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Can being behind get you ahead? Reference Dependence and Asymmetric Equilibria in an Unfair Tournament*

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1 Abstract

Everyone remembers a plot where a disadvantaged individual facing the prospect of failure, spends more effort, turns around the game and wins unexpectedly. Most tournament theories, however, predict the opposite pattern and see the disadvantaged agent investing less effort. We show that 'turn arounds', i.e. situations where the trailing player spends more effort and becomes the likely winner of the tournament, can be the outcome of a Nash equilibrium when the initial unevenness is known and players have reference-dependent preferences. Under certain conditions, they are the only pure strategy equilibrium. If the initial unevenness is large enough the advantaged player will always invest the most effort. We also show that equilibria in which the player behind catches up without becoming the likely winner do not exist.

Keywords: loss aversion, gain-loss utility, normal distribution, competition

JEL classification: C72, D63, D44

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2 Introduction

Rank order tournaments are a common mechanism for providing social and economic order. They are somewhat special, because they tie the privilege of receiving a certain good or benefit to the effort of performing best at some productive task. Politicians need to convince their constituents to be elected and business men need to create value for their company to be considered for promotion. Especially when high stakes are involved any indication of the likely outcome of a tournament is an asset. Consider for example the large betting industry that offers bets at dynamic quotes during the progress of many publicly fought contests.

Many times tournaments are not entirely fair with one player, for example, having more information or better relations with the tournament decider. However, even more often such unevenness occurs in dynamic contests. Most real world tournaments are dynamic in the sense that they require repeated decisions by the competitors between which contestants can process new information and recalibrate their tactics and effort investment. The most general piece of new information is the intermediate score which exists in most tournament settings. Politicians obtain interims feedback through opinion polls and direct contact, students write mid-term or mock exams and sports men can usually collect a time or score signalling their relative position in the tournament. That such feedback is entirely even seems to be the exception rather than the rule.[1]

Previous research like the work on dynamic tournaments by Chan et al. (2009) or Aoyagi (2010) found that equilibria are "effort-symmetric" with respect to feedback. This means that independent of the interims feedback, both competitors invest the same level of effort and only the sum of efforts decreases the more uneven the feedback is. Without any difference in effort provisions and the corresponding changes in the relative winning probabilities these tournaments are essentially decided by the initial unevenness and chance. Providing information about the intermediate state of the game does not matter for the tournament outcome. The implicit assumption here is that the interims feedback does not affect the agent's utility directly. In such a world, a victory against all odds that follows a drastic comeback after having been far behind initially is the same as any other victory in terms of utility.

Gill and Stone (2010) were the first to account for the direct effect of feedback on utility by introducing "fairness and desert" concerns in the form of reference-dependent preferences. They investigate the influence of experiencing something as deserved on equilibrium formation. Focussing on effort-symmetric equilibria they could show that for uneven games, symmetric effort equilibria, in which the interims score is immaterial

[1] How humans react to feedback is not yet fully understood. One potentially related idea is the concept of cognitive dissonance (Festinger, 1962), which proposes that anyone who holds contradictory beliefs will try to actively reduce this dissonance. Adjusting one's reference categories could be seen as one way to overcome the dissonance between the desire for a certain prize and the naturally limited resources to obtain it.

to the outcome of the tournament, do not exist. However, currently it is not clear whether asymmetric equilibria exist and if so whether they favour the victory of the player ahead or behind.[2]

Gill and Stone (2010) derive predictions for asymmetric equilibria, but only for the case where chance in the game is uniformly distributed. Uniformly distributed errors are commonly used in economic models and laboratory experiments. Indeed, without further knowledge of the situation at hand it may be as well-suited as any other distribution. However, when thinking about many examples of rank order tournaments the assumption that all random events, no matter how extreme, are equally likely and that at the same time more extreme events carry probability of zero appears odd. In many tournament applications like job or sport contests the notion that extreme events can happen, but do so with a low probability, has an intuitive appeal.[3]

With uniformly distributed errors Gill and Stone (2010) find that just one class of equilibria exists, in which the player ahead always spends more effort than the trailing player. Our paper makes new and very different predictions for the same setting when uncertainty is normally distributed. We find that, depending on the strength of the reference dependence, the tournament prize and the initial unevenness three different classes of equilibria exist. Remarkably, in two of these classes the player being behind overtakes the opponent and ends up with a higher probability of winning the tournament. In tournaments where the initial unevenness is strongly favourable for one party we find a unique equilibrium, in which the leading player extends the lead by investing more effort than the player behind. However, when the game is tight and the tournament prize is large enough to motivate the trailing player to overcome the initial disadvantage, equilibria where the player behind spends much more effort than the player ahead and obtains a higher probability of winning the tournament, always exist.

In the first class of what we call Turn Around Equilibria (TAE) the agent behind turns a marginal disadvantage ex ante, a 48 percent probability of winning, into a marginal advantage with slightly more than a 53 percent chance of winning. In the second class, the turn around can be much more pronounced. Here a trailing player starting with a winning probability of say around 30 percent may turn the game into one which yields almost certain victory with the winning probability exceeding 90 percent. We show that whenever the player behind catches up on the opponent the extra effort will be sufficient to overcome, and even exceed the entire initial disadvantage. Situations where the trailing player makes up some of the disadvantage without becoming the favourite winner do not exist in equilibrium. We show that depending on parameter values, the only possible pure strategy equilibrium is one in which the disadvantaged player turns the game. Lastly, we predict that equilibria where one agent catches up without taking

[2] In a new article Dato et al. (2015) further explore the circumstances under which symmetric equilibria arise.

[3] Stern (1991) investigates score differences in football and cannot reject that they are normally distributed.

the lead do not exist.

The model is set up as a version of the tournament formulated by Lazear and Rosen (1981) with the defining characteristic that the element of chance enters additively into the contest success function. Since we introduce reference-dependent preferences we use the notion of choice acclimating personal equilibrium, that was introduced by Kőszegi and Rabin (2007), in which the reference point is endogenous to maximisation process as a solution concept. This means that agents take into account that their effort choice affects their reference utility, i.e. that a high effort level makes winning more likely and, hence, increases the reference point.

We contribute to a growing literature taking an interest in dynamic and uneven tournaments. Contributions like Gill and Stone (2010) discuss that in a dynamic setting agents have time to emotionally react to events and deviate from standard rationality. How emotions within a sports game can impact the motivation and ability of players psychologically is described by Lazarus (2000). Klaassen and Magnus (2001) support this notion empirically by showing, with a large data set of tennis matches, that points in tennis are not individually and identically distributed. Gill and Prowse (2012) confirm experimentally the key economic concept of strategy functions where the effort of one agent crowds out the effort of the competitor. They introduce a dynamic frame by letting subjects choose their effort sequentially providing complete information about the choice of the first subject. Ederer (2010) studies asymmetric equilibria as a result of asymmetrically distributed ability between two agents. In his model interim feedback gives competitors the chance to update their beliefs about their opponent's ability. This leaves the agent who is ahead in the game more confident of the value of his own effort investment and results in relatively greater effort provision from the leading player.

Our model provides a theoretical explanation for the existence of turn arounds. Our results can explain the puzzling empirical evidence presented by Berger and Pope (2011), who investigate data from 18,060 American basketball games and find that teams which are slightly behind at half time have a significantly higher probability to win the game. As basketball is a complex sport it could be argued, for example, that their results are not directly linked to effort investment. However, they consolidate their finding by running an experiment in a controlled laboratory environment where participants had to compete in a real effort task that involved fast clicking and were told an intermediate score at half time. Those who were slightly behind at half time showed a marked increase in clicks in the second half compared to those who were ahead or to the no feedback control group. Previous literature was not able to explain this pattern.[4]

[4]In Ederer (2010) and Gill and Stone (2010), for example, the only type of asymmetric equilibrium is one, where the leading player exerts more effort than the disadvantaged opponent.

3 The Model

The model studies a contest with two players $j \in \{A, B\}$ who exert effort e^j. The initial unevenness is given by δ_1 which represents an advantage for Player A when positive and vice versa. The parameter δ_1 is exogenous and observable. The unobservable random noise parameter ϵ is not affected by effort and follows a normal distribution with mean 0 and variance σ^2.[5] The initial unevenness δ_1, the shock term ϵ and the two choice variables e^A and e^B constitute the final outcome which is determined by $\delta_2 = \delta_1 + e^A - e^B + \epsilon$.

The prize received by Player j is given by z_j. If the player wins the tournament the player receives the winner prize w, while the loser prize is normalised to zero. Therefore: Player A wins if $\delta_2 > 0$ and receives $z_A = w$, while Player B obtains $z_B = 0$ and vice versa. In this setting, the probability that Player A wins the contest equals $Prob(\delta_2 > 0)$ which implies $Prob(\epsilon > -\delta_1 + e^B - e^A)$. Using the fact that ϵ is normally distributed we can rewrite this as $1 - F(-\delta_1 + e^B - e^A)$ where $F(\cdot)$ is the cumulative distribution function of the normal distribution. From the symmetry of the normal distribution it follows that:

$$Prob_{\{\text{A wins}\}} = Prob(\delta_2 > 0) = 1 - F(-\delta_1 + e^B - e^A) = F(\delta_1 + \Delta e) \text{ where } \Delta e = e^A - e^B$$

3.1 Utility with reference-dependent preferences

In the first part of our analysis we make no assumptions about how the reference points $\{r^A, r^B\}$ are formed. Instead, we study the additional incentives reference dependence imposes on the players. Afterwards, we investigate how a reference point contributes to determine the tournament winner assuming that it is formed endogenously as described by Köszegi and Rabin (2006).

A player's utility under a reference point r^j is given by:

$$U^j = v(z_j) + m(z_j|r^j) - c(e^j) \text{ where } m(z_j|r^j) = \begin{cases} \eta(w - r^j) & \text{if Player j wins} \\ \eta(1+\theta)(0 - r^j) & \text{if Player j loses} \end{cases}$$

$$\text{and } v(z_j) = z_j \ , \ c(e^j) = \frac{1}{2}(e^j)^2 \ , \ \eta \geq 0 \ , \ \theta \geq 0$$

We assume $r^j \in [0, w]$ as the reference point for the tournament prize should give us a value between the lowest possible outcome and the highest possible outcome of the tournament. The utility is composed of a convenient consumption part v, for which a linear specification is used, the cost of effort provision $c(e^j)$ and a reference dependent term. The weight of the reference utility is calibrated by η, such that setting $\eta =$

[5] To ensure pure-strategy equilibria the variance has to be sufficiently large as described in Lazear and Rosen (1981).

0 returns the model without reference dependence. The parameter θ introduces loss aversion. It represents the difference between the disutility of falling short of the reference point and the utility of exceeding it by one unit. We assume that losses loom larger than gains and consequently take θ to be positive. We use quadratic costs of effort for simplicity.

Both players choose an effort level to maximise their expected utility given the unevenness δ_1. Player A maximises expected utility with respect to e^A. Consequently, the optimisation problem for Player A can be written as:

$$\max_{e^A} F(\delta_1 + \Delta e)(w + \eta(w - r^A)) + (1 - F(\delta_1 + \Delta e))(-\eta(1 + \theta)r^A) - c(e^A).$$

The first term $F(\delta_1 + \Delta e)(w + \eta(w - r^A))$ represents the utility in case the agent wins the tournament. It is added to the utility of losing $(1 - F(\delta_1 + \Delta e))(-\eta(1 + \theta)r^A)$ and the costs of effort which have to be paid independent of the outcome. We define P^j as winning-probability of player j, i.e. $P^A = F(\delta_1 + \Delta e)$ and $P^B = 1 - F(\delta_1 + \Delta e)$. The contribution of reference dependent utility lies in adding the term below to the standard objective function $wF(\delta_1 + \Delta e) - c(e^A)$:

$$R^j := \eta\Big(P^j w - r^j\big[1 + \theta\big(1 - P^j\big)\big]\Big)$$

Except for the potentially different reference points and the individual winning probability the term R^j is the same for both players. While the sign of R^j depends on the actual parameter values, it becomes apparent that a greater reference point reduces the agents' utility. This is intuitive as a higher reference point renders a victory less sweat, but a defeat all the more bitter. Moreover, reference dependence contributes an incentive effect which is given by

$$\frac{\partial R^A}{\partial e^A} = \eta\Big(f(\delta_1 + \Delta e)(w + r^A\theta) - \frac{\partial r^A}{\partial e^A}\big[1 + \theta(1 - P^A)\big]\Big).$$

The expression reveals the delicate nature of the effect which may take different sizes locally over the decision space. The first term $\eta f(\delta_1 + \Delta e)(w + r^A\theta)$ adds a positive incentive, that is caused by an increase of the effective prize spread. Since Lazear and Rosen (1981) it has been known that when there are no participation constraints an agent's effort decision is not affected by the absolute level of prizes, but by the spread between the winner and loser prize. Reference dependence increases the effective prize spread, making the valuation of both tournament outcomes more extreme. The strength of its impact, however, depends on the reference point r^A which may take different values for different $\{e^A, e^B, \delta_1\}$. The second term reduces to 0 in case of an exogenous reference point as the derivative $\frac{\partial r^A}{\partial e^A}$ remains 0.

3.2 Endogenous Reference Points

In the following we endogenise the reference points and employ the choice-acclimating personal equilibrium concept of Köszegi and Rabin (2006) to derive the player's first and second order conditions. After establishing a necessary and sufficient condition for the interiority of all solutions in Lemma 1, we show in Lemma 2 and Lemma 3 that both first and second order conditions can be reduced to one equivalent expression. We proceed to define the three classes of equilibria and derive conditions for their existence in Proposition 1 to Proposition 3. In Proposition 4 we give conditions for the uniqueness of a Turn Around Equilibrium. Finally, in Proposition 5 we discuss a fourth class of equilibria and interpret our results.

Modelling the reference point formation explicitly makes the precise effect of reference dependence tractable. We assume expectation based reference points, but remain agnostic about whether expectations are formed as in the reference dependence theory of Köszegi and Rabin (2006) or as in disappointment aversion theory developed by Bell (1985) and Loomes and Sugden (1986). Additionally, we will allow the reference point to adjust in the process. As solution concept we use choice-acclimating personal equilibria (CPE) that are defined "as a decision that maximises expected utility given that it determines both the reference lottery and the outcome lottery" (Kőszegi and Rabin, 2007). In consequence, the reference points are taken to be the endogenous winning probability of each player multiplied by the winner prize, which constitutes the expected gain of each player. Explicitly, the reference points are modelled as $r^A = F(\Delta e + \delta_1)w$ for Player A and $r^B = (1 - F(\Delta e + \delta_1))w$ for Player B.[6] The explicit reference point enables us to rewrite the contribution term R^A for both players to $-w\eta\theta F(\Delta e + \delta_1)(1 - F(\Delta e + \delta_1))$. The negative sign shows that each player has an incentive to minimise this term. For player A this results in the following incentive effect[7]:

$$\frac{\partial R^A}{\partial e^A} = w\eta\theta f(\delta_1 + \Delta e)(2F(\delta_1 + \Delta e) - 1)$$

The derivative above is positive if $\delta_1 + \Delta e > 0$ and negative if $\delta_1 + \Delta e < 0$. The absolute value of R^A is highest for close games when $\delta_1 + \Delta e$ is zero and falls steadily when the game gets less tight. In other words, players have an incentive to flee the middle and avoid the uncertainty associated with close games, which has also been described by Gill and Stone (2010). Note that the incentive does not point the player into a particular direction. Whether the player "gets ahead" or "falls behind" is not important. Evenness at the end of the period is unattractive for agents with reference points since it jointly maximises the size of the disutility from falling short of the reference point weighted by

[6] Like Gill and Stone (2010) we do not model a reference point in the effort domain. We believe that further conceptual work on what a reference point in the effort domain could be is interesting and could yield a valuable addition to this and other models. Yet with all its psychological and technical implications it exceeds the scope of this paper.

[7] The corresponding term for Player B is $\frac{\partial R}{\partial e^B} = w\eta\theta f(\delta_1 + \Delta e)(1 - 2F(\delta_1 + \Delta e))$.

the probability of its occurrence. With normally distributed chance in the game, this opens up the possibility for multiple equilibria.

To understand this last point better consider Figure 1 which sketches both player's marginal costs and benefits.[8] In the upper graph, which depicts the standard Lazear and Rosen (1981) tournament without reference dependence, both player's marginal benefit curves coincide with the equilibrium being reached at their peak. The effect of reference dependence in the lower graph of Figure 1 is to steepen and drive apart both player's marginal benefits. The peak of the marginal benefits of both players is now located in the area where they themselves are more likely to win. Intuitively, both players benefit most from their effort when they can use it not only to increase their own winning probability, but also to decrease the uncertainty of the game. Here, without further asymmetries (i.e. $\delta_1 \neq 0$) the same symmetric equilibrium continues to exist.

This can also be seen in two top panels of Figure 2 which plots both players' best response functions along with a 45 degree line for the given parameter values $\eta = 1, \theta = 1, \delta_1 = 0.2, \sigma = 2$ and $w = 3\pi$. Moving from the top panel to the middle one with reference dependence the symmetric equilibrium is preserved. However, we can now see that also two other potential asymmetric equilibria on either side do exist. Again both players peak best response effort lies on the side of the 45 degree line where they are more likely to win. When we introduce asymmetry in favour of Player A (i.e. $\delta_1 > 0$) we can see that A's peak best response effort moves towards the 45 degree line while Player B's moves away from it. From the intersections of the two functions we can thus identify three potential equilibrium candidates, two of whom lie above the 45 degree line which implies that the disadvantaged player behind spends more effort than the advantaged player.The best response functions have the simple structure:

$$
\begin{aligned}
e^A &= wf(\delta_1 + \Delta e)\big[1 + \eta\theta\big(2F(\delta_1 + \Delta e) - 1\big] = wf(x)\big[1 + \gamma G(x)\big] \\
e^B &= wf(\delta_1 + \Delta e)\big[1 - \eta\theta\big(2F(\delta_1 + \Delta e) - 1)\big] = wf(x)\big[1 - \gamma G(x)\big]
\end{aligned}
$$

We define $x = \delta_1 + \Delta e$, $\gamma = \eta\theta$ and $G(x) = 2F(x) - 1$. The variable x, thus, represents the state of the game just before the random shock ϵ is realised. Since the two conditions for e^A and e^B must be fulfilled in equilibrium they provide information about when equilibria are interior, i.e. when both agents provide strictly positive effort. From $wf(x) > 0$ we know that there is an interior solution whenever $(1 + \gamma G(x))$ and $(1 - \gamma G(x))$ are both strictly greater than zero. Small rearrangement implies that both conditions are fulfilled whenever $\gamma < |\frac{1}{G(x)}|$. Since the set of possible values of $|G(x)|$ which is bounded above by one,[9] a simple corollary is that for $\gamma \leq 1$ the condition is fulfilled and the

[8]The marginal benefits are given by $MB^A = MB^B = w * f(\delta_1 + \Delta e)$

[9]$|G(x)| = |2F(x) - 1|$ converges to 1, since the cdf of the normal distribution converges to 0 for $x \to -\infty$ and to 1 for $x \to \infty$.

corresponding equilibrium must be interior. This leads to the first lemma.

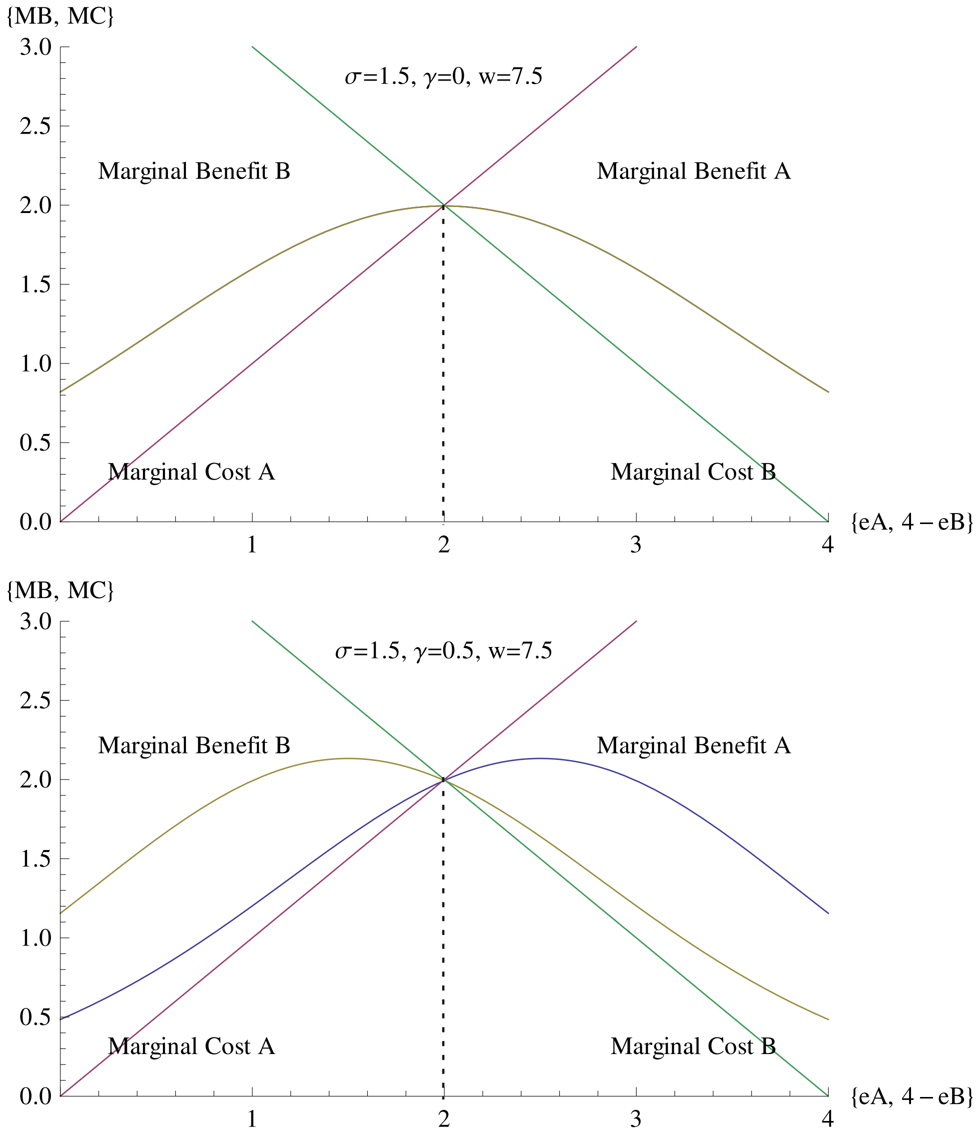

Figure 1: The Figure shows the **Marginal Costs and Benefits** of both players. Player B's effort increases from left to right. In the top panel without reference dependence ($\gamma = 0$) marginal benefits of both players are identical. Introducing reference dependence changes that. The Marginal Benefit curves are now only equal at the equilibrium effort levels, which without initial asymmetry are still symmetic (they are equal to two in this case).

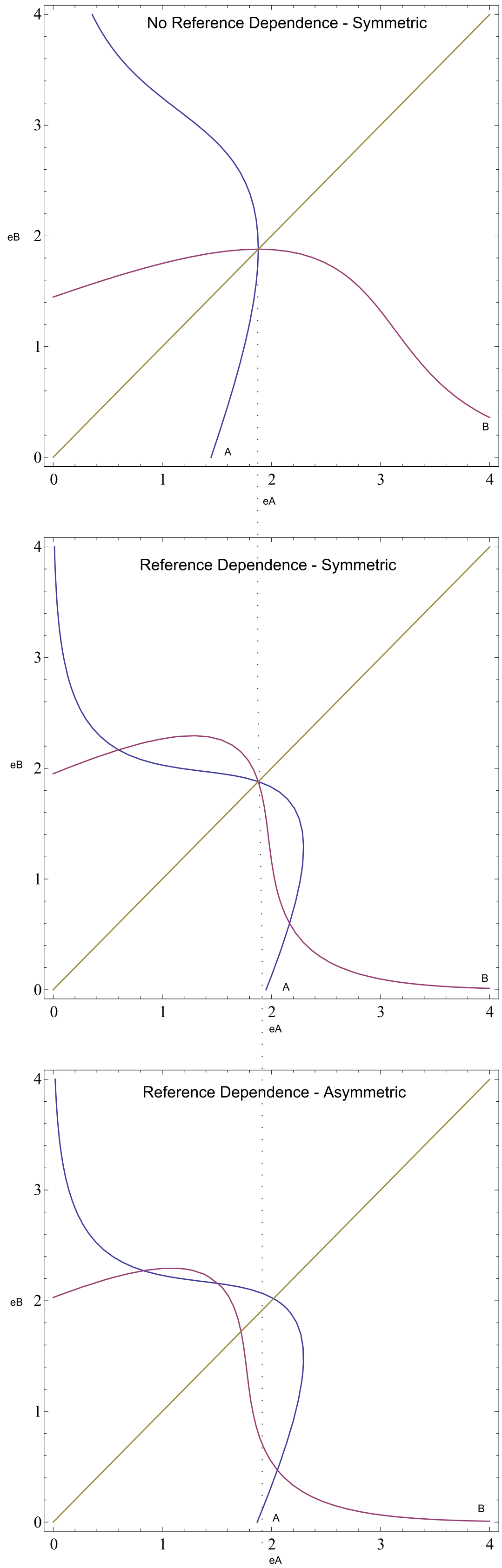

Figure 2: Effort combinations which fulfil each players First Order Condition. All intersections are potential equilibrium candidates. Below the 45 line Player A exerts greater effort, above it Player B exerts more. The top panel assumes $w = 3\pi$ and $\sigma = 2$. In the middle panel the reference dependence parameter $\gamma = 1$. In the lower panel Player A is additionally given an advantage of $\delta_1 = 0.2$.

Lemma 1. *An equilibrium is interior if* $\gamma < |\frac{1}{G(x)}|$*. Therefore all equilibria are interior whenever* $\gamma \leq 1$*.*

All lemmas and propositions are proven formally in Appendix 1. The term $\frac{1}{G(x)}$ is always defined as $G(x) \neq 0$ for all x that describe equilibria. To ensure that all equilibria are interior, we will assume $\gamma < |\frac{1}{G(x)}|$. This is not a restrictive assumption as for any x, $|G(x)|$ is always between zero at the origin and one as x becomes arbitrarily small or large. Hence, all moderate forms of loss aversion where $\gamma \leq 1$ are covered as well as many stronger versions depending on the degree of the state of the game x.

For simplification we proceed by combining both first order conditions as well as both second order conditions to obtain two new functions we term candidate and maximum condition function.

Lemma 2. *The system of first order conditions can be expressed as the candidate function* $\delta_1 = x - 2w\gamma f(x)G(x)$*. All combinations of* $\{e^A, e^B, \delta_1\}$ *which fulfil this equation are referred to as candidate points.*

Lemma 3. *If* x *fulfils the maximum condition* $0 < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2}$ *then at the corresponding vector* $\{e^A, e^B, \delta_1\}$ *both second order conditions are fulfilled.*

We call the function that describes the border of the inequality given in Lemma 3, $\frac{|x|}{\sigma^2} = \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x))$, the maximum condition function. We can plot the candidate function and the maximum condition functions into one system as illustrated in Figure 3. Both graphs depend on x which is given on the horizontal axis. The candidate function is depicted by the blue curve and every point on it represents an equilibrium in case the second order conditions are fulfilled for the same x-value. The second order conditions are jointly represented by the maximum condition function in red. In case this function has a positive value for a certain x both second order conditions are fulfilled. Remember that x was initially defined as $e^A - e^B + \delta_1$. For this reason we know that Player A has a higher winning probability for positive and Player B for negative x, but we also know that Player B must have chosen a significantly higher effort than A in case of a negative x-value and $\delta_1 > 0$. We can now read Figure 3 in a convenient way. The vertical axis is also a scale for δ_1; hence we can choose a particular initial unevenness δ_1, take the corresponding x-value from the candidate function and evaluate it using the maximum condition function. When it is positive at that point, the combination of x, δ_1 must be an equilibrium. Lemma 2 shows that with the help of the first order conditions the unique pair of $\{e^A, e^B\}$ can be retrieved.

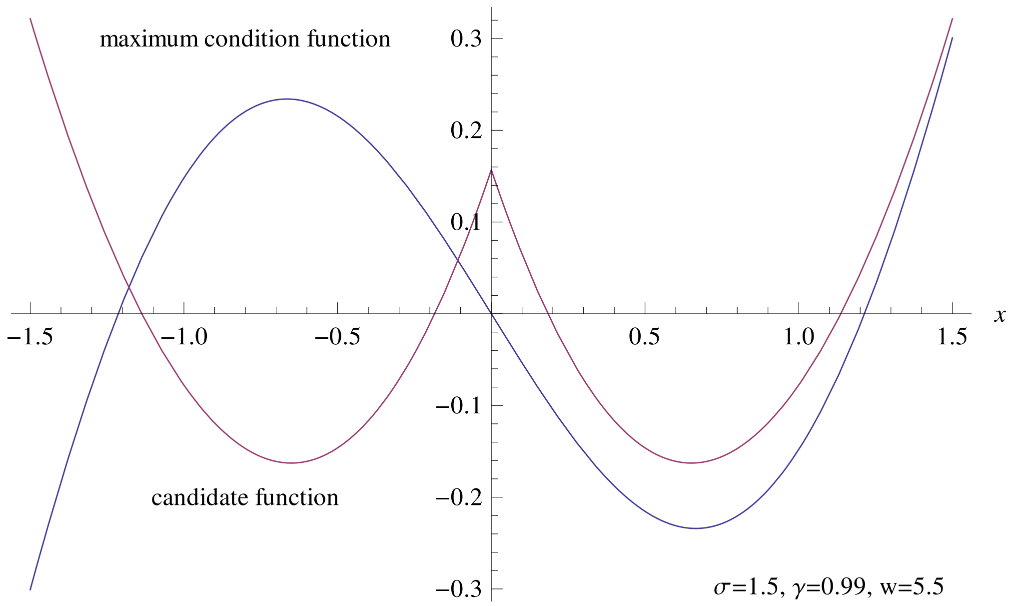

Figure 3: maximum condition and candidate function with positive intersections

4 Multiple Equilibria

In the following, we leave most technical details to the appendix but provide some intuition verbally and graphically for why asymmetric equilibria exist. We will assume throughout that $\delta_1 > 0$, i.e. Player A is ahead and benefits from the initial unevenness.[10]

4.1 Confirming Asymmetric Equilibria

As explained earlier, introducing reference dependence renders the middle ground, i.e. when $\delta_1 + \Delta e$ is close to zero, unattractive to both players. Without reference dependence Player B and Player A would always choose the same level of effort since both players have the same marginal costs and benefits. and are also the same due to the symmetry of the normal distribution's density. Therefore, the player ahead always maintains the same advantage in the relative winning probability. An extra incentive rewarding more unequal winning probabilities like reference-dependent preferences, in this setting, would just widen the already existing probability spread. To achieve this the player ahead needs to put in relatively more effort than the player behind. Thus, when reference dependence increases the effective prize spread, both players will invest more effort,

[10] Due to the symmetry of the problem all results also apply in case Player B is ahead.

but the player ahead claims a larger share of the extra contribution. In the following this is referred to as Confirming Asymmetric Equilibrium.

Definition 1. *Confirming Asymmetric Equilibrium (CAE)*
A Confirming Asymmetric Equilibrium is an equilibrium where the advantaged player spends more effort than the other player.

Figure 4 shows CAEs explicitly when Player A being initially advantaged. When δ_1 is positive, Player A ends up with a higher winning probability and the CAEs are located in the upper right quadrant of the coordinate system. We see that each $\{x, \delta_1\}$ combination, for which the candidate function lies in this quadrant, is a CAE in case the maximum condition function is positive for this x-value as well. In the depicted case there exists a CAE for all values of δ_1. However, this does not need to be the case.[11] While there will always be candidate CAEs for all values of δ_1, the maximum condition is not necessarily fulfilled. We prove the following proposition:

Proposition 1. *For δ_1 large enough there always exists one Confirming Asymmetric Equilibrium (CAE) that is a unique equilibrium.*

For tournaments without reference dependence, Lazear and Rosen (1981) show that symmetric equilibria do not necessarily exist and depend on the wage-schedule as well as the degree of uncertainty inherent to the tournament.[12] Proposition 1 shows that strong unevenness at the start of the tournament curbs the first point. For sufficient uncertainty, it eventually guarantees the existence of a pure strategy equilibrium. While equilibria in which a leading player extends the lead are not uncommon in the literature, we now introduce two further types of equilibria.

4.2 Type One Turn Around Equilibria

Reference dependence as described above introduces an incentive to "flee the middle", but this can be done in yet another way. As an alternative to the CAE the player behind may decide to outspend the leading player. Such an equilibrium is called Turn Around Equilibrium.

Definition 2. *Turn Around Equilibrium (TAE)*
A Turn Around Equilibrium is an equilibrium where the initially disadvantaged player

[11] It can happen, that the candidate function produces combinations of x and δ_1 at which the maximum condition function is still negative. In consequence CAEs are guaranteed for great x and δ_1, but given parameter values they may not exist for the whole range of δ_1.

[12] Imagine there was no uncertainty in the tournament. Then, each player would try to marginally overbid the opponent and no equilibrium in pure strategies would exist. Besides there would of course exist a symmetric mixed strategy equilibrium.

spends so much more effort that this player has higher probability to win the game than the opposing player.

Definition 3. *Type one Turn Around Equilibria (TAE1)*
Type one Turn Around Equilibria (TAE1) are TAEs that exist over an interval for $\{e^A - e^B + \delta_1\}$ *that is open and bounded above by* 0.

Suppose that Player B is initially disadvantaged and considers investing more effort than Player A. For Player A this could be an equilibrium since the player is willing to settle at a point where the marginal benefits together with the marginal reduction of the reference dependence cost meets the marginal costs. The key to understanding this intuition is to see that the incentive effect of reference dependence changes sign at $x = \delta_1 + \Delta e = 0$. When Player A backs off, the incentive effect $\frac{\partial R^A}{\partial x} = w\gamma f(x)(2F(x)-1)$ flips and the player will accept an equilibrium where the lower marginal benefit minus the reference cost of increasing effort equals the marginal cost. This intuition is intact as long as the unevenness is rather small and the wage level is high enough to motivate Player B to overcome the initial disadvantage, but not so high as to make it intolerable for Player A to back off.

This leads to the following proposition:

Proposition 2.

i) If $w > \frac{1}{4\gamma f(0)^2}$ *and* $w < \frac{1}{2\gamma f(0)^2}$*, a type one Turn Around Equilibrium (TAE1) always exists.*

ii) TAE1s are always interior.

The condition provided formulates a parameter range for the exogenous tournament prize w and the reference dependence variables $\gamma = \eta\theta$. Under the conditions of Proposition 2 no symmetric equilibria exist.[13]

In case of an initial disadvantage for Player B, TAEs are defined as equilibrium points where Player B spends sufficiently more effort than Player A to become the favourite for winning the tournament. In consequence, TAEs for positive δ_1 can be found in the upper left quadrant of Figure 4. When $\delta_1 > 0$, as we assume throughout without loss of generality, TAE1s are equilibria located in the negative x-domain bordering zero. Depending on the parameter values of w and γ these equilibria exist since the curvature of the candidate function is strong enough to reach into the positive range of δ_1 while the maximum condition function is still fulfilled for those x-values as can be seen in Figure 4.[14]

[13] This is also shown in Gill and Stone (2010).

[14] To verify that TAE1s are not only pathological cases, but appear over a range of x, we estimate an interval of x values over wich TAE1s exist. For this we use a linear approximation of the maximum

The TAE1s that follow from Proposition 2 occur only for tight games and the magnitude of the turn around is generally small. For illustration consider the example where the tournament prize $w = 5.5$, chance has standard deviation of $\sigma = 1.5$, the experience of losses is twice as strong as that of gains ($\theta = 1$), and reference utility is weighted equally strongly as consumption utility $\eta = 1$ such that $\gamma = 1$. Then, in a game where Player A is ahead by 0.033 standard deviations initially, Player B can overtake in equilibrium turning around a disadvantage of 0.06 standard deviations into a lead of roughly 0.06 standard deviations. In terms of winning probabilities Player B starts the tournament with a chance of winning of about 48.6 percent and ends it with about 52.4 percent. So the leading player has a 3.8 percentage points lower probability to win the game in the end. This is similar to the empirical evidence of Berger and Pope (2011) who conduct an experiment where participants compete against each other over two periods in a real effort task. They find that their subjects inserted most effort in the second period when being told that they were slightly behind their opponent and were more likely to win as a result. Berger and Pope (2011) also find a significant increase of winning probability for basketball teams that are slightly behind before the break compared to the leading team. Instead of having a lower probability to win, the team being behind by one point is more likely to win the game. In case of the NBA data the trailing team has 1.1 percentage points higher probability to win the game than the leading team. For the NCAA the result is even stronger: 5.6 percentage points. The difference in winning probability at the breakpoint is significant. Naturally, this field data result can have various explanations, one of which would be to describe it as a TAE1 under the premises of this model.

4.3 Type Two Turn Around Equilibria

While the TAE1s described above are tight in the sense that the initially disadvantaged player increases his winning probability only marginally above fifty percent, there can also be TAEs where the lagging player outspends the opponent sufficiently to increase the winning probability to much more than fifty percent.

Definition 4. *Type two Turn Around Equilibria (TAE2)*
Type two Turn Around Equilibria (TAE2) are TAEs that exist over intervals for $\{e^A - e^B + \delta_1\}$ *that are bounded above by some* $x_\delta \leq 0$.

In this second class of TAEs the leading player backs off much to benefit from the following reference point reduction. This equilibrium may also exist for greater values of w,

condition function. Because of the convexity of the maximum condition function we can evaluate a conservative estimation guarantees us TAE1 for $x \in \left[\frac{(w\sigma\gamma - 2\sigma^3\pi)}{\sqrt{2\pi}w}, 0\right)$. The maximum condition function is convex for the whole range of w used in this proposition. The proof is given in Lemma 5. The boundaries for the set are derived in the proof of Proposition 2.

which becomes apparent once we remember that the weight of the reference dependence effect, $\frac{\partial R^A}{\partial e^A} = \eta f(\delta_1 + \Delta e)w(\theta(2F(\delta_1 + \Delta e) - 1))$, increases in w. The stronger impact of reference dependence makes it more important in the turn around case for the leading player to reduce the effort and flee the middle. As a result, even for high w, TAE2s exist.

To construct the formal criterion we will use the point where the candidate function and the maximum condition function intersect. This point is given by $x^s = \frac{(2f(x^s)^2 w\gamma - 1 - 2f(x^s)^2 G(x^s) w^2 \gamma)\sigma^2}{f(x^s)w(1+G(x^s)\gamma - \sigma^2)}$. As x^s is exogenously determined by the parameters of the model the conditions for w and γ provided in the proposition are exogenous as well.

Proposition 3.

i) When $w \in \left(\frac{1}{4\gamma f(0)^2}, \frac{1}{2\gamma f(x^s)^2 - B}\right)$ where $\gamma \in \left[0.54, -\frac{1}{G(x^s)}\right)$, σ sufficiently large and $B = \sqrt{\gamma f(x^s)^2 \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2}} \leq 0$, a type two Turn Around Equilibrium (TAE2) in which the agent behind spends much more effort than the agent ahead exists. The parameter x^s determines the intersection between candidate function and maximum condition function exogenously.

ii) If there exist TAE2s there also exist Confirming Asymmetric Equilibria (CAEs) for small δ_1.

iii) If the maximum condition function and the candidate function intersect but there are no TAE2s also CAEs for small δ_1 do not exist.

The conditions in Proposition 3 appear more complex than they are. Unlike Proposition 2, Proposition 3 requires a minimum strength of reference dependence γ. If this condition is not met, it is never optimal for the leading player to back off as much as required in the TAE2. To illustrate this consider the following example: Suppose the tournament prize is $w = 10$, chance again enters with a standard deviation of $\sigma = 1.5$, experience of losses is twice as strong as that of gains ($\theta = 1$) and reference utility enters fully with $\eta = 1$ such that $\gamma = 1$. Then, TAE2 exist for any unevenness that is smaller or equal to 0.07 standard deviations. From an initial probability of winning of around 47.3 percent the lagging player in this equilibrium improves his chances to 87.8 percent. This will only be optimal for the player ahead if it is possible to benefit sufficiently from lowering the reference point and hence γ must exceed a certain value. The new condition for w has a similar spirit. While the lower bound coincides with the one in Proposition 2, the upper bound is tightened by $B \geq 0$ which added to the denominator. Again, the reason is that for large tournament prizes it is never optimal for the leading player to allow the other player to overtake. Imagine for example a student who is competing with a class mate over relative grades in a course that is not too important to both. After beating his mate in the midterms that student could still decide not to prepare much for the final exam. He knows that he will probably not come in first. Yet, that would not be too bad, because he also knows that it happened because he was not really trying and could not expect to do any better given his effort.

The definition of TAE2s includes all TAE1s, but TAE2s are potentially located further away from zero than TAE1s as illustrated in Figure 4. Due to the symmetry properties of the candidate and maximum condition function they can be seen as mirror images of certain CAEs. The maximum condition function is axis-symmetric whereas the candidate function is point-symmetric. Consequently, any intersection of the candidate function with the x-axes on the negative domain also exists in the positive domain and vice versa. For x-values larger than the positive root of the candidate function there are CAEs while for x-values above the negative root there exist TAE2s given that the maximum condition function to be positive. Due to axis-symmetry of the maximum condition function it returns the same value at both outer roots of the candidate function. Therefore, if the one equilibrium exist for small δ_1 the other does as well.[15]

4.4 Unique Turn Around Equilibria

The equilibria described spark questions about why the leading player may allow the other player to overtake. One conceivable explanation would be that Turn Arounds are somewhat "lazy equilibria" where the agent ahead has discovered that he greatly benefits from lowering its reference point. However, such an intuition does not truly capture the dynamics of the model. When there are three equilibria, TAE1s are the equilibria with the highest total effort investment. Only for the CAE and TAE2s large asymmetries are possible because one player benefits from lowering his reference point. Moreover, we show that for certain parameter values where the CAEs do not exist a TAE1 is the unique equilibrium.

Proposition 4. *When* $\frac{1}{4f(0)^2\gamma} < w < \frac{1}{2f(0)^2\gamma}$ *and* $\gamma \leq -\frac{f(x^s)^2 G(x^s)\pi^2}{\frac{2}{\sigma^2}+f(x^s)^2\pi(-2+G(x^s)^2\pi)}$ *then for small unevenness the unique equilibrium in pure strategies is a type one Turn Around Equilibrium (TAE1), where* x^s *exogenously determines the intersection between candidate function and maximum condition function.*

The condition for w ensures that TAE1s exist, while the condition for γ rules out the existence of TAE2s and even of CAEs for the particular interval of unevenness over which TAE1s exist. This surprising result is made possible by the missing guarantee for the existence of equilibria in Lazear and Rosen (1981) type tournaments. In a region where the second order conditions do not allow CAEs to exist, the TAE1 candidate point close to, but smaller than zero, satisfies them as illustrated in Figure 5.

Proposition 4 demonstrates that a TAE1 can be the only equilibrium in pure strategies. While we do not engage in equilibrium selection, this shows that at least among pure

[15]Because of continuity this is at least the case for an ϵ-ball around the root of the candidate function.

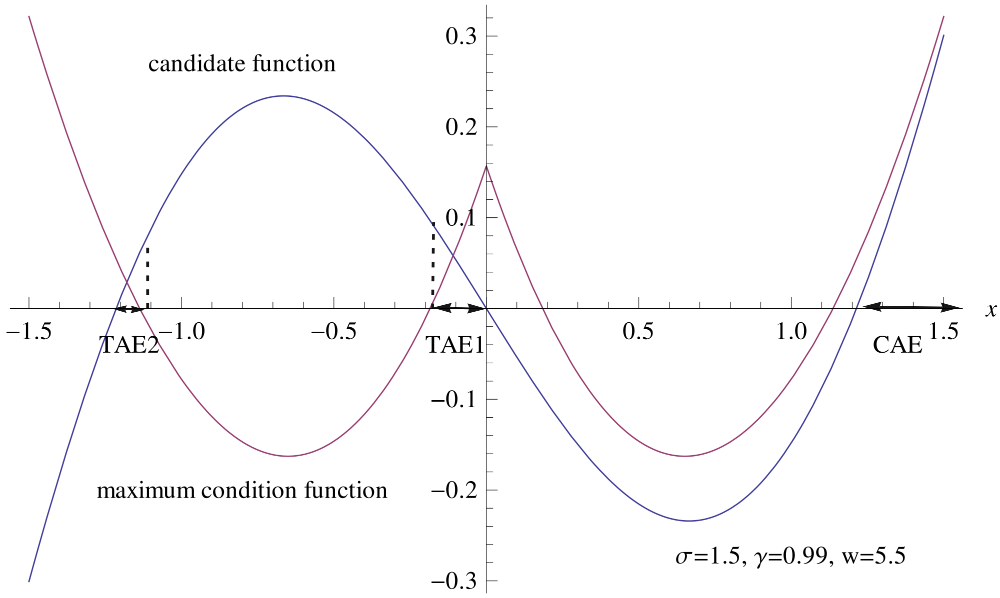

Figure 4: maximum condition and candidate function with all equilibria

strategies there are situations where TAE1s must be played, as no other equilibria exist. Thus, we show that it does not need differences in ability or imperfect information to obtain the unambiguous prediction that a trailing player wins a tournament. Having expectation based reference-dependent preferences can be sufficient for given parameter constellations.

4.5 Catching Up Equilibria

At first glance the notion of Turn Around Equilibria maybe appears (too) strict. It would have been possible to define TAEs as all asymmetric equilibria in which the initially disadvantaged player spends more effort than the advantaged player irrespective of whether the difference is significant enough to turn the game. We call this broader class of equilibria Catching Up Equilibria (CUE).

Definition 5. *Catching Up Equilibrium (CUE)*
A Catching Up Equilibrium is an equilibrium where the initially disadvantaged player spends more effort than the opposing player.

From the definition it is apparent that every TAE must also be a CUE. However, we show that the converse holds as well. Every equilibrium in which the player being behind

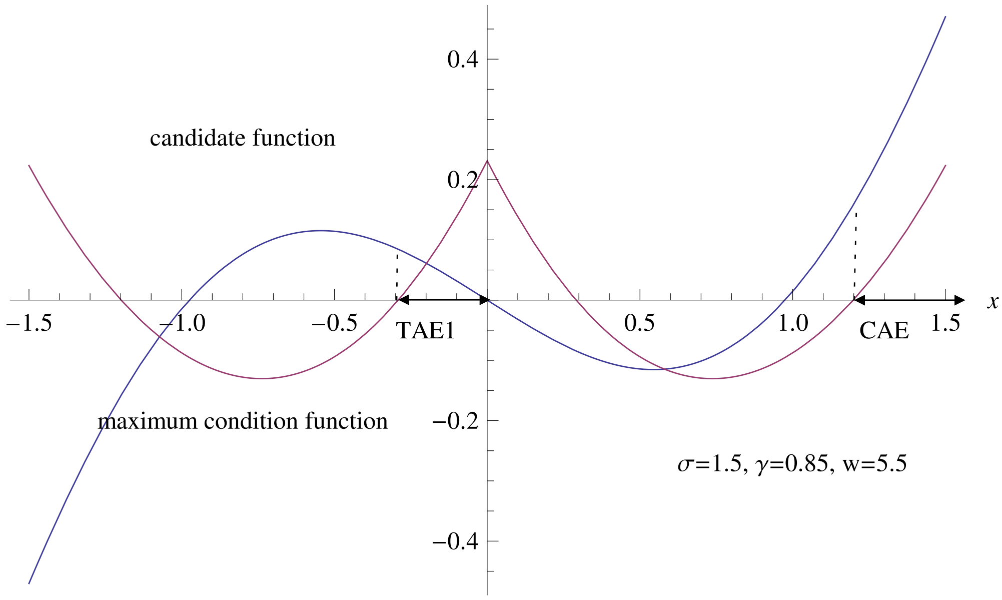

Figure 5: maximum condition and candidate function with only TAE1

invests more effort than the opponent is also a TAE. In other words, situations where trailing player catches up a little without turning the game do not exist.

Proposition 5. *Every Catching Up Equilibrium is also a Turn Around Equilibrium.*

For an intuition consider again the equilibrium in the model without reference dependence. Although one player is advantaged at the start of the tournament both players pick the same effort. Compared to the trailing player in this set-up, a player who tries to catch up, but not overtake, in the model with reference-dependent preferences faces greater marginal effort costs, larger marginal benefits[16] and a more negative marginal utility from reference comparison as the game becomes more even. If the agent had favoured the greater marginal benefits over the marginal effort cost, it would have chosen to insert more effort ex ante. Introducing an additional marginal cost in the form of reference dependence cannot motivate the agent to try catching up. Only when the sign of the marginal effect of reference dependence changes, as it is the case once one agent overtakes the other, this can be an equilibrium.

[16]Since the probability density function of the normal distribution is single peaked at $x = 0$.

5 Conclusion

Which factors motivate players to invest in contest success is still a vibrant topic of debate. While classical tournament theory as introduced by Lazear and Rosen (1981) focuses on the higher probability of winning as benefit and the unpleasantness of effort as a cost, a large recent literature indicates that players evaluate outcomes also along certain reference points. Such reference-dependent preferences are an economically powerful concept, as they can imply that an otherwise positive event causes negative utility if the reference category was even more positive and vice versa. As a result, theoretical predictions can change drastically once a model is augmented by reference dependence. In the context of tournaments, predicting a different winner could be considered such a change.

We add to the work of Gill and Stone (2010), who focus on symmetric equilibria when the half time score is even. For the large class of non even scores Gill and Stone (2010) show that symmetric equilibria do not exist. We find that depending on the strength of the reference dependence, the tournament prize and the initial unevenness three different classes of equilibria exist. In games where the initial unevenness is strongly favourable for one party we find a unique equilibrium, in which the leading player invests more effort than the player behind. However, when the game is tight and the tournament prize is large enough to motivate the lagging player to overcome the initial disadvantage, Turn Around Equilibria, where the player initially behind spends much more effort than the player ahead and has a higher probability of winning the tournament, always exist.

Our results can help to explain tournament outcomes that so far have been economically puzzling as presented by Berger and Pope (2011). Our results generate further testable predictions. We find that for all equilibria where the player behind spends more effort than the opponent, this player also has a greater chance of winning the tournament. Thus, we show that equilibria, in which the player behind merely catches up with the leading player do not exist. Furthermore, we can show that under certain conditions the TAE is the unique pure strategy equilibrium. While dynamic implications of this framework were only touched upon, future research adding a further optimisation period may provide interesting insights into how the anticipation of possible TAEs affects agents' incentives in an initial period.

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Appendix 1

In this appendix we prove all propositions and the lemmas stated in the main section. The proofs will make use of additional lemmas which are proven within the proposition where they are used first. Throughout we will assume without loss of generality that $\delta_1 > 0$, which implies that Player A is at an advantage. However, due to the symmetry of both players all results are also valid when $\delta_1 < 0$. All equilibria described assume that a solution to the tournament exists. As described by Lazear and Rosen (1981) this is always the case when precision of the random term given by $\frac{1}{\sigma}$ is sufficiently small.[17] The following proofs hold for $\sigma^2 \geq 1$.

Lemma 1. *An equilibrium is interior if* $\gamma < |\frac{1}{G(x)}|$. *Therefore all equilibria are interior whenever* $\gamma \leq 1$.

Proof. This follows directly from the first order conditions. Using $x = \Delta e + \delta$ and $\gamma = \eta\theta$ the first order conditions yield:

$$e^A = wf(x)(1 + \gamma G(x))$$
$$e^B = wf(x)(1 - \gamma G(x))$$

Since $wf(x)$ must be positive we will obtain interior solutions whenever $(1 + \gamma G(x))$ and $(1 - \gamma G(x))$ are also greater than zero. This implies that both conditions are fulfilled whenever $\gamma < |\frac{1}{G(x)}|$.

The term $G(x)$ will never be 0 for any equilibrium with $\delta_1 > 0$:
Suppose: $G(x) = 0 \Rightarrow 2F(x) - 1 = 0 \Leftrightarrow F(x) = \frac{1}{2} \Leftrightarrow 0 = x = \delta_1 + \Delta e$. From the first order conditions we know that in case of $x = 0$ $e^A = e^B = wf(0) \Rightarrow \Delta e = 0$. This leads to a contradiction with $\delta_1 > 0$. [18]

Since the function $|G(x)|$ is bounded above by one and open there, a simple corollary is that for $\gamma \leq 1$ the condition is fulfilled and the corresponding equilibrium must be interior. □

Lemma 2. *The system of first order conditions can be expressed as the candidate function* $\delta_1 = x - 2w\gamma f(x)G(x)$. *All combinations of* $\{e^A, e^B, \delta_1\}$ *which fulfil this equation are referred to as candidate points.*

Proof. Using $x = \Delta e + \delta$ and $\gamma = \eta\theta$ the first order conditions yield:

$$e^A = wf(x)(1 + \gamma(2F(x) - 1))$$

[17] See Lazear and Rosen (1981) p.845 for more information.
[18] This also reveals that there cannot exist symmetric equilibria with initial unevenness.

$$e^B = wf(x)(1 - \gamma(2F(x) - 1))$$

Subtracting both equations and substituting $G(x) = 2F(x) - 1$ leads to:

$$e^A - e^B = 2wf(x)\gamma G(x) \tag{1}$$

Since $x - \delta_1 = e^A - e^B$ we can reformulate the above expression as:

$$\delta_1 = x - 2wf(x)\gamma G(x)$$

The variable x describes an equilibrium uniquely whereas the exact corresponding effort combination can be revealed by inserting x back into the first order conditions. □

Lemma 3. *If x fulfils the maximum condition $0 < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2}$ then at the corresponding vector $\{e^A, e^B, \delta_1\}$ both second order conditions are fulfilled.*

Proof. The second order conditions for a local maximum are given by:

$$wf'(x) - wf'(x)\gamma + 2\gamma w\left[f'(x)F(x) + f(x)^2\right] - 1 < 0 \tag{2}$$

$$wf'(x)(-1) - wf'(x)\gamma + 2\gamma w\left[f'(x)F(x) + f(x)^2\right] - 1 < 0 \tag{3}$$

We use the following property of the normal distribution:

$$f'(x) = \frac{-x}{\sigma^2}f(x) \tag{4}$$

By substituting (4) into (2) and (3) we can derive new inequalities which include only the density and the distribution function of the normal distribution. Using that $G(x) = 2F(x) - 1$ we can solve for:

$$wf(x)\left\{2\gamma f(x) - \frac{x}{\sigma^2}\left[1 + \gamma G(x)\right]\right\} - 1 < 0$$

$$wf(x)\left\{2\gamma f(x) + \frac{x}{\sigma^2}\left[1 - \gamma G(x)\right]\right\} - 1 < 0$$

We will use the symmetry of the above two statements to condense their informational content into a single condition. Using $a = 2\gamma wf(x)^2$, $b = w\frac{x}{\sigma^2}f(x)$ and $c = \gamma G(x)$ we can reformulate the statements to:

$$a - b(1 + c) - 1 < 0$$

$$a + b(1 - c) - 1 < 0$$

which can be rewritten as:

$$-b < 1 - a + bc$$
$$b < 1 - a + bc$$

It is now clear that both conditions must be fulfilled whenever $|b| < 1 - a + bc$ holds. Substituting back we obtain $0 < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2}$.

□

5.1 Proof of Proposition 1:

Proposition 1.

For δ_1 large enough there always exists one Confirming Asymmetric Equilibrium (CAE) that is a unique equilibrium.

Proof. We first showed in Lemma 2 that we can rewrite the system of first order conditions to a simpler, but equivalent representation. Afterwards, using symmetry we derived a single bound from the second order conditions which will be necessary and sufficient to identify equilibria in Lemma 3.
We make use of the candidate function from Lemma 2 and the maximum condition derived in Lemma 3.

$$0 < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2}$$

$$\delta_1 = x - 2w\gamma f(x)G(x)$$

We know that $f(x)G(x) \to 0$ for $x \to \infty$ since $f(x) \to 0$ and $G(x) \to 1$. For this reason letting δ_1 go towards ∞ implies that $x \to \infty$.

As $x > 0$ we can simplify the maximum condition to:

$$1 < \frac{\sigma^2}{f(x)wx} - \frac{\sigma^2\gamma(2f(x) - \frac{x}{\sigma^2}G(x))}{x}$$

The second term on the RHS will converge to the constant γ as $x \to \infty$. The first term can be reformulated as

$$\frac{\sigma^2}{f(x)wx} = \frac{\sigma^3\sqrt{2\pi}e^{\frac{x^2}{2\sigma^2}}}{wx}$$

Following L'Hôpital's rule

$$\frac{\sigma\sqrt{2\pi}xe^{\frac{x^2}{2\sigma^2}}}{w*1} \to \infty \qquad \Rightarrow \qquad \frac{\sigma^2}{f(x)wx} \to \infty$$

So the maximum condition will be fulfilled for sufficiently large δ_1. It is not only unique in the class of asymmetric equilibria but for all equilibria as symmetric equilibria cannot exist for $\delta_1 \neq 0$ (see Proposition 4 in Gill and Stone (2010)).

□

5.2 Proof of Proposition 2:

To prove Proposition 2 we first show that under certain conditions candidate points in the sense of Lemma 2 exist that are potentially type one Turn Around Equilibria. We proceed by showing that the maximum condition function introduced in Lemma 3 is strictly convex over some interval.

Lemma 4. *For $w > \frac{1}{4f(0)^2\gamma}$, there always exist a positive δ_1 such that its corresponding extreme points include candidate Turn Around Equilibria (i.e. $x < 0$).*

Proof. We show that under the condition TAE candidates (i.e. points where both player's First Order Conditions are fulfilled s.t. $x < 0$) exist for small positive values of δ_1. The inverse of the candidate function Lemma 2 would yield the equilibrium candidates for each value of δ_1. Since it is not possible to express the inverse explicitly we show that the function produces a small positive $\delta_1(x)$ when given a small negative value for x as an argument. Note that for $x < 0$ the function $G(x) < 0$ as well.

$$\delta_1(x) = x - 2w\gamma f(x)G(x) \tag{5}$$

The derivative of this function with respect to x yields:

$$\frac{\partial\delta_1(x)}{\partial x} = 1 - 4w\gamma f(x)^2 + \frac{x}{\sigma^2}2w\gamma f(x)G(x)$$

When $x = 0$ and $w = \frac{1}{4f(0)^2\gamma}$ the above expression equals zero and is negative for any w larger than $\frac{1}{4f(0)^2\gamma}$. Given this negative slope at $x = 0$ the function must be positive for some small negative x.

□

Lemma 5. *The maximum condition function $\frac{1}{wf(x)} - \gamma(2f(x) - xG(x)) - \frac{|x|}{\sigma^2}$ is strictly convex for all $w \in \left[\frac{1}{4f(0)^2\gamma}, \frac{1}{f(0)^2\gamma}\right]$.*

Proof. The maximum condition $0 < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) + \frac{x}{\sigma^2}$ for $x < 0$ is convex if the second derivative is positive:

$$\frac{\sigma^2 + x^2}{2f(x)^2 w\sigma^2} + \left(3 - \frac{2x^2}{\sigma^2}\right)\gamma > 0 \tag{6}$$

To find the prize w for which this condition is always fulfilled we substitute $w = \frac{1}{f(0)^2\gamma * a}$ and obtain $\frac{af(0)^2}{2f(x)^2\sigma^2}(\sigma^2 + x^2) + (3 - 2\frac{x^2}{\sigma^2}) > 0$. Solving as an equality for a yields

$$a = \frac{4x^2 - 6\sigma^2}{\frac{f(0)^2}{f(x)^2}(\sigma^2 + x^2)} \tag{7}$$

We then find the maximum value for 7 using the following first order condition,

$$8\sigma^2 + x^2 - 2\frac{x^4}{\sigma^2} = 0$$

which is fulfilled whenever $x_{max} = -\frac{1}{2}\sigma\sqrt{1 + \sqrt{65}}$.[19] Then, at the maximum σ drops out and we obtain $a(x_{max}) = (9 - \sqrt{65})e^{-0.25(1+\sqrt{65})} \approx 0.97$. Consequently the second order condition must be fulfilled when $w < \frac{1}{f(0)^2\gamma}$.

□

Proposition 2.

i) If $w > \frac{1}{4\gamma f(0)^2}$ and $w < \frac{1}{2\gamma f(0)^2}$, a type one Turn Around Equilibrium (TAE1) always exists.

ii) TAE1s are always interior.

Proof. We showed in Lemma 4 that for certain values of w extreme point couples (for values of $\{e^A, e^B, \delta_1\}$) exist that could be type one Turn Around Equilibria (TAE1). Lemma 5 gives us the convexity of the maximum condition for certain values of w.

We will execute the proof of part *i)* by using Lemma 4 and by showing that given $w < \frac{1}{2\gamma f(0)^2}$ the maximum condition derived in Lemma 3 is fulfilled. From Lemma 3 we know that both second order conditions will be fulfilled whenever

$$0 < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2} \tag{8}$$

[19]The Second Order Condition at x_{max} is negative and yields $\frac{(7\sqrt{65}-65)8e^{-0.25(1+\sqrt{65})}}{5\sigma^2} \approx \frac{-1.42}{\sigma^2}$.

Since $w < \frac{1}{2\gamma f(0)^2}$ we know that

$$\frac{2\gamma f(0)^2}{f(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2} < \frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2}$$

Now suppose $x = 0$. We obtain:

$$2\gamma f(0) - 2\gamma f(0) + 0 - 0 = 0 < \frac{1}{wf(0)} - 2\gamma f(0)$$

Therefore we know that for all $w < \frac{1}{2\gamma f(0)^2}$ the maximum condition function $\frac{1}{wf(x)} - \gamma(2f(x) - \frac{x}{\sigma^2}G(x)) - \frac{|x|}{\sigma^2}$ will take up a value greater than zero when $x = 0$. Then, it follows by the continuity of the maximum condition function that for any such w there exist some ϵ close to zero such that $0 < \frac{1}{wf(\epsilon)} - \gamma(2f(\epsilon) - \frac{\epsilon}{\sigma^2}G(\epsilon)) - \frac{|\epsilon|}{\sigma^2}$.

To obtain a conservative estimate of an interval in which the TAE1s lie, we use the strict convexity of the maximum condition function shown in Lemma 5. Now we can derive the first order Taylor approximation around $x = 0$ for $x \leq 0$ which yields:

$$T_1(0) = \left(\frac{\sigma\sqrt{2\pi}}{w} - \frac{1}{\sigma\sqrt{2\pi}}\gamma\right) + \frac{x}{\sigma^2}$$

Given the positive slope and the convexity of the maximum condition function we know, that the intersection of the approximation with the abscissa will provide a conservative lower bound for the interval. The resulting interval of x-values in which TAE1s exist can be expressed as:

$$\left[\frac{(w\sigma\gamma - 2\sigma^3\pi)}{\sqrt{2\pi}w}, 0\right)$$

As $G(x) \to 0$ for $x \to 0$ all TAE1s close to zero are interior as stated in part *ii*).

□

5.3 Proof of Proposition 3:

To prove Proposition 3 we first show in Lemma 6 that there is only one convex interval for x over which candidate TAEs exist. We continue by showing in Lemma 7 that the candidate function and the maximum condition function have an intersection where $\delta_1 > 0$ or the maximum condition is always fulfilled. Then, we show in Lemma 8 that the maximum condition function cannot intersect the horizontal axis more than twice. Lastly, we establish in Lemma 9 that the maximum condition function may not have these two roots over an interval over which it is strictly greater than the candidate function.

Definition 6. *Intersection in positive/negative range*
We say that two function intersect in positive/negative range, when they return a positive/negative value at that intersection.

Lemma 6. *For* $w > \frac{1}{4f(0)^2\gamma}$*, the candidate function* $\delta_1(x) = x - 2w\gamma f(x)G(x)$ *has exactly one maximum on the domain* $x \in (-\infty, 0)$*. At this maximum the candidate function is positive. There exists some* $x^* < 0$ *such that* $\delta_1(x^*) = 0$*.*

Proof. We know from Lemma 4 that when $w > \frac{1}{4f(0)^2\gamma}$ Turn Around candidates with $x < 0$ and $\delta_1(x) > 0$ exist for some x close to zero. Moreover, it is easy to see that $\delta_1(x) \to -\infty$ when $x \to -\infty$ and that $\delta_1(0) = 0$. Since the candidate function is continuous there must be at least one maximum point for negative x. In the following we will show that there is only one. Consider the first and second derivative of the candidate function:

$$\frac{\partial \delta_1(x)}{\partial x} = 1 + \frac{2w\gamma x f(x)G(x)}{\sigma^2} - 4w\gamma f(x)^2 \tag{9}$$

$$\frac{\partial \delta_1(x)^2}{\partial^2 x} = \frac{8x\gamma w f(x)^2}{\sigma^2} + \frac{2w\gamma f(x)}{\sigma^2}(G(x) + x(2f(x) - \frac{xG(x)}{\sigma^2})) \tag{10}$$

Note that $|G(x)| < 0$ for $x < 0$ so that (10) is strictly negative and hence the first derivative is monotonously decreasing as long as $\frac{G(x)x}{\sigma^2} \leq 2f(x)$. This is fulfilled as long as $\frac{2f(x)\sigma^2}{G(x)} \leq x$ and $x < 0$. Inserting the boundary case $x = \frac{2f(x)\sigma^2}{G(x)}$ in (9) simplifies it to:

$$1 - 2w\gamma(-2f(x)^2 + 2f(x)^2) = 1 > 0$$

However when $x = 0$ equation (9) is smaller than zero if $w > \frac{1}{4f(0)^2\gamma}$. Thus the first derivative of the candidate function is below zero for $x = 0$ and greater than zero when $x = \frac{2f(x)\sigma^2}{G(x)}$ and it is monotonously decreasing over the interval $[\frac{2f(x)\sigma^2}{G(x)}, 0)$. Thus, the first derivative intersects the abscissa exactly once over that interval. Furthermore, when $x < \frac{2f(x)\sigma^2}{G(x)}$ condition (9) is always positive and therefore does not have another root for negative x. □

Lemma 7. *When* $w \in \left(\frac{1}{4\gamma f(0)^2}, \frac{\sigma}{\gamma f(x^s)^2 + \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}}\right)$,

$\gamma \in \left[0.54, -\frac{1}{G(x^s)}\right)$ *and* σ *large enough*

- *either the maximum condition function and the candidate function intersect and do so for $x < 0$ when $\delta_1 > 0$ only*

- *or in case of no intersection the maximum condition is fulfilled for all x where the candidate function has positive values.*

Proof. To derive the conditions for when the intersection is within positive range (as illustrated in Figure 3) we begin by setting both functions equal. The intersection point is endogenously described by $x^s = \frac{(2f(x^s)^2 w\gamma - 1 - 2f(x^s)^2 G(x^s) w^2 \gamma)\sigma^2}{f(x^s)w(1+G(x^s)\gamma - \sigma^2)}$ and is used as an argument for the maximum condition which, then, yields $0 < -\frac{2f(x^s)^2 w\gamma(G(x^s)w + G(x^s)^2 w\gamma - \sigma^2) + \sigma^2}{f(x^s)w(1+G(x^s)\gamma - \sigma^2)}$. Using $\sigma^2 \geq 1$ we can derive the following condition. The latter expression is larger than zero whenever either of the following hold:

$$w < \frac{1}{\gamma f(x^s)^2 - \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}} \tag{11}$$

$$w < \frac{1}{\gamma f(x^s)^2 + \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}} \tag{12}$$

To ensure that the equilibrium is interior we assume $\gamma < -\frac{1}{G(x)}$. When $\gamma < -\frac{1}{G(x)}$, (11) is always negative and is therefore neglected. Instead we employ (12) as an upper bound. To ensure that the lower bound $w > \frac{1}{4\gamma f(0)^2}$ is below (12) another restriction for γ is required which is obtained by solving the following for γ:

$$\frac{1}{4\gamma f(0)^2} < \frac{1}{\gamma f(x^s)^2 + \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}}$$

This can be rearranged as condition for γ:

$$\gamma > -\frac{f(x^s)^2 G(x^s)\pi^2}{\frac{2}{\sigma^2} + f(x^s)^2\pi(-2 + G(x^s)^2\pi)} \tag{13}$$

This expression appears to be complicated and restrictive. However, it can be simplified at little cost in terms of accuracy. Using that $G(x) < 0$ for negative x and that $f(x)^2 < \frac{1}{2\pi\sigma^2}$, one can quickly see that the denominator will always be larger than one. The numerator contains $G(x)$ which equals $2F(x) - 1 = Erf(\frac{x}{\sqrt{2}\sigma}) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{2}\sigma}} e^{-t^2}\mathrm{d}t$. It must hold that the actual area underneath the integrated function is smaller than the area of the rectangle formed by the global maximum of the function over the x-interval. The largest value e^{-t^2} may assume is one. Thus, it holds for negative x that $-G(x) = -Erf(\frac{x}{\sqrt{2}\sigma}) \leq -\frac{2}{\sqrt{2\pi}\sigma} x * 1$. For the entire numerator this implies that

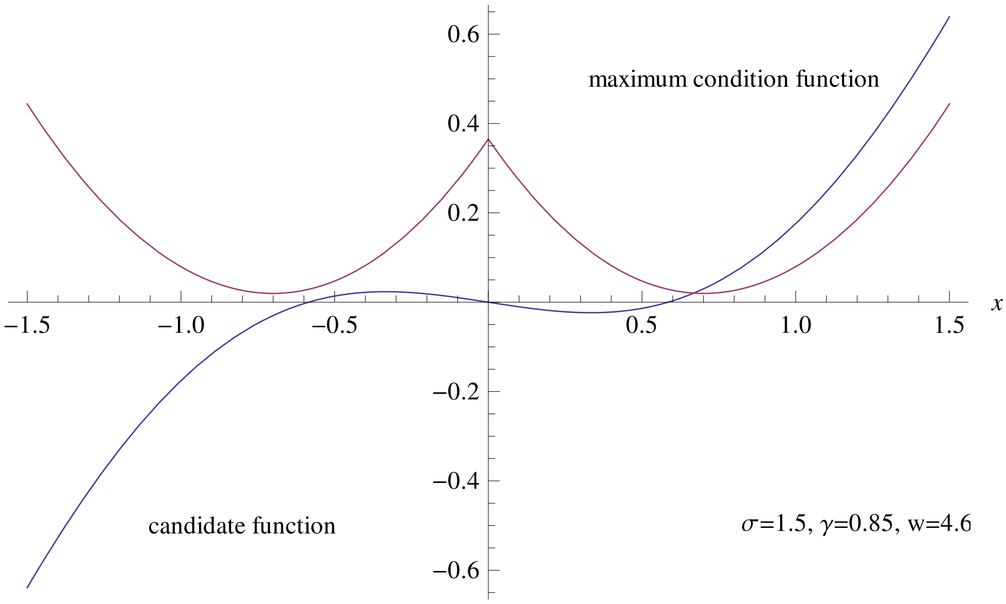

Figure 6: maximum condition and candidate function having no intersection points in the negative domain

$-f(x)^2G(x)\pi^2 \leq -\frac{\sqrt{\pi}}{\sqrt{2}\sigma}xe^{-\frac{x^2}{\sigma^2}}$, the maximum of which is at $x = -\frac{\sigma}{\sqrt{2}}$. Hence, the numerator will not exceed $\frac{\sqrt{\pi}}{\sqrt{4}}e^{-\frac{1}{2}} \approx 0.53$ and whenever $\gamma \geq 0.54$, condition (13) will also be satisfied. When both conditions are fulfilled any intersection between the maximum condition and the candidate function occurs in positive range.

Suppose no intersection between the candidate function and the maximum condition function and hence no x^s exists (as illustrated in Figure 6). For sufficiently small x we know that the maximum condition function is always positive while the candidate function is strictly negative. Without an intersection the continuity of both functions implies that the maximum condition function lies above the candidate function for all $x < 0$. However, when $w > \frac{1}{4f(0)^2\gamma}$ it is known from Lemma 4 that there are always values for $x < 0$ where the candidate function is positive. Since the maximum condition function must return greater values than the candidate function it also must be positive.

□

Lemma 8. *The maximum condition function has no more than two roots when $x < 0$ and $w < \frac{1}{2f(x)^2\gamma}$.*

Proof. Setting the maximum condition function equal to zero and solving for x yields

$x^R = \frac{\sigma^2(2w\gamma f(x)^2-1)}{wf(x)(1+\gamma G(x))} = x^R(x)$. This equation must be fullfilled for every root of the maximum condition function. We show that the maximum condition function has at most two roots by showing that this equation has at most two solutions for $x < 0$. For this we demonstrate in the remainder of the proof that the function $x^R(x) = \frac{\sigma^2(2w\gamma f(x)^2-1)}{wf(x)(1+\gamma G(x))}$ is strictly concave and can, thus, have at most one maximum. To understand why this implies the statement in the lemma, consider the following: We want to know for how many x the equation $x^R = x^R(x)$ can be fulfilled. We also know that x^R (the left-hand-side of the equation) is a straight line with slope one. If we now knew that $x^R(x)$ was strictly concave, we would know that it cannot intersect the straight line x^R more than twice (and hence that the maximum condition function may not have more than two roots). Thus, in the remainder of the proof we show that the second derivative of $x^R(x)$ ist strictly smaller than zero for $x < 0$. The second derivative of $x^R(x)$ is given by:[20]

$$\frac{\partial x^R(x)^2}{\partial^2 x} = \frac{8\gamma^2\sigma^2 f(x)(2w\gamma f(x)^2-1) + 2x\gamma(1+\gamma G(x))(6w\gamma f(x)^2+1)}{w(1+\gamma G(x))^3} - \frac{(\frac{1}{f(x)}(1+2w\gamma f(x)^2) + \frac{x^2}{\sigma^2 f(x)}(1-2w\gamma f(x)^2))}{w(1+\gamma G(x))} < 0$$

We now show that the term above is strictly negative. For this it suffices to look at the numerator of both fractions as the denominators are strictly positive under the assumption of Lemma 1 that $\gamma < |\frac{1}{G(x)}|$. The numerator of the first fraction is a sum of two elements. The first element must be negative, since $(2w\gamma f(x)^2 - 1) < 0$ when $w < \frac{1}{2f(x)^2\gamma}$. The second element is all positive except for the x which is taken to be smaller than zero. Thus, we know that the first fraction is negative. The second fraction, which gets substracted, is positive. It is also composed of two elements, the first of which is unambiguously positive while the second is positive as long as $w < \frac{1}{2f(x)^2\gamma}$. In consequence, the second derivative of $x^R(x)$ is strictly smaller than zero.

Therefore, the equation $x^R(x) = \frac{\sigma^2(2w\gamma f(x^R)^2-1)}{wf(x^R)(1+\gamma G(x^R))}$ has at most two solutions and the maximum condition function has at most two roots. □

Lemma 9. *The maximum condition function cannot have two roots within an interval over which it is strictly larger than the candidate function for* $w < \frac{1}{2f(x)^2\gamma}$.

Proof. Consider again the root of the maximum condition function as given by $x^R(x) = \frac{\sigma^2(2w\gamma f(x^R)^2-1)}{wf(x^R)(1+\gamma G(x^R))}$. We will show that its first derivative is strictly positive if the maximum condition function lies above the candidate function. The latter is true whenever:

[20]The first derivative is given by $\frac{\partial x^R(x)}{\partial x} = \frac{-x(1+\gamma G(x))(2wf(x)\gamma + \frac{1}{f(x)}) - 4w\sigma^2\gamma^2 f(x)^2 + 2\gamma\sigma^2}{w(1+\gamma G(x))^2}$

$$\frac{1}{wf(x)} - \gamma\left(2f(x) - \frac{x}{\sigma^2}G(x)\right) + \frac{x}{\sigma^2} > x - 2w\gamma f(x)G(x)$$

which can be rewritten as an upper bound for w:

$$w < \frac{\sigma^2}{f(x)^2\gamma\sigma^2 - \frac{f(x)x((1+G(x)\gamma)-\sigma^2)}{2} + \frac{\sqrt{f(x)^2(-8G(x)\gamma\sigma^4+(x+G(x)x\gamma-(x+2f(x)\gamma)\sigma^2)^2)}}{2}} = \tilde{w} \tag{14}$$

Now consider the first derivative of the root function $x^R(x)$:

$$\frac{\partial x^R(x)}{\partial x} = \frac{-x(1+\gamma G(x))(2wf(x)\gamma + \frac{1}{f(x)}) - 4w\sigma^2\gamma^2 f(x)^2 + 2\gamma\sigma^2}{w(1+\gamma G(x))^2} \tag{15}$$

As the denominator is positive it remains to show that the numerator is strictly positive. We start by rewriting the term to the following inequality:

$$\sigma^2(1 - 2w\gamma f(x)^2) - \frac{x}{2f(x)\gamma}(1+\gamma G(x)) - wxf(x)(1+\gamma G(x)) > 0 \tag{16}$$

The last two subtrahends of the numerator are negative for all $x < 0$ whereas the first summand is positive in case $w < \frac{1}{2\gamma f(x)^2}$. Thus, if condition (14) implies $w < \frac{1}{2\gamma f(x)^2}$, the lemma must be true. Consequently, we verify in the following that $w < \frac{1}{2\gamma f(x)^2}$ holds if the maximum condition function is bigger than the candidate function.

We begin by considering the large term under the root in the denominator of $\tilde{w}$ in condition (14):

$$\sqrt{f(x)^2(-8G(x)\gamma\sigma^4 + (x + G(x)x\gamma - (x + 2f(x)\gamma)\sigma^2)^2)} =$$
$$\sqrt{f(x)^2(4f(x)^2\gamma^2\sigma^4 + x^2((1+\gamma G(x)) - \sigma^2)^2 + C)}$$

Firstly, we show that the term $C = -8G(x)\gamma\sigma^4 - 4f(x)\gamma\sigma^2 x((1 + G(x)\gamma) - \sigma^2)$ is positive.

$$0 < -4(2G(x)\gamma\sigma^4 + f(x)\gamma\sigma^2 x((1 + G(x)\gamma) - \sigma^2))$$
$$\Leftrightarrow \quad 0 < -4\gamma\sigma^2(\sigma^2(2G(x) - f(x)x) + f(x)x(1 + \gamma G(x)))$$
$$\Leftrightarrow \quad 0 > \sigma^2(2G(x) - f(x)x) + f(x)x(1 + \gamma G(x))$$

It is easy to verify that $(2G(x) - f(x)x)$ is strictly negative[21] for all $x < 0$. Since the term $(1+\gamma G(x))$ is positive by the assumptions on γ, the statement above must be true

[21] Its derivative $f(x)(3 + \frac{x^2}{\sigma^2})$ is strictly positive. Moreover it is zero when $x = 0$ and approaches -2 when $x \to -\infty$.

and C is indeed positive. Having established that C is positive we can now overestimate $\tilde{w}$ by dropping C. Thus,

$$\tilde{w} < \frac{1}{f(x)^2\gamma\sigma^2 - \frac{f(x)x((1+G(x)\gamma)-\sigma^2)}{2} + \frac{\sqrt{f(x)^2[x((1+G(x)\gamma)-\sigma^2)+2f(x)\gamma\sigma^2]^2}}{2}}$$

which can be simplified to:

$$\tilde{w} < \frac{1}{f(x)^2\gamma\sigma^2 - \frac{f(x)x((1+G(x)\gamma)-\sigma^2)}{2} + \frac{2f(x)^2\gamma\sigma^2+f(x)x((1+G(x)\gamma)-\sigma^2)}{2}} = \frac{1}{2f(x)^2\gamma\sigma^2}$$

Hence, $\tilde{w} < \frac{1}{2\gamma f(x)^2}$ is true as well. Consequently, (16) holds for all $x < 0$ where the maximum condition function lies above the candidate function. □

Proposition 3.

i) When $w \in \left(\frac{1}{4\gamma f(0)^2}, \frac{1}{2\gamma f(x^s)^2 - B}\right)$ *where* $\gamma \in \left[0.54, -\frac{1}{G(x^s)}\right)$, σ *sufficiently large and* $B = \sqrt{\gamma f(x^s)^2 \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2}} \leq 0$, *a type two Turn Around Equilibrium (TAE2) in which the agent behind spends much more effort than the agent ahead exists. The parameter* x^s *determines the intersection between candidate function and maximum condition function exogenously.*

ii) If there exist TAE2s there also exist Confirming Asymmetric Equilibria (CAEs) for small δ_1.

iii) If the maximum condition function and the candidate function intersect, but there are no TAE2s, also no CAEs for small δ_1 *exist.*

Proof. i) To establish the existence of TAE2s, i.e. TAEs over an x-interval which is not necessarily adjacent to zero, one needs to show that over such an interval and under some conditions both the candidate function and maximum condition function return positive values. In Lemma 6 it was established that the candidate function has exactly one maximum and no other extreme points over the domain of strictly negative x. We also know from Lemma 4 that when $w > \frac{1}{4\gamma f(0)^2}$ the candidate function always returns positive values over the interval given by the roots of candidate function $(x^*, 0)$ where $x^* = 2w\gamma f(x^*)G(x^*)$. Lemma 7 implies that when candidate and maximum condition function do not intersect for $x < 0$ the maximum condition derived in Lemma 3 is fulfilled for all x where the candidate function is positive. Especially at the left root of the candidate function this leads to TAE2s that are rather 'far away' from $x = 0$. Additionally, given its conditions Lemma 7 implies that if intersections between the candidate and the maximum condition function exist for some $x < 0$, then both the maximum condition and the candidate function are positive at the intersection as illustrated in figure 4. It follows that around this intersection TAE2s exist.

Before we continue with part *ii*) we show that Lemma 8 and Lemma 9 also hold for the upper bound of w given in Proposition 3. We will proceed the proof for the general case using x so it will also hold for a specific x^s. We want to show that $\frac{1}{2f(x)^2\gamma} > \frac{1}{2f(x)^2\gamma - B}$ which holds if B is strictly negative. If this holds the lower bound used in Proposition 3 is always smaller than the one used in the two lemmas meaning they also hold for this proposition. The denominator of the upper bound for w can be reformulated as follows:

$$\gamma f(x)^2 + \sqrt{\gamma f(x)^2(\gamma f(x)^2 - \frac{2G(x)(1+\gamma G(x))}{\sigma^2})}$$

$$= \gamma f(x)^2 + \sqrt{\gamma^2 f(x)^4 - \frac{2\gamma f(x)^2 G(x)(1+G(x)\gamma)}{\sigma^2}} = \gamma f(x)^2 + \sqrt{\gamma^2 f(x)^4 - A)}$$

$$\geq \gamma f(x)^2 - \sqrt{A}) = 2\gamma f(x)^2 - B$$

The last step establishes by using Jensen's inequality. Solving for B leads to:

$$B \leq \gamma f(x)^2 - \sqrt{\gamma^2 f(x)^4 - A} = \sqrt{\gamma^2 f(x)^4 + A} - \sqrt{\gamma^2 f(x)^4} < 0$$

This holds as A is negative for $\gamma \leq |\frac{1}{G(x)}|$.

Since $B < 0$ it must be that $\frac{1}{2f(x)^2\gamma} > \frac{1}{2f(x)^2\gamma - B}$. This also holds for the case that $x = x^s$ so:

$$\frac{1}{2f(x)^2\gamma} > \frac{1}{\gamma f(x^s)^2 + \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}}$$

ii) To study the relationship between strong TAEs and CAEs for small δ_1 we make use of the symmetry property of the candidate function $\delta_1(x)$ as well as the maximum condition function $maxcond(x)$:

$$\delta_1(x) = -\delta_1(-x)$$

$$maxcond(x) = maxcond(-x)$$

Having CAEs means that $\delta_1(x) > 0$ for $x > 0$ and the $maxcond(x) > 0$. Using the symmetry this is equivalent to $\delta_1(-x) < 0$ while $maxcond(-x) < 0$.

From Lemma 6 we know that the candidate function has only one maximum on the negative domain and in Proposition 1 we derived that the candidate function approaches infinity if $x \to \infty$. By symmetry this implies that the candidate function goes towards minus infinity if $x \to -\infty$. Since $\delta_1(0) = 0$ it follows from continuity that candidate CAEs (not necessarily CAEs) exist for all values of δ_1.

To find CAEs we have to insure that the maximum condition is fulfilled. We use the previously derived lemmas to make a statement about the maximum condition for all $x < x_\delta$, where x_δ is the negative root of the candidate function, and, then, use the symmetry properties from above to apply it to the candidate CAEs. We know that the maximum condition as derived in Lemma 3 goes to infinity when $x \to -\infty$ and since we have shown the existence of TAE2s in part i) there also exist some $x < 0$ where maximum condition and candidate function are both positive.

In consequence for the maximum condition to become negative over $x < 0$ it has to have at least two roots on the same domain. Moreover, we know from Lemma 8 that the maximum condition function cannot not have more than 2 roots for $x < 0$.

One possibility would be that the maximum condition function could have one root below x_δ and one above. In this case there would be a negative intersection of the maximum condition function with the candidate function as the candidate function must be negative for $x < 2w\gamma f(x)G(x)$ by Lemma 4. Since our conditions for TAE2s ensure that all intersection points are positive for $x < 0$ this case can be excluded. Secondly, the roots of the maximum condition function could both be below x_δ. This, however, directly contradicts Lemma 9 as the maximum condition has to be bigger than the candidate function at x_δ. Otherwise this would be equivalent to the previous example. Using symmetry this implies the existence of CAEs for small δ_1.

Lastly we address part iii). Following the same argument we know that in cases where no TAE2s exist, but the candidate and maximum condition functions still intersect, the intersection point must lie in negative range. Since the candidate function is negative for sufficiently small x, we know from Lemma 6 that the candidate function will not have an intersection with the abscissa for $x < 0$. This implies, that, because of the symmetry property of the candidate function, CAEs for small δ_1 do also not exist.

□

5.4 Proof of Proposition 4:

Proposition 4. *When* $\frac{1}{4f(0)^2\gamma} < w < \frac{1}{2f(0)^2\gamma}$ *and* $\gamma \leq -\frac{f(x^s)^2G(x^s)\pi^2}{\frac{2}{\sigma^2}+f(x^s)^2\pi(-2+G(x^s)^2\pi)}$ *then for small unevenness the unique equilibrium in pure strategies is a type one Turn Around Equilibrium (TAE1), where* x^s *exogenously determines the intersection between candidate function and maximum condition function.*

Proof. We know from Proposition 3 and Lemma 7 that TAE2s only exist when:

$$\gamma > -\frac{f(x^s)^2G(x^s)\pi^2}{\frac{2}{\sigma^2}+f(x^s)^2\pi(-2+G(x^s)^2\pi)} \tag{17}$$

Moreover, we know from Proposition 3 that for δ_1 small enough CAEs only exist if

TAE2s exist as well. TAE1s as described in Proposition 2 on the other hand, always exist when

$$\frac{1}{4f(0)^2\gamma} < w < \frac{1}{2f(0)^2\gamma}$$

Since the lower bound for γ (17) is strictly positive and lower and upper bound for w cannot intersect, we know that when the condition for γ is not satisfied there is yet a prize level w for which a TAE1 exists and is the only equilibrium for small enough δ_1.

□

5.5 Proof of Proposition 5:

Proposition 5. *Every Catching Up Equilibrium (CUE) is also a Turn Around Equilibrium (TAE).*

Proof. Remember that x was defined as $x = \Delta e + \delta_1$. Suppose again without loss of generality that player 1 is initially ahead, i.e. $\delta_1 > 0$, and that at the CUE player 2 spends more effort than player 1 with $\Delta e < 0$, but not enough to turn the game, i.e. $\Delta e + \delta_1 > 0$. From Lemma 2 we know that the candidate function provides all possible equilibrium candidate points:

$$\delta_1(x) = x - 2f(x)w\gamma G(x)$$

To find a CUE that is no TAE we need to show that there exist candidate points where for $x > 0$ and $\Delta e = x - \delta_1 < 0$. We show that this can never be the case:

$$x - \delta_1 = 2f(x)w\gamma G(x)$$

For $x > 0$ the RHS cannot be negative since $G(x)$ is positive for all $x > 0$ and the other terms are always positive. So $x - \delta_1$ will be positive for all $x > 0$.

□

Appendix 2

6 Additional Pages for the Referee

We provide these pages as an additional aid for the verification of some expressions.

6.1 Derivation of (11) and (12) in Lemma 7

We want to find a condition for w that ensures that the intesection between the candidate and the maximum condition function occurs in positive range. We show in Lemma 7 that this must be the case when:

$$0 < \frac{-2f(x^s)^2 w\gamma(G(x^s)w + G(x^s)^2 w\gamma w - \sigma^2) + \sigma^2}{f(x^s)w(1 + G(x^s)\gamma - \sigma^2)}$$

Note that as $G(x^s)\gamma < 0$ and $\sigma^2 \geq 1$ the denominator is smaller zero. Collecting the w terms and multiplying with the negative denominator yields:

$$0 > w^2(-2f(x^s)^2\gamma G(x^s)(1 + G(x^s)\gamma)) + 2f(x^s)^2 w\gamma\sigma^2 - \sigma^2$$

Next, we solve the above inequality as a quadratic equation for w. This gives:

$$w = \frac{\sigma^2(2\gamma f(x^s)^2 \pm \sqrt{f(x^s)^2\gamma(f(x^s)^2\gamma - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}}{2f(x^s)^2\gamma G(x^s)(1 + \gamma G(x^s))} = \frac{\sigma^2(A \pm \sqrt{B})}{C}$$

We now get to (11) and (12) by recognising that $C = \sigma^2(A + \sqrt{B})(A - \sqrt{B})$. Thus

$$w = \frac{\sigma^2(A \pm \sqrt{B})}{\sigma^2(A + \sqrt{B})(A - \sqrt{B})} = \frac{1}{(A - \sqrt{B})} \text{ or } \frac{1}{(A + \sqrt{B})}$$

The first possible solution is equivalent to (11), the second to (12) in Lemma 7.

6.2 Derivation of (13) in Lemma 7

We want to derive the lower bound for γ given in (13). Starting with the inequality

$$\frac{1}{4\gamma f(0)^2} = \frac{\pi\sigma^2}{2\gamma} < \frac{1}{\gamma f(x^s)^2 + \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})}}$$

first rearranging leads to

$$\gamma f(x^s)^2 + \sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})} < \frac{2\gamma}{\pi\sigma^2}$$

Suqaring the next rearrangement

$$\sqrt{\gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2})} < \frac{2\gamma}{\pi\sigma^2} - \gamma f(x^s)^2$$

gives us

$$\begin{aligned}
& \gamma f(x^s)^2(\gamma f(x^s)^2 - \frac{2G(x^s)(1+\gamma G(x^s))}{\sigma^2}) < \gamma^2(\frac{2}{\pi\sigma^2} - f(x^s)^2)^2 \\
\Leftrightarrow \quad & f(x^s)^4 - \frac{2G(x^s)f(x^s)^2}{\sigma^2\gamma} - \frac{2f(x^s)^2G(x^s)^2}{\sigma^2} < (\frac{2}{\pi\sigma^2} - f(x^s)^2)^2 \\
\Leftrightarrow \quad & -\frac{2G(x^s)f(x^s)^2}{\sigma^2\gamma} < (\frac{2}{\pi\sigma^2} - f(x^s)^2)^2 - f(x^s)^4 + \frac{2G(x^s)f(x^s)^2}{\sigma^2\gamma}
\end{aligned}$$

In the next step, we need to solve for γ.

$$\frac{2G(x^s)f(x^s)^2}{\sigma^2((\frac{2}{\pi\sigma^2} - f(x^s)^2)^2 - f(x^s)^4 + \frac{2f(x^s)^2G(x^s)^2}{\sigma^2})} < \gamma$$

Simplifying the the LHS reveals (13).

$$-\frac{2G(x^s)f(x^s)^2}{\sigma^2(\frac{4}{\pi^2\sigma^4} - \frac{4f(x^s)^4}{\pi\sigma^2} + f(x^s)^4 - f(x^s)^4 + \frac{2f(x^s)^2G(x^s)^2}{\sigma^2})} = -\frac{f(x^s)^2G(x^s)\pi^2}{\frac{2}{\sigma^2} + f(x^s)^2\pi(-2 + G(x^s)^2\pi)} < \gamma$$

6.3 Derivation of (14) in Lemma 9

$$\frac{1}{wf(x)} - \gamma\left(2f(x) - \frac{x}{\sigma^2}G(x)\right) + \frac{x}{\sigma^2} > x - 2w\gamma f(x)G(x)$$

can be rewritten as:

$$w < \frac{\sigma^2}{f(x)^2\gamma\sigma^2 - \frac{f(x)x((1+G(x)\gamma)-\sigma^2)}{2} + \frac{\sqrt{f(x)^2(-8G(x)\gamma\sigma^4 + (x+G(x)x\gamma-(x+2f(x)\gamma)\sigma^2)^2)}}{2}} = \tilde{w}.$$

We begin by bringing all terms to the left side and multiplying by $wf(x)$:

$$w^2 2\gamma f(x)^2 G(x) + wf(x)(x(\frac{1}{\sigma^2} - 1) - \gamma(2f(x) - \frac{x}{\sigma^2}G(x))) + 1 > 0$$

Collecting all x together and multiplying by σ^2 we obtain:

$$w^2 2\gamma f(x)^2 G(x)\sigma^2 + wf(x)(x((1+\gamma G(x)) - \sigma^2) - 2\gamma f(x)\sigma^2) + \sigma^2 > 0$$

To solve for w we now use the quadratic formula for the equality $w = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where $a = 2\gamma f(x)^2 G(x)\sigma^2$, $b = f(x)(x((1+\gamma G(x)) - \sigma^2) - 2\gamma f(x)\sigma^2)$ and $c = \sigma^2$. Plugging in a, b and c yields:

$$\begin{aligned} w =& \frac{f(x)(x(\sigma^2 - (1+\gamma G(x))) + 2\gamma f(x)\sigma^2)}{4\gamma\sigma^2 f(x)^2 G(x)} \\ & \frac{\pm\sqrt{(f(x)(x((1+\gamma G(x)) - \sigma^2) - 2\gamma f(x)\sigma^2))^2 - 8\gamma\sigma^4 f(x)^2 G(x)}}{4\gamma\sigma^2 f(x)^2 G(x)} \end{aligned}$$

To deal with this big term we now temporarily express it as $\frac{s-\sqrt{t}}{4\gamma\sigma^2 f(x)^2 G(x)}$. We neglect the postive root, as we are looking for a conservative upper bound. The crucial step to obain (14) is to realise that the denominator $4\gamma\sigma^2 f(x)^2 G(x)$ can be rewritten as $\frac{s^2-t}{2\sigma^2}$, which we now show:

$$\begin{aligned} s^2 - t =& (f(x)(x((1+\gamma G(x)) - \sigma^2) - 2\gamma f(x)\sigma^2))^2 \\ & - (f(x)(x((1+\gamma G(x)) - \sigma^2) - 2\gamma f(x)\sigma^2))^2 + 8\gamma\sigma^4 f(x)^2 G(x) \\ =& 8\gamma\sigma^4 f(x)^2 G(x) \end{aligned}$$

Thus $\frac{s^2-t}{2\sigma^2} = 4\gamma\sigma^2 f(x)^2 G(x)$. We can now say $\frac{2\sigma^2(s-\sqrt{t})}{s^2-t} = \frac{2\sigma^2(s-\sqrt{t})}{(s-\sqrt{t})(s+\sqrt{t})} = \frac{2\sigma^2}{s+\sqrt{t}}$ which is equal to:

$$\frac{\sigma^2}{f(x)^2\gamma\sigma^2 - \frac{f(x)x((1+G(x)\gamma)-\sigma^2)}{2} + \frac{\sqrt{f(x)^2(-8G(x)\gamma\sigma^4 + (x+G(x)x\gamma-(x+2f(x)\gamma)\sigma^2)^2)}}{2}}$$