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Convertible Bonds: Default Risk and Uncertain Volatility

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Abstract

Within a default intensity approach we discuss the optimal exercise of the callable and convertible bonds. Pricing bounds for convertible bonds are derived in an uncertain volatility model, i.e. when the volatility of the stock price process lies between two extreme values.

1 Introduction

A callable and convertible bond refers to a bond which can be converted into a firm's common shares at a predetermined number at the bondholder's decision, while the bond is also callable by the issuer, i.e. the bondholder can be enforced to surrender the bond to the issuer for a previously agreed price. Two sources of risks are essential for the valuation of the contract, one stemming from the randomness of prices, the other stemming from the randomness of the termination time, namely the contract can be stopped by call, conversion and default.

Empirical research indicates that firms that issue convertible bonds often tend to be highly leveraged, the default risk may play a significant role. Moreover, the equity and default risk cannot be treated independently and their interplay must be modeled explicitly. In the case that the true complex nature of the capital structure of the firm and information asymmetry make it hard to model the firm's value and the capital structure, the reduced-form model would be a more proper approach for the study of convertible bonds. Stock prices, credit spreads and implied volatilities of options are used as model inputs.

One of the early models which treat the callable and convertible bond with a reduced-form approach is proposed by Davis and Lischka (1999). They construct a model framework that incorporate Black-Scholes stock price, Gaussian stochastic interest rate and stochastic default intensity driven by a Brownian motion that also governs the movement of the stock price. It is called *two-and-a-half factors model* and has found its application in the industry. A similar model has been developed by Ayache, Forsyth and Vetzal (2003). Linetsky (2006) and Duffie and Singleton (2003)(p.206ff) model the default intensity as a negative power function of the underlying stock price. Duffie and Singleton (2003) valuate a callable and convertible bond with the intensity-based default model. In Bielecki, Crèpey, Jeanblanc and Rutkowski (2007) and Kovalov and Linetsky (2008) the default intensity is modeled as a deterministic function of the underlying stock price. The valuation of callable and convertible bond is explicitly related to the defaultable game option and BSDE or PDE is applied to solve the optimization problem.

In this paper the stock price is described by a jump diffusion. It jumps to zero at the time of default. In order to describe the interplay of the equity risk and the default risk of the issuer, we adopt a parsimonious, intensity-based default model, in which the default intensity is modeled as a function of the pre-default stock price. This assumes, in effect, that the equity price contains sufficient information to predict the default event. To make the combined effect of the default and equity risk of the underlying tractable, it is assumed that the default intensity has two values, one is the normal default rate, and the other one is much higher if the current stock price falls beneath a certain boundary. Thus, during the life time of the bond, the more time the stock price spends below the boundary, the higher the default risk. This model has certain similarity with some structural models, e.g. in the first-passage approach, the firm defaults immediately when its value falls below the boundary, while in the excursion approach, the firm defaults if it reaches and remains below the default threshold for a certain period.

In the intensity-based default model the default time is modeled as the time of the first jump of a Poisson process and it is not adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the pre-default stock price process. To price a defaultable contingent claim we need not only the information about the evolution of the pre-default stock price but also the knowledge whether default has occurred or not which is described by the filtration $(\mathcal{H}_t)_{t\in[0,T]}$. The filtration $(\mathcal{G}_t)_{t\in[0,T]}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, contains the full information and is larger than the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. This problem can be circumvented with specific modeling of the default time, e.g. Lando (1998) shows that if the time of default is modeled as the first jump of a Poisson process with random intensity, which is called doubly stochastic Poisson process or Cox process and under some measurable conditions, the expectations with respect to $\mathcal{G}_t$ can be reduced to the expectation with respect to $\mathcal{F}_t$. With the help of the filtration reduction we move to the fictitious default-free market in which cash flows are discounted according to the modified discount factor which is the sum of the risk free discount factor and the default intensity. Hence the results of the game option in the default-free setting can be extended to the defaultable game option in the intensity model. The embedded option rights owned by both of the bondholder and the issuer can be exercised optimally according to the well developed theory on the game option. The optimization problem is not approximated with recursions on a tree as in the case of the structural approach, it is formulated and solved with help of the theory of doubly reflected backward stochastic differential equations (BSDE) which is a more general approach developed by Cvitanić and Karatzas (1996). The parabolic partial differential equation (PDE) related to the doubly reflected BSDE is provided by Cvitanić and Ma (2001) and it can be solved with finite-difference methods. Furthermore, pricing bound is derived under rational optimal behavior, if the stock volatility is assumed to lie in a certain interval.

2 Intensity-based Default Model

In the following we will formulate the default event according to Lando (1998), where the time of default is modeled directly as the time of the first jump of a Poisson process with random intensity, which is called Cox process.

2.1 Cox process and default time

A Cox process is a generalization of the Poisson process in which the intensity is allowed to be random but in such a way that if it is conditional on a particular realization $h(\cdot,\omega)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $h(s,\omega)$. The random intensity is often characterized as a function of the current level of a set of state variables

$$h(s,\omega) = h(X_s).$$

X is an $\mathbb{R}^d$-valued stochastic process in the filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, Q)$. And $h:\ \mathbb{R}^d \to [0,\infty)$ is a nonnegative, continuous function. According to this construction the Cox process has the following properties

$$\mathbb{E}_Q[dN] = h(t)dt$$

and given the realization (path) of the intensity h,

$$\begin{aligned} P[N_t - N_s = n] &= \mathbb{E}_Q\Big[P[N_t - N_s = n] \mid h\Big] \\ &= \mathbb{E}_Q\left[\frac{1}{n!}\left(\int_s^t h(u)du\right)^n \exp\Big\{-\int_s^t h(u)du\Big\}\right]. \end{aligned}$$

In particular, the probability of no jumps in $[s,t]$ equals

$$P[N_t - N_s = 0] = \mathbb{E}_Q\left[\exp\Big\{-\int_s^t h(u)du\Big\}\right]. \tag{1}$$

Lando (1998) models the default time as the first jump time of a Cox process with intensity process $h(X_t)$,

$$\tau = \inf\left\{t \ge 0 : \int_0^t h(X_s)ds \ge E_1\right\}.$$

where E_1 is an exponentially distributed random variable with parameter 1. The state variables X may include information about stock price, risk-free interest rate and other economical relevant factors which can predict the likelihood of default. Given that a firm survives till time t, its default probability within the next small time interval Δt equals $h(X_t)\Delta t + o(\Delta t)$. According to Equation (1) the survival probability of a firm thus equals

$$P[\tau > t] = \mathbb{E}_Q\left[\exp\Big\{-\int_0^t h(u)du\Big\}\right].$$

2.2 Defaultable stock price dynamics

The literature on stock options usually model the firm's stock price as geometric Brownian motions and preclude the possibility of default. Whereas modeling of default event and credit spread is an essential task of study on corporate bond. Apart from convertible bonds there are also other hybrid products which have both the characteristics of equity and debt. Facing these problems, the two strands of research have merged recently. Default risk is integrated in the diffusion of the stock prices. In the reduced-form framework, one specifies the default intensity as a decreasing function of the underlying stock price. The default event is modeled as the first jump time of a doubly stochastic Poisson process. For example, Linetsky (2006) and Duffie and Singleton (2003) (p.206ff) model the default intensity as a negative power function of the underlying stock price. This assumes, in effect, that the equity price conveys sufficient information for the prediction of the default probability.

We assume that the Brownian motion which governs the movement of the stock prices is 1-dimensional[1]. The model framework is established according to Linetsky (2006).

Assumption 2.1. A filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, Q)$ where $\mathbb{G} := \{\mathcal{G}_t\}_{t\in[0,T]}$ is assumed. It supports a 1-dimensional Brownian motion $\{W_t, t \geq 0\}$, and an exponentially distributed random variable E_1 with parameter 1. The random variable E_1 is independent of the Brownian motion W. The stock price process S is subject to default. The pre-default stock price is denoted as $\tilde{S}_t$. The default intensity is specified as a decreasing function of the underlying stock price, and is denoted as $h(\tilde{S})$ where $h : \mathbb{R} \to [0, \infty)$ is a nonnegative, continuous function. The default time τ is modeled as

$$\tau = \inf\left\{t \geq 0 : \int_0^t h(\tilde{S}_u)du \geq E_1\right\}. \tag{2}$$

It corresponds to the first jump time of a doubly stochastic Poisson process with intensity $h(\tilde{S}_t)$. Take an equivalent martingale measure Q as given. Under Q, the pre-default stock price $\tilde{S}_t$ is a diffusion process solving the following stochastic differential equation

$$d\tilde{S}_t = (r_t + h(\tilde{S}_t))\tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t, \tag{3}$$

where r_t is the risk-free instantaneous interest rate and σ_t is the volatility of the pre-default stock price. Furthermore, it is assumed that if the firm defaults the stock price jumps to zero. Therefore the price process of the defaultable stock S follows the jump diffusion

$$dS_t = S_{t-}(r_t dt + \sigma_t dW_t - dM_t), \tag{4}$$

with

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} h(\tilde{S}_u)du ,$$

which is a martingale with respect to the filtration $\mathbb{G}$.

Assumption 2.2. In particular, we assume that the intensity function $h(\tilde{S}_t)$ has two values

$$h(\tilde{S}_t) = \begin{cases} a & \text{if } \tilde{S}_t \leq K \\ b & \text{if } \tilde{S}_t > K \end{cases} \tag{5}$$

[1] It is a rough approximation of the reality but it makes the computation tractable.

where a, b and K are constant and $a > b > 0$.

The firm has a normal default intensity b. If the firm is in trouble, i.e. the stock price is lower than the constant level K, it has a higher default rate a. Thus, during the life time of the bond, the more time the stock price spends below the boundary, the higher the default risk. Thus, the default intensity is strongly influenced by the stock price but they are not perfectly correlated. Moreover, this model has certain similarity with some structural models, e.g. in the first-passage approach, the firm defaults immediately when its value falls below the boundary, while in the excursion approach, the firm defaults if it reaches and remains below the default threshold for a certain period.

Linetsky (2006) and Duffie and Singleton (2003)(p.206ff) model the default intensity as a negative power function of the underlying stock price. In Linetsky (2006) closed-form solutions in form of spectral expansions are derived for bonds and stock options. The expansions contain several special functions and integration of them. In both papers, the parameters of the negative power function are chosen in the way that, there is a small region, if the stock price is above it, the default probability is quite low. As soon as the stock price goes below this region, the default probability rises dramatically. Therefore our simple assumption can be seen as an approximation of the power function modeling.

2.3 Information structure and filtration reduction

At first, we will explain the information structure due to the interplay of the stock and default risk. According to assumption 2.1 on the stock price and the default intensity, the information about the aforementioned two risks is contained in the full-filtration $\mathbb{G}$, which is composed of two sub-filtrations

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H},$$

where $\mathbb{G} := \{\mathcal{G}_t\}_{t\in[0,T]}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. $\mathcal{F}_t = \sigma\{\tilde{S}_s : 0 \leq s \leq t\}$ contains information about the evolution of the pre-default stock price $\tilde{S}_t$. In our model the default intensity $h(\tilde{S}_t)$ depends only on the pre-default stock price $\tilde{S}_t$, and there are no other state variables involved, therefore, the information about the likelihood of the default is given by $\mathcal{F}_t$. $\mathcal{H}_t = \sigma\{\mathbf{1}_{\tau\leq s} : 0 \leq s \leq t\}$ holds the information whether there has been a default till time t. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ then corresponds to knowing the evolution of the stock price up to time t and whether default has occurred or not. E_1 is independent of sigma field $\mathcal{F}_T$ and $\mathcal{H}_t \subseteq \sigma(E_1)$. In this information setting,

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t \vee \sigma(E_1). \tag{6}$$

Under such construction of filtration, it has been shown in Lando (1998) that, under some measurable conditions, the expectations with respect to $\mathcal{G}_t$ can be reduced to the expectation with respect to $\mathcal{F}_t$. There are three basic components for the valuation of default contingent claims: *promised payment* X at expiry, *a stream of payments* $Y_s 1_{\tau>s}$ which stops when default occurs and *recovery payment* Z_τ at time of default. In particular for convertible bonds the expiry time can be the maturity date T, the conversion or call time τ_b or τ_s, which is written as $\tilde{T} = \tau_b \wedge \tau_s \wedge T$. For a given equivalent martingale measure Q, the expected value of these three basic components are:

$$\mathbb{E}_Q\left[\exp\Big(-\int_t^{\tilde{T}} r_s ds\Big) X 1_{\tau>T}\Big|\mathcal{G}_t\right] = 1_{\tau>t}\mathbb{E}_Q\left[\exp\Big(-\int_t^{\tilde{T}}(r_s+h_s)ds\Big)X\Big|\mathcal{F}_t\right], \quad (7)$$

$$\mathbb{E}_Q\left[\int_t^{\tilde{T}} Y_s 1_{\tau>s}\exp\Big(-\int_t^s r_u du\Big)ds\Big|\mathcal{G}_t\right] = 1_{\tau>t}\mathbb{E}_Q\left[\int_t^{\tilde{T}} Y_s\exp\Big(-\int_t^s(r_u+h_u)du\Big)ds\Big|\mathcal{F}_t\right], \quad (8)$$

and

$$\mathbb{E}_Q\left[\exp\Big(-\int_t^{\tau} r_s ds\Big) Z_\tau\Big|\mathcal{G}_t\right] = 1_{\tau>t}\mathbb{E}_Q\left[\int_t^{\tilde{T}} Z_s h_s\exp\Big(-\int_t^s(r_u+h_u)du\Big)ds\Big|\mathcal{F}_t\right], \quad (9)$$

Where X is $\mathcal{F}_{\tilde{T}}$ measurable[2], i.e. $X \in \mathcal{F}_{\tilde{T}}$. Y and Z are adapted processes, i.e. Y_t and Z_t are measurable for each $t \in [0, \tilde{T}]$. h_u is the abbreviation of $h(\tilde{S_u})$ and stands for the default intensity. The *lhs* (left hand sides) of Equations (7), (8) and (9) show that, in the original market subject to default risk, cash flows are discounted according to the risk free discount factor $\exp(-\int_s^t r_u du)$. With the help of filtration reduction we move to the fictitious default-free market in which cash flows are discounted according to the modified discount factor $\exp(-\int_s^t (r_u + h_u)du)$. This effect is demonstrated by the *rhs* (right hand sides) of Equations (7), (8) and (9).

Remark 2.3. If the market is complete, e.g. the defaultable stock and defaultable discount bond with maturity T are tradeable, there exists a unique martingale measure P^* for the valuation. In incomplete market, the equivalent martingale measure Q can e.g. be the so-called minimal martingale measure introduced by Föllmer and Schweizer (1990) or the minimal entropy martingale measure proposed by Frittelli (2000). The former measure emerges from the mean-variance optimal hedging strategy which minimizes the variance between the random payoff and the terminal wealth generated from a self-financing strategy. Whereas the latter minimizes the relative entropy to the original objective measure P. Both measures have the nice property that zero risk premium is associated with default timing risk, i.e. the risk-neutral intensity under Q remains the same as the original intensity under P. Details about these results can be found e.g. in Blanchet-Scalliet, El Karoui and Martellin (2005).

3 Contract Feature

The bondholder can stop and convert the bond into stocks according to the prescribed conversion ratio γ. The conversion time of the bondholder is $\tau_b \in [0, \tau]$, where τ is the default time. The issuer which is often the shareholder can stop and buy back the bond for a price given by the maximum of call level H and the current conversion price, where H can be constant or time dependent. The call time of the seller is $\tau_s \in [0, \tau]$.

The payoff of a defaultable callable and convertible bond can be distinguished in four cases. The principal of the bond is L, R_t stands for the recovery process, S_t is the stock price at time t and c the coupon rate.

[2] Note that τ_b and τ_s can be any time in the interval $[0, T]$. The measurable condition is satisfied because conversion and call payoff are adapted processes.

(i) Let $\tau_b < \tau_s \leq T$, such that the contract begins at time 0 and is stopped and converted by the bondholder. In this case, the discounted payoff $ccb(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through conversion

$$conv(0) = c\int_0^{\tau_b\wedge\tau} \beta(0,s)ds + R_\tau \cdot \beta(0,\tau)\mathbf{1}_{\{\tau\leq\tau_b\}} + \beta(0,\tau_b)\mathbf{1}_{\{\tau_b<\tau\}}\gamma S_{\tau_b}.$$

(ii) Let $\tau_s < \tau_b \leq T$, such that the contract is bought back by the issuer before the bondholder converts. In this case, the discounted payoff $call(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through call,

$$call(0) = c\int_0^{\tau_s\wedge\tau} \beta(0,s)ds + R_\tau \cdot \beta(0,\tau)\mathbf{1}_{\{\tau\leq\tau_s\}} + \beta(0,\tau_s)\mathbf{1}_{\{\tau_s<\tau\}} \max[H, \gamma S_{\tau_s}].$$

(iii) If $\tau_s = \tau_b < T$ the discounted payoff of the bond equals the smaller value, i.e. the discounted payoff with conversion.

(iv) For $\tau_b \geq T$ and $\tau_s \geq T$, the discounted payoff of a callable and convertible bond at time 0 is

$$term(0) = c\int_0^{\tau\wedge T} \beta(0,s)ds + R_\tau \cdot \beta(0,\tau)\mathbf{1}_{\{\tau\leq T\}} + \beta(0,T)\mathbf{1}_{\{T<\tau\}} \max[\gamma S_T, L].$$

Denote the minimum of conversion and call time by $\zeta = \tau_s \wedge \tau_b$. Then, the discounted payoff of a callable and convertible bond in all four cases can be expressed with one equation,

$$\begin{aligned} cbb(0) \quad := \quad & \mathbf{1}_{\{\zeta<\tau\}}\Big(c\int_0^{\zeta\wedge T} \beta(0,s)ds + \mathbf{1}_{\{\zeta=\tau_s<\tau_b\leq T\}}\beta(0,\zeta)\max\{H,\gamma S_\zeta\} \\ & \qquad + \mathbf{1}_{\{\zeta=\tau_b<\tau_s<T\}}\beta(0,\zeta)\gamma S_\zeta + \mathbf{1}_{\{\zeta=T\}}\beta(0,T)\gamma S_T\Big) \\ + \quad & \mathbf{1}_{\{\tau\leq\zeta\}}\Big(c\int_0^{\tau\wedge T} \beta(0,s)ds + \mathbf{1}_{\{\tau\leq T\}}\beta(0,\tau)R_\tau + \mathbf{1}_{\{T<\tau\}}\beta(0,T)L\Big). \end{aligned} \tag{10}$$

Theorem 3.1. The payoff of a callable and convertible bond can be decomposed into a straight bond and a defaultable game option component $g(0)$.

$$ccb(0) = d(0) + g(0) \tag{11}$$

with

$$d(0) := c\int_0^{\tau\wedge T} \beta(0,s)ds + \mathbf{1}_{\{\tau\leq T\}}\beta(0,\tau)R_\tau + \mathbf{1}_{\{T<\tau\}}\beta(0,T)L$$

and

$$\begin{aligned} g(0) \quad := \quad & \mathbf{1}_{\{\zeta<\tau\}}\beta(0,\zeta)\Big\{\mathbf{1}_{\{\zeta=\tau_b<\tau_s<T\}}\left(\gamma S_\zeta - \phi_\zeta\right) \\ & \qquad + \mathbf{1}_{\{\zeta=\tau_s<\tau_b\leq T\}}\left(\max\{H_\zeta,\gamma S_\zeta\} - \phi_\zeta\right) + \mathbf{1}_{\{\zeta=T\}}\left(\gamma S_T - L\right)^+\Big\}. \end{aligned}$$

where

$$\phi_\zeta := c\int_\zeta^{\tau\wedge T}\beta(0,s)ds + \mathbf{1}_{\{\tau\le T\}}\beta(\zeta,\tau)R_\tau + \mathbf{1}_{\{T<\tau\}}\beta(\zeta,T)L \tag{12}$$

is the discounted value (discounted to time ζ) of the sum of the remaining coupon payments and the principal payment of a straight coupon bond given that it has not defaulted till time ζ.

4 Optimal Strategies

As the call value is strictly larger than the conversion value prior to maturity and they are the same at the maturity, thus, we can apply the the theories of game option developed by Kallsen and Kühn (2005). Within the reduced-form approach, the *max-min* and *min-max* strategies are still valid for the callable and convertible bond but they are derived with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$. The optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This *max-min strategy* of the bondholder leads to the lower value of the convertible bond, whereas the *min-max strategy* of the issuer leads to the upper value of the convertible bond. For a given martingale measure, the assumption that the call value is always larger than the conversion value prior to the maturity and they are the same at maturity T ensures that the lower value equals the upper value such that there exists a unique solution.

Under an equivalent martingale measure Q, the no-arbitrage price of the callable and convertible bond at the inception of the contract, $CCB(0)$ is given by

$$CCB(0) = \sup_{\tau_b\in\mathcal{G}_{0T}}\inf_{\tau_s\in\mathcal{G}_{0T}}\mathbb{E}_Q[ccb(0)|\mathcal{G}_0] = \inf_{\tau_s\in\mathcal{G}_{0T}}\sup_{\tau_b\in\mathcal{G}_{0T}}\mathbb{E}_Q[ccb(0)|\mathcal{G}_0]. \tag{13}$$

where $\mathcal{G}_{0T}$ is the set of stopping times with respect to the filtration $\{\mathcal{G}_u\}_{0\le u\le T}$ with values in $[0,T]$. After the inception of the contract, the value process $CCB(t)$ satisfies

$$\begin{aligned} CCB(t) &= \text{esssup}_{\tau_b\in\mathcal{G}_{tT}}\text{essinf}_{\tau_s\in\mathcal{G}_{tT}}\mathbb{E}_Q[ccb(0)|\mathcal{G}_t] \\ &= \text{essinf}_{\tau_s\in\mathcal{G}_{tT}}\text{esssup}_{\tau_b\in\mathcal{G}_{tT}}\mathbb{E}_Q[ccb(0)|\mathcal{G}_t]. \end{aligned} \tag{14}$$

where $\mathcal{G}_{tT}$ is the set of stopping times with respect to the filtration $\{\mathcal{G}_u\}_{t\le u\le T}$ with values in $[t,T]$. Furthermore, the optimal stopping times for the equity holder and bondholder respectively are

$$\begin{aligned} \tau_b^* &= \inf\{t\in[0,T] \mid conv(0)\ge CCB(t)\} \\ \tau_s^* &= \inf\{t\in[0,T] \mid call(0)\le CCB(t)\}. \end{aligned} \tag{15}$$

It is optimal to convert as soon as the current conversion value is equal to or larger than the value function $CBB(t)$, while the optimal strategy for the issuer is to call the bond as soon as the current call value is equal to or smaller than the value function $CBB(t)$.

In general, the optimization problem formulated via equation(13) has no closed-form solution.[3] After the reduction of the filtration from $(\mathcal{G}_t)_{t\in[0,T]}$ to $(\mathcal{F}_t)_{t\in[0,T]}$ the no-arbitrage value can be formulated as adapted solution of backward stochastic differential equations (BSDE) with two reflecting barriers. In Section 6 we give a brief summary of the results on BSDE which are closely related to the financial market. At first we show the reduction of the filtration.

5 Expected Payoff

Applying the methodology of filtration reduction described in Section 2.3 expected payoffs related to a callable and convertible bond have simple and explicit expressions. For a given equivalent martingale measure Q, the no-arbitrage price of a straight coupon bond with face value L, constant continuous coupon rate c, maturity T and a constant recovery amount R upon default time τ is

$$\begin{aligned} D(t) &= 1_{\tau>t}E_Q\left[\exp\Big(-\int_t^T(r_s+h_s)ds\Big)L\Big|\mathcal{F}_t\right] \\ &\quad +1_{\tau>t}\,E_Q\left[\int_t^T(c+R\cdot h_s)\cdot\exp\Big(-\int_t^s(r_u+h_u)du\Big)ds\Big|\mathcal{F}_t\right]. \end{aligned} \tag{16}$$

In the fictitious default-free market, the sum of the discounted cash flows in equation(16) corresponds to a default-free coupon bond with face value L and variable coupon rate $\bar{c}+R\cdot h_s$. The modified discount factor amounts $\exp(-\int_s^t(r_u+h_u)du)$. At the inception of the contract, $t=0$, the expression can be simplified to

$$\begin{aligned} D(0) &= E_Q\left[\exp\Big(-\int_0^T(r_s+h_s)ds\Big)L\right] \\ &\quad +E_Q\left[\int_0^T(\bar{c}+R\cdot h_s)\cdot\exp\Big(-\int_0^s(r_u+h_u)du\Big)ds\right]. \end{aligned} \tag{17}$$

where $E_Q[\,.\,]$ is an abbreviation for $E_Q[\,.\,|\mathcal{F}_0]$.

Equations (13) and (14) can be reformulated as

$$CCB(0)=\sup_{\tau_b\in\mathcal{F}_{0T}}\inf_{\tau_s\in\mathcal{F}_{0T}}\mathbb{E}_Q[ccb(0)|\mathcal{F}_0]=\inf_{\tau_s\in\mathcal{F}_{0T}}\sup_{\tau_b\in\mathcal{F}_{0T}}\mathbb{E}_Q[ccb(0)|\mathcal{F}_0], \tag{18}$$

where $\mathcal{F}_{0T}$ is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0\le u\le T}$ with values in $[0,T]$. After the inception of the contract, the value process $CCB(t)$ satisfies

$$\begin{aligned} CCB(t) &= \operatorname{esssup}_{\tau_b\in\mathcal{F}_{tT}}\operatorname{essinf}_{\tau_s\in\mathcal{F}_{tT}}\mathbb{E}_Q[ccb(0)|\mathcal{F}_t] \\ &= \operatorname{essinf}_{\tau_s\in\mathcal{F}_{tT}}\operatorname{esssup}_{\tau_b\in\mathcal{F}_{tT}}\mathbb{E}_Q[ccb(0)|\mathcal{F}_t], \end{aligned} \tag{19}$$

[3]The continuous time problem can be approximated with a discrete time one and the no-arbitrage price of the callable and convertible bond can then be derived e.g. by recursion alongside the branches of a tree. But the dynamic of the stock price is modeled as jump diffusion with varying drifts and it is sometimes difficult to construct a recombining tree especially if the uncertain volatility is considered. Therefore we need to solve it with the help of BSDE.

where $\mathcal{F}_{tT}$ is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{t\le u\le T}$ with values in $[t,T]$, and

$$\begin{aligned} E_Q[ccb(0)|\mathcal{F}_t] &= \mathbf{1}_{\{\tau>t\}} E_Q\Big[\int_t^{\zeta\wedge T} (\bar{c}+R\cdot h_s)\cdot \exp\Big(-\int_t^s (r_u+h_u)du\Big)ds \\ &\quad +\mathbf{1}_{\{\zeta=\tau_b<\tau_s<T\}} \exp\Big(-\int_t^{\zeta}(r_s+h_s)ds\Big)\gamma\tilde{S}_\zeta \\ &\quad +\mathbf{1}_{\{\zeta=\tau_s<\tau_b<T\}} \exp\Big(-\int_t^{\zeta}(r_s+h_s)ds\Big)\max[H,\gamma\tilde{S}_\zeta] \\ &\quad +\mathbf{1}_{\{\zeta=T\}} \exp\Big(-\int_t^{T}(r_s+h_s)ds\Big)\max[L,\gamma\tilde{S}_T]\ \Big|\ \mathcal{F}_t\Big]. \end{aligned}$$

6 Excursion: Backward Stochastic Differential Equations

The study of non-linear BSDE is initiated by Pardoux and Peng (1990). The authors prove existence and uniqueness of the solution under suitable assumptions on the coefficient and the terminal value of the BSDE. Since then it has been recognized that the theory of BSDE is a useful tool to formulate and study many problems in finance, e.g. hedging and pricing of European contingent claims, see El Karoui and Quenez (1997). Further studies are carried out in El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) to BSDE's with reflection, i.e., the solution is forced to stay above a given stochastic process. Existence and uniqueness of the solution is proved. Moreover they show that in a special case the solution is the value function of a mixed optimal stopping and optimal stochastic control problem. Concrete examples are pricing of American option in complete and incomplete market. These results are further generalized in Cvitanić and Karatzas (1996) to the case of two reflecting barrier processes, i.e. the solution process of the BSDE has to remain between the prescribed upper- and lower-boundary processes. They prove the existence of the solution and show that the solution coincides with the value of a Dynkin game, therefore establish the uniqueness of the solution. There are numerous studies on theory and numerics of BSDE's. A comprehensive review will go out of the range of our study. We will only summarize the results closely related to financial market, especially the game option.

6.1 Existence and uniqueness

The existence and uniqueness of the backward stochastic differential equation was first treated in Pardoux and Peng (1990).

Definition 6.1. Let $T\in\mathbb{R}_+$. Given a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P)$. The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by a d-dimensional Brownian motion W. Consider the following BSDE

$$-dY_t = f(t,Y_t,Z_t)dt - Z_t^\top dW_t, \qquad Y_T=\xi, \tag{20}$$

or equivalently

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s^\top dW_s$$

where

- The terminal value ξ is an n-dimensional $\mathcal{F}_T$-measurable square integrable random vector.

- f maps $\Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{d\times n}$ into $\mathbb{R}^n$. f is assumed to be $\mathcal{P} \bigotimes \mathcal{B}^n \bigotimes \mathcal{B}^{d\times n}$ measurable. $\mathcal{P}$ denotes σ-algebra of $\mathcal{F}_t$-progressively measurable subsets of $\Omega \times \mathbb{R}_+$. Moreover f is uniformly Lipshitz, i.e. there exists $C > 0$ such that $dt \times dP$ a.s. for all y_1, z_1, y_2, z_2

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

- Y and Z are $\mathbb{R}^n$ and $\mathbb{R}^{d\times n}$ valued progressively measurable processes and Y is continuous. $Z^\top$ denotes the transpose of the matrix Z.

- f is called the *driver* of the BSDE.

There *exists* a *unique* pair of adapted process (Y, Z) satisfies equation (20).

6.2 Forward backward stochastic differential equation

A well-investigated class of BSDE's is of the following form, it is also called forward backward stochastic differential equation (FBSDE)

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s^\top dW_s$$

where g and f are deterministic functions and X satisfies the following SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)^\top dW_s$$

where b and σ are measurable functions. The adapted solution of Y is associated to the solution of a quasi-linear parabolic PDE

$$\begin{cases} u_t + \frac{1}{2}tr\{\sigma\sigma^\top u_{xx}\} + bu_x + f(t, x, u, u_x\sigma) = 0 \\ u(T, x) = g(x). \end{cases} \tag{21}$$

The explicit expression of the solution (Y, Z) is

$$Y_t = u(t, X_t), \qquad Z_t = \partial_x u(t, X_t)\sigma(t, X_t).$$

6.3 Financial market

Consider a *complete market* there are $n + 1$ primary assets which are denoted by the vector $S = (S^0, S^1, ..., S^n)^\top$. S^0 is a non-risky asset and has the following price dynamic

$$dS_t^0 = S_t^0 r_t dt$$

r_t is the deterministic interest rate. The price process for S^i, $i \in (1, ..., n)$ is modeled by the linear SDE driven by an n-dimensional Brownian motion W, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$,

$$dS_t^i = S_t^i \Big(b_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \Big).$$

P is the objective probability measure. Assume that the number of risky assets equals the dimension of the Brownian motion[4]. By absence of arbitrage there exists an n-dimensional bounded and progressively measurable vector θ such that

$$b_t - r_t \mathbf{1} = \sigma_t \theta_t, \qquad dt \times dP \quad a.s.,$$

where $\mathbf{1}$ denotes n-dimensional unit vector. σ_t is an $n \times n$ matrix and is assumed to have full rank. θ is called the premium of the market risk. Under these assumptions the market is complete.

For hedge of a European contingent claim in complete market a self-financing and replicating portfolio can be builded. At time t the trading strategy $\phi_t = (\phi_t^1, ..., \phi_t^n)^\top$ can be decided. And under the assumption of self-financing the investment in the risk-less asset must satisfy $\phi_t^0 S_t^0 = V_t - \sum_{i=1}^n \phi_t^i S_t^i$. Therefore the value of the self-financing portfolio has the following dynamic

$$\begin{aligned} dV_t &= r_t V_t dt + \pi_t^\top (b_t - r_t \mathbf{1}) dt + \pi_t^\top \sigma_t dW_t \\ &= r_t V_t dt + \pi_t^\top \sigma_t (dW_t + \theta_t dt). \end{aligned}$$

The vector $\pi_t = (\pi_t^1, ..., \pi_t^n)^\top$ with $\pi_t^i = \phi_t^i S_t^i$ denotes the amount of the money invested in risky assets i at time t. In expression of BSDE

$$V_t = \xi + \int_t^T f(s, V_s, Z_s) ds - \int_t^T Z_s^\top dW_s,$$

where ξ is the terminal value of contingent claim, $Z_t^\top = \pi_t^\top \sigma_t$ and

$$f(t, y, z) = -r_t y - z_t^\top \theta_t. \tag{22}$$

The driver in equation (22) is a *linear* function of y and z.

7 Hedging and Optimal Stopping Characterized as BSDE with Two Reflecting Barriers

In general, the optimization problem formulated via Equation (18) has no closed-form solution. Cvitanić and Karatzas (1996) show that, the no-arbitrage value can be formulated as adapted solution of backward stochastic differential equations (BSDE) with two reflecting barriers. The proper BSDE for valuation of callable and convertible bond is derived via hedging arguments. It has been shown in literatures that the most significant

[4] This assumption and the full rank of volatility matrix ensure the completeness of the market

risk factor for a typical convertible bond is the equity price subject to default risk. Interest rate risk is usually a secondary consideration. Therefore we assume that the default-free interest rate is deterministic. Another hypothesis which make the hedge possible, requires that two kinds of risky assets are traded in the market:

- *defaultable stock*, with its dynamic described by equation(4),
- *defaultable zero-coupon bond with zero recovery*, based on the assumption of absence of interest rate risk, its dynamic can be expressed as

$$d\bar{B}_t = \bar{B}_{t-}(r_t dt - dM_t), \tag{23}$$

with

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} h(\tilde{S}_u)du \ ,$$

equivalently, the pre-default bond price $\tilde{B}_t$ satisfies

$$d\tilde{B}_t = (r_t + h(\tilde{S}_t))\tilde{B}_t dt.$$

The bond holder pays the price, which is a non-random amount at time zero and is entitled to the cumulative coupon payments and the lump-sum settlement at conversion or call time, or at default. While the issuer receives the price, but must provide the aforementioned random payments to the bondholder. The issuer's objective is to hedge his short position by trading in the market in such a way as to make the necessary payments and still be solvent at the termination of the contract, almost surely. The price process of the callable and convertible bond is then associated with the following hedging strategy, with investment in risky zero bonds and stock,

$$dCCB(t) + (\bar{c} + R\cdot h_t)dt = (r_t + h_t)CCB(t)dt - dK^+(t) + dK^-(t) + \pi_t\sigma_t dW_t, \tag{24}$$

where $K^+(t)$ and $K^-(t)$ are two continuous, increasing and adapted processes satisfy

$$\int_0^T (CCB(t) - CV(t))dK^+(t) = \int_0^T (CCB(t) - Call(t))dK^-(t) = 0$$

where π_t denotes the amount of money invested in the risky stock, $CV(t)$ the conversion value, $Call(t)$ the call value.

Proposition 7.1. In standard expression of BSDE,

$$\begin{cases} CCB(t) = g(\tilde{S}_T) + \int_t^T f(s, \tilde{S}_s, CCB(s))ds - \int_t^T Z_s dW_s + \int_t^T dK_s^+ - \int_t^T dK_s^- \\ CV(\tilde{S}_t) \leq CCB(t) \leq Call(\tilde{S}_t) \qquad \forall 0 \leq t \leq T \\ \int_0^T (CCB(s) - CV(\tilde{S}_s))dK_s^+ = \int_0^T (Call(\tilde{S}_s) - CCB(s))dK_s^- = 0 \end{cases} \tag{25}$$

with

$$\begin{aligned} d\tilde{S}_t &= (r_t + h(\tilde{S}_t))\tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \\ f(t, CCB(t)) &= (\bar{c} + R\cdot h_t) - (r_t + h_t)CCB(t). \end{aligned}$$

where $Z_t = \pi_t\sigma_t$, and $f(t, CCB(t))$ is the *driver*.

The value process of the convertible bond is forced to stay between the upper- and lower-boundary, which are the call and conversion value respectively. This effect is achieved through the two reflection processes $K^+(t)$, and $K^-(t)$, , which push the value process of the callable and convertible bond upward or downward to prevent the boundary crossing. The "push" is minimal in the sense that it will only be carried out in the case that $CCB(t) = CV(t)$ or $CCB(t) = Call(t)$. According to Cvitanić and Karatzas (1996), the existence and uniqueness of the solution of equation(25) is ensured, if additional to the general conditions on terminal value and the driver defined in definition 6.1, the following conditions are satisfied

- K^+ and K^- are continuous, increasing and adapted processes.
- CV and $Call$ are two continuous, progressively measurable processes and satisfy

$$CV(t) < Call(t), \quad \forall\ 0 \le t \le T \quad \text{and} \quad CV(T) \le \xi \le Call(T)\ \ a.s.$$

Having formulated the no-arbitrage value of the callable and convertible bond as solution of BSDE with two reflecting barriers, our next task is to derive numerical solutions.

Remark 7.2. According to our assumptions, the bondholder can only exchange the bond against stock of one prescribed firm. However, BSDE with two reflecting barriers usually encompasses the more general case, where the bondholder can convert the bond into a basket of risky stocks, i.e. Z can be $\mathbb{R}^d$, $d \ge 1$ valued and the hedge portfolio contains positions in d different risky stocks.

8 Numerical Solution

There are basically two types of schemes for solving BSDE's. The first type is the numerical solution of a parabolic PDE related to the BSDE and the second type of algorithms works backwards and treats the stochastic problem directly via simulation. For financial problems with few random factors, the associated PDE provided by Cvitanić and Ma (2001) can be solved with finite-difference methods. For callable and convertible bond with more than three risky stock as underlying, a direct treatment with Monte Carlo method is a better method. A recursion algorithm is provided e.g. in Chassagneux (2007). Equation (25) belongs to a well-investigated class of BSDE's in a Markovian framework, the FBSDE.

Proposition 8.1. According to Cvitanić and Ma (2001) the solution of equation (25) is associated with the following PDE, which is called the obstacles problem,

$$\begin{cases} (Call - CV) \wedge \{(u - Call) \vee -[u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + (r + h_t)xu_x + f(t,x,u)]\} = 0 \\ u(T,x) = g(x). \end{cases} \tag{26}$$

For simplicity of the notations, x stands for $\tilde{S}$ and h_t the default intensity $h(\tilde{S}_t)$. The driver $f(t,x,u) = (c + R \cdot h_t) - (r_t + h_t)u$. The explicit expression of the solution (CCB, Z) is

$$CCB(t) = u(t, x_t), \qquad Z_t = \partial_x u(t, x_t)\sigma(t, x_t).$$

Here, we will not give an exact mathematical definition of the obstacle problem, and discuss the existence and uniqueness of its solution, for details see Cvitanić and Ma (2001). We apply explicit finite difference method for derivation of the numerical solution, i.e. we work step by step down the grid. Finite difference methods can be thought as a generalization of the binomial concept and is more flexible. In the finite-difference methods the grid is fixed but parameters change to reflect a changing diffusion. At first, we derive the value $\tilde{u}_i^k$ backwardly from the next time period, then compare it with the payoffs by conversion or call. If $\tilde{u}_i^k$ is greater or lesser than the call or conversion value, it will be replaced by the call or conversion value respectively. For each time step k and stock step i,

$$u_i^k = \min[Call, \max[CV, \tilde{u}_i^k]].$$

Example 8.2. As an illustrative example we compute the no-arbitrage price of a defaultable callable and convertible bond. The default intensity is modelled as piecewise constant function of the pre-default stock price.

$$h(\tilde{S}_t) = \begin{cases} a & \text{if} \quad \tilde{S}_t \leq K \\ b & \text{if} \quad \tilde{S}_t > K \end{cases}$$

In default case, the stock value jumps to zero, while the bond has a constant recovery rate of $R = 30\%$ of the face value. The convertible value is $CV_t = \gamma \tilde{S}_t$, and the call value is always lager than the convertible value and amounts $Call_t = \max[H, \gamma \tilde{S}_t]$. The model parameters are given as $T = 4$, $r = 0.06$, $S_0 = 70$, $a = 0.5$, $b = 0.02$, $K = 30$, $L = 100$, $c = 3$, $\gamma = 1.2$. The no-arbitrage values by different stock volatilities and the comparison with the default free case are summarized in table 1. The stability of numeric is ensured by proper choice and combination of the steps for the stock price and time.

	$H = 110$		$H = 120$		$H = 130$	
σ	defaultable	default-free	defaultable	default-free	defaultable	default-free
0.1	95.02	96.52	96.59	97.73	97.51	98.36
0.2	97.34	99.21	99.56	101.45	101.11	102.94
0.3	98.33	100.88	101.32	103.99	103.45	106.32
0.4	97.85	101.96	101.25	105.68	103.70	108.65
0.5	96.85	102.65	100.33	106.84	102.91	110.21

Table 1: No-arbitrage prices of callable and convertible bonds without and with default risk

The results in table 1 show that, in default free case, the price of callable and convertible bond increases in volatility. But if default risk is considered and the default intensity is explicitly linked to the stock price, the price increases at first with increasing volatility then decreases after the volatility excesses a certain value. The increasing volatility increase the conversion value but it also increases the default probability.

9 Uncertain Volatility

Suppose that the seller and buyer relax the assumption of constant volatility by the valuation and adopt the assumption of *uncertain volatility.* In this case the market is incomplete, i.e. there is no unique price of market risk, there is a set of possible equivalent martingale measures which are compatible with the no arbitrage requirement.

Proposition 9.1. Suppose that only a *buy-and-hold* strategy is allowed in the callable and convertible bond, while only the risky stock and defaultable zero-coupon bond can be traded dynamically. The set of initial no-arbitrage prices is determined by super hedging and lies in the interval $[CCB_{low}(0), CCB_{up}(0)]$ with

$$CCB_{low}(0) = \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] = \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \sup_{\tau_B \in \mathcal{F}_{0T}} \mathbb{E}_Q[ccb(0)], \quad (27)$$

$$CCB_{up}(0) = \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] = \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \inf_{\tau_A \in \mathcal{F}_{0T}} \mathbb{E}_Q[ccb(0)], \quad (28)$$

where $\mathcal{Q}$ is the family of equivalent martingale measures.

Proof 9.2. Applying theorem 2.2 of Kallsen and Kühn (2005).

The lower and upper bound are derived under the most pessimistic expectations of the buyer and seller respectively.

Theorem 9.3. Combine proposition 8.1 with proposition 9.1. The solution of equation (27) and (28) is associated with the following PDE

$$\begin{cases} (Call - CV) \wedge \Big\{ (u - Call) \vee - \Big[u_t + \frac{1}{2} \Sigma^2[u_{xx}] x^2 u_{xx} + (r + h_s) x u_x + f(t, x, u) \Big] \Big\} = 0 \\ u(T, x) = g(x). \end{cases} \quad (29)$$

where $\Sigma^2[x]$ stands for a volatility parameter which depends on x. CCB_{low} is derived by setting

$$\Sigma^2 [x] = \begin{cases} \sigma^2_{\max} & \text{if} \quad x \leq 0 \\ \sigma^2_{\min} & \text{else} \end{cases}$$

and CCB_{up} is derived by setting

$$\Sigma^2 [x] = \begin{cases} \sigma^2_{\max} & \text{if} \quad x \geq 0 \\ \sigma^2_{\min} & \text{else} \end{cases}$$

Example 9.4. The volatility of stock is supposed to lie within the interval $[0.2, 0.4]$. The other model parameters are the same as in example 8.2, with $T = 4$, $R = 30\%$, $r = 0.06$, $K = 30$, $S_0 = 70$, $L = 100$, and $c = 3$. The bid and ask prices are listed in Table 2.

Default risk reduces the price but explicit modeling of default risk does *not* enlarge the price spread. The reason is that default risk brings varying convexity and concavity to the value function. Moreover, both parties can decide when they exercise. Therefore each of them must bear the strategy of the other party in mind. The pricing bound is not only determined by the default risk and volatility but also depends on the optimal exercises.

	$a = 0.5,\ b = 0.02$			$a = 0,\ b = 0$		
H	lower	upper	spread	buyer	lower	upper
120	99.19	102.97	3.79	101.45	105.68	4.22
130	100.76	105.69	4.94	102.91	108.65	5.73
140	101.64	107.75	6.11	103.70	110.85	7.15
150	102.15	109.17	7.02	104.11	112.36	8.25

Table 2: No-arbitrage pricing bounds with stock price volatility lies within the interval $[0.2, 0.4]$

10 Summary

The exposure of callable and convertible bonds to both credit and equity risk and the corresponding optimal conversion and call strategies build the focus of our study. The interplay between equity and credit risk is taken into account by adopting an intensity-based default model in which the risk-neutral default intensity is linked to the equity price. The embedded option rights owned by both of the bondholder and issuer is treated by the well developed theories on the Dynkin game and is solved with help of the associated doubly reflected backward stochastic differential equations (BSDE). Valuation of callable and convertible bond as defaultable game option has been proposed by Bielecki et al. (2007). But our model framework is more simple and we give pricing bounds for uncertain stock volatility. Subject of future study could be inclusion of the mixed-strategies and their effects on the pricing bound. In this paper the stock price follows a simple jump diffusion and a further step could be modeling the stock price with a more flexible Lévy process.

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