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On the Solution of Markov-switching Rational Expectations Models*

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Abstract

This paper describes a method for solving a class of forward-looking Markov-switching Rational Expectations models under noisy measurement, by specifying the unobservable expectations component as a general-measurable function of the observable states of the system, to be determined optimally via stochastic control and filtering theory. Solution existence is proved by setting this function to the regime-dependent feedback control minimizing the mean-square deviation of the equilibrium path from the corresponding perfect-foresight autoregressive Markov jump state motion. As the exact expression of the conditional (rational) expectations term is derived both in finite and infinite horizon model formulations, no (asymptotic) stationarity assumptions are needed to solve forward the system, for only initial values knowledge is required. A simple sufficient condition for the mean-square stability of the obtained rational expectations equilibrium is also provided.

Keywords: Rational Expectations; Markov-switching dynamic systems; Dynamic programming; Time-varying Kalman filter

JEL Classification: C5; C61; C62; C63

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1 Introduction

A number of recent studies have made important progress toward connecting the reduced form econometric literature on regime switching autoregressive processes, which can be traced back to [19], with structural economic theory, by developing the notion of Markov-switching Rational Expectations (MSRE) models, that is dynamic forward-looking stochastic frameworks in which the parameters governing the behaviour of the system are taken to be functions of a discrete-state Markov chain.

Since able to account for parameter instability and yield quantitatively different responses of macroeconomic variables to fundamental shocks from those implied by fixed regime models, MSRE systems have recently been advocated to investigate the role of regime switching monetary policy in New-Keynesian frameworks (e.g. [12]) or rather to gauge the effects of uncertainty over structural parameters governing the optimal behavior of rational agents (e.g. [23]).

From a technical viewpoint, regime dependency engenders structural nonlinearities which prevent from employing standard solution tools for linear RE systems, such as [5]'s, [20]'s and [30]'s. In this respect, a number of authors have been interested in deriving determinacy (local uniqueness) conditions for RE equilibria to MSRE models. In their seminal contribution to the generalization of the Taylor principle, [12] study how regime switching alters determinacy properties of RE solution and provide analytical restrictions on monetary policy behavior to ensure local uniqueness of the equilibrium path. By focusing on bounded solutions, [12] find out that, while accounting for structural shifts noticeably enlarges the determinacy region relative to the constant parameter setup, regimes that fail to fulfil the generalized Taylor principle may well be characterized by improved time series properties as reaction to fundamental shocks, even when sunspot noise or nonfundamental uncertainty are ruled out. The nonlinearity problem is addressed by introducing a two-step solution method that consists in studying an augmented system which is linear in fictitious variables, the latter coinciding with the actual ones in some of the regimes, and then using the solution to the linear representation in order to construct solutions for the original nonlinear system.

In a more general perspective, [16] and [17] have provided a series of characterization results for the set of minimal state variable (MSV) solutions as well as the full set of RE equilibria - also sunspot ones - to MSRE frameworks, which satisfy a suitable stability concept. Their approach rests on expanding the state-space of the underlying stochastic system and to focus on an equivalent model in the expanded space that features state-invariant parameters. Furthermore, [17] demonstrate an equivalence property between determinacy for MSRE models and mean-square stability in a class of Markov jump autoregressive systems.

The aim of this paper is to extend the model reference adaptive technique developed in [7] to solution of dynamic MSRE models in state-space form, with past expectations and noisy measurement on the state vector. In the linear (time-varying) setup, [7] show that, for an important class of purely forward-looking RE systems, a solution can be obtained via a *causal* (controllable) system forced by a general-measurable function of the available information, estimated via a Kalman filter technique. More specifically, an exact solution of the RE system is determined by forcing it with the optimal minimum variance estimate of the future state, recursively computed on the autoregressive equation describing the perfect-foresight dynamics of the economy.

In this work, we define nearly perfect-foresight equilibrium dynamics in a class of MSRE models under noisy measurement, as the outcome of a Kalman filter-based mechanism of information processing and optimal state estimation. This approach, which can be traced back to [3], complies with a broader definition of rationally formed expectations, comparable to some form of dynamic optimizing behaviour, which does not require - as typically done in the RE macroeconomic literature - that economic agents possess a priori knowledge of the structure of the model's solution itself. Indeed, we depart from RE and specify the unobservable expectations component as a (square-integrable) random function adapted to the filtration - the actually available information - generated by the Markov switching system itself.

Formally, the novelty of our solution method is based on using dynamic programming arguments and optimal filtering techniques applied to a causal Markov jump system, where RE are replaced by a specific control sequence, the latter being measurable with respect to the observable states of the system. Such input is chosen optimally as the feedback regime-dependent control law minimizing the mean-square deviation of the equilibrium path from the corresponding autoregressive Markov jump state motion (the reference model). We present a recursive algorithm, based upon optimal stochastic control and Kalman filtering theory, for the design of the control law and show that the latter has the same structure of the conditional expectations operator featuring in the canonical (possibly regime switching) RE systems.

It is well known that the dimension of the solution set for RE models is closely related the stability properties of the latter, and that stability restrictions can be advocated in order to weaken the multiplicity issue (e.g. [5], [28]). However, as the agents' expectations in RE frameworks are typically obtained by recursively iterating the system into the future, (asymptotic) stationarity is needed for this process to be well-defined (e.g. [17]). While equilibrium stability is usually enforced by the existence of transversality conditions in the underlying (infinite horizon) dynamic economic frameworks, there exist models for which no such boundary conditions arise or rather, though present, they do not serve as necessary optimality requirements (e.g. [18]; [14]). In this respect, we emphasize that, by providing a readily computable expression of the RE component both in finite and infinite horizon model representations, our method need not invoke approximation hypotheses or stability concepts to solve forward the system, for only initial conditions knowledge is required. We also provide an easy-to-check sufficient condition for the mean-square stability of the obtained RE equilibrium.

While concerned with computational issues in MSRE models, our analysis also relates to studies on the process of expectation formation. Previous work on this subject differs from the present one in that it generally focuses on learning behavior, that is the way systematic forecasting biases are eliminated over time (e.g. [25], [15]). Specifically, the adaptive learning literature endows boundedly rational agents with a forecasting model - the perceived law of motion of the economy - which can be an arbitrary function of past endogenous and past and current exogenous variables, and has to be optimally parameterized based on new data and observable (past) forecast errors. RE equilibria are thus regarded as asymptotic outcomes of this learning process, whenever conditions for convergence of agents' beliefs to the equilibrium values hold. Though methodologically related, our method also differs from the Bayesian learning literature (e.g. [26], [6]), as these studies typically assume that agents employ filtering techniques to update estimates of (possibly time-varying) parameters within not fully rational forecasting functions. Rather, our approach posits that rational agents may be thought of as revising their (best) estimate of the (hidden) variables governing the dynamics of the economic system as new observations are generated, when only a reduced information set - the measurement process - is available to them.

The paper is organized as follows. In Section 2 the class of stochastic MSRE models we deal with is introduced. In Section 3, we develop the solution algorithm, whereas Section 4 is devoted to the stability analysis. Section 5 concludes.

2 The class of MSRE models

We study the following class of forward-looking MSRE models with noisy observations on the state vector, defined on a properly filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$:

$$x_{t+1} = \Gamma_{s(t)}^{-1}\mathbf{E}[x_{t+2}|\mathcal{F}_t] + \Gamma_{s(t)}^{-1}\Psi_{s(t)}v_t, \quad x_0 = \bar{x} \tag{1}$$

$$y_t = \Phi_{s(t)}x_t + w_t \tag{2}$$

where x_t is an n-dimensional real vector of random variables of economic interest, y_t is an l-dimensional real vector of observables, and the state error $v_t \in \Re^n$, the measurement noise $w_t \in \Re^l$ and the initial state $\bar{x} \in \Re^n$ are zero-mean white Gaussian processes. With no loss of generality, the covariances of the unobserved structural disturbance and of the measurement noise are normalized to the $I_{n\times n}$ and $I_{l\times l}$ identity matrices respectively, whereas $\bar{x}$ has covariance P_0. $\Gamma_{s(t)}$,$\Psi_{s(t)}$ and $\Phi_{s(t)}$ are conformable matrices holding the coefficients of the underlying economic model, with $\Gamma_{s(t)}$ assumed invertible, as in [17].

In (1)-(2), the regime switches are governed by an ergodic discrete-state Markov chain indexed by $s(t)$, with $s(t) \in \mathbb{S} := \{1, \ldots, S\}$. Let $\widetilde{\mathcal{S}}$ denote the σ-field of all subsets in $\mathbb{S}$, and $\widetilde{\mathcal{F}}_t$ the σ-field of $\Re^{n+l}$ in which (x_t, y_t) lie. We define:

$$\Omega := \prod_{t\in\mathcal{T}} \left(\Re_t^{n+l} \times \mathcal{S}_t\right)$$

where $\Re_t^{n+l}, \mathcal{S}_t$ are copies of $\Re^{n+l}, \mathcal{S}$ and $\mathcal{T}$ denotes a discrete-time set of interest. Let $\mathcal{T}_t := \{k \in \mathcal{T}; k \leq t\}$ for each $t \in \mathcal{T}$, then:

$$\mathcal{F} := \sigma\left\{\prod_{t\in\mathcal{T}} (\alpha_t \times \beta_t)\,; \alpha_t \in \widetilde{\mathcal{F}}_t, \beta_t \in \widetilde{\mathcal{S}}, \forall t \in \mathcal{T}\right\}$$

and for each $t \in \mathcal{T}$:

$$\mathcal{F}_t := \sigma\left\{\prod_{\iota\in\mathcal{T}_t} \alpha_\iota \times \beta_\iota \times \prod_{\tau\in\mathcal{T}\setminus\mathcal{T}_t} \Re_\tau^{n+l} \times \mathcal{S}_\tau; \alpha_\iota \in \widetilde{\mathcal{F}}_\iota, \beta_\iota \in \widetilde{\mathcal{S}}, \iota \in \mathcal{T}_t\right\}$$

with $\mathcal{F}_t \subset \mathcal{F}$. Then $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ defines a stochastic basis for (1)-(2), with $\mathcal{P}$ representing a probability measure such that:

$$\mathcal{P}\{s(t+1) = j|\mathcal{F}_t\} = \mathcal{P}\{s(t+1) = j|s(t)\} = p_{s(t)j}$$

with $p_{i,j} \geq 0$ for $i, j \in \mathbb{S}$ and $\sum_{j=1}^{S} p_{ij} = 1$ for each $i \in \mathbb{S}$. The initial conditions $(\bar{x}, s_0)$ are taken to be independent random variables.

More specifically, the information set available at time t, upon which conditional (rational) expectations $\mathbf{E}[\cdot|\mathcal{F}_t]$ in (1) are built, includes the complete filtrations generated by the output process (2), namely $\{y_k, k \leq t\}$, and by the Markov state realizations $\{s_k, k \leq t\}$. We thus allow for observable shifts in modes solely, as in most of the macroeconomic literature on regime switching RE models (e.g., [12], [17])[1]. Accordingly, while the current values of parameters are known, future ones are uncertain. As a working assumption, we also require that $\bar{x}$, v_t, w_t and s_t be mutually independent.

A *Rational Expectations Equilibrium* (REE) is any process $\{x_t, y_t\}$ which, for fixed initial conditions and in both finite and infinite model horizon, satisfies equations (1)-(2). The goal of this paper is to develop a model reference adaptive technique in order to solve (1)-(2) for a particular REE by directly computing the conditional expectations term from actually available information, without imposing any a priori stability concept. This will be accomplished by specifying the RE component as a (general-measurable) function of the t-dated filtration $\mathcal{F}_t$ of the stochastic economy, and employing optimal stochastic control and filtering techniques to adapt the actual system evolution to the perfect-foresight Markov jump autoregressive state motion (the reference model); we finally demonstrate equivalence to RE equilibrium of the obtained solution, both in finite and infinite horizon representations, by showing that the optimal feedback control has the same structure of the RE term.

3 The solution algorithm

In this work we are interested in reconsidering RE models from a point of view which, along the lines of [3] and [7], departs to some extent from the approaches typically adopted in the macroeconomic literature (e.g. [5]). To illustrate this point, we set the following Markov jump (controllable) system with linear noise corrupted observations:

$$x_{t+1} = \Gamma_{s(t)}^{-1} u_t + \Gamma_{s(t)}^{-1} \Psi_{s(t)} v_t, \quad x_0 = \bar{x} \tag{3}$$

$$y_t = \Phi_{s(t)} x_t + w_t \tag{4}$$

where u_t is an $\mathcal{F}_t$-measurable input process, and define the perfect-foresight (Markov jump autoregressive) dynamics where the two-step ahead values of the x_t variables are perfectly anticipated and no (endogenous) forecasting errors are made:

$$x_{t+1}^* = \Gamma_{s(t)}^{-1} x_{t+2}^* + \Gamma_{s(t)}^{-1} \Psi_{s(t)} v_t, \quad x_0^* = \bar{x},\, x_{-1}^* = 0 \tag{5}$$

$$y_t^* = \Phi_{s(t)} x_t^* + w_t \tag{6}$$

where both (3)-(4) and (5)-(6) are defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

Let us introduce:

$$\epsilon_t := x_t - x_t^*, \qquad z_t' := \left(\epsilon_t' \; x_t^{*'} \; x_{t+1}^{*'}\right)$$

[1]For theoretical work dealing with unobserved current regimes, see, among others, [2], [22] and [11].

and consider the problem of finding an input sequence $u = \{u_t\}_{t \in \mathbf{T}}$, $\mathbf{T} = [0, T] \subset \mathbb{N}$, $u_t \in U_t$ - with U_t denoting the space of all square-integrable $\mathcal{F}_t$-measurable random vectors - which minimizes the objective functional:

$$J(u) = \mathbf{E} \sum_{t=0}^{T+1} (z_t' M z_t) \tag{OF}$$

under the following state-space recursive constraints[2]:

$$z_{t+1} = A_{s(t)} z_t + B_{s(t)} u_t + C_{s(t)} v_t, \quad z_0 = \bar{z} \tag{7}$$

$$y_t = \bar{\Phi}_{s(t)} z_t + w_t \tag{8}$$

where M consists of the identity matrix $I_{n \times n}$ as first block on the main diagonal and 0's elsewhere. Expression (8) can be properly used as the observation equation for the augmented Markov jump system (7) in which the first n entries of the state vector z_t describe the evolution of the deviation from the autoregressive behavior of the MSRE model.

The design of an input sequence $\{\hat{u}_t\}$, $t \in \mathbf{T}$ minimizing (OF) subject to (7)-(8) is accomplished by employing an optimal Markov jump feedback controller in conjunction with the minimum mean-square estimate (MMSE) obtained by a time-varying Kalman filter. We indeed show that a separation principle holds for the system at issue - i.e., the optimal input sequence depends on the observed state only through the optimal estimate of the latter. In the classical literature on Markov jump linear quadratic (MJLQ) problems (e.g. [10]), it has been shown that the solution of such problems engenders a twofold set of coupled Riccati equations, each associated to the filtering and control programs respectively. Since these backward-recursive equations cannot be represented as a single higher-dimensional Riccati equation, structural concepts and algorithms from the classical linear theory are not directly applicable to Markov jump systems. While further requirements are generally needed to determine the existence of a steady-state solution for the coupled Riccati equations (e.g. [4], [8], [1]), we prove that, when applied to the solution method for MSRE models we propose in this paper, this issue vanishes for the Riccati gain is shown to admit a simple time-invariant and state-independent representation, both in finite and infinite horizon problems. The following statement clarifies this insight:

Theorem 1. *Given the system (3)-(4), the input sequence $\hat{u}_t := \{\hat{u}_t\}$ which produces for any $t = 0, 1, \ldots$ the mean-square minimum deviation from the Markov jump autoregressive state motion (5), is in the form:*

$$\hat{u}_t = \Gamma_{s(t)} \hat{x}^*_{t+1|t} \tag{9}$$

where the optimal estimate $\hat{x}^*_{t+1|t} := (0\ 0\ I\ 0)' \mathbf{E}[z_t | \mathcal{F}_t]$ is obtained recursively via a time-varying Kalman filter.

Proof. - See Appendix B. □

[2] See Appendix A.

The estimator of the one-step ahead perfect-foresight state, $\hat{x}^*_{t+1|t}$, is mean-square optimal with respect to the σ-algebra generated by the *actual* measurement process (2), the only available data[3]. Our final claim rests on showing that, for any t and the input (9), the optimal two-step ahead prediction of the perfect-foresight state x^*_t following the regime switching law of motion (5), given the measurement $(y_0, \ldots, y_t)$ and the filtration $\sigma(s^t) = (s_0, \ldots, s_t)$, is equal to that relative to the actual state x_t in (3):

Corollary 1. *Let $x = (x_t), y = (y_t)$ be the solution of (3)-(4) under the control law $\hat{u}_t$. Then, for any t and Markov state $s(t) = i \in \{1, 2, \ldots, S\}$, it holds:*

$$\hat{x}_{t+2|t} = \hat{x}^*_{t+2|t} \tag{10}$$

Proof. - It readily follows from Theorem 1 and the independence assumption between v_t and s_t. □

Consider now the perfect-foresight Markov jump state motion (5). It is easily verified that:

$$\Gamma^{-1}_{s(t)} E[x^*_{t+2}|\mathcal{F}_t] = \Gamma^{-1}_{s(t)} \hat{u}_t$$

which shows, in conjunction with the assertion of Corollary 1, that the optimal feedback controller (9) has the same structure of the conditional (rational) expectation term $E[x_{t+2}|\mathcal{F}_t]$, and hence the solution $x = (x_t), y = (y_t)$ of (3)-(4) with $\hat{u}_t \equiv E[x_{t+2}|\mathcal{F}_t]$ is an REE for the Markov-switching model (1)-(2). In other words, both in finite and infinite horizon formulations, there always exists an REE $x = (x_t)$ - that is, a stochastic sequence of (functions of) states and observables in $\mathcal{F}_t$ fulfilling the *noncausal* regime switching RE model (1)-(2) - which is computable via a *causal* Markov jump (controllable) system of the form (3)-(4) forced by the optimal Markov jump feedback control law $\hat{u}$ inducing the minimum variance displacement between the actual x-state and the perfect-foresight one x^*_t.

4 The stability of the RE equilibrium

As in [17], we focus on the concept of mean-square stability for RE equilibria under regime switching. We set $u_t = \hat{u}_t$ in (3)-(4), and consider the evolution equation for the RE solution as derived in Section 3:

$$x_{t+1} = \hat{x}^*_{t+1|t} + \Gamma^{-1}_{s(t)} \Psi_{s(t)} v_t \tag{11}$$

with:

$$\hat{x}^*_{t+1|t} = \Gamma_{s(t-1)} \hat{x}^*_{t|t-1} + \bar{K}_{t-1} \bar{\Phi}_{s(t-1)} \eta_{t-1} + \bar{K}_{t-1} w_{t-1} \tag{12}$$

where $\bar{K}_t$ is the precomputable filter gain and $\eta := z_t - \hat{z}_t$ denotes the estimation error[4].

We study the first two moments of the equilibrium process x_t, i.e. $m_t = \mathbf{E}[x_t]$ and $\gamma_t = \mathbf{E}[x_t x_t']$, which characterize its mean-square stability. Indeed, system (11) is mean-square stable if its first and second moments converge to finite (possibly zero) values in

[3]This result, presented in [7], improves upon [13]'s filtering approach to solution of linear RE models, as their MMSE estimator rather exploits the *fictitious* observations y^*_t according to (6), which are clearly not available. Moreover, [13] only find an approximate solution to the original (linear) RE model.

[4]See the optimal filter derivation in the proof of Theorem 1 (Appendix B).

the limit for $t \to \infty$. From (11) we have $m_t \to 0$ if and only if $m_t^* = \mathbf{E}[\hat{x}^*_{t+1|t}] \to 0$. Moreover, provided that the noise covariance is uniformly bounded with respect to t, i.e. there exists $L \in \Re$ such that:

$$\sum_{i=1}^{S} \|\Gamma_i^{-1}\Psi_i\Psi_i'\Gamma_i'^{-1}\|\mathcal{P}\{s(t)=i\} \leq L < +\infty, \quad \forall t \tag{13}$$

then $\gamma_t \to 0$ if and only if $\gamma_t^* = \mathbf{E}[\hat{x}^*_{t+1|t}\hat{x}^{*\prime}_{t+1|t}] \to 0$.

Taking expectations in (12) yields:

$$\mathbf{E}\left[\hat{x}^*_{t+1|t}\right] = \mathbf{E}\left[\Gamma_{s(t-1)}\right]\mathbf{E}\left[\hat{x}^*_{t|t-1}\right]$$

from which $m_t^* \to 0$ obtains if:

$$\max_i \max_j \left|\lambda_j(\Gamma_i)\right| < 1 \tag{14}$$

where $\lambda_j(\Xi)$ denotes the j-th eigenvalue of a matrix Ξ.

As to the second moment, since η_t is orthogonal to $\hat{x}^*_{t+1|t}$ and the measurement noise w_t proves independent of x_t and the σ-algebra $\{y_k, k \leq t\}$, we readily derive:

$$\begin{aligned}\gamma_t^* = \mathbf{E}\left[\Gamma_{s(t-1)}\gamma^*_{t-1}\Gamma'_{s(t-1)}\right] + \mathbf{E}\left[\bar{K}_{t-1}\bar{K}'_{t-1}\right]\\ +\mathbf{E}\left[\bar{K}_{t-1}\bar{\Phi}_{s(t-1)}\bar{P}_{t-1}\bar{\Phi}'_{s(t-1)}\bar{K}'_{t-1}\right]\end{aligned} \tag{15}$$

where $\bar{P}_t = \mathbf{E}[P_t]$ and $P_t := \mathbf{E}[(z_t - \hat{z}_t)(z_t - \hat{z}_t)'|s_k, k \leq t]$ is the mean-squared error covariance[5]. Thus, $\gamma_t^* \to 0$ for $t \to \infty$ obtains if:

$$\max_i \max_j \left|\lambda_j(\Gamma_i\Gamma_i')\right| < 1 \tag{16}$$

and:

$$\sum_{i=1}^{S} \|\Phi_i\Phi_i'\|\mathcal{P}\{s(t)=i\} \leq L < +\infty, \quad \forall t \tag{17}$$

$$\|\bar{P}_t\| \leq L < +\infty, \quad \forall t \tag{18}$$

In fact, (17) is always verified as $\mathcal{P}$ is a probability measure, and from (18) it follows that $\bar{K}_t\bar{K}_t'$ is bounded as well[6]. As for $\bar{P}_t$, its evolution is described by the following recursive equation[7]:

$$\begin{aligned}\bar{P}_{t+1} = \mathbf{E}[A_{s(t)}\bar{P}_tA'_{s(t)}] + \mathbf{E}[C_{s(t)}C'_{s(t)}]\\ -\mathbf{E}[A_{s(t)}\bar{P}_t\bar{\Phi}'_{s(t)}\left(I + \bar{\Phi}_{s(t)}\bar{P}_t\bar{\Phi}'_{s(t)}\right)^{\dagger}\bar{\Phi}_{s(t)}\bar{P}_tA'_{s(t)}]\end{aligned} \tag{19}$$

where $\bar{P}_0 = \mathrm{cov}(z_0, z_0)$. Riccati equations with Markov jump coefficients have been extensively studied in the engineering literature (e.g. [8], [9], [1], [10]). According to well-known results established in the mentioned references, condition (14) entails condition (18). It follows that requirement (14) - which also implies (16) - is sufficient for the mean-square stability of the obtained RE equilibrium.

[5]See Appendix B.

[6]Note that $\mathbf{E}[A_{s(t)}A'_{s(t)}]$ has the same eigenvalues as $\mathbf{E}[\Gamma_{s(t)}\Gamma'_{s(t)}]$ and in addition zero eigenvalues.

[7]Given a matrix Ξ, we denote its pseudoinverse by $\Xi^{\dagger}$.

5 Conclusion

In this paper, we describe a model reference adaptive approach to solution of noisily observed MSRE models containing past expectations of future states, which are replaced by a (general-measurable) function of the actually available information. By applying dynamic programming techniques along with a time-varying Kalman filtering algorithm, the evolution of a *causal* system is adapted to the corresponding Markov jump perfect-foresight state behaviour (the reference model); the resulting state motion is shown to be an RE equilibrium for the original *noncausal* regime switching model for the optimal feedback control features the same structure of the (unobservable) conditional expectations component. As a consequence, our equilibrium existence result does not rest on any stochastic stability requirements or approximation hypotheses, with no reference to model determinacy either.

APPENDIX

A. Definition of matrices for the augmented system

The matrices $A_{s(t)}$, $B_{s(t)}$, $C_{s(t)}$ and $\bar{\Phi}_{s(t)}$ appearing in (7)-(8) have the following block structure:

$$A_{s(t)} = \begin{pmatrix} 0 & 0 & -I \\ 0 & 0 & I \\ 0 & 0 & \Gamma_{s(t)} \end{pmatrix}; \quad B_{s(t)} = \begin{pmatrix} \Gamma_{s(t)}^{-1} \\ 0 \\ 0 \end{pmatrix}$$

$$C_{s(t)} = \begin{pmatrix} \Gamma_{s(t)}^{-1}\Psi_{s(t)} \\ 0 \\ -\Psi_{s(t)} \end{pmatrix}; \quad \bar{\Phi}_{s(t)} = \begin{pmatrix} \Phi_{s(t)} \\ \Phi_{s(t)} \\ 0 \end{pmatrix}$$

B. Proof of Theorem 1

To save notation, let us define $u_t^+ = [u_t', u_{t+1}', ..., u_T']'$, and let $\xi^t = [\xi_0', ..., \xi_t']'$ denote a sequence of random vectors $\xi_0, ..., \xi_T$. The σ-algebra generated by $\xi_0, ..., \xi_t$, namely $\sigma(\xi^t)$, will be for simplicity identified with the vector ξ^t.
We first derive the conditional expectations for the augmented state vector z_t. This is accomplished by employing a time-varying Kalman filter for the state-space system (7)-(8). Indeed, the objective is to identify at every time step t, an estimate $\hat{z}_t$ that minimizes the mean-squared error covariance:

$$P_t = \mathbf{E}\left[(z_t - \hat{z}_t)(z_t - \hat{z}_t)' | s^t\right]$$

A potential issue lies in that the noise provides information about the state since the regime switching matrices multiplying the two depend on the same underlying Markov state. However, as long as the current realization of the Markov chain is observable, the state variable z_t and the noise v_t become independent. Likewise, though the noise turns correlated, conditioned on the current state estimate and the Markov state, the next period noise remains (conditionally) zero-mean.
Since the estimator at time t has access to observations $(y_0, ..., y_t)$ and the Markov state values $(s_0, ..., s_t)$, the optimal linear MMSE filtering estimate $\mathbf{E}[z_t|\mathcal{F}_t]$ is obtained from a time-varying (sample path) Kalman filter (e.g. [9]). Let $s_t = i \in \mathbb{S}$ be the Markov state observed in time t, then:

$$\hat{z}_t = \hat{z}_{t|t-1} + \bar{K}_t\left(y_t - \bar{\Phi}_i\hat{z}_{t|t-1}\right), \quad \hat{z}_0 = \mathbf{E}\{z_0\} \tag{20}$$

$$\bar{K}_t = P_{t|t-1}\bar{\Phi}_i'\left(I + \bar{\Phi}_i P_{t|t-1}\bar{\Phi}_i'\right)^{\dagger} \tag{21}$$

$$\hat{z}_{t+1|t} = A_i\hat{z}_t + B_i u_t$$

$$P_t = P_{t|t-1} - \bar{K}_t C_i P_{t|t-1}$$

$$P_{t+1|t} := \mathbf{E}\left[(z_{t+1} - \hat{z}_{t+1|t})(z_{t+1} - \hat{z}_{t+1|t})'|s^t\right] = A_i P_t A_i' + C_i C_i'$$

where $P_0 = \text{cov}(z_0, z_0|s_0)$.
Using the measurement equation (8), (20) rewrites:

$$\hat{z}_{t+1} = A_i\hat{z}_t + B_i u_t \bar{K}_t \left(\bar{\Phi}_i(z_t - \hat{z}_t) + w_t\right)$$

which along with (7) yields the equation of the estimation error $\eta_t := z_t - \hat{z}_t$:

$$\eta_{t+1} = \left(A_i - \bar{K}_t\bar{\Phi}_i\right)\eta_t + C_i v_t - \bar{K}_t w_t \tag{22}$$

from which we observe that η_t is independent of u_t.
We turn now to the Markov jump LQG problem described by (OF)-(7)-(8). Let us define the cost-to-go at t:

$$J_t(u_t^+, \mathcal{F}_t) = \mathbf{E}\left\{\sum_{s=t}^{T+1} z_s^T M z_s \Big| \mathcal{F}_t\right\} \tag{23}$$

and the optimal cost-to-go (at t):

$$J_t^*(\mathcal{F}_t) = \min_{u\in U} J_t(u_t^+, \mathcal{F}_t), \tag{24}$$

where U readily follows from the above defined U_t, and the min is taken samplewise with respect to $\mathcal{F}_t$. Finally denote:

$$u_t^{+*} = \arg\min_{u\in U} J_t(u_t^+, \mathcal{F}_t) \tag{25}$$

The *optimality principle* ensures that $\left(u_t^{+*}\right)_{t+1}^{+} = u_{t+1}^{+\;*}$, i.e. the restriction of the optimal control sequence for the t-th instance of the sequence (24) of optimal control problems, is the optimal control for the $t+1$-th problem. Straightforward computation yields the following recursive relation between the optimal cost-to-go functionals (24):

$$J_t^*(\mathcal{F}_t) = \mathbf{E}\left\{z_t' M z_t|\mathcal{F}_t\right\} + \min_{u_t}\mathbf{E}\left\{J_{t+1}^*(\mathcal{F}_{t+1})\big|\mathcal{F}_t\right\} \tag{26}$$

which is the general equation of the Dynamic Programming Algorithm (DPA). Going backwards, at the last stage one has:

$$u_0^{+*} = \arg\min_{u\in U} J_0(u_0^+, \mathcal{F}_0)$$

hence a fortiori:

$$u_0^{+*} = \arg\min_{u\in U}\mathbf{E}\left\{J_0(u_0^+, \mathcal{F}_0)\right\} = \arg\min_{u\in U} J(u)$$

which delivers the desired solution.
As to the initial stage, we need $J_T^*(\mathcal{F}_T)$, which requires us to solve for:

$$u_T^* = \arg\min_{u_T} J_T(u_T, \mathcal{F}_T) = \arg\min_{u_T}\mathbf{E}\left\{z_T' M z_T + z_{T+1}' M z_{T+1}\big|\mathcal{F}_T\right\} \tag{27}$$

and then to substitute it into the functional:

$$\begin{aligned}
J_T^*(I_T) &= J_T(u_T^*, \mathcal{F}_T) \\
&= \mathbf{E}\left\{z_T' M z_T + z_{T+1}' M z_{T+1}\Big|\mathcal{F}_T\right\} \\
&= \mathbf{E}\Big\{z_T' M z_T + z_T' A_{s(T)}' M A_{s(T)} z_T + u_T^{*\prime} B_{s(T)}' M B_{s(T)} u_T^* \\
&\qquad +2 z_T' A_{s(t)}' M B_{s(T)} u_T^* + v_T' C_{s(T)}' M C_{s(T)} v_T \Big|\mathcal{F}_T\Big\}
\end{aligned} \tag{28}$$

where it has been used the independence of $z_T, s(T)$ and v_T, which implies:

$$\mathbf{E}\left\{z_T' A_{s(T)}' M C_{s(T)} v_T | \mathcal{F}_T\right\} = \mathbf{E}\left\{z_T' A_{s(T)}' M C_{s(T)} \mathbf{E}\{v_T\} | \mathcal{F}_T\right\} = 0 \tag{29}$$

as well as:

$$\mathbf{E}\left\{u_T' B_{s(T)}' M C_{s(T)} v_T | \mathcal{F}_T\right\} = \mathbf{E}\left\{u_T' B_{s(T)}' M C_{s(T)} \mathbf{E}\{v_T\} | \mathcal{F}_T\right\} = 0 \tag{30}$$

by the independence of s^T, y^T, hence of $u_T \equiv u_T(\mathcal{F}_T)$, and v_T.
Noting that u_T only affects the quadratic form of z_{T+1} in (27), thus using the system equation, it holds:

$$u_T^* = \arg\min_{u_T} \mathbf{E}\left\{z_{T+1}' M z_{T+1} \middle| \mathcal{F}_T\right\} \tag{31}$$

Using (29), (30), and noting that u_T does not affect the quadratic terms in z_T and v_T, we obtain:

$$\begin{aligned} u_T^* &= \arg\min_{u_T} \mathbf{E}\left\{u_T' B_{s(T)}' M B_{s(T)} u_T + 2 z_T' A_{s(t)}' M B_{s(T)} u_T \middle| \mathcal{F}_T\right\} \\ &= \arg\min_{u_T} \left\{u_T' B_{s(T)}' M B_{s(T)} u_T + 2 \hat{z}_T' A_{s(t)}' M B_{s(T)} u_T\right\} \end{aligned}$$

By setting to zero the derivative respect to u_T of the positive quadratic functional in the above equation, and solving with respect to u_T, we get u_T^*:

$$u_T^* = -\left(B_{s(T)}' M B_{s(T)}\right)^{-1} B_{s(T)}' M A_{s(T)} \hat{z}_T \tag{32}$$

and substituting (32) into (28), the following expression of the optimal cost at time T obtains:

$$J_T^*(I_T) = \mathbf{E}\left\{z_T' K_T z_T + (z_T - \hat{z}_T)' L_T (z_T - \hat{z}_T) + v_T' C_{s(T)}' M C_{s(T)} v_T \middle| \mathcal{F}_T\right\} \tag{33}$$

where:

$$L_T = A_{s(T)}' M A_{s(T)} \tag{34}$$
$$K_T = M - L_T + A_{s(T)}' M A_{s(T)} = M \tag{35}$$

Now, the DPA (26) for $t = T-1$ implies:

$$\begin{aligned} u_{T-1}^* &= \arg\min_{u_{T-1}} \mathbf{E}\left\{J_T^*(\mathcal{F}_T) \middle| \mathcal{F}_{T-1}\right\} \\ &= \arg\min_{u_{T-1}} \mathbf{E}\left\{z_T' K_T z_T \middle| \mathcal{F}_{T-1}\right\}, \\ &= \arg\min_{u_{T-1}} \mathbf{E}\left\{z_T' \mathbf{E}\{K_T\} z_T \middle| \mathcal{F}_{T-1}\right\} \end{aligned} \tag{36}$$

where the second equality comes from being the estimation error $(z_t - \hat{z}_t)$ not affected by u_t, and the third one from being $z_T, \mathcal{F}_{T-1}$ independent of $s(T)$. Equations (31) and (36) show the recursive representation of the problem at hand, thereby the following general characterization holds for the optimal control:

$$u_t^* = \arg\min_{u_t} \mathbf{E}\left\{z_t' \mathbf{E}\{K_t\} z_t \middle| \mathcal{F}_{t-1}\right\}$$

whose value is given by:

$$u_t^* = -\left(B'_{s(t)}\mathbf{E}\{K_t\}B_{s(t)}\right)^{-1} B'_{s(t)}\mathbf{E}\{K_t\}A_{s(t)}\hat{z}_t \tag{37}$$

where the gain K_t solves the backward-recursive equations:

$$L_t = A'_{s(t)}\mathbf{E}\{K_{t+1}\}B_{s(t)}\left(B'_{s(t)}\mathbf{E}\{K_{t+1}\}B_{s(t)}\right)^{-1} B'_{s(t)}\mathbf{E}\{K_{t+1}\}A_{s(t)} \tag{38}$$

$$K_t = \mathbf{E}\{K_{t+1}\} - L_t + A'_{s(t)}\mathbf{E}\{K_{t+1}\}A_{s(t)}, \quad K_{T+1} = M \tag{39}$$

As M is a square, idempotent matrix, from (35) it follows that $K_t = M$ for all periods $t = 1, \cdots, T$ and states $s(t) \in S$.
Finally, by substitution of K_t in (37) we derive[8]:

$$u_t^* = \Gamma_{s(t)}\hat{x}^*_{t+1|t} \tag{40}$$

Insofar as the expression for the feedback matrices does not depend on the finite horizon T, it yields the optimal control law for all the LQG control problems in the (OF)-(7)-(8) form for any $T = 1, 2, \ldots$.

[8]The third entry of $\hat{z}_t$ is $\mathbf{E}[x^*_{t+1}|\mathcal{F}_t]$.

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