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LAPLACE TRANSFORMS AND SUPREMA OF STOCHASTIC PROCESSES

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ABSTRACT. It is shown that moments of negative order as well as positive non-integral order of a nonnegative random variable X can be expressed by the Laplace transform of X. Applying these results to certain first passage times gives explicit formulae for moments of suprema of Bessel processes as well as strictly stable Lévy processes having no positive jumps.

KEY WORDS: Laplace transform, Bessel process, Lévy process.

0. INTRODUCTION

In the sequel (B_t) denotes a d-dimensional standard linear Brownian motion starting at $0 \in \mathbb{R}^d$ (denoted BM(d)). In Shiryaev (1999,p.251) a beautiful trick is used in order to show that if (B_t) is a BM(1),

$$\mathrm{E}\Big[\sup_{0\le s\le 1}|B_s|\Big] = \sqrt{\pi/2}. \tag{0.0.1}$$

In fact, the verification of (0.0.1) can be based on the stopping time

$$T(1) := \inf\{t \ge 0 |\ |B_t| = 1\} \tag{0.0.2}$$

and its Laplace transform

$$\varphi_1(t) = \mathrm{E}\big[\exp(-t\,T(1))\big] = \frac{1}{\cosh(\sqrt{2t})}, \quad t \ge 0. \tag{0.0.3}$$

The latter is easily obtained by applying the optional stopping theorem to the martingale

$$\cosh(sB_t)\exp(-s^2t/2) \quad (t \ge 0) \quad \text{for fixed } s \ge 0;$$

see, e.g., Revuz/Yor (1991,p.68) or Rogers/Williams (1994,p.19). Although there is no explicit inversion of the Laplace transform in (0.0.3) in any particularly useful form, it turns out, however, that (0.0.3) contains enough information in order to yield (0.0.1). In fact, putting

$$M(t) = \sup_{0\le s\le t}|B_s|, \quad t \ge 0, \tag{0.0.4}$$

we get by Brownian scaling, for any $t > 0$,

$$P\Big(M(1) \le t\Big) = P\Big(\sup_{0\le s\le 1} |B_{s/t^2}| \le 1\Big) = P\Big(M(1/t^2) \le 1\Big)$$
$$= P\big(T(1) \ge 1/t^2\big) = P\Big((T(1))^{-1/2} \le t\Big),$$

i.e.,

(0.0.5) $\qquad M(1)$ and $(T(1))^{-1/2}$ have the same distribution

which implies

$$\mathrm{E}[M(1)] = E\Big[(T(1))^{-1/2}\Big]. \tag{0.0.6}$$

Next, using the density of a normal distribution with mean 0 and variance $s^2/2$ we get

$$s = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-(t/s)^2)\, dt, \quad s > 0. \tag{0.0.7}$$

Hence if $X \ge 0$ is a random variable having Laplace transform φ_X we obtain from (0.0.7) by using the Fubini-Tonelli theorem,

$$\mathrm{E}\big[X^{-1/2}\big] = \frac{2}{\sqrt{\pi}} \int_0^\infty \varphi_X(t^2)\, dt. \tag{0.0.8}$$

Applying (0.0.8) to $X = T(1)$ and taking into account (0.0.6) as well as (0.0.3) we arrive at

$$\mathrm{E}[M(1)] = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{1}{\cosh(\sqrt{2}t)}\, dt.$$

Using the substitution $u = \exp(\sqrt{2}t)$ we end up with (0.0.1).

In the sequel we first extend (0.0.8) in two different ways (see Theorems 1.1 and 1.2 in the next section). Using the same pattern of proof as before allows us to obtain results similar to (0.0.1) for Bessel processes as well as for a certain class of Lévy processes.

1. Calculation of Moments via Laplace Transforms

We first derive an extension of (0.0.8). In order to achieve this it is natural to start with the identity

$$\Gamma(1/\tau) = \int_0^\infty u^{1/\tau - 1}\, \exp(-u)\, du, \quad \tau > 0.$$

Using the substitution $u = (t/s)^\tau$ ($s > 0, \tau > 0$ being constants) we get

$$(1.0.1) \qquad s = \frac{\tau}{\Gamma(1/\tau)} \int_0^\infty \exp\Big(-(t/s)^\tau \Big)\, dt, \quad s \geq 0, \quad \tau > 0.$$

1.1. **Theorem.** *Let $X \geq 0$ be a random variable having Laplace transform φ_X. Then*

$$(1.1.1) \qquad \mathrm{E}[X^{-r}] = \frac{1}{r\,\Gamma(r)} \int_0^\infty \varphi_X(t^{1/r})\, dt, \quad r > 0.$$

Proof. Apply (1.0.1) to $s = X^{-r}$, $\tau = 1/r$ and use the Fubini-Tonelli theorem. □

There are interesting connections between (1.1.1) and the "fractional calculus" (see Ross (1974) and Wolfe (1974)). It has been noticed in Wolfe (1974) that it is possible to calculate moments of positive non-integral order of a nonnegative random variable X by using the Laplace transform φ_X of X. In fact, we have the

1.2. **Theorem.** *Let $X \geq 0$ be a random variable having Laplace transform φ_X. Then,*

$$(1.2.1) \qquad E[X^r] = \frac{r}{\Gamma(1-r)} \int_0^\infty \frac{1-\varphi_X(t)}{t^{r+1}}\, dt, \quad 0 < r < 1.$$

More generally, we have for any integer $n \geq 0$ and $n < r < n+1$

$$(1.2.2) \qquad E[X^r] = \frac{r-n}{\Gamma(n+1-r)} \int_0^\infty \frac{(-1)^n(\varphi_X^{(n)}(0) - \varphi_X^{(n)}(t))}{t^{r+1-n}}\, dt.$$

Proof. In order to prove (1.2.1) we use the identity

$$(1.2.3) \qquad \int_0^\infty \frac{1-\exp(-st)}{t^{r+1}}\, dt = \frac{1}{r}\,\Gamma(1-r)\, s^r, \quad 0 < r < 1, \quad s \geq 0$$

which can be derived by using partial integration and the definition of the gamma function (see, e.g., Gradshteyn/Ryzhik (1965,p.333)). Applying (1.2.3) to $s = X$ and using the Fubini-Tonelli theorem we obtain (1.2.1). In a similar way we get (1.2.2), noting that

$$\varphi_X^{(n)}(t) = (-1)^n\, \mathrm{E}\big[X^n \exp(-tX)\big], \quad t \geq 0, \quad n \geq 0.$$

□

2. Bessel Processes

In the sequel (B_t) denotes a BM(d). Then $(|B_t|)$ is a realization of a d-dimensional Bessel process starting at $0 \in \mathbb{R}^d$ ($|B_t|$ denoting the Euclidean norm of B_t). Consider the stopping time

$$T(s) := \inf\{t \geq 0 |\ |B_t| = s\}, \quad s \geq 0. \tag{2.0.1}$$

The Laplace transform φ_s of $T(s)$ is given by

$$\varphi_s(t) = (s\sqrt{2t})^\lambda \big[2^\lambda\, \Gamma(\lambda+1)\, I_\lambda\, (s\sqrt{2t})\big]^{-1}, \quad s > 0, \quad t > 0 \tag{2.0.2}$$

(see, e.g., Getoor/Sharpe (1979) and Getoor(1979)). Here,

$$\lambda := (d-2)/2 \geq -1/2, \tag{2.0.3}$$

and I_ν is denoting the modified Bessel function of order ν given by

$$I_\nu(t) = \sum_{m=0}^{\infty} \frac{(t/2)^{\nu+2m}}{m!\ \Gamma(\nu+m+1)}\,, \quad t > 0, \quad \nu \geq -1/2. \tag{2.0.4}$$

In particular,

$$I_{-1/2}(t) = \sqrt{2/(\pi t)}\ \cosh(t), \quad t > 0. \tag{2.0.5}$$

(Note that, in the one-dimensional case, this yields the Laplace transform

$$\mathrm{E}\big[\exp(-t\, T(s))\big] = \frac{1}{\cosh(s\sqrt{2t})}, \quad s \geq 0, \quad t \geq 0.) \tag{2.0.6}$$

Putting

$$M(t) = \sup_{0 \leq s \leq t} |B_s|, \quad t \geq 0 \tag{2.0.7}$$

we obtain by Brownian scaling

(2.0.8) $\qquad M(1)$ and $(T(1))^{-1/2}$ have the same distribution.

By (1.1.1) this implies for any $r > 0$

$$E\big[(M(1))^r\big] = \mathrm{E}\big[(T(1))^{-r/2}\big] = \frac{2}{r\ \Gamma(r/2)} \int_0^\infty \varphi_1(t^{2/r})\, dt.$$

Using (2.0.2) and substituting $u = \sqrt{2}t^{1/r}$ yields, for any $r > 0$,

$$E\big[(M(1))^r\big] = \frac{4}{2^{(d+r)/2}\ \Gamma(d/2)\ \Gamma(r/2)} \int_0^\infty \frac{u^{d/2+r-2}}{I_\lambda(u)}\, du \tag{2.0.9}$$

$(\lambda = (d-2)/2)$. In the case $d = 1$ we get from (2.0.9) (using (2.0.5))

$$E\big[(M(1))^r\big] = \frac{2}{2^{r/2}\ \Gamma(r/2)} \int_0^\infty \frac{u^{r-1}}{\cosh(u)}\, du, \quad r > 0.$$

The substitution $t = \exp(u)$, i.e. $u = \log t$, yields for any $r > 0$

$$\mathrm{E}\big[(M(1))^r\big] = \frac{4}{2^{r/2}\,\Gamma(r/2)} \int_1^\infty \frac{(\log t)^{r-1}}{t^2+1}\,dt \quad (d=1) \tag{2.0.10}$$

In the special case $r = 2n+1$ ($n \geq 0$) the integral in (2.0.10) equals

$$\int_1^\infty \frac{(\log t)^{2n}}{t^2+1}\,dt = \int_0^1 \frac{(\log t)^{2n}}{t^2+1}\,dt = \frac{\pi^{2n+1}}{2^{2n+2}}\,|E_{2n}| \tag{2.0.11}$$

(see Gradshteyn/Ryzhik (1965,p.549)), $E_0, E_2, \ldots$ denoting the Euler numbers determined by

$$\frac{1}{\cosh(t)} = \sum_{n=0}^\infty E_{2n} \frac{t^{2n}}{(2n)!}.$$

The first Euler numbers are $E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, E_8 = 1385, E_{10} = -50521, E_{12} = 2702765, E_{14} = -199360981$. Combining (2.0.10) and (2.0.11) gives for $n = 0, 1, \ldots$

$$\mathrm{E}\big[(M(1))^{2n+1}\big] = \sqrt{\frac{\pi}{2}}\Big(\frac{\pi^2}{2}\Big)^n \frac{n!}{(2n)!}\,|E_{2n}| \quad (d=1) \tag{2.0.12}$$

In order to investigate the asymptotic behaviour of $\mathrm{E}\big[(M(1))^r\big]$ (as $r \to \infty$) we need

2.1. **Lemma.** *As* $t \to \infty$

$$\Gamma(t) \sim (t/e)^t\,\sqrt{2\pi/t} \tag{2.1.1}$$

and

$$I_\nu(t) \sim (2\pi t)^{-1/2}\,\exp(t) \tag{2.1.2}$$

independently of $\nu \geq -1/2$.

Proof. For a proof of (2.1.1) see Bender/Orszag (1978,p.275); a proof of (2.1.2) can be found in Courant/Hilbert (1966,p.526) (see also Bender/Orszag (1978,p.271)). □

2.2. **Proposition.** *As* $r \to \infty$

$$\mathrm{E}\big[(M(1))^r\big] \sim \frac{4\sqrt{\pi}\;r^{(d+r-1)/2}}{2^{d/2}\,\Gamma(d/2)\,e^{r/2}}. \tag{2.2.1}$$

In particular

$$E\big[(M(1))^r\big] \sim 2\sqrt{2}(r/e)^{r/2} \quad (d=1) \tag{2.2.2}$$

Proof. For the integral in (2.0.9) we get

$$\int_0^\infty \frac{u^{d/2+r-2}}{I_\lambda(u)}\,du \sim \frac{2\pi}{e^r}\,r^{d/2+r-1} \quad (r \to \infty)$$

by using some routine estimates and applying (2.1.2). Combining this with (2.1.1) gives (2.2.1). □

2.3. **Corollary.** *For any $d \geq 1$ the distribution of $M(1)$ is uniquely determined by its moments*

$$\mu_n = \mathrm{E}\big[(M(1))^n\big], \quad n = 0, 1, \ldots$$

Proof. It suffices to show that

$$\sum_{n=0}^{\infty} \mu_{2n}^{-1/(2n)} = \infty \tag{2.3.1}$$

(see Feller (1966,p.224)). But, by (2.2.1),

$$\mu_{2n}^{1/(2n)} \sim \sqrt{2n/e} \quad (n \to \infty)$$

which implies (2.3.1). □

2.4. **Remark.** Comparing (2.0.12) and (2.2.2) we obtain for the asymptotic behaviour of the Euler numbers E_{2n}

$$|E_{2n}| \sim \frac{8\sqrt{n}}{\sqrt{\pi}} \left(\frac{4n}{e\pi}\right)^{2n}. \tag{2.4.1}$$

3. Strictly Stable Lévy Processes

Let $X = (X_t)$ be a one-dimensional Lévy process starting at 0 such that X_1 is not a.s. constant. Furthermore X is assumed to be a.s. *càdlàg*, i.e. almost all paths of X are right continuous and have finite left-hand limits at every point. In the sequel α denotes a real number such that

$$1 < \alpha < 2. \tag{3.0.1}$$

The distribution μ of X_1 is assumed to be *strictly α-stable* with parameters (α, β, c) for constants $-1 \leq \beta \leq 1$ and $c > 0$, i.e. the characteristic function $\widehat{\mu}$ of μ is of the form

$$\widehat{\mu}(t) = \exp(-\Psi(t)), \quad t \in \mathbb{R}. \tag{3.0.2}$$

Here, Ψ is the *characteristic exponent* given by

$$\Psi(t) = c|t|^{\alpha}\Big[1 - i\beta(\operatorname{sgn} t)\tan\Big(\frac{\pi\alpha}{2}\Big)\Big], \quad t \in \mathbb{R} \tag{3.0.3}$$

where sgn t is equal to 1,0,-1 when t is $> 0, = 0, < 0$, respectively (see Bertoin (1996,p.217) or Sato (1999,p.86)). It follows from the above assumptions that, for any $a > 0$,

$$(X_{at}) \text{ has the same law as } \big(a^{1/\alpha}\, X_t\big). \tag{3.0.4}$$

From now on we will additionally assume

$$\beta = -1. \tag{3.0.5}$$

This implies (see Bertoin (1996,p.217) or Sato (1999,p.346)) that the Lévy measure of X has support in $]-\infty, 0]$, and X has *no positive jumps*, i.e.

$$P\Big(X_t \leq X_{t-} \text{ for all } t > 0\Big) = 1. \tag{3.0.6}$$

Furthermore (see Sato (1999,p.350))

$$P\Big(\limsup_{t\to\infty} X_t = \infty\Big) = 1. \tag{3.0.7}$$

Let $R(s)$ denote the first passage time defined by

$$R(s) = \inf\{t > 0 | X_t > s\}, \quad s \geq 0. \tag{3.0.8}$$

Note that, by (3.0.7), $R(s) < \infty$ a.s., $s \geq 0$. For the following result see Sato (1999,pp.346,347).

3.1. **Theorem.** *We continue to assume* (3.0.1) *and* (3.0.5). *Then the following results hold.*

(a) $P\Big(X_{R(s)} = s \textit{ for all } s \geq 0\Big) = 1.$

(b) *The function*

$$\psi(t) := -\Psi(-it) = -\frac{c}{\cos(\frac{\pi\alpha}{2})}\, t^{\alpha}, \quad t \geq 0 \tag{3.1.1}$$

(Ψ *given by* (3.0.3)) *is strictly increasing, continuous and satisfies* $\psi(0) = 0$ *and* $\psi(t) \to \infty \quad (t \to \infty)$.

(c) *The Laplace transforms of the first passage times* $R(s) \quad (s > 0)$ *are given by*

$$\mathrm{E}\big[\exp(-t\, R(s))\big] = \exp(-s\, \psi^{-1}(t)), \quad s > 0, \quad t \geq 0 \tag{3.1.2}$$

(ψ^{-1} *denoting the inverse function of* ψ).

It follows from (3.1.1) that ψ^{-1} is given by

$$\psi^{-1}(t) = \Big(\frac{1}{c}\,\Big|\cos\Big(\frac{\pi\alpha}{2}\Big)\Big|\Big)^{1/\alpha}\, t^{1/\alpha}, \quad t \geq 0. \tag{3.1.3}$$

Combining Theorems 3.1 and 1.1 gives

3.2. **Proposition.** *We continue to assume* (3.0.1) *and* (3.0.5). *Then the following results hold.*

(a) *For any* $r > 0$,

$$\mathrm{E}\Big[\big(\sup_{0\leq s\leq 1} X_s\big)^r\Big] = \frac{\alpha\, \Gamma(r)}{\Gamma(r/\alpha)}\, \bigg(\frac{c}{\big|\cos\big(\frac{\pi\alpha}{2}\big)\big|}\bigg)^{r/\alpha}. \tag{3.2.1}$$

(b) *We have, as* $r \to \infty$,

$$\mathrm{E}\Big[\big(\sup_{0\leq s\leq 1} X_s\big)^r\Big] \sim \sqrt{\alpha}\bigg(\frac{\alpha\, c\, r^{\alpha-1}}{\big|\cos\big(\frac{\pi\alpha}{2}\big)\big| e^{\alpha-1}}\bigg)^{r/\alpha}. \tag{3.2.2}$$

Proof. In order to prove (3.2.1) first note that it follows from (3.0.6) that

$$\widetilde{M}(t) := \sup_{0\le s\le t} X_s \le 1 \Longleftrightarrow R(1) \ge t \quad \text{a.s.}, \qquad t>0,$$

which, by (3.0.4), implies that $\widetilde{M}(1)$ and $(R(1))^{-1/\alpha}$ have the same distribution. Hence, by Theorem 1.1 and (3.1.2),

$$\begin{aligned}\mathrm{E}\big[(\widetilde{M}(1))^r\big] &= \mathrm{E}\big[(R(1))^{-r/\alpha}\big]\\ &= \frac{\alpha}{r\,\Gamma(r/\alpha)} \int_0^\infty \exp\Big(-\psi^{-1}(t^{\alpha/r})\Big)\,dt, \qquad r>0.\end{aligned}$$

Taking into account (3.1.3) and putting

$$\widetilde{c} := \left(\frac{1}{c}\left|\cos\left(\frac{\pi\alpha}{2}\right)\right|\right)^{1/\alpha}, \tag{3.2.3}$$

we get, using the substitution $u = \widetilde{c}\,t^{1/r}$,

$$\mathrm{E}\Big[(\widetilde{M}(1))^r\Big] = \frac{\alpha\,\Gamma(r)}{\Gamma(r/\alpha)\,\widetilde{c}^{\,r}},$$

i.e. (3.2.1). The assertion (3.2.2) is immediate from (3.2.1) and (2.1.1). □

As a final result we mention

3.3. **Proposition.** *We continue to assume* (3.0.1) *and* (3.0.5). *Then, for any* $s>0$,

$$\mathrm{E}\big[(R(s))^r\big] = \frac{\Gamma(1-\alpha r)}{\Gamma(1-r)} \left(\frac{\left|\cos\left(\frac{\pi\alpha}{2}\right)\right|}{c}\right)^r s^{\alpha r}, \qquad 0\le r<1/\alpha \tag{3.3.1}$$

and

$$\mathrm{E}\big[(R(s))^r\big] = \infty, \qquad r \ge 1/\alpha. \tag{3.3.2}$$

Proof. Put $\delta := s\,\widetilde{c}$ ($s>0$; $\widetilde{c}$ given by (3.2.3)). Taking into account (3.1.2), (1.2.1) and (3.1.3) we obtain, for any $0<r<1$,

$$\mathrm{E}\big[(R(s))^r\big] = \frac{r}{\Gamma(1-r)} \int_0^\infty \frac{1-\exp(-\delta t^{1/\alpha})}{t^{r+1}}\,dt.$$

Using the substitution $u = \delta t^{1/\alpha}$ we arrive at

$$\mathrm{E}\big[(R(s))^r\big] = \frac{\alpha\, r\, \delta^{\alpha r}}{\Gamma(1-r)} \int_0^\infty \frac{1-\exp(-u)}{u^{\alpha\,r+1}}\,du, \qquad 0<r<1.$$

This clearly entails (3.3.2) and (using partial integration) also (3.3.1). □

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