Bonn Econon Discussion Papers

Discussion Paper 02/2010

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by

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March 2010

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Financial support by the
Deutsche Forschungsgemeinschaft (DFG)
through the
Bonn Graduate School of Economics (BGSE)
is gratefully acknowledged.

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Market Structure and Matching with Contracts *

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Abstract

Ostrovsky [10] develops a theory of stability for a model of matching in exogenously given networks. For this model a generalization of pairwise stability, *chain stability*, can always be satisfied as long as agents' preferences satisfy same side substitutability and cross side complementarity. Given this preference domain I analyze the interplay between properties of the network structure and (cooperative) solution concepts. The main structural condition is an acyclicity notion that rules out the implementation of trading cycles. It is shown that this condition and the restriction that no pair of agents can sign more than one contract with each other are jointly necessary and sufficient for (i) the equivalence of group and chain stability, (ii) the core stability of chain stable networks, (iii) the efficiency of chain stable networks, (iv) the existence of a group stable network, and (v) the existence of an efficient and individually stable network. These equivalences also provide a rationale for chain stability in the unrestricted model. The (more restrictive) conditions under which chain stability coincides with the core are also characterized.

JEL classification: C71; C78; D85

Keywords: Matching with Contracts; Network Structure; Chain Stability; Acyclicity; Group Stability; Core; Efficiency

*This research was initiated while I was a Visiting Fellow at Harvard University. Its hospitality and financial support by the German Academic Exchange Agency (DAAD) as well as the Deutsche Forschungsgesellschaft (DFG) are gratefully acknowledged. I am indebted to Benny Moldovanu for invaluable guidance and support. I have greatly benefitted from discussions with and comments from Al Roth during my stay at Harvard and beyond. An associate editor and one anonymous referee provided very helpful suggestions for improving the quality of this paper. I would like to thank Peter Coles, Lars Ehlers, Itay Fainmesser, Thomas Gall, Fuhito Kojima, Konrad Mierendorff, Michael Ostrovsky, participants of the Social Choice and Welfare Meeting 2009 in Montreal, as well as seminar audiences in Bonn and Montreal for helpful comments and suggestions. Any remaining errors are, of course, my responsibility.

1 Introduction

Theoretical models of network formation and matching markets are concerned with predicting which outcomes are likely to emerge when self-interested agents interact. An important strand of this literature belongs to the area of cooperative game theory and "likely outcomes" are not defined by writing down an explicit negotiation protocol, but rather by postulating a set of stability constraints that one perceives to be relevant in the problem under study. In several cases such constraints have been an important guideline for the design of real-life mechanisms for *two-sided matching problems* in which a group of workers/students has to be assigned among a set of firms/schools (see [13] and the references therein). The literature has focused on the *pairwise stability* concept, which only considers the possibility of coordinated deviations by pairs of players. As long as workers can take at most one job and firms have substitutable preferences, a pairwise stable matching not only exists ([7]), but is also *group stable* ([14]): There is no group of agents who can obtain a strictly preferred matching by forming new partnerships only among themselves, possibly dropping some previously held partnerships. In particular, a pairwise stable matching is *efficient*. While these are encouraging results for a restricted class of assignment problems, many interesting applications do not fit the assumptions above: Some workers may demand multiple jobs in a labor market,[1] firms may not view workers as substitutes,[2] and markets are often not two-sided.[3]

Ostrovsky [10] shows how some of these features can be accommodated by a model in which agents are located in an exogenously given vertically ordered directed network and have preferences over sets of trading relationships, or *contracts*, with their neighbors. A set of contracts is *chain stable* if (i) it is *individually stable* in the sense that no agent would prefer to drop some of her contracts, and (ii) there is no downstream sequence of agents who can obtain a strictly preferred set of contracts by forming new contracts only with their direct neighbors in the sequence, possibly dropping some of their previously held contracts. Ostrovsky shows that chain stable outcomes exist as long as agents' preferences satisfy *same side substitutability* and *cross side complementarity*. However, unlike pairwise stable matchings in the simple matching models above, chain stable allocations may not be group stable. In fact, chain stable outcomes may even be inefficient and thus fail to be in the *core*.[4] I characterize the conditions under which these problems cannot occur.

A main contribution of this study is methodological. Instead of introducing further restrictions on preferences, I introduce restrictions on *who can contract with whom* and

[1]For example, Echenique and Oviedo [3] mention that 35 % of teachers in Argentina work for more than one school.

[2]Many tasks can only be accomplished by the combined workforce of a set of specialized workers. The construction of a building, for example, requires a structural engineer, a carpenter, and so on, so that complementarities between individual workers are likely.

[3]Brokers act as intermediaries between owners and potential tenants in housing markets, temporary employment agencies supply firms with short-term labor, some stores (e.g. Gamestop) allow customers to trade in used goods which they then sell to other customers, and so on.

[4]An outcome is in the *core* (defined by weak domination), if no group of agents can obtain a weakly preferred outcome for all involved by trading only among themselves.

how many relationships agents can form. Such constraints are common in two-sided matching models, where an agent cannot contract with another agent on her side of the market and it is often assumed that agents on one side of the market can all engage in at most one relationship. The restrictions that I develop can be interpreted as restricting the sets of acceptable contracts. However, unlike the usual preference restrictions in the matching literature, such as responsiveness or strong substitutability, my conditions do not place further restrictions on the ranking of acceptable (or unacceptable) sets of contracts. The main structural restriction in the paper is an acyclicity notion which rules out the implementation of trading cycles. I show that this condition and the restriction that no pair of agents can sign more than one contract with each other are jointly necessary and sufficient for (i) the equivalence of group and chain stability, (ii) the core stability of chain stable outcomes, (iii) the efficiency of chain stable outcomes, (iv) the existence of a group stable outcome, and (v) the existence of an efficient and individually stable outcome. The equivalences provide two justifications for the use of chain stability in the unrestricted model: First, whenever the minimal stability requirement of individual stability can be reconciled with efficiency, chain stable outcomes are also efficient. Thus, imposing the stronger chain stability concept does not lead to any additional efficiency loss. Second, if chain stable networks fail to be group stable, the very existence of a group stable outcome cannot be guaranteed. In this sense chain stable allocations are *as stable as it gets*. In the last part of the paper I characterize the (more restrictive) conditions under which core and chain stability coincide. This result subsumes a number of important existing results and provides a unified perspective on them.

From a methodological perspective, the paper most closely related to the present study is Abeledo and Isaak [1]. They start from a fixed structure of potential partnerships in a simple one-to-one matching model represented by an undirected graph that contains edges between mutually acceptable pairs of agents. Their main result is that a pairwise stable matching will exist for all preference profiles if and only if the market is two-sided. In this paper I restrict attention to the model introduced by Ostrovsky [10] for which the existence of a chain stable allocation is guaranteed. In contrast to [1], my focus is the relationship between cooperative solution concepts. Furthermore, their methodology has no direct extension to the model I consider where the set of acceptable allocations cannot be summarized by a simple undirected graph.

More closely related in focus is a line of research that has been concerned with stability concepts for two-sided many-to-many matching markets. If all preferences are substitutable, this model is a special case of the supply chain model so that existence of a pairwise stable allocation follows from the existence result in [10] (for this special case, existence had been previously established in [12]). Blair [2] was the first to note that in such markets the core can be empty. This implies in particular that (i) group stable allocations may fail to exist, and (ii) the set of pairwise stable allocations may be disjoint from both, the core and the set of group stable allocations. In light of these problems, most studies have focused on stability concepts which limit the set of allowable coalitional deviations. Important examples of this line of research are Roth [12], who considers the restriction that all members of a deviating coalition should obtain a subset of their most preferred set of contracts out of the set of previously held and newly formed contracts, Konishi and Ünver [9], who require that deviations have to be pairwise stable themselves, and Echenique and Oviedo [3], who consider the restriction that deviations have to be individually stable in the sense that no deviating agent wants to drop some of her partners after the deviation. There are two important differences between this line of research and my study: First, all these papers study stability notions that restrict the set of allowable coalitional deviations. A problem with this approach is that none of these concepts guarantee efficiency of the outcome. This leaves open the question whether there are other natural stability concepts compatible with efficiency. The results of my paper apply equally well to many-to-many matching models and, to the best of my knowledge, this is the first systematic study of the relationship between pairwise stability, group stability, the core, and efficiency. A second difference is that most of the papers in the above line of research introduce stronger restrictions on preferences than those needed to guarantee existence of a pairwise stable allocation and then show that for the restricted model pairwise stability coincides with some (restricted) notion of group stability. While this yields insightful foundations for pairwise stability in appropriately restricted models, it does not explain why it is the right stability criterion when all substitutable preferences are allowed.

Other papers that have studied the importance of the set of allowed potential interactions for cooperative solution concepts in other contexts include Pápai [11], who analyzes how to restrict allowable trades in a general indivisible goods exchange market in order to guarantee a singleton core, and Kalai, Postlewaite and Roberts [6], who compare core outcomes between an unrestricted market game and a game in which some players are not allowed to form coalitions.

The remainder of this paper is organized as follows: Section 2 briefly summarizes Ostrovsky's supply chain model and introduces the solution concepts studied. Section 3 contains all main results of this paper and section 4 concludes. The appendix contains an omitted proof and a discussion of the main results.

2 The Supply Chain Model

Consider a market consisting of a finite set of agents V. Agents trade discrete units of indivisible goods and trading relationships are represented by bilateral *contracts*. Each contract is of the form (s, b, a, p) and represents the sale of one (unit of a) *good* $a \in \mathbb{N}$ from seller $s \in V$ to buyer $b \in V$ at a *price* $p \in \mathbb{R}$.[5] The set of all possible contracts, denoted by X, is assumed to be exogenously given and finite. For $x \in X$, let s_x denote the seller in contract x and let b_x denote the buyer in contract x. It is assumed that there are no *directed trading cycles in* X, that is, there is no sequence of agents $v_1, \dots, v_m$ such that, for all $i \in \{1, \dots, m\}$, there exists a contract x_i such that $s_{x_i} = v_i$ and $b_{x_i} = v_{i+1}$ (where $m + 1 := 1$).[6] For future reference I now introduce some more terminology and

[5]This formulation of contracts follows [10] and is chosen for concreteness. My results do not depend on the exact nature of the set of contracts apart from the assumption that each contract is bilateral. For example, a labor market contract could specify wage, days of leave, retirement plans, and so on.

[6]This assumption corresponds to Ostrovsky's assumption in [10] that agents are located in a vertically ordered network.

notation: Agent $v \in V$ is *involved with contract* $x \in X$, if either $s_x = v$ or $b_x = v$. A contract in which agent v is the seller (buyer) is called a *downstream (upstream) contract* for v. Given a set of contracts $Y \subseteq X$, let $D_v(Y)$ denote the set of contracts (in Y) in which v is a seller, $U_v(Y)$ denote the set of contracts in which v is a buyer, and $Y(v)$ denote the set of all contracts that v is involved with. Agent $w \in V$ is a *downstream (upstream) agent relative to agent* $v \in V$, if there is a contract $x \in X(v)$ such that $b_x = w$ ($s_x = w$). An agent $v \in V$ with $U_v(X) \neq \emptyset$ and $D_v(X) \neq \emptyset$ is an *intermediary*. For each pair $v, w \in V$, $X(v, w)$ denotes the set of all possible contracts between v and w.

2.1 Preferences

Agents care only about the contracts they are directly involved with and are never indifferent between two distinct subsets of $X(v)$. Formally, $v \in V$ has a weak preference relation (i.e. a reflexive, transitive, and complete binary relation) R_v on 2^X such that $Y I_v Z$ for two subsets $Y, Z \in 2^X$ if and only if $Y(v) = Z(v)$. A set of contracts Y is *acceptable* (according to R_v) if $Y(v) R_v \emptyset$. Given a set of contracts $Y \subseteq X$ and a strict preference relation R_v, $Ch_v(Y)$ denotes v's most preferred subset of Y, v's *choice* from Y. Formally, it is defined by $Ch_v(Y) P_v Z$ for any $Z \subseteq Y$ with $Z \neq Ch_v(Y)$. The next two restrictions concern the choices of agents from various sets of contracts and were introduced in [10].

(i) Whenever some downstream (upstream) contract becomes unavailable, v does not reduce her demand for any still available downstream (upstream) contract. More formally, let $Y \subseteq X$ and either $\{x, x'\} \subseteq D_v(Y)$ or $\{x, x'\} \subseteq U_v(Y)$. Then $x \in Ch_v(Y)$ implies that also $x \in Ch_v(Y \setminus \{x'\})$ (**Same Side Substitutability**)

(ii) If an additional upstream (downstream) contract becomes available to an agent $v \in V$, she does not reduce her demand for any downstream (upstream) contract. More formally, let $Y \subseteq X$ and either $x \in D_v(Y)$ and $x' \in U_v(X) \setminus Y$, or $x \in U_v(Y)$ and $x' \in D_v(X) \setminus Y$. Then $x \in Ch_v(Y)$ implies that also $x \in Ch_v(Y \cup \{x'\})$ (**Cross Side Complementarity**)

A preference profile R is *admissible*, if it satisfies all of the assumptions above and $\mathcal{R}$ denotes the set of all admissible profiles. Note that in a supply chain model without intermediaries CSC is vacuously satisfied so that such a model reduces to a many-to-many two-sided matching model with *substitutable* preferences as studied in e.g. [12].

2.2 Networks and Solution Concepts

Given a preference profile in the domain introduced above, the aim of a supply chain model is to predict which contracts will be signed by the agents. In the supply chain model the relevant outcomes are sets of contracts, or *networks*. Networks will usually be denoted by μ and agent v's set of contracts under μ will be denoted by $\mu(v)$. Predictions take the form of (cooperative) solution concepts which require a network to be robust against certain deviations of individuals or groups.

A network is *individually rational* if no agent is assigned an unacceptable set of contracts. This assumes that if an individual wanted to deviate she has to discontinue all of her existing relationships. A network μ is *individually stable* if no agent v wants to drop some of her contracts in $\mu(v)$, that is, $Ch_v(\mu(v)) = \mu(v)$ for all $v \in V$. In contrast to individual rationality, individual stability thus allows an individual to delete some but also to keep other contracts. Next, I consider stability notions that rule out coordinated deviations by groups of agents.

The *core* (defined by weak domination) considers deviations by arbitrary groups of players, but deviating agents are not allowed to maintain existing relationships with "outsiders". More formally, a network μ' *weakly dominates network* μ *via coalition* $A \subseteq V$ if (i) $x \in \mu'$ implies that either $\{s_x, b_x\} \subseteq A$ or $\{s_x, b_x\} \cap A = \emptyset$, and (ii) $\mu'(a) R_a \mu(a)$ for all $a \in A$ with at least one strict preference. A network μ is in the core, or *core stable*, if it is not weakly dominated by any other network.[7] *Group stability*, on the other hand, considers any deviation that a coalition can implement by forming new contracts only among themselves while possibly dropping some previously held contracts. Coalition $A \subseteq V$ can *obtain* μ' *from* μ if $x \in \mu' \setminus \mu$ implies that $\{s_x, b_x\} \subseteq A$ and $x \in \mu \setminus \mu'$ implies that $\{s_x, b_x\} \cap A \neq \emptyset$. Network μ is *blocked by coalition* A via network μ' if (i) A can obtain μ' from μ, and (ii) $\mu'(a) P_a \mu(a)$ for all $a \in A$. A network is *group stable* if it is not blocked by any coalition. Note that group stability is a stronger solution concept than the core. Furthermore, note that if μ is core stable given some preference profile R then it is also *efficient*, that is, there is no other network μ' that makes all agents weakly and at least one agent strictly better off compared to μ.

A major problem is that group and core stable networks can fail to exist even under quite restrictive assumptions about preferences.[8] Thus, in order to guarantee existence the set of coalitional deviations has to be restricted. Ostrovsky [10] introduces a new stability criterion which generalizes the idea of pairwise stability in the sense that it considers (some) coordinated deviations by downstream sequences of agents instead of only deviations by pairs. A *chain* is a downstream sequence of contracts $\{x_1, \ldots, x_n\} \subseteq X$ such that for all $i < n$, $b_{x_i} = s_{x_{i+1}}$. Network μ is *blocked by the chain* $x_1, \ldots, x_n \notin \mu$ if (i) $x_1 \in Ch_{s_{x_1}}(\mu(s_{x_1}) \cup \{x_1\})$, (ii) $\{x_i, x_{i+1}\} \subseteq Ch_{s_{x_{i+1}}}(\mu(s_{x_{i+1}}) \cup \{x_i, x_{i+1}\})$ for all $i < n$, and (iii) $x_n \in Ch_{b_{x_n}}(\mu(b_{x_n}) \cup \{x_n\})$. A network μ is *chain stable*, if it is individually stable and if it is not blocked by any chain. One of the main results in [10] is that chain stable networks exist for all profiles in the domain $\mathcal{R}$ introduced in 2.1. Note that if there are no intermediaries chain stability reduces to *pairwise stability* so that his result generalizes the existence results from two-sided matching models with substitutable preferences.

Given the existence result, chain stability is a natural candidate for extending the theory of stable matchings to the supply chain setting. This concept, however, has two potential problems: First, requiring robustness against any possible chain block in complex supply chain models means that coordinated deviations by large groups of agents

[7] Analogously, one can define the core by strong domination by requiring that all members of the coalition A have to be strictly better off. Roth and Sotomayor [14, Ch. 5] show that even in many-to-one matching markets with responsive preferences, this core concept allows for matchings that are not pairwise stable.

[8] See e.g. Klaus and Walzl [8] and Konishi and Unver [9], who show that even when preferences are strongly substitutable and responsive, respectively, there may not exist a group or core stable network/matching in a two-sided matching model that is a special case of the supply chain model.

are thought to be possible. But a "small" coalition of agents might have a profitable deviation from a chain stable network that can only be implemented by e.g. a cyclical sequence of trades. It is not clear why robustness against such a deviation is irrelevant while all possible chain blocks have to be considered. Second, requiring robustness against chain blocks may come at the expense of efficiency. If some of the deviations considered by chain stability are implausible due to e.g. the size of the coalitions involved, less demanding stability concepts may reduce the efficiency loss while still being satisfactorily robust. In short, a foundation for the use of chain stability in the general supply chain model is needed.

3 Results

This section develops conditions under which chain stable networks are efficient as well as core and group stable. Furthermore, I provide two justifications for the use of the chain stability concept in the general supply chain model.

For the analysis it is useful to work with a graphical representation of the supply chain model. Let G_X be the simple directed graph that contains an edge from agent v to agent w if and only if X contains some contract x with $s_x = v$ and $b_x = w$. This is the *graph of potential interactions* that describes who can contract with whom. Note that even though there may be more than one possible contract between a pair of agents, G_X contains at most one edge between each pair of agents. The assumption that there are no directed trading cycles in X is equivalent to the assumption that G_X contains no *directed cycles*, that is, there is no sequence of agents $v_1, \ldots, v_m$ such that, for all $i \in \{1, \ldots, m\}$, $(v_i, v_{i+1}) \in G_X$ (where $m + 1 := 1$).

I restrict attention to supply chain models in which agents face fixed upper bounds on the number of contracts they can sign: Agent v can sign contracts with at most q_v^D downstream agents and at most q_v^U upstream agents. For example, an agent who owns k indivisible goods and whose only interest is to sell these goods can sign contracts with at most k agents. Furthermore, an arbitrary pair of agents $v, w \in V$ can sign at most $q_{v,w} \leq |X(v, w)|$ contracts with each other, where $q_{v,w} = q_{w,v}$. For example, $q_{v,w}$ could represent the capacity of the unique distribution channel between v and w. Let q be the vector containing all individual and pair capacity constraints. Given V and X, the *unrestricted model* is obtained by setting $q_{v,w} = q_v^U = q_v^D = |X|$, for all $v, w \in V$. The next definition connects capacity constraints and agents' preferences.

Definition 1 *A set of contracts $Y \subseteq X$* ***violates v's capacity constraints****, if $Y(v)$ either contains contracts with more than q_v^U upstream or more than q_v^D downstream agents relative to v, or if there is an agent $w \in V \setminus \{v\}$ such that $Y(v)$ contains more than $q_{v,w}$ contracts between v and w.*

Preferences ***conform to capacities*** *if no agent ever finds a set of contracts acceptable that violates one or more of her capacity constraints.*

As I discuss in the appendix one could place feasibility restrictions on the set of networks instead of requiring preferences to conform to the capacity vector. For the following, $\mathcal{R}_q \subseteq \mathcal{R}$ denotes the set of all admissible preference profiles conforming to the capacity vector q. Since all solution concepts I consider satisfy individual rationality, it is without loss of generality to assume that if $(v, w) \in G_X$ then $q_v^D \geq 1$, $q_w^U \geq 1$, and $q_{v,w} \geq 1$: Otherwise some contracts in X would play no role for the analysis and could be deleted from the problem.

The pair (G_X, q) is the *market structure* induced by the supply chain model (V, X) and the capacity constraints q. Note that the assumptions made so far ruled out only directed, but not *undirected cycles* in G_X, where an undirected cycle in G_X is a sequence of distinct agents $v_1, \ldots, v_m$ such that, for all $i \in \{1, \ldots, m\}$, either $(v_i, v_{i+1}) \in G_X$ or $(v_{i+1}, v_i) \in G_X$ (where $m + 1 := 1$). It turns out that the (im)possibility of actually implementing such a cycle of trading relationships is key to the relationship between the solution concepts studied in this paper. The following definition introduces the notion of capacity constraints on cycles.

Definition 2 *Let $v_1, \ldots, v_m$ be an undirected cycle in G_X. Agent v_i is*

(i) a ***source (of the cycle)*** *if $\{(v_i, v_{i+1}), (v_i, v_{i-1})\} \subset G_X$,*

(ii) a ***sink*** *if $\{(v_{i-1}, v_i), (v_{i+1}, v_i)\} \subset G_X$, and*

(iii) a ***passing node*** *if $\{(v_{i-1}, v_i), (v_i, v_{i+1})\} \subset G_X$.*

Agent v_i is ***capacity constrained*** *on the cycle $v_1, \ldots, v_m$ if either v_i is a source and $q_{v_i}^D = 1$, or v_i is a sink and $q_{v_i}^U = 1$.*

Note that since G_X is assumed not to include directed cycles, any undirected cycle in G_X must contain at least one source and at least one sink. An undirected cycle may or may not contain (multiple) intermediaries as well as multiple sources and/or sinks. The above definition does not consider the possibility of a capacity constrained passing node on a cycle. The reason is that such a capacity constraint would have to require that the agent can either sign no upstream or/and no downstream contracts, effectively rendering some of the edges in G_X irrelevant.[9] Now, in order to guarantee that trading cycles cannot be implemented, each cycle in G_X has to contain at least one capacity constrained agent, who cannot engage in more than one trading relationship. As I show below this restriction by itself is not sufficient for the equivalence of chain and group stability since a pair of agents might still have an incentive to block a chain stable network using a set of two contracts with each other. Thus, we also have to require that each pair of agents is capacity constrained in the sense that they can sign at most one contract with each other. The following definition summarizes these requirements.

Definition 3 *The market structure (G_X, q) satisfies* ***weak acyclicity*** *if every undirected cycle in G_X contains at least one capacity constrained agent, and it satisfies* ***bundling*** *if $q_{v,w} = 1$ for all $v, w \in V$.*

Note that bundling does not require that an agent can sell/buy at most one good to/from any given neighbor. Rather, it requires that all trading relationships between a given pair of agents can be bundled into one contract. To state the main result, I need the following additional notation: Given a preference profile $R \in \mathcal{R}_q$, let $\mathcal{CS}(R)$ denote the

[9] I thank an anonymous referee for pointing this out.

set of all chain stable networks, $\mathcal{IS}(R)$ denote the set of all individually stable networks, $\mathcal{GS}(R)$ denote the set of all group stable networks, $\mathcal{C}(R)$ denote the core, and $\mathcal{E}(R)$ denote the set of all efficient networks. We have the following.

Theorem 1 *The following are equivalent:*

(i) The market structure (G_X, q) satisfies weak acyclicity and bundling.

(ii) Chain stable networks are always group stable, that is, $\mathcal{CS}(R) = \mathcal{GS}(R)$ for all $R \in \mathcal{R}_q$.

(iii) Chain stable networks are always in the core, that is, $\mathcal{CS}(R) \subseteq \mathcal{C}(R)$ for all $R \in \mathcal{R}_q$.

(iv) Chain stable networks are always efficient, that is, $\mathcal{CS}(R) \subseteq \mathcal{E}(R)$ for all $R \in \mathcal{R}_q$.

(v) A group stable network always exists, that is, $\mathcal{GS}(R) \neq \emptyset$ for all $R \in \mathcal{R}_q$.

(vi) An efficient and individually stable network always exists, that is, $\mathcal{E}(R) \cap \mathcal{IS}(R) \neq \emptyset$ for all $R \in \mathcal{R}_q$.

Proof of Theorem 1:

The implications $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$, $(iv) \Rightarrow (vi)$, and $(v) \Rightarrow (vi)$ follow immediately from the definitions. That $(ii) \Rightarrow (v)$ follows directly from Ostrovsky [10]'s result that chain stable networks exist for all profiles in $\mathcal{R}$. I now show the two non-obvious implications $(i) \Rightarrow (ii)$ and $(vi) \Rightarrow (i)$, which completes the proof of Theorem 1.

$(i) \Rightarrow (ii)$ For agent $v \in V$ and a contract $x \in X(v)$, let $S_{x,v}$ denote the direction of the contract relative to v, that is, $S_{x,v} = U_v$ if and only if x is an upstream contract for v. Let $\overline{S}_{x,v}$ denote the complementary direction, that is, $\overline{S}_{x,v} = D_v$ if and only if $S_{x,v} = U_v$.

The proof is by contradiction: Suppose that (G_X, q) satisfies weak acyclicity and bundling but for some preference profile $R \in \mathcal{R}_q$ there exists a network $\mu \in \mathcal{CS}(R) \setminus \mathcal{GS}(R)$. By the definition of group stability there must then be a coalition A that blocks μ via network μ'. I show that $\mu' \setminus \mu$ must contain a blocking chain of μ. The following algorithm will be key to the proof.

Step 1: Let $v_1 \in A$ be arbitrary.

1.1 If $x \in Ch_{v_1}(\mu(v_1) \cup \{x\})$ for some $x \in \mu'(v_1) \setminus \mu(v_1)$, set $x_1 = y_1 = x$, $B_1 = \emptyset$ and let $v_2 \neq v_1$ be such that $\{v_1, v_2\} = \{s_{x_1}, b_{x_1}\}$.

1.2 Else, let $x \in \mu'(v_1) \setminus \mu(v_1)$ and $y \in \overline{S}_{x,v_1}(\mu'(v_1) \setminus \mu(v_1))$ be such that $\{x, y\} \subseteq Ch_{v_1}(\mu(v_1) \cup \{x, y\})$. Set $x_1 = x$, $y_1 = y$, $B_1 = \{v_1\}$ and let $v_2 \neq v_1$ be such that $\{v_1, v_2\} = \{s_{x_1}, b_{x_1}\}$.

⋮

Step k: **k.1** If $x_{k-1} \in Ch_{v_k}(\mu(v_k) \cup \{x_{k-1}\})$ and $B_{k-1} = \emptyset$, stop.

k.2 If $x_{k-1} \in Ch_{v_k}(\mu(v_k) \cup \{x_{k-1}\})$ and $B_{k-1} = \{v_l\}$ for some $l < k$, set $x_k = y_k = y_l$, $B_k = \emptyset$ and let $v_{k+1} \neq v_l$ be such that $\{v_l, v_{k+1}\} = \{s_{x_k}, b_{x_k}\}$.

k.3 If $x_{k-1} \notin Ch_{v_k}(\mu(v_k) \cup \{x_{k-1}\})$ and $x \in Ch_{v_k}(\mu(v_k) \cup \{x\})$ for some $x \in \mu'(v_k) \setminus \mu(v_k)$ set $x_k = y_k = x$, $B_k = \emptyset$, and let $v_{k+1} \neq v_k$ be such that $\{v_k, v_{k+1}\} = \{s_{x_k}, b_{x_k}\}$.

k.4 If $x \notin Ch_{v_k}(\mu(v_k) \cup \{x\})$ for all $x \in \mu'(v_k) \setminus \mu(v_k)$ and $x_{k-1} \in Ch_{v_k}(\mu(v_k) \cup \mu'(v_k))$, let $x \in \overline{S}_{x_{k-1},v_k}(\mu'(v_k) \setminus \mu(v_k))$ be a contract such that $\{x_{k-1}, x\} \subseteq Ch_{v_k}(\mu(v_k) \cup \{x_{k-1}, x\})$. Set $x_k = y_k = x$, $B_k = B_{k-1}$ and let $v_{k+1} \neq v_k$ be such that $\{v_k, v_{k+1}\} = \{s_{x_k}, b_{x_k}\}$.

k.5 If $x \notin Ch_{v_k}(\mu(v_k) \cup \{x\})$ for all $x \in \mu'(v_k) \setminus \mu(v_k)$ and $x_{k-1} \notin Ch_{v_k}(\mu(v_k) \cup \mu'(v_k))$, let $x \in \mu'(v_k) \setminus \mu(v_k)$ and $y \in \overline{S}_{x,v_k}(\mu'(v_k) \setminus \mu(v_k))$ be such that $\{x, y\} \subseteq Ch_{v_k}(\mu(v_k) \cup \{x, y\})$. Set $x_k = x$, $y_k = y$, $B_k = \{v_k\}$ and let $v_{k+1} \neq v_k$ be such that $\{v_k, v_{k+1}\} = \{s_{x_k}, b_{x_k}\}$.

$\vdots$

In the algorithm the sequence $\{B_k\}_{k \geq 1}$ contains agents who are marked for later processing. Note that B_k is either empty, or contains exactly one agent. I now show that the algorithm is well defined. This requires me to establish that the case distinction made by the algorithm is exhaustive and that it ends after a finite number of steps.

I show the first statement via induction on the number of steps k, starting with the induction base $k = 1$. Suppose that case 1.1 does not apply. I need to show that case 1.2 applies. Note that since $v_1 \in A$, we must have $\mu'(v_1) P_{v_1} \mu(v_1)$ by the group stability definition of blocking coalitions. Since $\mu \in \mathcal{CS}(R) \subseteq \mathcal{IS}(R)$ this implies that there exists a contract $x \in Ch_{v_1}(\mu(v_1) \cup \mu'(v_1)) \setminus \mu(v_1)$. By SSS, we must have $x \in Ch_{v_1}[\mu(v_1) \cup \{x\} \cup \overline{S}_{x,v_1}(\mu'(v_1) \setminus \mu(v_1))]$. If $Ch_{v_1}[\mu(v_1) \cup \{x\} \cup \overline{S}_{x,v_1}(\mu'(v_1) \setminus \mu(v_1))] \setminus (\mu(v_1) \cup \{x\}) = \emptyset$, we would obtain that $Ch_{v_1}[\mu(v_1) \cup \{x\} \cup \overline{S}_{x,v_1}(\mu'(v_1) \setminus \mu(v_1))] \subseteq \mu(v_1) \cup \{x\}$. By revealed preference, this would imply $x \in Ch_{v_1}[\mu(v_1) \cup \{x\} \cup \overline{S}_{x,v_1}(\mu'(v_1) \setminus \mu(v_1))] = Ch_{v_1}(\mu(v_1) \cup \{x\})$ so that case 1.1 would apply - a contradiction. So let $y \in Ch_{v_1}[\mu(v_1) \cup \{x\} \cup \overline{S}_{x,v_1}(\mu'(v_1) \setminus \mu(v_1))]$ be arbitrary and note that, again by SSS, we must have $y \in Ch_{v_1}(\mu(v_1) \cup \{x, y\})$. If $x \notin Ch_{v_1}(\mu(v_1) \cup \{x, y\})$, we would have, again by revealed preference, $Ch_{v_1}(\mu(v_1) \cup \{x, y\}) = Ch_{v_1}(\mu(v_1) \cup \{y\})$ yielding another contradiction to the assumption that case 1.1 did not apply. This completes the proof for the induction base. Now suppose the statement is true for all $l \leq k' - 1$ for some $k' \geq 2$ and suppose that cases $k'.1 - k'.3$ do not apply. Note that since $x_{k'-1} \in \mu'(v_{k'}) \setminus \mu(v_{k'})$, we must have $v_{k'} \in A$ as well. Suppose first that $x_{k'-1} \in Ch_{v_{k'}}(\mu(v_{k'}) \cup \mu'(v_{k'}))$. I need to show that there exists some contract $x \in \overline{S}_{x_{k'-1},v_{k'}}(\mu'(v_{k'}) \setminus \mu(v_{k'}))$ such that $\{x_{k'-1}, x\} \subseteq Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$. Since $x_{k'-1} \in Ch_{v_{k'}}(\mu(v_{k'}) \cup \mu'(v_{k'})) \setminus \mu(v_{k'})$, SSS implies $x_{k'-1} \in Ch_{v_{k'}}[\mu(v_{k'}) \cup \{x_{k'-1}\} \cup \overline{S}_{x_{k'-1},v_{k'}}(\mu'(v_{k'}) \setminus \mu(v_{k'}))]$. Since cases $k'.1 - k'.3$ do not apply, there cannot be a contract $x \in \mu'(v_{k'}) \setminus (\mu(v_{k'}) \cup \{x_{k'-1}\})$ such that either (i) $x \in Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$ and $x_{k'-1} \notin Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$, or (ii) $x \notin Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$ and $x_{k'-1} \in Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$. Now if $x \in Ch_{v_{k'}}[\mu(v_{k'}) \cup \{x_{k'-1}\} \cup \overline{S}_{x_{k'-1},v_{k'}}(\mu'(v_{k'}) \setminus \mu(v_{k'}))] \setminus (\mu(v_{k'}) \cup \{x_{k'-1}\})$, SSS would imply that $x \in Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$ and hence $\{x_{k'-1}, x\} \subseteq Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}, x\})$ by the above. If there was no such contract, we would necessarily obtain $Ch_{v_{k'}}(\mu(v_{k'}) \cup \{x_{k'-1}\}) = Ch_{v_{k'}}[\mu(v_{k'}) \cup \{x_{k'-1}\} \cup \overline{S}_{x_{k'-1},v_{k'}}(\mu'(v_{k'}) \setminus \mu(v_{k'}))]$. Since $x_{k'-1}$ belongs to the set on the right hand side, we obtain a contradiction to the assumption that cases $k'.1 - k'.3$ do not apply. Showing that case $k'.5$ holds provided that $k'.1 - k'.4$ do not apply amounts to repeating the arguments used for the induction base. The details are omitted.

Next, I show that the algorithm must terminate after a finite number of steps. Suppose to the contrary that this is not the case. Then, since the set of agents is finite there must exist indices k and l such that $k < l$ and $v = v_k = v_l$. I can assume w.l.o.g. that all agents between v_k and v_l are different. Since all agents considered by the algorithm are part of the blocking coalition A we must have $\mu'(v_j) P_{v_j} \mu(v_j) R_{v_j} \emptyset$ for all $j \in \{k, \ldots, l-1\}$. Note that we must have $l \geq k+2$. If $l \geq k+3$, $v_k, \ldots, v_{l-1}$ is a cycle in G_X. Since $\{x_{k+1}, \ldots, x_{l-1}\} \subseteq \mu'$ no agent on the cycle is capacity constrained so that the market structure cannot satisfy weak acyclicity. If $l = k+2$, note that since the algorithm does not stop in step $k+1$, we must have $x_{k+1} \in \overline{S}_{x_k, v_{k+1}}(\mu'(v_{k+1}) \setminus \mu(v_{k+1}))$ so that in particular $x_k \neq x_{k+1}$. But then $|\mu \cap X(v_k, v_{k+1})| \geq 2$ and, since preferences conform to capacities, $q_{v_k, v_{k+1}} \geq 2$, which contradicts the assumption that bundling is satisfied.

So let $K < \infty$ be the last step of the above algorithm and let $\{x_k, y_k, B_k\}_{k=1}^{K-1}$ be the corresponding sequences of contracts and stacks for later processing. I now show how to find a blocking chain of μ within $\{x_k\}_{k=1}^{K-1}$. By definition of K, $x_{K-1} \in Ch_{v_K}(\mu(v_K) \cup \{x_{K-1}\})$ and $B_{K-1} = \emptyset$. Let L be the last step $k \leq K$ for which case k.2 applies, that is, $B_L = \{v_M\}$ for some $M < L$ and $B_k = \emptyset$ for all $k \geq L+1$. Here I use the convention that $L = 1, M = 0$, and $x_0 = x_1$ if $B_k = \emptyset$ for all $k \leq K-1$. Suppose first that there is no step $k > L$ such that case k.3 applied. Then case k.4 has to apply for all steps $k \in \{M+1, \ldots, L-1\} \cup \{L+1, \ldots, K-1\}$: If there was a step $k \in \{M+1, \ldots, L\}$ such that one of the cases k.2, k.3, and k.5 applied, the stack would have been either emptied or modified before step L. Since, as shown above, no agent is considered twice by the algorithm this is a contradiction to $B_L = \{v_M\}$. By the definitions of L and K, no agent could have been put on the stack in some step $k \in \{L+1, \ldots, K-1\}$ so that neither case k.2 nor case k.5 could have applied for such k. Now if x_{K-1} is a downstream contract for agent v_K, this implies that $x_{K-1}, x_{K-2}, \ldots, x_L, x_M, x_{M+1}, \ldots, x_{L-1}$ is a blocking chain of μ. Otherwise, the sequence in reverse order is a blocking chain. If on the other hand $k \in \{L+1, \ldots, K-1\}$ is the last step $\geq L+1$ for which $k.3$ applies, the sequence $x_{K-1}, \ldots, x_k$ is a blocking chain for μ (it may again be necessary to reverse the order).

Thus, the above algorithm finds a chain block of μ within $\mu' \setminus \mu$. This contradicts the assumption that μ is chain stable and completes the proof that $(i) \Rightarrow (ii)$.[10]

$(vi) \Rightarrow (i)$ Suppose that either there is a cycle $v_1, \ldots, v_m$ in G_X which does not contain a capacity constrained agent, or that there is a pair v, w such that $q_{v,w} \geq 2$. In the second case, set $v_1 = v$, $v_2 = w$, and $m = 2$. I show that there exists a profile $R \in \mathcal{R}_q$ such that $\mathcal{E}(R) \cap \mathcal{IS}(R) = \emptyset$. Since I can always construct the preference

[10] Note that this proof does not rely on the CSC assumption. In the appendix I discuss which results continue to hold without the CSC assumption.

profile in such a way that none of $v_1, \ldots, v_m$ wants to sign a contract with the outside world and vice versa, I can assume that $V = \{v_1, \ldots, v_m\}$. Let $x_i \in X$ denote some contract between v_i and v_{i+1} for $i \in \{1, \ldots, m\}$, where $x_1 \neq x_2$ if $m = 2$ (remember that $2 \leq q_{v_1,v_2} \leq |X(v_1, v_2)|$ in this case). Agents' preferences are defined as follows: For all $i \in \{1, \ldots, m\}$, Agent v_i strictly prefers signing only contract x_i over signing the set of contracts $\{x_{i-1}, x_i\}$, over signing no contracts at all. These are the only acceptable sets of contracts. It is easy to check that the resulting profile belongs to $\mathcal{R}_q$.[11]

I show that for this profile any individually stable network must assign the empty set of contracts to all agents. By construction of the preference profile, individual stability demands that v_1 is assigned either the empty set of contracts or only x_1. So suppose that there is an individually stable network μ that includes x_1. But v_2 must be assigned either the empty set of contracts or only contract x_2 in an individually stable network. Hence, any individually stable network must assign the empty set of contracts to agent v_1. A simple repetition of this argument establishes that the unique individually stable network is the empty network. But this network is (strictly) Pareto dominated by the complete network. This completes the proof that $(vi) \Rightarrow (i)$.

□

One important special case of the implication $(i) \Rightarrow (ii)$ above is the many-to-one matching model with substitutable preferences. Before proceeding, I now consider two applications of Theorem 1 that have not been covered by the previous literature. The first application uses restrictions on the pattern of connections and the capacity vector, while the second application only introduces restrictions on the capacity vector.

Application 1 (Many-to-One Matching with a central intermediary) *Consider a matching market with a finite set of firms F, a finite set of workers W, and one central intermediary I. All workers can sign at most one contract and work for either a firm or the intermediary. The intermediary and the firms can have an arbitrary number of trading partners but no firm can sign more than one contract with the intermediary. This market structure satisfies bundling as well as weak acyclicity since each undirected cycle must contain at least one worker and each worker is capacity constrained on any cycle. Hence, group and chain stability coincide and, in particular, a group stable network always exists. If there is more than one intermediary, more restrictions on the set of allowable trades and/or capacities are necessary to guarantee weak acyclicity.*

Application 2 (Multiple Inputs - One Output) *Consider an arbitrary supply chain model (V, X). Assume that the capacity vector is such that $q_{v,w} = 1$ for all $v, w \in V$ and that no agent can sign more than one downstream contract, that is, $q_v^D = 1$ for all $v \in V$. This special case of the supply chain model can be thought of as describing a production*

[11] Note that these preferences are actually *same side responsive* in the sense that they are responsive when restricted to either upstream or downstream contracts. Hence, $(vi) \Rightarrow (i)$ continues to hold for this smaller domain of preferences. I thank Lars Ehlers for pointing this out to me.

process in which each agent combines indivisible inputs from (potentially) multiple sources into one indivisible output good, which she sells to one of her customers. The resulting market structure satisfies weak acyclicity since all sources are capacity constrained. Note that in this application only the capacity vector is restricted. This shows in particular that we do not necessarily have to require that the market structure resembles a (sequence of) many-to-one matching markets in order to guarantee the equivalence of chain and group stability. It is clear that instead of restricting downstream capacities one could also restrict upstream capacities of agents to obtain a weakly acyclic market structure.

Theorem 1 is related to the literature on network formation models since the supply chain model is a special case of the general network formation models studied in e.g. [5]. For these models the incompatibility between efficiency and stability is well known. On the other hand, the supply chain model contains the many-to-one matching model as a special case. As mentioned above efficiency and stability are compatible in this model. Theorem 1 thus identifies a point at which the positive results from the two sided matching literature break down and the general incompatibility results from the network formation literature obtain since even the minimal requirement of individual stability cannot in general be reconciled with efficiency.[12] While the above applications show that Theorem 1 extends the domain of models for which chain and group stability coincide, the characterizing conditions for the equivalence are quite restrictive. However, I will now argue that the above equivalences can be interpreted as justifying the use of chain stability also in the unrestricted supply chain model. The following is an immediate corollary of Theorem 1.

Corollary 1 *A group stable network always exists if and only if chain stable networks are always group stable.*

This can be seen as a justification for chain stability from a *robustness perspective*: The only reason for a chain stable network to fail the group stability criterion is that the existence of a group stable network cannot, in general, be guaranteed. In this sense a chain stable network is *as stable to coordinated deviations as it gets*. The following is another immediate corollary of Theorem 1.

Corollary 2 *An efficient and individually stable network always exists if and only if chain stable networks are always efficient.*

This can be seen as a justification for chain stability from an *efficiency perspective*: The only reason for a chain stable network to fail the efficiency criterion is that even the minimal requirement of individual stability cannot, in general, be reconciled with efficiency. This implies that there is *no additional efficiency loss from imposing the stronger chain stability concept*.

It is important to bear in mind that the above results are about solution concepts. As I show in the appendix, the non-trivial of the implications in Theorem 1 do not

[12]Individual stability is often viewed as a minimal stability requirement in the matching literature. For example, except for the core all stability concepts considered in [3], [8] and [9] require individual stability.

hold without the quantifiers. For example, it is not in general true that if for *some* profile $R \in \mathcal{R}_q$ an efficient and individually stable network exists, then all chain stable networks with respect to R are efficient. This may lead some readers to question the efficiency justification for chain stability given above. After all, a better compromise between efficiency and stability considerations might be achieved if we settle for individual stability whenever it is compatible with efficiency but chain stability is not, and otherwise require chain stability. Apart from the question whether such a concept is descriptively appealing, such a solution concept is likely to be (computationally) infeasible as one would need to check that given a particular problem, (a) there is an individually stable and efficient network, and (b) any chain stable network is inefficient. Ostrovsky [10] provides a reasonably fast algorithm to compute chain stable allocations so that chain stability is immune to this type of criticism. A similar remark applies to the robustness justification of chain stability.

3.1 Core Equivalence

The core is one of the most prominent solution concepts in cooperative game theory and it is interesting to know when core and chain stability coincide. By Theorem 1, we can restrict attention to the class of supply chain models that satisfy weak acyclicity and bundling. Given the equivalence of chain and group stability for this class of supply chain models, the question can be rephrased as: When is it irrelevant whether a stability concept allows deviating agents to maintain relationships with outsiders or not? It turns out that the absence of implementable trading cycles is not sufficient for this equivalence to hold and that we need the following stronger restriction.

Definition 4 *The market structure (G_X, q) is* ***strongly acyclic*** *if every undirected cycle in G_X contains at least two capacity constrained agents.*

Note that strong implies weak acyclicity. It is easy to see that the market structures in applications 1 and 2 satisfy weak but not strong acyclicity. The following result shows that for these applications chain stability is a strictly stronger solution concept than the core.

Theorem 2 *The following are equivalent:*
(i) The market structure (G_X, q) satisfies strong acyclicity and bundling.
(ii) Chain stability is equivalent to core stability, that is, $\mathcal{CS}(R) = \mathcal{C}(R)$ for all $R \in \mathcal{R}_q$.

Proof of Theorem 2

(i) $\Rightarrow$ *(ii)* Note that any network μ with $|\mu \cap X(v, w)| \leq 1$ for all $v, w \in V$ defines a (unique) subgraph G_μ of G_X that includes an edge from v to w if and only if μ contains a contract x with $s_x = v$ and $b_x = w$.

By Theorem 1 we know that, for all $R \in \mathcal{R}_q$, $\mathcal{CS}(R) \subseteq \mathcal{C}(R)$ if the market structure satisfies weakly acyclicity and bundling. Hence, I only need to show that $\mathcal{C}(R) \subseteq \mathcal{CS}(R)$ if the market structure satisfies strong acyclicity and bundling. The proof will be by contradiction: Assume that for some $R \in \mathcal{R}_q$ there exists a network $\mu \in \mathcal{C}(R) \setminus \mathcal{CS}(R)$. In the appendix, I prove the following Lemma.

Lemma 1 *If the market structure (G_X, q) satisfies weak acyclicity and bundling, core allocations are always individually stable, that is, $\mathcal{C}(R) \subseteq \mathcal{IS}(R)$ for all $R \in \mathcal{R}_q$.*

Since strong implies weak acyclicity, $\mathcal{C}(R) \subseteq \mathcal{IS}(R)$ by the above lemma. Hence, there must be a chain $x_1, \ldots, x_n \notin \mu$ that blocks μ. Consider the network μ' that results from μ when we add the contracts $x_1, \ldots, x_n$ and delete contracts in $\mu(s_{x_1}) \backslash Ch_{s_{x_1}}(\mu(s_{x_1}) \cup \{x_1\})$, $\mu(b_{x_n}) \backslash Ch_{b_{x_n}}(\mu(b_{x_n}) \cup \{x_n\})$, and $\mu(b_{x_i}) \backslash Ch_{b_{x_i}}(\mu(b_{x_i}) \cup \{x_i, x_{i+1}\})$ for all $i < n$. Note that μ' and μ can both contain at most one contract between each pair of agents since the market structure satisfies bundling. Now let $A \subseteq V$ be the set of agents who are in the same connected component of $G_{\mu'}$ as s_{x_1}, that is, the set of agents $v \in V$ such that G_X contains a sequence of edges connecting s_{x_1} and v. I claim that μ' weakly dominates μ via A.

Suppose to the contrary that there is an agent $\hat{v} \in A \setminus \{s_{x_1}, b_{x_1}, \ldots, b_{x_n}\}$ such that $\mu(\hat{v}) P_{\hat{v}} \mu'(\hat{v})$. Since we have only deleted some contracts involving agents on the blocking chain this means that there is a contract $x \in \mu \setminus \mu'$ which involves $\hat{v}$ and an agent $\tilde{v} \in \{s_{x_1}, b_{x_1} \ldots, b_{x_n}\}$. Note that $G_{\mu'}$ cannot contain a cycle due to weak acyclicity. Since $\hat{v}$ is in the same connected component of $G_{\mu'}$ as s_{x_1} this implies that $G_{\mu' \cup \{x\}}$ contains a cycle $v_1, \ldots, v_m$ with $\{\hat{v}, \tilde{v}\} \subset \{v_1, \ldots, v_m\}$. But on this cycle, $\tilde{v}$ is the only capacity constrained agent: For each agent $v \in \{v_1, \ldots, v_m\} \setminus (\{s_{x_1}, b_{x_1}, \ldots, b_{x_n}\} \cup \{\hat{v}\})$, $\mu'(v) \subseteq \mu(v)$ contains contracts with both of her neighbors on the cycle and, given the individual rationality of μ, $\mu(v) P_v \emptyset$. For each agent $v \in (\{v_1, \ldots, v_m\} \cap \{s_{x_1}, b_{x_1}, \ldots, b_{x_n}\}) \setminus \{\tilde{v}\}$, $\mu'(v)$ contains contracts with both of her neighbors on the cycle and $\mu'(v) P_v \mu(v) R_v \emptyset$. Finally, $\mu'(\hat{v}) \cup \{x\} \subseteq \mu(\hat{v})$ (as $\hat{v} \notin \{s_{x_1}, b_{x_1}, \ldots, b_{x_n}\}$) and $\mu(\hat{v}) P_{\hat{v}} \emptyset$ since $\mu \in \mathcal{C}(R)$. Hence, the market structure cannot satisfy strong acyclicity.

Hence, μ' weakly dominates μ via A if the market structure satisfies strong acyclicity and we obtain a contradiction to the assumption that $\mu \in \mathcal{C}(R)$.

$(ii) \Rightarrow (i)$ If either weak acyclicity or bundling is not satisfied, consider the counterexamples used to prove that $(vi) \Rightarrow (i)$ in Theorem 1: It is easy to check that the core consists of the complete network while the empty network is the unique chain stable network. Now suppose that weak acyclicity and bundling are satisfied but that there is a cycle $v_1, \ldots, v_m$ for which only the source v_1 is capacity constrained (the case where v_1 is a sink can be handled similarly). Let $x_1, \ldots, x_m$ be an accompanying sequence of contracts, that is, x_k is a contract between agents v_k and v_{k+1} (where $m + 1 := 1$). As in the proof of Theorem 1 I can assume that $V = \{v_1, \ldots, v_m\}$. I now define a preference profile for the agents starting with $R_{v_1} = \{x_1\}, \{x_m\}$. Let $k \in \{2, \ldots, m\}$ be arbitrary and set $R_{v_k} = \{x_{k-1}, x_k\}$ if v_k is a passing node, and $R_{v_k} = \{x_{k-1}, x_k\}, \{x_k\}, \{x_{k-1}\}$ in any other case. Let the resulting profile be denoted by R and note that $R \in \mathcal{R}_q$ since v_1 is the only capacity constrained agent among $v_1, \ldots, v_m$. Now let j be the smallest index in $\{2, \ldots, m\}$ such that v_j is a sink and consider the network $\mu = \{x_j, \ldots, x_m\}$. Clearly, $\mu \notin \mathcal{CS}(R)$ since it is blocked by the chain $x_1, \ldots, x_{j-1}$. I now show that $\mu \in \mathcal{C}(R)$.

Suppose to the contrary that μ is weakly dominated by some network μ' via a coalition $A \subseteq V$. Since agents $v_{j+1}, \ldots, v_m$ get their most preferred set of contracts under μ, $A \cap \{v_1, \ldots, v_j\} \neq \emptyset$. By definition of j, agents $v_2, \ldots, v_{j-1}$ are all passing nodes. It follows readily from the construction of R that $v \in \{v_1, \ldots, v_j\} \cap A$ and $\mu'(v) P_v \mu(v)$ imply $\{x_1, \ldots, x_{j-1}\} \subseteq \mu'$ as well as $\{v_1, \ldots, v_j\} \subseteq A$. Since $\{x_j\} P_{v_j} \{x_{j-1}\}$ this implies $x_j \in \mu'$ and thus $v_{j+1} \in A$. Continuing with this line of reasoning it is easy to see that we must have $\mu' = \{x_1, \ldots, x_m\}$ and $A = \{v_1, \ldots, v_m\}$. But then v_1 is worse off compared to μ since $\emptyset P_{v_1} \{x_1, x_m\}$, a contradiction.

□

Two special cases of Theorem 2 that have been studied in the previous literature are the many-to-one matching model with substitutable preferences ([14]) and the *unit capacity model*, in which each agent can sign at most one upstream and at most one downstream contract ([10, Theorem 5]). Another special case is a discrete version of the many-to-one matching model with substitutes and complements studied by Sun and Yang [15]: A set of workers can be decomposed into two sets W_1 and W_2 such that all firms view two workers from the same set as substitutes (in the sense of SSS) and two workers from different sets as complements (in the sense of CSC). Workers can work for at most one firm, while firms can hire an arbitrary number of workers. Ostrovsky [10] discusses how this can be formulated as a supply chain model in which the set of sellers of basic inputs comprises W_1, the set of consumers of final products comprises W_2, and the set of intermediaries comprises all firms. Note that since there are no edges connecting two firms or two workers, any undirected cycle must contain at least two workers. Since all workers are capacity constrained (on any cycle), the market structure is strongly acyclic and satisfies bundling so that the core coincides with the set of chain stable networks by Theorem 2. Hence, we obtain the non-emptiness of the core as a corollary to the existence of a chain stable network for this model.

4 Conclusion

This paper showed that the structural properties of supply chain models are important for the relationship between (cooperative) solution concepts. Weak acyclicity and bundling were shown to be necessary and sufficient for (i) the equivalence of chain and group stability, (ii) the core stability of chain stable networks, (iii) the efficiency of chain stable networks, (iv) the existence of group stable networks, and (v) the existence of an efficient and individually stable network. The second main result characterized the class of models for which the chain stable set coincides with the core by means of a stronger acyclicity condition and bundling. I have argued that the first main result can be interpreted as a justification of chain stability on basis of efficiency and robustness considerations. The cooperative foundation shows that this stability concept is a reasonable allocative goal for markets that fit the assumptions of the supply chain model. An important open question for future research is how such markets would have to be organized in order to reach this goal when agents act strategically.

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5 Appendix

5.1 Proof of Lemma 1

Suppose to the contrary that (G_X, q) satisfies weak acyclicity and bundling but that for some preference profile $R \in \mathcal{R}_q$ there exists a core stable network μ that is not individually stable. Let $v_0 \in V$ be an agent who would like to drop some of the contracts in $\mu(v_0)$, that is, $Ch_{v_0}(\mu(v_0)) \neq \mu(v_0)$. This implies in particular that $\mu(v_0) \neq \emptyset$. But then it has to be the case that $Ch_{v_0}(\mu(v_0)) \neq \emptyset$ since μ is at least individually rational. I denote by μ' the network that results from μ when contracts in $\mu(v_0) \setminus Ch_{v_0}(\mu(v_0))$ are deleted. For the following let $V_0 := \{v_0\}$.

Let $V_1 \subseteq V \setminus V_0$ be the (nonempty) set of agents who are involved with some contract in $Ch_{v_0}(\mu(v_0))$ and let W be the set of agents who are involved with one of the contracts v_0 wants to drop, that is, with one of the contracts in $\mu(v_0) \setminus Ch_{v_0}(\mu(v_0))$. We must have $V_1 \cap W = \emptyset$ since $q_{v,w} \leq 1$ for all $v, w \in V$ and μ is individually rational. Furthermore, μ cannot contain a contract between a pair of agents in $V_1 \times (W \cup V_1)$ if the market structure satisfies weak acyclicity. Otherwise, there would be two agents $w_1 \in V_1$ and $w_2 \in W \cup V_1$ such that μ contains a contract between w_1 and w_2. By definition of V_1 and W, μ also contains contracts between v_0 and both, w_1 and w_2. Since μ is individually rational, none of the three agents can be capacity constrained. Hence, we have found an undirected cycle in G_X that does not contain a capacity constrained agent. On the other hand, μ has to contain at least one contract between an agent in V_1 and an agent in $V \setminus (W \cup V_0 \cup V_1)$. Otherwise μ' weakly dominates μ via the coalition $V_0 \cup V_1$ since (i) all agents in V_1 would be indifferent between these two networks, (ii) v_0 strictly prefers μ' over μ, and (iii) μ' does not contain a contract between an agent in $V_0 \cup V_1$ and another agent in $V \setminus (V_0 \cup V_1)$. Thus, there has to be a nonempty set of agents $V_2 \subseteq V \setminus (W \cup V_0 \cup V_1)$ who sign a contract with some agent in V_1 under μ.

Now suppose that for some $k \geq 2$ we have shown that there is a sequence of sets of agents $V_1, \ldots, V_k$ such that, for all $l \in \{2, \ldots, k\}$, $V_l \subset V \setminus (W \cup V_0 \cup \ldots \cup V_{l-1})$ and the set of all agents who sign a contract with agents in V_{l-1} under μ is $V_{l-2} \cup V_l$. If the market structure satisfies weak acyclicity and bundling, μ cannot contain a contract between a pair of agents in $V_k \times (W \cup V_0 \ldots \cup V_k)$. The argument is similar to above and the details are omitted. On the other hand, μ has to contain at least one contract between an agent in V_k and an agent in $V \setminus (W \cup V_0 \ldots \cup V_k)$. Otherwise μ' weakly dominates μ via the coalition $V_0 \cup \ldots \cup V_k$ since (i) all agents in $V_1 \cup \ldots \cup V_k$ would be indifferent between these two networks, (ii) v_0 strictly prefers μ' over μ, and (iii) μ' does not contain a contract between an agent in $V_0 \cup \ldots \cup V_k$ and another agent in $V \setminus (V_0 \cup \ldots \cup V_{k-1})$. Thus, there has to be a nonempty set of agents $V_{k+1} \subseteq V \setminus (W \cup V_0 \cup \ldots \cup V_k)$ who sign a contract with some agent in V_k under μ.

The above argument is valid for any k and the procedure would thus run forever, contradicting the finiteness of V. This completes the proof.

□

Note that the converse is also true: If $\mathcal{C}(R) \subseteq \mathcal{IS}(R)$ for all $R \in \mathcal{R}_q$, the market structure must satisfy weak acyclicity and bundling. To see this note that individual stability of the core implies that an efficient individually stable network always exists. Hence, the statement follows from $(vi) \Rightarrow (i)$ in Theorem 1.

5.2 Feasibility Restrictions on Networks

Instead of requiring preferences to conform to an exogenously given capacity vector, one could also restrict the set of feasible networks. Given a capacity vector q, the set of feasible networks $\mathcal{M}_q$ can be defined as follows: $\mu \in \mathcal{M}_q$ if and only if μ does not violate any agent's capacity constraints (cf Definition 1 in section 3). Given some network $\mu \in \mathcal{M}_q$ let G^μ be the directed graph which contains one edge from v to w for each contract $x \in \mu$ such that $s_x = v$ and $b_x = w$. Note that in contrast to the graphs used in the main text, this graph may contain multiple edges between a given pair of agents. The following shows how the acyclicity condition developed in section 3 can be expressed in this framework.

Proposition 1 *The market structure (G_X, q) satisfies weak acyclicity and bundling if and only if G^μ is a forest for all $\mu \in \mathcal{M}_q$.*[13]

The proof is straightforward and omitted here. It is not clear how strong acyclicity could have been formulated in this framework. This is the main reason for requiring preferences to conform to capacities instead.

5.3 Discussion of the main results

For this Appendix I assume that there are no capacity constraints and consider the domain $\mathcal{R}$ of all admissible preference profiles. The following statements are easily seen to be true for any given $R \in \mathcal{R}$.

$\mathcal{CS}(R) = \mathcal{GS}(R) \Rightarrow \mathcal{CS}(R) \subseteq \mathcal{C}(R), \mathcal{CS}(R) \subseteq \mathcal{E}(R), \mathcal{GS}(R) \neq \emptyset, \mathcal{E}(R) \cap \mathcal{IS}(R) \neq \emptyset$
$\mathcal{CS}(R) \subseteq \mathcal{C}(R) \Rightarrow \mathcal{CS}(R) \subseteq \mathcal{E}(R), \mathcal{E}(R) \cap \mathcal{IS}(R) \neq \emptyset$
$\mathcal{CS}(R) \subseteq \mathcal{E}(R) \Rightarrow \mathcal{E}(R) \cap \mathcal{IS}(R) \neq \emptyset$
$\mathcal{GS}(R) \neq \emptyset \Rightarrow \mathcal{E}(R) \cap \mathcal{IS}(R) \neq \emptyset$

In this appendix I show that all other implications of Theorem 1 do not necessarily hold for any given preference profile. Chain stable networks in the examples can be calculated using the *T-algorithm* of [10]. All counterexamples use a supply chain model with five agents and the following graph of potential interactions:

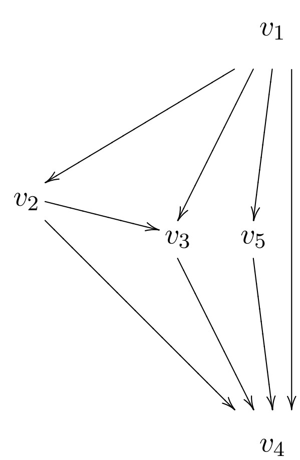

Figure 1: Graph of potential interactions for counterexamples.

[13] A forest is a directed graph containing no directed or undirected cycles. Note that since G^μ is not necessarily a simple graph a cycle may consist of two agents.

Throughout the Appendix I use the following notation: x_i^j denotes some contract in which agent i sells something to agent j. Agent v_5 will only be needed for the last example and her preferences will not be specified in the other examples.

1. There exist profiles $R \in \mathcal{R}$ such that $\mathcal{CS}(R) \subseteq \mathcal{C}(R)$, $\mathcal{CS}(R) \subseteq \mathcal{E}(R)$, and $\mathcal{E}(R) \cap \mathcal{IS}(R) \neq \emptyset$, but $\mathcal{CS}(R) \neq \mathcal{GS}(R)$ and $\mathcal{GS}(R) = \emptyset$.

   Consider the following preference profile:[14]

R^1	$R^1_{v_1}$	$R^1_{v_2}$	$R^1_{v_3}$	$R^1_{v_4}$
	$\{x_1^2\}$	$\{x_1^2, x_2^3\}$	$\{x_1^3, x_3^4\}$	$\{x_1^4, x_3^4\}$
	$\{x_1^2, x_1^3\}$		$\{x_1^3, x_2^3, x_3^4\}$	$\{x_1^4\}$
	$\{x_1^4\}$		$\{x_3^4\}$	$\{x_3^4\}$

   Using the T-Algorithm it is easy to show that the unique chain stable network is given by $\mu = \{x_1^4, x_3^4\}$.

   To see that $\mu \in \mathcal{C}(R^1)$ note that v_4 cannot be made better off and the only network which makes v_1 and v_3 better off without hurting v_2 is $\{x_1^2, x_1^3, x_2^3, x_3^4\}$. This network would make v_4 worse off and thus does not weakly dominate μ in the sense of the core. Hence, μ is in the core and thus in particular efficient.

   On the other hand, μ is blocked by $\{v_1, v_2, v_3\}$ via $\{x_1^2, x_1^3, x_2^3, x_3^4\}$ so that μ is not group stable. The other nonempty individually stable networks are $\{x_3^4\}$ and $\{x_1^4\}$, which are not even chain stable. Since a group stable matching has to be individually stable, this shows that $\mathcal{GS}(R^1) = \emptyset$.

2. There exist preference profiles $R \in \mathcal{R}$ such that $\mathcal{IS}(R) \cap \mathcal{E}(R) \neq \emptyset$, but $\mathcal{CS}(R) \cap \mathcal{C}(R) = \emptyset$ and $\mathcal{CS}(R) \cap \mathcal{E}(R) = \emptyset$.

   Consider the following preference profile:

R^2	$R^2_{v_1}$	$R^2_{v_2}$	$R^2_{v_3}$	$R^2_{v_4}$
	$\{x_1^4, x_1^3\}$	$\{x_1^2, x_2^3, x_2^4\}$	$\{x_1^3, x_2^3, x_3^4\}$	$\{x_1^4, x_2^4\}$
	$\{x_1^2, x_1^3\}$	$\{x_2^3\}$	$\{x_2^3\}$	$\{x_2^4, x_3^4\}$
	$\{x_1^4\}$	$\{x_2^4\}$	$\{x_1^3\}$	$\{x_1^4\}$
	$\{x_1^2\}$			$\{x_3^4\}$
	$\{x_1^3\}$			$\{x_2^4\}$

   The unique chain stable network is given by $\mu = \{x_1^4, x_2^3\}$. But the network $\mu' = \{x_1^2, x_1^3, x_2^3, x_2^4, x_3^4\}$ is individually stable as well as efficient, and makes all agents better off (note that this network is blocked by the chain x_1^4).

3. There exist preference profiles $R \in \mathcal{R}$ such that $\mathcal{CS}(R) \subseteq \mathcal{E}(R)$ but $\mathcal{CS}(R) \cap \mathcal{C}(R) = \emptyset$.

   Consider the following preference profile:

[14] As in all examples that follow it is easy to check that SSS and CSC are indeed satisfied.

R^3	$R^3_{v_1}$	$R^3_{v_2}$	$R^3_{v_3}$	$R^3_{v_4}$
	$\{x^2_1\}$	$\{x^2_1, x^3_2\}$	$\{x^3_1\}$	$\{x^4_1\}$
	$\{x^2_1, x^3_1\}$		$\{x^3_1, x^3_2\}$	
	$\{x^4_1\}$			

The unique chain stable network is the efficient network $\{x^4_1\}$. The unique core network, however, is given by $\{x^2_1, x^3_1, x^3_2\}$.

4. There exist preference profiles $R \in \mathcal{R}$ such that $\mathcal{GS}(R) \neq \emptyset$ but $\mathcal{CS}(R) \setminus \mathcal{E}(R) \neq \emptyset$, $\mathcal{CS}(R) \setminus \mathcal{GS}(R) \neq \emptyset$, and $\mathcal{CS}(R) \setminus \mathcal{C}(R) \neq \emptyset$.

   Preferences are given by:

R^4	$R^4_{v_1}$	$R^4_{v_2}$	$R^4_{v_3}$	$R^4_{v_4}$	$R^4_{v_5}$
	$\{x^4_1\}$	$\{x^2_1, x^4_2\}$	$\{x^3_1, x^4_3\}$	$\{x^4_5\}$	$\{x^5_1, x^4_5\}$
	$\{x^2_1\}$			$\{x^4_2, x^4_3, x^4_5\}$	
	$\{x^2_1, x^3_1, x^5_1\}$			$\{x^4_3\}$	
	$\{x^3_1\}$			$\{x^4_1\}$	

   For this profile there are two chain stable networks: $\{x^4_1\}$ and $\{x^3_1, x^4_3\}$.

   The first network is also group stable, but the second is not even efficient as the network $\{x^2_1, x^3_1, x^5_1, x^4_2, x^4_3, x^4_5\}$ makes all agents (weakly) better off.

5.4 Beyond the Supply Chain Model

Some of the main results of the paper continue to hold without the CSC assumption. Let (G_X, q) be a market structure and let $\hat{\mathcal{R}}_q$ be the set of all preference profiles that conform to capacities and satisfy all assumptions of section 2.1 except CSC. The following theorem summarizes the results that carry over to this more general setting (in which the existence of chain stable networks cannot be guaranteed).

Theorem 3 *1. The following are equivalent:*

(i) (G_X, q) satisfies weak acyclicity and bundling.
(ii) $\mathcal{CS}(R) = \mathcal{GS}(R)$ for all $R \in \hat{\mathcal{R}}_q$.
(iii) $\mathcal{CS}(R) \subseteq \mathcal{C}(R)$ for all $R \in \hat{\mathcal{R}}_q$.
(iv) $\mathcal{CS}(R) \subseteq \mathcal{E}(R)$ for all $R \in \hat{\mathcal{R}}_q$.

2. If $\mathcal{GS}(R) \neq \emptyset$ for all $R \in \hat{\mathcal{R}}_q$ then (G_X, q) satisfies weak acyclicity and bundling.

3. If $\mathcal{IS}(R) \cap \mathcal{E}(R) \neq \emptyset$ for all $R \in \hat{\mathcal{R}}_q$ then (G_X, q) satisfies weak acyclicity and bundling.

4. (G_X, q) satisfies strong acyclicity and bundling if and only if $\mathcal{C}(R) = \mathcal{CS}(R)$ for all $R \in \hat{\mathcal{R}}_q$.

The proof of this Theorem follows directly from the observation that the corresponding parts of the proofs of Theorems 1 and 2 do not rely on CSC. Since the existence of a chain stable network cannot be guaranteed for the larger domain of preferences, weak acyclicity and bundling are not in general sufficient for the existence of either a group stable or an efficient individually stable network.

One application of Theorem 3 is the well known *roommate problem* first analyzed by Gale and Shapley [4]. In this problem $2n$ agents have to be assigned among n rooms that each have place for 2 agents. Each agent can share a room with any other agent, has a strict preference relation over potential roommates, and does not care about which room she is assigned to (only the roommate matters). If we want to allow agents to have any (rational) preference relation over potential roommates, this model does not belong to the class of supply chain models with same side substitutable and cross side complementary preferences: In order to write this problem as a supply chain model, we would have to define a directed graph of potential trading relationships. In order to allow all potential roommate combinations we would need to introduce an arbitrarily directed edge between all pairs of agents. It is easy to see that if $n \geq 2$, at least one agent has to be an intermediary if we require the market structure to be free of directed cycles. The preferences of such an agent would be severely restricted by the assumption of CSC: The intermediary would be required to either declare all upstream or all downstream agents as unacceptable roommates. Given that the direction of the edges introduced is arbitrary, this is not a satisfactory embedding of the roommate problem. If we dispense with the assumption that preferences satisfy the CSC condition this problem does not occur since it is easy to see that SSS does not restrict the set of allowed preference relations. Hence, any roommate problem can be formulated as a supply chain model in which agents' preferences satisfy SSS (but are allowed to violate CSC). Note that since each agent is looking for at most one partner chain stability reduces to pairwise stability and (any) market structure satisfies strong acyclicity and bundling. The above Theorem thus implies the following.

Corollary 3 *For roommate problems the set of pairwise stable matchings coincides with the core.*