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Pricing Heterogeneous Goods under Ex Post Private Information

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Pricing Heterogeneous Goods under Ex Post Private Information

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This paper studies the role of exchange policies as a price discrimination device in a sequential screening model with heterogeneous goods. In the first period, agents are uncertain about their ordinal preferences over a set of horizontally differentiated goods, but have private information about their intensity of preferences. In the second period, each individual privately learns his preferences and consumption takes place. Revenue maximizing mechanisms are completely characterized. They partially restrict the flexibility between the goods in the second stage for consumers that care little about which variety they obtain while granting always the favorite good to consumers that care much. The optimal design of the partial restriction of flexibility can be implemented by offering Limited Exchange Contracts. A Limited Exchange Contract consists of an initial product choice and a subset of products to which free exchange is possible in the second period. The use of exchange fees in contracts is not optimal for the purpose of price discrimination.

Keywords: Sequential screening, dynamic mechanism design, heterogeneous goods
JEL codes: D42, D82, L12

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1 Introduction

In many situations, firms contract with consumers that know about their desire to buy a certain kind of good, but learn which variety they favor only later. An example is the sale of tickets for transportation means like planes, buses, trains, or ships when the customer's favorite departure time is unclear. Further instances include the sale of experience goods when there are several varieties that differ in a product feature as well as various procurement settings in which the contractor's favorite delivery time of the good or service is uncertain. Firms react to this uncertainty by designing exchange policies that are part of the sales contract. For example, in the airline industry consumers are typically offered comparatively cheap tickets, which, however, tie the customer to a specific flight in the sense that these tickets entail certain restrictions regarding refund and exchange. Alternatively, more flexible tickets for the same flight are offered at a higher price.

This paper identifies a common structure in the situations described above and uses it to provide a theory of exchange policies based on intertemporal price discrimination. Thereby it contributes to the literature on sequential screening that to the present primarily studies the design of refund policies.[1] Typically, these papers study the sale of homogeneous goods to customers that learn their valuations for the good gradually over time. To capture the motivated type of situation, I consider a firm which may offer a set of horizontally differentiated goods. This introduces a new dimension into the monopolist's maximization problem, which enables me to address among others the following questions: What drives firms to set up menus of contracts that give consumers differing flexibility between the goods? Why do we observe contracts that partially restrict changes between goods? How is the restriction of flexibility optimally designed – through fees, interdiction or other means? Do we need to treat product exchange separately from product return or can optimal exchange be implemented by refund policies?

I consider a revenue maximizing firm which may offer horizontally differentiated goods to consumers with unit demand and single-peaked preferences. Consumers learn their valuations for the goods in two stages. In the first period - in the following called ex ante stage - agents are uncertain about their favorite product, but differ in the privately known valuation of the favorite variety and the relative valuation loss of obtaining non-favorite products. This means the consumers are uncertain about their ordinal preferences among the goods, but have private information about their intensity of preferences.

[1] The canonical contribution is Courty and Li (2000). Further references are given in the literature part.

In the second period, the agents privately learn which product they prefer most, which means each individual privately learns his complete preferences, and consumption takes place.

In the airline example, tickets are differentiated by the departure time of the flight and travelers initially do not know the departure time they will actually need. A traveler's preference intensity is the cost from flying off the favorite departure time, which can naturally be interpreted as a measure of flexibility in terms of time. Only subsequently, but before flying, the customers learn their favorite departure time. In the framework of the purchase of experience goods, varieties are differentiated by some product feature like color or size. When buying a good in a store which needs to be experienced at home, the favorite variety is uncertain to the consumer, but he learns about it before usage. Despite this uncertainty, consumers already know how flexible they are between the goods, for example how important it is to them that colors fit well. In a procurement setting, any order specifies a time when to deliver the good or service. Upon conclusion of the contract, the contractor has uncertainty regarding his internal work-flow and hence about his preferred delivery time. He knows, however, how tightly operational procedures are packed in his company which influences the cost of amending delivery times.

A key assumption of my model is a positive relation between the valuation of the most preferred product - in the following called top valuation - and the preference intensity. This feature of the model reflects an observation typically present in the above cited examples: Compared to leisure travelers, business travelers value the flight at their favorite time more, but are less flexible concerning departure time. Shoppers that attach importance to clothing both enjoy wearing nice clothes much and pay special attention to wear colors that don't clash. And contractors with a tighter operational schedules use fewer resources for the same task and hence generate higher revenue, but rearranging processes to hold delivery times is comparatively costly.

In this framework, I provide an explanation for the observation that firms offer menus of contracts with different change policies based on price discrimination. The revenue-maximizing menu, which is found using a mechanism design approach without restrictions on contracts, consists of two offers. An expensive contract that allows for costless exchange to any other variety and a cheaper contract that restricts the change between products in the second period. Consumers that care much about obtaining the favorite variety choose the expensive contract, whereas agents who do not attach much importance to whether they consume the favorite variety take the cheaper one. This menu corresponds to common observations in the airline, train or ship industry. The driving force behind the establishment of the menu of contracts with differing exchange policies is a price discrimination motive. Exchange policies can be used as a price discrimination device in the following way: The utility customers derive from a contract which allows them to change variety arbitrarily equals their top valuation. If only this contract is offered, the firm has to trade off leaving rent to consumers with high top valuations and excluding those with low top valuations. The firm can, however, exploit a single crossing property with respect to the consumers' flexibility in product choice in the second stage. This makes it profitable to offer a second contract with little such flexibility in order to extract more rent from consumers with high top valuations.

The paper in particular explains the appearance of contracts that *partially* restrict exchanges - an observation that is commonly made for example in ticket pricing. By this is meant that the customer is neither granted free exchange to whatever variety he prefers nor is he restricted to definitively stay with the initially purchased good. This is unusual in environments that generate step solutions as optimal mechanisms. In particular, this result demonstrates that the restriction on contracts made by Gale (1993) in his pioneering work on contracting in situations with ex post private information, which is further sketched in the literature part, excludes the optimal solution. The 'partial distortion' of the cheaper contract results from the present sequential screening problem being a mechanism design problem with countervailing incentives in the sense of Lewis and Sappington (1989). In this kind of problem it may depend on the allocation whether upward or downward incentive constraints are binding. In my model this gives rise to a new source of pooling around the type where the binding constraints turn. The corresponding pooling contract is the one to be explained. The economic intuition for this non-standard situation emerges from the combination of ex post private information and the horizontal differentiation of goods. While in a usual setting with vertical differentiation the ordering of agents' types induced by the valuation of a good is independent of the variety considered, in the case with horizontally differentiated goods there is no such ordering. A consumer who values getting the right good more also suffers more from ending up with the wrong good. As a consequence, it depends on the specific contract whether it is in expectation valued more or less by consumers with higher top valuation. This paper provides a particularly tractable method how to solve such problems when equilibrium utilities are convex.

Finally, I answer the question how the partial restriction in flexibility is optimally designed. It is shown that any optimal mechanism can be implemented by offering a menu of Limited Exchange Contracts. A deterministic Limited Exchange Contract fixes a payment, a specific product the customer obtains, and a basket of varieties in the first period. In the second period, the consumer is given the possibility to exchange his variety with a product from the basket free of charge. Exchanges with products outside this basket of goods are not possible at all.[2] In particular, contracts that make use of exchange fees to limit product changes are not optimal. In the application to the transportation industry, a Limited Exchange Contract with partial flexibility restriction would for example allow consumers to exchange their departure time within a certain interval of time around the initially bought departure time for free. Many US airlines implement this by offering very cheap or costless same day exchanges even for their cheapest category of tickets. The use of menus of Limited Exchange Contracts is also widespread among European ferry companies; an example is P&O ferries.

The result on the optimality of Limited Exchange Contracts, in particular that exchange fees are not used as a price discrimination device, has an important implication for the literature on sequential screening: Since a canonical contribution by Courty and Li (2000), there is an increasing amount of papers that study how revenue maximizing mechanisms employ differing refund policies as a price discrimination device. This paper introduces a new kind of ex post information about the valuation of heterogeneous goods. The optimal contracts in this broader setting can *not* be interpreted as refund contracts, where the consumer can give back one product for a partial refund and purchase another variety. Instead, it is optimal to differentiate contracts by distorting the allocation and not by creating incentives through differing monetary payments.

Related Literature. There is small number of papers in which agents' preferences over differentiated goods are gradually learned over time. Gale (1993) studies intertemporal pricing policies in a setting similar to the two goods version of mine. In order to obtain a fruitful comparison between monopolistic and oligopolistic pricing behavior, Gale, however, restricts his considerations to two types of contracts: Late purchases with a single price for both goods and early purchases at an advance-purchase discount but without any exchange or refund possibilities. As I will show, allowing for the full range of possible contracts further raises the monopolist's revenue. A major contribution of my paper is the characterization of this new type of contract. Furthermore, Gale considers a two-product case with special attention to advance-purchase discounts. However, it turns out that it is specifically the richness of my setting which allows for an understanding of the underlying effects and enables a modeling of exchange-policies as such.

[2]Stochastic Limited Exchange Contracts are the straightforward extension to distributions over goods.

Two papers by Gale and Holmes (1992, 1993) start with the same basic framework as Gale (1993) but depart from price discrimination and focus on how intertemporal pricing rules can optimally resolve capacity problems. In their model, they assume that there is a peak-demand flight for which the exogenously given capacity does not suffice and an off-peak flight with capacity left. Discounts for first period purchases are then used in order to incentivize agents with small valuation differences to take the off-peak flight even if it turns out that they prefer the peak flight. In contrast to that, my focus is on pure price discrimination motives without any capacity constraints.

Recently, Möller and Watanabe (mimeo, 2013) rediscovered the early model on advance-purchase discounts with differentiated goods as a way to introduce oligopolistic competition. In order to obtain a tractable analysis of strategic interaction, they use a stylized two-goods model and restrict strategies to advance-purchase discounts as well.

Based on the contributions by Gale and Holmes and earlier insights by Baron and Besanko (1984), an extensive literature on advance-purchase discounts in settings with homogeneous goods has evolved. Examples are DeGraba (1995), Courty (2003a, b), Möller and Watanabe (2010) and Nocke, Peitz and Rosar (2011). In these papers, the possibility to buy a certain good at an early point in time at a discount but without any refund possibility is primarily used as a price discrimination device.

In the canonical paper by Courty and Li (2000), the authors set up a theory of intertemporal pricing to explain the prevalence of *partial* refund contracts. In their model there is one homogeneous type of tickets and customers initially have individual uncertainty about their final valuation for it. This uncertainty is then resolved in a second period, in which also consumption takes place. As both the initial valuation distributions and the final valuations differ among agents and are private information, a revenue maximizing monopolist sequentially screens the agents. Courty and Li show that for some cases revenue maximization occurs by offering menus of partial refund contracts.[3] A partial refund contract consists of an initial payment for receiving the good and a later option to return it and receive a partial refund. Since the contribution by Courty and Li (2000), partial refund contracts have been examined in many varieties and extensions. Deb and

[3] In their model results better match reality when a business traveler's valuation distribution differs from a leisure traveler's distribution by a spread, rather than by first order stochastic dominance - which they initially considered to be natural. My model provides a foundation to this assumption made by Courty and Li. Paralleling their initial intuition, my model exhibits a higher top valuation for business travelers. However, the relative valuation-loss from consuming unfavorable goods is also larger for them. When the variety that can be offered is fixed, as considered in Courty and Li, distances to the favorite product vary and the steeper loss function for business travelers leads to greater fluctuations in valuation for the offered variety.

Said (mimeo, 2014) add limited commitment of the monopolist and Inderst and Peitz (2012) as well as Akan, Ata and Dana (mimeo, 2011) study a model in which the time of resolution of an individual's uncertainty is type-dependent. However, all of these contributions have in common that there is just one homogeneous good involved and hence product choice is not an issue. My work contributes to this literature by expanding the problem to one with differentiated goods.

Further contributions to the literature on sequential screening focus on information rents and the question of whether disclosure of ex post private information to the agents is beneficial for the monopolist. Examples are Esö and Szentes (2007) and Krähmer and Strausz (mimeo, 2014). Important insights into why sequentially screening is optimal are provided by Krähmer and Strausz (mimeo, 2014), who investigate the role of ex post participation constraints on profits.

More generally, my paper adds to the literature on dynamic mechanism design in environments with long-lived agents. The design of incentive-compatible mechanisms in dynamic settings in which information gradually arrives over time has been studied by Pavan, Segal and Toikka (2014).

Finally, my analysis relates to mechanism design problems with continuous types and type-dependent outside options. The pioneering contribution is Lewis and Sappington (1989). A continuative analysis is done by Maggi and Rodriguez-Clare (1995) and a general exposition is Jullien (2000). While the latter two references apply results from optimal control theory to obtain a solution, Samuelson and Nöldeke (2007) provide an alternative approach.

In Section 2, I introduce a simple version of my model to outline a technical aspect that arises from the horizontal differentiation of goods, to show basic properties of optimal mechanisms and to clarify the relation to Gale (1993). A key feature of optimal contracts is that some consumers are partially restricted in their flexibility to change varieties in the second period. In Section 3, a more general model is introduced in order to study the optimal design of this partial limitation of flexibility and its implementation. Finally, Section 4 concludes.

2 The two goods model

2.1 Model

Consider a revenue maximizing monopolist with full commitment who can produce arbitrary amounts of goods one and two. Production costs are assumed to be constant and normalized to zero. There is a unit mass of consumers with unit demand. Each agent is characterized by an ex ante type r and an ex post type a. The ex ante type r determines the two valuations for the preferred and the alternative good. This means the ex ante type determines the intensity of preferences and thereby the loss of ending up with the wrong product. Ex post types a determine the ordinal preferences over the goods. I denote this information by ex post types equal to a_1 or a_2 for preferring good one and two respectively. For the sake of simplicity, I assume that valuations are linear in ex ante types. A consumer of type r values the favored good by $v^+(r) = v + \beta r$ and the other good by $v^-(r) = v + \gamma r$ with $\beta > 0$ and $\gamma < \beta$. $\beta > 0$ means that the valuation of the favorite good is increasing in ex ante types and $\gamma < \beta$ means that the comparative loss of ending up with the lesser valued product is increasing. Note that γ may take negative values, which means I allow for the valuation of the alternative good to be lower for higher ex ante types. Let the basic valuation v be high enough such that $v^+(r)$ and $v^-(r)$ are positive. For both the analytical part and interpretation it turns out to be helpful to rescale ex ante types in a way such that valuations are $v^+(r) = v - \delta r + r$ and $v^-(r) = v - \delta r - r$ with $\delta < 1$[4]. This notation disentangles the spread in valuations and a common trend in both valuations given by $-\delta r$. Ex ante types r are continuously distributed over the type space $R = [0, \overline{r}]$. The ex ante type space ranges from the type that is completely indifferent about which product he obtains to types with strong preference intensities. Denote the density function by $f(r)$ and the probability distribution function by $F(r)$. Let the distribution satisfy the standard assumption of increasing virtual values $r - \frac{1-F(r)}{f(r)}$.

There are two periods. In the first period, each agent privately learns his ex ante type, but is uncertain about his ex post type. Ex post types are equally likely, independent of ex ante types[5]. In the first period, there is an outside option of zero. In the second

[4]Rescale by multiplying ex ante types with $\frac{\beta-\gamma}{2}$. Then define $\delta = \frac{\beta+\gamma}{\gamma-\beta}$.

[5]The uniform distribution is imposed to isolate incentives for price discrimination created by the sequential setup from known incentives that stem from an ex ante vertical differentiation of the goods. Such differentiation would emerge if there was one good that is ex ante more likely to be favored. Given independence, the extension to any distribution is technically without further complications.

period, each consumer privately learns his ex post type and consumption takes place.

2.2 First best

In order to convey a basic economic intuition for the monopolist's problem, the analysis is started with a brief discussion of the firm's optimal behavior under complete information. In the absence of private information, the firm can extract the entire surplus. Thus the revenue maximizing firm maximizes welfare. The corresponding first best provision of goods is to always give any customer his preferred good. This is surplus maximizing as all valuations by assumption exceed production costs and the provision of all varieties is equally costly. The firm achieves first best profits if he implements this allocation rule and then extracts all rents by charging each consumer with ex ante type r his top valuation $v^+(r)$.

Next, consider implementation of this allocation rule when consumers have private information about ex ante and ex post types. If the consumer's top valuation was independent of the ex ante type r, which would be the case for $\delta = 1$, the firm could indeed achieve first best profits. A simple way to receive these profits would be to offer one type of contract in the first period exclusively. This contract entails an immediate payment equal to the top valuation and garantuees the customer to obtain his favorite variety in the second period. Every consumer would sign that contract and then choose his favorite product. However, by assumption the top valuation is increasing in r. To achieve first best profits, the firm would need to induce the consumers to sign contracts that only differ in prices. For obvious reasons this is not possible. Hence, when only offering contracts that garantuee the customer to obtain his favorite variety, the firm has to trade off leaving rents to high types and excluding low types. As shown in the full analysis of the problem, the monopolist can, however, profitably price discriminate by using the property that with increasing r also the valuation difference between the two goods is increasing.

2.3 Analysis

As the firm has full commitment power, the revelation principle applies (see Myerson, 1986), which allows me to concentrate on direct mechanisms. A direct mechanism specifies for any reported pair of types $(\hat{r}, \hat{a})$ a price p paid by the agent to the designer and an allocation. A general allocation is a probability distribution over all possible product sets that the agent can end up with. These are "only good one", "only good two", "both goods" and "no good". An allocation is completely described by $X = (x_1, x_2, x_{1\&2})$, where the three entries denote the probabilities for the first three product sets, respectively. In the following I will identify a direct mechanism directly with its outcome function $\big(X(\hat{r},\hat{a}), p(\hat{r},\hat{a})\big)$. Given a direct mechanism and a report about the ex ante type, I call the function that maps ex post type reports into allocations and prices a contract. The choice of the ex ante report then corresonds to the choice of a contract and the choice of an ex post report determines an option within that contract.

Given a pair of types (r, a_i), $i \in \{1,2\}$, and an allocation determined by a pair of reports $(\hat{r},\hat{a})$ the consumer's utility is

$$u(r,\hat{r},a_i,\hat{a}) = v^+(r)\cdot\big(x_i(\hat{r},\hat{a}) + x_{1\&2}(\hat{r},\hat{a})\big) + v^-(r)\cdot x_{3-i}(\hat{r},\hat{a}) - p(\hat{r},\hat{a}). \tag{1}$$

An agent's second period strategy is described by a function $\sigma : A \times R \times R \to A$, where $\sigma(a,r,\hat{r})$ denotes a customer's strategy for announcing a second period report, which may depend on his ex ante and ex post type as well as his ex ante report. An agent's first period expected utility is then[6]

$$U(\hat{r},r,\sigma) = \mathbb{E}_a[u(r,\hat{r},a,\sigma(a,r,\hat{r}))].$$

For truthtelling denote $\sigma(a,r,\hat{r}) \equiv a$ by the identity id_a. Define further $U(r,id_a) := U(r,r,id_a)$. Now the maximization problem $(\mathcal{P})$ can be formulated:

$$\max_{X,p} \int_0^{\overline{r}} f(r)\mathbb{E}_a[p(r,a)]dr$$

s.t.

$$U(r,id_a) \geq U(\hat{r},r,\sigma) \quad \forall r,\hat{r}\neq r,\sigma, \tag{IC_1}$$

$$U(r,id_a) \geq 0 \quad \forall r, \tag{IR}$$

$$u(r,r,a,a) \geq u(r,r,a,\hat{a}) \quad \forall r,a,\hat{a} \tag{IC_2}$$

$$x_1(\hat{r},\hat{a}), x_2(\hat{r},\hat{a}), x_{1\&2}(\hat{r},\hat{a}) \geq 0, \quad x_1(\hat{r},\hat{a}) + x_2(\hat{r},\hat{a}) + x_{1\&2}(\hat{r},\hat{a}) \leq 1 \quad \forall \hat{r},\hat{a}. \tag{F}$$

[6]Since a is uniformly distributed on the binary support $\{a_1, a_2\}$,

$$\mathbb{E}_a[u(r,\hat{r},a,\sigma(a,r,\hat{r}))] = \frac{1}{2}u(r,\hat{r},a_1,\sigma(a_1,r,\hat{r})) + \frac{1}{2}u(r,\hat{r},a_2,\sigma(a_2,r,\hat{r}))$$

In this application of the dynamic revelation principle, the second period incentive constraints (IC_2) ensure that any agent *who has truthfully reported his first period type* also truthfully reports his ex post type. The first period incentive constraints (IC_1) say that the expected utility of telling the truth in the first period, which is then followed by truthtelling in the second period by (IC_2), must be better than any combination of lying about the ex ante type potentially followed by another lie about the ex post type. This means the first period incentive constraints must ensure against double deviations. Furthermore, the individual rationality constraints (IR) hold in the first period and (F) is the feasibility constraint for the allocation.

The following Lemma simplifies the problem. It states that as there is unit demand I can restrict attention to allocations that assign at maximum one good.

Lemma 1. *For any direct mechanism that satisfies the constraints of $\mathcal{P}$, there exists a direct mechanism with the same payments that never assigns both goods and satisfies the constraints of $\mathcal{P}$.*

The proof follows by a simple replication argument[7]. The assignment of several goods at a time can be replaced by assigning the single good which is claimed to be preferred among those. In equilibrium, consumption and hence on-path utilities are unchanged. Incentive compatibility is preserved as well, because off-path utilities are weakly lowered. As in the modified mechanism payments are unchanged, the mechanisms are equivalent in terms of profit. Therefore it is without loss to only consider mechansims than assign at maximum one good. Formally, I from now on set $x_{1\&2}$ to zero for all reports $(\hat{r}, \hat{a})$ and drop it.

As ex post types are equally likely for each ex ante type, the expected valuation in the first period for any allocation with $x_1 + x_2 = 1$ is $v - \delta r$. Whether it increases in or decreases in ex ante types depends on δ, which represents the relation between the increase in top valuation and the potential decrease in $v^{-}(r)$. Since the solution technique and results differ for these two possibilities, the analysis of the problem is split into two cases.

[7] The proof of Lemma 1 as well as all subsequent ones are given in the appendix.

2.3.1 Decreasing mean

The analysis begins with the case in which the comparative loss from ending up with the wrong product outweighs the increase in valuation of the preferred alternative such that the expected valuation of a particular good assigned in the first period decreases in r. Formally this corresponds to the case $\delta \geq 0$.

A solution to this maximization problem will be found with the help of a technique that is common in the literature on sequential screening[8]: I consider the relaxed problem with publicly observable second period types and find the set of solutions to it. The relaxed problem differs from the original one through leaving out all IC_2 constraints as well as all those IC_1 constraints which insure against first period deviations which are followed by another lie. The profit generated by the solutions to the relaxed problem constitutes an upper bound on the profit that can be achieved in the full maximization problem. Then I show that each solution to the relaxed problem satisfies the constraints which are left out. Hence, these are solutions to the original problem.

The relaxed maximization problem ($\mathcal{P}_o$) is

$$\max_{X,p} \int_0^{\overline{r}} f(r) \cdot \mathbb{E}_a[p(r,a)]dr$$

s.t.

$$U(r, id_a) \geq U(\hat{r}, r, id_a) \quad \forall r, \hat{r}, a, \tag{IC_1'}$$

$$U(r, id_a) \geq 0 \quad \forall r, \tag{IR}$$

$$x_1(\hat{r}, \hat{a}), x_2(\hat{r}, \hat{a}) \geq 0, \quad x_1(\hat{r}, \hat{a}) + x_2(\hat{r}, \hat{a}) \leq 1 \quad \forall \hat{r}, \hat{a}. \tag{F}$$

In the next step, I exploit the model's symmetry with respect to the two goods. Instead of identifying goods by their name, I distinct between preferred and undesired goods. Denote $x_+(\hat{r}) = \frac{1}{2}x_1(\hat{r}, a_1) + \frac{1}{2}x_2(\hat{r}, a_2)$ and correspondingly $x_-(\hat{r}) = \frac{1}{2}x_2(\hat{r}, a_1) + \frac{1}{2}x_1(\hat{r}, a_2)$. $x_+(\hat{r})$ and $x_-(\hat{r})$ are probabilities themselves that are specific to the contract determined by ex ante report $\hat{r}$. Formed in the first period, they indicate the probability of the assignment of a preferred and an undesirable good in the second period given truthful revelation of second period types. With this notation, the expected utility can be rewritten as

[8]See for example Gale and Holmes (1993) and Esö and Szentes (2007)

$$
\begin{aligned}
U(\hat{r}, r, id_a) &= x_+(\hat{r}) \cdot v^+(r) + x_-(\hat{r}) \cdot v^-(r) - \mathbb{E}_a[p(r,a)] \\
&= v[x_+(\hat{r}) + x_-(\hat{r})] + r \cdot K(\hat{r}, \delta) - \mathbb{E}_a[p(r,a)]
\end{aligned} \tag{2}
$$

with $K(\hat{r}, \delta) = x_+(\hat{r}) - x_-(\hat{r}) - \delta(x_+(\hat{r}) + x_-(\hat{r}))$.

Lemma 2. *The first period incentive constraints IC_1' are satisfied if and only if*

$$\partial U(r, id_a)/\partial r = K(r, \delta) \text{ a.e.} \tag{ENV}$$

$$\text{and} \quad K(r, \delta) \text{ is mon. increasing in } r. \tag{MON}$$

Lemma 2 implies that maximizing with respect to the constraints (IC_1'), (IR) and (F) is equivalent to taking (ENV), (MON), (IR) and (F) as constraints. If (ENV) and (MON) hold, the ex ante utility is convex in types.

Even though Lemma 2 and its proof are familiar from the literature on static mechanism design, it is non-standard in the literature on sequential screening. In the standard sequential screening problem, the first and second order condition are not sufficient for incentive compatibility (for an exposition see Courty and Li (2000) and Esö and Szentes (2007)). Difficulties arise, because the first and second order condition hold only in expectation over the ex post type. In my model, this problem can be overcome by exploiting the symmetry that stems from the horizontal differentiation of goods: The utility level just depends on whether the obtained good is favorite or non-favorite, but the identity of goods does not play any role. This permits to rewrite the expected utility of an allocation as a probability distribution over utility levels as done in (2).

What distinguishes this maximization problem also from a standard static mechanism design problem with continuous types and linear utility is that $K(r, \delta)$ can take both positive and negative values. Hence, expected utility might monotonically increase or decrease on R, but it might also be the case that expected utility is decreasing with increasing ex ante types only for small r and expected utility as a function of ex ante types is U-shaped. Consequently, it is not clear which agent will have the lowest expected utility which will then be set to zero in the optimum by the individual rationality constraints.

The economic intuition for this non-standard situation emerges from the horizontal differentiation of goods. While in a usual setting with vertical differentiation the ordering of agents' types induced by the valuation of an allocation is independent of the allocation considered, in the case with horizontally differentiated goods there is no such clear ordering. In my model, an agent who values getting the right good more also suffers more from ending up unfavorably. As a consequence, any allocation that has a tendency towards assigning the 'wrong' good is valued *less* by higher types. On the contrary, allocations that have a tendency towards assigning the 'right' good, are valued more by higher types.

Models in which the type with binding individual rationality constraint is unclear have first been considered in the framework of type-dependent outside options (see Lewis and Sappington (1989), Maggi and Rodriguez-Clare (1995), Jullien (2000) and Samuelson and Nöldeke (2007) for an exposition). However, their results cannot directly be applied, because all of the latter papers by assumption exclude my case of pure revenue maximization. Furthermore, in my model the convexity of the equilibrium utilities in r, a consequence of Lemma 2, allows for a particularly tractable method of solving the maximization problem: It is known that in every solution, there is an ex ante type $z \in [0; \overline{r}]$ that has the lowest ex ante utility. Making use of this fact, in a first step I solve problem $\mathcal{P}_o$ with the additional constraint $U(r, id_a) \geq U(z, id_a)$ for all $r \in R$ and some arbitrary but fixed ex ante type z. Denote this problem by $\mathcal{P}_o^z$. This results in the description of an optimal allocation dependent on z for all $z \in R$, where z is the exogenously given ex ante type with the lowest expected utility. In a second step I then maximize the profit in z.
Having fixed ex ante type z, Lemma 2 can be employed to reformulate the maximization problem: In the optimum, z's expected utility is zero by binding individual rationality. By (ENV) and absolute continuity[9] any ex ante type's expected utility can then be written as

$$U(r, id_a) = U(z, id_a) + \int_z^r K(y, \delta) dy = \int_z^r K(y, \delta) dy. \tag{3}$$

By (2) and (3) prices can be written as a function of the allocation:

$$\mathbb{E}_a[p(r, a)] = v \cdot K(r, \delta) - \int_z^r K(y, \delta) dy. \tag{4}$$

[9]For a proof of absolute continuity see for example Theorem 2 in Milgrom and Segal (2002)

Plugging (4) into the objective reduces problem $\mathcal{P}_o^z$ to:

$$\max_x \int_0^{\overline{r}} f(r)\left(v \cdot K(r,\delta) - \int_z^r K(y,\delta)dy\right)dr$$

s.t. (MON), (F) and $U(z, id_a) \geq U(r, id_a) \quad \forall r \in R$.

By integration by parts and rearranging terms, the problem can be rewritten as

$$\begin{aligned} \max_x \int_0^z f(r)\left[v[x_+(r) + x_-(r)] + K(r,\delta) \cdot \left(r + \frac{F(r)}{f(r)}\right)\right] dr \\ + \int_z^{\overline{r}} f(r)\left[v[x_+(r) + x_-(r)] + K(r,\delta) \cdot \left(r - \frac{1-F(r)}{f(r)}\right)\right] dr \end{aligned} \tag{5}$$

s.t. (MON), (F) and $U(z, id_a) \geq U(r, id_a) \quad \forall r \in R$.

Define $b = sup\{r \in R | r - \frac{1-F(r)}{f(r)} \leq 0\}$.

Lemma 3. *Any solution to problem $\mathcal{P}_o^z$ has the following properties:*[10]

$x_+(r) = \frac{1+\delta}{2}$	*and*	$x_-(r) = \frac{1-\delta}{2}$	*if* $r \leq max\{b, z\}$,
$x_+(r) = 1$	*and*	$x_-(r) = 0$	*if* $r > max\{b, z\}$.

For any $z \in R$ a solution does exist.

Lemma 3 is proven by pointwise maximization of objective (4) for every ex ante type r, subject to a relaxed version of the constraints. The constraints are weakened in the sense that I only pay attention to the bounds on both $x_+(r) + x_-(r)$ and $K(r, \delta)$ that are implied by (MON), (F) and z to be the type with minimal utility. Note that unlike in standard pointwise maximization problems familiar from the literature on mechanism design[11], the monotonicity constraint cannot be entirely ignored at that point.
Pointwise maximization in this relaxation of $\mathcal{P}_o^z$ is not completely trivial. I show that contracts with the properties given in Lemma 3 lead to an upper bound on pointwise profits within that relaxation. Then it is shown that there exists a feasible allocation

[10] W.l.o.g. let b and types with $r - \frac{1-F(r)}{f(r)} = 0$ get the allocation of low types.
[11] See Myerson (1981)

rule with these properties. Finally, it can immediately be seen that allocations with the derived properties satisfy (MON) and z is the type with lowest expected utility, which completes the proof of the Lemma.

This allows me to turn to the second step now, the maximization with respect to the worst-off type z.

Lemma 4. *z is optimal if and only if $z \leq b$.*

Lemma 4 results from inserting the properties from Lemma 3 into objective (4). By the optimality conditions from Lemma 3, both the virtual value for types $r < z$ as well as the virtual value for types $z < r < b$ are multiplied by zero and hence do not influence profits. Thus for ex ante types $r \leq b$ the relative position to z, which determines the virtual value associated with the ex ante type, has no relevance even though for a given r virtual values are not equal. The virtual value for types $r > b$, however, enters the objective strictly positively if and only if $r > z$. Therefore, profit maximization demands that for all r with $r > b$ holds $r > z$.

Lemma 5. *The set of mechanisms which solve problem $\mathcal{P}_o$ is the following: Agents with ex ante types $r > b$ always obtain their favorite good and prices satisfy $\mathbb{E}_a[p(r,a)] = v + b(1-d)$. Agents with ex ante types $r \leq b$ obtain contracts with $\mathbb{E}_a[p(r,a)] = v$. There is a continuum of optimal allocation rules for ex ante types $r \leq b$ characterized by $\alpha \in [\delta, 1]$:*

$$x_1(r, a_1) = \alpha, \qquad x_2(r, a_1) = 1 - \alpha,$$
$$x_1(r, a_2) = \alpha - \delta, \qquad x_2(r, a_2) = 1 + \delta - \alpha.$$

The set of optimal allocation rules is obtained as the solution to a system of linear equations, which are the feasibility requirements and the optimality conditions given by Lemmas 3 and 4. As the relaxed problem takes into account only first period incentives, only expected prices matter. They are pinned down by (4).

Before turning to implementability in the original problem, the outcome for the relaxed problem is discussed. Any optimal contract, which is determined by an ex ante report r, satisfies $x_+(r) + x_-(r) = 1$. This means that every consumer always obtains a good and the event 'no assignment' does not occur. For this reason I call this property 'full assignment'.

For the sake of a thorough interpretation of optimal contracts, it pays off to insert the full assignment property into expected utility (2). This yields

$$U(\hat{r}, r, id_a) = v - r\delta + r[x_+(\hat{r}) - x_-(\hat{r})] - \mathbb{E}_a[p(r, a)]. \tag{6}$$

Given truthful reporting, the term $x_+(r) - x_-(r)$ is the difference between the ex ante probability to obtain the right and the ex ante probability to obtain the the wrong good and is central for the analysis. In the sequel, the term will be interpreted as 'responsiveness' of the corresponding contract. This can be seen as a measure for quality of the contract from an ex ante point of view. The responsiveness of a contract that maps any ex post type into the same allocation, which means the contract fixes an allocation in the first period that cannot be influenced by any second period report, is zero. If the responsiveness is positive, the contract is said to positively resond to the agent's needs and vice versa. Due to feasibility, responsiveness is bounded above by $x_+(r)-x_-(r) = 1$, the case in which the agent always obtains the good he prefers, and bounded below by $x_+(r) - x_-(r) = -1$, the case in which the agent never obtains the preferred good.

As the first best allocation rule implicates maximum responsiveness, contracts with lower values of responsiveness are considered as distorted and distortion is measured by the difference to one. The solution turns out to be a step function that bunches ex ante types into two groups as illustrated in Figure 2.3.1. High ex ante types above a certain threshold type b always receive the good they prefer $(x_+(r) - x_-(r) = 1)$, which implicates the classical 'no distortion at the top' result. All ex ante types lower than the critical type get a contract from the continuum of contracts that solve $x_+(r) - x_-(r) = \delta$. This common characteristic means that the contracts positively respond to the second period report in the sense that the likelihood to obtain a given good is increasing by δ when it is announced to be favorite. Consider for example the contract $\alpha = 1$ from Lemma 5. The contract would give the consumer variety one with certainty if he announces this good to be the favorite one. However, if the agent preferred the second good, there would be a chance of δ that he obtains the second good. For the special case $\delta = 0$ and hence $x_+(r) - x_-(r) = 0$ an optimal contract chosen by low ex ante types fixes an arbitrary allocation that always assigns a good in the first period and does not respond to second period reports in any way. Note that any contract with responsiveness unequal to one, minus one, or zero assigns nondegenerate allocations to at least some ex ante type. I call such contracts stochastic.

The set of contracts for low ex ante types is designed such that the expected utility of all ex ante types $r \leq b$ equals zero: By (6) expected utility net of payments is constant among low ex ante types and equals v, which exactly matches expected prices given through (4). In addition, ex ante types are indifferent among all the contracts for any $\hat{r} \leq b$. For high types $r > b$, expected utility is linearly increasing. In particular, this means that in the optimum, expected utility is not U-shaped, but monotonically increasing in ex ante types everywhere. The expected utility of ex ante types below the type with minimum expected utility is 'ironed' to zero.
The contracts for low ex ante types are the main object of study in this paper. They are non-standard in step solutions, because they are not maximally downward distorted. The appearance of such 'intermediately distorted' contracts is in line with the established literature on mechanism design with type-dependent outside options. The unusual type of allocations stems from the fact that the binding first period incentive constraints change from upward constraints to downward constraints at the interior ex ante type whose participation constraint is binding.[12] In the standard case, under certain assumptions the optimal incentive compatible contract gives zero rent from participating to an entire interval of types around this critical type.[13] In my model this interval is is $[0, b]$ and providing those types with their outside option is achieved via bunching on that interval.[14]

Having found the set of solutions to the relaxed problem with observable second period types ($\mathcal{P}_o$), I will finally address the implementability in the original problem with private ex post types ($\mathcal{P}$). When ex post types are private, two additional types of incentive constraints have to be satisfied: The second period incentive constraints (IC_2) have to hold, which means no agent who has truthfully revealed his ex ante type may have an incentive to lie about his ex post type. And importantly, in the first period there may not exist profitable double deviations (IC_1), i.e., a first period lie followed by another one in the next period. Proposition 1 shows that the entire set of optimal allocation rules from Lemma 4 is also implementable in the original problem with private ex post types.

[12]This is meant by 'countervailing incentives', see Lewis and Sappington (1989) and Maggi and Rodriguez-Clare (1995).
[13]See in particular the exposition by Jullien (2000).
[14]For the same reason bunching occurs in problems with linear type-dependent outside options and linear utility, see Maggi and Rodriguez-Clare (1995).

Proposition 1. *Let the mean be decreasing. The set of allocation rules which solve the problem with private ex ante and ex post types is the following:*

- *For* $r > b$*: The consumer always obtains his favorite good.*
- *For* $r \leq b$*:* $x_1(r, a_1) = \alpha, \quad x_1(r, a_2) = \alpha - \delta$ *with arbitrary* $\alpha \in [\delta, 1]$ *and* $x_2(r, a) = 1 - x_1(r, a) \quad$ *for* $a \in \{a_1, a_2\}$.

Necessary conditions for prices are:

- *For* $r > b$*:* $\mathbb{E}_a[p(r, a)] = v + b(1 - \delta)$.
- *For* $r \leq b$*:* $\mathbb{E}_a[p(r, a)] = v$.

Ex post type independent prices $p(r, a_1) = p(r, a_2)$ *always ensure incentive compatibility.*

In order to prove incentive compatibility, I show that a stronger condition than (IC_2) holds: Each agent has an incentive to truthfully report his second period type no matter what his first period report was. This is sufficient for incentive compatibility. The (IC_2) constraints are then trivially satisfied. However, (IC_1) is satisfied as well: I have shown that lying in the second stage and thus complex deviations are never optimal. Additionally, it is already known that the solutions satisfy the relaxed problem's (IC_1') constraints, which means unilateral first-period deviations are not profitable either.

For problem $\mathcal{P}_o$ only expected prices matter and hence by Lemma 5 for each ex ante type only the sum of the prices $p(r, a_1) + p(r, a_2)$ is pinned down. When dealing with implementability in the original problem, single ex post prices are relevant due to second period incentives. I set $p(r, a_1) = p(r, a_2)$ for all ex ante types, such that there are no additional incentives from the price scheme on how to announce the ex post type. As a possible implementation, one can imagine that the price has to be paid in the first period.

Now assume some type $r \in R$ has reported an arbitrary $\hat{r} \in R$. The price he pays for the chosen contract is then fixed independently of the report on his ex post type. The agent will then report honestly about a, because the contract will always provide him with the product which is announced to be the 'good' one with a higher probability. So lying would harm the agent, because he would then be less likely to get his preferred good. Hence, it is the tendency towards allocating the 'right' good that makes the entire set of solutions to the relaxed problem implementable in the original problem.

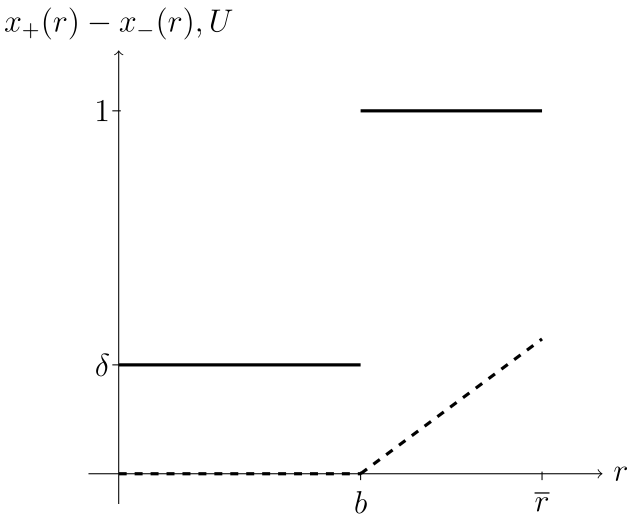

Figure 1: Solution to the decreasing mean case. The solid line represents the responsiveness of the contracts, the dashed line expected utility.

As the set of optimal allocation rules from Lemma 5 is identical to those of Proposition 1, all comments on Lemma 5 and Figure 1 carry over to Proposition 1. In particular, high ex ante types above the threshold type always get the variety they prefer. This is implementable, for example, by selling them the varieties for a uniform price in the second period or selling them a good in the first period but with an option for free exchange. The others do not always end up with their preferred good, but the allocation 'reacts' to the second period announcement in the sense that a 'mobile' probability mass of δ is shifted towards the preferred good. For example, in the contract with $\alpha = 1$, an agent will always obtain good one if he prefers it. In case of a preference for good two, there is a chance of δ that he receives good two, otherwise he ends up with the less preferred good. For the monopolist a possible implementation of this contract is to sell good one in the first period, but if the agent afterwards reports that he would prefer the other good, give him a chance of δ to exchange the good. In practice, the stochastic element can - for example - be implemented by allowing for exchange subject to availability.

The optimal menu of contracts is the result of price discrimination. When everybody is fully flexible between the goods, a higher type has an ex ante higher expected utility from the contract than lower types because his valuation of the preferred good, which he will get with certainty, is higher. When offering just one contract with full flexibility, the monopolist has to trade off leaving rent to high types and excluding low types. However, the monopolist can exploit a single-crossing property with respect to $x_+(r) - x_-(r)$ in order to price-discriminate: Given full assignment, the marginal expected utility with respect to $x_+(r) - x_-(r)$ is increasing in the ex ante type. This means the monopolist exploits the fact that flexibility between goods has a higher value to the higher ex ante types. Therefore, by offering less responsive contracts to lower ex ante types, the loss from the lower types is smaller than the gain from extracting rent from higher types. However, it is not optimal to distort contracts for low types to zero responsiveness. The critical type's rent from signing the high-type contract can already be entirely extracted without incurring the high cost of distorting contracts for low types to zero responsiveness.

This result shows that the restriction on contracts done by Gale (1993) is with consequences. Gale looked at a setting which is very close to the one examined here. However, he did not use a general mechanism design approach to derive the revenue maximizing menu of contracts. Instead, there is a restriction to selling in the first period without any later flexibility or selling the good in the second period, which corresponds to a contract with full flexibility. In his pioneering work, this restriction was imposed in order to obtain a fruitful comparison between monopolistic and oligopolistic behavior in a setting with individual demand uncertainty. For the monopolistic case, my analysis shows that by allowing for intermediately distorted contracts, which are stochastic contracts, it is possible to achieve even more. Analytically spoken, Gale allows for the contract including the first-best allocation, which is shown to actually be optimal for high types. However, as 'discrimination' alternative he only allows for a contract with $x_+(r) = x_-(r) = 1$, which differs from the optimal contract for any $\delta > 0$. $\delta = 0$ is the case in which the expected utility of a fixed allocation stays constant over ex ante types. For this case, my model also predicts a contract with zero responsiveness to be optimal for low types. Corollary 1 summarizes the relation to Gale (1993):

Corollary 1. *Whenever the expected utility of a given good is decreasing in ex ante types, the solution to the revenue-maximization problem includes stochastic contracts. They strictly improve upon a menu of contracts without possibility for exchange on the one hand and goods sold in the second period on the other hand.*

2.3.2 Increasing mean

The analysis is completed with the examination of the case in which the increase in top valuation dominates the increase in valuation loss from obtaining the wrong product in

the sense that the expected valuation of a particular good assigned in the first period, $v - r\delta$, is increasing in ex ante types. Formally, this corresponds to $\delta < 0$.
Define $e = sup\{r \in R | r - \frac{1-F(r)}{f(r)} \leq \frac{v}{\delta}\}$ whenever the supremum exists and $e = 0$ otherwise. As $\delta < 0$, the constant v/δ is negative and from the increasing virtual value assumption it follows that $e < b$.

Proposition 2. *Let the mean be increasing ($\delta < 0$). The set of allocation rules that solve the problem with private ex ante and ex post types is the following:*

- *For $r > b$:* *The consumer always obtains his favorite good.*
- *For $r \in [e, b]$:* $x_1(r, a) = \alpha, \quad x_2(r, a) = 1 - \alpha \quad \forall a$ *and $\alpha \in [0, 1]$ arbitrary.*
- *For $r < e$:* *No assignment.*

Necessary conditions for prices are:

- *For $r > b$:* $\mathbb{E}_a[p(r, a)] = v + b(1 - \delta)$.
- *For $r \in [e, b]$:* $\mathbb{E}_a[p(r, a)] = v - \delta e$.
- *For $r < e$:* $\mathbb{E}_a[p(r, a)] = 0$.

Ex post type independent prices $p(r, a_1) = p(r, a_2)$ always ensure incentive compatibility.

For $\delta < 0$, the problem $\mathcal{P}$ is solved again by considering a relaxed problem $\mathcal{P}_*$. $\mathcal{P}_*$ differs from $\mathcal{P}_o$ by the additional constraint

$$x_1(r, a_1) + x_2(r, a_2) \geq x_1(r, a_2) + x_2(r, a_1) \quad \forall r, \tag{*}$$

which is a necessary condition for second period incentive compatibility. In $\mathcal{P}_o$ second period incentives are completely ignored. As for positive δ the solution to $\mathcal{P}_o$ is implementable in $\mathcal{P}$, (*) is not strictly binding there[15]. This changes when δ is negative. The solution to $\mathcal{P}_o$, which is stated in Lemma 5, for negative δ violates (*) and hence second period incentive compatibility. The upper bound on profits attained by the solution to $\mathcal{P}_*$ is hence lower than the one derived from $\mathcal{P}_o$.
Due to the similar structure of incentive contraints in the problems $\mathcal{P}_o$ and $\mathcal{P}_*$, Lemma 2 applies. A difference, however, is that from (*) and $\delta < 0$ it follows that $K(r, \delta) \geq 0$

[15]This can also be seen directly from the properties of optimal contracts as stated in Proposition 1.

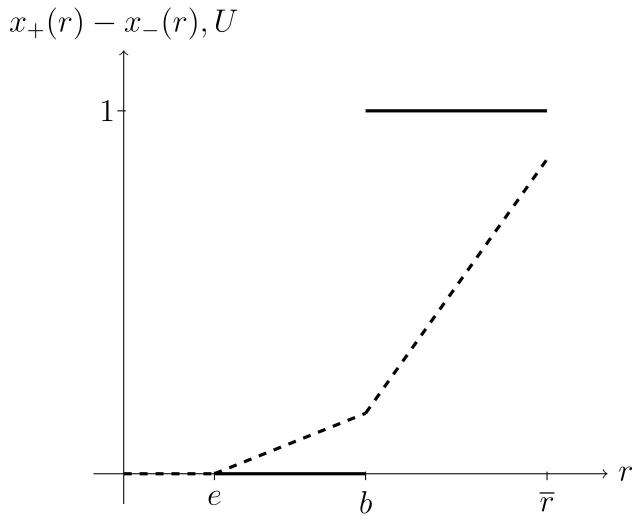

Figure 2: Solution to the increasing mean case with interior e. The solid line represents the responsiveness of the chosen contracts, the dashed line expected utility.

for all ex ante types. Consequently, in the optimum the lowest ex ante type's individual rationality constraint binds and a solution to $\mathcal{P}_*$ is found following standard steps including integration by parts, reformulations and pointwise maximization. By the same way as for the decreasing mean case, the allocation rule is then shown to be implementable in $\mathcal{P}$.

For $e > 0$ the solution is illustrated in Figure 2. Ex ante types above b again always obtain their preferred good. Types lower than b but sufficiently close to it ($r > e$) always obtain a good, the contract is however maximally distorted with the limit given by (*). This contract has responsiveness zero and hence does not respond to the announcement of second period types at all. Types lower than e are excluded, which means the full assignment property does not hold if $e > 0$. Ex ante expected utility is increasing over all types that obtain a good. This menu of contracts can be implemented using advance-purchase discounts, which are well known from numerous studies[16].
The interpretation of the result as one of price discrimination is the logical continuation of the decreasing mean case. Assume for a moment that second period incentive compatibility would be no binding constraint, as is the case when the mean is decreasing in ex ante types. When ignoring second period incentive constraints, the monopolist would like to distort the contracts for ex ante types lower than b such that $x_+(r) - x_-(r) = \delta < 0$.

[16]See for example Gale and Holmes (1993), Gale (1993), Möller and Watanabe (2010) or Nocke et al. (2011).

This is an immediate consequence of Lemma 5. Distortions would be comparatively large, as there is much rent from high ex ante types to be extracted. The upper bound derived from $\mathcal{P}_o$ would be achieved and ex ante utility for these low ex ante types would then be zero. However this contract violates (*), the necessary condition for second period incentive compatibility. By (*) the maximal distortion is $x_+(r) - x_-(r) = 0$. From the single crossing property it then follows that expected utility is increasing in ex ante types when agents obtain the maximally distorted contract. This creates an incentive for the monopolist to completely exclude very low ex ante types. To put it another way: For the increasing mean case, distortions in the quality of contracts are not sufficient to extract the high types' rents and therefore additional quantity distortions are used.

3 The continuous goods model

In the last section I solved a model of sequential screening with horizontally differentiated goods without ad-hoc restrictions on contracts. In the case in which the ex ante expected valuation of a certain good is decreasing in ex ante types, optimal contracts turned out to partially restrict low types' flexibility and can be implemented by an appropriate design of exchange policies. The aim of this part is to further study the optimal design of the partial restriction of flexibility for lower types and thereby further characterize optimal exchange policies. This is done by applying the presented technique to a more comprehensive model. The only sense in which the model presented in the sequel is more specific is that it focuses on the decreasing mean case exclusively.

3.1 Model

I introduce a continuum of horizontally differentiated goods $s \in S = [0, 1]$, which is a typical Hotelling line. There is again a unit mass of consumers with unit demand. Each agent has a utility function over support S depending on the agent's ex ante type r and his ex post type a. There is a continuum of ex post types $a \in A = [0, 1]$, where a determines the most preferred good. Thus each good could possibly be the favorite one. Ex post types are independently and uniformly distributed on A, which in particular means that the ex ante type does not provide any information on what the ex post type will be. This allows me to clearly distinguish the new aspect from incentives for price discrimination which arise when different groups of buyers systematically differ in their preferences among a set of differentiated goods. I assume the utility loss from getting a non-favorite good to be linear in the distance from the favorite good.[17] [18] Let the continuous distribution F of ex ante types r over type space $R = [0; \overline{r}]$ again satisfy the standard assumption of increasing virtual values $r - \frac{1-F(r)}{f(r)}$. I assume that both the utility derived from the favorite good and the comparative utility loss from getting a non-favorite good are linearly increasing in the ex ante type. As already explained, I focus on the case where the relative changes are such that the expected utility from any fixed allocation is decreasing in ex ante types.

A utility function with these properties can be written as $v_{r,a}(s) = v + kr - cr|a - s|$ with $k, c > 0$ and additional restrictions on k and c to ensure the decreasing expectation. By an appropriate normalization of the space of ex ante types, this utility can be rewritten as

$$v_{r,a}(s) = v + (1 - \delta)r - 4r|a - s| \tag{7}$$

with $\delta = 1 - 4k/c$. Normalization is chosen such that the assumption of decreasing expectations boils down to $\delta \geq 0$.[19] Given this normalization, analogies to the two goods model are obvious: The most preferred good is valued by $v + r - r\delta$ and the expected utility of a certain assignment of good $s = 1/2$ is $v - r\delta$ as was the expected utility of any fixed assignment in the two goods model. Note that in this more general model, the ex ante expected utility of prefixed assignments varies and is maximized by the assignment $s = 1/2$. The firm and timing are as before.

3.2 Analysis

Again, the revelation principle applies (see Myerson, 1986). The outcome function of a direct mechanism can be written as $(X(\hat{r}, \hat{a}), p(\hat{r}, \hat{a}))$, where $\hat{r}$ and $\hat{a}$ are the reported types. $p(\hat{r}, \hat{a})$ is the payment rule with payments defined from the agent to the monopolist and $X(\hat{r}, \hat{a})$ is the allocation rule. An allocation is a probability distribution over single elements from the set of products S and 'no assignment' depending on the two reported types. Hence $X(\hat{r}, \hat{a})[\tilde{s}]$ is the probability of an assignment of good $s \leq \tilde{s}$ and

[17] The restriction to the uniform distribution and linear deviation costs is for illustrative purposes only. Qualitatively, all results remain valid for any continuous and ex ante type-independent distribution $G(a)$, whose support is a subset of A with positive measure. Deviation costs can be any continuous function $c(a, s)$ that is quasi-convex in s for any a with the minimum at a where $c(a, a) = 0$. As the aim is to study the decreasing mean case, deviation cost must be sufficiently steep: $\min_k E_a[c(k, s)] \geq 1 - \delta$ for all $k \in S$. For the general case, equation (7) turns to $v_{r,a}(s) = v + (1 - \delta)r - rc(a, s)$.

[18] Note that in this model the valuation for a given good varies gradually in the ex-post type, a key property that helps to reveal more characteristics of exchange-policies.

[19] See Lemma 9 in the appendix for the formal proof.

$X(\hat{r},\hat{a})[1] \leq 1 \ \forall \hat{r},\hat{a}$. The probability of no assignment is $1 - X(\hat{r},\hat{a})[1]$. As already argued, the restriction to probability distributions over the assignment of the different goods and no assignment is without loss of generality. The notion of a contract is carried over from the previous section. Recall that $\sigma(a,r,\hat{r})$ is a customer's strategy for posting a second period type and truthtelling is denoted by the identity id_a.
An agent's first period expected utility is

$$U(\hat{r},r,\sigma) = \int_0^1 \left(\int_0^1 v_{r,a}(s) dX(\hat{r},\sigma(a,r,\hat{r}))[s] \right) - p(\hat{r},\sigma(a,r,\hat{r}))da.$$

Define further $U(r,id_a) := U(r,r,id_a)$. By an application of the revelation principle, the maximization problem $(\mathcal{P})$ can again be formulated:

$$\max_{X,p} \int_0^{\overline{r}} f(r) \int_0^1 p(r,a) da dr$$

s.t.

$$U(r,id_a) \geq U(\hat{r},r,\sigma) \quad \forall r,\hat{r} \neq r,\sigma, \tag{IC_1}$$

$$U(r,id_a) \geq 0 \quad \forall r, \tag{IR}$$

$$\int_0^1 v_{r,a}(s) dX(r,a)[s] - p(r,a) \geq \int_0^1 v_{r,a}(s) dX(r,\hat{a})[s] - p(r,\hat{a}) \quad \forall r,\hat{r},\hat{a}, \tag{IC_2}$$

$$0 \leq X(\hat{r},\hat{a})[s] \leq X(\hat{r},\hat{a})[s'] \leq 1 \quad \forall \hat{r},\hat{a},s,s' \text{ with } s \leq s'. \tag{F}$$

The structure of expected utility in this richer framework has important similarities to the expected utility (2) from Section 2. To make this explicit note that expected utility can be rewritten as

$$U(\hat{r},r,id_a) = v\left(\int_0^1 X(\hat{r},a)[1]da \right) + r \cdot \tilde{K}(\hat{r},\delta) - \int_0^1 p(\hat{r},a)da \tag{8}$$

with $\tilde{K}(\hat{r},\delta) = \int_0^1 \int_0^1 1 - \delta - 4|a-s| dX(\hat{r},a)[s]da$.

As both, the structure of maximization problem $\mathcal{P}$ and the form of expected utilities are

closely related to the simple version, I can follow steps known from the simple model to obtain a solution.
Hence, in order to solve this problem, I again consider the relaxed problem with observable ex post types ($\mathcal{P}_o$). Problem $\mathcal{P}_o$ differs from $\mathcal{P}$ by omitting all IC_2 constraints and relaxing IC_1 to

$$U(r, id_a) \geq U(\hat{r}, r, id_a) \quad \forall r, \hat{r}, a. \tag{IC_1'}$$

The set of solutions to the relaxed maximization problem with observable types is characterized by Lemma 6:

Lemma 6. *A mechanism solves problem $\mathcal{P}_o$ if and only if it satisfies (F) and has the following properties:*

$$\textit{For } r \leq b: \quad \int_0^1 X(r,a)[1]da = 1, \quad \tilde{K}(r,\delta) = 0 \qquad \textit{and} \int_0^1 p(r,a)da = v;$$

$$\textit{For } r > b: \quad \int_0^1 X(r,a)[1]da = 1, \quad \tilde{K}(r,\delta) = 1-\delta \qquad \textit{and} \int_0^1 p(r,a)da = v + b(1-\delta).$$

The proof proceeds as follows: Due to the parallel form of expected utility functions, incentive compatibility can be characterized along the lines of Lemma 2. The slope of expected utility in equilibrium is $\tilde{K}(r,\delta)$. As $\tilde{K}(r,\delta)$ can take negative values for some ex ante types, the solution strategy presented in Section 2.3.1. is applied. The resulting optimality conditions are given in Lemma 5.
To show necessity, I give the menu of deterministic contracts (LE) below that satisfies the conditions of Lemma 5 and whose feasibility is straightforward to see. High ex ante types $r > b$ always obtain their favorite good. For low ex ante types $r \leq b$, an interval of goods is specified. Whenever the favorite variety lies in this interval, it is assigned. Otherwise, the consumer obtains the good of the interval that is closest. Denote the

single mass point of mass one of $X(r,a)$ in (LE) by $x^{LE}(r,a)$:

$$\begin{aligned}
&\text{For } r > b \\
&\qquad x^{LE}(r,a) = a && \text{and} \quad p(r,a) = v + b(1-\delta) \quad \forall a; \\
&\text{For } r \leq b \\
&\qquad x^{LE}(r,a) = \sqrt{1-\delta}/2 && \text{for} \quad a < \sqrt{1-\delta}/2, \\
&\qquad x^{LE}(r,a) = a && \text{for} \quad a \in [\sqrt{1-\delta}/2; 1-\sqrt{1-\delta}/2], \\
&\qquad x^{LE}(r,a) = 1-\sqrt{1-\delta}/2 && \text{for} \quad a > 1-\sqrt{1-\delta}/2, \\
&\qquad p(r,a) = v \quad \forall a.
\end{aligned} \tag{LE}$$

Every solution to the problem $\mathcal{P}_o$ satisfies the full assignment property $\int_0^1 X(r,a)[1]da = 1$, which is equivalent to $X(r,a)[1] = 1$ for almost all a. This means that generically[20] each agent ends up with some good, independent of the reported pair of types.
Any optimal allocation rule is a step function in the ex ante type with threshold-type b. Types above b always get their most preferred good, which implies no distortion at the top. Agents with ex ante types lower than b get a contract which gives them utility v net of payments. This implies that these customers do not always end up with their favorite variety. For low ex ante types the optimality conditions for the problem with observable second period types leave a significant amount of variability to the contract.

In contrast to the two-goods model, the sets of solutions to the relaxed and the original problem do not coincide. In the sequel, a characterization of the set of solutions to the original problem is derived. By construction, maximal profits in $\mathcal{P}_o$ pose an upper bound to profits in $\mathcal{P}$. Differences may arise due to the stronger incentive compatibility requirements in $\mathcal{P}$. Lemma 7 states that the maximal profit in $\mathcal{P}_o$ can also be attained in $\mathcal{P}$. The set of solutions to problem $\mathcal{P}$ is hence a subset of the set solutions to problem $\mathcal{P}_o$. This is reformulated as a necessary condition for solutions to $\mathcal{P}$:

Lemma 7. *Any solution to the original problem satisfies the conditions of Lemma 6.*

It suffices to show that the suggested allocation rule (LE) is implementable in the original problem with private ex post types. As in Section 2, I show that every agent

[20]Except for a set on A of probability measure zero. Note that selling to a mass of consumers with probability measure zero on $R \times A$ has no impact on profits.

has an incentive to truthfully report his second period type no matter what his first period report was. As already argued, this is sufficient to show incentive compatibility. The intuition why in the menu of contracts (LE), truthtelling in the second period is always optimal is the following: Assume some arbitrary type $r \in R$ has reported $\hat{r} \leq b$. The price he pays is v independent of the report on his ex post type. Furthermore, the contract specifies the interval $[\sqrt{1-\delta}/2; 1-\sqrt{1-\delta}/2]$ of potentially assigned goods. Out of this set, the agent obtains the good which maximizes the utility of his reported ex post type. Therefore, it follows immediately that truthtelling about the second period type is optimal. Assume some arbitrary type $r \in R$ has reported $\hat{r} > b$. The price he pays is $v + b(1-\delta)$ and the agent simply gets what he claims to prefer. It can immediately be seen that the agent will report honestly about the ex post type.

A further important characterization of the set of solutions to the original problem is achieved by the following Lemma 8:

Lemma 8. *In any solution to the original problem, prices for low types $r \leq b$ generically do not depend on ex post types: $p(r,a) = p(r,a')$ for all $r \leq b$, for almost all $a, a' \in A$.*

Even though Lemma 8 is proved by contradiction, the argument gives valuable insights into the structure of incentives. Note first that the continuous set of low ex ante types $r \in [0,b]$ includes types that care arbitrarily little about which product they get. Assume, there is some low ex ante type $r' < b$ who is confronted with two different prices depending on the report about his ex post type. There will always be agents that care sufficiently little about which product they obtain, such that they would irrespectively of their preferences go for the smaller price, if they had the choice. This would imply lying about the ex post type. Any low ex ante type has this choice, when having reported to be of type r'.

From the optimality conditions we know that any ex ante type's expected utility from - possibly untruthfully - claiming to be of any type $r \leq b$ and then truthtelling about ex post types is zero. Consider the following double deviation for very low ex ante types: First, falsely report to be of ex ante type r' and then profitably deviate from truthfully reporting about the ex post type. This strategy would yield a positive expected utility for these agents because they profitably deviate from a strategy that gives them zero utility. In the optimum, this may not occur, because by incentive compatibility, their expected utility from truthful reporting would then have to be strictly positive as well, which has been shown to be not optimal.

Lemma 8 has important consequences for the characterization of optimal contracts for low ex ante types. As Lemma (6) pins down expected prices, Lemmas (6) and (8) together completely determine optimal prices for types $r \leq b$. This also significantly reduces the set of optimal contracts. The set of solutions is hence a strict subset to the set of solutions to the problem $\mathcal{P}_o$. Using(8), from the second period incentive constraints (IC_2) follows

$$\int_0^1 |a-s|dX(r,a)[s] \leq \int_0^1 |a-s|dX(r,a')[s] \; \forall r \leq b \text{ for almost all } a, a' \in A. \tag{9}$$

Equation (9) implies that given an agent has reported a low ex ante type $r \leq b$, for each ex post type he must be assigned his favorite allocation from the set of allocations in the chosen contract.

Given the previous results, Proposition 3 can be formulated:

Proposition 3. *The set of allocation rules which solve the original problem with private ex ante and ex post types is the following:*

- *For $r > b$: $X(r,a)$ puts probability mass one on a.*
- *For $r \leq b$: $X(r,a)$ s.t. (9), (F), full assignment and* $\int_0^1 \int_0^1 |a-s|dX(r,a)[s]da = \frac{1-\delta}{4}$.

Necessary conditions for prices are:

- *For $r > b$:* $\int_0^1 p(r,a)da = v + b(1-\delta)$.
- *For $r \leq b$: $p(r,a) = v$ for almost all a.*

Ex post type independent prices always support incentive compatibility.

Outline of the proof: Lemmas 6 to 8 show that the properties given in the four bullet points are necessary for a solution. The conditions of Proposition 3 imply all conditions of Lemma 6. Hence, if the properties and ex post type independent prices are sufficient for incentive compatibility, the proof is completed.

To prove incentive compatibility, it is sufficient to show that any type r, that has reported any $\hat{r}$, reports truthfully in the second period. First, assume an arbitrary type r has reported $\hat{r} > b$. From the ex post type independence of prices, it follows that $p(\hat{r}, a) = v + b(1-\delta)$ for any ex post type report. The corresponding contract gives the agent the good which is reported to be the favorite one. It is immediate that reporting truthfully about the ex post type is optimal. Second, assume an arbitrary type r has reported $\hat{r} \leq b$. Then $p(\hat{r}, a) = v$, which is again independent of the ex post type report. The corresponding contract satisfies (9). From this property follows that reporting truthfully about the ex post type is optimal: (9) guarantees an agent of arbitrary true type r to receive the allocation he reports (via $\hat{a}$) to like most among all allocations he can get given report $\hat{r}$. Note that it is crucial that the mechanism designer can do the latter without knowing the true ex ante type r. This is the case because the ordinal ranking of goods given one ex post type a does not depend on the ex ante type.

As the set of solutions is a subset of the set of solutions to the relaxed problem, the properties from Lemma 6 carry over to Proposition 3. Also in the more general model, each solution satisfies the full assignment property. This means the revenue maximizing monopolist writes contracts such that every agent ends up with some good. Again, there is a critical ex ante type. Ex ante types above that threshold-type always obtain the variety they prefer most, which implies the classical 'no distortion at the top' result. The expected price that is paid for this contract is fixed such that the critical type gets zero expected utility from this contract. Letting the price be $v + b(1-\delta)$ for any ex post type is incentive compatible.
Agents with ex ante types $r \leq b$ do not always end up with their preferred good. However, the contract positively 'responds' to the announcement of a in the following sense: An agent's expected utility from a given allocation is smaller or equal to $v - r\delta$, which is reached by assigning $s = 1/2$ with certainty. The optimal contract, however, gives the agents a higher expected utility before payments.
The contract associated with $r' \leq b$ specifies a set of allocations $\{X(r', a) : a \in A\}$. For each realization of ex post types a', the corresponding allocation $X(r', a')$ is the allocation type a' favors out of this set. When choosing the contract by the ex ante type report, which is in particular before knowing the ex post type, this contract induces an expected distance between the most liked good and the assigned good given truthful reporting about a. This expected distance is used as a measure for the quality of a contract and hence plays a role equivalent to that of 'responsiveness' in the two goods model. The first best contract always gives an agent his favorite variety and hence its expected distance between the favorite variety and the assigned variety is zero. The expected distance can be used as a measure of distortion of a contract away from first best. The least

distorted contract that always assigns the same allocation has an expected distance of $1/4$, which is achieved by always giving the agent $s = 1/2$. In the optimal contract, the set of allocations $\{X(r', a) : a \in A\}$ is designed such that the expected distance is equal to $(1-\delta)/4$. On the one hand, this means that types $r \leq b$ do not always end up with the most preferred good. On the other hand, the contract is better than assigning one allocation for all ex post types. Hence, it is again partially distorted.
If δ is close to one, which means the top valuation is increasing in ex ante types only slowly, the set of allocations for low ex ante types is chosen such that the expected distance to the favored good is small. This is intuitive, because in that case there is not much rent to be extracted from high types and therefore contracts do not have to be distorted strongly. The lower δ, the stronger is the top valuation increasing in ex ante types and hence the more rent can potentially be extracted from high types. Therefore, the expected distance to the favored good is larger. If $\delta = 0$, the expected utility of a fixed assignment of good $s = 1/2$ is constant among ex ante types. In this case, the contract (LE) would restrict the low types' choice completely by assigning them the same good independently of the realized ex post type.

For low ex ante types, a necessary condition for optimality is that the price equals v independently of ex post realizations. This means that all small ex ante types get an expected utility of zero, because the expected utility from the allocation is equal to v as well. Furthermore, from the properties of optimal contracts it follows that any type is indifferent between reporting any two low ex ante types $r, r' < b$. The expected utility from the allocation rule and the payment are the same for both reports. As a consequence, an optimal mechanism can be simplified by offering just one contract for all ex ante types below the threshold type.
For low ex ante types, in optimal contracts that are deterministic, the set of feasible allocations takes the form of a subset of goods. This set of varieties is chosen such that the distance between the ex post type and the closest element of the set is in expectation equal to $(1-\delta)/4$. Optimal mechanisms that are deterministic do exist; an example is contract (LE).

3.3 Implementation

The direct mechanisms described above are implementable by menus of Limited Exchange Contracts. A Limited Exchange Contract in the first period specifies an allocation the consumer obtains, a price that has to be paid, and a set of allocations to which the agent may exchange in the second period for free. There is no possibility at all to change to allocations which are not in this set. The following proposition states how any solution to $\mathcal{P}$ can be implemented as a menu of Limited Exchange Contracts:

Proposition 4. *Every optimal allocation rule $\{X(r,a) : r \in R, a \in A\}$ can be implemented by a menu of Limited Exchange Contracts:*

- *For $r > b$: Offer any good for the price $p_F = v + b(1-\delta)$ with an option for free exchange in the second period.*
- *For $r \leq b$: Choose the set of allocations $\{X(r,a) : a \in A\}$. Sell one allocation from this set for $p_{UF} = v$ and give the option to exchange to any other allocation from this set for free. Offer such a contract for each $r \leq b$.*

Contracts with exchange fees are never optimal.

Proof:
It is immediate that the suggested menu of Limited Exchange Contracts implements the optimal allocation rule. Contracts with exchange fees are not optimal, because this implies ex post type-dependent prices: If an agent decided to buy a certain distribution over goods for some price p in the first period, it depends on his ex post type whether he prefers to stay with the allocation or to pay an additional exchange fee p_e and get a preferred distribution over goods. This means for some ex post types the price is p and for some ex post types it is $p + p_e$. ■

The first best contract for high ex ante types can be seen as a special case of a Limited Exchange Contract, where the limitation on the exchange set is not binding in the sense that the agent never favors an allocation which does not belong to the set. As already argued for direct mechanisms, any agent gets the same utility from all optimal Limited Exchange Contracts with a restrictive exchange set. Therefore offering just one of these restrictive Limited Exchange Contracts is optimal as well. For the case of deterministic optimal mechanisms, an implementing menu of Limited Exchange Contracts would be the following example:

Example 1: Implementation of (LE)
Offer the following two Limited Exchange Contracts:
(i) Sell at price $p_F = v + b(1-\delta)$ any $s \in [0,1]$ with the option for free exchange to any good in the second period.
(ii) Sell at price $p_{UF} = v$ the 'average'-good $s = 1/2$ with the option for free exchange within $[\sqrt{1-\delta}/2, 1-\sqrt{1-\delta}/2]$ in the second period.

This example demonstrates that for δ close to one, contract (ii) allows for almost free exchange, which means the contract is only slightly distorted. The lower δ, the smaller is the set to which free exchange is possible and for $\delta = 0$ contract (ii) does not give any opportunity for exchange.

In the context of ticket pricing for transportation services, this means that, on the one hand, there are offered tickets with free exchange to any other departure time. On the other hand, for any departure time there are sold flight tickets that include the option to change departure time for free within a certain time span around the initially purchased departure time. Many airlines have explicitly designed such options by introducing costless same-day exchange possibilities and stand-by options. A same-day exchange option usually is an extra amendment to the terms and conditions of a flight ticket, which allows customers to change flight within the same day for free or at a symbolic price. Stand-by options are closely related amendments, which - upon availability - enable passengers to take an earlier flight if they arrive early at the airport or to take a later one if they miss their flight. An implicit equivalent to these contracts emerges when airlines create a reputation for being obliging concerning their refund and exchange policy. Importantly, these kinds of additional options mostly apply independently of the contract purchased initially. The use of Limited Exchange Contracts is also common among ferry companies; examples are P&O Ferries and DFDS Seaways. Both companies offer tickets which explicitly specify a time interval around the purchased departure time within which costless change is possible. Tickets that provide full flexibility can be obtained at higher prices. Note that the first best contract can also be implemented by offering expensive tickets for each variety at the point in time of consumption.

A Limited Exchange Contract leads to the consumption of non-favorite varieties by restricting the set of goods that can be chosen from in the second period. An alternative way to induce agents to not always consume the most preferred good is to charge exchange fees. In contrast to Limited Exchange Contracts, the agent in principle has the possibility to change to any good in such contracts. However, he does not want to do so if he is already close to the favorite good and hence the potential gain in valuation is outweighed by the exchange fee. Proposition 4 states that the use of exchange fees is not optimal for a revenue maximizing monopolist. As the model presented in this paper isolates the aspect of price discrimination, this result means that for the purpose of price discrimination, the use of varying exchange fees in contracts is not optimal.

The result on the optimality of Limited Exchange Contracts, in particular that exchange fees are not used as a price discrimination device, has an important implication for the literature on sequential screening. Since the canonical contribution by Courty and Li (2000), the literature on sequential screening has concentrated on firms that sell homogeneous goods to customers that learn their valuations for the good gradually over time. A natural consequence is that the primary focus has been on the use of refund policies as a price discrimination device. This guides the reader towards the thought that product exchange does not have to be treated independently, as giving one good back and buying a new one is essentially equivalent to exchange. This paper formally studies a setting with ex post information about the valuation of heterogeneous goods. The optimal contracts in this broader setting can *not* be interpreted as refund contracts, where the consumer can give back one product for a partial refund and purchase another variety. To see this, note first that when goods can be given back against a full refund, this is indeed equivalent to free exchange to any product. However, changing goods in restrictive Limited Exchange Contracts cannot be modeled by giving back a good for a partial refund and buying a new one. The latter procedure entails a cost, which is the money for the returned good which is not being refunded. This is equivalent to an exchange fee whose use is shown to be not optimal.

4 Conclusion

In this paper, I have characterized revenue maximizing contracts for situations in which agents learn their valuations for horizontally differentiated goods gradually over time. In the beginning, agents differ in terms of their preference intensity and their highest valuation. Let higher ex ante types have higher valuations for their favorite good and larger cost from consuming non-favorite goods. The agent's initial uncertainty is about which product he favors. The mechanism design approach without ad-hoc restrictions on contracts shows that agents with high ex ante types always receive their most preferred good. In the two-product case, the optimum involves stochastic contracts for agents with low ex ante types if the expected utility of an ex post type-independent allocation

is decreasing in ex ante types. If it is increasing, optimal contracts are maximally distorted for an intermediate range of types and the lowest types are excluded. A more comprising model with a continuum of products and ex post types shows the optimality of Limited Exchange Contracts as flexibility restriction device for lower ex ante types. A deterministic Limited Exchange Contract consists of an initial product offered in the first period at some price and the option to exchange it to some product out of a fixed subset of goods later on for free. The use of exchange fees as a price discrimination device is shown to be generally not optimal.
There are several versions of and extensions to the model which are worth being examined. This paper studies the benchmark case in which the ex ante type completely determines the shape of the valuation function - including the level of top valuation. Relaxing this assumption could have an impact on the model's predictions. A model in which the ex ante type leaves a sufficiently high degree of uncertainty about the top valuation may imply that optimal policies involve both exchanges and refunds. Furthermore, the question of how capacity constraints influence optional exchange policies in the presence of aggregate uncertainty deems interesting as well: The revenue maximizer then faces an additional trade-off between giving agents the optimal amount of flexibility and directing them towards available capacity.

5 Appendix

Proof of Lemma 1:

Consider any mechanism $\big(X(\hat r,\hat a),p(\hat r,\hat a)\big)$.
Construct an alternative mechanism $\big(\tilde X(\hat r,\hat a),\tilde p(\hat r,\hat a)\big)$ such that

- $\tilde p(\hat r,\hat a)=p(\hat r,\hat a) \quad \forall \hat r,\hat a,$
- $\tilde x_{1\&2}(\hat r,\hat a)=0 \quad \forall \hat r,\hat a,$
- $\tilde x_i(\hat r,a_i)=x_i(\hat r,a_i)+x_{1\&2}(\hat r,a_i) \quad \forall \hat r,\forall i\in\{1,2\},$
- $\tilde x_{3-i}(\hat r,a_i)=x_{3-i}(\hat r,a_i) \quad \forall \hat r,\forall i\in\{1,2\}.$

From (1) can immediately be seen that for any given $r,\hat r,$ and a, $u(r,\hat r,a,a)$ is equal for both mechanisms, whereas for $a\neq\hat a$ ex post utility $u(r,\hat r,a,\hat a)$ is weakly higher under mechanism $\big(X(\hat r,\hat a),p(\hat r,\hat a)\big)$. From this follows that the modified mechanism satisfies (IC_1) and (IC_2) if mechanism $\big(X(\hat r,\hat a),p(\hat r,\hat a)\big)$ does. ■

Proof of Lemma 2:

Remember $K(\hat{r},\delta) = x_+(\hat{r}) - x_-(\hat{r}) - \delta(x_+(\hat{r}) + x_-(\hat{r}))$.
Take r, r' with $r > r'$. By (IC_1'), it holds that

$$\begin{aligned} U(r, id_a) &\geq U(r', r, id_a) \\ &= v[x_+(r') + x_-(r')] + r' \cdot K(r',\delta) - \mathbb{E}_a[p(r',a)] \qquad + (r - r') \cdot K(r',\delta) \\ &= U(r', id_a) \qquad + (r - r') \cdot K(r',\delta). \end{aligned}$$

Analogeously,

$$U(r', id_a) \geq U(r, id_a) + (r' - r) \cdot K(r,\delta).$$

Combining the two inequalitities yields

$$(r - r') \cdot K(r,\delta) \geq U(r, id_a) - U(r', id_a) \geq (r - r') \cdot K(r',\delta).$$

Dividing by $(r - r')$ yields:

$$K(r,\delta) \text{ is monotonically increasing in } r \qquad (MON).$$

Letting r' converge to r yields:

$$\partial U(r, id_a)/\partial r = K(r,\delta) \text{ almost everywhere} \qquad (ENV).$$

To proof the invers direction, note that from (ENV) and absolute continuity follows for any r, r' with $r > r'$

$$U(r, id_a) = U(r', r, id_a) + \int_{r'}^{r} K(y,\delta) dy.$$

For a proof of absolute continuity see for example Theorem 2 in Milgrom and Segal (2002).

From the monotonicity condition (MON) follows

$$U(r', r, id_a) + \int_{r'}^{r} K(y, \delta) dy \geq U(r', r, id_a) + \int_{r'}^{r} K(r', \delta) dy$$
$$= U(r', id_a) + (r - r') \cdot K(r', \delta)$$
$$= U(r', r, id_a).$$

■

Proof of Lemma 3:

Maximization problem (5) is solved by pointwise maximization for every ex ante type r. As z by definition minimizes expected utility, from (ENV) and (MON) follows

$$\begin{aligned} & K(r, \delta) \leq 0 \quad \text{for } r < z \\ \text{and} \quad & K(r, \delta) \geq 0 \quad \text{for } r > z. \end{aligned} \tag{10}$$

Furthermore from the feasibility constraints (F) follows

$$\begin{aligned} & x_+(r) + x_-(r) \leq 1 \\ \text{and} \quad & K(r, \delta) \leq x_+(r) + x_-(r) - \delta(x_+(r) + x_-(r)) \leq 1 - \delta. \end{aligned} \tag{11}$$

<u>Case 1:</u> $r \geq z$

The optimal allocation rule maximizes

$$v[x_+(r) + x_-(r)] + K(r, \delta) \cdot \left(r - \frac{1 - F(r)}{f(r)} \right).$$

Case 1.1: $r > b$
The virtual value is positive. If there exists a contract such that $x_+(r) + x_-(r) = 1$ and $K(r, \delta) = 1 - \delta$, by (11) it is optimal at point r.
Case 1.2: $r \leq b$
The virtual value is weakly negative[21]. If there exists a contract such that $x_+(r) +$

[21] w.l.o.g. let b and types with $r - \frac{1-F(r)}{f(r)} = 0$ get the contract that is optimal for types with $r - \frac{1-F(r)}{f(r)} < 0$

$x_-(r) = 1$ and $K(r, \delta) = 0$, by (10) and (11) it is optimal at point r.

Case 2: $r < z$

The optimal allocation rule maximizes

$$v[x_+(r) + x_-(r)] + K(r, \delta) \cdot \left(r + \frac{F(r)}{f(r)} \right).$$

The virtual value $\left(r + \frac{F(r)}{f(r)} \right)$ is positive for all $r \in R$. If there exists a contract such that $x_+(r) + x_-(r) = 1$ and $K(r, \delta) = 0$, by (10) and (11) it is optimal at point r.

The two pairs of optimality conditions can be solved for $x_+(r)$ and $x_-(r)$:

$$\begin{aligned} x_+(r) + x_-(r) = 1, K(r, \delta) = 0 \qquad &\Leftrightarrow x_+(r) = \frac{1+\delta}{2}, \qquad x_-(r) = \frac{1-\delta}{2} \\ x_+(r) + x_-(r) = 1, K(r, \delta) = 1 - \delta \qquad &\Leftrightarrow x_+(r) = 1, \qquad x_-(r) = 0 \end{aligned}$$

The properties suggested above are stated in the Lemma. It is left to be proven that there exists an allocation rule with the determined properties that satisfies (F) and any allocation rule satisfying the properties satisfies (MON) and $U(z, id_a) \geq U(r, id_a) \quad \forall r \in R$. It can immediately be seen that (MON) and $U(z, id_a) \geq U(r, id_a) \quad \forall r \in R$ are satisfied for any allocation rule, as they directly follow from the properties. Existence of a feasible allocation rule with the desired properties is shown by construction of an example:

$$\begin{aligned} &\text{For } r > max\{b, z\}: \quad x_i(r, a_i) = 1, \quad x_i(r, a_{3-i}) = 0 \qquad && \forall i \in \{1, 2\} \\ &\text{For } r \leq max\{b, z\}: \quad x_i(r, a_i) = \frac{1+\delta}{2}, \quad x_i(r, a_{3-i}) = \frac{1-\delta}{2} \qquad && \forall i \in \{1, 2\} \end{aligned}$$

■

Proof of Lemma 4:

Insert the optimality conditions from Lemma 3 into objective (5):

$$\int_0^z f(r)\left[v[x_+(r)+x_-(r)]+K(r,\delta)\cdot\left(r+\frac{F(r)}{f(r)}\right)\right]dr$$

$$+\int_z^{\overline{r}} f(r)\left[v[x_+(r)+x_-(r)]+K(r,\delta)\cdot\left(r-\frac{1-F(r)}{f(r)}\right)\right]dr$$

$$=\int_0^z f(r)\left[v+0\cdot\left(r+\frac{F(r)}{f(r)}\right)\right]dr+\int_z^{\overline{r}} f(r)vdr$$

$$+\int_{max\{b,z\}}^{\overline{r}} f(r)(1-\delta)\cdot\left(r-\frac{1-F(r)}{f(r)}\right)dr$$

$$=v+\int_{max\{b,z\}}^{\overline{r}} f(r)(1-\delta)\cdot\left(r-\frac{1-F(r)}{f(r)}\right)dr$$

By definition, $r-\frac{1-F(r)}{f(r)}>0 \quad \forall r>b$. Hence, z is optimal if and only if $z\leq b$. ■

Proof of Lemma 5:

From Lemma 4 follows $max\{b,z\}=b$. The conditions from Lemma 3 are then

$$x_+(r)=\frac{1+\delta}{2} \qquad \text{and } x_-(r)=\frac{1-\delta}{2} \qquad \text{if } r\leq b,$$

$$x_+(r)=1 \qquad \text{and } x_-(r)=0 \qquad \text{if } r>b.$$

By (4), these characteristics determine the sum of prices $p(r,a_1)+p(r,a_2)$ for each ex ante type r.

Finally, it is left to state the nonempty set of feasible allocation rules that satisfy the optimality conditions. For all types $r>b$ from the optimality condition $x_+(r)=1$ and $x_-(r)=0$ together with feasibility (F), it follows $x_1(r,a_1)=x_2(r,a_2)=1$ and $x_2(r,a_1)=x_1(r,a_2)=0$. For $r\leq b$ feasibility and optimality can be described by the following system of equations:

$$x_1(r,a_1)+x_2(r,a_2)=1+\delta,$$
$$x_2(r,a_1)+x_1(r,a_2)=1-\delta,$$
$$x_1(r,a)+x_2(r,a)\le 1 \quad \forall a,$$
$$x_i(r,a)\in[0,1] \quad \forall a,i.$$

There is one degree of freedom and the non-empty set of solutions to this system is the following:

$$x_1(r,a_1)=\alpha\in[\delta,1],$$
$$x_2(r,a_1)=1-\alpha,$$
$$x_1(r,a_2)=\alpha-\delta,$$
$$x_2(r,a_2)=1+\delta-\alpha.$$

■

Proof of Proposition 1:

It is left to prove that for $p(r,a_1)=p(r,a_2) \quad \forall r$ any allocation satisfies (IC_1) and (IC_2). Proposition 1 follows then from Lemma 5.
Define the following strengthening of condition (IC_2):

$$u(r,\hat{r},a,a)\ge u(r,\hat{r},a,\hat{a}) \quad \forall r,\hat{r},a,\hat{a}. \qquad (IC_2^s)$$

(IC_2^s) states that in the second period truthtelling is optimal for any first period report.

Claim 1: *From (IC_2^s) and (IC_1') follows (IC_2) and (IC_1)*

(IC_2) trivially follows from (IC_2^s). (IC_1) is a consequence as well: Consider some agent and an arbitrary reporting strategy. By (IC_2^s), the agent can always weakly improve by reporting truthfully about his second period type. Given truthful reporting about the second period type, by (IC_1') the agent can then weakly improve by reporting truthfully about the first period type.

Claim 2: *Any element from the set of allocation rules from Proposition 1 satisfies (IC_2^s) if $p(r,a_1)=p(r,a_2) \quad \forall r$.*

The claim is shown by plugging an arbitrary element of the set and corresponding prices into (IC_2^s) using (1):

Case 1: $\hat{r} \leq b$

(IC_2^s) is satisfied, as

$$\alpha(v-\delta r+r)+(1-\alpha)(v-\delta r-r)-v \geq (\alpha-\delta)(v-\delta r+r)+(1-\alpha+\delta)(v-\delta r-r)-v \quad \forall r, \alpha$$

and

$$(1-\alpha+\delta)(v-\delta r+r)+(\alpha-\delta)(v-\delta r-r)-v \geq (1-\alpha)(v-\delta r+r)+\alpha(v-\delta r-r)-v \quad \forall r, \alpha$$

hold if and only if $\delta \geq 0$.

Case 2: $\hat{r} > b$

(IC_2^s) is satisfied, as

$$1\cdot(v-\delta r+r)+0\cdot(v-\delta r-r)-v-b(1-\delta) \geq 0\cdot(v-\delta r+r)+1\cdot(v-\delta r-r)-v-b(1-\delta) \quad \forall r, \alpha.$$

■

Proof of Proposition 2:

To proof the proposition, I first solve a relaxed problem, which gives an upper bound on profits, and then show that any solution to the relaxed problem is implementable in $\mathcal{P}$.

Define $\mathcal{P}_*$ as $\mathcal{P}_o$ with the additional constraint

$$x_1(r, a_1) + x_2(r, a_2) \geq x_1(r, a_2) + x_2(r, a_1) \quad \forall r. \tag{*}$$

<u>Claim 1:</u> *$\mathcal{P}_*$ is a relaxed problem of $\mathcal{P}$*

It is sufficient to show that (*) follows from IC_2. IC_2 states that $\forall r$ hold

$$u(r, r, a_1, a_1) \geq u(r, \hat{r}, a_1, a_2) \quad \text{and}$$
$$u(r, r, a_2, a_2) \geq u(r, \hat{r}, a_2, a_1).$$

This is equivalent to

$$\begin{aligned} & v^+(r) \cdot (x_1(r,a_2) - x_1(r,a_1)) + v^-(r) \cdot (x_2(r,a_2) - x_2(r,a_1)) \\ \leq\ & p(r,a_2) - p(r,a_1) \\ \leq\ & v^+(r) \cdot (x_2(r,a_2) - x_2(r,a_1)) + v^-(r) \cdot (x_1(r,a_2) - x_1(r,a_1)) \quad \forall r. \end{aligned}$$

An immediate consequence is

$$\begin{aligned} & v^+(r) \cdot [(x_2(r,a_2) - x_2(r,a_1)) - (x_1(r,a_2) - x_1(r,a_1))] \\ \geq\ & v^-(r) \cdot [(x_2(r,a_2) - x_2(r,a_1)) - (x_1(r,a_2) - x_1(r,a_1))] \quad \forall r, \end{aligned}$$

which is equivalent to

$$x_1(r,a_1) + x_2(r,a_2) \geq x_1(r,a_2) + x_2(r,a_1) \quad \forall r. \tag{*}$$

Define $e = sup\{r \in R | r - \frac{1-F(r)}{f(r)} \leq \frac{v}{\delta}\})$ whenever the supremum exists and $e = 0$ otherwise.

<u>Claim 2:</u> *The solution to $\mathcal{P}_*$ is the following:*

$$\begin{aligned} \textit{For } r > b: \quad & x_1(r,a_1) = x_2(r,a_2) = 1, \quad x_1(r,a_2) = x_2(r,a_1) = 0 \\ \textit{and} \quad & \mathbb{E}_a[p(r',a)] = v + b(1-\delta). \\ \textit{For } r \in [e,b]: \quad & x_1(r,a_1) = x_1(r,a_2) = \alpha, \quad x_2(r,a_2) = x_2(r,a_1) = 1-\alpha \quad \alpha \in [0,1] \\ \textit{and} \quad & \mathbb{E}_a[p(r',a)] = v - \delta e. \\ \textit{For } r < e: \quad & x_i(r,a_j) = 0 \quad \forall i,j \in 1,2 \\ \textit{and} \quad & \mathbb{E}_a[p(r',a)] = 0. \end{aligned}$$

As the first period incentive constraints are identical in $\mathcal{P}_o$ and $\mathcal{P}_*$, Lemma 2 applies.

Lemma 2: The first period incentive constraints IC_1' are satisfied if and only if

$$\partial U(r, id_a)/\partial r = K(r,\delta) \ \textit{a.e.} \tag{ENV}$$

$$\textit{and} \quad K(r,\delta) \textit{ is mon. increasing in } r. \tag{MON}$$

(*) is equivalent to $x_+(r) \geq x_-(r)$ and from (*) and $\delta < 0$ follows $K(r,\delta) \geq 0$. Hence in the optimum expected utility is increasing in ex ante types everywhere and the lowest ex ante type's participation constraint is binding. Following the standard approach, the problem can be restated as

$$\max_x \int_0^{\bar{r}} f(r)\left[v[x_+(r)+x_-(r)] + K(r,\delta)\left(1-\frac{1-F(r)}{f(r)}\right)\right]dr$$

s.t. MON, (F), (*).

A solution to this problem is found by pointwise maximization of the relaxed version without the monotonicity constraint.

Case 1: $r > b$

Virtual value is positive. Pointwise maximization gives $x_+(r) = 1$ and $x_-(r) = 0$. For a formal derivation see Lemma 3. The contract trivially satisfies (*).

Case 2: $r \leq b$

Virtual value is negative. Maximization is done in two steps:
First, for any fixed $x_+(r) + x_-(r) = m$, under the restriction (*) virtual surplus is maximized for $x_+(r) = x_-(r) = m/2$.
Second, m is chosen to satisfy (F) and maximize

$$v \cdot m - \delta \cdot m\left(1 - \frac{1-F(r)}{f(r)}\right).$$

The solution to this linear problem is

$$m = \begin{cases} 1, & \text{if } r - [1-F(r)]/f(r) \geq v/\delta \\ 0, & \text{if } r - [1-F(r)]/f(r) < v/\delta \end{cases}$$

Cases 1 and 2 give the allocation rules of Claim 2, which satisfy the monotonicity constraint. Expected prices are fixed by equation (4).

Claim 3: *Any solution to $\mathcal{P}_*$ is implementable in $\mathcal{P}$ with ex post type independent prices.*

It is left to prove that for $p(r, a_1) = p(r, a_2) \quad \forall r$ any allocation rule satisfies (IC_1) and (IC_2).
According to the proof of Proposition 1 it suffices to show that for $p(r, a_1) = p(r, a_2)$ any solution to $\mathcal{P}_*$ satisfies IC_2^s.

As shown in the proof of Proposition 1, the contract for types $r > b$ satisfies IC_2^s. The contract for types $r \leq b$ trivially satisfies IC_2^s, as the report about the ex post type has no influence on the allocation.

From claims one, two and three follows the proposition. ■

Lemma 9. $\partial/\partial r\left[\mathbb{E}_a(v_{r,a}(k))\right] \leq 0 \quad \forall k \in S$ *if and only if* $\delta \geq 0$.

Proof:
From straightforward algebraic reforumlations follows

$$\begin{aligned}
\frac{\partial}{\partial r}\mathbb{E}_a(v_{r,a}(k)) &= \frac{\partial}{\partial r}\left[\int_0^1 v + (1-\delta)r - 4r|a-k|da\right] \\
&= \frac{\partial}{\partial r}\left[v + r - \delta r - 4r\left(\frac{k^2}{2} + \frac{(1-k)^2}{2}\right)\right] \\
&= 1 - \delta - 4\left(\frac{k^2}{2} + \frac{(1-k)^2}{2}\right) \\
&\leq 1 - \delta - 4\left(\frac{(1/2)^2}{2} + \frac{(1-(1/2))^2}{2}\right) \\
&= -\delta.
\end{aligned}$$

An immediate consequence is $\partial/\partial r\left[\mathbb{E}_a(v_{r,a}(k))\right] \leq 0 \quad \forall k \in S$ if $\delta \geq 0$.
Necessity follows as $\partial/\partial r\left[\mathbb{E}_a(v_{r,a}(1/2))\right] > 0$ if $\delta < 0$. ■

Proof of Lemma 6:

Remember, $\tilde{K}(\hat{r}, \delta) = \int_0^1 \int_0^1 1 - \delta - 4|a-s|dX(\hat{r}, a)[s]da.$
An immediate consequence from the similar structure of the expected utilities (8) and (2) is the following characterization of incentive compatibility:

The first period incentive constraints (IC_1') are satisfied if and only if

$$\partial U(r, id_a)/\partial r = \tilde{K}(r, \delta) \;\; a.e. \qquad (ENV)$$

$$and \quad \tilde{K}(r, \delta) \;\; is\ mon.\ increasing\ in\ r. \qquad (MON)$$

Therefore maximizing with respect to the constraints (IC_1'), (IR) and (F) is equivalent to taking (ENV), (MON), (IR) and (F) as constraints. Depending on r, the term

$\tilde{K}(r,\delta)$ can take both negative and positive values.
The solution concept presented in the last part is applied here as well: It is known that in every solution, there exists an ex ante type $z \in [0, \overline{r}]$ such that $U(z, id_a) \leq U(r, id_a) \forall r \in R$. In the first step I arbitrarily fix z and solve problem $\mathcal{P}_o^z$, which is problem $\mathcal{P}_o$ with the additional constraint $U(r, id_a) \geq U(z, id_a) \forall r \in R$. In the second step profit is maximized in z.

<u>Preliminary Step:</u> Reformulation of $\mathcal{P}_o^z$

By (ENV) and individual rationality, in the optimum, $U(z, id_a) = 0$ and any ex ante type's expected utility can then be written as

$$U(r, id_a) = U(z, id_a) + \int_z^r \tilde{K}(y,\delta)dy = \int_z^r \tilde{K}(y,\delta)dy. \tag{12}$$

By (8) and (12) prices can be written as a function of the allocation:

$$\int_0^1 p(r,a)da = v\left(\int_0^1 X(r,a)[1]da\right) + r \cdot \tilde{K}(r,\delta) - \int_z^r \tilde{K}(y,\delta)dy. \tag{13}$$

Plugging (13) into the objective reduces problem $\mathcal{P}_o^z$ to

$$\max_x \int_0^{\overline{r}} f(r)\Big(v\int_0^1 X(r,a)[1]da + r \cdot \tilde{K}(r,\delta) - \int_z^r \tilde{K}(y,\delta)dy\Big)dr$$

s.t. (MON), (F) and $U(z, id_a) \geq U(r, id_a) \quad \forall r \in R$.

By partially integrating and reformulating the problem can be rewritten as

$$\begin{aligned} \max_x \int_0^z f(r)\left[v\int_0^1 X(r,a)[1]da + \tilde{K}(r,\delta) \cdot \left(r + \frac{F(r)}{f(r)}\right)\right]dr \\ + \int_z^{\overline{r}} f(r)\left[v\int_0^1 X(r,a)[1]da + \tilde{K}(r,\delta) \cdot \left(r - \frac{1-F(r)}{f(r)}\right)\right]dr \end{aligned} \tag{14}$$

s.t. (MON), (F) and $U(z, id_a) \geq U(r, id_a) \quad \forall r \in R$.

Claim 1: *A feasible allocation rule which satisfies condition (RO) is a solution to* $\mathcal{P}_o^z$. *If there exists a feasible allocation rule that satisfies (RO) then this condition is also necessary for an allocation rule to be an optimum to* $\mathcal{P}_o^z$.

$$\begin{aligned} \int_0^1 X(r,a)[1]da =& 1 \quad \text{and } \tilde{K}(r,\delta) = 0 \quad \text{if } r \leq max\{b,z\} \\ \int_0^1 X(r,a)[1]da =& 1 \quad \text{and } \tilde{K}(r,\delta) = 1-\delta \quad \text{if } r > max\{b,z\} \end{aligned} \tag{RO}$$

Maximization problem (14) is solved by pointwise maximization for every ex ante type r. Recall $b = sup\{r \in R | (r - \frac{1-F(r)}{f(r)} \leq 0\}$.
As z by definition minimizes expected utility, from (ENV) and (MON) follows

$$\begin{aligned} & \tilde{K}(r,\delta) \leq 0 \quad \text{for } r < z \\ \text{and} \quad & \tilde{K}(r,\delta) \geq 0 \quad \text{for } r > z. \end{aligned} \tag{15}$$

Furthermore from the feasibility constraints (F) follows

$$\int_0^1 X(r,a)[1]da \leq 1 \quad \text{and}$$
$$\tilde{K}(r,\delta) = \int_0^1 \int_0^1 (1-\delta-4|a-s|)dX(r,a)[s]da \leq \int_0^1 \int_0^1 (1-\delta)dX(r,a)[s]da \leq 1-\delta. \tag{16}$$

Case 1: $r > z$

The optimal allocation rule maximizes

$$v \int_0^1 X(r,a)[1]da + \tilde{K}(r,\delta) \cdot \left(r - \frac{1-F(r)}{f(r)}\right).$$

Case 1.1: $r > b$
The virtual value is positive. If there exists a contract such that $\int_0^1 X(r,a)[1]da = 1$ and $\tilde{K}(r,\delta) = 1-\delta$, by (16) it is optimal at point r.

Case 1.2: $r \leq b$
The virtual value is weakly negative.[22] If there exists a contract such that $\int_0^1 X(r,a)[1]da =$

[22]w.l.o.g. let b and types with $r - \frac{1-F(r)}{f(r)} = 0$ get the allocation of low types

1 and $\tilde{K}(r,\delta)=0$, by (15) and (16) it is optimal at point r.

Case 2: $r < z$

The optimal allocation rule maximizes

$$v\int_0^1 X(r,a)[1]da + \tilde{K}(r,\delta)\cdot\left(r+\frac{F(r)}{f(r)}\right).$$

The virtual value $\left(r+\frac{F(r)}{f(r)}\right)$is positive for all $r \in R$. If there exists a contract such that $\int_0^1 X(r,a)[1]da = 1$ and $\tilde{K}(r,\delta)=0$, by (15) and (16) it is optimal at point r.

Provided the existence of a feasible allocation rule with the determined characteristics, any solution to the relaxed problem has the properties (RO). Monotonicity is satisfied.

Claim 2: *Given feasible allocation rules that (RO) exist, z is optimal if and only if $z \leq b$.*

Insert the optimality conditions (RO) into objective (14):

$$\begin{aligned}
&\int_0^z f(r)\left[v\int_0^1 X(r,a)[1]da + \tilde{K}(r,\delta)\cdot\left(r+\frac{F(r)}{f(r)}\right)\right]dr\\
+&\int_z^{\bar{r}} f(r)\left[v\int_0^1 X(r,a)[1]da + \tilde{K}(r,\delta)\cdot\left(r-\frac{1-F(r)}{f(r)}\right)\right]dr\\
=&\int_0^z f(r)\left[v+0\cdot\left(r+\frac{F(r)}{f(r)}\right)\right]dr + \int_z^{\bar{r}} f(r)v dr + \int_{max\{b,z\}}^{\bar{r}} f(r)(1-\delta)\cdot\left(r-\frac{1-F(r)}{f(r)}\right)dr\\
=&v+\int_{max\{b,z\}}^{\bar{r}} f(r)(1-\delta)\cdot\left(r-\frac{1-F(r)}{f(r)}\right)dr
\end{aligned}$$

By definition, $r-\frac{1-F(r)}{f(r)} > 0 \quad \forall r > b$. Hence, z is optimal if and only if $z \leq b$.

Finally, I present a deterministic allocation rule (LE) which satisfies (RO).

Denote the single mass point of mass one of $X(r,a)$ in (LE) by $x^{LE}(r,a)$:

$$\begin{aligned}
&\text{For } r > b\\
&\qquad x^{LE}(r,a) = a && \text{and} \quad p(r,a) = v + b(1-\delta) \quad \forall a;\\
&\text{For } r \leq b\\
&\qquad x^{LE}(r,a) = \sqrt{1-\delta}/2 && \text{for} \quad a < \sqrt{1-\delta}/2,\\
&\qquad x^{LE}(r,a) = a && \text{for} \quad a \in [\sqrt{1-\delta}/2; 1-\sqrt{1-\delta}/2],\\
&\qquad x^{LE}(r,a) = 1-\sqrt{1-\delta}/2 && \text{for} \quad a > 1-\sqrt{1-\delta}/2,\\
&\qquad p(r,a) = v \quad \forall a.
\end{aligned} \tag{LE}$$

Feasibility is straight forward to see and (RO) can be easily checked as well. From existence of a feasible allocation rule together with Claims 1 and 2 follows Lemma 6.

■

Proof of Lemma 7:

It suffices to show that the suggested allocation rule (LE) is implementable in the original problem with private ex post types. Lemma 7 then follows from Lemma 5.
In line with Section 2 I show that (LE) satisfies the following strengthened version of IC_2:

$$\int_0^1 v_{r,a}(s)dX(\hat{r},a)[s] - p(\hat{r},a) \geq \int_0^1 v_{r,a}(s)dX(\hat{r},\hat{a})[s] - p(\hat{r},\hat{a}) \quad \forall r,\hat{r},a,\hat{a}. \tag{IC_2^s}$$

(IC_2^s) states that every agent has an incentive to truthfully report his second period type independent of his first period report. As argued in the proof of Proposition 1, this is sufficient to show incentive compatibility.

As (LE) satisfies $p(\hat{r},\hat{a}) = p(\hat{r},\hat{a}') \ \forall r \leq b \ \forall \hat{a},\hat{a}' \in A$, (LE) satisfies (IC_2^s) if and only if:

$$\int_0^1 v_{r,a}(s)dX(\hat{r},a)[s] \geq \int_0^1 v_{r,a}(s)dX(\hat{r},\hat{a})[s] \quad \forall r,\hat{r},a,\hat{a}.$$

As (LE) is deterministic, denote the allocation by $x^{LE}_{\hat{r},\hat{a}} \in S$. Then (LE) satisfies (IC^s_2) if and only if:

$$\begin{aligned} v_{r,a}(x^{LE}_{\hat{r},a}) &\geq v_{r,a}(x^{LE}_{\hat{r},\hat{a}}) \quad \forall r, \hat{r}, a, \hat{a} \\ \Leftrightarrow v + (1-\delta)r - 4r|a - x^{LE}_{\hat{r},a}| &\geq v + (1-\delta)r - 4r|a - x^{LE}_{\hat{r},\hat{a}}| \quad \forall r, \hat{r}, a, \hat{a}. \end{aligned} \tag{17}$$

A sufficient condition for (17) is:

$$\Leftrightarrow |a - x^{LE}_{\hat{r},a}| \leq |a - x^{LE}_{\hat{r},\hat{a}}| \quad \forall r, \hat{r}, a, \hat{a}. \tag{18}$$

Case 1: $\hat{r} \leq b$

Case 1.1: $a \in [\sqrt{1-\delta}/2; 1 - \sqrt{1-\delta}/2]$

By definition, $x^{LE}_{\hat{r},\hat{a}} = \hat{a} \quad \forall \hat{a} \in [\sqrt{1-\delta}/2; 1 - \sqrt{1-\delta}/2]$.
(18) is satisfied, as $|a - x^{LE}_{\hat{r},a}| = 0 \leq |a - x^{LE}_{\hat{r},\hat{a}}| \quad \forall r, a, \hat{a}$.

Case 1.2: $a < \sqrt{1-\delta}/2$

By definition, $x^{LE}_{\hat{r},\hat{a}} \geq \sqrt{1-\delta}/2 \quad \forall \hat{a}$.
(18) is satisfied, as $|a - x^{LE}_{\hat{r},a}| = |a - \sqrt{1-\delta}/2| \leq |a - x^{LE}_{\hat{r},\hat{a}}| \quad \forall r, a, \hat{a}$.

Case 1.3: $a > 1 - \sqrt{1-\delta}/2$

By definition, $x^{LE}_{\hat{r},\hat{a}} \leq \sqrt{1-\delta}/2 \quad \forall \hat{a}$.
(18) is satisfied, as $|a - x^{LE}_{\hat{r},a}| = |a - 1 - \sqrt{1-\delta}/2| \leq |a - x^{LE}_{\hat{r},\hat{a}}| \quad \forall r, a, \hat{a}$.

Case 2: $\hat{r} > b$

By definition, $x^{LE}_{\hat{r},\hat{a}} = \hat{a} \quad \forall \hat{a}$.
(18) is satisfied, as $|a - x^{LE}_{\hat{r},a}| = 0 \leq |a - x^{LE}_{\hat{r},\hat{a}}| \quad \forall r, a, \hat{a}$.

■

Proof of Lemma 8:

The proof works by contradiction:
Assume there is a solution such that $\exists r' < b$ and $A', A'' \subseteq A$ with positive probability

measure such that $p(r', a') > p(r', a'') \quad \forall a' \in A', a'' \in A''$.

Claim 1: *For all* $(a', a'') \in A' \times A'' \quad \exists r_{a',a''} > 0$ *such that* $\forall r \leq r_{a',a''}$ *truthtelling about the second period type is not optimal for at least one* $a \in \{a', a''\}$.

Take any pair a', a'' with $a' \in A'$ and $a'' \in A''$. An arbitrary ex ante type r that has reported r' will report honestly about his second period type only if the following inequalities hold:

$$\int_0^1 v_{r,a'}(s) dX(r', a')[s] - p(r', a') \geq \int_0^1 v_{r,a'}(s) dX(r', a'')[s] - p(r', a''), \tag{19}$$

$$\int_0^1 v_{r,a''}(s) dX(r', a'')[s] - p(r', a'') \geq \int_0^1 v_{r,a''}(s) dX(r', a')[s] - p(r', a'). \tag{20}$$

Using the full assignment property, (19) and (20) are equivalent to

$$\begin{aligned} &\left(\int_0^1 |a' - s| dX(r', a')[s] - \int_0^1 |a' - s| dX(r', a'')[s] \right) * 4r \\ &\leq p(r', a'') - p(r', a') \\ &\leq \left(\int_0^1 |a'' - s| dX(r', a')[s] - \int_0^1 |a'' - s| dX(r', a'')[s] \right) * 4r. \end{aligned} \tag{21}$$

Since $p(r', a'') - p(r', a') \neq 0$, $\exists r_{a',a''} > 0$ such that $\forall r \leq r_{a',a''}$ (21) does not hold. Hence, by (19) and (20) any type $r \leq r_{a',a''}$ that has claimed to be of type r' has a strict incentive to lie about his ex post type when being either a' or a''.

Claim 2: *Define* $r_{A',A''} = \inf \{r_{a',a''} | a' \in A', a'' \in A''\}$. *Any type* $r \leq r_{A',A''}$ *that has claimed to be of type* r' *has a strict incentive to lie on a set of ex post types that has positive probability measure.*

By construction, any type $r \leq r_{A',A''}$ has a strict incentive to lie when being either a' or a'' for any pair $(a', a'') \in (A', A'')$. Assume first $\exists a' \in A'$ and $A''_s \subseteq A''$ with positive probability measure such that types (r, a'), $r \leq r_{A',A''}$ have *no* strict incentive to deviate to any $a \in A''_s$. But then by Claim 1 the types $r \leq r_{A',A''}$ have an incentive to deviate on A''_s, which has positive measure. Second assume that for any a' there is no such subset

A''_s. But then by Claim 1 the types $r \leq r_{A',A''}$ have a strict incentive to deviate on the entire set A', which has positive measure.

Claim 3: *If an ex ante type r reports $r'' \leq b$ and then truthfully reveals his ex post type a, his ex ante expected utility is zero ($U(r'', r, id_a) = 0$).*

By assumption the allocation rule is optimal and therefore by Lemma 7 satisfies the properties of Lemma 6. Inserting the optimality properties from Lemma 6 into utility (8) reveals that

$$U(r'', r, id_a) = U(r'''', r''', id_a) \quad \forall r, r''' \in R, \ \forall r'', r'''' < b.$$

From this follows

$$U(r'', r, id_a) = U(r'', id_a) = U(z, id_a) = 0 \quad \forall r \in R, \ \forall r'' < b.$$

Final Step: By Claim 3 $U(r', r, id_a) = 0$. If an ex ante type r with $r \leq \min\{r_{A',A''}, b\}$ reports r', by Claim 2 he has a strict incentive to deviate from truthfully revealing his ex post type a on a set of ex post types with positive probability measure. From this follows $U(r', r, \sigma^*) > 0 \quad \forall r \leq r_{A',A''}$, where $\sigma^*(a, r, r')$ is an agent's optimal strategy about reporting ex post types as a function of his true a, when being of type r and having reported r'. From IC_1 follows then $U(r, id_a) \geq U(r', r, \sigma^*) > 0$. This contradicts optimality condition $U(r, id_a) = v - v = 0$, which follows from Lemma 6. ■

Proof of Proposition 3:

Lemmas 6 to 8 show that the properties given in the four bullet points are necessary for a solution. The conditions of Proposition 3 imply all conditions of Lemma 6. Hence, if the properties and ex post type independent prices are sufficient for incentive compatibility, the proof is completed.

Again, it is sufficient to show that (IC_2^s) is satisfied. Using ex post type independence of prices $p(r,a) = p(r,a') \; \forall r \in R \; \forall a, a' \in A$, (IC_2^s) can be reformulated:

$$\int_0^1 v_{r,a}(s) dX(\hat{r},a)[s] - p(\hat{r},a) \geq \int_0^1 v_{r,a}(s) dX(\hat{r},\hat{a})[s] - p(\hat{r},\hat{a}) \quad \forall r,\hat{r},a,\hat{a}$$

$$\Leftrightarrow \int_0^1 v + (1-\delta)r - 4r|a-s| dX(\hat{r},a)[s] \geq \int_0^1 v + (1-\delta)r - 4r|a-s| dX(\hat{r},\hat{a})[s] \quad \forall r,\hat{r},a,\hat{a}$$

$$\Leftrightarrow \int_0^1 |a-s| dX(\hat{r},a)[s] \leq \int_0^1 |a-s| dX(\hat{r},\hat{a})[s] \quad \forall r,\hat{r},a,\hat{a}$$

Case 1: $\hat{r} \leq b$

(IC_2^s) is independent of the true ex ante type r and relabeling $\hat{r}$ as r gives (9). By the second property of Proposition 3 (9) is satisfied.

Case 2: $\hat{r} > b$

By the first property follows:

$$\int_0^1 |a-s| dX(\hat{r},a)[s] = 0 \leq \int_0^1 |a-s| dX(\hat{r},\hat{a})[s] \quad \forall r,\hat{r},a,\hat{a}.$$

■

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