ such that in the neighborhood of $\lambda = 1$,

$$\phi_0'\underbrace{\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)(\overline{p} - p)}{(p - w)^3}dp}_{\equiv I_1} + M(\lambda)\underbrace{\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^2(\overline{p} - p)}{(p - w)^3}dp}_{\equiv I_2} = 0.$$

We have:

$$\begin{aligned}
I_1 &= \int_{\underline{p}}^{\overline{p}} \frac{-(\overline{p} - w)(p_0 - w) - (p - w)^2 + (\overline{p} + p_0 - 2w)(p - w)}{(p - w)^3}dp \\
&= \frac{(\overline{p} - w)(p_0 - w)}{2}\left(\frac{1}{(\overline{p} - w)^2} - \frac{1}{(\underline{p} - w)^2}\right) \\
&\quad - \log\frac{\overline{p} - w}{\underline{p} - w} + (\overline{p} + p_0 - 2w)\left(\frac{1}{\underline{p} - w} - \frac{1}{\overline{p} - w}\right)
\end{aligned}$$

$$= \frac{1}{\underline{p} - w} \left[\frac{\overline{p} - w}{2} \left(\frac{(\underline{p} - w)(p_0 - w)}{(\overline{p} - w)^2} - \epsilon \right) \right.$$
$$\left. -(\underline{p} - w) \log \frac{\overline{p} - w}{\underline{p} - w} + (\overline{p} + p_0 - 2w) \left(1 - \frac{\underline{p} - w}{\overline{p} - w} \right) \right],$$

It follows from Lemma D.4.1 that

$$(\underline{p} - w) I_1 = p_1 - p_0 - \frac{\overline{p} - w}{2} \epsilon + o(1).$$

Moreover,

$$I_2 = \int_{\underline{p}}^{\overline{p}} \frac{[(p-w)^2 - 2(p_0 - w)(p - w) + (p_0 - w)^2]\,[(\overline{p} - w) - (p - w)]}{(p - w)^3} dp$$
$$= \int_{\underline{p}}^{\overline{p}} \left[-1 + \frac{\overline{p} + 2p_0 - 3w}{p - w} - (p_0 - w) \frac{2\overline{p} + p_0 - 3w}{(p - w)^2} + \frac{(p_0 - w)^2 (\overline{p} - w)}{(p - w)^3} \right] dp$$
$$= -(\overline{p} - \underline{p}) + (\overline{p} + 2p_0 - 3w) \log \frac{\overline{p} - w}{\underline{p} - w} + (p_0 - w)(2\overline{p} + p_0 - 3w) \left[\frac{1}{\overline{p} - w} - \frac{1}{\underline{p} - w} \right]$$
$$+ (p_0 - w)^2 (\overline{p} - w) \frac{1}{2} \left[\frac{1}{(\underline{p} - w)^2} - \frac{1}{(\overline{p} - w)^2} \right]$$
$$= \frac{1}{\underline{p} - w} \left(-(\overline{p} - \underline{p})(\underline{p} - w) + (\overline{p} + 2p_0 - 3w)(\underline{p} - w) \log \frac{\overline{p} - w}{\underline{p} - w} \right.$$
$$\left. + (p_0 - w)(2\overline{p} + p_0 - 3w) \left[\frac{\underline{p} - w}{\overline{p} - w} - 1 \right] + (p_0 - w)(\overline{p} - w) \frac{1}{2} \left[\epsilon - \frac{(p_0 - w)(\underline{p} - w)}{(\overline{p} - w)^2} \right] \right),$$

which implies by Lemma D.4.1 that

$$(\underline{p} - w) I_2 = \frac{\overline{p} - w}{2} (p_0 - w) \epsilon + o(1).$$

Plugging the approximations of I_1 and I_2 into the first-order condition, we obtain:

$$\phi_0'(p_1 - p_0) + \left[-\phi_0' \frac{\overline{p} - w}{2} + M(\lambda) \frac{\overline{p} - w}{2} (p_0 - w) \right] \epsilon + o(1) = 0.$$

As the term inside square brackets tends to $-\phi_0'(p_1 - p_0)/2$ as λ goes to 1, it follows that $\epsilon \underset{\lambda \to 0}{\longrightarrow} 2$.

This implies that

$$p_0 - w = 2(\underline{p} - w) + o(\underline{p} - w).$$

Hence, a little-o of $p_0 - w$ is a little-o of $\underline{p} - w$ (and vice versa).

Next, we obtain a higher-order approximation of ϵ (and thus of $p_0 - w$). Applying again Lemma D.1.1 to equation (79), we obtain the existence of a bounded function $N(\lambda)$ such that in the neighborhood of $\lambda = 1$,

$$\phi_0'(\underline{p} - w)I_1 + \frac{\phi_0''}{2}(\underline{p} - w)I_2 + (\underline{p} - w)N(\lambda)\underbrace{\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^3(\overline{p} - p)}{(p - w)^3}dp}_{\equiv I_3} = 0. \tag{81}$$

Using the above expression for I_1 and the approximation of $p_0 - w$ yields:

$$(\underline{p} - w)I_1 = -\frac{\overline{p} - w}{2}\epsilon + (\underline{p} - w)\log(\underline{p} - w) + (\overline{p} + p_0 - 2w) + o\left((\underline{p} - w)\log(\underline{p} - w)\right),$$

where we have used the fact that a little-o of $(\underline{p} - w)$ is a little-o of $(\underline{p} - w)\log(\underline{p} - w)$. Next, we require an approximation of $\overline{p} - w$, which by Lemma D.4.1, is equal to $p^m(w) - w$. We have:

$$\begin{aligned}
\overline{p} - w &= p^m(p_0 + (w - p_0)) - p_0 + (p_0 - w) \\
&= p^m(p_0) - p_0 + p^{m\prime}(p_0)(w - p_0) - (w - p_0) + o(w - p_0) \\
&= p_1 - p_0 + (1 - \alpha_1)(p_0 - w) + o(p_0 - w) \\
&= p_1 - p_0 + 2(1 - \alpha_1)(\underline{p} - w) + o(\underline{p} - w) \\
&= p_1 - p_0 + o\left((\underline{p} - w)\log(\underline{p} - w)\right).
\end{aligned}$$

Plugging this into the approximation of I_1 yields:

$$(\underline{p} - w)I_1 = -\frac{p_1 - p_0}{2}\epsilon + (\underline{p} - w)\log(\underline{p} - w) + (p_1 - p_0) + o\left((\underline{p} - w)\log(\underline{p} - w)\right).$$

Next, we approximate the second term in equation (81). Using the above expression for I_2 and the approximations of $p_0 - w$ and $\overline{p} - w$ yields:

$$(\underline{p} - w)I_2 = -(p_1 - p_0)(\underline{p} - w)\log(\underline{p} - w) + o\left((\underline{p} - w)\log(\underline{p} - w)\right).$$

Finally, we argue that I_3 is bounded in the neighborhood of $\lambda = 1$. We have

$$|I_3| \leq \int_{\underline{p}}^{\overline{p}} \left|\frac{p - p_0}{p - w}\right|^3 (\overline{p} - p)dp \leq \max_{p \in [\underline{p}, \overline{p}]} \left|\frac{p - p_0}{p - w}\right| p^m(p_0)^2.$$

As the function $p \in [\underline{p}, \overline{p}] \mapsto (p - p_0)/(p - w)$ is strictly increasing, we have that

$$\max_{p \in [\underline{p}, \overline{p}]} \left|\frac{p - p_0}{p - w}\right| = \max\left\{\frac{p_0 - \underline{p}}{\underline{p} - w}, \frac{\overline{p} - p_0}{\overline{p} - w}\right\} \underset{\lambda \to 1}{\longrightarrow} 1,$$

where we have used Lemma D.4.1 and the above approximation of $p_0 - w$ to obtain the limit.

Hence, I_3 is bounded and $(\underline{p}-w)I_3 = o((\underline{p}-w)\log(\underline{p}-w))$.

Putting together our approximations of I_1, I_2, and I_3 yields the following approximation of Equation (81):

$$-\frac{p_1-p_0}{2}\epsilon+(\underline{p}-w)\log(\underline{p}-w)+p_1-p_0-\frac{\phi_0''}{2\phi_0'}(p_1-p_0)(\underline{p}-w)\log(\underline{p}-w)+o\left((\underline{p}-w)\log(\underline{p}-w)\right)=0.$$

Rearranging terms, this means that

$$\epsilon = 2+\left(\frac{2}{p_1-p_0}-\frac{\phi_0''}{\phi_0'}\right)(\underline{p}-w)\log(\underline{p}-w)+o\left((\underline{p}-w)\log(\underline{p}-w)\right).$$

Multiplying both sides by $(\underline{p}-w)$ proves the lemma. □

Approximation of $\pi(\overline{p}, w)$ and λ.

Lemma D.4.3. *For $s > \hat{s}$, in the neighborhood of $\lambda = 1$, we have:*

$$\pi(\overline{p},w)^2 = (p_1-p_0)^2D_1^2 + 2(p_1-p_0)D_1^2(p_0-w) + o\left((\underline{p}-w)^2\log(\underline{p}-w)\right),$$
$$\lambda = 1-2\frac{D_0}{(p_1-p_0)D_1}(\underline{p}-w)+o\left((\underline{p}-w)^2\log(\underline{p}-w)\right),$$
$$(\underline{p}-w)^2\log(\underline{p}-w) = \frac{\pi^2(p_1,p_0)}{4D_0^2}(1-\lambda)^2\log(1-\lambda)+o((1-\lambda)^2\log(1-\lambda)).$$

Proof. We begin by approximating $\overline{p} = p^m(w)$:

$$\begin{aligned}\overline{p} &= p^m(p_0)+p^{m\prime}(p_0)(w-p_0)+\frac{1}{2}p^{m\prime\prime}(p_0)(w-p_0)^2+o((w-p_0)^2)\\ &= p_1+\alpha_1(w-p_0)+\frac{1}{2}\beta_1(w-p_0)^2+o((p_0-w)^2)\\ &= p_1-\alpha_1(p_0-w)+o\left((\underline{p}-w)^2\log(\underline{p}-w)\right),\end{aligned}$$

where we have used the fact that $(w-p_0)^2$ is a little-o of $(\underline{p}-w)^2\log(\underline{p}-w)$ (see Lemma D.4.2).

Next, we derive the approximation of $\underline{\pi}(w) \equiv \pi(\underline{p}, w)$. We have:

$$\begin{aligned}\underline{\pi}(w) &= (\underline{p}-w)D(p_0+\underline{p}-p_0)\\ &= (\underline{p}-w)\left(D_0+D_0'(\underline{p}-p_0)\right)+o((\underline{p}-w)^2)\\ &= (\underline{p}-w)D_0+o\left((\underline{p}-w)^2\log(\underline{p}-w)\right),\end{aligned}$$

where we have again used Lemma D.4.2.

Next, we turn to the approximation of $\overline{\pi}(w) \equiv \pi(\overline{p}, w)$. Using the above approximation of $\overline{p}$ and Lemma D.4.2, we obtain:

$$\begin{aligned}\overline{p} - w &= p_1 - p_0 + (\overline{p} - p_1) + (p_0 - w) \\ &= p_1 - p_0 + (1 - \alpha_1)(p_0 - w) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right)\end{aligned}$$

and

$$\begin{aligned}D(\overline{p}) &= D(p_1 + \overline{p} - p_1) \\ &= D_1 + D_1'(\overline{p} - p_1) + \frac{D_1''}{2}(\overline{p} - p_1)^2 + o((\overline{p} - p_1)^2) \\ &= D_1 + D_1'(\overline{p} - p_1) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right) \\ &= D_1 + \frac{D_1}{p_1 - p_0}\alpha_1(p_0 - w) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right).\end{aligned}$$

It follows that

$$\overline{\pi}(w) = (p_1 - p_0)D_1 + D_1(p_0 - w) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right).$$

Next, we approximate λ:

$$\begin{aligned}\lambda &= \frac{\overline{\pi}(w) - \underline{\pi}(w)}{\overline{\pi}(w) + \underline{\pi}(w)} \\ &= 1 - 2\frac{\underline{\pi}(w)}{\overline{\pi}(w) + \underline{\pi}(w)} \\ &= 1 - 2\frac{(\underline{p} - w)D_0 + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right)}{(p_1 - p_0)D_1 + D_1(p_0 - w) + (\underline{p} - w)D_0 + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right)} \\ &= 1 - 2\frac{D_0}{(p_1 - p_0)D_1}(\underline{p} - w) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right).\end{aligned}$$

Finally, we approximate $(\underline{p} - w)^2 \log(\underline{p} - w)$ in terms of λ. Rewriting the above approximation of λ and using the fact that a little-o of $\underline{p} - w$ is a little-o of $1 - \lambda$, we have

$$\underline{p} - w = \frac{\pi(p_1, p_0)}{2D_0}(1 - \lambda) + o(1 - \lambda),$$

which implies that

$$\begin{aligned}(\underline{p} - w)^2 \log(\underline{p} - w) &= \left(\frac{\pi(p_1, p_0)}{2D_0}\right)^2 (1 - \lambda)^2 \log(\underline{p} - w) + o((1 - \lambda)^2 \log(\underline{p} - w)) \\ &= \left(\frac{\pi(p_1, p_0)}{2D_0}\right)^2 (1 - \lambda)^2 \log\frac{(1 - \lambda)\overline{\pi}(w)}{(1 + \lambda)D(\underline{p})}\end{aligned}$$

$$+ o\left((1-\lambda)^2 \log \frac{(1-\lambda)\overline{\pi}(w)}{(1+\lambda)D(\underline{p})}\right)$$
$$= \frac{\pi^2(p_1,p_0)}{4D_0^2}(1-\lambda)^2 \log(1-\lambda) + o((1-\lambda)^2 \log(1-\lambda)),$$

where we have used equation (3) to obtain the second line and the fact that $\overline{\pi}(w)/D(\underline{p})$ is bounded for λ close to 1 to obtain the third line. □

D.4.3 Proof of Proposition 10 for $s > \hat{s}$

Consumer surplus approximation.

Lemma D.4.4. *For $s > \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\Delta CS = \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2}\left[\alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0-c)\alpha^2\tilde{\mu}(s)\right]$$
$$\times (1-\lambda)^2 \left|\log(1-\lambda)\right| + o((1-\lambda)^2 \log(1-\lambda)).$$

Proof. It follows from equation (42) that $\Delta CS = \Psi/(4\lambda)$, where

$$\Psi \equiv -4\lambda \int_{p_0}^{\overline{p}} D(p)dp + (1+\lambda)^2 \int_{\underline{p}}^{\overline{p}} D(p)dp - (1-\lambda)^2\overline{\pi}(w)^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)}$$
$$= 4\lambda \int_{\underline{p}}^{p_0} D(p)dp + (1-\lambda)^2 \int_{\underline{p}}^{\overline{p}} D(p)dp - (1-\lambda)^2\overline{\pi}(w)^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)},$$

where we have used the fact that $(1+\lambda)^2 = (1-\lambda)^2 + 4\lambda$. We seek a Taylor approximation of Ψ up to order $(\underline{p}-w)^2 \log(\underline{p}-w)$.

By Lemma D.4.3,

$$(1-\lambda)^2 = \frac{4D_0^2}{(p_1-p_0)^2 D_1^2}(\underline{p}-w)^2 + o\left((\underline{p}-w)^3 \log(\underline{p}-w)\right),$$

and so $(1-\lambda)^2 = o((\underline{p}-w)^2 \log(\underline{p}-w))$. As $\int_{\underline{p}}^{\overline{p}} D(p)dp$ is bounded when λ is close to 1, this implies that

$$(1-\lambda)^2 \int_{\underline{p}}^{\overline{p}} D(p)dp = o\left((\underline{p}-w)^2 \log(\underline{p}-w)\right).$$

Next, we turn our attention to the term $4\lambda \int_{\underline{p}}^{p_0} D(p)dp$. By Lemmas D.1.1 and D.4.2, the integral can be approximated as

$$\int_{\underline{p}}^{p_0} D(p)dp = D_0(p_0-\underline{p}) - \frac{D_0'}{2}(p_0-\underline{p})^2 + o\left((p_0-\underline{p})^2\right)$$

$$= D_0(p_0 - \underline{p}) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right),$$

where we have used Lemma D.4.2 to obtain the second line. Combining this with the approximation of λ in Lemma D.4.3 and using the fact that $(\underline{p} - w)(p_0 - \underline{p}) = o((\underline{p} - w)^2 \log(\underline{p} - w))$, we obtain:

$$\begin{aligned}
4\lambda \int_{\underline{p}}^{p_0} D(p)dp &= 4D_0(p_0 - \underline{p}) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right) \\
&= 4D_0 \left(\underline{p} - w + \left[\frac{2}{p_1 - p_0} - \frac{\phi_0''}{\phi_0'}\right] (\underline{p} - w)^2 \log(\underline{p} - w)\right) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right),
\end{aligned}$$

where we have again used Lemma D.4.2.

Finally, we turn to the third term in the above expression for Ψ. By Lemma D.1.1, the integral can be approximated as[31]

$$\begin{aligned}
\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p - w)^2 D(p)} &= \frac{1}{D_0} \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p - w)^2} - \frac{D_0'}{D_0^2} \int_{\underline{p}}^{\overline{p}} \frac{p - p_0}{(p - w)^2} dp + O\left(\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^2}{(p - w)^2} dp\right) \\
&= \frac{1}{D_0} \left[\frac{1}{\underline{p} - w} - \frac{1}{\overline{p} - w}\right] + \frac{1}{r_0} \left[\log \frac{\overline{p} - w}{\underline{p} - w} + \frac{p_0 - w}{\overline{p} - w} - \frac{p_0 - w}{\underline{p} - w}\right] + O(1) \\
&= \frac{1}{D_0} \left(\frac{1}{\underline{p} - w} - \frac{1}{p_0 - c} \log(\underline{p} - w)\right) + O(1),
\end{aligned}$$

where we have used Lemma D.4.2.

Combining this with the above Taylor approximation of $(1 - \lambda)^2$, we obtain:

$$\begin{aligned}
&(1 - \lambda)^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p - w)^2 D(p)} = \\
&\quad \frac{4D_0}{(p_1 - p_0)^2 D_1^2} \left(\underline{p} - w - \frac{1}{p_0 - c} (\underline{p} - w)^2 \log(\underline{p} - w)\right) + o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right),
\end{aligned}$$

where we have used the fact that a big-o of $(\underline{p} - w)^2$ is a little-o of $(\underline{p} - w)^2 \log(\underline{p} - w)$, and a little-o of $(\underline{p} - w)^3 (\log(\underline{p} - w))^2$ is also a little-o of $(\underline{p} - w)^2 \log(\underline{p} - w)$. Combining this with the approximation of $\pi(\overline{p}, w)^2$ from Lemma D.4.3, it follows that

[31] To see why $\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^2}{(p - w)^2} dp$ is bounded in the expression below, recall from the proof of Lemma D.4.2 that

$$\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^2}{(p - w)^2} dp \leq \max_{p \in [\underline{p}, \overline{p}]} \left(\frac{p - p_0}{p - w}\right)^2 (\overline{p} - \underline{p}) \leq \max \left\{\left(\frac{p_0 - \underline{p}}{\underline{p} - w}\right)^2, \left(\frac{\overline{p} - p_0}{\overline{p} - w}\right)^2\right\} p^m(p_0)$$

and that both terms within the second maximum are bounded when λ is close to 1.

$$(1-\lambda)^2\overline{\pi}(w)^2\int_{\underline{p}}^{\overline{p}}\frac{dp}{(p-w)^2D(p)}=$$

$$4D_0\left(\underline{p}-w-\frac{1}{p_0-c}(\underline{p}-w)^2\log(\underline{p}-w)\right)+o\left((\underline{p}-w)^2\log(\underline{p}-w)\right).$$

We thus obtain the Taylor approximation of Ψ:

$$\Psi=4D_0\left[\frac{2}{p_1-p_0}-\frac{\phi_0''}{\phi_0'}+\frac{1}{p_0-c}\right](\underline{p}-w)^2\log(\underline{p}-w)+o\left((\underline{p}-w)^2\log(\underline{p}-w)\right).$$

Using the approximation of $(\underline{p}-w)^2\log(\underline{p}-w)$ from Lemma D.4.3 and the fact that $\Delta CS=\Psi/(4\lambda)$, we obtain:

$$\Delta CS=\frac{\pi^2(p_1,p_0)}{4D_0}\left[\frac{2}{p_1-p_0}-\frac{\phi_0''}{\phi_0'}+\frac{1}{p_0-c}\right](1-\lambda)^2\log(1-\lambda)+o((1-\lambda)^2\log(1-\lambda)).$$

Let us define

$$\zeta\equiv(p_0-c)\frac{\phi_0''}{\phi_0'}-1-\frac{2(p_0-c)}{p_1-p_0}$$

and rewrite it as a function of the pass-through and its derivative:

$$\begin{aligned}\zeta&=(p_0-c)\left(\frac{r_0'''}{r_0''}-2\frac{D_0''}{D_0'}+2\frac{D_0''}{D_0'}-4\frac{D_0'}{D_0}\right)-1-\frac{2(p_0-c)}{p_1-p_0}\\&=\frac{-\beta(p_0-c)}{\alpha^2}-2\frac{2\alpha-1}{\alpha}+3-\frac{2(p_0-c)}{p_1-p_0}\\&=-1-\frac{\beta(p_0-c)}{\alpha^2}+2\left(\frac{1}{\alpha}-\frac{p_0-c}{p_1-p_0}\right).\end{aligned}$$

Thus,

$$\Delta CS=\frac{\pi^2(p_1,p_0)}{4r_0}\left[-1-\frac{\beta(p_0-c)}{\alpha^2}+2\left(\frac{1}{\alpha}-\frac{p_0-c}{p_1-p_0}\right)\right]$$
$$\times(1-\lambda)^2\left|\log(1-\lambda)\right|+o((1-\lambda)^2\log(1-\lambda)).$$

As $\tilde{\mu}(s)=-\frac{1}{p_1-p_0}$ and $\tilde{p}(s)=p_1$ for $s>\hat{s}$, this concludes the proof of the lemma. □

Producer surplus approximation.

Lemma D.4.5. *For $s>\hat{s}$, in the neighborhood of $\lambda=1$, we have*

$$\Delta\Pi=-\frac{\pi^2(\tilde{p}(s),p_0)}{4r_0\alpha}(1-\lambda)^2\left|\log(1-\lambda)\right|+o((1-\lambda)^2\log(1-\lambda)).$$

Proof. Using equation (43), we see that the $\Delta\Pi = \Phi/(4\lambda)$, where

$$\begin{aligned}\Phi &\equiv 4\lambda\left(r(\overline{p}) - r_0\right) + (1-\lambda)^2\overline{\pi}(w)^2\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2} - (1+\lambda)^2\left(r(\overline{p}) - r(\underline{p})\right)\\ &= 4\lambda\left(r(\underline{p}) - r_0\right) + (1-\lambda)^2\overline{\pi}(w)^2\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2} - (1-\lambda)^2\left(r(\overline{p}) - r(\underline{p})\right).\end{aligned}$$

The third term is clearly a little-o of $(\underline{p}-w)^2\log(\underline{p}-w)$, as $r(\overline{p}) - r(\underline{p})$ is bounded and $(1-\lambda)^2 = o\left((\underline{p}-w)^2\log(\underline{p}-w)\right)$ by Lemma D.4.3. The first term is also negligible, as

$$r(\underline{p}) - r_0 = \frac{r_0''}{2}(\underline{p}-p_0)^2 + o\left((\underline{p}-p_0)^2\right) = o\left((\underline{p}-w)^2\log(\underline{p}-w)\right),$$

where the second equality follows by Lemma D.4.2.

Using Lemma D.1.1, we obtain an approximation of the integral in the second term:[32]

$$\begin{aligned}\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2} &= \frac{r_0''}{D_0^2}\int_{\underline{p}}^{\overline{p}} \frac{p-p_0}{(p-w)^2}dp + O\left(\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2}dp\right)\\ &= \frac{r_0''}{D_0^2}\left[\log\frac{\overline{p}-w}{\underline{p}-w} + \frac{p_0-w}{\overline{p}-w} - \frac{p_0-w}{\underline{p}-w}\right] + O(1)\\ &= -\frac{r_0''}{D_0^2}\log(\underline{p}-w) + O(1),\end{aligned}$$

where the last line follows by Lemmas D.4.1 and D.4.2. Combining this with Lemma D.4.3, we obtain:

$$(1-\lambda)^2\overline{\pi}(w)^2\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2} = -4r_0''(\underline{p}-w)^2\log(\underline{p}-w) + o\left((\underline{p}-w)^2\log(\underline{p}-w)\right).$$

As $r_0'' = -D_0/(\alpha(p_0-c))$, this implies that

$$\Phi = 4\frac{D_0}{p_0-c}\frac{1}{\alpha}(\underline{p}-w)^2\log(\underline{p}-w) + o\left((\underline{p}-w)^2\log(\underline{p}-w)\right).$$

Combining this with the approximation of $(\underline{p}-w)^2\log(\underline{p}-w)$ with respect to $(1-\lambda)^2\log(1-\lambda)$ from Lemma D.4.3 proves the lemma. □

Aggregate Surplus Approximation. Combining Lemmas D.4.4 and D.4.5, we obtain:

Lemma D.4.6. *For $s > \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

[32] For the argument why the integral inside the big-O on the first line is bounded, see footnote 31.

$$\Delta AS = \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2}\left[\alpha(1-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)\right]$$
$$\times (1-\lambda)^2 \left|\log(1-\lambda)\right| + o((1-\lambda)^2 \log(1-\lambda)).$$

D.4.4 Distributional Effects

We now separately derive the approximation of consumer surplus for online and offline consumers. The change in consumer surplus in the offline and the online markets, denoted by ΔCS_B and ΔCS_O, are given in equations (65) and (66) respectively.

Lemma D.4.7. *For $s > \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\Delta CS_B = -\frac{1}{2}\pi(\tilde{p}(s), p_0)(1-\lambda)^2 |\log(1-\lambda)| + o((1-\lambda)^2 \log(1-\lambda)) \quad (82)$$

$$\Delta CS_O = \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2}\left[\alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\left(\tilde{\mu}(s) + \frac{D_0}{\pi(p_1, p_0)}\right)\right]$$
$$\times (1-\lambda)^2 \left|\log(1-\lambda)\right| + o((1-\lambda)^2 \log(1-\lambda)). \quad (83)$$

Proof. From equation (65), the change in consumer surplus in the offline market is given by $\Delta CS_B = (1-\lambda)/(2\lambda)\Psi_B$, where

$$\Psi_B \equiv -2\lambda \int_{p_0}^{\overline{p}} D(p)dp + (1+\lambda)\int_{\underline{p}}^{\overline{p}} D(p)dp - (1-\lambda)\pi(\overline{p}, w)\log\frac{\overline{p} - w}{\underline{p} - w}$$
$$= (1+\lambda)\int_{\underline{p}}^{p_0} D(p)dp + (1-\lambda)\int_{p_0}^{\overline{p}} D(p)dp - (1-\lambda)\pi(\overline{p}, w)\log\frac{\overline{p} - w}{\underline{p} - w}.$$

We derive an approximation of $\Psi_B(\lambda)$ of order $(\underline{p} - w)\log(\underline{p} - w)$. By Lemma D.1.1, we have that

$$\int_{\underline{p}}^{p_0} D(p)dp = D_0(p_0 - \underline{p}) + o(p_0 - \underline{p}) = o((\underline{p} - w)\log(\underline{p} - w)),$$

implying that the first term of $\Psi_B(\lambda)$ is a little-o of $(\underline{p} - w)\log(\underline{p} - w)$. As $\int_{p_0}^{\overline{p}} D(p)dp$ is bounded in the neighborhood of $\lambda = 1$, Lemma D.4.3 implies that the second term of $\Psi_B(\lambda)$ is also a little-o of $(\underline{p} - w)\log(\underline{p} - w)$. Applying again Lemma D.4.3 to approximate the third term, we obtain

$$(1-\lambda)\pi(\overline{p}, w)\log\frac{\overline{p} - w}{\underline{p} - w} = -(1-\lambda)\pi(\overline{p}, w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w))$$
$$= -2D_0(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

It follows that

$$\Psi_B = 2D_0(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

Using the approximation of $1-\lambda$ from Lemma D.4.3, we obtain

$$\Delta CS_B = \frac{2D_0^2}{\pi(p_1,p_0)}(\underline{p}-w)^2\log(\underline{p}-w) + o((\underline{p}-w)^2\log(\underline{p}-w)).$$

Finally, using the approximation of $(\underline{p}-w)^2\log(\underline{p}-w)$ from Lemma D.4.3, we obtain the approximation of ΔCS_B in the statement of the lemma. Inserting this approximation and the approximation of ΔCS from Lemma D.4.4 into $\Delta CS - \Delta CS_B$, we obtain the approximation of ΔCS_O in the statement of the lemma. □

D.5 Proofs of Welfare Results when λ is High and $s < \hat{s}$

In this appendix, we study the welfare effects of banning dual pricing when λ is close to 1 and $s < \hat{s}$, thus proving Proposition 10 for low search costs. The approach is similar to that in Appendix D.4. In Section D.5.1, we describe the limiting equilibrium behavior when λ tends to 1. In Section D.5.2, we derive some auxiliary Taylor approximations required to approximate ΔCS and ΔAS. Section D.5.3 is devoted to the approximations of ΔCS and ΔAS. In Section D.5.4, we study the properties of the function $\tilde{\mu}(s)$. In Section D.5.5, we study the distributional effects of a ban on dual pricing. In Section D.5.6, we show that the manufacturers deals with both retailers in equilibrium provided λ is high enough.

D.5.1 Basic Properties of the Equilibrium for High λ and $s < \hat{s}$

We already characterized the limiting equilibrium behavior as λ tends to 1 in Lemma D.4.1. For λ high enough, the upper bound of the equilibrium CDF of prices, $\overline{p}$, solves $H(\overline{p}, w) = s$. It converges to $\hat{p}$ as λ tends to 1, where $\hat{p}$ is the unique solution to

$$\int_{p_0}^{\hat{p}} D(p)dp = s.$$

D.5.2 Taylor Approximation of Equilibrium Behavior

Define $\hat{D} \equiv D(\hat{p})$, $\hat{D}' \equiv D'(\hat{p})$, $\hat{r} = \pi(\hat{p}, p_0)$, $\hat{r}' \equiv \hat{D}'(\hat{p}-p_0) + \hat{D}$, and $\hat{\mu} \equiv (\hat{r}'\frac{D_0}{\hat{D}} - \hat{D})/\hat{r}$.

Limit of $\overline{p}'(w)$ as λ tends to 1.

Lemma D.5.1. *We have:* $\lim_{\lambda\to 1}\overline{p}'(w) = D_0/\hat{D}$.

Proof. Manipulating equation (3), we obtain

$$(1-\lambda)\log\frac{\overline{p}-w}{\underline{p}-w} = (1-\lambda)\log\left[(1+\lambda)\frac{D(\underline{p})}{D(\overline{p})}\right] - (1-\lambda)\log(1-\lambda) \underset{\lambda\to 1}{\longrightarrow} 0,$$

$$(1-\lambda)\frac{\overline{p}-w}{\underline{p}-w} = (1+\lambda)\frac{D(\underline{p})}{D(\overline{p})} \underset{\lambda\to 1}{\longrightarrow} 2\frac{D_0}{\hat{D}}.$$

The lemma follows by taking the limit as λ tends to 1 in equation (14). □

First-order approximation of $p_0 - w$. Let $\epsilon \equiv \frac{p_0-w}{\underline{p}-w}$. As in Appendix D.4.2, we define $\phi(p) = r'(p)/D(p)^2$ to rewrite equation (20) as

$$\int_{\underline{p}}^{\overline{p}} \phi(p)\frac{1}{(p-w)^3}dp + \mu(\overline{p}, w)\int_{\underline{p}}^{\overline{p}} \phi(p)\frac{1}{(p-w)^2}dp = 0, \tag{84}$$

where

$$\mu(\overline{p}, w) = \frac{\pi_1'(\overline{p}, w)\frac{\partial \overline{p}}{\partial w} - D(\overline{p})}{\overline{\pi}(w)} \quad \text{and} \quad \overline{\pi}(w) \equiv \pi(\overline{p}, w).$$

Recall from Appendix D.4.2 that $\phi_0' \equiv \phi'(p_0) = r_0''/D_0^2$ and

$$\phi_0'' \equiv \phi''(p_0) = \frac{r_0'''}{D_0^2} - 4\frac{D_0' r_0''}{D_0^3} = \phi_0'\left(\frac{r_0'''}{r_0''} - 4\frac{D_0'}{D_0}\right).$$

Lemma D.5.2. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$p_0 - w = 2(\underline{p} - w) + o\left(\underline{p} - w\right). \tag{85}$$

Proof. Applying Lemma D.1.1 to the two integrals in equation (84), we obtain the existence of bounded functions $M(\lambda)$ and $N(\lambda)$ such that

$$\begin{aligned}0 = \phi_0' \overbrace{\int_{\underline{p}}^{\overline{p}} \frac{p-p_0}{(p-w)^3}dp}^{\equiv I_1} + \mu(\overline{p}, w)\phi_0' \overbrace{\int_{\underline{p}}^{\overline{p}} \frac{p-p_0}{(p-w)^2}dp}^{\equiv I_2} \\ + M(\lambda)\underbrace{\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^3}dp}_{\equiv I_3} + \mu(\overline{p}, w)N(\lambda)\underbrace{\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2}dp}_{\equiv I_4}.\end{aligned} \tag{86}$$

We can explicitly compute I_1 and I_2:

$$\begin{aligned}I_1 &= \left(\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2} - (p_0 - w)\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^3}\right) \\ &= \frac{1}{\underline{p}-w}\left[1 - \frac{\underline{p}-w}{\overline{p}-w} - \frac{p_0-w}{2}\left(\frac{1}{\underline{p}-w} - \frac{\underline{p}-w}{(\overline{p}-w)^2}\right)\right], \\ I_2 &= \int_{\underline{p}}^{\overline{p}} \frac{dp}{p-w} - (p_0-w)\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2}\end{aligned} \tag{87}$$

$$= \frac{1}{\underline{p} - w} \left((\underline{p} - w) \log \left(\frac{\overline{p} - w}{\underline{p} - w} \right) - (p_0 - w) \left(1 - \frac{\underline{p} - w}{\overline{p} - w} \right) \right). \tag{88}$$

As $\mu(\overline{p}, w)$ is bounded (by Lemmas D.4.1 and D.5.1), it follows that

$$(\underline{p} - w)\left(I_1 + \mu(\overline{p}, w) I_2\right) = 1 - \frac{\epsilon}{2} + o(1).$$

We can also explicitly compute I_3 and I_4 as

$$\begin{aligned} I_3 &= \int_{\underline{p}}^{\overline{p}} \frac{dp}{p - w} - 2 \int_{\underline{p}}^{\overline{p}} \frac{p_0 - w}{(p - w)^2} dp + \int_{\underline{p}}^{\overline{p}} \frac{(p_0 - w)^2}{(p - w)^3} dp, \\ &= \frac{1}{\underline{p} - w} \left((\underline{p} - w) \log \left(\frac{\overline{p} - w}{\underline{p} - w} \right) \right. \\ &\quad \left. -2(p_0 - w) \left(1 - \frac{\underline{p} - w}{\overline{p} - w} \right) + \frac{(p_0 - w)^2}{2} \left(\frac{1}{\underline{p} - w} - \frac{\underline{p} - w}{(\overline{p} - w)^2} \right) \right) \end{aligned} \tag{89}$$

$$\begin{aligned} I_4 &= \int_{\underline{p}}^{\overline{p}} dp - 2 \int_{\underline{p}}^{\overline{p}} \frac{p_0 - w}{p - w} dp + \int_{\underline{p}}^{\overline{p}} \frac{(p_0 - w)^2}{(p - w)^2} dp, \\ &= \frac{1}{\underline{p} - w} \left((\underline{p} - w)(\overline{p} - \underline{p}) \right. \\ &\quad \left. -2(p_0 - w)(\underline{p} - w) \log \left(\frac{\overline{p} - w}{\underline{p} - w} \right) + (p_0 - w)^2 \left(1 - \frac{\underline{p} - w}{\overline{p} - w} \right) \right). \end{aligned} \tag{90}$$

Multiplying by $\underline{p} - w$ and taking limits, we obtain:

$$(\underline{p} - w) I_3 = \frac{p_0 - w}{2} \epsilon + o(1) \quad \text{and} \quad (\underline{p} - w) I_4 = o(1).$$

Plugging the approximations of I_1, I_2, I_3, and I_4 into equation (86), we obtain:

$$\phi_0' + \left[-\frac{1}{2} \phi_0' + M(\lambda) \frac{p_0 - w}{2} \right] \epsilon + o(1) = 0.$$

As the term inside square brackets tends to $-\phi_0'/2$ as λ goes to 1, we have that $\epsilon \underset{\lambda \to 1}{\longrightarrow} 2$. □

The lemma implies in particular that a little-o of $\underline{p} - w$ is a little-o of $p_0 - w$ (and vice versa).

Approximation of $\overline{p} - \hat{p}$. Define $\underline{\pi}(w) \equiv \pi(\underline{p}, w)$.

Lemma D.5.3. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\overline{p} - \hat{p} = -\frac{D_0}{\hat{D}} (\underline{p} - w) \log(\underline{p} - w) + o((\underline{p} - w) \log(\underline{p} - w)) \tag{91}$$

Proof. Let

$$\nu \equiv \frac{\overline{p} - \hat{p}}{(\underline{p} - w)\log(\underline{p} - w)}.$$

Manipulating equation (3), we obtain that

$$\lambda = \frac{\overline{\pi}(w) - \underline{\pi}(w)}{\overline{\pi}(w) + \underline{\pi}(w)}.$$

Plugging this into condition $H(\overline{p}, w) - s = 0$ yields:

$$\begin{aligned} 0 &= \frac{1+\lambda}{2\lambda} \int_{\underline{p}}^{\overline{p}} D(p)dp - \frac{1-\lambda}{2\lambda} \overline{\pi}(w) \log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) - s \\ &= \frac{\overline{\pi}(w)}{\overline{\pi}(w) - \underline{\pi}(w)} \left(\int_{\underline{p}}^{\overline{p}} D(p)dp - \underline{\pi}(w) \log\left(\frac{\overline{p} - w}{\underline{p} - w}\right)\right) - s. \end{aligned}$$

Dividing by $\overline{\pi}(w)/(\overline{\pi}(w) - \underline{\pi}(w))$ and using the definition of $\hat{p}$, we obtain:

$$\begin{aligned} 0 &= \int_{\underline{p}}^{\overline{p}} D(p)dp - \underline{\pi}(w) \log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) - \int_{p_0}^{\hat{p}} D(p)dp + \frac{\underline{\pi}(w)}{\overline{\pi}(w)} s \\ &= \int_{\hat{p}}^{\overline{p}} D(p)dp + \int_{\underline{p}}^{p_0} D(p)dp - \underline{\pi}(w) \log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) + \frac{\underline{\pi}(w)}{\overline{\pi}(w)} s. \end{aligned} \tag{92}$$

Applying Lemma D.1.1 to the two integrals, we obtain the existence of bounded functions $M(\lambda)$ and $N(\lambda)$ such that for λ close enough to 1, we have:

$$\hat{D}(\overline{p} - \hat{p}) + M(\lambda)(\overline{p} - \hat{p})^2 + D_0(p_0 - \underline{p}) + N(\lambda)(p_0 - \underline{p})^2 - \underline{\pi}(w) \log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) + \frac{\underline{\pi}(w)}{\overline{\pi}(w)} s = 0.$$

Rearranging terms and using Lemma D.5.2, we obtain

$$(\hat{D}\nu + D(\underline{p}))(\underline{p} - w)\log(\underline{p} - w) + M(\lambda)(\overline{p} - \hat{p})\nu(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)) = 0,$$

or, equivalently,

$$\hat{D}\nu + D(\underline{p}) + M(\lambda)(\overline{p} - \hat{p})\nu + o(1) = 0.$$

It follows that ν converges to $-D_0/\hat{D}$ when λ tends to 1, which proves the lemma. □

Lemma D.5.3 implies that a little-o of $\overline{p} - \hat{p}$ is a little-o of $(\underline{p} - w)\log(\underline{p} - w)$ (and vice versa). Moreover, combining equation (91) with the fact that $p_0 - w$ is a little-o of $(\underline{p} - w)\log(\underline{p} - w)$ we have

$$\overline{p} - w = \overline{p} - \hat{p} + \hat{p} - p_0 + p_0 - w$$

$$= \hat{p} - p_0 - \frac{D_0}{\hat{D}}(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)),$$

$$\overline{p} - \underline{p} = \overline{p} - w - (\underline{p} - w)$$

$$= \hat{p} - p_0 - \frac{D_0}{\hat{D}}(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

For what follows, it is also useful to compute

$$\log(\overline{p} - w) = \log(\hat{p} - p_0) + \frac{1}{\overline{p} - p_0}(\overline{p} - w - (\hat{p} - p_0)) + o((\underline{p} - w)\log(\underline{p} - w))$$

$$= \log(\hat{p} - p_0) - \frac{D_0}{\hat{r}}(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

A higher order approximation of $p_0 - w$.

Lemma D.5.4. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$p_0 - w = 2(\underline{p} - w) - 2(\hat{\mu} + \gamma)(\underline{p} - w)^2 \log(\underline{p} - w) + o((\underline{p} - w)^2 \log(\underline{p} - w)). \qquad (93)$$

Proof. By Lemma D.1.1 applied to the two integrals in equation (84), there exist bounded functions $M(\lambda)$ and $N(\lambda)$ such that

$$\phi_0' \overbrace{\int_{\underline{p}}^{\overline{p}} (p - p_0)\left(\frac{\mu(\overline{p}, w)}{(p - w)^2} + \frac{1}{(p - w)^3}\right) dp}^{\equiv B_1} + \phi_0'\gamma \overbrace{\int_{\underline{p}}^{\overline{p}} (p - p_0)^2 \left(\frac{\mu(\overline{p}, w)}{(p - w)^2} + \frac{1}{(p - w)^3}\right) dp}^{\equiv B_2}$$
$$+ \underbrace{\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^3}{(p - w)^3} \left[M(\lambda)\mu(\overline{p}, w)(p - w) + N(\lambda)\right] dp}_{\equiv B_3} = 0,$$

where ϕ_0' was defined above and

$$\gamma \equiv \frac{1}{2}\frac{r_0'''}{r_0''} - 2\frac{D_0'}{D_0}.$$

We seek the approximations of $(\underline{p} - w)B_i$, for every i. We start with $(\underline{p} - w)B_1$. Note that $(\underline{p} - w)B_1 = \mu(\overline{p}, w)(\underline{p} - w)I_2 + (\underline{p} - w)I_1$, where I_1 and I_2 were defined in the proof of Lemma D.5.2. Using equations (87) and (88), we obtain:

$$(\underline{p} - w)I_1 = 1 - \frac{\epsilon}{2} + o((\underline{p} - w)\log(\underline{p} - w)),$$

$$(\underline{p} - w)I_2 = -(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

It follows that

$$(\underline{p} - w)B_1 = 1 - \frac{\epsilon}{2} - \hat{\mu} \times (\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

Next, we approximate $(\underline{p} - w)B_2 = \mu(\overline{p}, w)(\underline{p} - w)I_4 + (\underline{p} - w)I_3$, where I_3 and I_4 were also defined in the proof of Lemma D.5.2. Using equation (89), we obtain:

$$\begin{aligned}(\underline{p} - w)I_3 &= -(\underline{p} - w)\log(\underline{p} - w) + \frac{p_0 - w}{2}\epsilon + o((\underline{p} - w)\log(\overline{p} - w)) \\ &= -(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)),\end{aligned}$$

where the second line follows as $(p_0 - w)\epsilon$ is a little-o of $(\underline{p} - w)\log(\underline{p} - w)$ by Lemma D.5.2. Moreover, using equation (90), we immediately obtain that $(\underline{p} - w)I_4$ is a little-o of $(\underline{p} - w)\log(\underline{p} - w)$, so that

$$(\underline{p} - w)B_2 = -(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)).$$

Let $\overline{M}$, $\overline{N}$, and m be upper bounds for $|M(\lambda)|$, $|N(\lambda)|$, and $|\mu(\overline{p}, w)|$ in the neighborhood of $\lambda = 1$. For high enough λ, we have:

$$\begin{aligned}B_3 &\leq \int_{\underline{p}}^{\overline{p}} \left|\frac{p - p_0}{p - w}\right|^3 \left(m\overline{M}(p - w) + \overline{N}\right) dp \\ &\leq \int_{\underline{p}}^{\overline{p}} \left(\max_{p' \in [\underline{p}, \overline{p}]} \left|\frac{p' - p_0}{p' - w}\right|\right)^3 \left(m\overline{M}p_1 + \overline{N}\right) dp \\ &\leq \left(m\overline{M}p_1 + \overline{N}\right) p_1 \left(\max\left\{\frac{p_0 - \underline{p}}{\underline{p} - w}, \frac{\overline{p} - p_0}{\overline{p} - w}\right\}\right)^3,\end{aligned}$$

where we have used the fact that $p - w \leq \overline{p} \leq p^m(w) \leq p^m(p_0) = p_1$ to obtain the second line and $p \mapsto (p - p_0)/(p - w)$ is strictly increasing to obtain the third line. Since the two terms inside the maximum are bounded (see Lemma D.5.2), it follows that $(\underline{p} - w)B_3 = o((\underline{p} - w)\log(\underline{p} - w))$.

Combining the above approximations, we obtain:

$$1 - \frac{\epsilon}{2} - (\hat{\mu} + \gamma)(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)) = 0,$$

which implies that

$$\epsilon = 2 - 2(\hat{\mu} + \gamma)(\underline{p} - w)\log(\underline{p} - w) + o((\underline{p} - w)\log(\underline{p} - w)),$$

proving the lemma. □

A higher-order approximation of $\overline{p} - \hat{p}$. Define $\eta \equiv \log(\hat{p} - p_0) - s/\hat{r}$. We have:

Lemma D.5.5. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\overline{p} - \hat{p} = -\frac{D_0}{\hat{D}}(\underline{p} - w)\log(\underline{p} - w) + \frac{D_0\eta}{\hat{D}}(\underline{p} - w) + o(\underline{p} - w). \tag{94}$$

Proof. Applying Lemma D.1.1 to the two integrals in equation (92) above, we obtain the existence of bounded functions $M(\lambda)$ and $N(\lambda)$ such that, for λ high enough, we have:

$$\hat{D}(\overline{p} - \hat{p}) + \frac{1}{2}\hat{D}'(\overline{p} - \hat{p})^2 + M(\lambda)(\overline{p} - \hat{p})^3 + D_0(p_0 - \underline{p}) + \frac{1}{2}D_0'(p_0 - \underline{p})^2 \\ + N(\lambda)(p_0 - \underline{p})^3 - \underline{\pi}(w)\log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) + \frac{\underline{\pi}(w)}{\overline{\pi}(w)}s = 0.$$

By Lemmas D.5.2 and D.5.3, the terms $(p_0 - \underline{p})^2$, $(p_0 - \underline{p})^3$, $(\overline{p} - \hat{p})^2$, and $(\overline{p} - \hat{p})^3$ are all little-os of $\underline{p} - w$. Moreover, using the definition of ν from the proof of Lemma D.5.3, we have that

$$\hat{D}(\overline{p} - \hat{p}) = \hat{D}\nu(\underline{p} - w)\log(\underline{p} - w).$$

Finally,

$$-\underline{\pi}(w)\log\left(\frac{\overline{p} - w}{\underline{p} - w}\right) + \frac{\underline{\pi}(w)}{\overline{\pi}(w)}s = D_0(\underline{p} - w)\log(\underline{p} - w) - D_0\eta(\underline{p} - w) + o(\underline{p} - w),$$

where we have used Lemma D.5.2 and the definition of η from above.

Combining the above approximations, we obtain:

$$(D_0 + \hat{D}\nu)(\underline{p} - w)\log(\underline{p} - w) - D_0\eta(\underline{p} - w) + o(\underline{p} - w) = 0.$$

It follows that

$$\nu = -\frac{D_0}{\hat{D}} + \frac{D_0\eta}{\hat{D}}\frac{1}{\log(\underline{p} - w)} + o\left(\frac{1}{\log(\underline{p} - w)}\right),$$

which proves the lemma. □

Approximation of $\overline{\pi}(w)$ and λ.

Lemma D.5.6. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have:*

$$\overline{\pi}(w)^2 = \hat{r}^2 - 2\hat{r}\hat{r}'\frac{D_0}{\hat{D}}(\underline{p} - w)\log(\underline{p} - w) + o\left((\underline{p} - w)^2\log(\underline{p} - w)\right),$$

$$\lambda = 1 - 2\frac{D_0}{\hat{r}}(\underline{p} - w) - 2\frac{\hat{r}'D_0^2}{\hat{r}^2\hat{D}}(\underline{p} - w)^2\log(\underline{p} - w) + o\left((\underline{p} - w)^2\log(\underline{p} - w)\right),$$

$$(\underline{p}-w)^2\log(\underline{p}-w) = \frac{\hat{r}^2}{4D_0^2}(1-\lambda)^2\log(1-\lambda) + o((1-\lambda)^2\log(1-\lambda)).$$

Proof. Combining Lemmas D.5.4 and D.5.5, we obtain:

$$\begin{aligned}
\overline{p}-w &= \overline{p}-\hat{p}+\hat{p}-p_0+p_0-w \\
&= \hat{p}-p_0-\frac{D_0}{\hat{D}}(\underline{p}-w)\log(\underline{p}-w)+\left(\frac{D_0\eta}{\hat{D}}+2\right)(\underline{p}-w)+o(\underline{p}-w).
\end{aligned}$$

Moreover, by Lemma D.5.5, we have:

$$\begin{aligned}
D(\overline{p}) &= \hat{D}+\hat{D}'(\overline{p}-\hat{p})+\frac{\hat{D}''}{2}(\overline{p}-\hat{p})^2+o((\overline{p}-\hat{p})^2) \\
&= \hat{D}-\frac{\hat{D}'D_0}{\hat{D}}(\underline{p}-w)\log(\underline{p}-w)+\frac{\hat{D}'D_0\eta}{\hat{D}}(\underline{p}-w)+o(\underline{p}-w).
\end{aligned}$$

It follows that

$$\begin{aligned}
\overline{\pi}(w) &= (\overline{p}-w)D(\overline{p}) \\
&= \hat{r}-\hat{r}'\frac{D_0}{\hat{D}}(\underline{p}-w)\log(\underline{p}-w)+\left(2\hat{D}+\hat{r}'\frac{D_0\eta}{\hat{D}}\right)(\underline{p}-w)+o(\underline{p}-w).
\end{aligned}$$

Taking the square and discarding higher-order terms, we obtain the first approximation in the statement of the lemma.

Next, we turn to the approximation of λ. By Lemma D.5.2, we have:

$$\underline{\pi}(w) = (\underline{p}-w)\left(D_0+D_0'(\underline{p}-p_0)\right)+o((\underline{p}-w)^2) = D_0(\underline{p}-w)+o((\underline{p}-w)^2\log(\underline{p}-w)).$$

Combining this with the above approximation of $\overline{\pi}(w)$ yields:

$$\begin{aligned}
\lambda &= \frac{\overline{\pi}(w)-\underline{\pi}(w)}{\overline{\pi}(w)+\underline{\pi}(w)} \\
&= 1-2\frac{\underline{\pi}(w)}{\overline{\pi}(w)+\underline{\pi}(w)} \\
&= 1-2\frac{D_0(\underline{p}-w)+o((\underline{p}-w)^2\log(\underline{p}-w))}{\hat{r}-\hat{r}'\frac{D_0}{\hat{D}}(\underline{p}-w)\log(\underline{p}-w)+\left(2\hat{D}+\hat{r}'\frac{D_0\eta}{\hat{D}}+D_0\right)(\underline{p}-w)+o(\underline{p}-w)} \\
&= 1-2\frac{D_0}{\hat{r}}(\underline{p}-w)-2\frac{\hat{r}'D_0^2}{\hat{r}^2\hat{D}}(\underline{p}-w)^2\log(\underline{p}-w)+o((\underline{p}-w)^2\log(\underline{p}-w)),
\end{aligned}$$

as stated.

Finally, we approximate $(\underline{p}-w)^2\log(\underline{p}-w)$ in terms of λ. Rewriting the above approximation of λ and using the fact that a little-o of $\underline{p} - w$ is a little-o of $1 - \lambda$, we have

$$\underline{p} - w = \frac{\hat{r}}{2D_0}(1-\lambda) + o(1-\lambda),$$

which implies that

$$\begin{aligned}
(\underline{p} - w)^2 \log(\underline{p} - w) &= \left(\frac{\hat{r}}{2D_0}\right)^2 (1-\lambda)^2 \log(\underline{p} - w) + o((1-\lambda)^2 \log(\underline{p} - w)) \\
&= \left(\frac{\hat{r}}{2D_0}\right)^2 (1-\lambda)^2 \log \frac{(1-\lambda)\overline{\pi}(w)}{(1+\lambda)D(\underline{p})} + o\left((1-\lambda)^2 \log \frac{(1-\lambda)\overline{\pi}(w)}{(1+\lambda)D(\underline{p})}\right) \\
&= \frac{\hat{r}^2}{4D_0^2}(1-\lambda)^2 \log(1-\lambda) + o((1-\lambda)^2 \log(1-\lambda)),
\end{aligned}$$

where we have used equation (3) to obtain the second line and the fact that $\overline{\pi}(w)/D(\underline{p})$ is bounded for λ close to 1 to obtain the third line. □

D.5.3 Proof of Proposition 10 for $s < \hat{s}$

Consumer surplus approximation

Lemma D.5.7. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\Delta CS = \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2}\left[\alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\tilde{\mu}(s)\right] \\
\times (1-\lambda)^2 \left|\log(1-\lambda)\right| + o((1-\lambda)^2 \log(1-\lambda)).$$

Proof. In the proof of Proposition D.4.4, we established that $\Delta CS = \Psi/(4\lambda)$, where

$$\Psi \equiv \underbrace{4\lambda \int_{\underline{p}}^{p_0} D(p)dp}_{\equiv A_1} + \underbrace{(1-\lambda)^2 \int_{\underline{p}}^{\overline{p}} D(p)dp}_{\equiv A_2} - \underbrace{(1-\lambda)^2 \overline{\pi}(w)^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p-w)^2 D(p)}}_{\equiv A_3}.$$

We approximate Ψ up to order $(\underline{p} - w)^2 \log(\underline{p} - w)$. Applying Lemmas D.1.1 and D.5.4 to the integral in A_1, we obtain:

$$\begin{aligned}
\int_{\underline{p}}^{p_0} D(p)dp &= D_0(p_0 - \underline{p}) - \frac{D_0'}{2}(p_0 - \underline{p})^2 + o((p_0 - \underline{p})^2) \\
&= D_0\left((\underline{p} - w) - 2(\hat{\mu} + \gamma)(\underline{p} - w)^2 \log(\underline{p} - w)\right) + o((\underline{p} - w)^2 \log(\underline{p} - w)).
\end{aligned}$$

Multiplying this by the approximation of λ from Lemma D.5.6?yields

$$A_1 = 4D_0(\underline{p} - w) - 8D_0(\hat{\mu} + \gamma)(\underline{p} - w)^2 \log(\underline{p} - w) + o((\underline{p} - w)^2 \log(\underline{p} - w)).$$

Next, we approximate A_2. By Lemma D.5.6,

$$(1 - \lambda)^2 = 4\frac{D_0^2}{\hat{r}^2}(\underline{p} - w)^2 + 8\frac{\hat{r}' D_0^3}{\hat{r}^3 \hat{D}}(\underline{p} - w)^3 \log(\underline{p} - w) + o((\underline{p} - w)^3 \log(\underline{p} - w)). \tag{95}$$

This implies that $(1 - \lambda)^2$ is a little o of $(\underline{p} - w)^2 \log(\underline{p} - w)$. As the integral $\int_{\underline{p}}^{\overline{p}} D(p)dp$ is bounded when λ is close to 1, we have that A_2 is a little-o of $(\underline{p} - w)^2 \log(\underline{p} - w)$.

Finally, we approximate A_3. By Lemma D.1.1, the integral in A_3 can be approximated as follows:[33]

$$\begin{aligned}
\int_{\underline{p}}^{\overline{p}} \frac{dp}{(p - w)^2 D(p)} &= \frac{1}{D_0} \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p - w)^2} - \frac{D_0'}{D_0^2} \int_{\underline{p}}^{\overline{p}} \frac{p - p_0}{(p - w)^2} dp + O\left(\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^2}{(p - w)^2} dp \right) \\
&= \frac{1}{D_0}\left(\frac{1}{\underline{p} - w} - \frac{1}{\overline{p} - w} \right) - \frac{D_0'}{D_0^2}\left(\log\left(\frac{\overline{p} - w}{\underline{p} - w} \right) + \frac{p_0 - w}{\overline{p} - w} - \frac{p_0 - w}{\underline{p} - w} \right) + O(1) \\
&= \frac{1}{D_0(\underline{p} - w)} + \frac{D_0'}{D_0^2} \log(\underline{p} - w) + O(1),
\end{aligned}$$

where we used Lemma D.5.4 to obtain the third line. Multiplying this by the approximation of $(1 - \lambda)^2$ from equation (95) and using the fact that a big-o of $(\underline{p} - w)^2$ is a little-o of $(\underline{p} - w)^2 \log(\underline{p} - w)$, and a little-o of $(\underline{p} - w)^3 \left(\log(\underline{p} - w)\right)^2$ is a little-o of $(\underline{p} - w)^2 \log(\underline{p} - w)$, we obtain

$$\begin{aligned}
&(1 - \lambda)^2 \int_{\underline{p}}^{\overline{p}} \frac{dp}{(p - w)^2 D(p)} = \\
&\qquad 4\frac{D_0}{\hat{r}^2}(\underline{p} - w) + \left(4\frac{D_0'}{\hat{r}^2} + 8\frac{\hat{r}' D_0^2}{\hat{r}^3 \hat{D}} \right) (\underline{p} - w)^2 \log(\underline{p} - w) + o((\underline{p} - w)^2 \log(\underline{p} - w)).
\end{aligned}$$

Using the approximation of $\overline{\pi}^2(w)$ from Lemma D.5.6, it follows that

$$A_3 = 4D_0(\underline{p} - w) - \frac{4D_0}{p_0 - c}(\underline{p} - w)^2 \log(\underline{p} - w) + o((\underline{p} - w)^2 \log(\underline{p} - w)).$$

[33]To obtain the second line, we used the fact that $\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2} dp$ is bounded when λ is close to 1. To see this, recall from the proof of Lemma D.5.4 that

$$\int_{\underline{p}}^{\overline{p}} \frac{(p - p_0)^2}{(p - w)^2} dp \leq \max_{p \in [\underline{p}, \overline{p}]} \left(\frac{p - p_0}{p - w} \right)^2 (\overline{p} - \underline{p}) \leq \max \left\{ \left(\frac{p_0 - \underline{p}}{\underline{p} - w} \right)^2, \left(\frac{\overline{p} - p_0}{\overline{p} - w} \right)^2 \right\} p^m(p_0)$$

By Lemmas D.5.4 and D.4.1, both terms within the second maximum are bounded when λ is close to 1.

Combining the terms, we obtain the approximation of Ψ:

$$\Psi = -4\frac{D_0}{p_0 - c}\left(2(p_0 - c)(\hat{\mu} + \gamma) - 1\right)(\underline{p} - w)^2 \log(\underline{p} - w) + o((\underline{p} - w)^2 \log(\underline{p} - w)).$$

Using the approximation of $(\underline{p} - w)^2 \log(\underline{p} - w)$ from Lemma D.5.6 and the fact that $\Delta CS = \Psi/(4\lambda)$, we have:

$$\Delta CS = \frac{\hat{r}^2}{4r_0}\underbrace{(2(p_0 - c)(\hat{\mu} + \gamma) - 1)}_{\equiv \zeta}(1 - \lambda)^2 |\log(1 - \lambda)| + o((1 - \lambda^2)\log(1 - \lambda)).$$

Plugging the expression of γ from the proof of Lemma D.5.4, we can rewrite ζ as a function of the pass-through and its derivative:

$$\begin{aligned} 2(p_0 - c)(\gamma + \hat{\mu}) - 1 &= (p_0 - c)\left(\frac{r_0'''}{r_0''} - 2\frac{D_0''}{D_0'} + 2\frac{D_0''}{D_0'} - 4\frac{D_0'}{D_0}\right) + 2(p_0 - c)\hat{\mu} - 1 \\ &= -\frac{\beta(p_0 - c)}{\alpha^2} - 2\frac{2\alpha - 1}{\alpha} + 3 + 2(p_0 - c)\hat{\mu} \\ &= -\frac{\beta(p_0 - c)}{\alpha^2} + \frac{2}{\alpha} - 1 + 2(p_0 - c)\hat{\mu}. \end{aligned}$$

Inserting this into the approximation of ΔCS and using the fact that $\tilde{\mu}(s) = \hat{\mu}$ and $\tilde{p}(s) = \hat{p}(s)$ for $s < \hat{s}$, we obtain the approximation in the statement of the lemma. □

Producer surplus approximation

Lemma D.5.8. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\Delta\Pi = -\frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha}(1 - \lambda)^2 |\log(1 - \lambda)| + o((1 - \lambda)^2 \log(1 - \lambda)).$$

Proof. In the proof of Proposition D.4.5, we established that $\Delta\Pi = \Phi/(4\lambda)$, where

$$\Phi \equiv 4\lambda\left(r(\underline{p}) - r_0\right) + (1 - \lambda)^2 \overline{\pi}(w)^2 \int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p - w)^2} - (1 - \lambda)^2\left(r(\overline{p}) - r(\underline{p})\right).$$

We approximate Φ up to order $(\underline{p} - w)^2 \log(\underline{p} - w)$. By Lemma D.5.4, we have:

$$r(\underline{p}) - r_0 = \frac{r_0''}{2}(\underline{p} - p_0)^2 + o\left((\underline{p} - p_0)^2\right) = o\left((\underline{p} - w)^2 \log(\underline{p} - w)\right).$$

Applying Lemma D.1.1, the integral in the second term can be approximated as:[34]

$$\begin{aligned}\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2} &= \frac{r_0''}{D_0^2}\int_{\underline{p}}^{\overline{p}} \frac{p-p_0}{(p-w)^2}dp + O\left(\int_{\underline{p}}^{\overline{p}} \frac{(p-p_0)^2}{(p-w)^2}dp\right)\\ &= \frac{r_0''}{D_0^2}\left[\log\frac{\overline{p}-w}{\underline{p}-w} + \frac{p_0-w}{\overline{p}-w} - \frac{p_0-w}{\underline{p}-w}\right] + O(1)\\ &= -\frac{r_0''}{D_0^2}\log(\underline{p}-w) + O(1),\end{aligned}$$

where the last line follows by Lemmas D.4.1 and D.5.4. Combining this with Lemma D.5.6, we obtain the approximation of the second term:

$$(1-\lambda)^2\overline{\pi}(w)^2\int_{\underline{p}}^{\overline{p}} \frac{r'(p)}{D(p)^2}\frac{dp}{(p-w)^2} = -4r_0''(\underline{p}-w)^2\log(\underline{p}-w) + o\left((\underline{p}-w)^2\log(\underline{p}-w)\right).$$

The third term is a little-o of $(\underline{p}-w)^2\log(\underline{p}-w)$, as $r(\overline{p})-r(\underline{p})$ is bounded and $(1-\lambda)^2 = o\left((\underline{p}-w)^2\log(\underline{p}-w)\right)$ by Lemma D.5.6.

Combining the terms and using the fact that $r_0'' = -D_0/(\alpha(p_0-c))$, we obtain

$$\Phi = 4\frac{D_0}{p_0-c}\frac{1}{\alpha}(\underline{p}-w)^2\log(\underline{p}-w) + o\left((\underline{p}-w)^2\log(\underline{p}-w)\right).$$

Using the approximation of $(\underline{p}-w)^2\log(\underline{p}-w)$ with respect to $(1-\lambda)^2\log(1-\lambda)$ from Lemma D.5.6 proves the lemma. □

Aggregate Surplus Approximation.

Lemma D.5.9. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have*

$$\Delta AS = \frac{\pi^2(\tilde{p}(s),p_0)}{4r_0\alpha^2}\left[\alpha(1-\alpha) - \beta(p_0-c) + 2(p_0-c)\alpha^2\tilde{\mu}(s)\right] \times (1-\lambda)^2\left|\log(1-\lambda)\right| + o((1-\lambda)^2\log(1-\lambda)).$$

Proof. This follows immediately from combining Lemmas D.5.7 and D.5.8. □

D.5.4 Properties of the Function $\tilde{\mu}$

Lemma D.5.10. *The function $\tilde{\mu}(s)$ is continuous, strictly negative, and identically equal to $1/(p_1-p_0)$ on $(\hat{s},\infty)$. Moreover, it is strictly decreasing on $(0,\hat{s})$ if the monopoly pass-through function, $p^{m\prime}(w)$, is non-increasing in w on $[c,p_0]$.*

[34] For the argument why the integral inside the big-O on the first line is bounded, see footnote 33.

Proof. The fact that $\tilde{\mu}$ is continuous in s and constant above $\hat{s}$ is immediate. Let us study the properties of $\tilde{\mu}$ on $(0, \hat{s})$. Define the function

$$\mu : p \in (p_0, p_1) \mapsto \frac{(p - p_0)\pi'(p, p_0)D_0}{\pi^2(p, p_0)} - \frac{1}{p - p_0},$$

and note that $\tilde{\mu}(s) = \mu(\hat{p}(s))$. As $\hat{p} : (0, \hat{s}) \to (p_0, p_1)$ is strictly increasing in s, $\tilde{\mu}$ inherits and monotonicity properties of μ. Observe that μ can be rewritten as

$$\mu(p) = \frac{D_0}{p - p_0}\left[\frac{1}{D(p)} - \frac{1}{D_0} - \left(\frac{1}{D(p)}\right)'(p - p_0)\right].$$

As $1/D$ is strictly convex on $[p_0, p_1]$, its graph lies above its tangent lines: for every $p \in (p_0, p_1)$,

$$\frac{1}{D_0} > \frac{1}{D(p)} + \left(\frac{1}{D(p)}\right)'(p_0 - p).$$

Therefore, μ is strictly negative.[35]

Next, we turn to the monotonicity of μ. We have:

$$\mu'(p) = \frac{D_0}{(p - p_0)^2}\underbrace{\left[\frac{1}{D_0} - \frac{1}{D(p)} + \left(\frac{1}{D(p)}\right)'(p - p_0) - \left(\frac{1}{D(p)}\right)''(p - p_0)^2\right]}_{\equiv \vartheta(p)}.$$

Clearly, $\vartheta(p_0) = 0$. Let us show that ϑ is strictly decreasing on (p_0, p_1). We have:

$$\vartheta'(p) = (p - p_0)\left[-\left(\frac{1}{D(p)}\right)'' - \left(\frac{1}{D(p)}\right)'''(p - p_0)\right].$$

For every $p \in (p_0, p_1)$, let $\alpha(p) \equiv p^{m\prime}\left((p^m)^{-1}(p)\right)$. By assumption, $\alpha(p)$ is non-increasing in p. As, for every $w \in (c, p_0)$,

$$p^{m\prime}(w) = \frac{1}{2 - \frac{D''(p^m(w))D(p^m(w))}{D'(p^m(w))^2}},$$

[35] To see why $1/D$ is strictly convex, note that

$$D(p)^3\left(\frac{1}{D(p)}\right)'' = 2D'(p)^2 - D''(p)D(p) \geq \frac{-D'(p)D(p)}{p}\left(\frac{-pD'(p)}{D(p)} - 1\right) > 0,$$

where the first inequality follows by differentiating the price elasticity of demand, using Marshall's second law of demand, and rearranging terms; and the second inequality holds because the price elasticity of demand strictly exceeds 1 whenever $p > p_0$, again due to Marshall's second law.

we can express $(1/D)''$ as

$$\left(\frac{1}{D(p)}\right)'' = \frac{1}{\alpha(p)}\frac{D'(p)^2}{D(p)^3}.$$

Differentiating once more, we obtain

$$\begin{aligned}\left(\frac{1}{D(p)}\right)''' &= -\frac{\alpha'(p)}{\alpha(p)^2}\frac{D'(p)^2}{D(p)^3} + \frac{1}{\alpha(p)}\frac{2D'(p)D''(p)D(p)^3 - 3D'(p)^3D(p)^2}{D(p)^6} \\ &= -\frac{\alpha'(p)}{\alpha(p)^2}\frac{D'(p)^2}{D(p)^3} + \frac{1}{\alpha(p)}\frac{D'(p)^3}{D(p)^4}\left(1 - \frac{2}{\alpha(p)}\right).\end{aligned}$$

Inserting these expressions into $\vartheta'(p)$ yields

$$\begin{aligned}\frac{\vartheta'(p)}{p - p_0} &= \frac{1}{\alpha(p)}\frac{D'(p)^2}{D(p)^3}\left[-1 - \frac{D'(p)}{D(p)}\left(1 - \frac{2}{\alpha(p)}\right)(p - p_0) + \frac{\alpha'(p)}{\alpha(p)}(p - p_0)\right] \\ &\leq \frac{1}{\alpha(p)}\frac{D'(p)^2}{D(p)^4}\left[-D(p) - D'(p)\left(1 - \frac{2}{\alpha(p)}\right)(p - p_0)\right] \\ &= \frac{1}{\alpha(p)}\frac{D'(p)^2}{D(p)^4}\left[-\left(D(p) + (p - p_0)D'(p)\right) + D'(p)\frac{2}{\alpha(p)}(p - p_0)\right],\end{aligned}$$

where the second line follows because $\alpha'(p) \leq 0$. As $p \mapsto (p - p_0)D(p)$ has a strictly positive derivative on (p_0, p_1) due to Marshall's second law of demand, it follows that $\vartheta'(p) < 0$ for every $p \in (p_0, p_1)$. □

D.5.5 Distributional Effects

We now separately derive the approximation of consumer surplus for online and offline consumers.

Lemma D.5.11. *For $s < \hat{s}$, in the neighborhood of $\lambda = 1$, we have:*

$$\Delta CS_B = -\frac{1}{2}\pi(\tilde{p}(s), p_0)(1 - \lambda)^2|\log(1 - \lambda)| + o((1 - \lambda)^2 \log(1 - \lambda)), \tag{96}$$

$$\begin{aligned}\Delta CS_O = \frac{\pi^2(\tilde{p}(s), p_0)}{4r_0\alpha^2}&\left[\alpha(2 - \alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\left(\tilde{\mu}(s) + \frac{D_0}{\pi(\tilde{p}(s), p_0)}\right)\right] \\ &\times (1 - \lambda)^2 |\log(1 - \lambda)| + o((1 - \lambda)^2 \log(1 - \lambda)).\end{aligned} \tag{97}$$

Proof. From equation (65), we have that $\Delta CS_B = (1 - \lambda)/(2\lambda)\Psi_B$, where

$$\Psi_B = (1 + \lambda)\int_{\underline{p}}^{p_0} D(p)dp + (1 - \lambda)\int_{p_0}^{\overline{p}} D(p)dp - (1 - \lambda)\pi(\overline{p}, w)\log\frac{\overline{p} - w}{\underline{p} - w}.$$

We approximate Ψ_B up to order $(\underline{p} - w)\log(\underline{p} - w)$. Applying Lemmas D.1.1 and D.5.2, we obtain

$$\int_{\underline{p}}^{p_0} D(p)dp = D_0(p_0 - \underline{p}) + o(p_0 - \underline{p}) = o((\underline{p} - w)\log(\underline{p} - w)),$$

implying that the first term of Ψ_B is a little-o of $(\underline{p}-w)\log(\underline{p}-w)$. As $\int_{p_0}^{\overline{p}} D(p)dp$ is bounded , the approximation of $1-\lambda$ from Lemma D.5.6 implies that the second term of Ψ_B is also a little-o of $(\underline{p}-w)\log(\underline{p}-w)$. Applying again Lemma D.5.6, we obtain the approximation of the third term of Ψ_B:

$$\begin{aligned}(1-\lambda)\pi(\overline{p}, w)\log\frac{\overline{p}-w}{\underline{p}-w} &= -(1-\lambda)\pi(\overline{p}, w)\log(\underline{p}-w) + o((\underline{p}-w)\log(\underline{p}-w)) \\ &= -2D_0(\underline{p}-w)\log(\underline{p}-w) + o((\underline{p}-w)\log(\underline{p}-w)).\end{aligned}$$

Combining the three terms yields

$$\Psi_B = 2D_0(\underline{p}-w)\log(\underline{p}-w) + o((\underline{p}-w)\log(\underline{p}-w)).$$

Using the approximation of $1-\lambda$ from Lemma D.5.6, this implies:

$$\Delta CS_B = \frac{2D_0^2}{\pi(\hat{p}, p_0)}(\underline{p}-w)^2\log(\underline{p}-w) + o((\underline{p}-w)^2\log(\underline{p}-w)).$$

Finally, using the approximation of $(\underline{p}-w)^2\log(\underline{p}-w)$ from Lemma D.5.6, we obtain the approximation of ΔCS_B in the statement of the lemma. Using the fact that $\Delta CS_O = \Delta CS - \Delta CS_B$, the approximation of ΔCS_O is obtained by combining this approximation with the approximation of ΔCS from Lemma D.5.7. □

Combining Lemmas D.4.7 and D.5.11, we obtain:

Lemma D.5.12. *Suppose that $s \neq \hat{s}$. Then, for λ high enough, offline consumers are harmed by a ban on dual pricing. If, in addition, the monopoly pass-through function, $p^{m\prime}(w)$, is non-increasing in w on $[c, p_0]$, then online consumers benefit from a ban on dual pricing for λ high enough.*

Proof. The results for offline consumers follows immediately from Lemmas D.4.7 and D.5.11. These lemmas also imply that, for λ close enough to 1, ΔCS_O has the same sign as

$$A(s) \equiv \alpha(2-\alpha) - \beta(p_0 - c) + 2(p_0 - c)\alpha^2\left[\tilde{\mu}(s) + \frac{D_0}{\pi(\tilde{p}(s), p_0)}\right].$$

By Lemma D.5.10, $\tilde{\mu}(\cdot)$ is decreasing on $(0, \hat{s})$. Moreover, Marshall's second law of demand implies that $p \in (p_0, p_1) \mapsto \pi(p, p_0)$ is strictly increasing. As $\tilde{s} \in (0, \hat{s}) \mapsto \tilde{p}(s) \in (p_0, p_1)$ is increasing, it follows that $D_0/\pi(\tilde{p}(s), p_0)$ is decreasing in s on $(0, \hat{s})$. Hence, $A(s) \geq A(\hat{s})$ for every $s > 0$, and all we need to do is show that $A(\hat{s}) > 0$.

At $s = \hat{s}$, $\tilde{\mu}(s) = -1/(p_1 - p_0)$, and so

$$A(\hat{s}) = 2\alpha - \beta(p_0 - c) + \alpha^2 \left[2\left(\frac{r_0}{r_1} - \frac{p_0 - c}{p_1 - p_0} \right) - 1 \right]. \tag{98}$$

By revealed profitability, we have that

$$r_0 \geq (p_1 - c)D_1 = \frac{p_1 - c}{p_1 - p_0} r_1 = \left(1 + \frac{p_0 - c}{p_1 - p_0} \right) r_1.$$

Combining this with the fact that $\beta \leq 0$, we obtain that $A(\tilde{s}) \geq \alpha^2 + 2\alpha$, which concludes the proof. □

D.5.6 On the Optimality of Supplying Both Retailers when λ is High

The following lemma leverages our Taylor approximations of equilibrium behavior to show that the manufacturer deals with both retailers in equilibrium when λ is close to 1:

Lemma D.5.13. *The manufacturer strictly prefers supplying both retailers provided λ is sufficiently close to 1.*

Proof. If the manufacturer supplies a single retailer, then it optimally sets $w = c$ and fully extracts the retailer's profit, earning

$$\pi_{one} = r_0 - \frac{1 - \lambda}{2} r_0.$$

If it supplies both retailers, then its expected profit is $\pi_{two} = r_0 + \Delta\Pi$, where $\Delta\Pi$ was defined in equation (43). Applying Lemmas D.4.5 and D.5.8, the difference in profits is:

$$\pi_{two} - \pi_{one} = \frac{1 - \lambda}{2} r_0 \left[1 - \frac{\pi(\tilde{p}(s), p_0)^2}{2\alpha r_0^2} (1 - \lambda) |\log(1 - \lambda)| \right] + o((1 - \lambda)^2 \log(1 - \lambda)),$$

As λ tends to one, the term inside square brackets tends to 1, implying that the manufacturer strictly prefers supplying both retailers. □