Bonn Econon Discussion Papers

Discussion Paper 08/2013

Assignment Games with Externalities And Matching-Based Cournot Competition

by

Bo Chen

October 2013

Bonn Graduate School of Economics
Department of Economics
University of Bonn
Kaiserstrasse 1
D-53113 Bonn

Financial support by the
Deutsche Forschungsgemeinschaft (DFG)
through the
Bonn Graduate School of Economics (BGSE)
is gratefully acknowledged.

Deutsche Post World Net is a sponsor of the BGSE.

Assignment Games with Externalities And Mactching-based Cournot Competition*

Bo Chen†

October 9, 2013

Abstract

We develop a model of assignment games with pairwise-identity-dependent externalities. A concept of conjectural equilibrium is proposed, and the universal conjecture is shown to be the necessary and sufficient condition for the general existence of equilibrium. We then apply the solution concept to a matching-based Cournot model in which the unit production cost of a firm depends on both the technology of the firm and the human capital of the manager hired, and show that if technology and human capital are complementary, the positive assortative matching (PAM) is a stable matching under rational expectations, or even if firm technology and human capital are substitutable yet the substitutive effect is dominated by the marginal effects of technology and human capital, the PAM is still a rational stable matching. However, if the substitutive effect on the unit production cost is sufficiently strong or the market demand is sufficiently high, the negative assortative matching is a rational stable matching.

Keywords: Cooperative Games, Two-sided Matching, Assignment Games, Externalities, Transferable Utility, Cournot Competition

JEL classication: C71, C78, D43, D47, D62, L13

---

*I would like to thank Deniz Dizdar, Yen Teik Lee, Benny Moldovanu, Phillip Strack, and participants at the BGSE Brownbag seminar. I am very grateful to Andreas Kleiner for the encouragement and help during the entire project. To make this paper more concise, I cut off some (minor) parts about characterizing equilibrium and the core of a general assignment game with externalities from the initial draft, but they are available upon request.

†Bonn Graduate School of Economics, University of Bonn, Kaiserstrae 1, D-53113 Bonn, Germany. Email: econbc@gmail.com.

1 Introduction

In the standard model of two-sided matchings introduced by Gale and Shapley (1962) and the standard model of assignment games (matchings with transferable utilities) introduced by Shapley and Shubik (1972), the welfare of each agent only depends on who he/she is going to pair up with, and thus the agents do not care about how the other agents are going to match with each other.[1] However, in many situations people do care about the partners of the others. In another word, externalities present. In the marriage market, agents do not only care about their own marriages but also others' (e.g., a brother's marriage). In the labor market, a firm may not only care about the quality of its own employees (CEO) but also about the quality of the employees (CEOs) of the competitors. In the housing market, agents care about the quality of houses they are going to buy as well as who the neighbors will be. In the college admission problem, a student may not only care about the quality of the school that he is going to attend but also who else are going to attend this school and which are the schools some other students are going to attend.

In this paper, we are particularly interested in the problem of one-to-one assignment between managers and firms, competing in the same goods market, on a labor market. When the firms produce the same goods and sell to the same market, the output quantity of a firm has some influence on the market price of the goods, and hence on the other firms' surpluses as well, which has some effect on the other firms' hiring decisions. Hence, this is an assignment game with externalities. To the best of my knowledge, there has been no theoretical work incorporating any specific form of externalities into the study of matchings, and this paper is the first attempt to link the assignment problem on a labor market to imperfect competition on a goods market.

To solve the above problem, we need to have proper concepts of equilibrium and stable matchings. Hence, we start by studying general one-to-one assignment games with externalities. We adapt Sasaki and Toda's (1996) concept of conjectural equilibrium (Hahn, 1987) and assume that agents are pessimistic (Aumann and Peleg, 1960). Then, an equilibrium we call is an outcome such that: 1) it is consistent with agents' conjectures (expectations about the matchings to be formed), i.e., the underlying matching is in the expectation of each agent; 2) no pair of agents either paired-up with each other or not can block the outcome. The first main result found is that an equilibrium is ensured to exist if and only if each agent considers all matchings are possible.

Then we apply the solution concept to a matching-based Cournot competition model in which each firm has to hire one manager to produce some

[1] See Roth and Sotomayor (1992) for a detailed discussion of literature.

homogeneous goods and the per unit production cost of a firm depends on the technology of the firm and the human capital of the manager hired. We assume that the market demand function is linear, per unit cost is a constant for each firm-manager pair, and this per unit cost is decreasing in both firm type and manager type. Under these assumptions, it is shown that if the per unit cost function is submodular (i.e., the cross derivative is nonpositive), then a positive assortative matching (PAM) is a stable matching if each agent believes that if he/she pairs up with some agent on the other side of the market, the other agents will form a positive assortative matching. That is, if firm technology and human capital are complementary, it will be an equilibrium in which a good manager works for a good firm and a bad manager works for a bad firm. Even if the the unit cost function is supermodular (i.e., the cross derivative is positive), as long as the cross derivative is sufficiently smaller than the product of the partial derivatives, a PAM is still a stable matching. That is, if firm technology and human capital are substitutable, yet the substitutive effect is dominated by the marginal effects of technology and human capital, it would still be a stable matching that good firms hire good managers. However, if the substitutive effect on the cost is sufficiently strong or the market demand is sufficiently high, the negative assortative matching is a stable matching. Further more, the same sufficient conditions for the existence of these assortative stable matchings also ensure them to be stable matchings under rational conjectures. However, when technology and human capital are complementary, a PAM, which is a stable matching, may, surprisingly, not be the most efficient matching.

The construction of the paper is as follows: After the discussion of literature in Section 1.1, we introduce notation in Section 2. In Section 3, the solution concept of equilibrium will be defined, and the main existence theorem will be proved. We apply the solution concept to the matching-based Cournot game in Section 4. The sufficient conditions for the existence of assortative (rational) stable matchings will be provided, and the efficiency of the matchings will be discussed. Section 5 concludes. Two simple examples and the proofs are collected in the appendix.

1.1 Literature

There are three papers closely related to this one. Firstly, the model in this paper differs from the one of Shapley and Shubik (1996) in the way that this model involves identity dependent external effects. Whilst, the Linear Programming method is borrowed from that paper to show the existence of equilibrium.

Secondly, the model in this paper is, in some sense, an extension to the one of Jehiel and Moldovanu (1996). In their model, there is only one seller with one indivisible good and several buyers, while in this model there are multiple sellers, each with one indivisible good. They emphasized on strategic play and showed that there are situations in which agents might be better off not participate in a game and that the core is most probably empty. However, we do not emphasize on strategic play (on the matching market).

Thirdly, the most closely related paper is by Sasaki and Toda (1996). In their model of one-to-one matching (with ordinal preferences), externalities are captured by preferences over the set of complete matchings rather than the set of agents on the other side of the market. Each agent, when considering about pairing up with an agent at the other side of the market, has a conjecture, called *an estimation function*[2], about the possible matching that would be formed among the other agents. An outcome is a conjectural equilibrium if no agent has an incentive to deviate under a given conjecture about the reactive behaviors of the others and if the conjecture is consistent with the current outcome. The formulation is non-Bayesian−that is, players do not assign probabilities to the different matchings, but rather deviate only if the deviation is profitable for worst case that he/she considers possible. However, they show that a stable matching is guaranteed to exist if and only if each agent considers all matchings to be possible.[3]

In addition, we are also aware that there is a very recent working paper by Gudmundsson and Habis (2013). Their model and our model for general assignment games are very similar, while the major difference in the settings is about specifying the values to the characteristic functions (i.e., minimum payoffs required by a pair of agents to block a matching). In our model each agent, when considering about pairing to some agent, has a conjecture (expectation) about how the other agents will pair up, and a characteristic value is based on a pair of conjectures. A stable matching must be consistent with the conjectures as well. Whilst, in their model, instead of deriving characteristic values from conjectures, they directly assume for each pair of agents a threshold value for deviation, i.e., a value in between the lowest possible surplus and the highest possible surplus for a pair if they pair up with each other, and thus there is no consistency required on matchings with conjectures. Their paper complements our paper by showing properties of equilibria of general two-sided one-to-one assignment games with externalities.[4] Our paper emphasizes on applying the solution concepts to

[2]Sasaki and Toda (1996) awared that the shortcoming of their own approach is that the conjectures are exogenously given. Hafalir (2008) perfects Sasaki and Todas model by endogenizing the set of matchings that a deviating pair considers possible on the preferences of the other agents.

[3]The proof is for the case of two-sided externalities. By modifying their proof, it can be shown that the same negative result holds even under one-sided externalities. By modifying example 1 in Hafalir (2008) a little bit, we can show that rational conjectures do not guarantee the existence of stable matchings as well.

[4]These two papers' first main results are the same. But we derived a result which was different from the Proposition 3.4 in their paper. In their proof, the tricky case that when a pair of agents want to increase their payoffs they must take (back) some money

concrete games (the Cournot game in this paper) and analyzing equilibria of the games.

Besides the papers mentioned above, the problem of two-sided one-to-one matching under externalities in the marriage markets were studied by Hafalir (2008), Mumcu and Saglam (2008) , Roy chowdhury (2004).[5] The college admission problem with externalities was explored by Echenique and Yenmez (2007), Dutta and Masso (1997), and Alcalde and Revilla (2004). Housing market with externalities was studied by Mumcu and Saglam (2007). Recently, there has also been some work emphasizing on applications and methodologies for quantifying externality effects in two-sided matching problems. Baccara , Imrohoroglu, Wilson and Yariv (2012) studied a problem of matching faculty to offices. They developed some method to quantifying the effects of network externalities. Uetake and Watanabe (2012) developed a two-sided matching model to study the problem in which some firms would like to enter a market by mergering with an incumbent firm.

Apart from the study of matchings, we would also like to mention a study by Brander and Spencer (1985). The Stackelberg-Cournot version of their model is, in some sense, a two stage game. Before the start of Cournot competition, there is a stage at which a government can choose to subsidize the production of the domestic firm. Under some conditions, this subsidy may increase the domestic social welfare. In our matching-based Cournot model, there are also two stages – a matching stage before the Cournot competition stage. Comparing to a bad manager, a good manager can relatively reduce the unit production cost of a firm. Thus, it can be regarded as a model in which the good managers choose the firms to subsidize before the start of Cournot competition among the firms.

2 The Model And Notion

Let I be the set of agents on one side (man) of the market and J be the set of agents on the other (woman). I and J are finite and disjoint. The market is denoted by $M = I \cup J$. To simplify the exposition, we assume (i) that I and J have the same cardinality (i.e., $n = |I| = |J| \geq 2$) and (ii) that each of the agents must be matched to one agent on the other side of the market. (These two assumptions are basically without loss of generality for the existence of equilibrium.) Let i and j denote a generic man and woman, respectively.

---

from the other agents was overlooked, and hence the third sentence of the proof may not always hold. Example A.1 in Appendix A of this paper could also be regarded as a counter example.

[5] Sasaki and Toda (1996) were said to be the first to consider matching problems with externalities. And Sasaki and Toda (1996) and Hafalir (2008) were said to be the only two papers that investigate a general model of two-sided matching model with externalities.

A **matching** is a bijection function $\mu : I \cup J \to I \cup J$ such that (i) for each $m \in I \cup J$ we have $\mu \circ \mu(m) = m$ and (ii) $\mu(i) \in J$, $\mu(j) \in I$ for all $i \in I$ and for all $j \in J$. $\mu(i) = j$ (or equivalently, $\mu(j) = i$) is written as $(i, j) \in \mu$. Let $A(I, J)$ be the set of all possible matchings. For each $(i, j) \in I \times J$, $A(i, j) = \{\mu \in A(I, J) | (i, j) \in \mu\}$.

Let $\pi : A(I, J) \times I \times J \to R_+$ denote a **valuation function**. $\pi(\mu, i, j)$ is interpreted as agent i and j's valuation of their production output in matching μ. Sometimes, we denote $\pi(\mu)$ as the **total valuation** of the agents on a matching μ, i.e.,

$$\pi(\mu) := \sum_{(i,j)\in\mu} \pi(\mu, i, j).$$

$(u, v) \in R_+^{|I|} \times R_+^{|J|}$ is a payoff profile.[6] For simplicity, we assume both the range of π and the payoffs to be nonnegative.[7]

Definition 2.1. *A triplet (μ, u, v), $\mu \in A(I, J)$ and $(u, v) \in R_+^{|I|} \times R_+^{|J|}$, is called a feasible outcome for (I, J, π) if*

$$\sum_{i\in I} u_i + \sum_{j\in J} v_j \le \sum_{(i,j)\in\mu} \pi(\mu, i, j).$$

In this case we say (u, v) and μ are **compatible** with each other, and we call (μ, u, v) a **feasible outcome**. Note that, as the usual case, a feasible payoff vector may involve monetary transfers between agents who are not matched to one another. The triplet (I, J, π) is called an **assignment game (matching problem with transferable utilities) with externalities**.

3 Assignment Games with Externalities

In this setting, there are different ways to define stability of matching. The aims in this section are forming a concept of equilibrium and showing a necessary and sufficient condition that ensures the general existence of equilibrium. When side payments are allowed in a matching, two types of pairwise deviations need to be considered, namely, deviation by paired-up agents and deviation by unpaired-agents. The first type of deviations is clear. A pair of agents matched would not like to transfer (too much) money to other

[6] We normalize the lower bound of the payoffs to be 0. In the general set up, an agent can be single. Although an agent's valuation of being single still depends on how the remaining agents are matched, we can normalize the lowest possible valuation among all the agents to be 0, then the payoff to each agent must be nonnegative. Typically, in some market games, if an agent quits from the market, this agent gets 0 profit and the matching among the remaining agents do not generate externality on this agent.

[7] So that we can directly apply the linear programming method without modification.

agents.[8] But it is not necessary to consider about this case, and thus we ignore it in this section.

For the second type of deviations, when a pair of man and woman are trying to divorce their current spouses and marry each other, they need to take into account about how the other agents other than them two will behave. We here adapt the non-Bayesian set up proposed by Sasaki and Toda (1996). Let $\varphi_i(j) \subset A(i,j)$ be the set of matchings that agent i considers *possible* when j is paired with him. Similarly, let $\varphi_j(i) \subset A(i,j)$ be the set of matchings that j considers possible when i is paired with her. φ_i and φ_j are called **estimation functions** by Sasaki and Toda (1996). The **estimation profile** is denoted by $\varphi = \{\varphi_m : \ m \in M\}$. Agents are assumed to be "pessimistic". That is, a pair of agents not matched would like to pair up if both of them think that they can gain by pairing to each other. Let us denote the value of deviation $V(ij|\varphi)$ as

$$\min\{\min_{\mu\in\varphi_i(j)}[\pi(\mu,i,j)], \ \min_{\mu\in\varphi_i(j)}[\pi(\mu,i,j)]\}.$$

Then, the minimum requirement for an equilibrium is described is follows.

Definition 3.1. *Given an estimation profile φ, a feasible outcome (μ, u, v) is a φ-pseudo-equilibrium if*

1. *φ-admissibility: for any pair $(i,j) \in \mu$,*

$$\mu \in \varphi_i(j) \cap \varphi_j(i);$$

2. *(partial) φ-stability: there is no pair of agents (i,j) for whom*

$$V(ij|\varphi) > u_i + u_j.$$

And this matching μ is called a φ-pseudo-equilibrium matching

φ-admissibility requires a matching to be consistent with agents conjectures, i.e., if a matching is going to be an equilibrium matching, it must be in the expectations of the agents. (Partial) φ-stability requires that the outcome cannot be blocked by any pair of agents. A pseudo-equilibrium payoff profile was called a *"stable payoff profile"* by Sasaki and Toda (1996). However, we know that such a profile may not indeed be stable since in a pseudo-equilibrium the total payoff to the agents could be less then the total valuation, which does not make sense. We here define a concept of (strict) equilibrium.

[8] This was not taken into account in the definition of stability proposed by Sasaki and Toda (1996).

Definition 3.2. *A φ-pseudo equilibrium (μ, u, v) is a (strict) φ-equilibrium if*

$$u_i + v_j = \pi(\mu, i, j), \text{ for all } (i, j) \in \mu.$$

And this matching μ is called a φ-equilibrium matching or a φ-stable matching.

In such an equilibrium, there is no monetary transfer from one pair of matched agents to another pair, and no agent can gain for sure by pairwise deviation. In general, of course, we can construct a weaker concept of equilibrium in which some pair of matched agents transfer money to another pair yet no pair of agents can gain by pairwise deviation. But the solution concept of (strict) equilibrium is sufficient for our analysis.[9]

Let $PE_\varphi(I, J, \pi)$ denote the set of all φ-pseudo-equilibria and $E_\varphi(I, J, \pi)$ denote the set of all φ-equilibria .

3.1 The LP Problem and Equilibria

We now study the existence of φ-equilibrium. Like what Shapley and Shubik (1972) did, we recast the problem into linear programming (LP) terminology. The *primal* problem directly deals with matchings, and we can construct some matchings from the solutions; the *dual* problem directly deals with payoffs.

Consider the *primal* problem (PP):

$$\begin{aligned} \max_x z = & \sum_{i \in I} \sum_{j \in J} V(ij|\varphi) x_{ij} \qquad (1) \\ s.t. \quad & \sum_{i \in I} x_{ij} \leq 1 \quad \forall j \in J, \\ & \sum_{j \in J} x_{ij} \leq 1 \quad \forall i \in I, \\ & x_{ij} \geq 0 \quad \forall i \in I \text{ and } j \in J. \end{aligned}$$

The maximum value z_{max} is attained with all $x_{ij} = 0$ or 1 (see Danzig, p. 318). Thus the each extreme solution corresponds to a matching. Given an extreme solution $\{x^*_{ij}\}$, We construct a matching μ^* such that it satisfies that

$$(i, j) \in \mu^* \text{ if and only if } x^*_{ij} = 1.$$

[9] Cross-pair monetary transfers have only some effects on the extreme equilibrium payoffs to the agents, but have nothing to do with the existence of equilibrium. A small part about a weak concept of equilibrium is available upon request.

Consider the *dual* problem (DP):

$$\min_{u,v} w = \sum_{i\in I} u_i + \sum_{j\in J} v_j \tag{2}$$

$$st.\ \ u_i + v_j \geq V(ij|\varphi)\ \ \forall i \in I \text{ and } \forall j \in J, \tag{3}$$
$$u_i \geq 0 \text{ and } v_j \geq 0\ \ \forall i \in I \text{ and } \forall j \in J.$$

Let us denote (u^*, v^*) as one solution to the *dual* problem. The fundamental duality theorem tells us that $w_{min} = z_{max}$.

Remark.

1. *A matching μ is a φ-pseudo-equilibrium matching only if*[10]

   i. *μ is admissible and*

   ii. *$z_{max} \leq \pi(\mu)$.*

2. *For any extreme solution x^* to the primal problem and any solution (u^*, v^*) to the dual problem,*

$$x^*_{ij} = 1 \Rightarrow u^*_i + v^*_j = V(ij|\varphi).$$

It is then easy to see the following intermediate results as well.

Lemma 3.1. *(a). If a μ^* is admissible, then the feasible outcome (μ^*, u^*, v^*) is a φ-pseudo-equilibrium.*

(b). There is a φ-pseudo-equilibrium if and only if there is a φ-equilibrium.

(c). A matching μ is a φ-pseudo-equilibrium matching if and only if it is a φ-stable matching.

Proof. See Appendix B. □

However, problems arise if there is no *admissible* matching with a total valuation of the agents being larger than or equal to z_{max}. The estimation profile φ plays the key role in this problem. In the following we present the first main result – a negative result that is parallel to the one found by Sasaki and Toda (1996) in the cases with ordinal preferences.

Theorem 3.1. *The set $E_\varphi(I, J, \pi)$ of φ-equilibria is non-empty for any valuation function π if and only if $\varphi_i(j) = \varphi_j(i) = A(i, j)$ for all i and j.*

Proof. See Appendix B. □

[10] However, unlike in assignment games with no externalities, these two conditions are not sufficient for the existence of pseudo equilibrium. Moreover, it is also possible that for two admissible matchings, the less efficient (with lower total valuation) matching is a stable matching while the more efficient matching is not. It is illustrated by Example A.1 in Appendix A.

4 Application: The Matching-based Cournot Model

In this section we apply the solution concept to a two-stage game in the framework of a Cournot competition model. The multi-dimensionality brought by externalities may cause this simple model to be too complicated to be solved. However, I will show that under some mild assumptions, some nice results can be derived.

Suppose there is a market for a consumer goods which could be produced by n firms. The n firms differ in production technology characteristics (e.g., per unit cost type). There is a labor market in which there are n potential managers who differ in quality (i.e., human capital), and each firm must employ one manager so that it could produce the consumer goods (and the per unit production cost of a firm depends on both the production technology of the firm and the human capital of the manager hired by the firm). Firms want to maximize profits, and each manager goes to the firm that would pay her the highest salary. The time line is as in the following graph.

$$\boxed{\text{Firms match managers}} \longrightarrow \boxed{\text{Cournot competition}}$$

It is a two-stage game. At the first stage, firms and managers form matches, and at the second stage firms simultaneously decide the output quantity after observing the matching formed at the first stage. Since each firm's output quantity has some influence on the market price the of goods, the surplus of a firm-manager pair depends on the full matching realized.

The question is would high-type (with good production technology) firms hire high-type managers (with high human capital) in this game. It seems ambiguous because there are two effects. On the one hand, if high-type managers work for high-type firms, the total output would be high, which results in a low market price, and in low profits, thus a high-type firm may not be willing to pay an amount of salary a high-type manager desires. On the other hand, a high-type manager with a high-type firm could lower the production cost and have the advantage to produce more goods and thus take a larger share of the market.

Formally, let I and J be the set of firms and managers respectively. Firms and managers are assigned to types via maps $f : I \to F$ and $s : J \to S$,where F and S are compact subsets of R. To simplify the exposition, we again assume that $|I| = |J| = n$ (This is basically without loss of generality as well) and that no two agents are of the same type. W.l.o.g., we permute the indices so that

$$\begin{aligned} &f_i > f_{i+1} > 0 \ for \ i = 1, 2, ..., n-1 \ \ and \\ &s_j > s_{j+1} > 0 \ for \ j = 1, 2, ..., n-1, \end{aligned}$$

where f_i is firm i's (technology) type and s_j is manager j's (human capital) type. Then we make the following assumptions.

Assumption 4.1. *The inverse demand function is linear, i.e.,*

$$p(\mu) := \max\{\ H - \alpha \sum_{(i,j)\in\mu} q_{ij},\quad 0\}$$

in which H is a constant, μ is the matching on the labor market, and $q_{ij} \geq 0$ is the amount of the goods firm i who employed manager j produced and sells in the goods market.

Assumption 4.2. *The unit production cost of each firm-manager pair is constant.*

That is the unit production cost function depends only on firm type and manager type. Let us denote the unit output cost function denoted by

$$c(\cdot,\cdot) : F \times S \to R_{++},$$

where $c(a, b)$ is the unit production cost of a type-a firm who hired a type-b manager.

Assumption 4.3. *The unit production cost function $c(\cdot,\cdot)$ is twice continuously differentiable and decreasing in both type indices (i.e., $c_1' < 0$ and $c_2' < 0$).*[11]

Assumption 4.4. *The value of c is bounded from above by $\frac{H}{n\alpha}$, i.e.,*

$$c(\underline{f}, \underline{s}) < \frac{H}{n\alpha},$$

where $\underline{f}$ is the lowest firm type (with the highest cost) and $\underline{s}$ is the lowest manager type.

By imposing Assumption 4.4, we restrict our attention to the cases in which no manager or firm will leave the market or produce 0 output. While this saves us some tedious computational work, this is basically without loss of generality for our analysis.[12] Further more, we say firm technology and human capital are substitutable if $c_{12}'' > 0$; complementary if $c_{12}'' \leq 0$.[13]

Hence, the surplus[14] created by firm i and manager j in matching μ is

$$\pi(\mu, i, j) = \max_{q_{ij}\geq 0} q_{ij} \cdot [p(\mu) - c_{ij}] = \max_{q_{ij}\geq 0} q_{ij} \cdot [H - \alpha \sum_{(i',j')\in\mu} q_{i'j'} - c_{ij}].$$

[11]We assume the unit cost to be decreasing because high-type means low cost.

[12]If we don't impose the upper bound on the c value, it may be, in some cases, that some low-type firms and managers will produce 0 output since the other firm-manager pairs will produce a large amount of output to make the market price to be low. For these cases, we can simply exclude these agents (not at once) and do the analysis on the reduced market, and the following results still hold.

[13]c_{12}'' means the cross derivative of $c(\cdot,\cdot)$.

[14]Here we need to notice that although other firms' output quantities are also involved in the surplus function of a pair of agents (i, j), this does not mean that there are indeed externalities. A simple example is that if $c(f, s) = f + s$, then the output and the surplus of a pair of agents (i, j) do not depend on how the other agents are going to pair up with each other since $\sum_{(i,j)\in\mu} f_i + s_j$ does not depend on what μ is.

Before solving the above problem, we need to review the following definition.

Definition 4.1. *A matching μ on market M is a positive assortative matching (PAM) if for any two pairs $(i, j), (i', j') \in \mu$,*

$$f_i > f_i' \Rightarrow s_j \geq s_j' \ \ and \ \ s_j > s_j' \Rightarrow f_i \geq f_i'.$$

A negative assortative matching (NAM) is defined similarly.[15]

Let us denote μ_+^M as a PAM on M and $\mu_+^{M(ij)} = (i, j) \cup \mu_+^{M\backslash\{i,j\}}$ as a matching in which the submatching $\mu^{M\backslash\{i,j\}}$ is a PAM on $M\backslash\{i, j\}$. Similarly denote μ_-^M as a NAM on M and $\mu_-^{M(ij)} = (i, j) \cup \mu_-^{M\backslash\{i,j\}}$ as a matching in which the submatching $\mu^{M\backslash\{i,j\}}$ is a NAM on $M\backslash\{i, j\}$. Since any two PAM's (NAM's) are equivalent in terms of type pairs, we just denote $\{\mu_+^M\}$ ($\{\mu_-^M\}$) as the set of PAM's (NAM's). In particular, if there are no two agents that are of the same type, then there is only one unique PAM (NAM). Similarly, we denote $\{\mu_+^{M(ij)}\}$ and $\{\mu_-^{M(ij)}\}$ as the sets for conditional PAM set and conditional NAM set, respectively. Let us further specify the solution concept for this model.

Definition 4.2. *Given an assignment game with externalities $< I, J, \pi >$ in which each agent is of some type, if each agent's conjectures are the sets of PAM's on the reduced market, i.e.,*

$$\varphi_j(i) = \varphi_i(j) = \{\mu_+^{M(ij)}\} \ for \ all \ i, j \in M = I \cup J,$$

and μ_+^M is a stable matching under these conjectures, then we call μ_+^M an positive assortative (conjectural) stable matching (PASM); a negative assortative (conjectural) stable matching (NASM) is defined similarly.

Then, the following theorem states when a PASM or NASM is ensured to or not to exist in our model.

Theorem 4.1. *In the above matching-based Cournot model with Assumption 4.1-4.4,*

1. *if the cross derivative $c_{12}'' \leq 0$, or*
2. *if $c_{12}'' > 0$ and $c_1'c_2' \geq c_{12}''(\frac{H}{n\alpha} - c + \frac{n-1}{n}c_{max})$,*

then the PAM μ_+^M is a PASM, while the NAM μ_-^M is not a NASM;

3. *if $c_{12}'' > 0$ and*

$$\frac{(c_1')_{min} \cdot (c_2')_{min}}{(c_{12}'')_{min}[\frac{H}{n\alpha} - c_{max} + \frac{n-1}{n}c_{min}]} \leq \frac{n^2}{(2n-1)(n+1)},$$

[15] For more about assortative matchings, we refer readers to Legros and Newman (2007).

then the NAM μ_-^M is a NASM, while the PAM μ_+^M is not a PASM.[16]

These are the sufficient conditions and necessary conditions for the general existence of PASM and NASM for this matching-based Cournot game. In the first scenario in which the two production factors – firm technology and human capital–are complementary, a PASM is ensured to exist.[17] The proof for this scenario is not trivial, but the intuition is simple. Let us take Scenario 1 as an example for analysis. Firstly, a good firm has always some incentive to hire a good manager, and a good worker has incentive to work for a good firm. Suppose, keep the other variables constant, the human capital of the manager hired by the best firm (i.e., Firm 1) increases by $\Delta > 0$. Then, firm 1's best response to this is to produce more goods, and the other firm's best response to firm 1's action is to produce fewer goods. In the matching framework, if the human capital of the manager hired by the best firm increases, there must be some other manager (hired by some other firm i) whose human capital now decreases by Δ. Firm i would produce even fewer goods, and the other firms will response to this by producing more goods. However, when firm technology and human capital are complementary, for firms other than firm 1 and firm i, firm 1's effect dominates, hence by hiring the best manager (i.e., manager 1) firm 1 is the only firm that gains and all the other firms suffer. With the same logic, the manager 1 is willing to work for the best firm. Then repeat this logic recursively on firm 2 and manger 2, firm 3 and manager 3, and so on. As a consequence, the game ends up with a positive assortative matching in this scenario.

The second and the third scenarios are more complicated, since there are counter effects – the marginal effects (i.e., c_1' and c_2') and the substitutive effect (i.e., c_{12}'') – on the unit production costs. To give the intuition, let us first look at another version of the game. Suppose each firm can only produce one unit and the market price of the goods is fixed, but the production cost still depends on firm type and manager type. Then, this is a usual assignment game with no externalities. In this game, the separation point is $c_{12}'' = 0$: when $c_{12}'' < 0$, then the PAM is a stable matching; when $c_{12}'' > 0$, the NAM is a stable matching. Now, let us take into account the effects of $c_1', c_2' < 0$. $c_2' < 0$ is effectively equivalent to that by hiring a (marginally) better manager, a firm can produce (marginally) more goods, and thus earn more profit. Hence, even when $c_{12}'' > 0$, as long as c_{12}'' is not too large, a good firm is still willing to spend a little more money to hire a better manager, and the higher the $|c_2'|$ value is the more the firm is willing to pay. Similar logic applies to c_1'. Therefore, the separation point, in terms of the

[16] c_{min}, $(c_1')_{min}$, $(c_2')_{min}$, and $(c_{12}'')_{min}$ are the minimum values of c, c_1', c_2' and c_{12}'', respectively; c_{max} is the maximum value of c.

[17] One thing needs to be noticed is that although $\mu_+^M(ij)$ is the worse case for agent i and agent j when they are paired up, we can not directly apply Theorem 3.1 to prove the result for this case.

c''_{12} value, for the cases in which the PAM a stable and the cases in which NAM s stable moves upward. As a consequence, the ratio of the products of the partial derivatives and the cross derivative determines which stable matching it will be or not be. That is, when the marginal effects dominate the substitutive effect, good firms should still hire good managers; when the substitutive effect dominates the marginal effects, good firms should hire bad managers and bad firms should hire good managers.

It also shows that a NASM cannot exists when the two production factors are complementary and that PASM can not survive when the substitutive effect dominates the marginal effect. In addition, market demand (indicated by $\frac{H}{\alpha}$) also plays an role here. If the market demand is sufficiently large while the two production factors are substitutive, a NASM can also exist.

4.1 Rational Equilibrium

In this part we show the motivation for us to choose the assortative conjectural stable matching as our solution concept for this model. It is because in many cases, the conjectures consisting of assortative matchings on the reduced market are rational.

We say an φ-stable matching is **rational** if the conjectures in φ are all rational; a conjecture, w.l.o.g, assume it to be $\varphi_i(j)$, is **rational**, if for any matching $\mu \in \varphi_i(j)$, $\mu \backslash \{(i,j)\}$ is an $\varphi^{M(ij)}$-stable matching on the reduced market $M(ij) = \{M \backslash \{i,j\} \mid (i,j)\}$ conditional on (i,j) are paired up, in which $\varphi^{M(ij)}$ is the profile of rational conjectures on the reduced market.[18,19] We illustrate this concept by the following example.

Example 4.1. *Consider a matching market with 5 agents at each side of the market. We show an example to form rational conjectures by backward deduction as follows.*

Firstly, suppose conditional on that (i_1, j_1), (i_2, j_2), and (i_3, j_3) are paired up respectively, it is an equilibrium[20] *that i_4 pairing up to j_4 and i_5 pairing up to j_5 on the reduced market $\{i_3, i_4, j_3, j_4\}$. Then, consider the reduced market $\{i_3, i_4, i_5, j_3, j_4, j_5\}$ conditional on (i_1, j_1) and (i_2, j_2) are paired up respectively. With a little abuse of notation, we denote this **conditional reduced market** by*

$$M'' = \{i_3, i_4, i_5, j_3, j_4, j_5 \mid (i_1, j_1), (i_2, j_2)\}.$$

If the conjectures are rational, it must be the case that $\varphi_{i_3}^{M''}(j_3) = \varphi_{j_3}^{M''}(i_3) = \{\{(i_3, j_3), (i_4, j_4), (i_5, j_5)\}\}$.

[18] Notice that we can not directly kick i and j out, since they still have an role in the Cournot competition at the second stage of the game.

[19] Li (1993) applied the idea of rational expectations to his model of a one-to-one matching with externalities and showed that the existence of the equilibrium is ensured when externalities are very weak.

[20] Since there are only 4 agents, a stable matching must exist.

Next, suppose $\{(i_3, j_3), (i_4, j_4), (i_5, j_5)\}$ is indeed a $\varphi^{M''}$-stable matching on market M'' conditional on (i_1, j_1) and (i_2, j_2) are paired up respectively.[21] *Consider the conditional reduced market*

$$M' = \{i_2, i_3, i_4, i_5, j_2, j_3, j_4, j_5 \mid (i_1, j_1)\}$$

in which (i_1, j_1) are paired up. if the conjectures are rational, it must be the case that $\varphi_{i_2}^{M'}(j_2) = \varphi_{j_2}^{M'}(i_2) = \{\{(i_2, j_2), (i_3, j_3), (i_4, j_4), (i_5, j_5)\}\}$.

Finally, suppose $\{(i_2, j_2), (i_3, j_3), (i_4, j_4), (i_5, j_5)\}$ is indeed a $\varphi^{M'}$-stable matching on market M' conditional on (i_1, j_1) are paired up. Consider the full market

$$M = \{i_1, i_2, i_3, i_4, i_5, j_1, j_2, j_3, j_4, j_5\}.$$

If the conjectures are rational, it must be the case that $\varphi_{i_1}(j_1) = \varphi_{j_1}(i_1) = \{\{(i_1, j_1), (i_2, j_2)), (i_3, j_3), (i_4, j_4), (i_5, j_5)\}\}$. That is, when i_1 is considering pairing up with j_1, i_1 believe that this matching will be the resulting matching, and same for j_1.

In brief, a rational conjecture, e.g., $\varphi_{i_1}(j_1)$, requires agent i_1 to have a correct belief about what i_2, when considering about pairing to j_2 conditional on (i_1, j_1) are paired up, believes about what i_3, when considering about pairing to j_3 conditional on (i_1, j_1) and (i_2, j_2) are paired up respectively, believes about what matching will be formed among $\{i_4, i_5, j_4, j_5\}$ conditional on(i_1, j_1), (i_2, j_2), and (i_3, j_3) are paired up respectively.

Let $M^c := I^c \cup J^c$ denote a submarket (in which there are the same number of firms and managers), and $M^s := I^s \cup J^s = M \backslash M^c$ be the complementary set of M^c.

Let μ^c denote a generic matching on M^c, and μ^s a generic matching on M^s. For a given conditional reduced market

$$M^r := \{M^s \mid \mu^c\},$$

denote

$$\varphi_+^{M^r} := \{\varphi_i^{M^r}(j), \varphi_j^{M^r}(i) \mid \varphi_i^{M^r}(j) = \varphi_j^{M^r}(i) = \mu_+^{M^s(ij)},\ i, j \in M^s\}$$

as the conjecture profile which consists of all the conjectures in each of which it is a PAM on the corresponding further reduced market. Similarly, denote

$$\varphi_-^{M^r} := \{\varphi_i^{M^r}(j), \varphi_j^{M^r}(i) \mid \varphi_i^{M^r}(j) = \varphi_j^{M^r}(i) = \mu_-^{M^s(ij)},\ i, j \in M^s\}.$$

The following result shows that in our model the sufficient conditions for the existence of PASM and NASM also ensure $\varphi_+^{M^r}$ and $\varphi_-^{M^r}$ to be rational in the corresponding scenarios.

[21] Such a stable matching may not exist as shown by Example A.2 in Appendix A.

Theorem 4.2. *In the above matching-based Cournot model with Assumption 4.1-4.4,*

1. *if the cross derivative* $c''_{12} \leq 0$*, or*

2. *if* $c''_{12} > 0$ *and* $c'_1 c'_2 \geq c''_{12}(\frac{H}{n\alpha} - c + \frac{n-1}{n} c_{max})$*,*

then $\varphi_{+}^{M^r}$ *is rational for any conditional reduced market* M^r*, and hence the PASM is rational;*

3. *if* $c''_{12} > 0$ *and*

$$\frac{(c'_2)_{min} \cdot (c'_1)_{min}}{(c''_{12})_{min}[\frac{H}{n\alpha} - c_{max} + \frac{n-1}{n} c_{min}]} \leq \frac{n^2}{(2n-1)(n+1)},$$

then $\varphi_{-}^{M^r}$ *is rational for any conditional reduced market* M^r*, and hence the NASM is rational.*

Proof. See Appendix C.2. □

4.2 Efficient Matchings

We end the application by doing some analysis about the efficiency of the matchings. We say a matching μ is **more efficient** than a matching μ' if the social surplus (i.e., total consumer benefit minus total production cost) generated in the Cournot game conditional on μ is formed is higher than the social surplus induced by matching μ', and μ is the **most efficient** matching if no other matching induces a higher social surplus than μ does. The result is describes as follows.

Proposition 4.1. *In the above matching-based Cournot model with Assumption 4.1-4.4,*

1. *when* $c''_{12} < 0$*,* μ_{+}^{M} *is not ensured to be the most efficient matching;*

2. *when* $c''_{12} < 0$*, while* $\frac{(c'_2)_{max} \cdot (c'_1)_{max}}{(c''_{12})_{min}} \leq -\frac{n}{2(n+1)^2} H$*, then* μ_{+}^{M} *is the most efficient matching;*

3. *when* $c''_{12} > 0$*, while* $\frac{(c'_2)_{max} \cdot (c'_1)_{max}}{(c''_{12})_{max}} \geq \frac{3n+5}{2(n+1)^2} H$*, then* μ_{+}^{M} *is the most efficient matching, and thus* μ_{-}^{M} *is not the most efficient matching;*

4. *when* $c''_{12} > 0$*, while* $c_{max} > c_{min} \geq \frac{n^2+2n+2}{2n^2+3n} \frac{H}{n}$ *and* $\frac{(c'_2)_{min} \cdot (c'_1)_{min}}{(c''_{12})_{min}}$ *is sufficiently close to* 0*, then* μ_{-}^{M} *is the most efficient matching;*

5. *when* $c''_{12} = 0$*,* μ_{+}^{M} *is the most efficient matching.*

Proof. See Appendix C.3. □

The result is quite surprising. What we expected was that when firm technology and human capital are complementary, the positive assortative matching μ_+^M should be the most efficient matching, since the total output is the highest under the PAM and the low cost agents produces more goods; when substitutable, the positive assortative matching should never be the most efficient matching. However, Statement 1 and 3 say that this guess is not right. In the proof for the Statement 1, we constructed an example to show that when the gap between of the unit cost by the worst two firm-manager pairs and that of the other pairs is too large while the marginal effects are sufficiently smaller than the complementary effect ($\frac{(c_2')_{min}\cdot(c_1')_{min}}{(c_{12}'')_{max}} \to 0$), there is some other matching which generates higher social welfare than PAM does. Statement 3 says that when the marginal effects are sufficiently higher than the substitutive effect, the PAM can still be the most efficient matching. Statement 2 says that the PAM is ensured to be the most efficient matching when the marginal effects are sufficiently larger than the complementary effect; Statement 3 says that the NAM is ensured to be the most efficient matching when the lowest cost is sufficiently high and the marginal effects are sufficiently smaller than the substitutive effect. Statement 5 is boring since no externalities present in this case.

Comparing the above sufficient conditions for PAM and NAM to be efficient and the sufficient conditions for the existence of PASM and NASM, we notice that existence of assortative stable matching and efficiency do not totally contradict each other. PASM and NASM can be efficient, although they may not be efficient.

5 Discussion And Concluding Remarks

In this paper, we showed that the necessary and sufficient condition for the general existence of equilibrium in an assignment game with externalities is that each agent counts on all possibilities of the actions of the others. The solution concept was applied to one industrial organization problem. A PAM can be a stable matching under rational expectations when firm technology and human capital are either complimentary or substitutable, while a NAM can be rational stable matching under rational expectations only when the two production factors are substitutable and the substitutive effect is strong.

Nothing about the core was mentioned in this paper.[22] More definitions need to be given. Firstly, for example, in our matching-base Cournot model the members of a coalition must not only have conjectures about what kind of matchings the agents outside of the coalition will form but also what kind

[22]Some results about the core are available upon request. But it may not be worthwhile to fully examine the core of general assignment games with externalities.

of actions they will play at the second stage.[23] Hence, we have to be more careful about specifying values to the characteristic functions. We cannot take the values by simply adding up the pairwise values of the coalition members and taking the minimum of such sums.[24] Secondly, the conjectures of an agent have in different coalitions must be consistent. However, the core is not ensured to be non-empty even under universal estimations.

Finally, let us point out some possible extensions. From the application perspective, some other models (e.g., voting) can be reexamine to see whether it can be modified in matching-based framework to apply the solution concept to analyze the problems just as what we did in this paper.

From the theoretical perspective, firstly, besides the core, the equilibrium payoffs and stable matchings were not characterized in this paper as well, and hence further work might be done.[25] Secondly, since the estimation functions in the general theory part were assumed to be exogenous, we might be able to find a way to (partially) endogenize these functions and find some sufficient conditions for the existence of equilibrium, just as what Hafalir (2008) did. Lastly, one-sided market models and Non-transferable utilities models may also be explored.

References

1. Alcalde, J. and Revilla, P., 2004. Researching with Whom? Stability and Manipulation. Journal of Mathematical Economics, 40, 869-887.

2. Aumann, R. and Peleg B., 1960. Von Neumann-Morgenstern Solutions to Cooperative Games Without Side Payments. Bulletin of the American Mathematical Society, 66, 173-179

3. Baccara, M., Imrohoroglu, A., Wilson, Alistair J. and Yariv, L., 2012. A Feild Study on Matching with Network Externalities. American Economic Review, 102 (5), 1773-1804.

[23] A coalition at the matching stage can play either non-cooperatively or cooperatively at the Cournot competition stage. Even when they play cooperative, there are two scenarios – they can play as one firm, or they can ask the best firm (to hire the best manager) to produce at the lowest unit cost for all the firms in the coalition.

[24] In the special cases in which we can do that, the core is included in the equilibrium payoff set.

[25] Some results about the characterization of one-sided optimal payoffs are available upon request.

4. Brander, J. and Spencer, B., 1985. Export Subsidies and International Market Share Rivalry. Journal of International Economics, 18.

5. Danzig, G. B., 1963. Linear Programming and extensions. Princeton University Press, Princeton.

6. Dutta, B. and Masso, J., 1997. Stability of Matchings When Individuals Have Preferences over Colleagues. Journal of Economic Theory, 75, 464-475.

7. Echenique, F. and Yenmez, B., 2007. A Solution to Matching with Preferences over Colleagues. Games and Economic Behavior, 59(1), 46-71.

8. Gale, D. and Shapley, L.S., 1962. College Admissions and the Stability of Marriage. American Mathematical Monthly, 69, 9-15.

9. Gudmundsson, J. and Habis, H., 2013. Assignment Games with Externalities. Lund University, Department of economics, Working Paper, 2013:27

10. Hafalir, Isa E., 2008. Stability of Marriage with Externalities. International Journal of Game Theory, 37 (3), 353-369.

11. Hahn, F. H. 1987. Conjectural equilibria, in "The New Palgrave: A Dictionary of Economics" (Eatwell, J., Milgate, M., and Newman, P. Eds.), Macmillan Press, London.

12. Jehiel, P. and Moldovanu, B., 1996. Strategic Non-Participation. RAND Journal of Economics, 27 (1), 84-98.

13. Legros, P. and Newman, A. F., 2007. Beauty Is A Beast And Frog Is a Prince: Assortive Matching with Nontransferable Utilities. Econometrica, 75 (4), 1073-1102.

14. Li, S., 1993. Competitive matching equilibrium and multiple principal-agent models, mimeo, University of Minnesota.

15. Mumcu, A., and Saglam, I., 2007. The Core of a Housing Market with Externalities. Economics Bulletin, 3(55); 1-5.

16. Mumcu, A., and Saglam, I., 2008. Characterizing stable one-to-one matchings under interdependent preferences. Mimeo.

17. Osborne, M. J. and Rubinstein, A., 1995. A Course In Game Theory. The MIT Press, London.

18. Roth, A. and Sotomayor, M. A. O., 1992. Two-sided matching, in "Handbook of Game Theory with Economic Application, Vol. 1" (R. J. Aumann and S. Hart, Eds.), North-Holland, Amsterdam, Netherlands.

19. Roy Chowdhury, P., 2004. Marriage Markets with Externalities. Indian Statistical Institute, Planning Unit, New Delhi, Discussion Papers, 04-11.

20. Sasaki, H. and Toda, M., 1996. Two-Sided Matching Problems with Externalities. Journal of Economic Theory, 70 (1), 93-108.

21. Shapley, L. S. and Shubik, M., 1969. On the core of an economic system with externalities. America Economic Review, 59, 678-684.

22. Shapley, L. S. and Shubik, M., 1972. The assignment game. I. The core, International Journal of Game Theory, 1, 111-130.

23. Uetake, Kosuke and Watanabe, Yasutora, August 12, 2012. Entry by Merger: Estimates from a Two-Sided Matching Model with Externalities. Available at SSRN: http://ssrn.com/abstract=2188581 or http://dx.doi.org/10.2139/ssrn.2188581

Appendix

A. Examples.

Example A.1. *(More efficient matching may not be stable)*: Let $n = 3$. Then we have six possible matchings. Consider a valuation function which assigns values to the pairs of agents in the six matchings as follows.

$$\mu_1 \begin{array}{ccc} 100 & 0 & 0 \\ i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{array} \qquad \mu_2 \begin{array}{ccc} 1 & 1 & 1 \\ i_1 & i_2 & i_3 \\ j_2 & j_3 & j_1 \end{array} \qquad \mu_3 \begin{array}{ccc} 1 & 1 & 1 \\ i_1 & i_2 & i_3 \\ j_3 & j_1 & j_2 \end{array}$$

$$\mu_4 \begin{array}{ccc} 1 & 1 & 1 \\ i_1 & i_2 & i_3 \\ j_1 & j_3 & j_2 \end{array} \qquad \mu_5 \begin{array}{ccc} 1 & 1 & 1 \\ i_1 & i_2 & i_3 \\ j_3 & j_2 & j_1 \end{array} \qquad \mu_6 \begin{array}{ccc} 1 & 1 & 1 \\ i_1 & i_2 & i_3 \\ j_2 & j_1 & j_3 \end{array}$$

Let the universal conjectures hold: $\varphi_i(j) = \varphi_j(i) = A(i,j)$ for all $i \in I, j \in J$. Then, all the matchings are admissible, and the values for the characteristic function are as following:

$$\begin{aligned} V(i_2 j_2) = V(i_3 j_3) = 0 \\ V(ij) = 1 \; for \; all \; (i,j) \notin \{(i_2,j_2),(i_3,j_3)\}. \end{aligned}$$

Hence, the maximum value to the primal problem is $z_{max} = 1$, and $\mu_2, \mu_3, \mu_4, \mu_5,$ *and* μ_6 are all stable matchings. But matching μ_1, the most efficient matching, is not a stable matching since it is blocked by (i_2, j_3) and (i_3, j_2).

Example A.2. *(Non-existence of Equilibrium Under Rational Conjectures):* Let $n = 3$. Consider a valuation function which assigns values to the pairs of agents in the six matchings as follows.[26]

$$\mu_1 \begin{array}{ccc} 4 & 7 & 7 \\ i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{array} \qquad \mu_2 \begin{array}{ccc} 7 & 6 & 8 \\ i_1 & i_2 & i_3 \\ j_2 & j_3 & j_1 \end{array} \qquad \mu_3 \begin{array}{ccc} 6 & 10 & 5 \\ i_1 & i_2 & i_3 \\ j_3 & j_1 & j_2 \end{array}$$

$$\mu_4 \begin{array}{ccc} 7 & 8 & 9 \\ i_1 & i_2 & i_3 \\ j_1 & j_3 & j_2 \end{array} \qquad \mu_5 \begin{array}{ccc} 7 & 10 & 7 \\ i_1 & i_2 & i_3 \\ j_3 & j_2 & j_1 \end{array} \qquad \mu_6 \begin{array}{ccc} 3 & 9 & 8 \\ i_1 & i_2 & i_3 \\ j_2 & j_1 & j_3 \end{array}$$

[26] In this example, $r(i_p j_q)$ and $V(i_p j_q)$ are written as $r(pq)$ and $V(pq)$, respectively, for short.

Let $r(ij)$ denote the rational conjectural stable matchings in the market $M \setminus \{i, j\}$. For $n = 2$, there are no externalities, and so rational conjectures can be easily obtained by looking at the maximal total valuation of the four agents. Thus, we have

$$\begin{array}{lll} r(11) = \{\mu_4\} & r(12) = \{\mu_6\} & r(13) = \{\mu_5\} \\ r(21) = \{\mu_3\} & r(22) = \{\mu_5\} & r(23) = \{\mu_4\} \\ r(31) = \{\mu_5\} & r(32) = \{\mu_3\} & r(33) = \{\mu_6\}. \end{array}$$

These are also agents' estimations. Consequently, the values for the characteristic function are obtained as follows.

$$\begin{array}{lll} V(11) = 7 & V(12) = 3 & V(13) = 7 \\ V(21) = 10 & V(22) = 10 & V(23) = 8 \\ V(31) = 7 & V(32) = 5 & V(33) = 8. \end{array}$$

By putting these values into the primal problem, we obtain the maximum value $z_{max} = V(11) + V(22) + V(33) = 25$. This value is not achievable by any (admissible) matching, since the maximal total valuation of the six agents is 24 at matching μ_5.[27] Therefore, there is no equilibrium in this example.

Notice that matching $\mu_1 = \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$ is not admissible and only gives a total valuation 18 which is much smaller than 25.

B. Proofs for Section 3

Proof for Lemma 3.1. (a). Since μ^* is admissible, we have

$$\begin{aligned} \pi(\mu^*, i, j) &\geq V(ij|\varphi) \quad \forall (i, j) \in \mu^*, \text{ and therefore} \\ \sum_{(i,j)\in\mu^*} \pi(\mu^*, i, j) &\geq \sum_{(i,j)\in\mu^*} V(ij|\varphi) = z_{max} = w_{min}. \end{aligned}$$

Thus, (u^*, v^*) is feasible, and (μ^*, u^*, v^*) is a φ-pseudo-equilibrium.

(b). For the "if" part, simply use the fact that a φ-equilibrium is a φ-pseudo-equilibrium. For the "only if" part, suppose (μ, u, v) is φ-pseudo-equilibrium. (μ, u^*, v^*) is apparently a φ-pseudo-equilibrium. Since μ is admissible, we have $\pi(\mu, i, j) \geq V(ij|\varphi)$ for all $(i, j) \in \mu$, and thus we can construct a φ-equilibrium (μ, u', v'), in which

$$(u', v') > (u^*, v^*) \textit{ and } u'_i + v'_j = \pi(\mu, i, j) \textit{ for all } (i, j) \in \mu.$$

(c). Follows from (b).

□

[27] Notice that although matching μ_4 also gives a total valuation 24, it is not admissible.

Proof for Theorem 3.1. Firstly, we show that, for any given problem (I, J, π), if $\varphi_i(j) = \varphi_j(i) = A(i,j)$ for all $i \in I$ and all $j \in J$, the set $PE_\varphi(I,J,\pi)$ of φ-pseudo-equilibria is non-empty. To prove that, we construct a μ^* and a (u^*, v^*), in which μ^* is induced by an extreme solution to the primal problem and (u^*, v^*) is a solution to the dual problem. This μ^* is apparently admissible. Then, by Lemma 3.1, (μ^*, u^*, v^*) is a pseudo-equilibrium.[28]

Secondly, we show that, for a problem (I, J, π), if either $\varphi_i(j) \neq A(i,j)$ or $\varphi_j(i)$ for some pair of $i \in I$ and for $j \in J$, then there exists a valuation function π such that $PE_\varphi(I,J,\pi) = \emptyset$. Notice that this can not happen if there are only two agent on each side of the market, we only need to look at the cases in which $n = |I| = |J| \geq 3$. The proof is divided into several steps.

Step 1. Let us first consider a sub-LP problem of a pair of agents (i', j'), by ignoring the characteristic values related to each of them. Let $z_{max}(i', j')$ denote the maximum value to the following *primal* problem:[29]

$$\begin{aligned}
\max_x z = &\sum_{i\in I'}\sum_{j\in J'} V(ij|A(i,j)) && \\
s.t. \quad &\sum_{i\in I'} x_{ij} \leq 1 && \forall j \in J', \\
&\sum_{j\in J'} x_{ij} \leq 1 && \forall i \in I', \\
&x_{ij} \geq 0 && \forall i \in I' \text{ and } j \in J',
\end{aligned}$$

where $I' = I \setminus \{i'\}$ and $J' = J \setminus \{j'\}$.

Let $w_{min}(i', j')$ denote the minimum value to the following *dual* problem:

$$\begin{aligned}
\min_{u,v} w = &\sum_{i\in I'} u_i + \sum_{j\in J'} v_j && \\
st. \quad &u_i + v_j \geq V(ij|A(i,j)) && \forall i \in I' \text{ and } \forall j \in J', \\
&u_i \geq 0 \text{ and } v_j \geq 0 && \forall i \in I' \text{ and } \forall j \in J'.
\end{aligned}$$

Again, the fundamental duality theorem tells us that $w_{min}(i', j') = z_{max}(i', j')$, where $w_{min}(i', j')$ is the minimum value of the sum of the payoffs that the other agents demand when (i', j') are going to pair up.

Step 2. W.l.o.g., assume that $\varphi_{i_1}(j_2) \neq A(i_1, j_2)$. Then there is a $\mu \in A(i_1, j_2) \setminus \varphi_{i_1}(j_2)$.Let us denote $j'_k = \mu(i_k)$ for all $k \geq 2$. For all $k \geq 3$, let π

[28] Or see Sasaki and Toda (1996) for a constructed example.

[29] $V(ij|A(i,j)) = min\{\pi(\mu, i, j)| \ (i,j) \in \mu, \mu \in A(i,j)\}$. That is, although we ignore i' and j', the characteristic value of a pair (i,j) is still taken from all the full matchings in $A(i,j)$, but not necessarily in $A(i,j) \cap A(i', j')$.

satisfy

$$\pi(\tilde{\mu}, i_k, j'_k) + z_{max}(i_k, j'_k) > \pi(\tilde{\mu}')$$

for all $\tilde{\mu} \in A(i_k, j'_k)$ and for all $\tilde{\mu}' \notin A(i_k, j'_k)$. This is equivalent to

$$\pi(\tilde{\mu}, i_k, j'_k) > \pi(\tilde{\mu}') - w_{min}(i_k, j'_k)$$

for all $\tilde{\mu} \in A(i_k, j'_k)$ and for all $\tilde{\mu}' \notin A(i_k, j'_k)$. Then any matchings other than μ and $\mu' = \{(i_1, j'_2), (i_2, j_2), (i_3, j'_3), ..., (i_n, j'_n)\}$ are blocked by some pair (i_k, j'_k) for $k \geq 3$. Then, similarly, let π satisfy

$$\pi(\bar{\mu}, i_1, j'_2) > \pi(\mu) - w_{min}(i_1, j'_2)$$

for all $\bar{\mu} \in A(i_1, j'_2)$. Then μ is blocked by (i_1, j'_2). Lastly, let π satisfy

$$\pi(\hat{\mu}, i_1, j_2) > \pi(\mu') - w_{min}(i_1, j_2)$$

for all $\hat{\mu} \in \varphi_{i_1}(j_2)$. Then we have that μ' is blocked by (i_1, j_2). Therefore, $PE_\varphi(I, J, \pi) = \emptyset$ for the problem (I, J, π) constructed in this way.

Combining the above results with Lemma 3.1(b), we get our desired result. □

C. Proofs for Section 4

C.1: Existence of Assortative Stable Matchings

Lemma 5.1. *Given a twice continuously differentiable function $u(\cdot,\cdot) : D \times G \to R_{--}$, suppose, $d_1 > d_2 >, ..., > d_n \in D$ and $g_1 > g_2 >, ..., > g_n \in G$, and $2 \leq p \leq q \leq n$. For any $k = 2, 3, ..., p$, if $u''_{12} \geq 0$, then*

$$(u_{11} - u_{1q}) - (u_{p1} - u_{pq}) \geq (u_{k-1k-1} - u_{k-1k}) - (u_{kk-1} - u_{kk}),$$

and if $u''_{12} \leq 0$, then

$$(u_{11} - u_{1q}) - (u_{p1} - u_{pq}) \leq (u_{k-1k-1} - u_{k-1k}) - (u_{kk-1} - u_{kk}),$$

Proof.

$$\begin{aligned}
&[(u_{11} - u_{1q}) - (u_{p1} - u_{pq})] - [(u_{k-1k-1} - u_{k-1k}) - (u_{kk-1} - u_{kk})] \\
=&[(u_{11} - u_{1q}) - (u_{k-1k-1} - u_{k-1k})] + [(u_{kk-1} - u_{kk}) - (u_{p1} - u_{pq})] \\
=&\,[[(u_{11} - u_{1q}) - (u_{1k-1} - u_{1k})] + [(u_{1k-1} - u_{1k})) - (u_{k-1k-1} - u_{k-1k})]] \\
&[-[(u_{p1} - u_{pq}) - (u_{pk-1} - u_{pk})] + [(u_{kk-1} - u_{kk}) - (u_{pk-1} - u_{pk})]] \\
=&\,[[(u_{11} - u_{1k-1}) + (u_{1k} - u_{1q})] + [(u_{1k-1} - u_{1k})) - (u_{k-1k-1} - u_{k-1k})]] \\
&[-[(u_{p1} - u_{pk-1}) + (u_{pk} - u_{pq})] + [(u_{kk-1} - u_{kk}) - (u_{pk-1} - u_{pk})]] \\
=&[(u_{11} - u_{1k-1}) - (u_{p1} - u_{pk-1})] + [(u_{1k} - u_{1q}) - (u_{pk} - u_{pq})] \\
&+ [(u_{1k-1} - u_{1k})) - (u_{k-1k-1} - u_{k-1k})] + [(u_{kk-1} - u_{kk}) - (u_{pk-1} - u_{pk})] \\
\geq&0
\end{aligned}$$

The last inequality holds, since by supermodularity the terms in the 4 brackets on the left hand side of the inequality are all nonnegative. $\square$

Lemma 5.2. *Given a twice differentiable function $u(\cdot,\cdot) : D \times G \to R_{--}$, suppose $u_1' > 0$, $d_1 > d_2 >, ..., > d_n \in D$ and $g_1 > g_2 >, ..., > g_n \in G$, and $p \leq q \leq n$, then for any $k = 2, 3, ..., p$,*

$$(u_{11} + u_{1q}) - (u_{p1} + u_{pq}) > (u_{kk-1} + u_{kk}) - (u_{k-1k-1} + u_{k-1k})$$

Proof.

$$\begin{aligned}
&[(u_{11} + u_{1q}) - (u_{p1} + u_{pq})] - [(u_{kk-1} + u_{kk}) - (u_{k-1k-1} + u_{k-1k})] \\
=&[(u_{11} - u_{p1}) + (u_{1q} - u_{pq})] + [(u_{k-1k-1} - u_{kk-1}) + (u_{k-1k} - u_{kk})] \\
>&0
\end{aligned}$$

The last inequality follows because the terms in the brackets are all nonnegative. $\square$

Lemma 5.3. *Given a twice differentiable function $u(\cdot,\cdot)$, suppose $\frac{Y}{n}+u > 0$.*

1. *If $u_1'u_2' \geq u_{12}''(-\frac{Y}{n} - u)$, $u_1' > 0, u_2' > 0$, and $u_{12}'' \leq 0$, or*
2. *if $u_1'u_2' \leq u_{12}''(-\frac{Y}{n} - u)$, $u_1' > 0$, $u_2' < 0$, and $u_{12}'' \geq 0$,*

then

$$\frac{u(a,b) - u(a,b')}{u(a',b) - u(a',b')} \geq \frac{\frac{Y}{n} + \frac{1}{2}[u(a',b) + u(a',b')]}{\frac{Y}{n} + \frac{1}{2}[u(a,b) + u(a,b')]},$$

where $a > a'$ and $b > b'$.

Proof. We show for Case 1. The analysis for Case 2 is similar, and thus is omitted.

$$\begin{aligned}
&u_1'u_2' \geq u_{12}''(-\frac{Y}{n} - u) \\
\Rightarrow& u_2'(a',\cdot)[u(a,\cdot) - u(a',\cdot)] \geq [u_2'(a,\cdot) - u_2'(a',\cdot)][-\frac{Y}{n} - u(a,\cdot)] \\
\Rightarrow& u_2'(a,\cdot)u(a,\cdot) - u_2'(a',\cdot)u(a',\cdot) \geq [u_2'(a,\cdot) - u_2'(a',\cdot)][-\frac{Y}{n}] \\
\Rightarrow& u_2'(a,\cdot)[\frac{Y}{n} + u(a,\cdot)] \geq u_2'(a',\cdot)[\frac{Y}{n} + u(a',\cdot)] \\
\Rightarrow& [u(a,b) - u(a,b')]\frac{Y}{n} + \frac{1}{2}[u^2(a,b) - u^2(a,b')] \geq [u(a',b) - u(a',b')]\frac{Y}{n} + \frac{1}{2}[u^2(a',b) - u^2(a',b')] \\
\Rightarrow& \frac{u(a,b) - u(a,b')}{u(a',b) - u(a',b')} \geq \frac{\frac{Y}{n} + \frac{1}{2}[u(a',b) + u(a',b')]}{\frac{Y}{n} + \frac{1}{2}[u(a,b) + u(a,b')]}.
\end{aligned}$$

The second inequality holds because $u_{12}'' \leq 0$, $u_2' > 0$, and $u_1' > 0$. The last inequality follows from $u_2' > 0$ and $\frac{Y}{n} + u > 0$. $\square$

Lemma 5.4. *For given real numbers $a, b, c, d, e, f > 0$,*

1. *if $b \geq a$, then $\frac{a+c}{b+c} \geq \frac{a}{b}$;*

2. *if $\frac{c}{d} \geq \frac{a}{b}$, then $\frac{a+c}{b+d} \geq \frac{a}{b}$;*

3. *if $a - c > 0$, $b - d > 0$, and $\frac{a-c}{b-d} > \frac{e}{f}$, then*

$$\frac{a}{b} \geq \frac{c}{d} \Rightarrow \frac{a+e}{b+f} > \frac{c+e}{d+f}.$$

Lemma 5.5. *Given a twice continuously differentiable function $u(\cdot,\cdot) : D \times G \to R_{--}$, suppose D and G are compact, $u_1', u_2' > 0$, $\frac{X}{n} + u > 0$, and $d_1 > d_2 >, ..., > d_n \in D$ and $g_1 > g_2 >, ..., > g_n \in G$.*

1. *If $u_{12}'' \geq 0$, or*

2. *if $u_{12}'' \leq 0$ and $u_1' u_2' \geq u_{12}''(-\frac{X}{n} - u + \frac{n-1}{n} u_{min})$,*

then for $p = 2, 3, ..., n$ and $q = 2, 3, ..., n$,

$$\left[X + (n+1)u_{11} - \sum_{(i,j) \in \mu_+^M} u_{ij} \right]^2 + \left[X + (n+1)u_{pq} - \sum_{(i,j) \in \mu_+^{M(pq)}} u_{ij} \right]^2$$
$$> \left[X + (n+1)u_{1q} - \sum_{(i,j) \in \mu_+^{M(1q)}} u_{ij} \right]^2 + \left[X + (n+1)u_{p1} - \sum_{(i,j) \in \mu_+^{M(p1)}} u_{ij} \right]^2.$$

Proof. Before doing the analysis, we first, w.l.o.g., assume $p \leq q$, and derive the following equations.

$$\begin{aligned}
&\left(\left[X+(n+1)u_{11}-\sum_{(i,j)\in\mu_+^M}u_{ij}\right]^2+\left[X+(n+1)u_{pq}-\sum_{(i,j)\in\mu_+^{M(pq)}}u_{ij}\right]^2\right)\\
&-\left(\left[X+(n+1)u_{1q}-\sum_{(i,j)\in\mu_+^{M(1q)}}u_{ij}\right]^2+\left[X+(n+1)u_{p1}-\sum_{(i,j)\in\mu_+^{M(p1)}}u_{ij}\right]^2\right)\\
&=\left((n+1)(u_{11}-u_{1q})-\left[\left(u_{11}+\sum_{k=2}^{q}u_{kk}\right)-\left(u_{1q}+\sum_{k=2}^{q}u_{kk-1}\right)\right]\right)\\
&\quad\cdot\left(2X+(n+1)(u_{11}+u_{1q})-\left[\sum_{(i,j)\in\mu_+^M}u_{ij}+\sum_{(i,j)\in\mu_+^{M(1q)}}u_{ij}\right]\right)\\
&-\left((n+1)(u_{p1}-u_{pq})-\left[\left(u_{p1}+\sum_{k=1}^{p-1}u_{kk+1}+\sum_{k=p+1}^{q}u_{kk}\right)-\left(u_{pq}+\sum_{k=1}^{p-1}u_{kk}+\sum_{k=p+1}^{q}u_{kk-1}\right)\right]\right)\\
&\quad\cdot\left(2X+(n+1)(u_{pq}+u_{p1})-\left[\sum_{(i,j)\in\mu_+^{M(pq)}}u_{ij}+\sum_{(i,j)\in\mu_+^{M(p1)}}u_{ij}\right]\right)\\
&=\left[n(u_{11}-u_{1q})+\sum_{k=2}^{q}(u_{kk-1}-u_{kk})\right]\\
&\quad\cdot\left(2X+(n+1)(u_{11}+u_{1q})-\left[\sum_{(i,j)\in\mu_+^M}u_{ij}+\sum_{(i,j)\in\mu_+^{M(1q)}}u_{ij}\right]\right)\\
&-\left[n(u_{p1}-u_{pq})+\sum_{k=1}^{p-1}(u_{kk}-u_{kk+1})+\sum_{k=p+1}^{q}(u_{kk-1}-u_{kk})\right]\\
&\quad\cdot\left(2X+(n+1)(u_{pq}+u_{p1})-\left[\sum_{(i,j)\in\mu_+^{M(pq)}}u_{ij}+\sum_{(i,j)\in\mu_+^{M(p1)}}u_{ij}\right]\right)
\end{aligned}$$

For convenience, denote T^1, T^2, T^3, T^4 as the terms in the 4 big brackets on the right hand side of the last equality, respectively. We want to show that in both cases

$$T^1T^2 - T^3T^4 > 0.$$

Firstly, we have

$$
\begin{aligned}
u_2' > 0 \ &\Rightarrow T^1, T^3 > 0, \quad and\\
\frac{X}{n} + u > 0 \ &\Rightarrow T^2, T^4 > 0.
\end{aligned}
$$

For Case 1, it is sufficient to show that $T^1 \geq T^3$ and $T^2 > T^4$ separately.

$$
\begin{aligned}
&T^1 - T^3\\
=&\left[n[(u_{11} - u_{1q}) - (u_{p1} - u_{pq})] - \sum_{k=2}^{p}[(u_{k-1k-1} - u_{k-1k}) - (u_{kk-1} - u_{kk})]\right]\\
\geq&0,
\end{aligned}
$$

where the inequality follows from Lemma 5.1, the fact that $n \geq p$, and that $u_{12}'' \geq 0$. Strict inequality holds if $u_{12}'' > 0$.

$$
\begin{aligned}
&T^2 - T^4\\
=&\left(n(u_{11} + u_{1q}) - \left[\left(\sum_{k=2}^{q} u_{kk}\right) + \left(\sum_{k=2}^{q} u_{kk-1}\right)\right]\right)\\
&-\left(n(u_{pq} + u_{p1}) - \left[\left(\sum_{k=2}^{p} u_{k-1k-1} + \sum_{k=p+1}^{q} u_{kk-1}\right) + \left(\sum_{k=2}^{p} u_{k-1k} + \sum_{k=p+1}^{q} u_{kk}\right)\right]\right)\\
=&\left[n(u_{11} + u_{1q}) - \sum_{k=2}^{p}(u_{kk} + u_{kk-1})\right] - \left[n(u_{pq} + u_{p1}) - \sum_{k=2}^{p}(u_{k-1k-1} + u_{k-1k})\right]\\
=&n[(u_{11} + u_{1q}) - (u_{pq} + u_{p1})] - \left[\sum_{k=2}^{p}[(u_{kk} + u_{kk-1}) - (u_{k-1k-1} + u_{k-1k})]\right]\\
>&0.
\end{aligned}
$$

The inequality follows from Lemma 5.2, the facts that $n \geq p$, and that the term in the first big bracket on the left hand side of the inequality is positive because $u_1' > 0$.

For Case 2, it is sufficient to show that $\frac{T^1}{T^3} \geq \frac{T^4}{T^2}$. Firstly, we have

$$
\begin{aligned}
&u_{12}'' \leq 0 \ and \ u_2' > 0\\
&\qquad\Rightarrow 0 < \frac{u_{11} - u_{1q}}{u_{p1} - u_{pq}} \leq 1 \quad and \quad \frac{u_{kk-1} - u_{kk}}{u_{k-1k-1} - u_{k-1k}} \geq 1;\\
&u_1' > 0\\
&\qquad\Rightarrow 0 < -[u_{k-1k-1} + u_{k-1k}] < -[u_{kk-1} + u_{kk}].
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\frac{T^1}{T^3} =& \frac{[n(u_{11}-u_{1q})] + \left[\sum_{k=2}^{p}(u_{kk-1}-u_{kk})\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1}-u_{kk})\right]}{[n(u_{p1}-u_{pq})] + \left[\sum_{k=2}^{p}(u_{k-1k-1}-u_{k-1k})\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1}-u_{kk})\right]} \\
=& \frac{[(n+1-p)(u_{11}-u_{1q})] + \left[\sum_{k=2}^{p}[(u_{11}-u_{1q}) + (u_{kk-1}-u_{kk})]\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1}-u_{kk})\right]}{[(n+1-p)(u_{p1}-u_{pq})] + \left[\sum_{k=2}^{p}[(u_{p1}-u_{pq}) + (u_{k-1k-1}-u_{k-1k})]\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1}-u_{kk})\right]} \\
\geq& \frac{[(n+1-p)(u_{11}-u_{1q})] + \left[\sum_{k=2}^{p}[(u_{11}-u_{1q}) + (u_{kk-1}-u_{kk})]\right]}{[(n+1-p)(u_{p1}-u_{pq})] + \left[\sum_{k=2}^{p}[(u_{p1}-u_{pq}) + (u_{k-1k-1}-u_{k-1k})]\right]} \\
=& \frac{[n(u_{11}-u_{1q})] + \left[\sum_{k=2}^{p}(u_{kk-1}-u_{kk})\right]}{[n(u_{p1}-u_{pq})] + \left[\sum_{k=2}^{p}(u_{k-1k-1}-u_{k-1k})\right]} \\
\geq& \frac{n(u_{11}-u_{1q})}{n(u_{p1}-u_{pq})} \\
\geq& \frac{2Y + n(u_{pq}+u_{p1})}{2Y + n(u_{11}+u_{1q})} \\
>& \frac{[2X + n(u_{pq}+u_{p1})] + [2(n-1)(-u_{min})]}{[2X + n(u_{11}+u_{1q})] + [2(n-1)(-u_{min})]} \\
\geq& \frac{[2X + n(u_{pq}+u_{p1})] - \left[\sum_{k=2}^{p}(u_{k-1k-1}+u_{k-1k})\right] - \left[\sum_{k=p+1}^{q}(u_{kk-1}+u_{kk})\right] - 2\left[\sum_{k=q+1}^{n}u_{kk}\right]}{[2X + n(u_{11}+u_{1q})] - \left[\sum_{k=2}^{p}(u_{kk-1}+u_{kk})]\right] - \left[\sum_{k=p+1}^{q}(u_{kk-1}+u_{kk})\right] - 2\left[\sum_{k=q+1}^{n}u_{kk}\right]} \\
=& \frac{T^4}{T^2}.
\end{aligned}
$$

The first and second inequalities follow from Lemma 5.1, Lemma 5.4.1, and 5.4.2; the third inequality follows from Lemma 5.3 with $Y = X-(n-1)u_{min}$; the fifth inequality follows from Lemma 5.4.3. $\square$

Lemma 5.6. *Given a twice continuously differentiable function $u(\cdot,\cdot) : D \times G \to R_{--}$, suppose D and G are compact, $u_1' > 0, u_2' < 0$, $\frac{X}{n} + u > 0$, and $d_1 > d_2 >, ..., > d_n \in D$ and $g_1 > g_2 >, ..., > g_n \in G$.*

1. *If $u_{12}'' \leq 0$, or*
2. *if $u_{12}'' > 0$ and $u_1'u_2' \leq u_{12}''(\frac{X}{n} - u + \frac{n-1}{n}u_{min}\}$,*

then for $p = 2, 3, ..., n$ and $q = 2, 3, ..., n$,

$$
\begin{aligned}
&\left[X + (n+1)u_{11} - \sum_{(i,j)\in\mu_+^M} u_{ij}\right]^2 + \left[X + (n+1)u_{pq} - \sum_{(i,j)\in\mu_+^{M(pq)}} u_{ij}\right]^2 \\
<& \left[X + (n+1)u_{1q} - \sum_{(i,j)\in\mu_+^{M(1q)}} u_{ij}\right]^2 + \left[X + (n+1)u_{p1} - \sum_{(i,j)\in\mu_+^{M(p1)}} u_{ij}\right]^2.
\end{aligned}
$$

Proof. The proof is basically the same as the the one for Lemma 5.5, and thus is omitted. $\square$

Lemma 5.7. *Given a twice continuously differentiable function $u(\cdot,\cdot): D \times G \to R_{--}$, suppose D and G are compact, $u_1' > 0, u_2' < 0$, $u_{12}'' > 0$, $\frac{X}{n} + u > 0$, and $d_1 > d_2 >, ..., > d_n \in D$ and $g_1 > g_2 >, ..., > g_n \in G$. If*

$$\frac{(u_2')_{min} \cdot (u_1')_{max}}{(u_{12}'')_{min}[-\frac{X}{n} - u_{min} + \frac{n-1}{n} u_{max}]} \leq \frac{n^2}{(2n-1)(n+1)},$$

then for any $l = 1, 2, ..., n-1$, $p = 2, 3, ..., n$, $q = 2, 3, ..., n$, and $l < p, l < q$.

$$\left[X + (n+1)u_{ll} - \sum_{(i,j)\in\mu_+^M} u_{ij}\right]^2 + \left[X + (n+1)u_{pq} - \sum_{(i,j)\in\mu_+^{M(pq)}} u_{ij}\right]^2$$
$$> \left[X + (n+1)u_{lq} - \sum_{(i,j)\in\mu_+^{M(lq)}} u_{ij}\right]^2 + \left[X + (n+1)u_{pl} - \sum_{(i,j)\in\mu_+^{M(pl)}} u_{ij}\right]^2.$$

Proof. We omit the similar computation as of the one at the beginning of the proof for Lemma 5.5, yet we denote T^1, T^2, T^3, T^4 as the similar corresponding terms in the proof of Lemma 5.5, and $T^1, T^3 < 0$ by Lemma 5.1 and $T^2, T^4 > 0$ by Lemma 5.2. We want to show that $\frac{T^1}{T^3} < \frac{T^4}{T^2}$. W.l.o.g., assume $p \leq q$.

$$\begin{aligned}
\frac{T^1}{T^3} =& \frac{[(n+l-p)(u_{ll} - u_{lq})] + \left[\sum_{k=l+1}^{p}[(u_{ll} - u_{lq}) + (u_{kk-1} - u_{kk})]\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1} - u_{kk})\right]}{[(n+l-p)(u_{pl} - u_{pq})] + \left[\sum_{k=l+1}^{p}[(u_{pl} - u_{pq}) + (u_{k-1k-1} - u_{k-1k})]\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1} - u_{kk})\right]} \\
<& \frac{[(n+l-p)(u_{ll} - u_{lq})] + \left[\sum_{k=l+1}^{p}[(u_{pl} - u_{pq}) + (u_{k-1k-1} - u_{k-1k})]\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1} - u_{kk})\right]}{[(n+l-p)(u_{pl} - u_{pq})] + \left[\sum_{k=l+1}^{p}[(u_{pl} - u_{pq}) + (u_{k-1k-1} - u_{k-1k})]\right] + \left[\sum_{k=p+1}^{q}(u_{kk-1} - u_{kk})\right]} \\
<& \frac{n(u_{ll} - u_{lq}) + (q-l)(u_{ql} - u_{qq})}{n(u_{pl} - u_{pq}) + (q-l)(u_{ql} - u_{qq})} \\
<& \frac{n[(u_{ll} - u_{lq}) - (u_{pl} - u_{pq})]}{(n+q-l)(u_{ql} - u_{qq})} + 1 \\
=& \frac{n}{n+q-l} \cdot \frac{\int_{d_p}^{d_l}\int_{g_q}^{g_l} u_{12}''(d,g)\, dddg}{\int_{g_q}^{g_l} u_2'(d_q, g) dg} + 1 \\
\leq& \frac{n}{n+q-l} \cdot \frac{(u_{12}'')_{min} \cdot (d_l - d_p) \cdot (g_1 - g_q)}{(u_2')_{min} \cdot (b_1 - b_q)} + 1 \\
=& \frac{n}{n+q-l} \cdot \frac{(u_{12}'')_{min} \cdot (d_l - d_p)}{(u_2')_{min}} + 1,
\end{aligned}$$

where the first inequality follows from Lemma 5.1, and the second inequality follows from Lemma 5.4 and $u''_{12} > 0$.

$$
\begin{aligned}
\frac{T^4}{T^2} =& \frac{\left[2X+n(u_{pq}+u_{pl})\right]-\left[\sum_{k=l+1}^{p}(u_{k-1k-1}+u_{k-1k})\right]-\left[\sum_{k=p+1}^{q}(u_{kk-1}+u_{kk})\right]-2\left[\sum_{k=q+1}^{n}u_{kk}+\sum_{k=1}^{l-1}u_{kk}\right]}{\left[2X+n(u_{ll}+u_{lq})\right]-\left[\sum_{k=l+1}^{p}(u_{kk-1}+u_{kk})\right]\right]-\left[\sum_{k=p+1}^{q}(u_{kk-1}+u_{kk})\right]-2\left[\sum_{k=q+1}^{n}u_{kk}+\sum_{k=1}^{l-1}u_{kk}\right]} \\
=& \frac{n[(u_{pq}+u_{pl})-(u_{ll}+u_{lq})]+\left[\sum_{k=l+1}^{p}[(u_{kk-1}+u_{kk})-(u_{k-1k-1}+u_{k-1k})]\right]}{[2X+n(u_{ll}+u_{1q})]-\left[\sum_{k=l+1}^{q}(u_{kk-1}+u_{kk})]\right]-2\left[\sum_{k=q+1}^{n}u_{kk}+\sum_{k=1}^{l-1}u_{kk}\right]}+1 \\
>& \frac{-n\left[\int_{d_p}^{d_l}u'_1(d,g_q)dd+\int_{d_p}^{d_l}u'_1(d,g_l)dd\right]-\sum_{k=l+1}^{p}\left[\int_{d_k}^{d_{k-1}}u'_1(d,g_{k-1})dd+\int_{d_k}^{d_{k-1}}u'_1(d,g_k)dd\right]}{2X+2nu_{min}-2(n-1)u_{max}}+1 \\
\geq& \frac{-2n[(u'_1)_{max}\cdot(d_l-d_p)]-2[(u'_1)_{max}\cdot(d_l-d_p)]}{2X+2nu_{min}-2(n-1)u_{max}}+1 \\
=& \frac{-(n+1)[(u'_1)_{max}\cdot(d_l-d_p)]}{X+nu_{min}-(n-1)u_{max}}+1.
\end{aligned}
$$

Hence, a sufficient condition to ensure $\frac{T^1}{T^3} < \frac{T^4}{T^2}$ is that for any $l = 1, 2, ..., n-1$, $p = 2, 3, ..., n$, $q = 2, 3, ..., n$, and $l < p, l < q$,

$$
\begin{aligned}
&\frac{n}{n+q-l}\cdot\frac{(u''_{12})_{min}\cdot(d_l-d_p)}{(u'_2)_{min}} \leq \frac{-(n+1)[(u'_1)_{max}\cdot(d_l-d_p)]}{X+nu_{min}-(n-1)u_{max}} \\
\Leftrightarrow&\frac{n^2}{(n+q-l)(n+1)} \geq \frac{(u'_2)_{min}\cdot(u'_1)_{max}}{(u''_{12})_{min}[-\frac{X}{n}-u_{min}+\frac{n-1}{n}u_{max}]}.
\end{aligned}
$$

Since the minimum value on the left side of the last inequality is $\frac{n^2}{(2n-1)(n+1)}$, our statement holds. In addition, it also holds for any $X' > X$. □

Lemma 5.8. *Given a twice continuously differentiable function $u(\cdot,\cdot) : D \times G \to R_{--}$, suppose D and G are compact, $u'_1 > 0, u'_2 < 0$, $u''_{12} < 0$, $\frac{X}{n} + u > 0$, and $d_1 > d_2 >, ..., > d_n \in D$ and $g_1 > g_2 >, ..., > g_n \in G$. If*

$$
\frac{(u'_2)_{max}\cdot(u'_1)_{max}}{(u''_{12})_{max}[-\frac{X}{n}-u_{min}+\frac{n-1}{n}u_{max}]} \leq \frac{n^2}{(2n-1)(n+1)},
$$

then for any $l = 1, 2, ..., n-1$, $p = 2, 3, ..., n$, $q = 2, 3, ..., n$, and $l < p, l < q$.

$$
\begin{aligned}
&\left[X+(n+1)u_{ll}-\sum_{(i,j)\in\mu_+^M}u_{ij}\right]^2+\left[X+(n+1)u_{pq}-\sum_{(i,j)\in\mu_+^{M(pq)}}u_{ij}\right]^2 \\
<&\left[X+(n+1)u_{lq}-\sum_{(i,j)\in\mu_+^{M(lq)}}u_{ij}\right]^2+\left[X+(n+1)u_{pl}-\sum_{(i,j)\in\mu_+^{M(pl)}}u_{ij}\right]^2.
\end{aligned}
$$

Proof. The proof is basically the same as the proof for Lemma 5.7, and therefore we omitted most part of the proof.

In this case $T^1, T^2, T^3, T^4 > 0$, and we want to show $\frac{T^1}{T^3} < \frac{T^4}{T^2}$. By the same computation as before, we have

$$\frac{T^1}{T^3} < \frac{n}{n+q-l} \cdot \frac{(u''_{12})_{max} \cdot (d_l - d_p)}{(u'_2)_{max}} + 1,$$

and

$$\frac{T^4}{T^2} > \frac{-(n+1)[(u'_1)_{max} \cdot (d_l - d_p)]}{X + nu_{min} - (n-1)u_{max}} + 1,$$

Hence, it is sufficient if

$$\frac{(u'_2)_{max} \cdot (u'_1)_{max}}{(u''_{12})_{max}[-\frac{X}{n} - u_{min} + \frac{n-1}{n}u_{max}]} \leq \frac{n^2}{(2n-1)(n+1)}.$$

□

Proposition 5.1. [30] *Suppose in an assignment game with no externalities* $< I, J, u >$,$|I| = |J| = n$, *each agent* $i \in I$ *is of some type* $a_i \in F$ *and agent* j *of some type* $b_j \in S$, *in which* F *and* S *are compact intervals and no two agents on the same side are of the same type, and* $u(\cdot, \cdot) : F \times S \to\in R_{--}$ *is twice continuously differentiable with* $u'_1, u'_2 > 0$ *and* $\frac{H}{n} + u > 0$. *(i.e.,* $u(a, b)$ *is the surplus of a pair of paired-up agents whose types are* a *and* b *respectively, and it is increasing in both* a *and* b.*) Then*

1. *if the cross derivative* $u''_{12} \geq 0$, *or*
2. *if* $u''_{12} < 0$ *and* $u'_1 u'_2 \geq u''_{12}(-\frac{H}{n} - u + \frac{n-1}{n}u_{min})$,

then

$$(i.) \qquad \mu^M_+ = \arg\max_{\mu \in A(I,J)} V^+(\mu), \quad \textit{and}$$

$$(ii.) \qquad \mu^M_- \neq \arg\max_{\mu \in A(I,J)} V^-(\mu);$$

3. *if*

$$u''_{12} < 0 \quad \textit{and} \quad \frac{(u'_2)_{max} \cdot (u'_1)_{max}}{(u''_{12})_{max}[-\frac{H}{n} - u_{min} + \frac{n-1}{n}u_{max}]} \leq \frac{n^2}{(2n-1)(n+1)},$$

[30] There might be two thought for proving this proposition. Look at the supermodular case. Take a matching μ such that $(i, j'), (i', j) \in \mu$ and $a_i \geq a_{i'}, b_j \leq b_{j'}$. Take $\mu' := \{(i, j), (i', j'), \mu \backslash \{(i, j'), (i', j)\}\}$. One thought is that firstly show $V + (\mu) \leq \pi(\mu)$, and then show $\pi(\mu) \leq \pi(\mu')$. Another thought is that show that the values of the terms in $V^+(\mu')$ is a mean increasing spread of that of $V^+(\mu)$. However, neither of these thoughts work out since we can find counter examples that they are not always true.

then

$$(iii.) \qquad \mu_-^M = \arg\max_{\mu\in A(I,J)} V^-(\mu), \quad \textit{and}$$

$$(iv.) \qquad \mu_+^M \neq \arg\max_{\mu\in A(I,J)} V^+(\mu).$$

In the above,

$$V^+(\mu) = \sum_{(p,q)\in\mu}\left[H + (n+1)u(a_p, b_q) - \sum_{(i,j)\in\mu_+^{M(pq)}} u(a_p, b_q)\right]^2;$$

$$V^-(\mu) = \sum_{(p,q)\in\mu}\left[H + (n+1)u(a_p, b_q) - \sum_{(i,j)\in\mu_-^{M(pq)}} u(a_p, b_q)\right]^2.$$

Proof. W.l.o.g., we assume $a_1 > a_2 > ... > a_n$ and $b_1 > b_2 > ... > b_n$. We just show the supermodularity case, and the proof is proceeded by analyzing the agents at one side of the market one by one sequentially.

For Case (i), we show that under the conditions agent (i_1, j_1) must be to with each other, and then (i_2, j_2),..., so on so forth until (i_n, j_n).

Take a matching $\mu_1 \in A(I, J)$. If $(i_1, j_1) \notin \mu_1$, we can show that for matching $\mu_1' := \{(i_1, j_1), (\mu_1(j_1), \mu_1(i_1))\} \cup [\mu_1 \backslash \{(i_1, \mu_1(i_1)), (\mu_1(j_1), j_1)\}]$,

$$V^+(\mu_1') > V^+(\mu_1),$$

by directly using Lemma 5.5 with setting X to be H.

Similarly, for $k = 2, 3, ..., n-1$, take a matching $\mu_k \in A(i_1, j_1) \cap A(i_2, j_2) \cap .... \cap A(i_k - 1, j_k - 1)$. If $(i_k, j_k) \notin \mu_k$, we can show that for matching $\mu_k' := \{(i_k, j_k), (\mu_k(j_k), \mu_k(i_k))\} \cup [\mu_k \backslash \{(i_k, \mu_k(i_k)), (\mu_k(j_k), j_k)\}]$,

$$V^+(\mu_k') > V^+(\mu_k),$$

by using Lemma 5.5 with the variable X being properly substituted.

As a result, $\mu_+^M = \arg\max_{\mu\in A(I,J)} V^+(\mu)$, since we cannot find another matching that induces a higher V^+ value than μ_+^M does.

For Case (ii), we define a function $w(a, \beta) := u(a, b(\beta))$ in which $b(\beta) := -\beta$, and apply Lemma 5.6 to function $w(\cdot, \cdot)$. Then, it is easy to see that

$$V^-(\mu_-^M) < V^-(\mu')$$

in which $\mu' := \{(i_1, j_1), (i_n, i_n)\} \cup [\mu_-^M \backslash \{(i_1, j_n), (i_1, j_n)\}$.

For Case (iii), we again define a function $w(a, \beta) := u(a, b(\beta))$ in which $b(\beta) := -\beta$, apply Lemma 5.7 to function $w(\cdot, \cdot)$, and then apply the same method in the analysis of Case 1, it can be show that the μ_-^M indeed induces the maximum value of V^-.

Similarly for Case (iv), by applying Lemma 5.8, we directly have

$$V^+(\mu_+^M) < V^+(\mu')$$

in which $\mu' := \{(i_l, j_k), (i_k, j_l)\} \cup [\mu_+^M \backslash \{(i_l, j_l), (i_k, j_k)\}$ for any $l, k = 1, 2, 3, .., n$ and $l \neq k$.

$\square$

Proof for Theorem 4.1. The above question is a quadratic problem and thus can be solved by a linear system. W.l.o.g., let us assume α to be 1.

We first derive for each realized matching the solution to the Cournot game (i.e., the output each firm-manager pair produces for a realized matching μ). For any pair $(i, j) \in \mu$, (i, j) chooses $q_{ij}(\mu)$ (also wrote as $q_{f_i, m_j}(\mu)$) such that

$$q_{ij}(\mu) = \arg\max_{q \geq 0} q \cdot [H - (q + \sum_{(i',j') \in \mu, i' \neq i, j' \neq j} q_{i'j'}) - c(f_i, s_j)]$$

The first order condition for each pair $(i, j) \in \mu$ gives us the following linear system.

$$\begin{bmatrix} H - c(f_1, s_{\mu(f_1)}) \\ H - c(f_2, s_{\mu(f_2)}) \\ \vdots \\ H - c(f_n, s_{\mu(f_n)}) \end{bmatrix} = \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 2 \end{bmatrix} \begin{bmatrix} q_{f_1, \mu(f_1)} \\ q_{f_2, \mu(f_2)} \\ \vdots \\ q_{f_n, \mu(f_n)} \end{bmatrix}$$

Thus

$$\begin{bmatrix} q_{f_1, \mu(f_1)} \\ q_{f_2, \mu(f_2)} \\ \vdots \\ q_{f_n, \mu(f_n)} \end{bmatrix} = \begin{bmatrix} \frac{n}{n+1} & -\frac{1}{n+1} & \dots & -\frac{1}{n+1} \\ -\frac{1}{n+1} & \frac{n}{n+1} & -\frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{n+1} & \dots & -\frac{1}{n+1} & \frac{n}{n+1} \end{bmatrix} \begin{bmatrix} H - c(f_1, s_{\mu(f_1)}) \\ H - c(f_2, s_{\mu(f_2)}) \\ \vdots \\ H - c(f_n, s_{\mu(f_n)}) \end{bmatrix}$$

Therefore, for a pair $(i, j) \in \mu$, their optimal output quantity for a realized matching μ is

$$q_{ij}(\mu) = \frac{1}{n+1}[H - (n+1)c_{ij} + \sum_{(i',j') \in \mu} c_{i'j'}] > 0.$$

Substituting back the above solution to the surplus functions, we get

$$\pi(i, j, \mu) = \frac{1}{(n+1)^2}[H - (n+1)c_{ij} + \sum_{(i',j') \in \mu} c_{i'j'}]^2 = q_{ij}^2(\mu).$$

Here we emphasize that $q_{ij}(\mu)$ is the amount of output that (i,j) are going to produce when (i,j) believe that if they pair up with each other, μ is going to be the resulting matching and all the firms will play the Cournot game in the second stage. That is, $q_{ij}(\mu)$ is (i,j)'s best response to their conjectures, and thus $\pi(i,j,\mu)$ is the corresponding surplus for (i,j).

Next, we define a function $u(\cdot,\cdot) := -c(\cdot,\cdot)$. Then the theorem can be proved simply by applying Propsition 5.1 to $< I, J, u >$. We do not go through the cases one by one here, but just illustrate that by showing the PASM case.

Under positive assortative conjectures (i.e., $\varphi_i(j) = \{\mu_+^{M(ij)}\}$), μ_+^M is admissible. Thus, to show μ_+^M is an stable matching, we only need to show that μ_+^M is induced by a solution to the primal problem of the game. Firstly,

$$c_1' < 0, c_2' < 0, \quad and \quad c_{12}'' \leq 0$$

implies

$$u_1' > 0, u_2' > 0, \quad and \quad u_{12}'' \geq 0,$$

and secondly,

$$\begin{aligned}
z(\mu) &= \sum_{(i,j)\in\mu} V(ij|\varphi) \\
&= \sum_{(i,j)\in\mu} \pi\left(i,j,(i,j)\cup\mu_+^{M(ij)}\right) \\
&= \frac{1}{(n+1)^2} \sum_{(i,j)\in\mu} [H-(n+1)c_{ij} + \sum_{(i',j')\in\mu_+^{M(ij)}} c_{i'j'}]^2 \\
&= \frac{1}{(n+1)^2} \sum_{(i,j)\in\mu} [H+(n+1)u_{ij} - \sum_{(i',j')\in\mu_+^{M(ij)}} u_{i'j'}]^2.
\end{aligned}$$

By applying Proposition 5.1, we know that z is maximized at μ_+^M. Therefore, μ_+^M is an equalibrium matching, and hence is a PASM as desired.

□

C.2: Rational Equilibrium

Proof for Theorem 4.2. The proof is basically a repetition of the proof for Theorem 4.1, thus we only give a sketch here, readers can go through Lemma 5.5, Lemma 5.7, and Proposition 5.1 again to derive the result.

For Case 1 and Case 2, we want to show that for any conditional reduced market

$$M^r := \{M^s \mid \mu^c\},$$

$$
\begin{aligned}
\mu_+^{M^s} &= \arg \max_{\mu^s \in A(I^s,J^s)} z^{M^r}(\mu^s|\varphi_+^{M^r}) \\
&= \arg \max_{\mu^s \in A(I^s,J^s)} \sum_{(i,j)\in\mu^s} V(ij|\varphi_+^{M^r}) \\
&= \arg \max_{\mu^s \in A(I^s,J^s)} \sum_{(i,j)\in\mu^s} \pi\left(i,j,\mu^c \cup \mu_+^{M^s(ij)}\right) \\
&= \arg \max_{\mu^s \in A(I^s,J^s)} \frac{1}{(n+1)^2} \sum_{(i,j)\in\mu^s} \left[H - (n+1)c_{ij} + \sum_{(i',j')\in\mu^c\cup\mu_+^{M(ij)}} c_{i'j'} \right]^2 .
\end{aligned}
$$

Next, define $u(\cdot,\cdot) = -c(\cdot,\cdot)$ again. By exactly the same proof for Lemma 5.5, we could show

$$
\begin{aligned}
&\left[X + (n+1)u_{11} - \sum_{(i,j)\in\mu^c\cup\mu_+^{M^s}} u_{ij} \right]^2 + \left[X + (n+1)u_{pq} - \sum_{(i,j)\in\mu^c\cup\mu_+^{M^s(pq)}} u_{ij} \right]^2 \\
> &\left[X + (n+1)u_{1q} - \sum_{(i,j)\in\mu^c\cup\mu_+^{M^s(1q)}} u_{ij} \right]^2 + \left[X + (n+1)u_{p1} - \sum_{(i,j)\in\mu^c\cup\mu_+^{M^s(p1)}} u_{ij} \right]^2 .
\end{aligned}
$$

Then by the same logic in the proof for Proposition 5.1, we could show that $\mu_+^{M^s}$ is the one maximizes the value of z^{M^r}.

For Case 3, using the same method with Lemma 5.7 and Proposition 5.1, we can show that

$$
\mu_-^{M^s} = \arg \max_{\mu^s \in A(I^s,J^s)} z^{M^r}(\mu^s|\varphi_-^{M^r})
$$

as well.

□

C.3: Efficient Matching

Proof for Proposition 4.1. W.l.o.g., assume again $\alpha = 1$. For a given matching μ, the social benefit is

$$
\begin{aligned}
B(\mu) :=& \int_0^{Q(\mu)} H - q \; dq \\
=& HQ(\mu) - \frac{Q^2(\mu)}{2} \\
=& \frac{H}{n+1}[nH - c(\mu)] - \frac{1}{2(n+1)^2}[nH - c(\mu)]^2 \\
=& \frac{1}{n+1}\left([n - \frac{n^2}{2(n+1)}]H^2 - \frac{1}{n+1}Hc(\mu) - \frac{1}{2(n+1)}c^2(\mu) \right),
\end{aligned}
$$

and the total cost of production is

$$
\begin{aligned}
C(\mu) :=& \frac{1}{n+1} \sum_{(i,j)\in\mu} c_{ij}[H - (n+1)c_{ij} + c(\mu)] \\
=& \frac{1}{n+1}[Hc(\mu) - (n+1) \sum_{(i,j)\in\mu} c_{ij}^2 + c^2(\mu)].
\end{aligned}
$$

Hence the social surplus is

$$
\begin{aligned}
& B(\mu) - C(\mu) \\
=& \frac{1}{n+1}\left([n - \frac{n^2}{2(n+1)}]H^2 - \frac{n+2}{n+1}Hc(\mu) - [1 + \frac{1}{2(n+1)}]c^2(\mu) + (n+1) \sum_{(i,j)\in\mu} c_{ij}^2 \right).
\end{aligned}
$$

Then, for any two matching μ and μ', in which $\mu \backslash \mu' = \{(i,j),(i',j')\}$ and $\mu' \backslash \mu = \{(i.j'),(i',j)\}$, we have

$$
\begin{aligned}
& [B(\mu) - C(\mu)] - [B(\mu') - C(\mu')] \\
=& -\frac{n+2}{n+1}H[c(\mu) - c(\mu')] - \left[1 + \frac{1}{2(n+1)}\right][c(\mu) - c(\mu')][c(\mu) + c(\mu')] \\
& + (n+1)[(c_{ij}^2 + c_{i'j'}^2) - (c_{ij'}^2 + c_{i'j}^2)] \\
=& -\frac{n+2}{n+1}H[(c_{ij} + c_{i'j'}) - (c_{ij'} + c_{i'j})] - \left[1 + \frac{1}{2(n+1)}\right][(c_{ij} + c_{i'j'}) - (c_{ij'} + c_{i'j})][c(\mu) + c(\mu')] \\
& + (n+1)[(c_{ij} - c_{ij'})(c_{ij} + c_{ij'}) - (c_{i'j} - c_{i'j'})(c_{i'j'} + c_{i'j})] \\
=& -\frac{n+2}{n+1}H[(c_{ij} + c_{i'j'}) - (c_{ij'} + c_{i'j})] - \left[1 + \frac{1}{2(n+1)}\right][(c_{ij} + c_{i'j'}) - (c_{ij'} + c_{i'j})][c(\mu) + c(\mu')] \\
& + (n+1)\left[(c_{ij} - c_{ij'})[(c_{ij} + c_{ij'}) - (c_{i'j'} + c_{i'j})] + [(c_{ij} - c_{ij'}) - (c_{i'j} - c_{i'j'})](c_{i'j'} + c_{i'j})\right].
\end{aligned}
$$

For Case 1, to show that μ_+^M can be inefficient when $c''_{12} < 0$, we construct the following example. We know that $c_{nn} = c_{max}$. Let us take two values $\tilde{c}$ and $\tilde{\tilde{c}}$ such that

$$c_{ij} \leq \tilde{c} < \tilde{\tilde{c}} \leq c_{nn-1} < c_{nn} \quad for\ all\ (i,j) \notin \{(n,n),(n,n-1)\}.$$

Take matching $\mu' := \{(i_n, j_{n-1}), (i_{n-1}, j_n)\} \cup [\mu_+^M \backslash \{(i_{n-1}, j_{n-1}), (i_n, j_n)\}]$. In the following we will show that there are some scenarios in which μ' is more efficient than μ_+^M. Firstly, $(c_{n-1n-1} - c_{n-1n}) - (c_{nn-1} - c_{nn}) < 0$ since $c''_{12} < 0$, and we have

$$\begin{aligned}
&\frac{[B(\mu_+^M) - C(\mu_+^M)] - [B(\mu') - C(\mu')]}{(c_{n-1n-1} - c_{n-1n}) - (c_{nn-1} - c_{nn})} \\
=& -\frac{n+2}{n+1}H - \left[1 + \frac{1}{2(n+1)}\right][c(\mu_+^M) + c(\mu')] + (n+1)(c_{n-1n-1} + c_{n-1n}) \\
&\qquad + (n+1)\frac{(c_{nn-1} - c_{nn})[(c_{n-1n-1} + c_{n-1n}) - (c_{nn} + c_{nn-1})]}{(c_{n-1n-1} - c_{n-1n}) - (c_{nn-1} - c_{nn})} \\
>& -\frac{n+2}{n+1}H - \left[1 + \frac{1}{2(n+1)}\right] \cdot 2(n-1)\tilde{c} + [n - \frac{1}{2(n+1)}] \cdot 2\tilde{\tilde{c}} \\
&\qquad + (n+1)\frac{(c_{i'j} - c_{i'j'})[(c_{ij} + c_{ij'}) - (c_{i'j'} + c_{i'j})]}{(c_{n-1n-1} - c_{n-1n}) - (c_{nn-1} - c_{nn})} \\
\geq& -\frac{n+2}{n+1}H - \frac{2n+3}{n+1} \cdot (n-1)\tilde{c} + \frac{2n^2+2n-1}{n+1}\tilde{\tilde{c}} + 2(n+1) \cdot \frac{(c'_2)_{min} \cdot (c'_1)_{min}}{(c''_{12})_{max}} \\
=& \frac{2n^2+2n-1}{n+1}\left[\tilde{\tilde{c}} - \frac{n^2+2n}{2n^2+2n-1} \cdot \frac{H}{n} - \frac{2n^2+n-3}{2n^2+2n-1}\tilde{c}\right] + 2(n+1) \cdot \frac{(c'_2)_{min} \cdot (c'_1)_{min}}{(c''_{12})_{max}} \\
>& \frac{2n^2+2n-1}{n+1}\left[(\tilde{\tilde{c}} - \tilde{c}) - \frac{n^2+2n}{2n^2+2n-1} \cdot \frac{H}{n}\right] + 2(n+1) \cdot \frac{(c'_2)_{min} \cdot (c'_1)_{min}}{(c''_{12})_{max}}.
\end{aligned}$$

It is not hard to see that there are certain circumstances in which the term on the right hand side of the last inequality can be larger than or equal to 0. For instance, when $\tilde{\tilde{c}}$ is sufficiently larger than $\frac{n^2+2n}{2n^2+2n-1} \cdot \frac{H}{n}$ (but of course $\tilde{\tilde{c}}$ has to be less than $\frac{H}{n}$), and $\tilde{c}$ and $\frac{(c'_2)_{min} \cdot (c'_1)_{min}}{(c''_{12})_{max}}$ are sufficiently close to 0, then the term is larger than 0. Hence in this scenario

$$[B(\mu_+^M) - C(\mu_+^M)] - [B(\mu') - C(\mu')] < 0,$$

and thus μ_+^M is less efficient than μ'.

For Case 2, take a matching $\mu' \neq \mu_+^M$. W.l.o.g, assume $(i,j'),(i',j) \in \mu'$, in which $i > i'$ and $j > j'$. Take another matching $\mu = \{(i,j),(i',j')\} \cup$

$[\mu'\backslash\{(i,j'),(i,j')\}]$. Then we have

$$
\begin{aligned}
&[B(\mu)-C(\mu)]-[B(\mu')-C(\mu')]\\
=&[(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})]\Big(-\frac{n+2}{n+1}H-\left[1+\frac{1}{2(n+1)}\right][c(\mu)+c(\mu')]+(n+1)(c_{i'j'}+c_{i'j})\\
&\qquad+(n+1)\frac{(c_{ij}-c_{ij'})[(c_{ij}+c_{ij'})-(c_{i'j'}+c_{i'j})]}{(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})}\Big)\\
>&[(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})]\Big(-\frac{n+2}{n+1}H+n(c_{i'j'}+c_{i'j})\\
&\qquad+(n+1)\frac{(c_{ij}-c_{ij'})[(c_{ij}+c_{ij'})-(c_{i'j'}+c_{i'j})]}{(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})}\Big)\\
>&[(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})]\Big(-\frac{n+2}{n+1}H+2n\frac{H}{n}\\
&\qquad+(n+1)\frac{(c_{ij}-c_{ij'})[(c_{ij}+c_{ij'})-(c_{i'j'}+c_{i'j})]}{(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})}\Big)\\
>&[(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})]\Big(\frac{n}{n+1}H+(n+1)\cdot\frac{\left[\int_{s_{j'}}^{s_j}c_2'(f_i,s)ds\right]\cdot\left[\int_{f_{i'}}^{f_i}c_1'(f,s_j)df+\int_{f_{i'}}^{f_i}c_1'(f,s_{j'})df\right]}{\int_{f_{i'}}^{f_i}\int_{f_{i'}}^{f_i}c_{12}''(f,s)dfds}\Big)\\
>&[(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})]\Big(\frac{n}{n+1}H+(n+1)\cdot\frac{\left[(c_2')_{max}(s_j-s_{j'})\right]\cdot 2\left[(c_1')_{max}(f_i-f_{i'})\right]}{(c_{12}'')_{min}(s_j-s_{j'})(f_i-f_{i'})}\Big)\\
&[(c_{ij}-c_{ij'})-(c_{i'j}-c_{i'j'})]\Big(\frac{n}{n+1}H+2(n+1)\cdot\frac{(c_2')_{max}\cdot(c_1')_{max}}{(c_{12}'')_{min}}\Big)\\
\geq&0.
\end{aligned}
$$

That is, for any matching $\mu'\neq\mu_+^M$, there is another matching which is more efficient than μ'. Hence μ_+^M is the most efficient matching.

For Case 3, take a matching $\mu'\neq\mu_+^M$. W.l.o.g, assume $(i,j'),(i',j)\in\mu'$, in which $i>i'$ and $j>j'$. Take another matching $\mu=\{(i,j),(i',j')\}\cup[\mu'\backslash\{(i,j'),(i,j')\}]$. Then we have

$$\begin{aligned}
&\frac{[B(\mu_+^M)-C(\mu_+^M)]-[B(\mu')-C(\mu')]}{(c_{n-1n-1}-c_{n-1n})-(c_{nn-1}-c_{nn})}\\
=&-\frac{n+2}{n+1}H-\left[1+\frac{1}{2(n+1)}\right][c(\mu_+^M)+c(\mu')]+(n+1)(c_{n-1n-1}+c_{n-1n})\\
&\qquad\qquad+(n+1)\frac{(c_{nn-1}-c_{nn})[(c_{n-1n-1}+c_{n-1n})-(c_{nn}+c_{nn-1})]}{(c_{n-1n-1}-c_{n-1n})-(c_{nn-1}-c_{nn})}\\
>&-\frac{n+2}{n+1}H-\left[1+\frac{1}{2(n+1)}\right]\cdot 2H+(n+1)\frac{(c_{i'j}-c_{i'j'})[(c_{ij}+c_{ij'})-(c_{i'j'}+c_{i'j})]}{(c_{n-1n-1}-c_{n-1n})-(c_{nn-1}-c_{nn})}\\
\geq&-\frac{3n+5}{n+1}H+2(n+1)\cdot\frac{(c_2')_{max}\cdot(c_1')_{max}}{(c_{12}'')_{max}}\\
\geq&0.
\end{aligned}$$

Thus, $[B(\mu_+^M)-C(\mu_+^M)]-[B(\mu')-C(\mu')]>0$, and μ_+^M must be the most efficient matching.

For Case 4, take a matching $\mu\neq\mu_-^M$. W.l.o.g, assume $(i,j),(i,j)\in\mu'$, in which $i>i'$ and $j>j'$. Take another matching $\mu'=\{(i',j),(i,j')\}\cup[\mu'\backslash\{(i,j),(i',j')\}]$. Then we have

$$\begin{aligned}
&\frac{[B(\mu_+^M)-C(\mu_+^M)]-[B(\mu')-C(\mu')]}{(c_{n-1n-1}-c_{n-1n})-(c_{nn-1}-c_{nn})}\\
<&-\frac{n+2}{n+1}H-\left[1+\frac{1}{2(n+1)}\right]\cdot 2c_{min}+(n+1)\cdot 2\frac{H}{n}+2(n+1)\cdot\frac{(c_2')_{min}\cdot(c_1')_{min}}{(c_{12}'')_{min}}\\
=&\frac{2n+3}{n+1}\left[\frac{n^2+2n+2}{2n^2+3n}\frac{H}{n}H-c_{min}\right]+2(n+1)\cdot\frac{(c_2')_{min}\cdot(c_1')_{min}}{(c_{12}'')_{min}}
\end{aligned}$$

Hence the right hand side of the last inequality is less than or equal to 0, if $c_{max}>c_{min}\geq\frac{n^2+2n+2}{2n^2+3n}\frac{H}{n}$ and $\frac{(c_2')_{min}\cdot(c_1')_{min}}{(c_{12}'')_{min}}$ is sufficiently close to 0. Then μ_-^M must be the most efficient matching by the same logic as in the previous cases.

For Case 5, take a matching $\mu'\neq\mu_+^M$. W.l.o.g, assume $(i,j'),(i',j)\in\mu'$, in which $i>i'$ and $j>j'$. Take another matching $\mu=\{(i,j),(i',j')\}\cup[\mu'\backslash\{(i,j'),(i,j')\}]$. Then we have

$$\begin{aligned}
&[B(\mu)-C(\mu)]-[B[\mu']-C(\mu')]\\
=&(c_{i'j}-c_{i'j'})(c_{ij}+c_{ij'})-(c_{i'j}-c_{i'j'})(c_{i'j'}+c_{i'j})\\
>&0.
\end{aligned}$$

Hence, μ_+^M is the most efficient matching.

□