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An analysis of the German university admissions system*

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Abstract

This paper analyzes the *sequential* admissions procedure for medical subjects at public universities in Germany. Complete information equilibrium outcomes are shown to be characterized by a *stability* condition that is adapted to the *institutional constraints* of the German system.

I introduce *matching problems with complex constraints* and the notion of *procedural stability*. Two simple assumptions guarantee existence of a student optimal procedurally stable matching mechanism that is strategyproof for students. In the context of the German admissions problem, this mechanism weakly Pareto dominates all equilibrium outcomes of the currently employed procedure. Applications to school choice with affirmative action are also discussed.

Keywords University Admissions · Matching · Stability · Strategyproofness · Complex Constraints

JEL classification C78 · D02

1 Introduction

According to German legislation, every student who obtains the *Abitur* (i.e., successfully finishes secondary school) or some equivalent qualification is entitled to study any subject at any

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public university. Given capacity constraints at educational institutions and the ensuing need to reject some applicants, this principle has long been reinterpreted as meaning that everyone should have a *chance* of being admitted into the program of his or her choice. In order to implement this requirement, places in those fields of study that are most prone to overdemand have been allocated by a centralized nationwide assignment procedure for over 25 years. In the first part of this paper I analyze the most recent version of this procedure that is currently used to allocate places for medicine and three specialities (dentistry, pharmacy, and veterinary medicine). In the winter term 2010/2011 more than 56 000 students applied for one of the less than 13 000 places available in these four subjects, meaning that ultimately three in four applicants had to be rejected. What sets this part of my study apart from previous investigations of real-life centralized clearinghouses is the *sequential* nature of the German admissions procedure: In the first step, the well known *Boston mechanism* is used to allocate *up to* 40 percent of the total capacity of each university among *special applicant groups*, consisting of applicants who have either obtained excellent school grades or have had to wait a long time since finishing school. About one month later, all remaining places - this includes in particular all places that could have been but were not allocated to special student groups - are assigned among *remaining* applicants according to criteria chosen by the universities using the *college (university) proposing deferred acceptance algorithm* (CDA). Applicants belonging to special student groups, who were not assigned one of the seats initially reserved for them, have another chance of obtaining a seat in this part of the procedure. Despite the complexities of the admissions procedure and the strategic incentives induced by it, I show that, under reasonable assumptions, the set of (complete information) equilibrium outcomes can be characterized by a *stability condition* with respect to the *true* preferences of participants. The concept of stability used here differs from the previous literature since it accounts for the fact that (i) each university has different types of places (corresponding to the places allocated in the different parts of the procedure) with differing admissions criteria, and (ii) vacant seats can be redistributed across quotas. I show that, even in the absence of information asymmetries, the procedure supports *Pareto dominated equilibrium outcomes* (with respect to the true preferences of applicants).

Motivated by this finding, I develop an alternative assignment procedure in the second part of my paper. For the redesign I interpret the current procedure's sequential design as reflecting

the constraint that a fraction of total capacity of each university should initially be reserved for special student groups (excellent and wait-time applicants) and should be allocated on basis of universities' criteria if (and only if) there is insufficient demand from these groups.[1] Similar constraints can play an important part in applications to *school choice*: For example, schools are sometimes required to offer guaranteed access to specific student groups,[2] and may have preferences over how any potentially remaining capacity should be distributed among applicants (e.g. a strict preference for an equal distribution among sexes). Any mechanism that implements such constraints by sequentially allocating parts of the total capacity, such as the current German procedure, and that depends on submitted preferences in a reasonable way, e.g. by producing a stable matching, necessarily induces incentives to manipulate the assignment procedure. Given that incentive compatibility is an important goal, it is thus important to study when and how complex constraints of the above type can be implemented by mechanisms that simultaneously assign all available places and achieve a satisfactory allocative performance. To study these questions in sufficient generality, I introduce a model of *matching with complex constraints*, where college preferences are given by *choice protocols* that can be thought of as an explicit representation of how a college chooses from a given set of applicants. Abstracting from details, a choice protocol consists of (i) a sequence in which student groups are to be considered, and (ii) an associated sequence of capacities, which, for each point of the sequence, describes how many places can be allocated as a function of the numbers of seats left empty by previously considered groups. I introduce a concept of *procedural stability* for this class of matching problems and show that as long as choice protocols satisfy two simple restrictions of *monotonicity* and *non-excessive reduction*, a student optimal procedurally stable matching always exists and the associated direct mechanism is (group) strategyproof for students. These results rest on a transformation of a matching problem with complex constraints to an associated "standard" two-sided matching problem, where constraints are not explicitly present, for which well known results (see Section 2 for references) apply. In case of the German admissions system,

[1] While some changes to the current quota system may be feasible, at least the quota for applicants with a long waiting time is generally seen as necessary. It ensures that that every applicant with the necessary qualification (i.e. finishing secondary school) has a *chance* to study any subject she wants. This has been put forth as a basic requirement for university admissions by the German constitutional court in its *Numerus Clausus Urteil* from 1972.

[2] Most public schools in Boston have to admit any student with an older sibling already attending the school (see Abdulkadiroglu et al 2006), while some public schools in New York City have to admit all students whose performance in a standardized English language exam is among the top 2 percent (see Abdulkadiroglu et al 2009).

procedural stability reduces to the stability notion that characterizes complete information equilibrium outcomes of the current admissions procedure studied in the first part of this paper. The choice protocols needed to implement the constraints of the German admissions system turn out to be monotonic and satisfy non-excessive reduction. Hence, a redesign based on the above ideas would provide (groups of) applicants with dominant strategy incentives to submit preferences truthfully and would thus lead to an assignment procedure that Pareto dominates all equilibrium outcomes of the current procedure with respect to applicants' true preferences.

Complementary to the analysis of this paper, Braun et al (2010) study the German university admissions system from an empirical perspective. Using data for the winter term 2006/2007, for which the rules of the centralized admissions procedure were slightly different from the rules of the procedure analyzed here, they find considerable support for the hypothesis that applicants try to manipulate the centralized admissions procedure. My paper, which was drafted independently of this empirical study, complements this research since it shows precisely how these findings can be explained by applicants' strategic incentives. A major benefit of the more theoretical approach is that I am not only able to design a promising alternative but can also compare it directly to the equilibrium outcomes of the current procedure.

Given that the German admissions system and my proposal for a redesign are closely related to the deferred acceptance algorithms and the Boston mechanism, the theoretical and applied literatures on these mechanisms are of course related to this paper. Excellent comprehensive surveys of some of these applications are Roth and Sotomayor (1990), Roth (2008) and Sönmez and Ünver (2010).

Variants of the college proposing deferred acceptance (CDA) algorithms have been used to allocate medical students to their first professional position in the US (Roth 1984a; see Roth and Peranson 1999 for an account of the efforts to redesign the resident matching procedure) and to assign students to public universities in Turkey (Balinski and Sönmez 1999). Variants of the student proposing deferred acceptance (SDA) algorithm have been and are used to assign students to public schools in Boston and New York City (Abdulkadiroglu et al 2006, Abdulkadiroglu et al 2009), to allocate on campus housing among students of the Massachusetts Institute of Technology (Guillen and Kesten 2012), and to assign medical graduates to residency programs in Japan (Kamada and Kojima 2011).

The Boston mechanism has been studied extensively since Abdulkadiroglu and Sönmez (2003) seminal paper on school choice mechanisms. Ergin and Sönmez (2006) show that if each school has a strict priority ranking of all potential students, the set of complete information equilibrium outcomes coincides with the set of stable matchings with respect to the true preferences of participants. Pathak and Sönmez (2008) show that when some students are naive in the sense that they always submit preferences truthfully, equilibrium outcomes of the Boston mechanism coincide with the set of matchings that are stable for a modified economy in which naive students lose their priorities to students who behave strategically. Chen and Sönmez (2006) and Abdulkadiroglu et al (2006) provide experimental and empirical evidence that students or their parents try to manipulate the Boston mechanism and that strategic behavior can lead to welfare losses. A more positive perspective on the Boston mechanism is provided by Abdulkadiroglu et al (2011b) who show that when schools have no priorities and all students have the same ordinal preferences but differ in their (privately known) preference intensities, any symmetric Bayes-Nash equilibrium of the Boston mechanism weakly ex-ante Pareto dominates the SDA mechanism with any symmetric tie-breaking rule. Arguments in support of the Boston mechanism in similar environments are also provided by Miralles (2008) and Featherstone and Niederle (2008). While the results in favor of the Boston mechanism are certainly relevant for school choice problems, where priorities are typically very coarse, they are arguably less so for my application to university admissions since evaluation criteria such as average grades or standardized tests typically yield a much finer ranking of students.

To the best of my knowledge, this paper is the first study of a *dynamic* assignment procedure that *combines* the Boston and deferred acceptance algorithms.[3] Apart from the practical relevance of this study for the current German admissions system, redesigning it poses new theoretical challenges due to its *complex constraints* and the proposed solutions yield new insights that are applicable outside of the specific context of the German system.

Additional literature is reviewed in context below (see the next section and Section 4.2, where I compare matching problems with complex constraints to the existing literature on matching problems with constraints).

[3]Chen and Kesten (2011) have recently studied a family of school choice mechanisms which contains the Boston and student proposing deferred acceptance (SDA) mechanisms as special cases. In contrast to the German admissions system, the mechanisms studied by Chen and Kesten (2011) are simultaneous in the sense that all available school seats are allocated within one coherent procedure.

The remainder of this paper is organized as follows. In Section 2, I introduce three known matching algorithms that are important for the analysis. Section 3 contains a simplified version of the current admissions procedure for German universities and an analysis of the induced revelation game. In Section 4, I introduce a theory of matching with complex constraints and then show how this theory can be applied to develop a redesign of the German admissions system. Section 5 conclude and Section 6 contains proofs of the main results. The supplementary appendix, Section 7, contains some omitted proofs as well as details and data on the current procedure.

2 Basic model and assignment algorithms

A *college admissions problem* consists of

- a finite set of students I
- a finite set of colleges C,
- a profile of student preferences $P_I = (P_i)_{i \in I}$, where P_i is a strict ordering of $C \cup \{i\}$, and
- a profile of college preferences $P_C = (P_c)_{c \in C}$, where P_c is a strict ordering of 2^I.

Given a strict ordering P_i for a student $i \in I$, let R_i denote the associated weak ordering, that is, cR_ic' if and only if either cP_ic' or $c = c'$. Define associated weak orderings for colleges analogously. College c is acceptable to student i according to P_i, if cP_ii, and group of students J is acceptable to college c according to P_c if $JP_c\emptyset$.

A *matching* is a mapping μ from $I \cup C$ into itself such that (1) for all students i, $\mu(i) \in C \cup \{i\}$ for all $i \in I$, (2) for all colleges c, $\mu(c) \subseteq I$, and (3) $i \in \mu(c)$ if and only if $\mu(i) = c$. I make the usual assumption that agents only care about their own partner(s) in a matching so that their preferences over matchings coincide with their preferences over (sets of) potential partners. The sets of students and colleges are assumed to be fixed so that a college admissions problem is given by a profile of student and college preferences $P = (P_I, P_C)$. Given c's strict ordering P_c, its *choice from* $J \subseteq I$, denoted $Ch_c(J|P_c)$, is the P_c-most preferred subset of J.

The key allocative criterion in the literature is *(pairwise) stability* as introduced by [18]. Given a college admissions problem P, a matching μ is pairwise stable if (1) no student is

matched to an unacceptable college, that is, for all $i \in I$, $\mu(i)R_i i$, (2) no college prefers to reject some of its assigned students, that is, $Ch_c(\mu(c)) = \mu(c)$, and (3) there is no student-college pair that blocks μ, that is, there is no pair (i, c) such that $cP_i\mu(i)$ and $i \in Ch_c(\mu(c) \cup \{i\})$.

Next, I introduce three important restrictions on college preferences over groups of students. College c's strict ranking P_c

(a) is *substitutable* (Kelso and Crawford 1982), if for all subsets $J \subseteq I$ whenever $i \in Ch_c(J|P_c) \cap \tilde{J}$ for some $\tilde{J} \subseteq J$ then $i \in Ch_c(\tilde{J}|P_c)$.

(b) satisfies *the law of aggregate demand* (Hatfield and Milgrom 2005), if $J \subseteq \tilde{J} \subseteq I$ implies $|Ch_c(J|P_c)| \leq |Ch_c(\tilde{J}|P_c)|$.

(c) is *responsive* (Roth 1985), if there is a strict ordering P_c^* of $I \cup \{c\}$ and a quota $q_c \in \mathbb{N}$ such that

(c1) $\emptyset P_c J$ for all $J \subseteq I$ such that $|J| > q_c$, and

(c2) for all $J \subseteq I$ with $|J| < q_c$ and all $i, j \in I \setminus J$, $J \cup \{i\}P_c J \cup \{j\}$ if and only if $iP_c^* j$ and $J \cup \{i\}P_c J$ if and only if $iP_c^* c$.

An important special case of the college admissions problem with responsive preferences is the *priority based allocation problem* in which colleges' rankings of individual students and their capacities are exogenously assigned. In particular, a colleges' ranking of individual students does not represent its preferences and thus has no intrinsic meaning for welfare evaluations. To emphasize this important interpretational difference, college c's strict ranking of $I \cup \{c\}$ will be called a *priority ranking* and denoted by $\succ_c$ if we are dealing with a priority based allocation problem.[4]

A (direct) *matching mechanism* is a mapping f from the set of feasible preference profiles to the set of possible matchings. In a college admissions problem, the set of feasible preference profiles consists of all preference profiles of colleges *and* students that satisfy appropriate restrictions (e.g. substitutability or responsiveness). In a priority based allocation problem, the "preferences" of colleges are fixed, so that the set of feasible preference profiles consists of all

[4]If all students are acceptable to all colleges and there is no shortage of total seats, a priority based allocation problem is usually referred to as a *school choice problem*, following the seminal article by Abdulkadiroglu and Sönmez (2003). See Erdil and Ergin (2008), Abdulkadiroglu et al (2009), Ehlers and Erdil (2010), and Ehlers and Westkamp (2011) for analyses of the priority based allocation problem when priorities are not necessarily strict.

profiles of student preferences. Given a matching mechanism f and a feasible preference profile P, $f_i(P)$ denotes the assignment of agent $i \in I \cup C$. A matching mechanism f is *stable*, if it selects a stable matching for each profile in its domain. A matching mechanism f is *group strategyproof for students*, if for all feasible preference profiles P and all sets of students $J \subseteq I$, there is no joint manipulation $P'_J = (P'_j)_{j \in J}$ such that $f_j(P'_J, P_{-J}) P_j f_j(P)$ for all $j \in J$. If this condition holds for all singleton subsets of students, f is *strategyproof for students*. I now describe three assignment procedures that play an important role in the literature and the remainder of this paper.

2.1 The student proposing deferred acceptance algorithm

The *student proposing deferred acceptance algorithm* (SDA), developed by Gale and Shapley (1962) (and extended to college admissions problems with substitutable preferences by Roth and Sotomayor 1990), will play an important role in my proposal for a redesign of the German admission system and proceeds as follows

> In the first round, every student applies to her favorite acceptable college. Each college c temporarily accepts its choice from the set of applicants in this round and rejects all other applicants.
>
> In the tth round, every student applies to her most preferred acceptable college (if any) among those that have not rejected her in any previous round of the algorithm. Each college c temporarily accepts its choice from the set of applicants in this round and rejects all other applicants.

Given some preference profile P, let $f^I(P)$ denote the matching chosen by the SDA. If all colleges have substitutable preferences, $f^I(P)$ is the most preferred stable matching for all students and the least preferred stable matching for all colleges (Roth 1984b).[5] If, in addition, colleges' preferences satisfy the law of aggregate demand, f^I is group strategyproof for students

[5]Student optimality of the SDA was first established by Gale and Shapley (1962) and then later extended to matching problems with substitutable preferences by Roth (1984b).

(Hatfield and Kojima 2009).[6][7]

2.2 The college proposing deferred acceptance algorithm

The *college proposing deferred acceptance algorithm* (CDA) lets colleges take an active role. This algorithm plays an important role in the current German admission procedure and proceeds as follows:

> In the first round, each college offers admission to its choice from the set of all students. Each student i temporarily holds on to her most preferred offer and rejects all other offers.
>
> In the tth round, each college offers admission to its choice from the set of all students that have not rejected one of its offers in any previous round. Each student i temporarily holds on to her most preferred offer and rejects all other offers.

Given a feasible preference profile P, let $f^C(P)$ denote the matching chosen by the CDA. For any preference profile P such that colleges' preferences are substitutable, $f^C(P)$ is the most preferred stable matching for all colleges and the least preferred stable matching for all students (Kelso and Crawford 1982). Furthermore, the set of pure strategy complete information Nash equilibrium outcomes of the preference revelation game induced by $f^C(\cdot, P_C)$ coincides with the set of stable matchings with respect to P if colleges' preferences are substitutable (Sotomayor 2007).[8]

[6]Strategyproofness of the SDA for the proposing side was first established independently by Dubins and Freedman (1981) and Roth (1982). Hatfield and Milgrom (2005) then extended this result to matching problems (with contracts) for which college preferences are substitutable and satisfy the law of aggregate demand. For the same setting, Hatfield and Kojima (2009) showed that these assumptions actually guarantee group strategyproofness.

[7]As shown by Roth (1982) there is no stable matching mechanism that is strategyproof for agents on *both* sides of the market. Consequently, the literature has studied the possibility of implementing stable matchings in Nash equilibrium when both sides of the market act strategically. Kara and Sönmez (1996) show that the stable correspondence is Nash implementable in one-to-one matching markets. This finding was extended to college admissions problems with responsive preferences by Kara and Sönmez (1997) and to many-to-one matching problems with substitutable preferences (and contracts) by Haake and Klaus (2009).

[8]Haeringer and Klijn (2009) show that there is no corresponding result for the SDA if students are allowed to rank more than one college: Unstable equilibrium outcomes can exist. However, if students are allowed to rank as many colleges as they want, any such equilibrium would involve some students playing (weakly) dominated strategies.

2.3 The Boston mechanism

The *Boston mechanism* is a popular real-life assignment procedure for priority based allocation problems. Given the fixed priority structure $\succ_C$ and a profile of strict student preferences P_I, the following algorithm is used to determine assignments

> In the first round, every student applies to her top choice college. Each college c admits the q_c highest priority students who apply in this round (or all those students if there are fewer than q_c). All other students are rejected. Let q_c^2 denote the remaining capacity of college c.
>
> In the tth round, every remaining student applies to her tth most preferred acceptable college (if any). Each college c admits the q_c^t highest priority students who apply in this round (or all those students if there are fewer than q_c^t). All other students are rejected. Let q_c^{t+1} denote the remaining capacity of college c.

Fix a priority structure $\succ_C$ and given some profile of student preferences P_I, let $f^B(P_I)$ denote the matching chosen by the Boston mechanism. As first pointed out by Abdulkadiroglu and Sönmez (2003), the outcome of the Boston mechanism is guaranteed to be efficient with respect to reported preferences but may fail to be efficient with respect to true preferences, since the Boston mechanism typically gives students strong incentives to *misrepresent* their preferences. For any profile of student preferences P_I, the set of pure strategy complete information Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism coincides with the set of stable matchings with respect to P_I (Ergin and Sönmez 2006). Note that this result implies that for *any* equilibrium outcome of the Boston mechanism, all students weakly prefer the outcome of the SDA under truth-telling. An example in Ergin and Sönmez (2006) shows that this dominance relation does not carry over to the case of incomplete information.

3 The German university admissions system

The centralized admissions procedure for public universities in Germany is used to allocate places in human medicine, dentistry, veterinary medicine, and pharmacy.[9] The basic outline of the admissions procedure is as follows:

1. In the first part, *up to* 40 percent of total capacity is allocated among *special* applicants.
2. In the second part, all remaining places, are allocated according to universities' preferences among those applicants not assigned in the first part.

In the actual procedure, the group of special applicants consists of applicants who have earned an excellent, which in Germany means very *low*, average grade in school leaving examinations, and of applicants who have waited a long time since finishing high school. There is a separate procedure for each of these two groups and in each of the two procedures up to 20 percent of total capacity are allocated. To simplify the description and analysis, I assume that the group of special applicants consists only of those with exceptional average grades and that 20 percent of total capacity is allocated in the first part. Under reasonable assumptions, all of the analysis below extends to the case where the set of special applicants also includes wait-time applicants. This and some other omitted details of the German admissions procedure are discussed in the supplementary appendix, Section 7.

I now introduce some notation and assumptions that will be used in the analysis below. There is a finite set of applicants A and a finite set of universities U. University u has q_u places to allocate among applicants. To avoid integer problems, assume throughout that q_u is a multiple of five and let $q_u^1 = \frac{1}{5}q_u$. The average grade of applicant a will be denoted by $g(a) \in \mathbb{R}_+$. Applicant a has a better average grade than applicant a' if $g(a) < g(a')$. I assume throughout that there are no ties in average grades so that $g(a) \neq g(a')$ whenever $a \neq a'$. Only the applicants with the $q^1 = \sum_u q_u^1$ lowest/best average grades are allowed to participate in the first part of the procedure. In the following, these applicants will be called *top-grade applicants*.

I assume throughout that all universities have *objective evaluation procedures*. By this I mean that universities' preferences for the second part of the procedure are based on objective criteria such as, but not restricted to, average grades, results from study-specific tests, practical experience, and so on. More formally, I assume that for each university u there is a strict

[9]The main reference for this section is the *Verordnung über die zentrale Vergabe von Studienplätzen durch die Stiftung für Hochschulzulassung (VergabeVO Stiftung)*, which can be found at www.hochschulstart.de.

ranking $\succ_u$ of $A \cup \{u\}$ that is fixed and known *prior* to the application deadline. University u's preferences over any set of applicants $B \subseteq A$ that remains in the procedure until the second part are then given by the restriction of $\succ$ to B, $\succ_u |_{B\cup\{u\}}$. This essentially assumes that universities do not act strategically.[10] While this assumption is certainly not without loss of generality, the majority of universities rely on objective evaluation procedures. For example, in the assignment procedure for places in Pharmacy, only 2 out of 22 universities chose to employ any subjective criteria for the winter term 2010/2011 (see Section **??**).

We are now in place to provide a formal description of the German admissions procedure. In order to participate, applicants have to submit two strict preference lists. There is no consistency requirement across the two lists and the list submitted for part $t \in \{1, 2\}$ is used only to determine assignments in part t. All preference lists are submitted simultaneously before any assignments are determined. Let $Q_a = (Q_a^1, Q_a^2)$ denote the profile of preference lists submitted by applicant $a \in A$ and $Q_A = (Q_A^1, Q_A^2) = ((Q_a^1)_{a\in A}, (Q_a^2)_{a\in A})$ denote the profile of reports by all applicants. Given Q_A and $\succ_U = (\succ_u)_{u\in U}$, the admissions procedure works as follows:

The German admissions procedure

Part 1: Assignment for top-grade applicants

Apply the Boston mechanism to determine assignments of top-grade applicants: University u can admit at most q_u^1 applicants, the preference relation of a top-grade applicant a is Q_a^1, and an applicant's priority for a university is determined by her average grade.
Denote the matching produced in part 1 by $f^{G1}(Q_A^1)$, the set of admitted applicants by A_1, the number of empty seats at university u by $r_{(u,1)} = q_u^1 - |f_u^{G1}(Q_A^1)|$, and the set of remaining applicants by $A^2 = A \setminus A_1$.

Part 2: Assignment according to universities' preferences

Apply the university proposing deferred acceptance algorithm to assign remaining places among remaining applicants: University u can admit at most $q_u^2 := \frac{4}{5}q_u + r_{(u,1)}$ applicants, the

[10] More formally, under this assumption the admissions procedure induces a sequential revelation game where in the first stage universities choose their evaluation procedures and in the second stage applicants choose their application strategies. My analysis focuses on the second stage of this revelation game.

preference relation of an applicant $a \in A^2$ is given by Q_a^2, and u's preferences are responsive to $\succ_u$. Denote the matching produced in the second part of the procedure by $f^{G2}(Q_A^2, \succ_U)$.

□

Given Q_A and $\succ_U$, let $f^G(Q_A, \succ_U) = (f^{G1}(Q_A^1), f^{G2}(Q_A^2, \succ_U))$ denote the pair of matchings chosen by the German admissions procedure. Since I will take $\succ_U$ to be fixed throughout, the dependency of f^G on $\succ_U$ will usually be omitted in the following.

Example 1. Suppose that $A = \{a_1, \ldots, a_6\}$ and $U = \{u_1, u_2, u_3\}$. For simplicity, assume that each university has two places to allocate among students and that one place at each university is available in both parts of the assignment procedure.[11] Applicants are indexed in increasing order of their average grades, so that a_i has the ith best/lowest average grade among $a_1, \ldots, a_6$. Given the above assumptions, applicants a_1, a_2, a_3 are the top-grade applicants. Universities' preferences are as follows

$\succ_{u_1}$	a_1	a_4	a_5	a_2	a_6	a_3
$\succ_{u_2}$	a_1	a_6	a_2	a_3	a_5	a_4
$\succ_{u_3}$	a_1	a_4	a_5	a_2	a_6	a_3

Finally, applicants' (true) preferences are given by

P_{a_1}	P_{a_2}	P_{a_3}	P_{a_4}	P_{a_5}	P_{a_6}
u_1	u_1	u_3	u_2	u_2	u_3
u_2	u_3	u_2	u_1	u_3	u_2
u_3	u_2	u_1	u_3	u_1	u_1

I now calculate the assignment chosen by the German admissions procedure under the assumption that all participants submit their preferences truthfully for each step of the procedure. This assumption is made for illustrative purposes only and as we will shortly see, applicants can in fact profit from misrepresenting their preferences. In the above example, the German

[11] It is unproblematic to enlarge the example so that each university's capacity is some multiple of five. Larger examples do not facilitate understanding of the assignment procedure and all points made below apply equally well to larger, more realistic settings.

admissions procedure yields the following assignments

$$f^{G1}(P_A) = \begin{matrix} u_1 & u_2 & u_3 \\ a_1 & a_2 & a_3 \end{matrix} \qquad f^{G2}(P_A) = \begin{matrix} u_1 & u_2 & u_3 \\ a_4 & a_6 & a_5 \end{matrix}$$

It is easy to see that applicants a_2 and a_5 could both secure a place at a more preferred university *within* the part of the procedure that they were assigned in. For example, applicant a_2 would obtain a place at her second most preferred university u_3 in part 1 of the procedure if only she had ranked it first. While these problems are known (cf Section 2.3), applicant a_2 could do even better than securing a place at u_3 in the first part: Suppose that a_2 *truncated* her ranking and declared only u_1 as acceptable for all parts of the procedure. Assuming all others continue to submit their preferences truthfully, applicants a_2, a_4, a_5, a_6 will remain in the procedure by the beginning of the third part, while the assignments of the other applicants are left unchanged. This implies in particular that the one place at u_2 initially reserved for applicants with excellent average grades is left unassigned in the first part and thus becomes available again in the second part of the procedure. It is easy to see that the assignment chosen in the second part of the procedure is then

$$\begin{matrix} u_1 & u_2 & u_3 \\ a_2 & \{a_4, a_6\} & a_5. \end{matrix}$$

Hence, a_2 can obtain a place at her most preferred university u_1 if she declares all other universities unacceptable. This suggests that in particular for applicants who can expect to be eligible for the first part of the procedure, finding an optimal application strategy is more difficult than in a one-shot application of the Boston mechanism: Such applicants first need to figure out what the best possible assignment is that they could get in the first part of the procedure. As in the example above, achieving this best possible assignment will often involve *over-reporting* preferences for some universities. This part of an applicant's optimization program is exactly the same as in a one-shot application of the Boston mechanism. However, in the German admissions system applicants face an additional problem since they also have to consider the trade-off between being assigned in the first part of the procedure and participating in the third part of the procedure. As seen above, *truncating* preferences increases the chances

of participating in the third part, where an even better assignment might be obtained. This is, however, often risky since applicants who fail to secure a place in the first part lose their guaranteed priority over others. In the next section, I analyze this trade-off and characterize equilibrium outcomes of the revelation game induced by the German admissions procedure.

□

3.1 Analysis

The above example shows that applicants sometimes have strong incentives to manipulate the German admissions procedure by submitting a ranking of universities that does not correspond to their true preferences. Strategic behavior is encouraged by the ability to submit different preference lists for the two parts of the procedure since study conditions are the same no matter in which part of the procedure an applicant receives a place at a given university. Furthermore, applicants are explicitly advised that stating preferences truthfully may not be in their best interest. On the official website of the administrator of the centralized procedure, www.hochschulstart.de, applicants are cautioned that (1) for the first part, their chances of admission at a university may significantly decrease if they do not rank it at the top, and (2) they should truncate their preference lists for the first part of the procedure if they want to be considered in the second part. Using data on submitted preference lists for the winter term 2006/2007, Braun et al (2010) provide empirical evidence that applicants do act upon the incentives to manipulate the admissions procedure.[12] In particular, a significant percentage of applicants reported different preference lists for the two parts of the procedure.

In order to evaluate the performance of the university admissions system it is thus important to analyze the strategic incentives induced by the assignment procedure. In this section, I provide a full characterization of complete information Nash-equilibrium outcomes of the revelation game induced by the current assignment procedure. While complete information is admittedly a strong assumption, data on previous outcomes of the admissions procedure are publicly available on the website of the centralized procedure's administrator. Assuming some stationarity, this provides applicants with relatively detailed information on the environment.[13]

[12]The admissions procedure for the winter term 2006/2007 is slightly different from the one used for the winter term 2010/2011 in terms of the rule for redistributing unused capacity. However, the basic structure and incentives of applicants have remained the same.

[13]While the assumption of complete information becomes less compelling without the assumption of ob-

For the analysis, I need some auxiliary definitions and assumptions. First, I define an outcome (of the centralized admissions procedure).

Definition 1. *An **outcome (of the centralized admissions procedure)** is a pair of matchings of applicants and universities $\mu = (\mu_1, \mu_2)$ such that*

(i) for $t \in \{1,2\}$, capacity constraints of part t are respected, i.e. $r_{(u,t)}(\mu) \geq 0$ for $t \in \{1,2\}$, where $r_{(u,1)}(\mu) = \frac{1}{5}q_u - |\mu_t(u)|$ and $r_{(u,2)}(\mu) = \frac{4}{5}q_u + r_{(u,1)}(\mu) - |\mu_2(u)|$, and

(ii) each applicant is matched to at most one university, that is, $|(\cup_t \mu_t(a)) \cap U| \leq 1$.

Given an outcome of the centralized procedure $\mu = (\mu_1, \mu_2)$, let applicant a's *assignment under* μ, $\mu(a)$, be $\mu_t(a)$ if $\mu_t(a) \in U$, and a if $(\mu_1(a) \cup \mu_2(a)) \cap U = \emptyset$. I assume that top-grade applicants have *lexicographic preferences* in the sense that when comparing two assignments a top-grade applicant primarily cares about the university she is assigned to and if two outcomes assign her to the same university, but in different parts of the procedure, she prefers the outcome that assigns her in the first part of the procedure. More formally, this means that applicant a has a strict ranking P_a of $U \cup \{a\}$ such that she strictly prefers an outcome (of the centralized procedure) μ over an outcome ν if and only if either $\mu(a) P_a \nu(a)$, or $\mu_1(a) = \nu_2(a) = u$ for some $u \in U$. This assumption is reasonable since assignments in the first part are determined more than one month before the second part is conducted. Since study conditions do not depend on the particular part of the procedure in which an applicant is admitted by a given university, the additional time to search for an apartment, prepare to move, and so on, motivates strict preference for early assignment. Finally, it will prove useful to define for each university $u \in U$ one strict ordering $\succ_{(u,1)}$ of $A \cup \{u\}$ by setting $a \succ_{(u,1)} u$ if and only if a is a top-grade student and $a \succ_{(u,1)} a'$ if and only if $g(a) < g(a')$, and a second strict ordering of $A \cup \{u\}$ by setting $\succ_{(u,2)} = \succ_u$. Here, $\succ_{(u,t)}$ is the preference/priority ranking that is used to allocated u's places in the tth part of the admissions procedure ($t \in \{1,2\}$).

We are now ready to analyze the revelation game induced by the German admissions procedure. In this game, applicants simultaneously submit two preference rankings which are then

jective evaluation procedures, no university relies exclusively on subjective criteria for its evaluation process. In particular, it is required that *in the determination of an applicant's position* [in universities' rankings] *average grade has to be a decisive factor* (*Merkblatt M09: Auswahlverfahren der Hochschulen*, http://hochschulstart.de/fileadmin/downloads/Merkblaetter/M09.pdf; translation by the author). This should allow applicants to form relatively reliable estimates of universities' preferences, so that the subsequent analysis should, to some extent, remain valid even when the partly subjective nature of evaluation procedures is acknowledged.

used to compute an outcome according to the rules explained above. Throughout the analysis, I assume that all of the assumptions introduced above (no ties in average grades, objective evaluation procedures, lexicographic preferences over assignments) are satisfied. I will now characterize equilibrium outcomes of the revelation game induced by the German admissions procedure by means of the following stability condition.

Definition 2. *An outcome $\mu = (\mu_1, \mu_2)$ is **stable** with respect to P_A if*

(i) no applicant is matched to an unacceptable university, that is, $\mu(a) R_a a$, for all $a \in A$,

(ii) no applicant receives a place for which she is unacceptable, that is, $a \succ_{(u,t)} u$ for all $a \in \mu_t(u)$ and $t = 1, 2$,

(iii) there is no applicant-university pair that blocks μ, that is, there is no pair (a, u) such that $u P_a \mu(a)$ and for some $t \in \{1, 2\}$ such that $a \succ_{(u,t)} u$ either $r_{(u,t)}(\mu) > 0$ or $a \succ_{(u,t)} a'$ for some $a' \in \mu_t(u)$,

(iv) applicants are matched as early as possible, that is, if $\mu_2(a) = u$ and $a \succ_{(u,1)} u$, then $r_{(u,1)}(\mu) = 0$ and $a' \succ_{(u,1)} a$ for all $a' \in \mu_1(u)$.

This definition of stability takes into account that different criteria are used to regulate admission in the two parts of the assignment procedure. Part (iv) of this definition ensures that in case of multiple possibilities of admission at a university, an applicant takes the place that was intended for her. Finally, note that this definition of stability takes into account that places reserved for, but not taken by top-grade applicants can be allocated according to universities' criteria. The following is the main result of this section.

Theorem 1. *Let P_A be an arbitrary profile of strict applicant preferences.*

The set of pure strategy Nash equilibrium outcomes of the game induced by the German admissions procedure coincides with the set of stable outcomes with respect to P_A.

While this result is related to the equilibrium characterizations for the Boston mechanism by Ergin and Sönmez (2006) and for the revelation game induced by the CDA by Sotomayor (2007), its proof is significantly more difficult due to the sequential nature of the German admissions procedure and the possibility of capacity redistribution. To get some intuition (the formal proof is in the Appendix) for this result, suppose a top-grade applicant a is matched in the first part of the German admissions procedure but could be admitted at some strictly

preferred university u in part 2 (given the set of applicants admitted in that part). In this case a could profitably deviate by ranking only u for both parts of the procedure: Since a was matched in the first part under her original report, only a subset of applicants have to wait for part 2 when she deviates in the just described way. But this implies that all universities make more offers in the CDA of part 2 and in particular a must receive an offer by u given the assumed instability. Next, I calculate the set of stable matchings in Example 1.

Example 2. Consider again the setting of Example 1. The following are the only two stable outcomes:

$$\mu_1 = \begin{matrix} u_1 & u_2 & u_3 \\ a_1 & \emptyset & a_3 \end{matrix}, \qquad \mu_2 = \begin{matrix} u_1 & u_2 & u_3 \\ a_2 & \{a_4, a_5\} & a_6 \end{matrix},$$

and

$$\nu_1 = \begin{matrix} u_1 & u_2 & u_3 \\ a_1 & a_3 & a_2 \end{matrix}, \qquad \nu_2 = \begin{matrix} u_1 & u_2 & u_3 \\ a_4 & a_6 & a_5 \end{matrix}.$$

By Theorem 1, there are thus two pure strategy equilibrium outcomes of the revelation game among applicants. Note that all applicants weakly prefer μ over ν. One strategy profile that implements the first matching is the following: All top-grade applicants (a_1, a_2, a_3) rank only their most preferred university for part 1 and submit their true ranking for part 2. The remaining three applicants (a_4, a_5, a_6) rank only their most preferred university for part 2. For a_2 this means that she *truncates* her true preferences so that she will stay in the procedure until part 2, where she can be assigned a place at her most preferred university u_1 given the reports of the others. The empirical analysis in Braun et al (2010) suggests that such strategies are indeed used since top-grade applicants' lists for the first part are relatively short.

However, note that a_2 is guaranteed to obtain a place at u_3 in part 1 if she ranks this university first - irrespective of the reports of the other applicants. On the other hand, a_2 has to rely on others to follow the right equilibrium strategy in order to reach the Pareto dominant equilibrium. If coordination fails, this strategy might even lead to a_2 being assigned to her third choice u_2.[14] In this sense the Pareto dominant equilibrium is more risky for a_2 so that

[14]This would happen, if e.g. a_6 ranks u_2 higher than u_3. In contrast to standard college admission problems, this is not necessarily a dominated strategy for a_6 here since u_2 may consider only students who ranked it

she might be inclined to use the safe strategy of over-reporting her preference for u_3 in part 1.

□

Theorem 1 shows that the potential instabilities of the centralized admissions procedure we saw in Example 1 are "corrected" by the strategic behavior of applicants. This will prove to be a useful benchmark for a comparison between my proposed redesign and the current procedure in Section 4.1. We will later see that the set of stable outcomes for the university admissions problem coincides with the set of stable matchings for a related college admissions problem with substitutable preferences.[15] Since for such problems there exists a student/applicant optimal stable matching that all students/applicants (weakly) prefer to any other stable matching, this implies that the German admission procedure supports Pareto dominated outcomes, as was already exemplified above.

4 Towards a New Design: Matching with complex constraints

The analysis of the current German admission system showed that it provides students with strong incentives to try to manipulate the procedure by strategically misrepresenting their preferences and that strategizing by students can lead to inefficiencies. But are there matching mechanisms that are immune to strategic manipulation by students and at the same time achieve a satisfactory allocative performance? Since top-grade applicants are, in principle, also eligible to obtain one of the places that universities are allowed to assign according to their own criteria, immunity to manipulations by applicants requires all places at each university to be allocated simultaneously. For example, even when the two assignment procedures in the current German admissions procedure were abandoned in favor of the student proposing deferred acceptance algorithm, the resulting direct mechanism would not be strategy-proof for top-grade applicants if the sequential structure is maintained: These applicants would still sometimes have an incentive to submit short preference lists in order to not forfeit their

first. Such *ranking constraints* are popular with German universities. As discussed in the Online Appendix, the presence of such constraints does not change the equilibrium characterization.

[15] This will also establish the existence of pure strategy equilibria of the game induced by the German admissions procedure.

chances in the second part of the procedure. An important problem in developing an appealing alternative admission system is thus to find ways of reconciling the simultaneous allocation of all seats with the institutional constraint that some places are reserved for special student groups and should only become available to other students if there is insufficient demand from these groups. This institutional constraint is an important cornerstone of the German university admissions system since quota for top-grade students represents the political will to give prioritized access to the very best high school graduates.[16] Constraints similar to those of the German admissions system also play an important role in school choice. Here, public schools often have a desire to achieve some target distribution of student types in their entering classes (e.g. an even distribution of sexes) but cannot afford to waste capacity to achieve this distributional goal, so that they sometimes have to accept violations of their affirmative action policies. However, a school may prefer some violations over others and it is thus important to provide it with enough flexibility to express such preferences. This issue has not been studied in the previous matching literature. Major differences to other matching problems with constraints that have been studied in the existing literature are discussed in Section 4.2.

I now introduce a general class of *matching problems with complex constraints* that can accommodate both of the applications mentioned above. Formally, a matching problem with complex constraints consists of

- a finite set of colleges C,
- a finite set of students I,
- for each student $i \in I$, a strict ordering P_i of $C \cup \{i\}$
- for each college c a *choice protocol* consisting of
  - (i) a vector $\succ_c = (\succ_{(c,t)})_{t=1}^{T_c}$, where for all $t \leq T_c$, $\succ_{(c,t)}$ is a strict ordering of $I \cup \{c\}$,
  - (ii) a sequence of capacities $q_c = (q_{(c,t)})_{t=1}^{T_c}$, where $q_{(c,1)} \in \{0, \ldots, |I|\}$ and $q_{(c,t)} : \{0, \ldots, |I|\}^{t-1} \to \{0, \ldots, |I|\}$ for all $t \geq 2$.

The idea behind this class of problems is that a college c may decide or be required to reserve certain parts of its capacity for special student groups (e.g. siblings, students with excellent

[16] The quota for wait-time students on the other hand, which I have abstracted from in the main body of this paper, is necessary in order for the admissions process to satisfy the constitutional requirement that every applicant with the appropriate qualification (i.e. having successfully finished secondary school) should have a *chance* of studying any subject she wants.

average grades, and so on) and may want to make some of these reserved seats available to other student groups to accommodate to the characteristics of applicants. Such constraints are encoded in the choice protocol of c: First, it specifies an order in which special student groups are to be considered, where the tth group to be considered is the set of acceptable students with respect to $\succ_{(c,t)}$. Note that a student may belong to multiple special student groups, so that $(\succ_{(c,t)})_t$ may not partition the set of students into disjoint acceptable subgroups. In particular, a given student may be considered multiple times by a choice protocol. Secondly, it dictates how much capacity is reserved for each group as a function of seats left vacant by groups considered earlier, starting from some fixed value for the first group to be considered. I will usually refer to the sequence $(q_{(c,t)}(0,\dots,0))_{t=1}^{T_c}$ as the *target distribution of college c*. The idea behind this is that college c initially intends to allocate $\overline{q}_c := \sum_t q_{(c,t)}(0,\dots,0)$ places and has a strict preference for filling these places according to its target distribution. If its target distribution cannot be achieved because too few students from one or more of the T_c student groups apply, a college can express its preferences over possible alternate distributions of student groups by specifying how capacity is to be redistributed through its choice of the functions $q_{(c,t)}$. For my purpose it is without loss of generality to assume that $T_c = T_{c'}$ for all colleges c, c' (since we can always set $q_{(c,t)}(r_1,\dots,r_{t-1}) \equiv 0$ if $t > T_c$) and I will henceforth let T denote the common final step of all colleges' choice protocols. Finally, note that the above formulation implicitly assumes that there are no specific advantages or disadvantages associated with being admitted because one belongs to a particular special student group so that students do not care about the type of place they receive, but only about their assigned colleges. I now illustrate these concepts by a simple but important practical example

Example 3 (Siblings and affirmative action constraints)**.** Consider a school (college) c with a fixed number of total seats $\overline{q}_c$. Suppose the school has to offer admittance to students who have a sibling already attending the school. If possible, the school then wants to distribute any remaining capacity equally among male and female students.

This can be implemented by a choice protocol as follows: First, let $\succ_1$ be some strict ordering of $I \cup \{c\}$ such that only students with siblings attending c are acceptable. Next, let $\succ_2$ be a strict ranking such that only male students are acceptable and $\succ_3$ be a strict ranking such that only female students are acceptable. To complete the description of the choice

protocol, let $q_1 \equiv \overline{q}_c$. If $l_1 < q_1$ students with siblings apply, $r_1 := q_1 - l_1$ can be allocated among remaining applicants. To ensure that seats not taken by students with siblings are distributed (approximately) equally between male and female students, set $q_2(r_1) = \lfloor \frac{r_1}{2} \rfloor$ and $q_3(r_1, r_2) = \lceil \frac{r_1}{2} \rceil$.

□

I now proceed to define a concept of stability adapted to matching problems with complex constraints. A *matching sequence* is a finite sequence of matchings $\mu = (\mu_1, \ldots, \mu_T)$ such that for all $t \leq T$, μ_t is a matching of students and colleges. Given a matching sequence μ, define the *associated sequence of empty seats* by first setting $r_{(c,1)}(\mu) := q_{(c,1)} - |\mu_1(c)|$, and then, assuming that $r_{(c,1)}(\mu), \ldots, r_{(c,t-1)}(\mu)$ have already been defined, $r_{(c,t)}(\mu) := q_{(c,t)}(r_{(c,1)}(\mu), \ldots, r_{(c,t-1)}(\mu)) - |\mu_t(c)|$. I now define a notion of feasibility with respect to a given profile of choice protocols.

Definition 3. *Given a profile of choice protocols $(\succ_c, q_c)_{c \in C}$, a matching sequence $\mu = (\mu_1, \ldots, \mu_T)$ is* feasible*, if*

(i) for all t and c, $r_{(c,t)}(\mu) \geq 0$, and

(ii) for all $i \in I$, $|(\cup_t \mu_t(i)) \cap C| \leq 1$.

Feasibility requires that each student is matched to at most one college and that all capacity constraints of all colleges are satisfied. Given a feasible matching sequence μ let student i's *assignment under* μ, $\mu(i)$, be $\mu_t(i)$ if $\mu_t(i) \in C$, and i, if $(\cup_t \mu_t(i)) \cap C = \emptyset$. Student i *receives a place in quota* t of college c (und μ), if $i \in \mu_t(c)$ and the set of students matched to c is $\mu(c) := \cup_t \mu_t(c)$. I now define a concept of stability that is adapted to the specific structure of a matching problem with complex constraints.

Definition 4. *Fix a profile of choice protocols $(\succ_c, q_c)_{c \in C}$.*

A feasible matching sequence $\mu = (\mu_1, \ldots, \mu_T)$ is ***procedurally stable*** *with respect to a profile P_I of student preferences, if*

(i) $\mu(i) R_i i$,

(ii) $i \succ_{(c,t)} c$ for all $i \in \mu_t(c)$,

(iii) *if* $cP_i\mu(i)$ *then there is no* t *such that* $i \succ_{(c,t)} c$ *and either* $r_{(c,t)}(\mu) > 0$, *or* $i \succ_{(c,t)} j$ *for some* $j \in \mu_t(c)$, *and*

(iv) *if* $i \in \mu_t(c)$, *there is no* $s < t$ *such that* $i \succ_{(c,s)} c$ *and either* $r_{(c,s)}(\mu) > 0$, *or* $i \succ_{(c,s)} j$ *for some* $j \in \mu_s(c)$.

Condition (i) is the standard individual rationality constraint for students. Condition (ii) requires that a college's desire to restrict certain parts of its capacity to special student groups is respected. Condition (iii) requires that a matching sequence is not blocked by a student-college (i, c) pair in the sense that i strictly prefers c to her allocation under μ, and that c would strictly prefer to admit i in at least one of its quotas given the matching sequence. Note that whether i and c form a blocking pair does not only depend on the set of students matched to c, but also on the particular distribution of these students across c's different quotas. Finally, condition (iv) requires that a student who receives a place in quota t of college c could not have been admitted in an earlier quota given the set of applicants $\mu(c)$. This requirement can be understood as a desire to stay as close as possible to the target distribution since the more students take the places that were intended for them (i.e. the earliest quota in which they can be admitted), the less capacity redistribution. The concept of procedural stability is a natural and desirable allocative criterion in the spirit of the stability concepts that have been used in theory and practical applications of matching models. The next example shows that a procedurally stable matching does not always exist.

Example 4. There are two colleges c_1, c_2, and two students i_1, i_2. The choice protocols of the two colleges are given by

$\succ_{(c_1,1)}$	$\succ_{(c_1,2)}$	q_{c_1}	$\succ_{(c_2,1)}$	$\succ_{(c_2,2)}$	q_{c_2}
i_1	i_2	$q_{(c_1,1)} = 1$	i_2	c_2	$q_{(c_2,1)} = 1$
c_1	c_1	$q_{(c_1,2)}(0) = 1$	i_1	i_2	$q_{(c_2,2)}(0) = 0$
i_2	i_1	$q_{(c_1,2)}(1) = 0$	c_2	i_1	$q_{(c_2,2)}(1) = 0$

Note that the target capacity of c_1 is two, while the target capacity for c_2 is one. Now suppose students' preferences are given by $P_{i_1} : c_2, c_1$ and $P_{i_2} : c_1, c_2$.

Then no procedurally stable matching exists: Suppose first that both students are matched to c_1. Then i_1 would strictly prefer to be matched to c_2 and c_2 has one place available, so that

the matching could not have been stable. If only one student is matched to c_1, it must be i_1. In this case, however, i_2 and c_1 would block the matching. Finally, if no student is matched to c_1, we must have that i_2 is matched to c_2, as otherwise they would block the matching. But then i_1 and c_1 would block the matching.[17]

□

Thus, in order to guarantee the existence of a procedurally stable matching, one has to restrict choice protocols. I now introduce two independent restrictions, which guarantee that situations such as in the example cannot occur.

Definition 5. *(i) A choice protocol $(\succ_c, q_c)$ is **monotonic**, if for all t and all pairs of sequences $(r_s, \tilde{r}_s)_{s=1}^{t-1}$ such that $\tilde{r}_s \geq r_s$ for all $s \leq t-1$,*

$$q_{(c,t)}(\tilde{r}_1, \ldots, \tilde{r}_{t-1}) \geq q_{(c,t)}(r_1, \ldots, r_{t-1}).$$

*(ii) A choice protocol $(\succ_c, q_c)$ satisfies **non-excessive reduction**, if for all t and all pairs of sequences $(r_s, \tilde{r}_s)_{s=1}^{t-1}$ such that for $s \leq t-1$, $\tilde{r}_s = r_s + k_s$ for some $k_s \geq 0$,*

$$\sum_{s=1}^{t} [q_{(c,s)}(\tilde{r}_1, \ldots, \tilde{r}_{s-1}) - q_{(c,s)}(r_1, \ldots, r_{s-1})] \leq \sum_{s=1}^{t-1} k_s.$$

The first requirement is that whenever weakly more seats are left unassigned in *every* quota from 1 to $t-1$, weakly more quota t seats should be made available to applicants. The second requirement says that in response to greater demand in steps 1 through $t-1$, students should not suffer too much in the sense that the total capacity reduction in steps 1 up to and including t, $\sum_{s=1}^{t} [q_{(c,s)}(\tilde{r}_1, \ldots, \tilde{r}_{s-1}) - q_{(c,s)}(r_1, \ldots, r_{t-1})]$, should not exceed the increase in demand in quotas 1 up to t, $\sum_{s=1}^{t-1} k_s$. The first requirement rules out a situation such as the one in Example 4. The second requirement basically rules out applications in which a college would like to decrease total capacity in response to increased demand in some quotas. This may be relevant in particular for applications to school choice, where a school may reserve some part of its capacity for students with, say, low reading scores. If students with lower reading scores

[17] The problem in this example is that students i_1 and i_2 are *complements* for college c_1 in the sense that it wants to admit i_2 only if it is able to attract i_1 as well. It is by now well known (see Hatfield and Milgrom 2005, and Hatfield and Kojima 2008) that in presence of complementarities stable matchings may fail to exist.

demand more attention than students with higher reading scores, a school may prefer to reduce the number of seats available to other students by, say, two for each additional student with low reading scores. The following is the main result of this section.

Theorem 2. *(i) Suppose all choice protocols are monotonic.*

Then for each profile of student preferences there exists a unique student optimal procedurally stable matching.

(ii) Suppose all choice protocols are monotonic and satisfy non-excessive reduction.

Then the student optimal procedurally stable matching mechanism is group strategy-proof for students.

The proof of this theorem, the formal argument can be found in Section 6, rests on a transformation of a matching problem with complex constraints to an associated college admissions problem (CAP), where choice protocols are eliminated from the description of the problem. For any profile of choice protocols, stability in this associated CAP turns out to be equivalent to procedural stability. Hence, given a profile of student preference P_I the student optimal stable matching for the associated college admissions problem, $f^I(P_I)$,[18] should it exist, would also be the student optimal procedurally stable matching. If choice protocols are monotonic, college preferences in the associated CAP satisfy substitutability. If choice protocols are monotonic and satisfy non-excessive reduction, college preferences also satisfy the law of aggregate demand. Hence, Theorem 2 can be derived using the results of Roth (1984b) and Hatfield and Kojima (2009) described in Section 2.1.

Note that the choice protocol in Example 3 is monotonic and satisfies non-excessive reduction: Any place initially reserved for siblings becomes available to other students if there is insufficient demand from these groups, so that the choice protocol is monotonic. For each additional sibling admitted, the number of remaining places available for male and female students is reduced by at most one, so that the choice protocol also satisfies non-excessive reduction. I now discuss how to embed the German system into the framework developed in this section.

[18]Of course, this matching also depends on the choice protocols/induced choice functions. Given that I take choice protocols to be exogenous and only consider students' incentives for preference manipulations, I chose to suppress this dependency.

4.1 Application to the German System

The exogenous inputs of the German admissions system are the sets of universities U and applicants A, the vector of capacities q, average grades of applicants $g(\cdot)$, and universities' preferences $\succ_U$. The main constraint here was that

(i) 20 % of total capacity at each university is reserved for top-grade applicants, and that

(ii) all places reserved for, but not demanded by, top-grade applicants should be allocated on basis of the criteria chosen by universities (as represented by $\succ_U$).

This policy can be implemented as a matching problem with complex constraints where the choice protocol of a university u is given by $\succ_u := (\succ_{(u,1)}, \succ_{(u,2)})$, $q_{(u,1)} = \frac{1}{5}q_u$, and $q_{(u,2)}(r_1) = \frac{3}{5}q_u + r_1$. These choice protocols are clearly monotonic and satisfy non-excessive reduction. Since procedural stability is equivalent to the notion of stability introduced in Definition 2, we obtain the following corollary to Theorems 1 and 2.

Corollary 1. *(i) For all profiles of applicant preferences P_A,*

(i.1) there exists a unique applicant optimal procedurally stable matching $f^A(P_A)$,

(i.2) the set of pure strategy equilibrium outcomes of the game induced by the German admissions procedure is non-empty, and

(i.3) the applicant optimal procedurally stable matching Pareto dominates any pure strategy equilibrium outcome of the current German admissions procedure with respect to applicants' true preferences, that is, for all pure strategy equilibria Q of the game induced by the German admissions procedure, $f_a^A(P_a) R_a f_a^G(Q)$ for all $a \in A$.

(ii) The direct mechanism f^A is group strategy-proof for applicants.

Thus, applicants are unambiguously better off under the applicant optimal procedurally stable matching mechanism than under the current German matching mechanism. Note that in the alternative mechanism, applicants submit only one preference list, opposed to the two lists they can submit in the current German admissions procedure.

To develop some intuition for the above result, it is helpful to consider how a university u determines who to (temporarily) accept when applicants in $B \subseteq A$ apply to it in some step of the SDA used to compute the applicant optimal procedurally stable matching:

(1) Among applicants in B, temporarily accept the $q_{(u,1)}$ top-grade applicants with the best average grades.

Let B_1 be the set of temporarily accepted applicants, $B^2 = B \setminus B_1$, and $r_1 = q_{(u,1)} - |B_1|$.

(2) Among applicants in B^2, temporarily accept the $q_{(u,2)}(r_1) = \frac{3}{5}q_u + r_1$ highest ranking acceptable applicants with respect to $\succ_{(u,2)}$.

Let B_3 be the set of temporarily admitted applicants.

This mimics the current admissions procedure in the sense that an applicant's admission chances are always checked in the order top-grade/general admissions. However, in contrast to the current German admissions procedure top-grade applicants can claim one of the places initially reserved for them in *any* round of the assignment procedure. In particular, those applicants never have an incentive to over-report their preference for a university in order to "match early" and thus avoid losing their guaranteed priority. This may free up additional capacity that can be allocated on basis of universities' criteria, potentially leading to better assignments for those applicants not eligible for top-grade places. While one may worry that increased competition from top-grade applicants can also be harmful to other applicants, this can never happen: In an equilibrium of the current German admissions procedure, it can never be beneficial for a top-grade applicant matched in the first part to displace an applicant who was matched in the second part of the procedure due to Theorem 1.

4.2 Other matching problems with constraints

I now discuss some other approaches to and models of matching models with constraints to outline the most important differences. For the discussion, fix a profile of monotonic and consistent choice protocols $(\succ_c, q_c)_{c\in C}$ where for each $c \in C$, $(\succ_c, q_c) = (\succ_{(c,t)}, q_{(c,t)})_{t=1}^{T}$.

Suppose first that for all $c, c' \in C$ and each t, the set of acceptable students with respect to $\succ_{(c,t)}$ is the same as the set of acceptable students with respect to $\succ_{(c',t)}$. In this case, say that student s has *characteristic* t, if he or she is acceptable with respect to $\succ_{(c,t)}$ for all $c \in C$. Note that at least $\overline{q}_{(c,t)} := q_{(c,t)}(0, \ldots, 0)$ places are exclusively reserved for students having characteristic t given the monotonicity of choice protocols. Even for this special case a matching problem with complex constraints cannot be reduced to a college admissions problem with responsive preferences. First, a procedurally stable outcome cannot in general be achieved by splitting

each college into T "mini" colleges $c_1, \ldots, c_T$ with capacities $\overline{q}_{(c,1)}, \ldots, \overline{q}_{(c,t)}$ and "preferences" $\succ_{(c,1)}, \ldots, \succ_{(c,T)}$, and then running separate assignment procedures for each characteristic. This approach is only guaranteed to work if students have *at most one* characteristic and if capacity *cannot* be redistributed. Second, it is not in general possible to eliminate the possibility of capacity redistribution by modifying colleges' priority/preference orderings. This would only be feasible if, for all characteristics t and all colleges c, there is an ordering $t = t_0 < t_1 < \ldots < t_n$ such that, for all j, *all* places that c initially reserved for students with characteristic t are supposed to become available to students with characteristic t_{j+1} whenever there is insufficient demand from students with characteristics $t_0, \ldots, t_j$. In this case capacity redistribution can be accommodated by replacing $\succ_{(c,t)}$ with a ranking $\succ'_{(c,t)}$ which ranks all students with characteristics $t_0, \ldots, t_n$ as acceptable and ranks all students with characteristic t_j who do not have characteristics $t_0, \ldots, t_{j-1}$ below all students who have at least one of those characteristics. This approach is no longer feasible if, as in Example 3, remaining seats for some characteristics are supposed to be split between several other characteristics.

Next, consider the *matching problem with affirmative action constraints* of Abdulkadiroglu and Sönmez (2003) and Abdulkadiroglu (2005). In this problem, each school c has a fixed upper bound q_c on the total number of students it can admit and evaluates all individual students according to the same preference ordering P_c. Each student i has *one* characteristic $\tau(i)$ (e.g. being male or female or belonging to a minority) and for each characteristic $t \in T := \cup_{i \in I}\{\tau(i)\}$ each school c has a fixed *upper bound* q_c^t on the number of students with characteristic t it is willing to admit. Abdulkadiroglu (2005) shows how the SDA can be modified to cope with such affirmative constraints while maintaining its desirable allocative and incentive properties. The main differences between matching problems with affirmative action constraints and matching problems with complex constraints are that in the latter (1) seats are exclusively reserved for students that have the corresponding characteristic, (2) students may have more than one characteristic, and (3) capacity may be redistributed between different type-specific quotas. This is not meant to imply that the matching problem considered in Abdulkadiroglu and Sönmez (2003) and Abdulkadiroglu (2005) is a special case of the matching problem with complex constraints considered here. Rather, matching problems with complex constraints are better suited for applications where the main concern is to *achieve a particular distribution* of

student characteristics and schools have preferences over how capacity is to be redistributed between quotas if the desired distribution cannot be met. Matching problems with affirmative action constraints on the other hand are better suited when the main concern is to *limit the number* of students with a given characteristic.

Recently, Hafalir et al (2011) have analyzed a model of school choice with minority and majority students in which each school reserves a number of its seats for minority students. They consider a stability concept that differs in the blocking opportunities it allows for minority and majority students, respectively. Minority students are allowed to block a non-wasteful matching whenever some strictly preferred school has not met its minority reserve or has admitted at least one strictly less preferred student. Majority students, on the other hand, are only allowed to block a non-wasteful matching if a strictly preferred school has admitted at least one strictly less preferred student who could be rejected without violating its minority reserve. All students are allowed to block a wasteful matching, however. Hafalir et al (2011) show that there exists a mechanism that always finds a student optimal stable matching with minority reserves and that is group-strategyproof for students. This result is a special case of my Theorem 2. To see this note that a minority reserve of r_c can be implemented by a choice protocol that first considers only minority students and allocates up to r_c places to these students and then allocates all remaining places among remaining applicants. It is not hard to see that these choice protocols are monotonic and satisfy non-excessive reduction. Furthermore, procedural stability with respect to these choice protocols is equivalent to the concept of stability with respect to minority reserves that was explained above. A formal argument for these claims can be found in Section 7.1.2. I should emphasize at this point that the main focus in Hafalir et al (2011) is to compare hard and soft affirmative action constraints and that they also consider these questions for efficient mechanisms. In another paper, Ehlers et al (2011) consider school choice problems with upper and lower bounds on the number of students of a given type. They show that while the existence of "stable" and strategyproof matching mechanisms cannot be guaranteed when these bounds are "hard", such mechanisms do exist for soft bounds that allow violations of type-specific quotas under certain conditions. The affirmative action constraints considered in Ehlers et al (2011) can not be implemented by the type of choice protocols that I consider. The reason is that in presence of (soft or hard) upper bounds on more than one

student type, choice protocols would have to condition the number of available seats in each step not only on the *numbers* of previously unfilled seats, but also on the *types* of previously admitted students. However, the model in Ehlers et al (2011) is not more general than my model of matching with complex constraints since they do not allow schools to have preferences over the way in which capacity is redistributed.

Finally, I compare matching problems with complex constraints to the model of matching with regional caps introduced by Kamada and Kojima (2011). In their model each college (hospital in their paper) belongs to a *region*, each region has a cap on the total number of students that can be admitted by its colleges, and students have strict preferences over colleges (and the option of remaining unmatched). In addition, each region has preferences over how its capacity is to be distributed among its colleges. This problem is conceptually similar to a matching problem with complex constraints in the sense that regional capacities are fixed, but the capacity of a given college c depends on how many students apply to the different colleges in c's region. However, a crucial difference is that in the application of Kamada and Kojima, students have strict preferences over individual colleges, whereas in a matching problem with complex constraints students are indifferent about which *type* of place they receive. In particular, their techniques for transforming a matching model with constraints to a two-sided matching problem with contracts cannot be applied to matching problem with complex constraints. It should be stressed at this point, however, that their model is neither more nor less general than the model I consider here.

5 Conclusion and Discussion

This paper analyzed the assignment procedure that is used to allocate places at public universities for medicine and related subjects in Germany. The procedure uses two mechanisms, the Boston and the college optimal stable mechanism, that have been studied extensively in the matching literature. Assuming universities are not strategic, it was shown that complete information equilibria are characterized by a stability notion which takes the specific constraints of the German university admissions system into account.

To develop an alternative assignment mechanism, I introduced matching problems with complex constraints where college preferences are represented by choice protocols. It was shown

that if these protocols are monotonic and satisfy non-excessive reduction, a matching problem with complex constraints gives rise to a well defined associated college admissions problem for which a group strategy-proof (for students) and stable assignment procedure exists. I then showed that the German university admissions problem can be understood as matching problem with complex constraints in which colleges' choice protocols are monotonic and consistent. This implies in particular that the applicant optimal stable mechanism for the associated college admissions problem (i) provides (groups of) applicants with dominant strategy incentives for truthful preference revelation, and (ii) produces a matching that Pareto dominates (with respect to applicant preferences) any pure strategy equilibrium of the current admissions procedure.

Since the assignment mechanism developed in this paper allocates all places simultaneously, it requires universities to evaluate all applicants prior to the start of the procedure. This might be problematic if evaluation is costly, for example since it is based on interviews with applicants. In this case a university may interview an applicant who ends up taking one of the places reserved for top-grade applicants. While such wasteful investment into the evaluation of applicants can occur in the alternative mechanism, universities will usually have a good estimate of whether it makes sense for them to interview top-grade applicants. Furthermore, the above mechanism could be augmented by allowing applicants to send a *signal* to one of the universities that they are interested in being interviewed. As in Abdulkadiroglu et al (2011a), this would preserve ordinal strategyproofness of the alternative mechanism and would alleviate the problem of wasteful investments since universities could restrict attention to evaluating those top-grade applicants who signaled their interest.

6 Appendix

Proof of Theorem 1

I show first that for any profile of strict applicant preferences P_A, if Q is a pure strategy Nash equilibrium of the game induced by the German admissions procedure then $f^G(Q)$ must be a stable outcome at P_A. Note that stability condition (i) is satisfied since applicants can only be matched to a university that is on their submitted preference lists and it can thus never be optimal for an applicant to end up matched to an unacceptable university. Stability condition

(ii) is satisfied since only top-grade applicants are considered in the first part of the procedure and since the CDA never assigns an unacceptable applicant to a university. Stability conditions (iv) and (iii) for $t = 1$ follow since applicants prefer to be matched as early as possible and since a top-grade applicant who ranks a university u first for part 1 is guaranteed to receive a place at u unless q_u^1 or more top-grade applicants with better average grades also rank it first for part 1.

Now suppose that $f^G(Q) := (\mu_1, \mu_2)$ violates (iii) for $t = 2$ and satisfies all remaining stability conditions. Let (a, u) be a pair such that $uP_a\mu(a)$ and either ($a \succ_{(u,2)} u$ and $r_{(u,2)}(\mu) > 0$) or ($a \succ_{(u,2)} \tilde{a}$ for some $\tilde{a} \in \mu_2(u)$). I will show that Q cannot be a Nash-equilibrium. Let $\tilde{Q}_a$ be an alternative report for applicant a that lists only u for both parts of the procedure. Let $\tilde{Q} = (\tilde{Q}_a, Q_{-a})$ and $f^G(\tilde{Q}) = (\tilde{\mu}_1, \tilde{\mu}_2)$. It is clear that unless $\tilde{\mu}(a) = a$, $\tilde{Q}_a$ is a profitable deviation for a. I now show that $\tilde{\mu}(a) = a$ is impossible. Assume the contrary and let A^2 and $\tilde{A}^2$ denote the sets of applicants apart from a who remain in the procedure by the beginning of part 2 under Q and $\tilde{Q}$, respectively. Similarly, let q^2 and $\tilde{q}^2$ be the vectors of remaining capacities for the second part under Q and $\tilde{Q}$, respectively. Note that $\tilde{\mu}(a) = a$ implies $\tilde{A}^2 \subseteq A^2$, $\tilde{q}_u^2 = q_u^2$, and $\tilde{q}_v^2 \geq q_v^2$ for all $v \in U \setminus \{u\}$. This implies that if a does not receive an offer by u in the course of the CDA under $\tilde{Q}$, all applicants in $\tilde{A}^2$ receive a superset of the set of offers they got when the profile of submitted preferences was Q. In particular, any applicant in $\tilde{A}^2$ who received and declined an offer by u in the CDA under Q will also decline an offer by u in the CDA under $\tilde{Q}$. Hence, if $r_{(u,2)}(\mu) > 0$, we would have $r_{(u,2)}(\tilde{\mu}) > 0$ as well. But if $r_{(u,2)}(\tilde{\mu}) > 0$, u must have made an offer to a in the CDA under $\tilde{Q}$ provided that (a, u) block μ. This contradicts $\tilde{\mu}(a) = a$. Hence, we must have $r_{(u,2)}(\mu) = 0$. Since (iii) is violated, there has to be an applicant $\tilde{a} \in \mu_2(u)$ such that $a \succ_{(u,2)} \tilde{a}$. As shown above, under $\tilde{Q}$ all applicants in $\tilde{A}^2 \subseteq A^2$ receive a superset of the set of offers they received under Q. This implies that for all $\hat{a}$ such that $\hat{a} \succ_{(u,2)} \tilde{a}$ and $\hat{a} \notin \mu_2(u)$ we must also have $\hat{a} \notin \tilde{\mu}_2(u)$. But then $\tilde{\mu}_2(u)$ has to contain at least one applicant who ranks strictly lower on $\succ_{(u,2)}$ than a and u would have made an offer to a in the CDA under $\tilde{Q}$, a contradiction.

Now let $\mu = (\mu_1, \mu_2)$ be a stable outcome with respect to P_A. If $\mu(a) = a$, let a submit her true preferences for both parts of the procedure. If $\mu(a) = u$, let a rank only u for both parts of the procedure. Let Q be the resulting strategy profile.

I show first that $f^G(Q) = \mu$. Let $f^G(Q) = (\tilde{\mu}_1, \tilde{\mu}_2)$. It is easy to see that (iii) and (iv) imply $\tilde{\mu}_1 = \mu_1$. Given this, any applicant a with $\mu_2(a) \in U$ will not be assigned in parts 1 or 2 of the centralized procedure under Q. Since all of these applicants rank only their assigned university under μ_2 for part 2 while all other unassigned applicants submit their true preferences, (iii) would be violated if one of the unassigned applicants received a place in part 2 of the centralized procedure under Q. Hence, we must have $\mu_2 = \tilde{\mu}_2$.

Next, I show that Q is a Nash equilibrium profile. Let $\tilde{Q}_a$ be an alternative report for applicant a, $\tilde{Q} = (\tilde{Q}_a, Q_{-a})$, and $f^G(\tilde{Q}) = (\tilde{\mu}_1, \tilde{\mu}_2)$. Note that it cannot be the case that $\tilde{\mu}_1(a) = u = \mu_2(a)$ since all applicants in $\mu_1(u)$ apply to u in the first round of the Boston mechanism under $\tilde{Q}$ so that μ could not satisfy (iii) or (iv) otherwise. A similar argument shows that a cannot obtain a university strictly preferred to $\mu(a)$ according to P_a in part 1. It remains to be shown that a cannot strictly prefer $\tilde{\mu}_2(a)$ over $\mu(a)$. Consider first an applicant a such that $\mu_1(a) = a$. Given that μ satisfies (iii) and (iv) (and given the construction of Q), no alternative report $\tilde{Q}_a$ that leads to different assignments in part 1 can be profitable for a. We can hence assume w.l.o.g. that $\tilde{Q}_a^1 = Q_a^1$. But then if $\tilde{\mu}_2(a) P_a \mu_2(a)$ we obtain an immediate contradiction to (iii) for $t = 2$ since all applicants in $\mu_2(u)$ are available in part 2 of the centralized procedure under $\tilde{Q}$ and rank only u. Now consider an applicant a such that for some $u \in U$, $\mu_1(a) = u$. By (iii) and the construction of Q, there is no alternative report for a such that she obtains a strictly preferred university in part 1 according to P_a. Thus the only way that a could potentially improve upon her assignment under μ is that $\tilde{\mu}_1(a) = a$. But the only applicants who could take the leftover seat at u in part 1 are those who are either unassigned under μ or who are matched to u under μ_2. In particular, for all universities $v \neq u$, all applicants in $\mu_2(v)$ remain in the procedure by the beginning of part 2 under $\tilde{Q}$ and $\tilde{q}_v^2 = q_v^2$. If $\tilde{\mu}_2(a) P_a u$, we must thus obtain a contradiction to (iii). This completes the proof.

□

Proof of Theorem 2

The proof proceeds in four steps: In the first step, I construct an associated college admissions problem (CAP) for an arbitrary matching problem with complex constraints. Next, I show that stability in the associated CAP is equivalent to procedural stability in the original

matching problem with complex constraints. In the third step, monotonicity of choice protocols is shown to imply substitutability of colleges' choice functions in the associated CAP. Together with the second step and the student optimality of the SDA for college admissions problems with substitutable preferences (Roth 1984b; cf Section 2.1), this establishes part (i) of Theorem 2. In the fourth step, I show that monotonicity and non-excessive reduction imply that colleges' choices also satisfy the law of aggregate demand. An application of the result by Hatfield and Kojima (2009) (cf Section 2.1) then yields part (ii) of Theorem 2 and completes the proof.

Step 1: Construction of the associated college admissions problem (CAP)

Given a choice protocol $(\succ_c, q_c)$, college c's choice from a set of applicants $J \subseteq I$ is determined as follows:

> In the first step admit the $q_{(c,1)}$ highest ranking acceptable students in J according to $\succ_{(c,1)}$ (or all acceptable students in J if there are fewer than $q_{(c,1)}$). Denote the set of admitted students by $I_{(c,1)}(J)$, the number of unused seats by $r_{(c,1)}(J) = q_{(c,1)} - |I_{(c,1)}(J)|$, and let $I^{(c,2)}(J) = J \setminus I_{(c,1)}(J)$ be the set of remaining students.
>
> $\vdots$
>
> In the tth step admit the $q_{(c,t)}(r_{(c,1)}(J), \ldots, r_{(c,t-1)}(J))$ highest ranking acceptable students in $I^{(c,t)}(J)$ according to $\succ_{(c,t)}$. Denote the set of admitted students by $I_{(c,t)}(J)$, the number of unused seats by $r_{(c,t)}(J) = q_{(c,t)}(r_{(c,1)}(J), \ldots, r_{(c,t-1)}(J)) - |I_{(c,t)}(J)|$, and let $I^{(c,t+1)}(J) = J \setminus I_{(c,t)}(J)$ be the set of remaining students.
>
> $\vdots$

College c's *choice from* $J \subseteq I$ *induced by* $(\succ_c, q_c)$ is then $Ch_c(J) = Ch(J|\ \succ_c, q_c) := I_{(c,1)}(J) \cup \ldots \cup I_{(c,T)}(J)$.

Given some matching problem with complex constraints $(C, I, P_I, (\succ_c, q_c)_{c \in C})$, define the *associated college admissions problem* by $(C, I, P_I, (Ch_c(\cdot))_{c \in C})$.

Step 2: Equivalence of stability concepts

Given a feasible matching sequence $\mu = (\mu_1, \ldots, \mu_T)$, define a matching ν^μ for the associated CAP by setting $\nu^\mu(c) := \mu(c)$ for all $c \in C$ and $\nu^\mu(i) := \mu(i)$ for all $i \in I$. Similarly, given a matching ν for the associated CAP, define a feasible matching sequence μ^ν by setting $\mu^\nu_t(c) = I_{(c,t)}(\nu(c))$ for all $c \in C$ and, for all $i \in I$, $\mu^\nu_t(i) = c$ if $i \in I_{(c,t)}(\nu(c))$ for some $c \in C$ and $\mu^\nu_t(i) = i$ if $i \notin \cup_{c \in C} I_{(c,t)}(\nu(c))$.

Given these definitions, it is straightforward to check that if μ is procedurally stable, then ν^μ is stable for the associated simple problem, and that if ν is stable for the associated simple problem, then $\mu^\nu = (\mu^\nu_1, \ldots, \mu^\nu_T)$ is procedurally stable (a formal proof can be found in the Online Appendix).

This implies that if ν is the student optimal stable matching in the associated CAP, then μ^ν is the student optimal procedurally stable matching for the matching problem with complex constraints.

Step 3: Monotonicity of choice protocols implies substitutability of colleges' choice functions in the associated CAP

For the following, fix some college $c \in C$ with choice protocol $(\succ_c, q_c)$. Suppose $\tilde{J} \subseteq J \subseteq I$. Let $(\tilde{J}_t, \tilde{r}_t, \tilde{J}^{t+1})_t$ and $(J_t, r_t, J^{t+1})_t$ denote the associated sequences of admitted students, remaining capacities, and remaining students for each step of c's admission process from $\tilde{J}$ and J, respectively.

I show first that if $(\succ_c, q_c)$ is monotonic, then $Ch(\cdot | \succ_c, q_c)$ is substitutable. To prove this, I show by induction that $(J_t \cap \tilde{J}^t) \subseteq \tilde{J}_t$.

The statement is trivial for $t = 1$. Furthermore, it is easy to see that we must have $r_1 \leq \tilde{r}_1$. So suppose that for some $t \geq 1$, $(J_s \cap \tilde{J}^s) \subseteq \tilde{J}_s$ and $r_s \leq \tilde{r}_s$ for all $s \leq t$. I show that the same statements hold for $t+1$ as well.

Note first that by the inductive assumption we must have $\tilde{J}^{t+1} \subseteq J^{t+1}$: Otherwise there would be an agent $i \in (J_1 \cup \ldots \cup J_t) \cap \tilde{J}^{t+1}$, which is impossible given the inductive assumption of $(J_s \cap \tilde{J}^s) \subseteq \tilde{J}_s$ for all $s \leq t$. By monotonicity and the inductive assumption we obtain that $q_{t+1} := q_{(c,t+1)}(r_1, \ldots, r_t) \leq q_{(c,t+1)}(\tilde{r}_1, \ldots, \tilde{r}_t) =: \tilde{q}_{t+1}$. This already yields $(J_{t+1} \cap \tilde{J}^{t+1}) \subseteq \tilde{J}_{t+1}$, since $\tilde{J}^{t+1} \subseteq J^{t+1}$ together with $\tilde{q}_{t+1} \geq q_{t+1}$ implies that the $\tilde{q}_{t+1}$st lowest ranking applicant in $\tilde{J}_{t+1}$ with respect $\succ_{(c,t+1)}$ must rank weakly lower than the

q_{t+1}st lowest ranking applicant in J_{t+1}.

It remains to be shown that $r_{t+1} \leq \tilde{r}_{t+1}$. By definition $\tilde{r}_{t+1} - r_{t+1} = \tilde{q}_{t+1} - q_{t+1} + |J_{t+1}| - |\tilde{J}_{t+1}|$. If $|J_{t+1}| < q_{t+1}$, note that $\tilde{J}^{t+1} \subseteq J^{t+1}$ implies $|J_{t+1}| = |\{i \in J^{t+1} : i \succ_{(c,t+1)} c\}| \geq |\{i \in \tilde{J}^{t+1} : i \succ_{(c,t+1)} c\}|$. Since $\tilde{q}_{t+1} \geq q_{t+1}$ this implies $|J_{t+1}| \geq |\{i \in \tilde{J}^{t+1} : i \succ_{(c,t+1)} c\}| = |\tilde{J}_{t+1}|$ and we obtain $\tilde{r}_{t+1} - r_{t+1} \geq \tilde{q}_{t+1} - q_{t+1} \geq 0$. If $|J_{t+1}| = q_{t+1}$, we obtain $\tilde{r}_{t+1} - r_{t+1} = \tilde{q}_{t+1} - |\tilde{J}_{t+1}| \geq 0$.

Step 4: Monotonicity and non-excessive reduction imply that colleges' choice functions in the associated CAP satisfy the law of aggregate demand

Next, I show that if $(\succ_c, q_c)$ is monotonic and satisfies non-excessive reduction, $Ch(\cdot| \succ_c, q_c)$ also satisfies the law of aggregate demand. Let t be arbitrary. For $s \leq t$, let $q_s := q_{(c,s)}(r_1, \ldots, r_{s-1})$ and $q_s := \tilde{q}_{(c,s)}(\tilde{r}_1, \ldots, \tilde{r}_{s-1})$. As shown above, monotonicity implies that, for all $s \leq t$, we must have $\tilde{r}_s = r_s + k_s$ for some $k_s \geq 0$, if $\tilde{J} \subseteq J$. The definitions of $\tilde{r}_s$ and r_s imply $|J_s| = q_s - \tilde{q}_s + |\tilde{J}_s| + k_s$. Hence,

$$\begin{aligned}
\sum_{s=1}^{t} |J_s| &= \sum_{s=1}^{t} |\tilde{J}_s| + \sum_{s=1}^{t} [k_s - (\tilde{q}_s - q_s)] \\
&\geq \sum_{s=1}^{t} |\tilde{J}_s| + \tilde{q}_{t+1} - q_{t+1} \\
&\geq \sum_{s=1}^{t} |\tilde{J}_s|,
\end{aligned}$$

where the first inequality follows from non-excessive reduction and the second inequality follows from monotonicity. Since t was arbitrary, this implies $|Ch_c(J)| = \sum_{s=1}^{T} |J_s| \geq \sum_{s=1}^{T} |\tilde{J}_s| = |Ch_c(\tilde{J})|$, which proves the statement.

□

References

[1] Abdulkadiroglu, A.: College admissions with affirmative action. Int J Game Theor **33**, 535–549 (2005)

[2] Abdulkadiroglu, A., Che, Y.-K., Yasuda, Y.: Expanding "Choice" in School Choice. Columbia University, Mimeo (2011a)

[3] Abduladiroglu, A., Che, Y.-K., Yasuda, Y.: Resolving conflicting preferences in School Choice: The Boston mechanism reconsidered. Am Econ Rev **101**, 399–410 (2011b)

[4] Abdulkadiroglu, A., Pathak, P. A., Roth, A., Sönmez, T.: Changing the Boston School Choice Mechanism. Harvard University, Mimeo (2006)

[5] Abdulkadiroglu, A., Pathak, P. A., Roth, A.: Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC High School Match. Am Econ Rev **99**, 1954–1978 (2009)

[6] Abdulkadiroglu, A., Sönmez, T.: School Choice - A Mechanism Design approach. Am Econ Rev **93**, 729–747 (2003)

[7] Balinski, M., Sönmez, T.: A tale of two mechanisms: Student placement. J Econ Theor **84**, 73–94 (1999)

[8] Braun, S., Kübler, D., Dwenger, N.: Telling the truth may not pay off: An empirical study of centralized university admissions in Germany. B.E. J of Econ Anal and Policy (Advances) **10**, article 22 (2010)

[9] Chen, Y., Kesten, O.: From Boston to Shanghai to Deferred Acceptance: Theory and experiments on a family of school choice mechanisms. University of Michigan, Mimeo (2011)

[10] Chen, Y., Sönmez, T.: School Choice: An experimental study. J Econ Theor **127**), 202–231 (2006)

[11] Dubins, L., Freedman, A.: Machiavelli and the Gale-Shapley algorithm. Am Math Mon **88**, 485–494 (1981)

[12] Ehlers, L., Erdil, A.: Efficient assignment respecting priorities. J Econ Theor **145**, 1269–1282 (2010)

[13] Ehlers, L., Hafalir, I., Yenmez, B., Yildirim, M.: School Choice with controlled choice constraints: Hard bounds versus soft bounds. Tepper School of Business, Mimeo (2011)

[14] Ehlers, L., Westkamp, A.: Strategyproof tie-breaking. Universite de Montreal, Mimeo (2011)

[15] Erdil, A., Ergin, H.: What's the matter with tie-breaking? Improving efficiency in School Choice. Am Econ Rev **98**, 669–689 (2008)

[16] Ergin, H., Sönmez, T.: Game of School Choice under the Boston mechanism. J Public Econ **90**, 215–237 (2006)

[17] Featherstone, C., Niederle, M.: Ex ante efficiency in School Choice mechanisms: An experimental investigation. Stanford University, Mimeo (2008)

[18] Gale, D., Shapley, L.: College admissions and the stability of marriage. Am Math Mon **69**, 9–15 (1962)

[19] Guillen, P., Kesten, O.: Matching markets with mixed ownership: The case for a real-life assignment mechanism. Int Econ Rev (2012, forthcoming)

[20] Haake, C.-J., Klaus, B.: Monotonicity and Nash implementation in matching markets with contracts. Econ Theor **41**, 393–410 (2009)

[21] Haeringer, G., Klijn, F.: Constrained School Choice. J Econ Theor **144**, 1921–1947 (2009)

[22] Hafalir, I., Yenmez, B., Yildirim, M.: Effective affirmative action in School Choice. Tepper School of Business, Mimeo (2011)

[23] Hatfield, J., Kojima, F.: Matching with contracts: Comment. Am Econ Rev **98**, 1189–1194 (2008)

[24] Hatfield, J., Kojima, F.: Group incentive compatibility for matching with contracts. Games Econ Behav **67**, 745–749 (2009)

[25] Hatfield, J., Milgrom, P.: Matching with contracts. Am Econ Rev **95**, 913–935 (2005)

[26] Kamada, Y., Kojima, F.: Improving efficiency in matching markets with regional caps: The case of the Japan Residency Matching Program. Stanford University, Mimeo (2011)

[27] Kara, T., Sönmez, T.: Nash implementation of matching rules. J Econ Theor **68**, 425–439 (1996)

[28] Kara, T., Sönmez, T.: Implementation of college admission rules. Econ Theor **9**, 197–218 (1997)

[29] Kelso, A., Crawford, V.: Job matching, coalition formation, and gross substitutes. Econometrica **50**, 1483–1503 (1982)

[30] Miralles, A.: School Choice: The case for the Boston mechanism. Boston University, Mimeo (2008)

[31] Pathak, P. A., Sönmez, T.: Leveling the playing field: Sincere and sophisticated players in the Boston mechanism. Am Econ Rev **98**, 1636–1652 (2008)

[32] Roth, A.: The economics of matching: Stability and incentives. Math Oper Res **7**, 617–628 (1982)

[33] Roth, A.: The evolution of the labor market for medical interns and residents: A case study in game theory. J Polit Econ **92**, 991–1016 (1984a)

[34] Roth, A.: Stability and polarization of interests in job matching. Econometrica **52**, 47–57 (1984b)

[35] Roth, A.: The college admissions problem is not equivalent to the marriage problem. J Econ Theor **36**, 277–288 (1985)

[36] Roth, A.: Deferred acceptance algorithms: History, theory, practice, and open questions. Int J Game Theor **36**, 537–569 (2008)

[37] Roth, A, Peranson, E.: The redesign of the matching market for American physicians. Am Econ Rev **89**, 748–780 (1999)

[38] Roth, A., Sotomayor M.: Two-Sided Matching: A study in game-theoretic modeling and analysis. Econometric Society Monographs. Cambridge: Cambridge University Press (1990)

[39] Sönmez, T., Ünver, U.: Matching, allocation, and exchange of discrete resources. in J. Benhabib, A. Bisin, and M. Jackson, eds.: Handbook of Social Economics, Vol. 1A. The Netherlands: North-Holland, 781-852 (2011)

[40] Sotomayor, M.: The stability of equilibrium outcomes of the admission games induced by stable matching rules. Universidade de Sao Paulo, Mimeo (2007)

7 Supplementary Appendix

7.1 Omitted proofs

7.1.1 Equivalence of stability concepts

As in the proof of Theorem 2, given a feasible matching sequence $\mu = (\mu_1, \ldots, \mu_T)$, define a matching ν^μ for the associated CAP by setting $\nu^\mu(c) := \mu(c)$ for all $c \in C$ and $\nu^\mu(i) := \mu(i)$ for all $i \in I$. Similarly, given a matching ν for the associated CAP, define a feasible matching sequence μ^ν by setting $\mu^\nu_t(c) = I_{(c,t)}(\nu(c))$ for all $c \in C$ and, for all $i \in I$, $\mu^\nu_t(i) = c$ if $i \in I_{(c,t)}(\nu(c))$ for some $c \in C$ and $\mu^\nu_t(i) = i$ if $i \notin \cup_{c \in C} I_{(c,t)}(\nu(c))$. We have the following.

Proposition 1. *(i) If μ is a procedurally stable feasible matching sequence, ν^μ is a stable matching for the associated college admissions problem.*

(ii) If ν is a stable matching for the associated college admissions problem, μ^ν is a procedurally stable feasible matching sequence.

Proof:

(i) I show first that $Ch_c(\nu^\mu(c)) = \nu^\mu(c)$ (individual rationality for students is obvious). By (iv) of Definition 4, $\mu_1(c)$ contains the $q_{(c,1)}$ highest ranking acceptable students in $\nu^\mu(c)$ with respect to $\succ_{(c,1)}$ (or all such acceptable students if there are fewer than $q_{(c,1)}$). This implies $I_{(c,1)}(\nu^\mu(c)) = \mu_1(c)$ and $r_{(c,1)}(\nu^\mu(c)) = q_{(c,1)} - |\mu_1(c)|$. Proceeding inductively, suppose $I_{(c,s)}(\nu^\mu(c)) = \mu_s(c)$ for all $s \leq t$. This implies $q_{(c,t+1)}(r_{(c,1)}(\nu^\mu(c)), \ldots, r_{(c,t)}(\nu^\mu(c))) = q_{(c,t+1)}(r_{(c,1)}(\mu), \ldots, r_{(c,t)}(\mu))$. Again by (iv) of Definition 4, $\mu_{t+1}(c)$ contains the $q_{(c,t+1)}(r_{(c,1)}(\mu), \ldots, r_{(c,t)}(\mu))$ highest ranking acceptable students in $\nu^\mu(c) \setminus (\mu_1(c) \cup \ldots \cup \mu_t(c))$ with respect to $\succ_{(c,t+1)}$ (or all such acceptable students if there are fewer than $q_{(c,t+1)}(r_{(c,1)}(\mu), \ldots, r_{(c,t)}(\mu))$). This implies $I_{(c,t+1)}(\nu^\mu(c)) = \mu_{t+1}(c)$.

Next, I show that if $cP_i\nu^\mu(i)$ for some student $i \in I$, then $i \notin Ch_c(\nu^\mu(c) \cup \{i\})$. Note that if $i \succ_{(c,1)} c$ and $i \in I_{(c,1)}(\nu^\mu(c) \cup \{i\})$, we must have either $|\mu_1(c)| < q_{(c,1)}$ or $i \succ_{(c,1)} j$ for some $j \in \mu_1(c)$, so that (iii) of Definition 4 must be violated with respect to $t = 1$. This and the individual rationality of ν^μ in the associated college admissions problem imply $I_{(c,1)}(\nu^\mu(c) \cup \{i\}) = \mu_1(c)$. Proceeding inductively, suppose $I_{(c,s)}(\nu^\mu(c) \cup$

$\{i\}) = \mu_s(c)$ for all $s \leq t$. This implies $q_{(c,t+1)}(r_{(c,1)}(\nu^\mu(c) \cup \{i\}), \ldots, r_{(c,t)}(\nu^\mu(c) \cup \{i\})) = q_{(c,t+1)}(r_{(c,1)}(\mu), \ldots, r_{(c,t)}(\mu))$. Hence, if $i \succ_{(c,t+1)} c$ and $i \in I_{(c,t+1)}(\nu^\mu(c) \cup \{i\})$, we must obtain a contradiction to (iii) of Definition 4 with respect to $t+1$ as above. This inductive argument shows $i \notin I_{(c,1)}(\nu^\mu(c) \cup \{i\}) \cup \ldots \cup I_{(c,T)}(\nu^\mu(c) \cup \{i\}) = Ch_c(\nu^\mu(c) \cup \{i\})$ and completes the proof of (i).

(ii) I show first that if $Ch_c(\nu(c)) = \nu(c)$, then μ^ν satisfies condition (iv) of Definition 4 (that conditions (i) and (ii) are satisfied is obvious). To see this, consider an arbitrary $t \leq T$ and note that by the rules of c's admission process, $\mu^\nu_t(c) = I_{(c,t)}(\nu(c))$ consists of the $q_{(c,t)}(r_{(c,1)}(\nu(c)), \ldots, r_{(c,t-1)}(\nu(c))) = q_{(c,t)}(r_{(c,1)}(\mu^\nu), \ldots, r_{(c,t-1)}(\mu^\nu))$ highest ranking acceptable applicants in $\nu(c) \setminus (I_{(c,1)}(\nu(c)) \cup \ldots \cup I_{(c,t-1)}(\nu(c))) = (\mu^\nu_t(c) \cup \ldots \cup \mu^\nu_T(c)) \setminus (\mu^\nu_1(c) \cup \ldots \cup \mu^\nu_{t-1}(c))$ with respect to $\succ_{(c,t)}$ (or all such acceptable students if there are fewer than $q_{(c,t)}(r_{(c,1)}(\mu^\nu), \ldots, r_{(c,t-1)}(\mu^\nu))$). Hence, for all $s < t$, any student in $\mu_t(c)$ must rank lower than all students in $\mu_s(c)$ with respect to $\succ_{(c,s)}$.

Next, suppose that (iii) is violated for some t so that $cP_i\nu(i)$, $i \succ_{(c,t)} c$, and either $|\mu^\nu_t(c)| < q_{(c,t)}(r_{(c,1)}(\mu^\nu), \ldots, r_{(c,t-1)}(\mu^\nu))$ or $i \succ_{(c,t)} j$ for some $j \in \mu^\nu_t(c)$. The stability of ν for the associated CAP implies $i \notin I_{(c,1)}(\nu(c) \cup \{i\}) \cup \ldots \cup I_{(c,t-1)}(\nu(c) \cup \{i\})$. Hence, we must have $I_{(c,s)}(\nu(c) \cup \{i\}) = I_{(c,s)}(\nu(c)) = \mu^\nu_s(c)$ for all $s \leq t-1$. This implies in particular that $q_{(c,t)} := q_{(c,t)}(r_{(c,1)}(\nu(c)), \ldots, r_{(c,t-1)}(\nu(c))) = q_{(c,t)}(r_{(c,1)}(\mu^\nu), \ldots, r_{(c,t-1)}(\mu^\nu))$. Given the assumed violation of (iii) for t, this implies that we must have $i \in I_{(c,t)}(\nu(c) \cup \{i\})$. But then $i \in Ch_c(\nu(c) \cup \{i\})$, contradicting the assumed stability of ν for the associated CAP.

□

7.1.2 The connection to stability under minority reserves

Given a problem with minority reserves $(C, S, (\succ_i)_{i \in C \cup S}, (q_c, r^m_c)_{c \in C})$, as introduced by Hafalir et al (2011), define an associated problem with complex constraints $(C, S, (\succ_s)_{s \in S}, (\succ_{(c,1)}, \succ_{(c,2)})_{c \in C}, (q_{(c,1)}, q_{(c,2)})_{c \in C})$ as follows: For any school/college $c \in C$,

- for all $s, s' \in S^m$, set $s \succ_{(c,1)} s' \succ_{(c,1)} c$ if and only if $s \succ_c s'$,
- for all $s \in S^M$, set $c \succ_{(c,1)} s$,

- set $\succ_{(c,2)}=\succ_c$, and
- set $q_{(c,1)} = r_c^m$ and $q_{(c,2)}(r_1) = q_c - r_c^m + r_1$ for all $r_1 \in \{0, \ldots, r_c^m\}$.

Given a matching μ for the problem with minority reserves, define a matching for the problem with complex constraints by

- letting $\nu_1^\mu(c)$ consist of the r_c^m highest ranking students in $\mu(c) \cap S^m$ with respect to $\succ_c$ (or all students in $\mu(c) \cap S^m$ if there are less than r_c^m) and
- setting $\nu_2^\mu(c) = \mu(c) \setminus \nu_1^\mu(c)$

for all colleges $c \in C$.

Similarly, given some matching $\nu = (\nu_1, \nu_2)$ for problem with complex constraints, define a corresponding matching μ^ν for the problem with minority reserves by setting $\mu^\nu(c) = \nu_1(c) \cup \nu_2(c)$ for all colleges $c \in C$.

A matching μ is *stable with respect to minority reserves* (Hafalir et al 2011)[19] if

(a) it is individually rational for students,

(b) whenever $c \succ_s \mu(s)$ for some $s \in S^m$ and $c \in C$, then $|\mu(c)| = q_c$, $|\mu(c) \cap S^m| \geq r_c^m$, and $s' \succ_c s$ for all $s' \in \mu(c)$,

(c) whenever $c \succ_s \mu(s)$ for some $s \in S^M$ and $c \in C$, then either

(c.1) $|\mu(c) \cap S^m| > r_c^m$ and $s' \succ_c s$ for all $s' \in \mu(c)$, or

(c.2) $|\mu(c) \cap S^m| \leq r_c^m$ and $s' \succ_c s$ for all $s' \in \mu(c) \cap S^M$.

A matching $\nu = (\nu_1, \nu_2)$ for the problem with complex constraints is *procedurally stable* if

(a') it is individually rational for students,

(b') $s \succ_{(c,t)} c$ for all $s \in \nu_t(c)$

(c') whenever $c \succ_s \nu(s)$ then there is no t such that $s \succ_{(c,t)} c$ and either $r_{(c,t)}(\nu) > 0$, or $s \succ_{(c,t)} s'$ for some $s' \in \nu_t(c)$, and

[19] This is exactly the same definition as in Hafalir et al (2011). I just reformulated it slightly to make it easier to read.

(d') whenever $s \in \nu_t(c)$, there is no $t' < t$ such that $s \succ_{(c,t')} c$ and either $r_{(c,t')}(\nu) > 0$, or $s \succ_{(c,t')} s'$ for some $s' \in \nu_{t'}(c)$.

Proposition 2. *(i) If μ is stable with respect to minority reserves, then ν^μ is procedurally stable.*

(ii) If ν is procedurally stable, then μ^ν is stable with respect to minority reserves.

Proof.

(i) That ν^μ satisfies (a') and (b') of procedural stability follows immediately from the construction of ν^μ and the corresponding properties of μ. Property (d') also follows immediately from the construction of ν^μ, since $\nu_1^\mu(c)$ always contains the highest priority minority students from $\mu(c)$.

To see that (c') must be satisfied, consider first a minority student $s \in S^m$ and assume that $c \succ_s \nu^\mu(s) = \mu(s)$ for some $c \in C$. By (b) of stability with respect to minority reserves, we must have $|\mu(c) \cap S^m| \geq r_c^m$ and $s' \succ_c s$ for all $s' \in \mu(c)$. But then, we must clearly have $|\nu_1^\mu(c)| = r_c^m$ and $s' \succ_c s$ for all $s' \in \nu_1^\mu(c) \cup \nu_2^\mu(c) = \mu(c)$.

Next, consider a majority student $s \in S^M$ and assume that $c \succ_s \nu^\mu(s) = \mu(s)$ for some $c \in C$. We need to show that $s' \succ_c s$ for all $s \in \nu_2^\mu(c)$. If $|\mu(c) \cap S^m| > q_c$, then (c.1) of stability with respect to minority reserves implies that $s' \succ_c s$ for all $s' \in \mu(c) = \nu_1^\mu(c) \cup \nu_2^\mu(c)$. If $|\mu(c) \cap S^M| \leq q_c$, the construction of ν^μ implies that we must have $\nu_2^\mu(c) \subset S^M$ and (c.2) implies $s' \succ_c s$ for all $s' \in \mu(c) \cap S^M = \nu_2^\mu(c)$.

(ii) If ν is procedurally stable, μ^ν must be individually rational for students by (a').

To see that (b) must be satisfied, consider a minority student $s \in S^m$ and assume that $c \succ_s \mu^\nu(s)$ for some $c \in C$. Since this implies that $c \succ_s \nu_1(s)$, we must have that $|\nu_1(c)| = r_c^m$ and $s' \succ_c s$ for all $s' \in \nu_1(c)$ by (c') for $t = 1$. On the other hand, (c') for $t = 2$ implies that $s' \succ_c s$ for all $s' \in \nu_2(c)$. Hence, $|\mu^\nu(c) \cup S^m| \geq |\nu_1(c)| = r_c^m$ and $s' \succ_c s$ for all $s' \in \mu^\nu(c) = \nu_1(c) \cup \nu_2(c)$.

To see that (c) must be satisfied, consider a majority student $s \in S^M$ and assume that $c \succ_s \mu^\nu(s)$ for some $c \in C$. Suppose first that $|\mu^\nu(c) \cap S^m| \leq r_c^m$. Since ν satisfies (d') of procedural stability, this implies that $\nu_2(c) \subset S^M$. By (c') for $t = 2$, we must have

$s' \succ_c s$ for all $s' \in \nu_2(c) = \mu(c) \cap S^M$. Next, assume that $|\mu^\nu(c) \cap S^m| > r_c^m$. By (c') for $t = 2$, we must have $s' \succ_c s$ for all $s' \in \nu_2(c)$. Since $|\nu_1(c)| \leq r_c^m$, there must be at least one $s'' \in \nu_2(c) \cap S^m$. Since ν also satisfies (d'), we must have $s'' \succ_c s'$ for all $s'' \in \nu_2(c)$, so that by transitivity of $\succ_c$, $s'' \succ_c s$ for all $s'' \in \nu_2(c)$.

□

7.2 A more detailed model of the German admissions procedure

In this section I provide a more detailed model of the German admissions procedure. The most important difference between the model presented in the main body of my paper and the actual assignment procedure is that the latter consists of three parts. In the first part, the Boston mechanism is used to allocate up to 20 percent of total capacity among applicants with excellent average grades. In the second part, the same mechanism is used to allocate another 20 percent of total capacity among applicants who have waited a long time since finishing high-school. Finally, all remaining capacity is allocated among all remaining applicants in the third part of the procedure on basis of criteria chosen by the universities using the university proposing deferred acceptance algorithm. I now describe the system in more detail and then argue that under reasonable assumptions the analysis from the main body of my paper carries through.

In order to participate in the centralized assignment procedure applicants have to submit one ordered (preference) list of universities for each part of the procedure. There is no consistency requirement across the three lists and the list submitted for part $t \in \{1, 2, 3\}$ is used only to determine assignments in part t. All preference lists are submitted simultaneously before any assignments are determined. For the first and third part of the procedure at most six universities can be ranked, while for the second part any number of universities can be ranked. Let $Q_a := (Q_a^1, Q_a^2, Q_a^3)$ denote the profile of strict preference lists submitted by applicant $a \in A$ and $Q_A := (Q_A^1, Q_A^2, Q_A^3) := ((Q_a^1)_{a \in A}, (Q_a^2)_{a \in A}, (Q_a^3)_{a \in A})$ denote the profile of reports by all applicants. An applicant *applies for a place in part* t, if she ranks at least one university for part t of the procedure. Let q_u denote the total number of places that university u has to offer. To avoid integer problems, I assume that all capacities are multiples of five. Let $q_u^1 = q_u^2 = \frac{1}{5} q_u$ and $q^1 = q^2 = \frac{1}{5} \sum_{u \in U} q_u$ denote the number of places at university u and the total number of places available in the first two parts of the procedure. As in the main body

of the paper, I assume throughout that all universities have *objective evaluation procedures*: for each university $u \in U$, there is a strict ranking $\succ_u$ of $A \cup \{u\}$ that is fixed and known prior to the application deadline. University u's preferences over any set of applicants $B \subseteq A$ who remain in the procedure until the third part are then given by the restriction of $\succ_u$ to B, $\succ_u |_{B\cup\{u\}}$. With these preparations, the German admissions procedure can be described as follows.

The German admissions procedure

Part 1: Assignment for top-grade applicants

(Selection) Select q^1 applicants from those who applied for a place in part 1. If there are more than q^1 such applicants, order applicants lexicographically according to (i) average grade, (ii) time since obtaining qualification, (iii) completion of military or civil service, (iv) lottery.[20] Select the q^1 highest ranked applicants in this ordering.

(Assignment) Apply the Boston mechanism to determine assignments of selected applicants: University u can admit at most q_u^1 applicants, the preference relation of a selected applicant a is Q_a^1, and an applicant's priority for a university is determined lexicographically by (i) average grade, (ii) social criteria,[21] (iii) lottery. Denote the matching produced in part 1 by $f^{G1}(Q_A^1)$, the set of admitted applicants by A_1, the number of empty seats at university u by $r_{(u,1)} = q_u^1 - |f_u^{G1}(Q_A^1)|$, and the set of remaining applicants by $A^2 = A \setminus A_1$.

Part 2: Assignment for wait-time applicants

(Selection) Select q^2 applicants from those in A^2 who applied for a place in part 2. If there are more than q^2 such applicants, order applicants lexicographically according to (i) time

[20] This means that an applicant A ranks higher than an applicant B if and only if either (a) A has a better (which in Germany means lower) average grade than B, (b) A and B have the same average grade, but A has a longer waiting time than B, (c) A and B have the same average grade and waiting time, but A has completed military or civil service and B has not, or (d) A and B do not differ with respect to the first three criteria and A was assigned higher priority than B by lottery. Analogous comments apply to the priority rankings in the other parts of the procedure.

[21] In this category, applicants are ordered lexicographically according to the following criteria: 1. Being severely disabled. 2. Main residence with spouse or child in the district or a district-free city associated to the university. 3. Granted request for preferred consideration of top choice. 4. Main residence with parents in the area associated with the university. Note that, in contrast to the selection stage, an applicant's priority may thus differ across universities.

since obtaining qualification, (ii) average grade, (iii) completion of military or civil service, (iv) lottery. Select the q^2 highest ranked applicants in this ordering.

(Assignment) Apply the Boston mechanism to determine assignments of selected applicants: University u can admit at most q_u^2 applicants, the preference relation of a selected applicant a is Q_a^2, and an applicant's priority for a university is determined lexicographically by (i) social criteria, (ii) average grade,(iii) lottery. Denote the matching produced in part 2 by $f^{G2}(Q_A^2)$, the set of admitted applicants by A_2, the number of empty seats at university u by $r_{(u,2)} = q_u^2 - |f_u^{G2}(Q_A^2)|$, and the set of remaining applicants by $A^3 = A^2 \setminus A_2$.

Part 3: Assignment according to universities' preferences

Apply the university proposing deferred acceptance algorithm to determine an assignment of applicants in A^3 to universities: University u can admit at most $q_u^3 = \frac{3}{5}q_u + r_{(u,1)} + r_{(u,2)}$ applicants, the preference relation of an applicant $a \in A^3$ is given by Q_a^3, and u's preferences are responsive to $\succ_u$. Denote the matching produced in the third part of the procedure by $f^{G3}(Q_A^3, \succ_U)$.

□

Given Q_A and $\succ_U$, let $f^G(Q) = (f^{G1}(Q_A^1), f^{G2}(Q_A^2), f^{G3}(Q_A^3, \succ_U))$ denote the three tuple of matchings chosen by the German admissions procedure.

I now introduce some additional assumptions. First, as in the main body of the text, I assume that the priority rankings used in the first two parts of the procedure are strict. Applicants who are selected in the first part of the procedure will be called top-grade applicants and applicants who are selected in the second part of the procedure will be called wait-time applicants. Second, all applicants are assumed to always rank at least one university for each part of the procedure. This is important since otherwise the sets of top-grade and wait-time applicants would depend on the profile of submitted preferences since an applicant is considered for part $t \in \{1, 2\}$ only if she has ranked at least one university for part t. While this is not without loss of generality, as there are rare cases in which it can be in an applicant's best interest to list no university for some part of the procedure (an example can be found in Section 7.2.2), the empirical evidence in Braun et al (2010) offers strong support in favor of this assumption. Third, I assume that an applicant can be eligible for a place in *at most one* of the first two

parts. Given that all potentially eligible applicants always apply for a place in the respective part of the procedure, it is reasonable to assume that top-grade applicants will not be eligible for a place reserved for wait-time applicants, since all universities (have to) rely partly on the average grades of applicants. Applicants with excellent average grades will typically find a place to study before they would become eligible in the wait-time quota, where the minimal waiting time to be eligible ranges from 2.5 years (veterinary medicine) to 5 years (general medicine). Finally, I assume that top-grade and wait-time applicants have lexicographic preferences in the sense that when comparing two outcomes such an applicant primarily cares about the university she is assigned to and, if the two outcomes assign her to the same university but in different parts of the procedure, she prefers the outcome that assigns her in the first (top-grade applicants) or second part (wait-time applicants) of the procedure.

Under these four assumptions, Theorem 1 from the main body of the text continues to hold for the more detailed model of the German admissions procedure presented above. The only change in the proof is that now one needs to take care of two special applicant groups and corresponding reserved quotas. Given the assumptions introduced above, these groups are disjoint and their composition is independent of the submitted preferences. It is straightforward to establish that then the presence of two special applicant groups poses no problem for the validity of Theorem 1. Details are available upon request.

One other potential problem for the validity of Theorem 1 is that in the actual procedure universities are allowed to use their rank in the preference lists that applicants submitted for part 3. For example, a university may decide to consider only applicants who ranked it first and order these applicants according to their average grade. The assumption of objective evaluation procedures above can be interpreted as saying that the other criteria a university u uses apart from such *ranking constraints* induces the strict ranking $\succ_u$. In the just mentioned example $\succ_u$ would simply list applicants in decreasing order of average grade. The ranking $\succ_u$ can thus be understood as the ranking that would result if all applicants had ranked u sufficiently high to satisfy its ranking constraint. While I have abstracted from ranking constraints in the above, Theorem 1 remains valid without this restriction. This follows since the proof of this result shows that any stable matching can be achieved by a strategy profile in which matched applicants only rank their assigned university and that the deviations used to show that

any equilibrium outcome must be stable similarly use a preference list of length one. Thus, the ranking constraints set by universities never come into play in a complete information equilibrium. An interesting corollary of Theorem 1 is therefore that neither ranking constraints nor the constraint that applicants can rank at most six universities for parts 1 and 3, have any effect on the set of matchings that are attainable as complete information equilibrium outcomes. The application strategies used in the proof are of course very risky since they entail a potentially high probability of being left unassigned by the end of the procedure if applicants are only slightly mistaken about the preferences of universities and other applicants. The discussion is meant to point out that if applicants had a very reliable estimate of their chances of admission at each university then constraints would be irrelevant.

7.2.1 Further details

The main legal text that governs the centralized assignment procedure is the *Verordnung über die zentrale Vergabe von Studienplätzen durch die Stiftung für Hochschulzulassung (VergabeVO Stiftung) (Stand: Wintersemester 2010/2011)*,[22] available at http://hochschulstart.de/fileadmin/downloads/Gesetze/G03.pdf. The text is only available in German and I now briefly summarize the sections of the text that are most relevant for the presentation of the procedure that is found in the main body of this text.

In §3, the text states that for the first and third part of the procedure applicants should list up to six universities in decreasing order of preference, and that for the second part they should list universities in decreasing order of preference. The division of the total capacity of each university is described in detail in §6 of the text and §6, section 6, sentence 2 describes capacity redistribution between the three quotas. The timing of the procedure is described in §7. The selection of applicants for the first two parts of the procedure is described in §11 - 13 (first part) and §14 (second part). The rules for the assignment stages of the first two parts are summarized in §20, which says that the stated preference rank is paramount for admission and describes how to determine rejections if more applicants have listed a given university at the same preference rank than can be admitted. The rules for the last part of the procedure are described in §10, section 5. Here it is stated that once universities' preference lists have been

[22]This can be translated as *Decree about the centralized allocation of places of study by the foundation for university admissions.*

received, the foundation for university admissions *aligns* these lists by iteratively deleting all but the most preferred (according to the applicants' submitted preference lists) possibility of admission for each applicant. This means that the foundation first checks who can be admitted at which university given the rankings submitted by universities and the number of available places at each university. Each applicant is temporarily assigned his or her most preferred university and is deleted from the lists of all other universities at which he or she could have been admitted. In the second round, the foundation checks who can be admitted at which university on basis of the modified lists, applicants are again temporarily assigned the most preferred university at which they can be admitted and are deleted from the lists of all other universities at which they could have been admitted. Iterating this procedure, we arrive at the description of the college/university proposing deferred acceptance algorithm as stated in the main body of this paper.

I now describe some of the additional simplifications made in the main body of the text. Readers interested in all details of the current German admissions procedure may still want to consult the *Verordnung über die zentrale Vergabe von Studienplätzen durch die Stiftung für Hochschulzulassung (VergabeVO Stiftung) (Stand: Wintersemester 2010/2011).*

Capacities: The total number of places at each university is determined by the application of federal laws. For each state there is a so called *Kapazitätsverodnung* (KaPVO) (available at http://hochschulstart.de/fileadmin/downloads/Gesetze/G05.pdf) which prescribes a formula for calculating the number of applicants a university can admit on basis of the number of professors, available teaching facilities, and so on.[23]

Special Quotas: Up to approximately fifteen percent of total available places are allocated in advance among foreign applicants, applicants pursuing a second university degree, and so on. These applicants are not allowed to participate in the regular assignment procedure.

Part 1: The education system in Germany is federalized and the general opinion is that average grades are not directly comparable across federal states. For this reason, there are actually sixteen separate assignment procedures in part 1, one for each federal state. This is achieved by splitting the 20 percent of (remaining) capacity available in part 1 into sixteen

[23]There has been some discussion about the KaPVO in recent years, see e.g. *Die fiese Formel* in Die Zeit, Nr. 39(2007).

quotas for the different federal states. Within each of these federal quotas only those applicants are considered who have received their high school diploma in this state.

Part 3: – Once assignments are determined by the centralized admissions procedure, successful applicants have to enroll at their assigned university. If some applicants fail to do so, their places are allocated according to the rules of part 3. Here, only those applicants are considered who did not receive a place in previous rounds of the assignment procedure. Again, students have to enroll at their assigned university (if any) and if they fail to do so, another round of part 3 is used to allocate remaining places (again only students who were not previously assigned a place are considered). Any places that remain after all of this are allocated via lottery by universities.

– In order to prevent multiple rounds of the assignment procedure in part 3, a university can demand the foundation for university admissions to overbook its capacities. Thus, a university with, say 100 places, may to be assigned 150 applicants since it expects some students not to accept their assigned places.

Lotteries: If a university does not fill its capacity in the centralized procedure, remaining places are allocated on basis of lottery. Each university conducts its own lottery and applicants have to apply directly to universities in order to participate.

7.2.2 An example showing that "No Empty Lists" is restrictive

The following example shows that the "No Empty Lists" assumption is restrictive, so that applicants may sometimes benefit from not participating in (one of) the first two steps of the German admissions procedure. There are seven applicants $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ indexed in order of increasing average grades. For simplicity, I assume that there are only two universities u and u' who have one place to allocate in each of the three steps of the admissions procedure. Preferences of applicants are as follows:

P	P_{a_1}	P_{a_2}	P_{a_3}	P_{a_4}	P_{a_5}	P_{a_6}	P_{a_7}
	u	u	u'	u'	u'	u	u'
			u				

Assume that a_6 and a_7 are the applicants with the longest waiting time, $a_3 \succ_{(u,3)} a_2$ as well

as $a_5 \succ_{(u',3)} a_4 \succ_{(u',3)} a_3$, and that all applicants submit their true ranking of universities for each step of the procedure. In this case the outcome of the admissions procedure is

$$f^{G1}(P_A) = \begin{matrix} u & u' \\ a_1 & \emptyset \end{matrix}, \qquad f^{G2}(P_A) = \begin{matrix} u & u' \\ a_6 & a_7 \end{matrix}, \qquad f^{G3}(P_A) = \begin{matrix} u & u' \\ a_3 & \{a_4, a_5\} \end{matrix}.$$

Note that if we keep the profile of reports by everyone but a_2 fixed, a_2 cannot obtain a place at u if she applies for a place in part 1. Suppose then that she decides to apply only for part 3 and submits $Q^3_{a_2} = u$. If everyone else submits the same preferences as before, a_3 would be a top-grade applicant and could obtain a place at her most preferred university u' in part 1 of the admissions procedure. But then a_2 would receive a place at u in part 3 of the procedure since no one else will apply to u in that part. Thus, she benefits from not applying for a place in part 1.

□

7.3 Evaluation of applicants in the German admissions procedure

In this section of the supplementary appendix I provide some further details on the different evaluation procedures used by universities in part 3 of the current German admissions procedure. A comprehensive (German language) overview can be found at http://hochschulstart.de/fileadmin/downloads/Studienangebot/adh-kriterien-ws10.pdf. All of the below concerns the procedure for the winter term 2010/2011. The evaluation process takes place after assignments in parts 1 and 2 have been determined and only those applicants who did not receive a place in these parts are considered. In principle, the administrator of the centralized procedure informs each university about all remaining applicants who have listed the university in their ranking for part 3.

A university may, however, limit the set of applicants it will consider for part 3 in advance on basis of its rank in the preference lists submitted for part 3, average grades, or a combination of the two criteria. For example, a university with, say, a hundred seats to be allocated in part 3 may consider only the 300 applicants with the best average grades among those who ranked it first. This practice is called *pre-selection* and universities are only informed about those applicants who "survived" its pre-selection process.

In case an applicant is not rejected in the pre-selection process of a university, the university is provided with detailed information including its rank in the submitted preference list (for part 3), average grade, waiting-time, and so on. Universities can then use average grades, interviews, statements of purpose, completion of on-the-job training in a relevant field, prizes in scientific competitions, and so on, to evaluate remaining applicants. Rules of the evaluation procedures of *all* universities are made available to applicants on the website of the administrator of the centralized admissions procedure and the websites of universities prior to the application deadline. Descriptions list all relevant criteria and specify a rule for using these criteria to calculate a *score* for each applicant, which is then used to rank order applicants. As an example, consider the evaluation procedure that was used by the university of Mainz to evaluate applicants for medicine:

> Applicants who had taken a standardized test for medical subjects (the *TMS*) and had achieved a better grade in this test than the average grade of their school leaving examinations, were assigned a score of $0.51 \times$ (average grade of school leaving examinations) + $0.49 \times$ (grade of TMS).
>
> The score for all other applicants coincided with their average grades.

While universities are, in principle, free to rank order applicants in any way they want, the official information brochure of the centralized procedure's administrator states that *in the determination of an applicant's position [in universities' rankings] average grade has to be a decisive factor.*[24] In practice, this is usually implemented by assigning a weight of at least 50 percent to the average grade of an applicant for the calculation of an applicant's score.

In the following, I provide some aggregate data on the evaluation procedures of universities. Say that a university uses an *objective evaluation procedure*, if it relies only on characteristics of applicants that are known prior to the application deadline, such as average grades, completion of on-the-job training etc. In the first table, $\#U$ lists the number of universities, $\#Pre_k$ lists the number of universities that consider only applicants who ranked them at least kth ($k = 1, \ldots, 4$), $\#O$ lists the number of universities that use only objective evaluation criteria, and $\#AG$ lists the number of universities that exclusively rely on average grades to evaluate applicants in the third part of the procedure.

[24] *Merkblatt M09: Auswahlverfahren der Hochschulen*, available at http://hochschulstart.de/fileadmin/downloads/Merkblaetter/M09.pdf. Translation by the author.

Subject	# U	#Pre 1	#Pre 2	#Pre 3	#Pre 4	# O	# AG
Vet. Med.	5	3	0	0	0	3	0
Pharmacy	22	0	3	4	0	20	7
Dentistry	29	4	4	5	1	23	10
Medicine	34	11	5	5	0	25	9

Table 1: Preselection and objective evaluation

Subject	$\#S$	$\#S_{\leq 0.5}$	$\#S_{>0.5}$	$\#S_{1.0}$
Vet. Med.	2	0	0	2
Pharmacy	2	0	0	2
Dentistry	6	2	2	2
Medicine	9	3	3	3

Table 2: Subjective criteria in the evaluation process

Note that for all subjects more than two thirds of all universities use objective evaluation procedures. Furthermore, out of those universities who rely on objective evaluation procedures, less than half rely exclusively on average grades.

The next table provides more detailed information on universities which rely on subjective criteria to allocate at least part of their total capacity in the third part of the centralized procedure. Universities are allowed to split their available capacity for the third part of the procedure into several quotas and to apply different admission criteria across these quotas. For example, a university may decide to allocate 50 percent of places (available in the third part of the admissions procedure) according to average grade and 50 percent on basis of performance in an interview. In this case, the university has to specify which place an applicant receives if she could be admitted in more than one of these quotas. In the following table, $\#S$ is the number of universities that use subjective criteria to allocate at least part of their total capacity, $\#S_{\leq 0.5}$ is the number of universities that assign at most half and at least one of their seats on basis of subjective criteria, $\#S_{>0.5}$ is the number of universities that assign more than half but less than all of their seats on basis of subjective criteria, and $\#S_{1.0}$ is the number of universities that assign all of their seats on basis of subjective criteria.