Discussion Paper Series – CRC TR 224

Discussion Paper No. 691
Project C 05

Can Public Debt Crowd in Private Investment?

Christian Bayer[1]
Fabio Stohler[2]

June 2025

[1] University of Bonn, CEPR, CESifo, and IZA. Email: christian.bayer@uni-bonn.de
[2] University of Bonn. Email: fabio.stohler@uni-bonn.de

Support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 is gratefully acknowledged.

Collaborative Research Center Transregio 224 - www.crctr224.de
Rheinische Friedrich-Wilhelms-Universität Bonn - Universität Mannheim

Can Public Debt Crowd in Private Investment?

by Christian Bayer and Fabio Stohler*

June 18, 2025

Abstract

If households self-select into a risky high-income state through investment, increased government debt can stimulate investment and improve welfare. In a heterogeneous agent endogenous growth model, government debt helps households smooth consumption and encourages investment in risky, high-return assets, crowding in aggregate growth. However, when debt becomes excessive, capital crowding out and distortionary taxation negate these benefits. Using a model calibrated to U.S. data, we show that this crowding-in effect suggests a higher optimal debt-to-GDP ratio than currently observed.

Keywords— Incomplete Markets, Public Debt, Endogenous Growth, Portfolio Choice
JEL codes— D31, E21, G11, H63, O43

1 Introduction

The past multiple crisis have led to a sharp rise in public debt in most developed countries. This has sparked concerns about debt sustainability in some cases, but even where not warranted, the increase in public debt makes it necessary to re-assess the undesirable consequences, but also the desirable ones of persistently higher public debt. An important traditional argument against maintaining high levels of public debt is the risk of crowding out private investment and the labor supply distortions implied by the taxes that are needed in order to service debt.

However, at least since the seminal papers by Huggett (1993) and Aiyagari and McGrattan (1998) the economic literature argued that positive or even elevated levels of

*Bayer: University of Bonn, CEPR, CESifo, and IZA (email: christian.bayer@uni-bonn.de); Stohler: University of Bonn (email: fabio.stohler@uni-bonn.de). Support by the German Research Foundation (DFG) through CRC TR 224 (Project C05) is gratefully acknowledged.

government debt can still be desirable. They facilitate self-insurance by households and the economic value of this facilitated self-insurance can outweigh the negative effects of crowding out capital and distorting labor supply.[1] While the trade-offs involved are clear and important, it is still an empirical question, depending in particular on the parameters of the income process and the elasticity of labor supply, whether the debt level should be higher or lower.[2]

Importantly, this literature has mostly used a framework of exogenous growth. Yet, from a dynamic point of view, the distortions and benefits at the level of activity can easily be dwarfed by the potential effects of government debt and taxation on endogenous growth. However, the direction of the effect of debt on growth is not a priori clear. The distortionary effect of taxation will also discourage growth-enhancing investment. At the same time, better insured households will be more willing to take more risks and risk-taking and innovation go hand in hand.

For this reason, we integrate public debt policy in a model of incomplete markets and risky innovation. We show that, for a wide range of debt levels, the risk-tolerance effect dominates crowding out and distortions. This implies that high levels of public debt are growth-enhancing. We carry out this analysis in an almost standard incomplete markets framework. Households face uninsurable income risks against which they self-insure. In contrast to the standard setting, households can invest not only in (risk-free) physical assets and/or government bonds but also in risky but growth-enhancing projects.[3]

In detail, we model growth projects as investments that add new varieties to the economy. These varieties each earn profits through monopolistic competition, but they also expand the set of choices available to consumers, making the economy as a whole more efficient. Through an externality on the invention of new varieties, adding varieties promotes *growth* and does not only shift the *level* of productivity.

We assume that households cannot diversify the portfolio of varieties they own. They operate the production of their varieties in a form of a backyard enterprise which we assume to fail from time to time. If it fails, all its products become obsolete and the households forfeits all its variety investments and has to start accumulating new product varieties.

As with their labor income risks, households cannot insure against these investment risks because markets are incomplete. Therefore, households trade off the returns to growth investments against the risks of these investments and their need to self-insure through

1 Among others see Aiyagari and McGrattan (1998), Challe and Ragot (2010), Heathcote (2005), or Woodford (1990).

2 See Krueger and Perri (2011), Röhrs and Winter (2015), Röhrs and Winter (2017), Bayer, Born and Luetticke (2022), and Dyrda and Pedroni (2023).

3 This links our paper to Krebs (2003) and Krebs, Kuhn and Wright (2015) that discuss human capital investment in endogenous growth models.

more appropriate alternative assets. The supply of assets by the government interacts with these trade-offs that households face. Higher levels of government debt facilitate self-insurance. Importantly, as we show, this allows households to take greater risks in their investments, thereby promoting growth.

To sharpen the understanding of the mechanism, we first develop a stylized three-period version of our model before presenting the full quantitative framework. In the stylized model, ex ante identical households invest in productivity-enhancing risky assets or risk-free government bonds. Since the productivity-enhancing investments are developments of new varieties, they acquire only a part of the public return of the investment, we have underinvestment in the *laissez-faire* equilibrium. This is exacerbated by the uninsurability of investment risk. After a successful initial investment, the investing household expects its return on the risky investment to decline over time. Therefore, it would like to save in a safe asset to smooth consumption over time. When liquidity is scarce, this is not possible and the effective return on the risky asset falls. As a result, when government debt is scarce, increasing the supply of government debt increases welfare because it increases the economic value of the return on the risky investment in good times.

Building on this intuition, we develop a quantitative heterogenous agent model. In this framework, we find the well-known result of Aiyagari and McGrattan (1998) that higher levels of government debt crowd out capital and distort the supply of labor through the need for taxation. At the same time, we show that households, on average, are willing to invest more in risky projects. In a high interest rate environment, they find it easier to build up a buffer of wealth that helps them manage the risk of equity.

Calibrating the model to long-run U.S. averages for income risk, government debt, equity returns, and economic growth, we find that a 20 percent increase in government debt necessitates a 25 basis point (annual) increase in the return on liquid assets, while also boosting annual growth by 10 basis points. At this higher level of public debt, households allocate more investment toward growth-enhancing projects but reduce their capital investment, resulting in a 3.1 percent decline in the aggregate capital stock. Our findings indicate that the optimal level of government debt is 200% of initial GDP—substantially higher than estimates in much of the existing literature. At this level, the economy features 13.6% less capital and 2.3% less labor, yet achieves an annual growth rate that is 39 basis points higher. The crowding-in of innovation-driven growth outweighs the crowding-out effects on capital and labor in an environment with endogenous growth. We further show that in the absence of distortionary taxation, both the optimal debt level and the growth-enhancing effect of public debt would be even greater. In our baseline scenario, where public debt significantly crowds out physical capital, the resulting contraction in market size also dampens investment in product variety.

Our results contribute to three different strands of literature. First, they are related to the heterogeneous agent model literature, which examines the role of heterogeneity in shaping policy outcomes.[4] To our knowledge, we are the first to integrate the analysis of fiscal policy within a heterogeneous agent model framework that includes portfolio choice and endogenous growth dynamics. Earlier contributions as Krebs (2003) and Krebs, Kuhn and Wright (2015) also feature endogenous growth with heterogenous agents through a portfolio choice to invest in human capital besides a risk-free asset. However, they do not focus on the role that the supply of the risk-free asset has on endogenous growth, but emphasize alternative factors. We share similarities with the work of Gaillard and Wangner (2022). They use a model with heterogeneous agents and endogenous growth, but with the restriction that only a fixed subset of households, the innovators, contribute to aggregate growth, while the remaining cohort, the workers, face idiosyncratic risk. They emphasize the role of stabilization policy in reallocating resources between these different groups, thereby highlighting the trade-off between short-run demand stabilization and long-run growth stabilization. In contrast, our contribution diverges in its mechanism by emphasizing the critical interplay between idiosyncratic risk and growth-enhancing investment. Angeletos and Calvet (2006) solve a model where each household has access to an idiosyncratic background technology and study the impact of interest rate changes on the economy. While they find that future higher interest rates depress investment today, we find the opposite. The model is solved with CARA preferences and does not allow for an analysis of the importance of wealth inequality for the accumulation decision of households and the interaction between portfolio choice and wealth distribution, which are key to our results.

Second, we contribute to the literature examining the role of government debt in heterogeneous agent models. Following the seminal work of Aiyagari and McGrattan (1998), recent contributions have revisited the question on the optimal level of government debt and the optimal conduct of fiscal policy.[5] However, much of this literature has primarily focused on determining the optimal level of public debt under the assumption of exogenous growth. Our analysis contributes to this literature by endogenizing the growth rate and quantifying the importance of liquidity provision and budget insurance in a framework that is consistent with both micro and macro data. While our focus on liquidity parallels

[4] See, for example, Woodford (1990), Heathcote (2005), Kitao (2008), Challe and Ragot (2010), Kaplan and Violante (2014), McKay and Reis (2016), and Bayer, Born and Luetticke (2022).

[5] For instance, see Flodén (2001), Krueger and Perri (2011), Gomes, Michaelides and Polkovnichenko (2012), Röhrs and Winter (2015), Bhandari et al. (2016), Röhrs and Winter (2017), Dyrda and Pedroni (2023), among others.

that of Bayer, Born and Luetticke (2022), we enrich their model by incorporating an endogenous growth mechanism.

Finally, our work contributes to the endogenous growth literature.[6] The mechanism driving our main result is similar to the reduced-form equity financing shock used in Bianchi, Kung and Morales (2019) or the liquidity demand shock in Anzoategui et al. (2019). In both papers, these shocks lead to a reduction in the representative household's R&D investment. In our model, changes in households' insurance against idiosyncratic risk affect their demand for liquidity and risky equity. Thus, our model structure provides a microfoundation for the shocks used in the aforementioned papers. While papers within the entrepreneurial literature[7] also incorporate heterogeneity, they primarily examine the impact of frictions on total factor productivity. While these models incorporate transitional dynamics, they do not feature endogenous long-run growth, which distinguishes our contribution.

This remainder of this paper is organized as follows. Section 2 develops a simple three period model to sharpen intuition. Section 3 develops the full infinite horizon model and describes the steady state endogenous growth equilibrium. Section 4 illustrates the calibration of the model to US data, followed by the presentation of results from our policy experiments. Finally, Section 5 concludes. An Appendix follows.

2 A simple analytical model

To form intuition, we first consider a three-period model of precautionary saving and portfolio choice.

2.1 Model Description

In the first period, ex ante identical households decide to invest in a government bond or in the creation of new product varieties, a risky equity investment. In the second period, the risk is realized and some households enjoy high returns on their equity investments, temporary monopoly rents on the varieties they created, while other households have to live exclusively on labor income. In the third period, the right to earn monopoly rents is extinguished and all households have only labor income. Since the right to monopoly rents vanished in the third period, the investing households do not earn the full economic

[6] For example, see Romer (1990), Kung and Schmid (2015), Okada (2022), among others.

[7] For example, see Kitao (2008), Buera and Shin (2013), and Midrigan and Xu (2014). For a comprehensive overview, see Buera, Kaboski and Shin (2015).

rent from their investment, and thus there is a nonpecuniary externality. A government provides a risk-free asset and repays for it by collecting lump-sum taxes.

Consumption, Investment and Portfolio Choices: Household utility depends only on consumption. We assume that the household's utility function is three times continuously differentiable and strictly concave. For simplicity, we abstract from discounting between periods one and two, but allow discounting of period three to capture the relative importance of the two future periods.

In period one, all households are endowed with ω goods. In addition, they receive government transfers (a negative lump-sum tax), $-\tau_1$. They decide to consume the endowment, c, to invest in government bonds, b_2, or to invest in the creation of new varieties, e.

In period two, households are endowed with a unit of labor $N = 1$ from which they receive labor income w_2. In addition, government debt is repaid with gross interest R_1. For a fraction of φ households, the investment in the creation of varieties was successful ("H-state") and they receive monopoly rents $\tilde{\pi}$ for each unit they invested, in addition to their labor income. The other household ("L-state") receives only labor income. In period two, households either consume c_2 or invest in government bonds b_2.

In period three, households are again endowed with a unit of labor $N = 1$ from which they receive labor income w_3. They no longer have any rights to monopoly rents, regardless of whether their investment in period two was successful or not. In neither period can households borrow in the risk-free asset (bonds) or equity. Figure 1 summarizes the timeline of the analytical model.

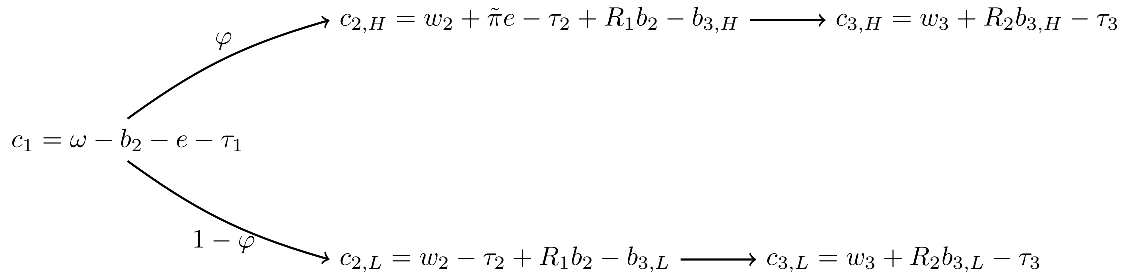
Production and factor incomes: A continuum of firms produces output from labor inputs with the production function $Y_t = Z\mathcal{E}N$. Z is a productivity scalar. The efficiency of labor is proportional to the total risky investment that was successful, $\mathcal{E}$.

In period two, owners of varieties can claim a share $1 - \phi$ of the output, so that the per-unit profit is $\tilde{\pi} = (1-\phi)Y_2/\mathcal{E}$, while the rest goes to labor income, implying a wage rate $w_2 = \phi Y_2$.[8] In the third period, all income is paid as wages to households, $w_3 = Y_3 = Z\mathcal{E}N$.

This implies that, in both periods, investment in risky assets increases factor incomes, but the variety of investors do not fully appropriate the economic rents they produce. In the full quantitative model, we micro-found this externality of investment in varieties through the preferences of variety-loving consumers.

[8] This payoff structure can be microfounded in a standard two-level production structure with symmetric intermediate goods producers enjoying monopolistic competition and a final goods bundler. The quantitative section provides a microfoundation for a production structure with labor and capital. The production structure obtained here is a special case without capital.

Figure 1 Timeline of the analytical model



NOTE - Subscripts on consumption denote the respective period and household groups. b_2 denotes savings in the risk-free asset between periods one and two, while $b_{i,2}$ denotes savings between periods two and three of household group i. e denotes risky savings between periods one and two.

Government: The government must balance its budget in each period. It issues $\mathcal{B}$ units of government bonds in period one, rolls these over in period two and pays them back in period three. In period one, the government pays out the proceeds of the government debt in the form of transfers. In periods two and three, it collects lump-sum taxes to finance the interest on the debt, i.e., the net interest rate in period two and the gross interest rate in period three, since all of the debt is then repaid.

$$\tau_1 = -\mathcal{B}, \quad \tau_2 = (R_1 - 1)\mathcal{B}, \quad \text{and } \tau_3 = R_2\mathcal{B}.$$

The government has no productive role except to provide liquidity to households.

Market Clearing: Market clearing requires that the asset, labor, and goods markets clear in all periods. The asset markets include the market for risk-free assets in periods one and two, and the market for risky assets in period one. The labor market exists in periods two and three, while the goods market exists in all three periods.

For the liquid asset market to clear, households must hold all government debt. Therefore, $\mathcal{B} = \int_0^1 b_{i,t} di$ must hold. In the first period this simplifies to $\mathcal{B} = b$ and $\mathcal{B} = \varphi b_{3,H} + (1 - \varphi) b_{3,L}$ in the second period. In period three, all debt is repaid.

Market clearing in the risky asset market requires $\mathcal{E} = \varphi \int_0^1 e_i di$, where e_i is the policy function in period one for the risky asset. Since all households are identical ex ante, this simplifies to $\mathcal{E} = \varphi e$.

Market clearing in the labor market requires that the labor demanded by the firm and used in production equals the labor supplied by households. Since households supply one unit of labor inelastically, labor market clearing requires $N_2 = N_3 = 1$.

Finally, goods market clearing requires that all endowments or produced goods are used by households. This yields the goods market clearing conditions, which we state in the order of the periods

$$\omega = c_1 + e, \text{ and } Y_t = \varphi c_{t,H} + (1-\varphi)c_{t,L} \text{ for } t = 2, 3.$$

Equilibrium: In equilibrium, all markets clear and households' consumption and investment decisions are optimal. This implies that government debt policy affects period one decisions only through its effect on future allocations and equilibrium prices, as $-\tau_1 = \mathcal{B}$.

2.2 Crowding in risky investment

In the following, we show that if government debt is small enough, an increase in government debt fosters investment in the risky asset. At zero debt, the H-type household expects income to fall between periods two and three, while the L-type household expects it to rise; the former wants to save, while the latter wants to borrow, but is strictly constrained to do so. By continuity, the same H-type-are-savers structure emerges even for small positive levels of government debt.[9]

Thus, for sufficiently small levels of debt, the H-type household buys up all government debt in period two, and we obtain as allocations for the H-type:

$$\begin{aligned} c_1 &= \omega - e^* && (1) \\ c_{2,H} &= w_2 + \tilde{\pi} e^* - \frac{1-\varphi}{\varphi}\mathcal{B} && (2) \\ c_{3,H} &= w_3 + R_2 \frac{1-\varphi}{\varphi}\mathcal{B}, && (3) \end{aligned}$$

where e^* is the optimal risky investment. Since only in the H-state there is a positive payoff from investing in e, this investment is determined by the Euler equation

$$u'(c_1) = \varphi\tilde{\pi}u'(c_{2,H}) \quad \Leftrightarrow \quad u'(\omega - e^*) = \varphi\tilde{\pi}u'\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right). \tag{4}$$

This immediately implies our

Proposition 1. *There exists $\mathcal{B}^*$ such that if $\mathcal{B} < \mathcal{B}^*$, the direct effect, i.e., keeping wages and profits fixed, of $\mathcal{B}$ on optimal risky investment e^* is positive.*

Proof. Taking the total differential of (4), but keeping $\mathcal{E}$ fixed (partial equilibrium), we obtain

$$-u''(\omega - e^*)de^* = \varphi\tilde{\pi}u''\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right)\left(\tilde{\pi}de^* - \frac{1-\varphi}{\varphi}d\mathcal{B}\right)$$

[9] In Appendix I we formally derive the existence of the debt level.

which implies for the partial equilibrium effect of a change in bonds on risky investment, assuming a fixed number of varieties:

$$\left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} = \frac{1-\varphi}{\varphi}\frac{\varphi\tilde{\pi}u''\left(w_2+\tilde{\pi}e^*-\frac{1-\varphi}{\varphi}\mathcal{B}\right)}{u''(\omega-e^*)+\varphi\tilde{\pi}^2u''\left(w_2+\tilde{\pi}e^*+\frac{1-\varphi}{\varphi}\mathcal{B}\right)} > 0 \tag{5}$$

where the last inequality follows from $u''(\cdot) < 0$. □

The intuition for this result is simple: with more government debt, the investor expects to be able to smooth the returns on her investment better over periods two and three (from the point of view of an individual investor that looks at prices instead of market clearing quantities: the interest rate between periods two and three is higher). This increases the marginal value of the returns to the investment in the case of success. In this sense, our result shows some structural similarity to the point made by Woodford (1990) that government debt can crowd in investment when liquidity is scarce.

However, if all households invest more in the risky asset, this changes output in period two as $\mathcal{E} = \varphi e^*$ and $Y_2 = Z\mathcal{E}$. Households that are richer in period two will save and invest less in period one. Yet, the direct effect dominates this indirect one as we show in

Proposition 2. *The direct effect of a change in government debt dominates the indirect one from changing output (wages and profits) for $\mathcal{B} < \mathcal{B}^*$ (from Proposition 1).*

Proof. First, we observe that the per-variety profit is $\tilde{\pi} = (1-\phi)Z$ and thus independent of the total number of varieties, $\mathcal{E}$. The overall effect of a change in government bonds on the investment into the risky asset is therefore the sum of the direct effect and the indirect effect through a wage change in period two:

$$de^* = \left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} d\mathcal{B} + \left.\frac{de^*}{dw_2}\right|_{\mathcal{E}} dw_2 = \left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} d\mathcal{B} + \left.\frac{de^*}{dw_2}\right|_{\mathcal{E}} \phi\varphi Z de^*$$

$$\Longrightarrow \frac{de^*}{d\mathcal{B}} = \left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} \left[1-\phi Z\varphi \left.\frac{de^*}{dw_2}\right|_{\mathcal{E}}\right]^{-1}$$

Here $\phi Z\varphi = \frac{dw}{d\mathcal{E}}\frac{d\mathcal{E}}{de^*}$. The total effect $\frac{de^*}{d\mathcal{B}}$ is positive, if $\left.\frac{de^*}{dw_2}\right|_{\mathcal{E}}\phi Z\varphi < 1$.

Next, we calculate the effect of a wage change in period two on investment in period one, for otherwise fixed prices, as we did for the effect of a change in government debt and obtain

$$\left.\frac{de^*}{dw_2}\right|_{\mathcal{E}} = -\frac{\varphi\tilde{\pi}u''\left(w_2+\tilde{\pi}e^*-\frac{1-\varphi}{\varphi}\mathcal{B}\right)}{u''(\omega-e^*)+\varphi\tilde{\pi}^2u''\left(w_2+\tilde{\pi}e^*-\frac{1-\varphi}{\varphi}\mathcal{B}\right)} < 0, \tag{6}$$

where the last inequality follows from $u''(\cdot) < 0$. $\square$

In Appendix I, we show that such an increase in government debt is also welfare increasing for log utility functions. Government debt is neutral in period 1, and the tax to repay period 1 debt (cum interest) does not create any redistribution. The reissuance of government debt in period 2 allows the L-type to effectively borrow against its future income and at an interest rate lower than the type's discount rate. Similarly, the H-type can smooth its consumption more because it is less constrained to save. The additional investment in the varieties adds to the welfare outcome because it generates an externality on income in periods two and three.

The model is kept simplistic on purpose. To illustrate the key mechanism we kept the environment Ricardian while abstracting from income and ex-ante wealth heterogeneity. We address these important aspects of reality in the next section.

3 Model environment

Next, we extend the model to a fully dynamic setting, an economy with incomplete markets and endogenous growth with infinitely lived agents. In this economy, we allow taxes to distort labor supply. We focus on its stationary equilibrium around the endogenous growth path.

The production side embeds a model of vertical innovation into an otherwise standard growth model. Instead of assuming an exogenous growth process, we assume that new varieties can be added to the economy in the tradition of Romer (1990). We assume roundaboutness in production such that the varieties are produced by differentiating the final product and the varieties are then used as intermediate inputs in the production of final goods. Each variety is subject to monopolistic competition and generates rents for the innovative sector, which are passed on to households. Beyond these intermediate inputs, capital and labor also enter production.

Households face idiosyncratic labor productivity risk. Markets are incomplete but households self-insure. They can invest in physical capital, government bonds or acquire varieties to start or expand a buinsess (equity). In doing so, they solve a portfolio choice problem, because equity investment carries non-diversifiable idiosyncratic risk, while capital and government debt are perfectly diversified and thus riskless in the steady state.

In addition to the firm and household sectors, we model a government, a fiscal authority that levies taxes on labor income and profits, provides lump-sum transfers, issues government bonds, and uses some of the tax revenue for (wasteful) government consumption.

We begin by describing the production sector of the model, before turning to a detailed description of the household side of the modified growth economy and the government.

3.1 Firms

The production side has four sectors. The final goods sector has a representative final goods producer who produces under perfect competition using physical capital K_t, labor L_t, and a composite of intermediate inputs Q_t. These intermediate inputs are themselves a composite of differentiated varieties Q_{jt} among which there is monopolistic competition. These varieties are produced by households in some form of backyard enterprise, differentiating final goods. This adds some roundaboutness to the production structure. The set of all varieties offered $\mathcal{E}_t$ is thus partitioned by the varieties each household i offers e_{it}. An innovation sector produces ideas for new varieties that households can buy and irreversibly add to their backyard production technology. Next, we discuss the production structure in more detail.

3.1.1 Intermediate Inputs

Intermediate inputs are a bundle over the goods that each household produces in its backyard technology

$$Q_t \equiv \left[\int Q_{it}^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}.$$

This backyard technology is itself a bundle of the varieties that household i has acquired the knowledge to produce:

$$Q_{it} \equiv \left[\int_0^{e_{it}} Q_{ijt}^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}, \tag{7}$$

where Q_{ijt} is the quantity of good j produced by household i at time t.

For simplicity, we assume that both substitution elasticities are equal, $\eta = \epsilon > 1$, so that we can write the aggregate bundling technology more compactly in terms of individual goods that households drop

$$Q_t = \left[\int_0^{\mathcal{E}_t} Q_{jt}^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}.$$

Here Q_{jt} is now the quantity offered in some variety j without specifying the household offering that variety, and $\mathcal{E}_= \int e_{it} di$ is the measure of varieties available in the economy.

The bundler buys each individual variety at the price P_{jt}. Minimizing the cost $\int_0^{\mathcal{E}_t} P_{jt} Q_{jt} dj$

to produce Q_t, she will therefore demand

$$Q_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\epsilon} Q_t, \tag{8}$$

with $P_t = \left(\int_0^{\mathcal{E}_t} P_{jt}^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$ as the price of the composite good. Equation (8) denotes the demand for each individual variety Q_{jt}.

Conversely, this implies that all households set a constant mark-up on the price of each variety. Using the final good as the numeraire, we get

$$P_{jt} = \frac{\epsilon}{\epsilon - 1} \quad \forall j, t.$$

and therefore each variety makes the same profit

$$\pi_{jt} = \frac{1}{\epsilon - 1} Q_{jt}. \tag{9}$$

Using the fact that all producers of existing varieties supply the same amount, and that each uses one unit of the final good for one unit of Q_{jt}, we can also express the production of Q_t in terms of the final good $X_t = \int Q_{jt} dj$ used in its production as

$$Q_t = \mathcal{E}_t^{\frac{1}{\epsilon-1}} X_t,$$

where $\mathcal{E}_t^{\frac{1}{\epsilon-1}}$ reflects the productivity-enhancing aspect of adding varieties. Given the markup on each variety, this implies that the bundler offers the intermediate input at price $P_t = \frac{\epsilon}{\epsilon-1}\mathcal{E}_t^{\frac{1}{1-\epsilon}}$

3.1.2 Final Goods Producer

Final goods are produced by a representative firm using capital K_t, labor N_t, and intermediate goods Q_t according to the gross output production technology.

$$Z_t = \bar{A}\left(K_t^{\alpha} N_t^{1-\alpha}\right)^{1-\nu} Q_t^{\nu}. \tag{10}$$

Where α is the physical capital share and ν is the share of intermediate goods in production, $\bar{A}$ is a scaling for productivity.

We can use the results from the last subsection to translate this output production function into value added in final goods production $Y_t = Z_t - X_t$ using the optimality condition for intermediate inputs:

$$P_tQ_t = \frac{\epsilon}{\epsilon - 1}X_t = \nu Z_t.$$

Then, using $Y_t = \left(1 - \nu\frac{(\epsilon-1)}{\epsilon}\right) Z_t$, after some algebra and normalizing the productivity scale $\bar{A}$ in the right way, we get

$$Y_t = K_t^{\alpha} N_t^{1-\alpha} \mathcal{E}_t^{\frac{\nu}{(\epsilon-1)(1-\nu)}},$$

which, under the further assumption $1 - \alpha = \frac{\nu}{(\epsilon-1)(1-\nu)}$, we can write as

$$Y_t = K_t^{\alpha}(\mathcal{E}_t N_t)^{1-\alpha}. \tag{11}$$

The latter assumption is necessary to obtain a balanced growth path.

Since there are mark-ups in intermediate goods, only a fraction of $\phi = \frac{\epsilon(1-\nu)}{\epsilon(1-\nu)+\nu}$ of this value added compensates capital and labor as factors of production, and $1 - \phi$ goes to the differentiated intermediate goods as profit.

Let W_t, r_t and δ refer to the wage rate, the interest rate and depreciation. With these definitions, the usual first-order conditions determine factor demands/factor prices

$$r_t + \delta = \phi\alpha\frac{Y_t}{K_t} = \phi\alpha\left(\frac{\mathcal{E}_t N_t}{K_t}\right)^{1-\alpha}, \tag{12}$$

$$w_t/\mathcal{E}_t = \phi(1-\alpha)\frac{Y_t/\mathcal{E}_t}{N_t} = \phi(1-\alpha)\left(\frac{\mathcal{E}_t N_t}{K_t}\right)^{-\alpha}, \tag{13}$$

$$\text{and } \pi_t = (1-\phi)Y_t. \tag{14}$$

The last equation determines the total profit from goods production, which is a fixed proportion of output.

3.1.3 Innovation Sector

Varieties are invented by a continuum of perfectly competitive innovators. The innovators k do so by conducting research, using R&D expenditures $S_{k,t}$ (in terms of final goods). Each innovator produces new varieties $\Delta_{k,t}$ according to the linear production function

$$\Delta_{k,t} = \chi_t S_{k,t}, \tag{15}$$

where χ_t is the productivity of the innovation sector, taken as exogenous by the innovators. We adopt the modeling assumption of Comin and Gertler (2006) and assume that the productivity coefficient χ_t has an aggregate externality that innovators do not internalize.

$$\chi_t = \chi \frac{\mathcal{E}_t}{\mathcal{E}_t^{\rho} S_t^{1-\rho}}, \tag{16}$$

where χ and $0 \leq \rho \leq 1$ are scalars controlling the importance of idea investment for growth. As in Romer (1990), there is a positive spillover of the aggregate stock of varieties $\mathcal{E}_t$ on individual productivity. However, we additionally model a congestion externality via the factor $\mathcal{E}_t^{\rho} S_t^{1-\rho}$. The congestion externality raises the cost of developing new varieties as the aggregate R&D intensity $S_t = \int S_{k,t} dk$ increases. For $\rho \to 0$, χ becomes the exogenous growth rate of ideas. It is straightforward to show that in equilibrium with symmetric innovators, the variety elasticity of R&D expenditure becomes ρ under this functional assumption. As a way to ensure that the growth rate of new intermediate products is stationary, we also assume that the congestion effect depends positively on the already existing number of varieties $\mathcal{E}_t$. This is equivalent to assuming that, all else equal, the marginal return to R&D investment declines as the economy becomes more sophisticated, as measured by the number of varieties.[10]

Households acquire varieties in an innovation market. We will discuss details when we describe the household's consumption-savings problem. Importantly, in the market for product ideas, there is a price q_t at which innovators can sell new varieties. They will add varieties until the marginal cost of a new variety equals the price q_t. Assuming free entry and perfect competition, the price of new varieties is fixed at the marginal cost of production. Expressed in terms of new and existing varieties and output, this gives the price per variety

$$q_t = \chi^{-\frac{1}{\rho}} \left(\frac{\Delta_t}{\mathcal{E}_t} \right)^{\frac{1-\rho}{\rho}} \frac{Y_t}{\mathcal{E}_t}, \tag{17}$$

where we have used that all innovators behave identically, i.e. $\Delta_{k,t} = \Delta_{j,t} = \Delta \, \forall j, k$.

3.1.4 Per-Variety Notation

Later we want to solve the model around a balanced growth path. To do this, we express the capital stock and output relative to the number of varieties, denoting the per-variety

[10] Endogenous growth models such as Romer (1990), which use labor as the only input factor in R&D production, also feature procyclical R&D costs. The relevant cost of producing a unit of new varieties is the real wage rate, and the wage rate is procyclical in these models.

variables by tilda. This gives us factor prices:

$$r_t + \delta = \phi\alpha \frac{\tilde{Y}_t}{\tilde{K}_t} = \phi\alpha \left(\frac{N_t}{\tilde{K}_t}\right)^{1-\alpha}, \tag{18}$$

$$\tilde{w}_t = \phi(1-\alpha)\frac{\tilde{Y}_t}{N_t} = \phi(1-\alpha)\left(\frac{N_t}{\tilde{K}_t}\right)^{-\alpha}, \tag{19}$$

$$\text{and } \tilde{\pi}_t = (1-\phi)\tilde{Y}_t. \tag{20}$$

The last equation is the profit per variety.

3.2 Households

The household side is similar to the setup in Aiyagari and McGrattan (1998): households face idiosyncratic income risk and self-insure against it. We extended the asset market by an illiquid investment option into a risky asset. We model the illiquidity in the sense of Bayer, Born and Luetticke (2022) as a random market participation. The illiquid asset is risky because the entire amount invested can be lost in any period. As before, all variables with a tilde are expressed relative to the total number of varieties in the economy $\mathcal{E}_t$.

3.2.1 Productivity, Preferences, and Income

There is a continuum of households $i \in [0,1]$. Households earn income from work, they earn interest income on their financial assets (consisting of claims on physical capital and bonds), and they earn profit income from their entrepreneurial activities.

They face risks in these activities as well as in the labor market. We model the latter as fluctuations in a household's human capital h_{it}, i.e. we focus on long-term labor market risks rather than, say, the risk of unemployment. Human capital evolves according to

$$\log h_{it} = \rho_h \log h_{it-1} + \epsilon_{it}, \tag{21}$$

where ϵ_{it} are normally distributed shocks with variance σ_ϵ^2 and mean $\mu_\epsilon = -(1-\rho_h)(1-\rho_h^2)\sigma_\epsilon^2/2$, so we normalize average productivity to unity: $\mathbb{E}(h_{it}) = 1$.

As described in the last subsection, we assume that in addition to offering labor, each household also engages in some entrepreneurial activity, offering a range of intermediate inputs. The number of varieties offered by household i is denoted by e_{it}. We think of these varieties as distinct but related products. Occasionally, the range of products offered by household i becomes obsolete. The household then loses all of its varieties, and the household must start over by accumulating new ideas of varieties. This obsolescence of the varieties offered by the household is the risk of investing in ideas. Moreover, we assume that investing in ideas is only possible from time to time. This makes them illiquid and implies that some households do not produce any varieties at all if they never had the opportunity and resources to invest in them.

Households have time-separable King, Plosser and Rebelo (1988) (KPR) type preferences and derive felicity from consuming the final good c_{it} and disutility from supplying labor n_{it}. Households discount felicity with a time discount factor β and maximize the discounted sum

$$V = \mathbb{E}_0 \max_{\{c_{it}, n_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(c_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} \right]$$

where ω is a scaling parameter that determines the average labor supply and γ is the inverse of the Frisch elasticity. The preference specification allows us to recast the household planning problem as a choice over labor supply and per-variety consumption $\tilde{c}_{it}$.

$$\mathbb{E}_0 \max_{\{\tilde{c}_{it}, n_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(\tilde{c}_{it}) + \ln(\mathcal{E}_t) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} \right], \tag{22}$$

which allows us later to solve the planning problem in stationary recursive form.

We assume a linear tax schedule for labor and profit income, so that a household's net labor income is given by

$$y_{it} = (1 - \tau_t^L) \tilde{w}_t n_{it} h_{it} \mathcal{E}_t, \tag{23}$$

where $\tilde{w}_t$ is the wage rate per variety, as derived in the last section, and τ_t^L is the linear labor tax. Given net labor income, the first-order condition on labor supply implies the optimal ratio of

$$n_{it}^{\gamma} = \frac{(1 - \tau_t^L) \tilde{w}_t h_{it}}{\tilde{c}_{it} \omega}, \tag{24}$$

which defines the optimal supply of labor. In addition to receiving income from labor, households receive profit income from their entrepreneural activities, asset income, and potentially (non-distortionary) transfers from the government $\mathcal{T}_{it} = \mathcal{T}_t(h_{it})$. Since each variety e_{it} earns the same profit $\tilde{\pi}$ and each unit of wealth a_{it} earns the return r_t, a household's total after-tax income is given by

$$\left(r_t \tilde{a}_t + \tilde{\pi}_t \tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it} \right) \mathcal{E}_t,$$

where $\tilde{e}_{it}$ is the number of varieties of household i relative to the average number in the economy. We assume that only labor is taxed as our baseline and consider a synthetic income tax as an alternative.

3.2.2 Household Maximization Problem

Given incomes and the functional form in (22), households face a consumption ($\tilde{c}_{it}$) and a portfolio choice over liquid asset holdings $\tilde{a}_{it+1}$ and equity/ideas $\tilde{e}_{it+1}$ to intertemporally optimize eq. (22).

To model the fact that most households are not entrepreneurial, we assume that only a random fraction λ of households can buy new equity in a given period. The remaining $(1-\lambda)$ households will only be able to adjust their portfolio of liquid assets. Below we present the budget constraints faced by households in each case.

The first group of households does not participate in the equity market and cannot adjust its equity holding. Although the households do not participate in the market, they can still earn profits and adjust their holdings of liquid assets.[11] Therefore, we have the following budget constraint:

$$\left(\tilde{c}_{it} + \tilde{a}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\mathcal{E}_t = (\tilde{a}_{it}R(\tilde{a}_{it}, R_t) + \hat{\pi}_t\tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it})\mathcal{E}_t,$$

in short

$$\begin{aligned} &\tilde{c}_{it} + \tilde{a}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} = \tilde{a}_{it}R(\tilde{a}_{it}, R_t) + \hat{\pi}_t\tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it}, \\ &\tilde{a}_{it+1} \geq \underline{A}. \end{aligned} \tag{25}$$

where $\tilde{a}_{it}$ is real wealth and $R(\tilde{a}_{it}, R_t)$ is the real interest rate on wealth, which depends on whether the household borrows or lends and on the market-clearing rate $R_t = 1 + r_t$. An intermediation cost creates a wedge $\overline{R}$ between the return on liquid assets R_t and the interest paid by households R_t. Therefore, we specify

$$R(\tilde{a}_{it}, R_t) = \begin{cases} R_t & \text{if } \tilde{a}_{it} \geq 0 \\ R_t + \overline{R} & \text{if } \tilde{a}_{it} < 0. \end{cases} \tag{26}$$

This unsecured credit wedge creates a mass of households with zero unsecured credit but with the ability to borrow at a penalty rate. Asset holdings (relative to the number of varieties) must exceed an exogenous debt limit $\underline{A}$.

[11] If the household is unable to adjust its equity position between periods, the value of equity relative to the total number of varieties decreases over time. In solving the model numerically, we assume that households that are not allowed to participate in the stock market withhold an amount of their profits to ensure that they have the same number of varieties in the next period as in the current period. Thus, the profits received by households that cannot adapt are $\hat{\pi}_t = \tilde{\pi}_t - q_t g_{t+1}$. This prevents us from having to interpolate over multiple dimensions when calculating the continuation values.

The second group of households participates in the stock market so that the household can adjust liquid assets $\tilde{a}_{it}$ and buy equity $\tilde{e}_{it+1}$ from the innovator at price q_t. Households face a trade-off between saving in the liquid asset $\tilde{a}_{it+1}$ to insure against idiosyncratic income risk, or investing in the lucrative but risky equity $\tilde{e}_{it+1}$. The corresponding budget constraint is

$$\left(\tilde{c}_{it} + \tilde{a}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} + q_t\tilde{e}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\mathcal{E}_t = \left(\tilde{a}_{it})R(\tilde{a}_{it}, R_t) + (q_t + \tilde{\pi}_{it})\tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it}\right)\mathcal{E}_t,$$

which simplifies to

$$\tilde{c}_{it} + \tilde{a}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} + q_t\tilde{e}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} = \tilde{a}_{it}R(\tilde{a}_{it}, R_t) + (q_t + \tilde{\pi}_{it})\tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it},, \tag{27}$$

$$\tilde{e}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \geq \tilde{e}_{it}, \quad \tilde{a}_{it+1}, \geq \underline{A}. \tag{28}$$

Thus, households have to choose between allocating their resources between consumption $\tilde{c}_{it}$ and the two investment possibilities $\tilde{a}_{it+1}, \tilde{e}_{it+1}$. The limited participation in the equity market renders equity investment an illiquid asset. We model the risk of equity investment by assuming that in each period, with probability $1-\varphi$, all of the equity held by the household is lost because the partition of varieties offered by the household becomes obsolete. Thus, equity investment has an idiosyncratic risk that cannot be diversified away given the existing market structure. The combination of these two assumptions makes equity a profitable but illiquid and risky investment.

Since a household's decisions will be nonlinear functions of its wealth a_{it}, its equities e_{it}, and its productivity h_{it}, all prices will be functions of the joint distribution Θ_t of (a_{it}, e_{it}, h_{it}) in t. This makes Θ_t a state variable of the household's planning problem, leaving us with three functions that characterize the household's problem: the value functions V_t^a and V_t^n for the two cases illustrated above, and the continuation value $\mathbb{W}_{t+1}$. Omitting individual indices, tildas for the per-variable notation, and letting variables with prime denote the next period value, we can characterize these value functions in the following equations:

$$V_t^a(a, e, h) = \max_{a', e'} u[c(a, a', e, e', h), n(a, a', e, e', h)] + \beta\mathbb{W}_{t+1}(a', e', h) \tag{29}$$

$$V_t^n(a, e, h) = \max_{a'} u[c(a, a', e, e, h), n(a, a', e, e, h)] + \beta\mathbb{W}_{t+1}(a', e, h) \tag{30}$$

$$\begin{aligned}\mathbb{W}_{t+1}(a, e, h) =& \varphi\Big(\lambda\mathbb{E}_t[V_{t+1}^a(a, e, h')] + (1-\lambda)\mathbb{E}_t[V_{t+1}^n(a, e, h')]\Big) \\ &+ (1-\varphi)\Big(\lambda\mathbb{E}_t[V_{t+1}^a(a, 0, h')] + (1-\lambda)\mathbb{E}_t[V_{t+1}^n(a, 0, h')]\Big)\end{aligned} \tag{31}$$

The notation here is that all a, e, a', e' are expressed relative to the number of varieties at time t. Expectations about the continuation value are made with respect to the productivity state h_{it} conditional on the current states. Maximization is subject to the appropriate budget constraints (e.g., 25, and 27), as well as non-negativity constraints (e.g., 28).

3.3 Government

The government operates a fiscal authority that issues government bonds B_t to finance deficits, chooses tax rates in the economy, provides transfers to households, and has (wasteful) government consumption. It is thus summarized by the budget equation (in per-variable notation)

$$\tilde{G}_t + \tilde{\mathcal{T}}_t + R_t^b \tilde{B}_t = \tilde{B}_{t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} + \tilde{T}_t. \tag{32}$$

The government is assumed to run a budget deficit and chooses debt $\tilde{B}_{t+1}$. Besides issuing bonds, the government uses tax revenues less transfers $\tilde{T}_t$, defined below, to finance government expenditures $\tilde{G}_t$ and interest on debt $(R_t^b - 1)\tilde{B}_t$. The government sets the linear tax τ_t^L to control the tax burden on the economy. Total taxes are then $\tilde{T}_t = \tau_t^L \mathbb{E}_{it}(\tilde{w}_t h_{it} n_{it})$, which is the cross-sectional average. In the baseline, the government adjusts the tax rate τ_t^L to satisfy the budget constraint.

3.4 Aggregates, Growth, Market Clearing and Equilibrium Definition

This section deals with the aggregates in this economy and their behavior. We begin by describing all the aggregates, illustrate the determinants of the growth rate in the economy, continue by specifying the market clearing conditions, and finally define the dynamic equilibrium in the economy.

3.4.1 Aggregation

Each household holds assets $\tilde{a}_{it}$ and shares $\tilde{e}_{it}$. We define the corresponding aggregates:

$$\tilde{A}_t = \int_0^1 \tilde{a}_{it} di, \tag{33}$$

$$\mathcal{E}_t = \int_0^1 \tilde{e}_{it} di. \tag{34}$$

Similarly, the aggregate effective labor supply is

$$N_t = \int_0^1 h_{it} n_{it} di.. \tag{35}$$

Having defined the aggregates, we can define the law of motion for capital

$$\tilde{K}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} = (1-\delta)\tilde{K}_t + \tilde{I}_t, \tag{36}$$

where $(1-\delta)$ is the fraction of the capital stock that has not been depreciated and I_t refers to the investment in total physical capital.

3.4.2 Aggregate Growth

The economy features endogenous growth through the endogenous accumulation of capital $\tilde{e}_{it}$ by households. In each period, the activities of $(1-\varphi)$ households become obsolete and these households lose all their varieties/equity. Thus, only φ households remain with their stock of e_{it} in the next period. Thus, we can write the total number of varieties $\mathcal{E}_t$ as

$$\mathcal{E}_{t+1} = \varphi\mathcal{E}_t + \Delta_t,$$

where Δ_t is the new number of varieties as in equation (15). φ is the fraction of varieties that have not become obsolete. This implies the following expression for the growth rate of new technologies:

$$1 + g_{t+1} := \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} = \varphi + \frac{\Delta_t}{\mathcal{E}_t}. \tag{37}$$

Having established the aggregates and the growth rate, we can define market clearing in all markets.

3.4.3 Market clearing

The labor market clears at the competitive wage given in (19) with aggregate labor as in (35). The asset market clears whenever

$$\tilde{B}_{t+1} + \tilde{K}_{t+1} = \tilde{A}^d(R_t, \tilde{\pi}_t, \tilde{w}_t, q_t, \tau_t^L, \Theta_t) := \mathbb{E}_t[\lambda\tilde{a}^*_{a,t} + (1-\lambda)\tilde{a}^*_{n,t}]$$

holds, where $\tilde{a}^*_{a,t}$, and $\tilde{a}^*_{n,t}$ are policy functions of the states $(\tilde{a}_{it}, \tilde{e}_{it}, h_{it})$, and depend on the current set of prices and tax rates $(R_t, \tilde{\pi}_t, \tilde{w}_t, q_t, \tau_t^L)$. $\tilde{K}_{t+1}$ denotes the supply of assets from the firm side such that the left-hand side above represents the total supply of liquid assets in which households can save in. Expectations on the right-hand-side expression are taken over the distribution Θ_t. An equilibrium requires the total net amount of assets households demand $\tilde{A}^d$ to equal the supply of government bonds $\tilde{B}_{t+1}$ and the supply of capital $\tilde{K}_{t+1}$. To ensure the market clearing, the interest rate on liquid assets R_t adjusts. In gross terms, more liquid assets are circulated as some households borrow up to $\underline{A}$.

The market for equities/varieties clears when

$$\tilde{\Delta}_t = \frac{\mathcal{E}_{t+1} - \varphi \mathcal{E}_t}{\mathcal{E}_t} = \mathbb{E}_t[\lambda \tilde{e}^*_{a,t} + (1-\lambda)\tilde{e}_{it}] - \varphi.$$

The first expression on the left side of the equation (3.4.3) is the supply of new varieties, whereas the expression on the right determines households demand for new varieties. For the market for new varieties to clear, the price of buying a new variety q_t adjusts.

Finally, the goods market clears when

$$Y_t = C_t + I_t + G_t + S_t + \mathcal{R}_t, \tag{38}$$

where $\mathcal{R}_t$ are the revenues that are earned from the interest rate spread between borrowing and lending. The goods market clears due to Walras' law, whenever the labor, capital, equity and bond market clear.

3.4.4 Dynamic equilibrium

At the beginning of each period, agents are heterogeneous in three dimensions summarized by the state vector $\tilde{s}_{it} = (\tilde{a}_{it}, \tilde{e}_{it}, h_{it})$, i.e. asset holdings $\tilde{a}_{it}$, equity holding $\tilde{e}_{it}$, and labor productivity h_{it}. An equilibrium in this model features sequences of prices $\mathcal{P} = \{g_t, R_t, w_t, q_t, \pi_t, \tau^L\}_{t=0}^{\infty}$, sequences of capital, and labor $\{K_t, L_t\}_{t=0}^{\infty}$, sequences of policy functions $\{\tilde{c}^*_{a,t}, \tilde{c}^*_{n,t}, \tilde{a}^*_{a,t}, \tilde{a}^*_{n,t}, \tilde{e}^*_{a,t}, n^*_{a,t}, n^*_{n,t}\}_{t=0}^{\infty}$, value functions $\{\tilde{V}_t, \tilde{V}^n_t, \tilde{\mathbb{W}}_{t+1}\}_{t=0}^{\infty}$, a law of motion $\Gamma_{\mathcal{P}_t}$, and a sequence of distributions $\{\Theta_{\mathcal{P}_t}\}_{t=0}^{\infty}$ over individual asset holdings, quality, and productivity, such that

1. The policy functions $\{\tilde{c}^*_a, \tilde{c}^*_n, \tilde{a}^*_a, \tilde{a}^*_n, \tilde{e}^*_a, n^*_a, n^*_n\}$ solve the households' planning problem given prices and the continuation values $\tilde{\mathbb{W}}$.

2. Together with the transition matrices of the exogenous states s, the policies induce a law of motion $\Gamma_{\mathcal{P}}$.

3. The distribution solves the forward equation $\Theta_{\mathcal{P}_{t+1}} = \Theta_{\mathcal{P}_t}\Gamma_{\mathcal{P}_t}$.

4. The value functions $\{\tilde{V}_t, \tilde{V}^n_t, \tilde{\mathbb{W}}_{t+1}\}$ solve the equations (29), (30), and (31) in every period.

5. The labor, the final goods, the market for new varieties and the asset market clear in every period.

6. The interest rate clears the asset market, returns on capital are determined by the marginal product of capital, the wage rate is determined as the marginal product of labor, profits are determined by the optimal behavior of the intermediate producer, the price of new varieties is determined by the optimality condition of the innovator, and government expenditure adjusts to clear the government budget constraint.

7. Capital K_t accumulates according to eq. (36), and the growth rate is determined as in eq. (37)

We solve the economy around a balanced growth path, where all detrended aggregate variables are constant over time. Hence, we solve for the steady state of the detrended economy.

4 Crowding in and out in the long run

We calibrate the model to the U.S. economy and revisit the question posed by Aiyagari and McGrattan (1998) regarding the optimal level of government debt. To illustrate the quantitative significance of various mechanisms, we explore several distinct cases. First, we determine the welfare-maximizing level of debt within the complete model. We then examine the roles tax distortions and endogenous growth play in shaping our results.

As a welfare measure for all experiments, we use the long-run utilitarian welfare function

$$W^* = \sum W(a, e, h) d\Theta(a, e, h). \tag{39}$$

$W(a, e, h)$ denotes the continuation value from the household's problem, while Θ represents the stationary distribution. We use the continuation value rather than the value functions because the participation constraint in asset markets necessitates weighting the obtained value functions.[12] We utilize the consumption equivalent variation for logarithmic utility to compare different utility levels

$$CE(B_t) = \exp\left((1-\beta)\left(W^*(B_t) - W_0^*\right)\right) - 1, \tag{40}$$

where W_0^* denotes welfare at the baseline level of government debt, and $W^*(\tilde{B})$ denotes welfare at the debt level $\tilde{B}$. This measure represents the maximum fraction of consumption that the average household would be willing to forgo to remain on the baseline balanced growth path.

[12] We do not compute the value function directly, as we solve the stationary version of the model. Instead, we use the detrended value function and subsequently add the long-run growth rate according to equation (3.2.1). Appendix (III) illustrates the calculation.

Table 1 Calibration Details (Quarterly Frequency)

Parameter	Value	Description	Source / Target
Households			
β	0.986	Discount factor	$K/Y = 9.0$ Auclert et al. (2021)
γ	3	Inverse Frisch	Chetty et al. (2011)
λ	0.3%	Portfolio adj. prob.	Income Gini $= 0.5$
ω	0.88	Scale labor disutility	$N_t = 1.0$ along BGP
ρ_h	0.98	Labor income persistence	Storesletten, Telmer and Yaron (2004)
σ_h	0.16	Labor income std.	Storesletten, Telmer and Yaron (2004)
Firms			
α	0.31	Capital share	62% labor income
ϵ	1.19	Substitution elasticity	profit share of 10%
δ	1.75%	Depreciation rate	Bayer, Born and Luetticke (2022)
ρ	0.1	Growth to equity inv.	see text
χ	0.1	New varieties scalar	Growth rate of 0.5% qtly.
φ	92.5%	Prob. keeping equity	Guvenen, Kaplan and Song (2014)
Government			
τ^L	37.8%	Tax rate level	$G/Y = 0.2$

NOTE - All parameters in the table are calibrated to a quarterly frequency. Probabilities represent the likelihood within a single quarterly period. Interest rates are reported quarterly.

4.1 Calibration

The model's parameters are either drawn from standard values commonly used in the literature or calibrated to match key targets in the baseline steady state. Table 1 presents the parameters for households, firms, and the government.

On the household side, we calibrate the discount rate β to match a capital-to-output ratio $K/Y = 9.0$, as in Auclert et al. (2021). This target is lower than alternative targets, however, our economy features intangible capital in the form of varieties, which is normally included in the calculation of the capital stock. Therefore, our calibration is more in line with the calibration of Domeij and Ellingsen (2018) that excludes intangible capital. We use a standard value of $1/3$ for the Frisch elasticity based on Chetty et al. (2011) yielding $\gamma = 3$. We calibrate the adjustment probability for varieties to $\lambda = 0.3\%$ to match the income Gini. Moreover, we calibrate the scalar on labor disutility ω to normalize labor supply to unity $N = 1$. The income process parameters $\rho_h = 0.98$ and $\sigma_h = 0.16$ are taken from Storesletten, Telmer and Yaron (2004). The latter parameter is at the upper bound of the values for labor income volatilities the authors report but matches reasonably well the average volatility of labor income as in Song et al. (2019).

On the firm side, we set the capital share in production $\alpha = 0.31$ and the elasticity of substitution between different to varieties $\epsilon = 1.19$, to obtain a standard labor share value of 62% and a profit share of 10% equal to average post-war values. We use a standard value for depreciation at the quarterly frequency of $\delta = 1.75\%$ as in Bayer, Born and Luetticke (2022). The elasticity of new varieties to equity investment is set to $\rho = 0.1$. This value is lower than former estimates of Kung and Schmid (2015), Bianchi, Kung and Morales (2019) and Gaillard and Wangner (2022) that obtain an estimate of $\rho \approx 0.5$. However, these models estimate the elasticities of the technology stock on corporate investment, while we focus on the elasticity to household investment. We argue that the elasticity of the aggregate technology stock to household investment is lower than to corporate investment into R&D. We set $\chi = 0.1$ to achieve a quarterly growth rate of 0.5% along the balanced growth path. We use the estimate of Guvenen, Kaplan and Song (2014) on the average probability that households drop out of the top income decile. With $\varphi = 92.5\%$, this implies that 30% of the households drop out of the top decile per year by losing all their investment. This number is in the range of estimates provided by Quadrini (2000) and Kitao (2008) for households that quit business activities. Finally, we set the labor income tax $\tau^L = 37.8\%$ to match an average government expenditure to GDP ratio of $G/Y = 0.2$.

4.2 The optimal level of debt

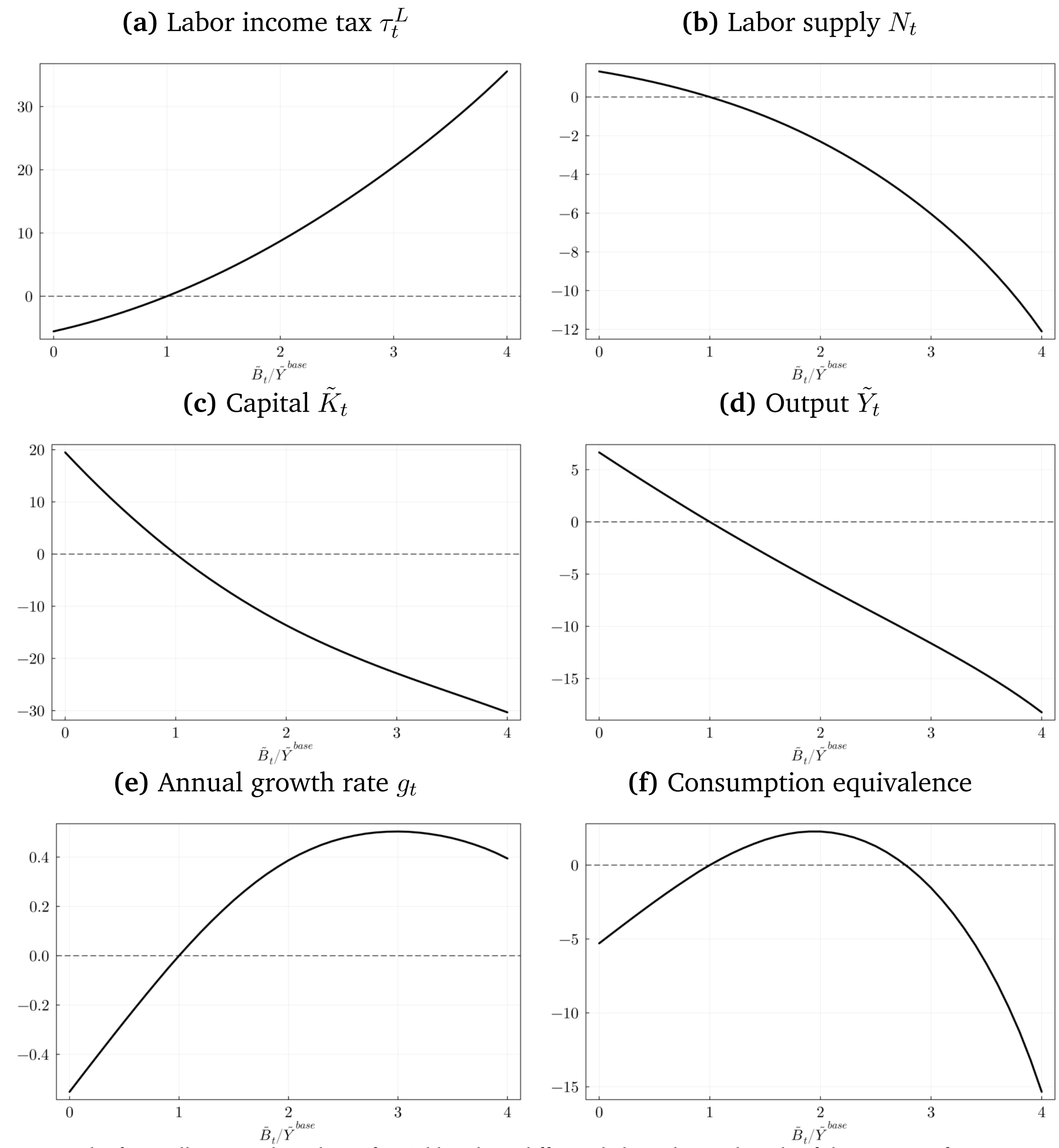
We now examine the welfare effects of changing the long-term government debt-to-GDP ratio using our baseline model. In this analysis, we assume the government balances the budget by adjusting the labor tax rate. The results of these policy experiments are shown in Figure 2. All variables, except for the growth rate and consumption equivalence, are expressed as percentage deviations from the calibrated balanced growth path. The growth rate is displayed in percentage points difference, while consumption equivalence is shown as a percentage.

As the debt-to-GDP ratio grows, labor income taxes increase sharply, crowding out labor and capital and eventually lowering output. Initially, higher debt boosts growth and improves consumption equivalence, suggesting a crowding in of risky investment. However, once the debt-to-GDP ratio surpasses 200%, consumption equivalence declines. At even higher debt levels, risky investment falls, slowing economic growth. Moderate debt increases improve welfare and growth, but excessive debt ultimately harms both. The experiment indicates that an increase in debt-to-GDP would improve welfare where the optimal amount of government debt is at 200% of debt-to-GDP.

The crowding out of capital and labor follows classical economic intuition. To accommodate the higher level of government debt, households demand a higher interest rate, reducing capital demand. As capital declines and the labor income tax rises, the after-tax wage rate falls, discouraging households from supplying labor. As a result, two well-established classical outcomes emerge: higher government debt leads to a crowding out of capital and labor, ultimately causing a decline in output.

The change in the growth rate can be explained using Figures 3 and 4. We begin by

Figure 2 Varying government debt and adjusting labor income tax residually

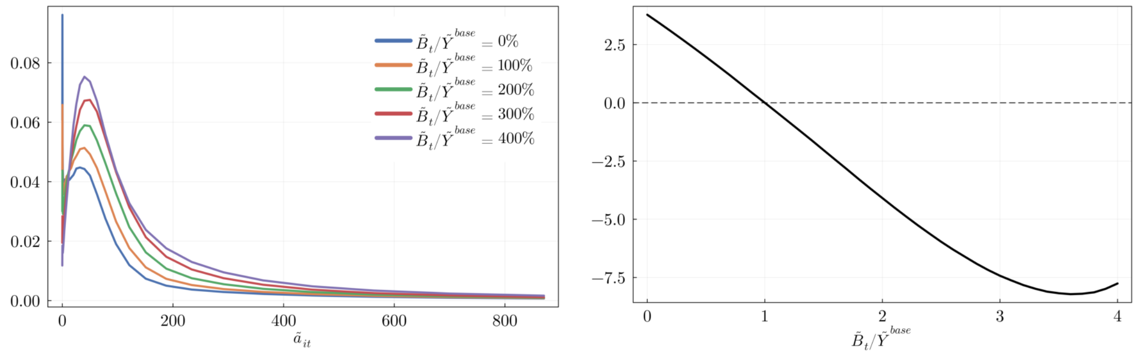


NOTE - The figure illustrates the values of variables along different balanced growth paths if the amount of government debt to the baseline GDP level $\tilde{B}_t/\tilde{Y}^{base}$ is increased. The x-axis refers to the ratio of debt-to yearly GDP such that the 1 refers to 100% of debt-to-GDP. Changes in the labor income tax, the growth rate, and consumption equivalence are given in percentage points, while all other variables are illustrated in percent change.

discussing the former. Figure 3 illustrates the shifting distribution of liquid assets $\tilde{a}_{it}$ as the debt-to-GDP ratio varies. Specifically panel (a) depicts the equilibrium distribution of liquid wealth $\tilde{a}_{it}$ across different debt-to-GDP levels. The figure shows that as the debt-to-GDP ratio increases, the distribution of liquid wealth shifts to the right. However, we observe a reversal for higher debt-to-GDP ratios, with the distribution moving leftward again. Panel (b) illustrates the changes in percentage points of the Gini coefficient of total wealth as a

Figure 3 Movement of the wealth distribution

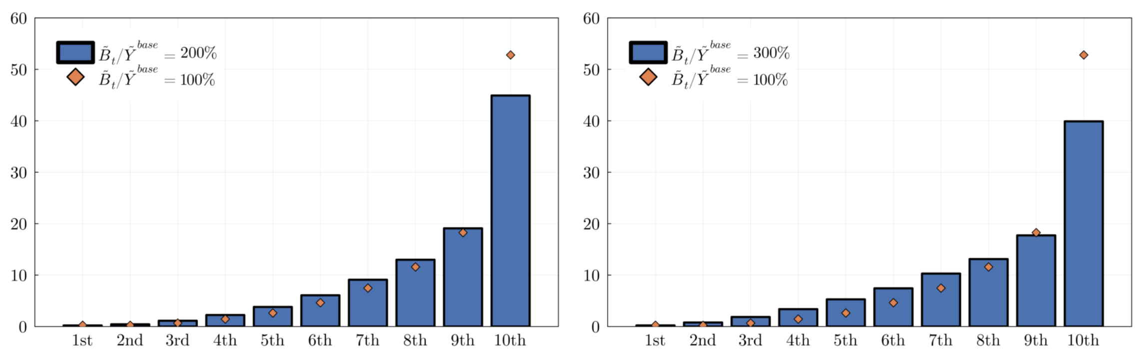
(a) Liquid asset distribution **(b)** Wealth gini



NOTE - Figure 3 (a) shows the liquid wealth distribution for different levels of government debt. Figure 3 (b) shows percentage points change in total wealth Gini coefficient for different levels of government debt.

Figure 4 Relative investment contributions along wealth deciles

(a) Investment shares at 200% debt-to-GDP ratio **(b)** Investment shares at 300% debt-to-GDP ratio



NOTE - Figure 4 (a) and 4 (b) show the contribution of households in a specified liquid wealth decile to total risky investment. The blue bars illustrate the shares of each decile, while the orange diamonds illustrate the investment shares at the baseline calibration at a debt-to-GDP ratio of 100%.

function of the debt-to-GDP ratio. The wealth Gini mirrors the result of Panel (a). Initially, the wealth Gini decreases with rising debt-to-GDP levels, before increasing slightly after reaching a ratio of 350%.

Figure 4 illustrates how the reduction in wealth inequality impacts investment in risky equity across different deciles of the wealth distribution. The figure shows the contribution of each wealth decile to the total amount of risky investment in the economy. Specifically, panel (a) presents the relative shares for a debt-to-GDP ratio of 200%, while panel (b) shows the shares for a 300% debt-to-GDP ratio. The shares in the calibrated balanced growth path with a debt-to-GDP ratio of 100% are illustrated as orange diamonds. As the debt-to-GDP ratio rises, the share of risky investment by the top decile decreases from 52.8% in the baseline to 44.9% at 200% and 39.9% at 300% debt-to-GDP. Households in the

3rd to 7th wealth deciles increase their relative share of investment in risky assets. As the economy's growth rate rises until a 300% debt-to-GDP ratio, a higher total absolute investment is required. Consequently, the increased share of investment by these households implies that they also invest more in absolute terms.

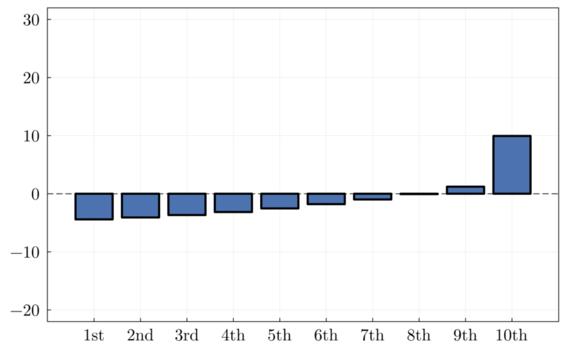
The intuition behind this result rests on two key factors. First, increasing government debt reduces wealth inequality by enabling poorer households to accumulate wealth through higher interest rates. As their wealth grows, these households invest more in risky projects. Second, higher interest rates enhance the utility of holding risky, high-return assets, following the logic of the toy model. Households with risky equity experience temporary high-income streams and save a large portion to smooth consumption over time. By raising the real interest rate, higher government debt improves their ability to transfer profit income across periods. Additionally, these savings generate more interest income after potential losses on equity investments. While higher interest rates also raise the opportunity cost of risky equity investment, this effect is outweighed by the increased utility from income smoothing and the shift in the wealth distribution. Consequently, an increase in government debt encourages poorer households to invest more due to their increased wealth, while also incentivizing all households to invest more by enhancing the marginal utility of successful investments.

Finally, we turn to the concept of consumption equivalence. Figures 5 (a) - (c) illustrate consumption equivalence across liquid wealth deciles for various debt-to-GDP ratios.[13] As the debt-to-GDP ratio rises from 150% to 250%, the consumption equivalence for households in the top wealth decile nearly triples, while for lower deciles, it quadruples in negative value. This can be explained since income sources differ across the wealth distribution. Figure 5 (d) illustrates the relative income shares across different wealth deciles. Households in lower wealth deciles derive a significant portion of their income from labor, whereas households in the top decile receive much of their income from assets and equity. As capital is crowded out, the real interest rate rises with higher debt-to-GDP levels, while after-tax wages fall. With small increases in the debt-to-GDP ratio, the crowding-out effect is limited, allowing utility losses in lower-wealth deciles to be outweighed by utility gains among wealthier households. However, as the debt-to-GDP ratio continues to rise, households in the lower-wealth deciles experience substantial utility losses due to more pronounced crowding out, while the utility gains at the top of the wealth distribution are no longer sufficient to offset these losses.

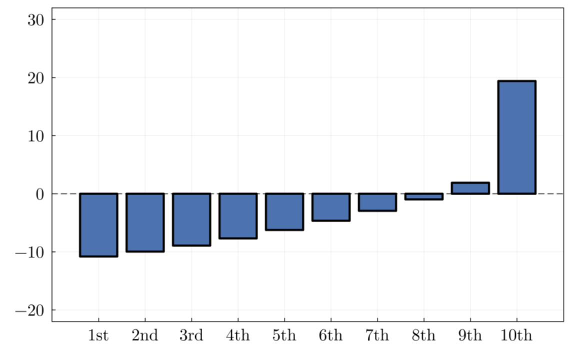
[13] Note that the calculation of consumption equivalence for different wealth deciles makes the measures not add up to total consumption equivalence.

Figure 5 Consumption equivalence and income shares along wealth deciles

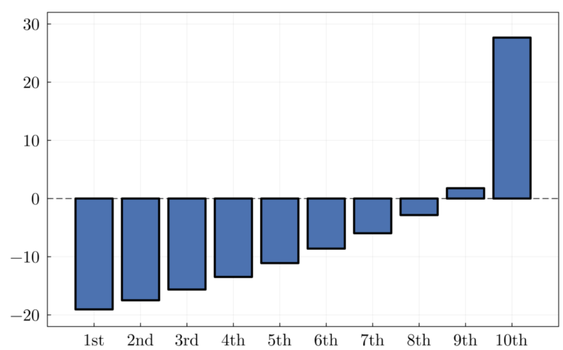
(a) Consumption equivalence along wealth deciles at 150% debt-to-GDP ratio



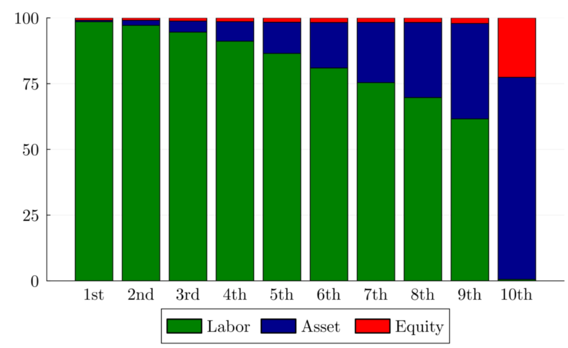
(b) Consumption equivalence along wealth deciles at 200% debt-to-GDP ratio



(c) Consumption equivalence along wealth deciles at 250% debt-to-GDP ratio



(d) Shares of income sources along liquid wealth deciles



NOTE - Figures 5 (a) - (c) show the average consumption equivalence for households in a specific wealth decile for increasing debt-to-GDP ratios. Figure 5 (d) illustrates the shares of different income sources along the liquid wealth deciles.

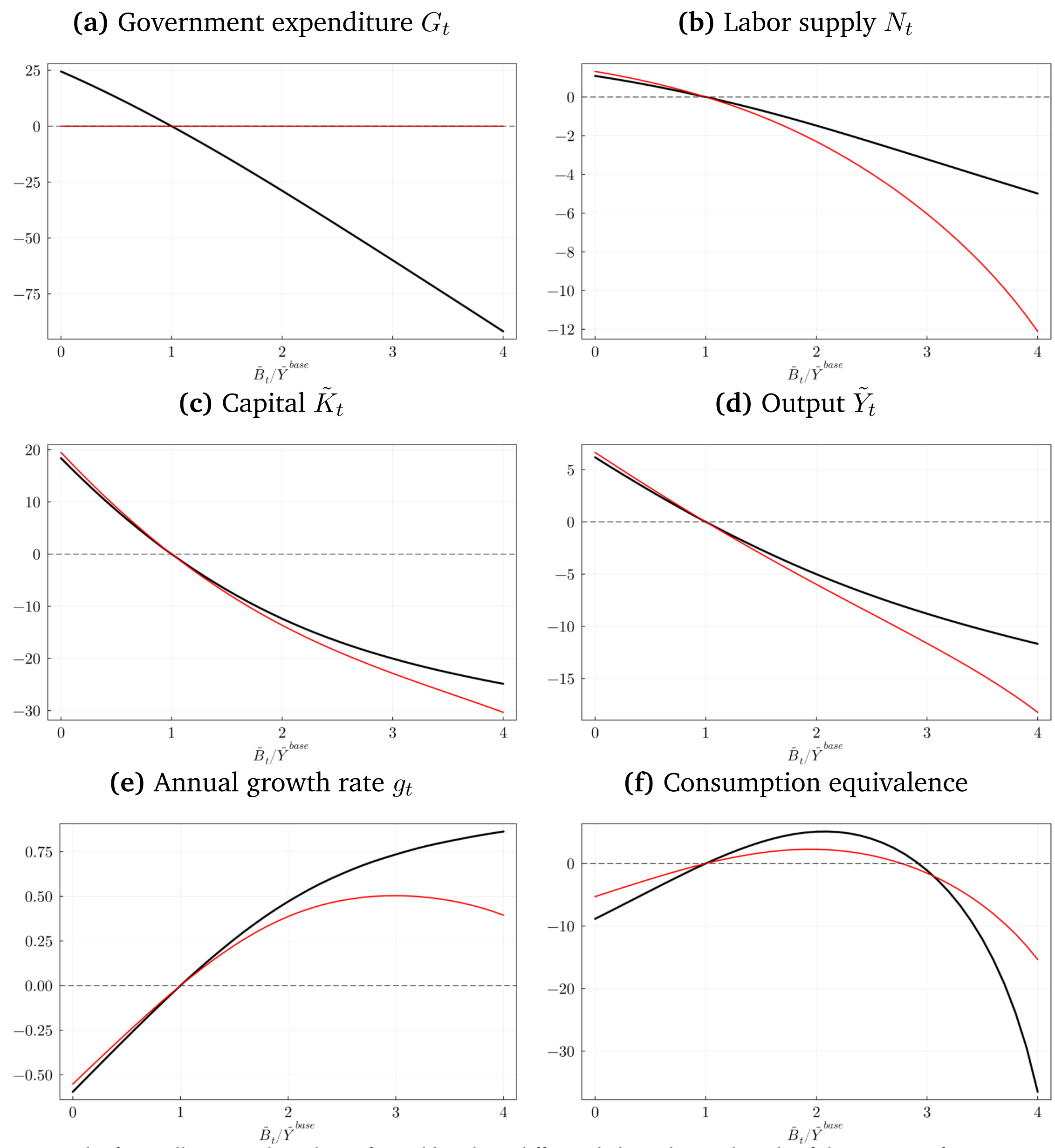
4.3 Dissecting the mechanism

After having illustrated the baseline result, we inspect the mechanism in modified environments. First, we illustrate how the results change if we adjust government expenditure instead of distortionary labor income taxes. Thereafter, we investigate the role of endogenous growth.

4.3.1 Adjusting government expenditures

To highlight the importance of tax distortions for the optimal level of debt and the crowding in of growth, we consider a version of our model, where the government adjusts government expenditures instead of taxes. To have meaningful welfare results, however, we replace the assumption of wasteful government expenditures with the assumption that households value government expenditures G_t. Here, we augment each household's felicity function by a term $\zeta \log(G_t)$ and choose the weight ζ such that a modified Samuelson

Figure 6 Varying government debt and adjusting government expenditure residually



NOTE - The figure illustrates the values of variables along different balanced growth paths if the amount of government debt to the baseline GDP level $\tilde{B}_t/\tilde{Y}^{base}$ is increased. The x-axis refers to the ratio of debt-to-yearly GDP such that 1 refers to 100% of debt-to-GDP. Changes in the labor income tax, the growth rate, and consumption equivalence are given in percentage points, while all other variables are illustrated in percent change. The red line illustrates the baseline experiment's results with endogenous growth.

condition holds in our original steady state.[14] Figure 6 illustrates the results of this experiment and compares them with the baseline results that are illustrated in red.

As debt rises, the government reduces expenditures to meet its budget constraint. While

[14] We modify the Samuelson condition in the sense that there is no welfare gain if all households switch to a new balanced growth path with marginally more of the public good financed by an increase in public debt. We derive the modified Samuelson condition in Appendix (IV.1).

labor supply, capital, and output still decrease, the decline is less severe than in the baseline case. The growth rate increases steadily, unlike its evolution when adjusting the labor tax rate. Consumption equivalence follows an inverse U-shape similar to the baseline but with a maximum of 210% debt-to-GDP and a larger magnitude. Therefore, without distortionary labor income taxation, the optimal debt-to-GDP ratio increases to 210% and the associated welfare gains increase to 5.1% of consumption equivalence.

As in the baseline case, rising debt-to-GDP crowds out capital and labor, but the effect is less pronounced. This is because, rather than increasing labor taxes, the government cuts expenditures, which helps maintain a higher after-tax real wage. As a result, households are incentivized to supply more labor. While the income effect from higher after-tax wages reduces the crowding out of capital, it does so to a lesser extent than it mitigates the labor supply reduction. As a result, output declines less than in the baseline. When debt increases without the distortion from labor taxes, households invest more in risky assets than in the baseline scenario, leading to a higher growth rate in comparison. Furthermore, households continue to invest in risky assets beyond a 400% debt-to-GDP ratio. This contrasts with the baseline scenario where risky investment and growth are crowded in only until a debt-to-GDP ratio of 200% due to the intensifying crowding out of labor and capital. Consumption equivalence rises more sharply without labor tax distortions due to larger crowding in of growth. Welfare starts to decline after a 210% debt-to-GDP ratio due to expenditure cuts that harm welfare.

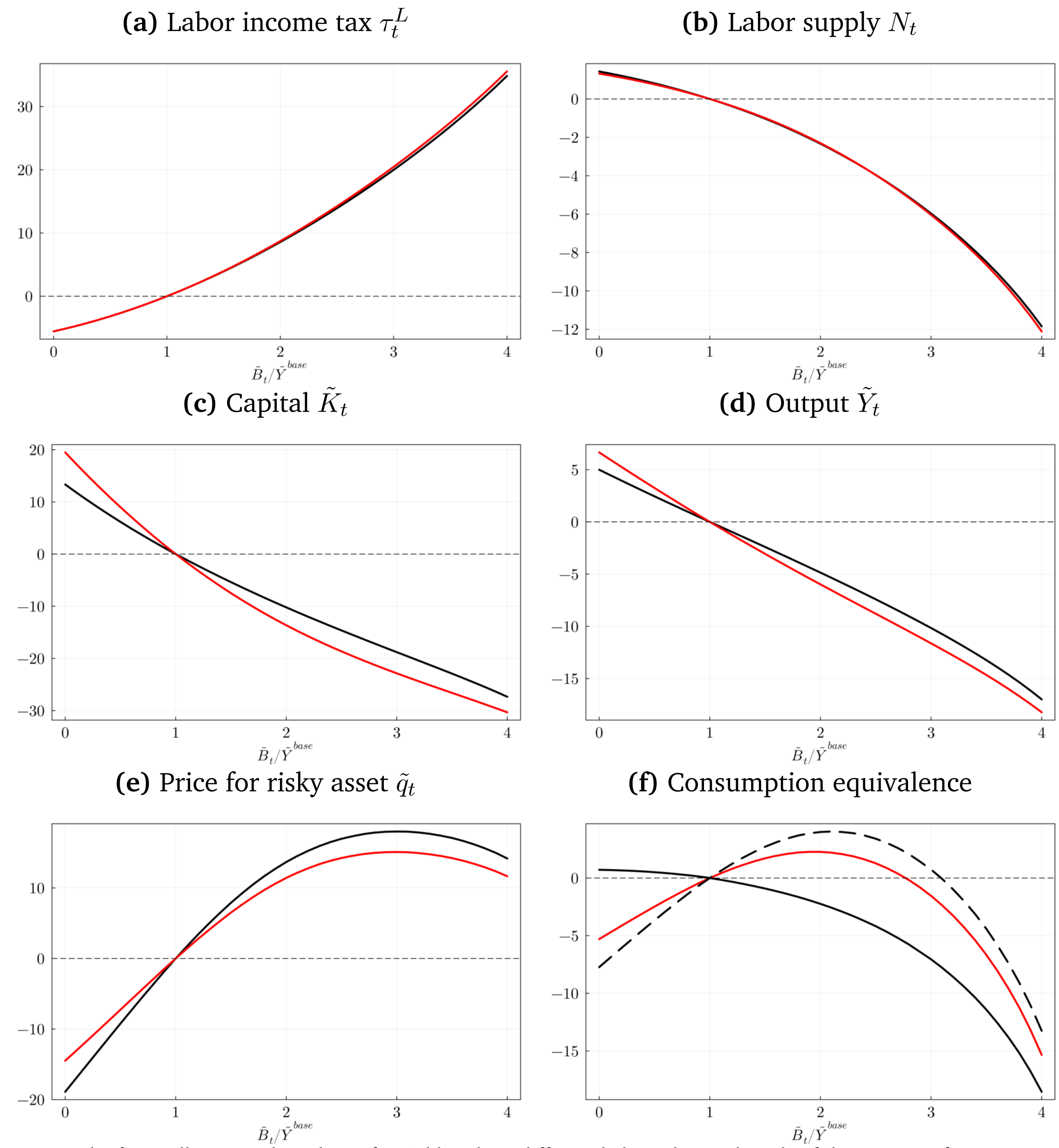
In summary, without the distortionary impact of labor taxes, the crowding out of capital and labor is reduced. Moreover, without the counteracting distortionary effect of labor taxation, households do not decrease their investment in risky assets. As a result, crowding in and welfare gains are amplified relative to the baseline case and the optimal debt-to-GDP ratio is elevated.

4.3.2 Exogenous growth

To highlight the importance of the endogenous growth effect for the optimal level of debt, we choose $\rho \approx 0$, which implies that a fixed amount of new varieties are added to the economy every period. These varieties are traded as before, so no other element of the model changes. Figure 7 illustrates the results with exogenous growth compared to the baseline results.

As the debt-to-GDP ratio rises, labor income taxes increase, and labor supply declines sharply, similar to the baseline case. However, capital decreases less than in the baseline, leading to a slightly smaller reduction in output. The price of risky investment rises more than in the baseline. Consumption equivalence decreases monotonically with the debt-to-

Figure 7 Varying government debt and adjusting labor income tax with fixed growth



NOTE - The figure illustrates the values of variables along different balanced growth paths if the amount of government debt to the baseline GDP level $\tilde{B}_t/\tilde{Y}^{base}$ is increased. The x-axis refers to the ratio of debt-to-yearly GDP such that 1 refers to 100% of debt-to-GDP. Changes in the labor income tax, the growth rate, and consumption equivalence are given in percentage points, while all other variables are illustrated in percent change. The red line illustrates the baseline experiment's results with endogenous growth. The dotted line in panel (f) also pictures the welfare of the exogenous growth economy when we readjust utility by the growth rate from the endogenous growth baseline.

GDP ratio and differs significantly from the baseline.

The crowding out of labor follows the same logic as in the baseline case. However, the difference in capital supply arises from the fixed growth rate. Growth affects the household's budget constraint by increasing the price of future asset holdings. To maintain the same amount of assets in efficiency units tomorrow, households must invest more today.

Consequently, higher growth leads to higher investment costs, intensifying capital crowding out. With a fixed growth rate, this effect is absent, leading to less capital crowding out compared to the baseline scenario with endogenous growth. As a result, the reduction in output is smaller, despite a similar decrease in labor supply. To clear the market for risky assets, households need to be disincentivized from investing, which requires a higher asset price. Since crowding out is lower and output drops less than in the baseline case, households have more resources available for investment. Therefore, the price of risky investments must increase more than in the baseline scenario to absorb the additional investment demand that households exhibit under a fixed growth rate.

The differential welfare effects are a result of the fixed growth assumption. Panel (f) illustrates consumption equivalence for the exogenous growth, the baseline case, and a counterfactual in a dashed black line. The counterfactual calculates welfare using the consumption allocation from the exogenous growth balanced growth equilibrium but uses the endogenous growth rate from the baseline case. Therefore, the difference between the black solid and the black dashed line illustrates the growth rate contribution to welfare. The differences in welfare from the baseline to the fixed growth rate scenario arise largely from the alternated growth assumption. The counterfactual consumption equivalence is even higher for high government debt since the crowding out of capital is muted in the exogenous growth case, positively impacting consumption levels.

In the fixed growth scenario, the optimal level of government debt is lower than in the other two cases. The welfare impact is most significant for wealth-poor, labor-dependent households. In scenarios with endogenous growth, increasing debt stimulates growth, compensating wealth-poor households for utility losses from crowding out capital and labor. However, with a fixed growth rate, this compensatory effect is absent, so an increase in debt reduces the welfare of wealth-poor households by negatively affecting their labor income. Conversely, when debt is reduced under fixed growth, growth remains unaffected, but capital and labor are still crowded in. As the debt-to-GDP ratio falls, after-tax labor income rises due to crowding in capital and a lower labor income tax. These combined effects benefit the majority of households, as all but the top 2 wealth deciles rely primarily on labor income.

To sum up, with exogenous growth, the crowding out of capital and labor is reduced. Due to the absence of crowding in of growth, the welfare effects turn sign and the optimal debt-to-GDP ratio is lower than in the baseline case with only small welfare increases.

5 Conclusion

We develop a heterogeneous agent model with portfolio choice and endogenous growth. In our model, households face a portfolio decision between a risk-free asset, which offers insurance against idiosyncratic risk, and a risky asset that not only provides high returns but also contributes to economic growth. We show in this class of models that an increase in government debt crowds in growth and welfare. In a toy model, starting from a low value of debt, an increase in the latter raises the value of a successful risky investment for households, thereby crowding in investment as well as aggregate growth and increasing aggregate welfare. In a quantitative model calibrated to U.S. time series data, we revisit the question of the optimal level of public debt. Our findings confirm the intuition from the toy model. For small increases in government debt, the increased value from risky investment crowds in growth and welfare. For higher levels of government debt, classical crowding out effects on capital and labor outweigh crowding in effects. When revisiting the question of the optimal level of public debt in the quantitative environment, our new channel results in a higher socially optimal level of public debt compared to previous studies. Dissecting the mechanism shows that the endogenous growth component is the main driver of the welfare result, whereas distortionary taxation is key for the hump-shaped growth pattern.

References

Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics* **109** (3), 659–684.

Aiyagari, S. R. and McGrattan, E. R. (1998). The optimum quantity of debt. *Journal of Monetary Economics* **42** (3), 447–469.

Angeletos, G.-M. and Calvet, L.-E. (2006). Idiosyncratic production risk, growth and the business cycle. *Journal of Monetary Economics* **53** (6), 1095–1115.

Anzoategui, D., Comin, D., Gertler, M. and Martinez, J. (2019). Endogenous Technology Adoption and R&D as Sources of Business Cycle Persistence. *American Economic Journal: Macroeconomics* **11** (3), 67–110.

Auclert, A., Bardóczy, B. and Rognlie, M. (2023). MPCs, MPEs, and Multipliers: A Trilemma for New Keynesian Models. *Review of Economics and Statistics* **105** (3), 700–712.

Auclert, A., Rognlie, M., Souchier, M. and Straub, L. (2021). Exchange Rates and Monetary Policy with Heterogeneous Agents: Sizing up the Real Income Channel. Tech. rep.

Bayer, C., Born, B. and Luetticke, R. (2022). The Liquidity Channel of Fiscal Policy. *Journal of Monetary Economics*.

Bayer, C. and Luetticke, R. (2020). Solving Heterogeneous Agent Models in Discrete Time with Many Idiosyncratic States by Perturbation Methods. *Quantitative Economics* **11** (4), 1253–1288.

Bhandari, A., Evans, D., Golosov, M. and Sargent, T. J. (2016). Fiscal Policy and Debt Management with Incomplete Markets*. *The Quarterly Journal of Economics* **132** (2), 617–663.

Bianchi, F., Kung, H. and Morales, G. (2019). Growth, slowdowns, and recoveries. *Journal of Monetary Economics* **101**, 47–63.

Buera, F. J., Kaboski, J. P. and Shin, Y. (2015). Entrepreneurship and Financial Frictions: A Macrodevelopment Perspective. *Annual Review of Economics* **7** (1), 409–436.

Buera, F. J. and Shin, Y. (2013). Financial Frictions and the Persistence of History: A Quantitative Exploration. *Journal of Political Economy* **121** (2), 221–272.

Carroll, C. D. (2006). The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters* **91** (3), 312–320.

Challe, E. and Ragot, X. (2010). Fiscal Policy in a Tractable Liquidity-Constrained Economy. *The Economic Journal* **121** (551), 273–317.

Chetty, R., Guren, A., Manoli, D. and Weber, A. (2011). Are Micro and Macro Labor Supply Elasticities Consistent? A Review of Evidence on the Intensive and Extensive Margins. *American Economic Review* **101** (3), 471–475.

Comin, D. and Gertler, M. (2006). Medium-Term Business Cycles. *American Economic Review* **96** (3), 523–551.

Dávila, J., Hong, J. H., Krusell, P. and Ríos-Rull, J.-V. (2012). Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks. *Econometrica* **80** (6), 2431–2467.

Domeij, D. and Ellingsen, T. (2018). Rational bubbles and public debt policy: A quantitative analysis. *Journal of Monetary Economics* **96**, 109–123.

Dyrda, S. and Pedroni, M. (2023). Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Income Risk. *The Review of Economic Studies* **90** (2), 744–780.

Flodén, M. (2001). The effectiveness of government debt and transfers as insurance. *Journal of Monetary Economics* **48** (1), 81–108.

Gaillard, A. and Wangner, P. (2022). Inequality, Business Cycles, and Growth: A Unified Approach to Stabilization Policies. *SSRN Electronic Journal*.

Gomes, F., Michaelides, A. and Polkovnichenko, V. (2012). Fiscal Policy and Asset Prices with Incomplete Markets. *Review of Financial Studies* **26** (2), 531–566.

Greenwood, J., Hercowitz, Z. and Huffman, G. (1988). Investment, Capacity Utilization, and the Real Business Cycle. *American Economic Review* **78** (3), 402–417.

Guvenen, F., Kaplan, G. and Song, J. (2014). How Risky Are Recessions for Top Earners? *American Economic Review* **104** (5), 148–153.

Heathcote, J. (2005). Fiscal Policy with Heterogeneous Agents and Incomplete Markets. *The Review of Economic Studies* **72** (1), 161–188.

Hintermaier, T. and Koeniger, W. (2010). The method of endogenous gridpoints with occasionally binding constraints among endogenous variables. *Journal of Economic Dynamics and Control* **34** (10), 2074–2088.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control* **17** (5-6), 953–969.

Kaplan, G. and Violante, G. (2014). A Model of the Consumption Response to Fiscal Stimulus Payments. *Econometrica* **82** (4), 1199–1239.
King, R. G., Plosser, C. I. and Rebelo, S. T. (1988). Production, growth and business cycles. *Journal of Monetary Economics* **21** (2-3), 195–232.
Kitao, S. (2008). Entrepreneurship, taxation and capital investment. *Review of Economic Dynamics* **11** (1), 44–69.
Krebs, T. (2003). Human Capital Risk and Economic Growth. *The Quarterly Journal of Economics* **118** (2), 709–744.
Krebs, T., Kuhn, M. and Wright, M. L. J. (2015). Human Capital Risk, Contract Enforcement, and the Macroeconomy. *American Economic Review* **105** (11), 3223–3272.
Krueger, D. and Perri, F. (2011). Public versus private risk sharing. *Journal of Economic Theory* **146** (3), 920–956.
Kung, H. and Schmid, L. (2015). Innovation, Growth, and Asset Prices. *The Journal of Finance* **70** (3), 1001–1037.
McKay, A. and Reis, R. (2016). The Role of Automatic Stabilizers in the U.S. Business Cycle. *Econometrica* **84** (1), 141–194.
Midrigan, V. and Xu, D. Y. (2014). Finance and Misallocation: Evidence from Plant-Level Data. *American Economic Review* **104** (2), 422–458.
Okada, T. (2022). Endogenous Technological Change and the New Keynesian Model. *The Review of Economics and Statistics* **104** (6), 1224–1240.
Quadrini, V. (2000). Entrepreneurship, Saving, and Social Mobility. *Review of Economic Dynamics* **3** (1), 1–40.
Röhrs, S. and Winter, C. (2015). Public versus private provision of liquidity: Is there a trade-off? *Journal of Economic Dynamics and Control* **53**, 314–339.
— (2017). Reducing government debt in the presence of inequality. *Journal of Economic Dynamics and Control* **82**, 1–20.
Romer, P. (1990). Endogenous Technological Change. *Journal of Political Economy* **98** (5, Part 2), 71–102.
Samuelson, P. A. (1954). The Pure Theory of Public Expenditure. *The Review of Economics and Statistics* **36** (4), 387.
Song, J., Price, D. J., Guvenen, F., Bloom, N. and Wachter, T. von (2019). Firming Up Inequality. *The Quarterly Journal of Economics* **134** (1), 1–50.
Storesletten, K., Telmer, C. I. and Yaron, A. (2004). Cyclical Dynamics in Idiosyncratic Labor Market Risk. *Journal of Political Economy* **112** (3), 695–717.
Woodford, M. (1990). Public Debt as Private Liquidity. *The American Economic Review* **80** (2), 382–88.
Young, E. R. (2010). Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm and non-stochastic simulations. *Journal of Economic Dynamics and Control* **34** (1), 36–41.

Appendix

I Appendix: Derivations of the Toy model

The following subsection illustrates the derivation of the result in the main text. We start by solving the model depending on the debt level. First, we solve the model with a low debt level such that only one household can save in the second period. Thereafter, we solve the model, with government debt high enough that both households can save in the second period. Based on the solution of the model, we then can proof the existence of a debt level for which the results in the main text hold.

Description of the toy model

Household side: This section derives the solution to the household problem illustrated in section 2. Given prices, the households face the following maximization problem:

$$\begin{aligned} V(\omega) = \max_{b_2, e, b_{3,H}, b_{3,L}} & \ln(\omega - \tau_1 - b_2 - e) \\ & + \varphi \Bigg[\ln(\tilde{\pi} e + R_1 b_2 + w_2 - \tau_2 - b_{3,H}) + \beta \ln(w_3 + R_2 b_{3,H} - \tau_3) \Bigg] \\ & + (1-\varphi) \Bigg[\ln(R_1 b_2 + w_2 - \tau_2 - b_{3,L}) + \beta \ln(w_3 + R_2 b_{3,L} - \tau_3) \Bigg] \\ \text{s.t. } b_2 \geq & 0, b_{3,H} \geq 0, b_{3,L} \geq 0, e \geq 0, \end{aligned}$$

To solve the model, we proceed recursively, by characterizing the optimal savings decision in period two. Since period three marks the end of the lifetime, households optimally consume all their available resources.

Households enter period two with risk-free asset holdings b_2 and risky asset holdings e. We assume no heterogeneity in period two, such that households only differ in period two due to their realization of investment risk. We separate households depending on their realizations into groups $i \in \{1, 2\}$, where $i = 1$ denotes households in the high-income group, while $i = 2$ denotes households in the low-income group. The high-income households have $\tilde{\pi}_1 = \tilde{\pi}$, while the low-income households have $\tilde{\pi}_2 = 0$. The asset positions b_2 and e, as well as the household group i are state variables for the household's decision problem. The optimal behavior of households in period two is characterized as a solution to the following problem:

$$\begin{aligned} V(b_2, e, i) = & \max_{b_{3,i}} \ln(\tilde{\pi}_i e + R_1 b_2 + w_2 - \tau_2 - b_{3,i}) + \beta \ln(w_3 + R_2 b_{3,i} - \tau_3) \\ \text{s.t. } & b_{3,i} \geq 0, \end{aligned}$$

where the household takes assets b_2 and e, as well as the realization π_i as given. The first-order condition of the problem is a standard Euler equation with Lagrange multiplier λ that is associated with the borrowing constraint.

$$\frac{1}{\pi_i e + R_1 b_2 + w_2 - \tau_2 - b_{3,i}} = \frac{\beta R_2}{w_3 + R_2 b_{3,i} - \tau_3} + \lambda(b_2, e, i) \tag{41}$$

where $\lambda(b_2, e, i)$ denotes that the Lagrange multiplier depends on the individual states.

In the first period, households face a portfolio problem between the risk-free asset b_2 and the risky asset e. The solution to the portfolio problem is determined by the set of Euler equations

$$\frac{1}{\omega - b_2 - e - \tau_1} = \frac{\varphi\tilde{\pi}}{\tilde{\pi}e + R_1 b_2 + w_2 - \tau_2 - b2, H} + \mu(\omega), \tag{42}$$

$$\frac{1}{\omega - b_2 - e - \tau_1} = R_1 \left[\frac{\varphi}{\tilde{\pi}e + R_1 b_2 + w_2 - \tau_2 - b_{3,H}} + \frac{1-\varphi}{R_1 b_2 + w_2 - \tau_2 - b_{3,L}} \right] + \kappa(\omega), \tag{43}$$

where $\mu(\omega)$ and $\kappa(\omega)$ denote the Lagrange multipliers associated with the nonnegativity constraints of the risky and the risk-free asset. Having defined the optimality conditions for the households, we continue with the other agents in the economy.

Production side: The representative firm produces in periods two and three according to the production function

$$Y = Z\mathcal{E}N.$$

In period two, the firm rents labor at the wage $w_2 = \phi Y$ and generates profits $\pi = (1 - \phi)Y$ that are paid out to households that hold the risky asset in period two. The total profits are distributed among all ideas $\mathcal{E}$, such that the individual return that households obtain is $\tilde{\pi} = \frac{\pi}{\mathcal{E}}$. The payout structure of the firm emerges, for example, from a two-level production structure with intermediate goods producers that produce under monopolistic competition and a perfectly competitive final goods producer. In this structure, ϕ denotes the inverse of the markup that the intermediate goods firms charge. In period three, the payout structure changes in the sense that the firm only has to pay labor such that $w_3 = Y$. This payout structure emerges if the intermediate goods producers lose their ability to exploit the monopoly. Hence, we assume that in the long run (which period three represents), product markets become perfectly competitive.

Government: In period one, the government issues government debt $\mathcal{B}$ that matures in period two to finance transfers $-\tau_1$ to the household in period one. Therefore

$$-\tau_1 = \mathcal{B}.$$

In period two, the government has to repay its outstanding government debt plus the interest on it $R_1\mathcal{B}$ through lump-sum taxes τ_2 and issuing new debt $\mathcal{B}$. Therefore the government budget constraint is

$$\tau_2 = (R_1 - 1)\mathcal{B}.$$

In period three the government repays its outstanding government debt and the interest payment associated with it $R_2\mathcal{B}$ through lump-sum taxes τ_3. The budget constraint in period three, therefore, is

$$\tau_3 = R_2\mathcal{B}.$$

Market clearing: Market clearing requires that goods markets, labor markets, risk-free and risky asset markets clear. Asset market clearing requires

$$\mathcal{B} = \int_0^1 b_2 di = b_2, \quad \text{and} \quad \mathcal{B} = \varphi b_{3,H} + (1-\varphi) b_{3,L}$$

in the first period and the second period for the risk-free asset, and

$$\mathcal{E} = \varphi \int_0^1 e_i di = \varphi e$$

for the risky asset in the first period. Market clearing in the labor market requires $N = 1$ in both periods since households inelastically supply one unit of labor. Finally, goods market clearing requires

$$\omega = \int (c_i + e_i) di$$
$$Y = \varphi c_{2,H} + (1-\varphi) c_{2,L},$$
$$\text{and } Y = \varphi c_{3,H} + (1-\varphi) c_{3,L}.$$

Equilibrium: An equilibrium in this economy consists of policy functions $\{b_2^*, e^*, b_{3,H}^*, b_{3,L}^*\}$, pricing functions $w_2, \tilde{\pi}, w_3$, aggregate labor and equity functions $\{\mathcal{E}, L\}$ such that the following statements holds:

1. Given prices, the policy functions solve the household planning problem.
2. The labor, the bond, the equity, and the goods market clear, the return on equity and the wage rate are determined competitively (i.e. by the firm's problem), while the interest rate on bonds is determined via bond market clearing.

After the description of the model and its equilibrium, we turn to the solution of it.

Solution of the toy model

Solving the model requires solving the household problem while imposing market-clearing conditions in the asset markets and substituting the prices from the firm side. All assumptions allow us to solve the model in closed form. However, the model features a solution that depends on the level of government debt. If government debt is scarce, the low-income households are constrained in period two such that only the high-income households are willing to save. On the contrary, if government debt is abundant both households can save in period two, effectively completing asset markets.

Case with scarce liquidity

To start, we are interested in the case where liquidity is scarce such that the high-households are the only households willing to save in government debt.[15] In the following, we solve the model by determining the optimal behavior of the unconstrained high-income households. Lemma 1 summarizes the solution of the household problem in period two

Lemma 1. *Assume that only the high-income households can save in the risk-free asset in period two. In equilibrium, consumption in period two is*

$$c_{2,H} = \tilde{\pi}e + w_2 - (\frac{1-\varphi}{\varphi})\mathcal{B} \tag{44}$$

$$c_{2,L} = w_2 + \mathcal{B}. \tag{45}$$

Consumption in period three is

$$c_{3,H} = w_3 + (\frac{1-\varphi}{\varphi})R_2\mathcal{B} \tag{46}$$

$$c_{3,L} = w_3 - R_2\mathcal{B}. \tag{47}$$

Finally, the real interest rate between periods two and three is

$$R_2 = \frac{w_3}{\beta\left(\tilde{\pi}e + w_2\right) - (1+\beta)(\frac{1-\varphi}{\varphi})\mathcal{B}}. \tag{48}$$

Proof. To obtain the consumption in the two periods, we use that consumption in periods two and three is $c_{2,i} = \pi_i e + R_1 b_2 + w_2 - \tau_2 - b_{3,i}$ and $c_{3,i} = w_3 + R_2 b_{3,i} - \tau_3$.

The assumption that the high-income household is the only household to be able to save implies that the other household groups is borrowing constrained $b_{3,L} = 0$ and face autarky. Market clearing in period two then implies that high-income households have to save $b_{3,H} = \frac{\mathcal{B}}{\varphi}$. We then obtain the consumption functions of households, by using the budget constraints of the government in periods two and three to substitute the tax rates out of the budget constraints of the households. The real interest rate is then determined

[15] Corollary 1 states the conditions that are necessary for the high-income households to be the only households on the Euler equation.

via the Euler equation (41) of the high-income household. □

Lemma 1 summarizes the solution of the consumption saving problem in the second period in the case with scarce liquidity, i.e. that only the high-income household saves. (44), (45), and (46) show that albeit being scarce, in equilibrium, government debt helps to smooth the consumption of households. Low-income households can use government debt to increase consumption in period two, effectively borrowing against income in period three in which they need to pay taxes. The high-households use government debt in period two to transfer some resources from the high-income state in period two to the lower-income state in period three, thereby smoothing consumption. Effectively, while being scarce, government debt reduces the dispersion of marginal utility of consumption in period two.

Finally, we can characterize the real interest rate analytically in equation (48). In the special case, the real interest rate R_2 increases with the amount of government debt. Moreover, the expression for the real interest rate allows us to characterize a condition on parameters such that high households are indeed the only household that saves between periods two and three. We derive the condition that implies $\lambda(e, y) > 0$ in the Euler equation for the low-income households.

Corollary 1. *If*

$$\frac{\beta}{1+\beta}\varphi\tilde{\pi}e > \mathcal{B}, \tag{49}$$

then low-income households are constraints and high-income households are the only households that are not constrained in period two.

Proof. We use the consumption functions in periods two and three and substitute them into the inequality

$$\frac{1}{w+\mathcal{B}} > \frac{\beta R_2}{w_3 - R_2\mathcal{B}},$$

that determines that the low-income-households are constrained. Substituting for the real interest rate (48) and rearranging terms yields the upper bound (49). □

Corollary 1 provides an upper bound on government debt for the high-income households to be the only households to save. Intuitively the condition states that the amount of government debt is not allowed to be higher than the income difference between high-income households and low-income households in period two. If this condition were violated, low-income households would accept a lower interest rate to save than high-income households, such that they would become the new marginal savers. The condition does not yet acknowledge that the rents the firm pays depend on the amount invested in the risky asset $\mathcal{E}$. We restate the condition considering this below. To be able to do so, however, we require an expression for the amount of investment in the risky asset.

After having specified the optimal behavior of the household in the second period, we turn to the optimal behavior of households in the first period. To solve the household decision problem we use the Euler equations (42) and (43) together with market clearing conditions to obtain an analytical solution for the choices of b_2 and e. Since we know the exact expressions for consumption in the second period, and since all households are ex-ante identical in the economy, we can exactly characterize the equilibrium policies. Lemma 2 characterizes the portfolio choices in equilibrium.

Lemma 2. *Assume that government debt is scarce in the sense of Corollary 1. Then in period one, households invest*

$$b_2 = \mathcal{B} \tag{50}$$

into the risk-free asset and

$$\mathcal{E} = e = \frac{\varphi(1-\phi)\omega + (1-\varphi)\frac{\mathcal{B}}{Z}}{(1+\varphi-\phi)} \tag{51}$$

into the risky asset. Consumption in period one is

$$c_1 = \omega - e. \tag{52}$$

The risk-free interest rate R between periods one and two is defined as

$$R_1 = \frac{\frac{1}{c_1}}{\frac{\varphi}{c_{2,H}} + \frac{(1-\varphi)}{c_{2,L}}}, \tag{53}$$

where $c_{2,i}$ denotes the consumption of household group i in period two.

Proof. Since all households are identical in the first period, market clearing requires each of the households to hold the total amount of government debt such that $b_2 = \mathcal{B}$, deriving equation (50). Since the government budget constraint in the first period requires $-\tau_1 = \mathcal{B}$, this implies that transfers and government debt cancel each other in the first-period budget constraint. This implies that consumption in period one is $c_1 = \omega - e$, deriving (52).

To obtain the optimal saving in the risky asset e, we substitute consumption for the high-income household in period two into the Euler equation for the risky asset (42). Solving the equation for e and substituting in yields equation (51). With all households being identical in the first period, this implies $e = \mathcal{E}/\varphi$. Finally, the risk-free interest rate follows from the Euler equation for the risk-free asset and from market clearing. We abstract from stating the exact expression here but only state the dependence of the interest rate on the consumption policies in period two from Lemma 1, and consumption in period one that is a function of optimal savings in e defined in equation (51). □

Lemma 2 provides us with the last policy functions to characterize the solution to the household problem in the case of scarce government debt. It is worth mentioning that the optimal investment of households in the risky asset e is an increasing function in government debt $\mathcal{B}$. This will be integral for the proof of the propositions one and two. The reason for this is that in period two, consumption decreases in the amount of government debt since the high-income household uses scarce debt to smooth consumption between periods two and three. Therefore, if the government increases the amount of government debt, all households will increase their risky investment, since they anticipate that they will have fewer resources in period two if they become high-income households.

Having specified the optimal portfolio choice, we can make our condition in Corollary 1 more concrete and relate the upper bound to model parameters. The following Corollary 2 summarizes the condition as function of the model parameters.

Corollary 2. *If*

$$\frac{\varphi(1-\phi)^2\omega}{\frac{\beta}{1+\beta}(1+\varphi-\phi)^2-(1-\varphi)(1-\phi)(1+\varphi-\phi)} > \mathcal{B}, \tag{54}$$

high-income households are the only households willing to save in the second period. The upper bound is positive as long as

$$\varphi > \frac{1-\phi}{1+2\beta-\phi(1-\beta)}. \tag{55}$$

Proof. The upper bound follows from substituting the optimal portfolio choice (51) into equation (49) and rearranging. For the upper bound to be positive, the denominator of (54) has to be positive. Ensuring a positive denominator yields expression (55). □

Corollary 2 provides an upper bound on government debt such that households that experience a positive asset income realization have sufficiently higher income than households with low income. Moreover, we obtain a condition for the upper bound to be positive. Condition (55) ensures in the proof of the propositions in the main text that there exists a nonnegative threshold for which the derivative of household welfare with respect to debt changes.

Case with abundant liquidity

Next, we derive the solution to the household problem if government debt is abundant in the sense that Corollary 2 does not hold and both households save in period two. This impacts consumption smoothing and the portfolio choice in period one. To obtain the solution of the problem in this case, we repeat the steps from above.

Lemma 3. *Assume that Corollary 2 does not hold. Then in period two, each household group i saves*

$$b_{3,i} = \mathcal{B} + \frac{\beta}{1+\beta}((2-i)\tilde{\pi}e + w_2 - w_3), \tag{56}$$

yielding consumption of the two household groups to be

$$c_{2,H} = \frac{\tilde{\pi}e + w_2}{1+\beta} + \frac{\beta}{1+\beta}w_3 \tag{57}$$

$$c_{2,L} = \frac{w_2}{1+\beta} + \frac{\beta}{1+\beta}w_3. \tag{58}$$

Finally, the real interest rate between periods two and three is

$$R_2 = \frac{1}{\beta} \tag{59}$$

Proof. If both groups are on the Euler equation (41), then optimal consumption and savings is related by

$$\frac{1}{c_{2,i}} = \frac{\beta R_2}{c_{3,i}}, \tag{60}$$

where we can substitute in the budget constraints for periods two and three. Solving the Euler equation for the optimal savings policy yields

$$b_{3,i} = \frac{\beta}{1+\beta}((2-i)\tilde{\pi}e + w_2 + R_1 b_2 - \tau_2) - \frac{w_3 - \tau_3}{(1+\beta)R_2}, \tag{61}$$

where the i denotes the dependence on the household group. We can impose that all households are identical in the first period such that $b_2 = \mathcal{B}$, as well as the budget constraints of the government to obtain that the policy function for savings is

$$b_{3,i} = \mathcal{B} + \frac{\beta}{1+\beta}\left((i-2)\tilde{\pi}e + w_2\right) - \frac{w_3}{(1+\beta)R_2}. \tag{62}$$

Using the market clearing condition of the asset market $\varphi b_{3,H} + (1-\varphi)b_{3,L} = \mathcal{B}$, as well as the expressions for prices w_2, w_3 and $\tilde{\pi}$, as well as the fact that output Y is constant over time yield the expression for the real interest rate (59). We can use this expression to simplify (62) to obtain the expression for savings (56). We obtain the expressions for consumption by substituting the optimal amount of savings into the budget constraints of the households. □

Lemma 3 states the solution to the second period's household problem. Since government debt is abundant, both household groups are on the Euler equation and save in the risk-free asset. This enables them to smooth their consumption between periods two and three making consumption of both households independent from the level of government debt. This has the important implication for the portfolio problem that investment in the risky asset becomes independent from the amount of government debt, as well. We state this result formally in Lemma 4

Lemma 4. *Assume that Corollary 2 does not hold. Then in period one, households in equilibrium invest*

$$b_2 = \mathcal{B} \tag{63}$$

into the risk-free asset and

$$\mathcal{E} = e = \frac{1-\phi}{1+\left(\frac{1+\beta\varphi}{1+\beta}\right)\left(\frac{1-\phi}{\varphi}\right)}\omega \tag{64}$$

into the risky asset. The risk-free interest rate R between periods one and two is defined as in expression (58).

Proof. Identical to the former argument, since all households are identical in the first period, market clearing requires each of the households to hold the total amount of government debt such that $b_2 = \mathcal{B}$, deriving equation (63). To obtain the optimal saving in the risky asset e, we substitute consumption for the high-income household in period two into the Euler equation for the risky asset (42). Solving the equation for e yields equation (64). With all households being identical in the first period, this implies $e = \mathcal{E}/\varphi$. □

Lemma 4 finalizes the solution of the model version with abundant government debt. If Corollary 2 does not hold, then there exists sufficient liquidity in the economy for both household groups to save in the second period and turn the model effectively Riccardian. This disables the formerly active channel that government debt crowds in risky investments. The reason for this is that government debt does not facilitate consumption smoothing for the high-income household, since household decisions are independent of the level of government debt.

Derivation of welfare results

Having solved the model in its two versions, we can now derive some welfare results. To do so, we follow a similar approach as Dávila et al. (2012), however within a three-period model.

Welfare effects of an increase in risky investment $\mathcal{E}$

First, we evaluate the welfare effect that an increase in the number of risky investments has on the economy. To do so, we take the derivate of the value function $V(\omega)$ with respect to the amount of risky investment $\mathcal{E}$. Thereafter, we use the optimality conditions, as well as market clearing to derive the sign of the derivative.

Proof. The derivative of the value function with respect to the amount of risky investment is

$$\begin{aligned}\frac{\partial V(\omega)}{\partial \mathcal{E}} &= u'(c_1)[-1 - \frac{\partial \tau_1}{\partial \mathcal{E}} - \frac{\partial b_2}{\partial \mathcal{E}}] \\ &+ \varphi\Bigg[u'(c_{2,H})\left(\tilde{\pi}\frac{\partial e}{\partial \mathcal{E}} + \frac{\partial \tilde{\pi}}{\partial \mathcal{E}}e + \frac{\partial R_1}{\partial \mathcal{E}}a + R_1\frac{\partial b_2}{\partial \mathcal{E}} + \frac{\partial w_2}{\partial \mathcal{E}} - \frac{\partial \tau_2}{\partial \mathcal{E}} - \frac{\partial b_{3,H}}{\partial \mathcal{E}}\right) \\ &+ \beta u'(c_{3,H})\left(\frac{\partial w_3}{\partial \mathcal{E}} - \frac{\partial \tau_3}{\partial \mathcal{E}} + \frac{\partial R_2}{\partial \mathcal{E}}b_{3,H} + R_2\frac{\partial b_{3,H}}{\partial \mathcal{E}}\right)\Bigg] \\ &+ (1-\varphi)\Bigg[u'(c_{2,L})\left(\frac{\partial R_1}{\partial \mathcal{E}}b_2 + R_1\frac{\partial b_2}{\partial \mathcal{E}} + \frac{\partial w_2}{\partial \mathcal{E}} - \frac{\partial \tau_2}{\partial \mathcal{E}} - \frac{\partial b_{3,L}}{\partial \mathcal{E}}\right) \\ &+ \beta u'(c_{3,L})\left(\frac{\partial w_3}{\partial \mathcal{E}} - \frac{\partial \tau_3}{\partial \mathcal{E}} + \frac{\partial R_2}{\partial \mathcal{E}}b_{3,L} + R_2\frac{\partial b_{3,L}}{\partial \mathcal{E}}\right)\Bigg]\end{aligned} \tag{65}$$

From the Euler equations for assset choices, we know that the the red, blue and green terms cancel. Moreover we know from the solution of the household problem that the orange terms are zero (in the case with scarce government debt) or cancle each other due to the Euler equation of the low-income household group (in the abundant government debt case). We can simplify the expression even further, by using the government budget constraints. $\tau_2 = (R_1 - 1)\mathcal{B}$ implies $\frac{\partial \tau_2}{\partial \mathcal{E}} = \frac{\partial R_1}{\partial \mathcal{E}}$, and $\tau_3 = R_2\mathcal{B}$ implies $\frac{\partial \tau_3}{\partial \mathcal{E}} = \frac{\partial R_2}{\partial \mathcal{E}}\mathcal{B}$. Finally, we know that the transfers $-\tau_1$ have to be equal the exogenous supply of government debt $\mathcal{B}$ such that $\frac{\partial \tau_1}{\partial \mathcal{E}} = 0$. We can use these implications together with the asset market clearing condition in the first period $b_2 = \mathcal{B}$ to obtain:

$$\frac{\partial V(\omega)}{\partial \mathcal{E}} = \varphi \left[u'(c_{2,H}) \left(\frac{\partial \tilde{\pi}}{\partial \mathcal{E}} e + \frac{\partial w_2}{\partial \mathcal{E}} \right) + \beta u'(c_{3,H}) \left(\frac{\partial w_3}{\partial \mathcal{E}} + \frac{\partial R_2}{\partial \mathcal{E}} (b_{3,H} - \mathcal{B}) \right) \right]$$
$$+ (1-\varphi) \left[u'(c_{2,L}) \frac{\partial w_2}{\partial \mathcal{E}} + \beta u'(c_{3,L}) \left(\frac{\partial w_3}{\partial \mathcal{E}} + \frac{\partial R_2}{\partial \mathcal{E}} (b_{3,L} - \mathcal{B}) \right) \right] \quad (66)$$

This condition allows us to evaluate the welfare implications of an increase in risky investment. We start with the terms multiplying the marginal utility of the household groups in the second period. From the definition of the factor prices, we know that $\frac{\partial \tilde{\pi}}{\partial \mathcal{E}} = 0$, $\frac{\partial w_2}{\partial \mathcal{E}} = \phi Z N > 0$, and $\frac{\partial w_3}{\partial \mathcal{E}} = Z N > 0$. This implies that the terms that multiply the marginal utility in the second period of both household groups is positive.

This leaves us with the terms that multiply the marginal utility in the third period. From Lemmas 1 and 3 we know the real interest rate between period two and three for the cases with scarce and abundant government debt. We analyze both cases in isolation.

The case with abundant government debt is the easier case of the two. Since $R_2 = \beta^{-1}$ this implies that $\frac{\partial R_2}{\partial \mathcal{E}} = 0$, such that all terms that multiply the marginal utility of households in period three are positive. This implies that with abundant government debt, an increase in risky investment $\mathcal{E}$ always increases households' welfare.

The case of scarce government debt requires more involved derivations. First, we have that $(b_{3,H} - \mathcal{B}) = (\frac{1-\varphi}{\varphi})\mathcal{B}$ and $(b_{3,L} - \mathcal{B}) = -\mathcal{B}$ have different signs in equilibrium. Consequently, we have that an increase in risky investment affects the consumption of the two household groups in period three differently by scaling the change in the risk-free rate between periods two and three differently. Second, we have that the interest rate between periods two and three depends on the amount of risky investment

$$R_2 = \frac{w_3}{\beta\left(\tilde{\pi} e + w_2\right) - (1+\beta)(\frac{1-\varphi}{\varphi})\mathcal{B}} = \frac{Z\mathcal{E}N}{\beta\left(\phi + \frac{1-\phi}{\varphi}\right) Z\mathcal{E}N - (1+\beta)(\frac{1-\varphi}{\varphi})\mathcal{B}}, \quad (67)$$

from which follows that

$$\frac{\partial R_2}{\partial \mathcal{E}} = \frac{-(1+\beta)\frac{1-\varphi}{\varphi}\frac{\mathcal{B}}{Z\mathcal{E}^2}}{\left[\left(\phi + \frac{1-\phi}{\varphi}\right)\beta - (1+\beta)(\frac{1-\varphi}{\varphi})\frac{\mathcal{B}}{Z\mathcal{E}}\right]^2} \leq 0. \quad (68)$$

These two things together imply that for a positive amount of government debt, the sign of the derivate of welfare with respect to the number of varieties cannot be determined exactly due to offsetting effects.

However, we can derive two conditions. First, we can determine that for $\mathcal{B} = 0$ all terms are positive such that in the case where the conditions of Corollary 2 holds and $\mathcal{B} = 0$, that welfare increases with an increase in risky investment. Second, we can determine that the overall expression remains positive for sure as long as

$$\frac{\partial w_3}{\partial \mathcal{E}} + \frac{\partial R_2}{\partial \mathcal{E}}(b_{3,H} - \mathcal{B}) = Z + \frac{1-\varphi}{\varphi}\frac{\partial R_2}{\partial \mathcal{E}}\mathcal{B} > 0. \tag{69}$$

Condition (69) implicitly defines a debt level B^*, an upper bound for the multiplier of the marginal utility of the second household group in the third period to be positive and hence for welfare to increase with the amount of risky investment in the economy. This condition constitutes a lower bound on the region of possible values for government debt that still allows for welfare increases through an increase in the amount of risky investment. Since we desire to show that such a region exists in general, the lower bound on government debt does not limit the content of Proposition 1.

To summarize, we have illustrated two cases. First, we have illustrated that in the case of abundant government debt, welfare always increases with the amount of risky investment. Second, we have illustrated that in the case of scarce government debt, if the amount of government debt is positive and below an implicitly defined positive threshold B^*, then we can be sure that welfare increases with the amount of risky investment. Hence, as long as government debt is below B^*, it is certain that welfare increases with the amount of risky investment. This derivation concludes the proof of Proposition 1. □

Welfare effects of an increase in government debt

Finally, we evaluate the welfare effect that an increase in government debt $\mathcal{B}$ has on the economy. Again, we take the derivate of the value function, but with respect to the amount of government debt. Thereafter, we simplify the expression with optimality conditions and market clearing conditions.

Proof.

$$\begin{aligned}
\frac{\partial V(\omega)}{\partial \mathcal{B}} &= u'(c_1)[-\frac{\partial e}{\partial \mathcal{B}} - \frac{\partial \tau_1}{\partial \mathcal{B}} - \frac{\partial b_2}{\partial \mathcal{B}}] \\
&+ \varphi \Bigg[u'(c_{2,H}) \left(\tilde{\pi} \frac{\partial e}{\partial \mathcal{B}} + \frac{\partial \tilde{\pi}}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} e + \frac{\partial R_1}{\partial \mathcal{B}} a + R_1 \frac{\partial b_2}{\partial \mathcal{B}} + \frac{\partial w_2}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_2}{\partial \mathcal{B}} - \frac{\partial b_{3,H}}{\partial \mathcal{B}} \right) \\
&+ \beta u'(c_{3,H}) \left(\frac{\partial w_3}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_3}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}} b_{3,H} + R_2 \frac{\partial b_{3,H}}{\partial \mathcal{B}} \right) \Bigg] \\
&+ (1-\varphi) \Bigg[u'(c_{2,L}) \left(\frac{\partial R_1}{\partial \mathcal{B}} a + R_1 \frac{\partial b_2}{\partial \mathcal{B}} + \frac{\partial w_2}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_2}{\partial \mathcal{B}} - \frac{\partial b_{3,L}}{\partial \mathcal{B}} \right) \\
&+ \beta u'(c_{3,L}) \left(\frac{\partial w_3}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_3}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}} b_{3,L} + R_2 \frac{\partial b_{3,L}}{\partial \mathcal{B}} \right) \Bigg] \qquad (70)
\end{aligned}$$

With some minor modifications, we know that from the Euler equations the red, blue and green terms cancel. The orange terms are either zero because the second household group is constrained, or they cancel each other due to the households being on the Euler equation. Again imposing the budget constraint of the government and the market clearing conditions yields that $\frac{\partial - \tau_1}{\partial \mathcal{B}} = 1$ and $b_2 = \mathcal{B}$, such that $\frac{\partial b_2}{\partial \mathcal{B}} = 1$. Moreover, $\frac{\partial \tau_3}{\partial \mathcal{B}} = \frac{\partial R_2}{\partial \mathcal{B}} \mathcal{B} + R_2$ and $\frac{\partial \tau_2}{\partial \mathcal{B}} = \frac{\partial R_1}{\partial \mathcal{B}} \mathcal{B} + (R_1 - 1)$. This yields the expression

$$\begin{aligned}
\frac{\partial V(\omega)}{\partial \mathcal{B}} &= u'(c_1) \\
&+ \varphi \Bigg[u'(c_{2,H}) \left(\frac{\partial \tilde{\pi}}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} e + \frac{\partial w_2}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - R_1 + 1 \right) \\
&+ \beta u'(c_{3,H}) \left(\frac{\partial w_3}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}} (b_{3,H} - \mathcal{B}) - R_2 \right) \Bigg] \\
&+ (1-\varphi) \Bigg[u'(c_{2,L}) \left(\frac{\partial w_2}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - R_1 + 1 \right) \\
&+ \beta u'(c_{3,L}) \left(\frac{\partial w_3}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}} (b_{3,L} - \mathcal{B}) - R_2 \right) \Bigg], \qquad (71)
\end{aligned}$$

where the red and blue terms cancel, because these households always are on the Euler equation. The green part is equal to the Lagrange multiplier $\lambda(\mathcal{B}, \mathcal{E}, 2)$. If the second household group is on the Euler equation then the multiplier is zero and if the households are constraint, the multiplier is positive. Therefore, we simplify the expression to

$$\frac{\partial V(\omega)}{\partial \mathcal{B}} = \varphi\left[u'(c_{2,H})\left(\frac{\partial \tilde{\pi}}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}e + \frac{\partial w_2}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}\right) + \beta u'(c_{3,H})\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,H} - \mathcal{B})\right)\right]$$
$$+ (1-\varphi)\left[u'(c_{2,L})\left(\frac{\partial w_2}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}\right) + \beta u'(c_{3,L})\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,L} - \mathcal{B})\right) + \lambda(\mathcal{B}, \mathcal{E}, 2)\right]. \tag{72}$$

We obtain a similar structure for the derivate of household welfare with respect to the amount of government debt $\mathcal{B}$ than we have obtained for the derivative with respect to risky investment $\mathcal{E}$. We have some slight differences. We consider begin analyzing the case with abundant liquidity and then turn to the case with scarce liquidity.

First, with abundant liquidity, we have that the real interest rate is constant such that $\frac{\partial R_2}{\partial \mathcal{B}} = 0$. Moreover, we have that the investment in the risky asset is independent of the government debt level. Hence, we have $\frac{\partial \mathcal{E}}{\partial \mathcal{B}} = 0$. Therefore we have for the case with abundant government debt that $\frac{\partial V(\omega)}{\partial \mathcal{B}} = 0$. Therefore, an increase in government debt does not alter the welfare of the households, due to Ricardian equivalence that effectively holds in this economy.

Second, with scarce liquidity, the derivative of the real interest rate between periods two and three R_2 with respect to the amount of government debt $\mathcal{B}$ is

$$\frac{\partial R_2}{\partial \mathcal{B}} = \frac{(\frac{1+\beta}{Z\mathcal{E}})(\frac{1-\varphi}{\varphi})\left(1 - \frac{\mathcal{B}}{\mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}\right)}{\left[\beta\left(\phi + (\frac{1-\phi}{\varphi})\right) - (1+\beta)(\frac{1-\varphi}{\varphi})\frac{\mathcal{B}}{Z\mathcal{E}}\right]^2} \lesseqgtr 0. \tag{73}$$

We want to continue with the same conservative approach as above. We want to make sure that all multipliers of the marginal utilities are positive. We know that the derivatives of the prices with respect to the amount of risky investment are positive, as well as that the derivative of investment in the risky asset with respect to the amount of government debt is positive. Therefore, the only term with an ambiguous sign is the multiplier of the marginal utility of the high- and low-income households in the third period. For the high-income households the condition is

$$\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,H} - \mathcal{B}) = Z\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}\left(\frac{1-\varphi}{\varphi}\right)\mathcal{B} > 0 \tag{74}$$

and for low-income households the condition is

$$\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,L} - \mathcal{B}) = Z\frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial R_2}{\partial \mathcal{B}}\mathcal{B} > 0. \tag{75}$$

To derive the condition, remember that in the case with scarce liquidity $b_{3,L} = 0$ and $b_{3,H} = \mathcal{B}/\varphi$. For the case with $\mathcal{B} = 0$, we can easily see that both expressions are positive, since all terms except the derivatives of the prices with respect to risky investment drop out. Consequently, the expression is positive with zero government debt. Since all expressions are continuous, either inequality (74) or inequality (75) implicitly define a government debt level $\hat{B}$ below which both inequalities hold.

To summarize, we have shown that for zero government debt, the derivative is always positive. Moreover, for government debt below the implicitly defined level, the derivative is also positive. Therefore, we have found an interval of an amount of government debt for which the derivative of welfare with respect to government debt is positive. This concludes the proof of Proposition 2.

□

II Appendix: Solution of the household's problem

Given the structure mentioned above, we now characterize its solution. In the following, we skip all time indexes of value functions and refer to future realizations of variables with a prime. We can write the first-order conditions as functions of the shadow values. These conditions read

$$e_a^*: \quad q \times \frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} = \beta \times \frac{\partial \mathbb{W}_{t+1}(a^a(a,e,h), e^a(a,e,h), h)}{\partial e'} \tag{76}$$

$$a_a^*: \quad \frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} = \beta \times \frac{\partial \mathbb{W}_{t+1}(a^a(a,e,h), e^a(a,e,h), h)}{\partial a'} \tag{77}$$

$$a_n^*: \quad \frac{u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} = \beta \times \frac{\partial \mathbb{W}_{t+1}(a_n(a,e,h), e, h)}{\partial a^n(a,e,h)} \tag{78}$$

$$n_a^*: \quad -\frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial n} = (1-\tau^L) w_t h_{it} \frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} \tag{79}$$

$$n_n^*: \quad -\frac{u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial n} = (1-\tau^L) w_t h_{it} \frac{u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} \tag{80}$$

Therefore, to solve the model numerically, we require expressions for the continuation values. In a first step, we require the Envelope conditions:

$$\frac{\partial V_a(a,e,h)}{\partial a} = R(a, R^a) \times \frac{\partial u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} \tag{81}$$

$$\frac{\partial V_n(a,e,h)}{\partial a} = R(a, R^a) \times \frac{\partial u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} \tag{82}$$

$$\frac{\partial V_a(a,e,h)}{\partial e} = (q + \Pi) \times \frac{\partial u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} \tag{83}$$

$$\frac{\partial V_n(a,e,h)}{\partial e} = (\Pi - gq) \times \frac{\partial u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} + \beta \times \frac{\partial \mathbb{W}_{t+1}(a_n^*, e, h)}{\partial e} \tag{84}$$

With these envelope conditions, we can derive the derivative of $\mathbb{W}$ with respect to the states a and e. These derivatives can be interpreted as the shadow value of the liquid asset and the equity asset for a household at the specified position of the state space.

$$\frac{\partial \mathbb{W}_{t+1}}{\partial a}(a,e,h) = R(a, R^a) \times \left\{ \varphi \mathbb{E}\left[\lambda_a \frac{\partial V_a(a,e,h)}{\partial a} + (1-\lambda_a) \frac{\partial V_n(a,e,h)}{\partial a} \right] + (1-\varphi)\mathbb{E}\left[\lambda_a \frac{\partial V_a(a,0,h)}{\partial a} + (1-\lambda_a) \frac{\partial V_a(a,0,h)}{\partial a} \right] \right\} \tag{85}$$

$$\frac{\partial \mathbb{W}_{t+1}}{\partial e}(a,e,h) = \varphi \mathbb{E}\left[\lambda_a \frac{\partial V_a(a,e,h)}{\partial e} + (1-\lambda_a) \frac{\partial V_n}{\partial e}(a,e,h) \right]. \tag{86}$$

The shadow value for equity has a recursive structure due to its dependence on its own derivative via the term $\frac{\partial V_n}{\partial e}$.

Finally, note that if households are constrained, they still optimally supply labor. To determine the optimal labor supply, households solve the following static optimization problem:

$$\max_{n_{it}} \ln(c_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma}$$
$$\text{s.t. } c_{it} = (1-\tau^L) w_t h_{it} n_{it} + T_{it} + (1+r) a_{it} + (\mathcal{I}_{\text{adj.}} q_t + \pi_t) e_{it} - \underline{a}$$

Solving the first-order condition of the constraint problem gives us the following expression for leisure

$$n_{it} = \left(\frac{(1-\tau^L) w_t h_{it}}{\tilde{c}_{it} \omega} \right)^{\frac{1}{\gamma}}, \tag{87}$$

that implicitly defines labor supply. We precompute this expression before solving the household problem. Given the optimality conditions we can characterize a solution algorithm for the individual problem.

III Appendix: Solution method

We use an algorithm similar to Aiyagari (1994), Aiyagari and McGrattan (1998) to compute the stationary equilibrium. The model will be solved by guessing the capital stock K_t, labor supply N_t, the growth rate of the economy g_t, and the tax rate on labor income τ_t^L. Given these guesses, we compute the policy functions via the endogenous grid method originally developed by Carroll (2006) and subsequently developed by Hintermaier and Koeniger (2010), and Bayer and Luetticke (2020) to a two-asset structure. The solution of the model with labor supply is based on the appendix of Auclert, Bardóczy and Rognlie (2023). We aggregate the economy via the histogram method of Young (2010). If aggregate supply matches aggregate demand in all markets, the algorithm has converged. The pseudo-code goes as follows:

1. Guess $\{K_t, N_t, g_t, \tau_t^L\}$

   (a) Compute the interest rate R_t, the wage rate W_t, and the price for new equity q_t.

   (b) Guess the policy functions c_a^*, c_n^*, n_a^*, n_n^* and value functions V_a, and V_n. In the first iteration guess the shadow value $\frac{\partial \mathbb{W}_{t+1}(a,e,h)}{\partial e} = \mathbf{0}$. Precompute the optimal labor supply if households are at the budget constraint.

   (c) Use equation (78) to solve for an updated policy function for a_n^*, n_n^* and c_n^* using standard endogenous grid methods. We elaborate below in detail on how to solve for leisure.

   (d) Combine equations (76) and (77) and find off-grid values for $\hat{a}(e,h)$ related to the exogenous grid of e'

$$0 = \frac{\partial \mathbb{W}_{t+1}(\hat{a}(e',h'),e',h)}{\partial e'} - \frac{\partial \mathbb{W}_{t+1}(\hat{a}(e',h'),e',h)}{\partial a'}. \tag{88}$$

   (e) From equation (77) compute an update for the marginal utility of consumption today. Use the marginal utility to update c_a^* and n_a^*. Use the policy functions and the optimal choice of $\hat{a}'$ obtained in step 3 for a fixed grid of e_a^* to construct an endogenous grid. Interpolate the policy functions from the endogenous grid on the exogenous grid.

   (f) Check whether the borrowing constraints are binding in any of the two cases. If they are binding, force households to allocate all their income into consumption and leisure according to eq. (87).

(g) Update the value for the shadow value of liquid assets $\frac{\partial \mathbb{W}_{t+1}(a,e,h)}{\partial a}$ with the new policy functions. Update the shadow value of equity $\frac{\partial \mathbb{W}_{t+1}(a,e,h)}{\partial e}$ with the new policy functions and the former guess for the shadow value of equity based on equation (86).

(h) Repeat steps b) to g) until convergence in all policy functions occurs.

2. Aggregate the economy up and check for market clearing given the guesses from step 1.

3. If market clearing is achieved, stop. If not, iterate on the guesses $\{K_t, N_t, g_t, \tau_t^L\}$.

To obtain the labor supply, note that labor is given as a function of the marginal utility of consumption as defined in equation (113). This relation holds for the adjustment case, as well as the non-adjustment case. We obtain the marginal utility defined on the endogenous grid from the Euler equation in both cases. We then calculate the policy function for labor supply on the endogenous grid. Given labor supply, we can construct the labor income that households obtain. With the labor income of households, we can compute the endogenous grid and interpolate back onto the exogenous grid.

Having found an equilibrium in the economy, we then proceed to compute the continuation value $\mathbb{W}$, which we use for welfare evaluations. Based on (22), the value functions can be written as

$$\begin{aligned} & V^a(a,e,h) = u(c_a^*, n_a^*) + \ln(1+g) + \beta \mathbb{W}(a_a^*, e_a^*, h) \\ \text{and} \quad & V^n(a,e,h) = u(c_n^*, n_n^*) + \ln(1+g) + \beta \mathbb{W}(a_n^*, e, h), \end{aligned}$$

where we have assumed that $\mathcal{E}_{-1} = 1$ such that in this period $\mathcal{E} = 1 + g$ along a balanced growth path. Consequently, we can decompose the value functions V^a and V^n into a stationary component and a growth component. Let $\tilde{V}(a,e,h)$ denote the stationary component, then we can write

$$V^i(a,e,h) = \tilde{V}^i(a,e,h) + \frac{\beta}{(1-\beta)^2} \ln(1+g) \text{ for } i \in \{a,n\}, \tag{89}$$

where the last term is the growth component that follows from an infinite sum. To obtain the value functions, we can first calculate the stationary value functions $\tilde{V}^i$ and then add the growth component. The continuation value then follows from (31).

IV Appendix: Model variants

This subsection includes the description of the alternative model versions we solve. Besides the model version with King, Plosser and Rebelo (1988)(KPR)-preferences in the main text, this section describes the household problem with Greenwood, Hercowitz and Huffman (1988) (GHH)-preferences. After illustrating a modification of the household side, we portray the alternative economies with exogenous growth rates and an illiquid aggregate capital stock. We solve all economies by adjusting the labor income tax to clear the government's budget constraint.

IV.1 Adjusting Government debt

When adjusting government expenditure instead of the labor income tax, we augment the utility function (3.2.1) by a term $\zeta \ln(G_t)$ such that households obtain utility from government expenditure. This enables us to obtain non-trivial welfare results. Writing utility as a function of detrended variables

$$\mathbb{E}_0 \max_{\{\tilde{c}_{it}, n_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(\tilde{c}_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} + \zeta \ln(\tilde{G}_t) + (1+\zeta) \ln(\mathcal{E}_t) \right], \tag{90}$$

households obtain additional utility from detrended government expenditure and experience stronger utility gains from the number of varieties in the economy. The latter effect appears since government expenditure G_t grows by the growth rate of the economy, as does individual consumption c_{it}. Using the identical calculation as in Appendix III, we can decompose the value function when valuing government expenditure $V_G^i, i \in \{a, n\}$ into the component without government expenditure V^i and the component that is added through government expenditure:

$$V_G^i(a, e, h) = V(a, e, h) + \zeta \frac{\beta}{(1-\beta)^2} \ln(1+g) + \zeta \frac{\ln(\tilde{G}_t)}{1-\beta}, \tag{91}$$

where the last two terms follow due to the utility of government expenditure and infinite sums. This implies that the continuation value with utility from government expenditure is

$$\mathbb{W}_G(a, e, h) = \mathbb{W}(a, e, h) + \zeta \frac{\beta}{(1-\beta)^2} \ln(1+g) + \zeta \frac{\ln(\tilde{G}_t)}{1-\beta}, \tag{92}$$

For the welfare results to be meaningful, we derive a modified Samuelson condition. Samuelson (1954) shows that the optimal supply of a public good is determined by the condition that the marginal rate of substitution between the public good and a consumption good is equal to their relative price. While the original condition is derived in a static environment we extend the condition to a dynamic setting. We modify the Samuelson condition in the sense that on average there is no welfare gain if all households switch to a new balanced growth path with marginally more of the public good financed by an increase in public debt. To achieve this, we set the parameter ζ. We impose the condition

$$\frac{\partial \mathbb{W}_G(a,e,h)}{\partial B} = 0$$

and solve for ζ, which yields

$$\zeta = -(1-\beta)\left(\frac{\beta}{1-\beta}\frac{\partial g/\partial \tilde{B}}{(1+g)} + \frac{\partial \tilde{G}^*/\partial \tilde{B}}{\tilde{G}^*}\right)^{-1}\int \frac{\partial \mathbb{W}(b_i,e_i,h_i)}{\partial \tilde{B}}di. \tag{93}$$

Intuitively, ζ is set such that around the calibrated balanced growth path, the welfare effect that a small change in the amount of government debt has via changing the growth rate or detrended government expenditure is compensated on average. We approximate each of the derivatives numerically using a symmetric derivative for a small perturbation around the baseline debt level.

IV.2 Fixed exogenous growth rate

We solve for an alternative model version, where we set $\rho \to 0$. We fix χ such that the growth rate in the economy is 0.5%. The sum of the assumptions implies:

$$\Delta_t = \chi \mathcal{E}_t, \tag{94}$$

and

$$q_t = 1/\chi \tag{95}$$

To achieve the growth rate of 0.5%, this implies that we have to set $\chi = 1.005 - \varphi$ in the baseline. When conducting all policy experiments, the price q_t adjusts in such a way as to keep the growth rate of the economy fixed.

IV.3 GHH-preferences

Households have time-separable preferences of the Greenwood, Hercowitz and Huffman (1988) (GHH) type with time-discount factor β and derive felicity from consuming the final good c_{it} and dislike supplying labor n_{it}. We assume GHH preferences to maximize the negative effect of crowding out the capital stock. We scale the disutility of work by the productivity level of a household $h_{it}\mathcal{E}_t$ to generate balanced growth and homogeneous labor supply. Households discount felicity by β and maximize the discounted sum

$$\mathbb{E}_0 \max_{\{c_{it},n_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u[c_{it} - G(n_{it})h_{it}\mathcal{E}_t]. \tag{96}$$

The maximization is subject to the budget constraints described in the main text. The felicity function u exhibits a constant relative risk aversion (CRRA) with risk aversion parameter $\xi > 0$,

$$u(x_{it}) = \frac{x_{it}^{1-\xi} - 1}{1-\xi}, \tag{97}$$

where $x_{it} = \tilde{c}_{it}\mathcal{E}_t - G(n_{it})h_{it}\mathcal{E}_t$ is household $i's$ composite demand for per-variety goods consumption c_{it} and leisure, where G measures the disutility from work.

This means, we can recast the household planning problem as choices over labor and per-variety-consumption c_{it}

$$\mathbb{E}_0 \max_{\{c_{it},n_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \mathcal{E}_t^{1-\xi} u[\tilde{c}_{it} - G(n_{it})h_{it}] + \beta^t \frac{\mathcal{E}_t^{1-\xi} - 1}{1-\xi}, \tag{98}$$

allowing us later to solve the planning problem in stationary recursive form. The last term expresses the love of variety that households have.

The tax schedule remains unchanged, such that a household's net labor income is given by

$$y_{it} = (1 - \tau_t^L)\tilde{w}_t n_{it} h_{it}\mathcal{E}_t, \tag{99}$$

with identical interpretations of the variables. Given net labor income, the first-order condition for labor supply is

$$\frac{\partial G(n_{it})}{\partial n_{it}} h_{it}\mathcal{E}_t = (1 - \tau_t^L)\tilde{w}_t h_{it}\mathcal{E}_t. \tag{100}$$

Therefore, The scaling of the disutility by productivity $h_{it}\mathcal{E}_t$, means that these terms drop from the FOC. In turn, labor supply is independent of individual productivity and depends only on the after-tax per-variety wage w_t. We assume a constant Frisch elasticity of labor supply by setting $G(n) = \frac{\omega n^{1+\gamma}}{1+\gamma}$ such that all households then supply

$$N_t := n_{it} = \left[\frac{(1-\tau_L)\tilde{w}_t}{\omega}\right]^{1/\gamma} \tag{101}$$

units of labor. We use ω to rescale labor in the steady state to unity as in our benchmark case. In turn, this implies that at optimal labor supply, the disutility from labor is a constant fraction of after-tax labor income per variety $\tilde{y}_{it}$:

$$G(N_t)h_{it}\mathcal{E}_t = \frac{1}{1+\gamma}\frac{\partial G(N_t)}{\partial n_{it}}h_{it}\mathcal{E}_t N_t \overset{!}{=} \frac{1}{1+\gamma}\tilde{y}_{it}$$

.

IV.4 Illiquid capital

Finally, we solve a model version where the capital stock is illiquid in the sense that households cannot trade it. This implies that the only variable production factor is labor supply. We fix the capital stock at the benchmark calibration level such that $K_t = \bar{K}_t$. Therefore, factor payouts are given as follows:

$$d_t + \delta = \phi\alpha \left(\frac{N_t}{\bar{K}_t}\right)^{\alpha-1}, \tag{102}$$

$$w_t = \phi(1-\alpha)\left(\frac{N_t}{\bar{K}_t}\right)^{-\alpha} \tag{103}$$

note that we use d_t as the payout that households obtain net of depreciation for providing the capital $\bar{K}$. The FOC of the firm still defines the wage rate w_t. Although the households cannot trade capital, they still obtain income from it. We assume two alternative approaches. First, we assume that each households owns the total capital stock, but is not able to trade it. This implies that the household obtains returns $(d_t - \delta)\bar{K}_t$ each period. Second, we assume that households obtain capital income proportional to their idiosyncratic labor income risk $(d_t - \delta)\bar{K}_t \frac{h_{it}}{\int_0^1 h_{it} di}$, such that capital income increases labor income risk. Both cases represent economies in which we drastically reduce the liquidity in the economy. Moreover, the second case additionally increase the income risk in the economy.

New equilibrium condition:

$$p_t K_t + A_t = \int_0^1 a_{it} di \tag{104}$$

$$1 = \mathbb{E}_t[\frac{d_t + p_t}{R_t}] \tag{105}$$

V Appendix: Characteristics of the KPR preferences used

This section displays the properties of the preferences used in the main part of the text, displayed below again

$$u(c_{it}, n_{it}) = \ln(c_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma}, \tag{106}$$

where ω is a weight between consumption and leisure. This utility formulation has an invertible marginal utility of consumption u_c. First, we illustrate the general properties of the utility specification. We start with monotonicity and curvature of the utility function

$$\frac{\partial u}{\partial c_{it}}(c_{it}, n_{it}) = u_c = \frac{1}{c_{it}} \tag{107}$$

$$\frac{\partial^2 u}{\partial c_{it}^2}(c_{it}, n_{it}) = u_{cc} = -\frac{1}{c_{it}^2} \tag{108}$$

$$\frac{\partial u}{\partial n_{it}}(c_{it}, n_{it}) = u_n = \omega n_{it}^{\gamma} \tag{109}$$

$$\frac{\partial^2 u}{\partial n_{it}^2} = u_{nn} = \omega \gamma n_{it}^{\gamma-1} \tag{110}$$

$$\frac{\partial^2 u}{\partial c_{it} \partial n_{it}} = u_{cn} = 0 \tag{111}$$

where equations (107) and (108) indicate positive but decreasing marginal utility for consumption. Equations (109) and (110) indicate increasing disutility from labor. Overall concavity of the utility function is ensured since the hessian H

$$H = \begin{bmatrix} u_{cc} & u_{cn} \\ u_{nc} & u_{nn} \end{bmatrix} \tag{112}$$

is negative semi-definite. Finally, the first-order condition of leisure demand implies the optimal ratio of consumption and leisure

$$(1-\tau_L) w_t h_{it} u_c = u_l$$

$$c_{it} = \frac{(1-\tau_L) w_t h_{it}}{\omega n_{it}^{\gamma}}$$

which enables us to invert the marginal utility for consumption to obtain an expression for leisure

$$n_{it} = \left(u_c \frac{(1 - \tau_L) \, w_t h_{it}}{\omega} \right)^{\frac{1}{\gamma}}. \tag{113}$$

Finally, we want to characterize the Frisch elasticity, the elasticity of intertemporal substitution (EIS), and the relative risk aversion of the preferences.

$$\text{Frisch} = \frac{\partial \ln n_{it}}{\partial \ln w} = \frac{1}{\gamma} \tag{114}$$

defines the Frisch elasticity of labor supply. Moreover, the elasticity of intertemporal substitution is

$$\text{EIS} = \frac{\partial \ln c_{it}}{\partial \ln \lambda} = 1 \tag{115}$$

Finally, the relative risk-aversion (RRA) follows a standard definition and gives

$$\text{RRA} = -c \frac{u_{cc}}{u_c} = 1. \tag{116}$$