Bonn Econon Discussion Papers

Discussion Paper 10/2009

Treatment of Double Default Effects within the Granularity Adjustment for Basel II

by

Sebastian Ebert and Eva Lütkebohmert

July 2009

Bonn Graduate School of Economics
Department of Economics
University of Bonn
Kaiserstrasse 1
D-53113 Bonn

Financial support by the
Deutsche Forschungsgemeinschaft (DFG)
through the
Bonn Graduate School of Economics (BGSE)
is gratefully acknowledged.

Deutsche Post World Net is a sponsor of the BGSE.

Treatment of Double Default Effects within the Granularity Adjustment for Basel II*

Sebastian Ebert
Bonn Graduate School of Economics
University of Bonn
Adenauerallee 24-42
53113 Bonn, Germany
Tel.: +49 228 739225
Email: sebastianebert@uni-bonn.de

Eva Lütkebohmert
Institute for Social Sciences and Economics
University of Bonn
Adenauerallee 24-42
53113 Bonn, Germany
Tel.: +49 228 736301
Email: eva.luetkebohmert@uni-bonn.de

10 July 2009

*Both authors acknowledge the financial support from the Bonn Graduate School of Economics. We are particularly grateful to Michael Gordy for many discussions on this topic which helped to improve this work. We would also like to thank Christian Gouriéroux and Alexander Szimayer for many helpful comments and suggestions.

Abstract

Within the Internal Ratings-Based (IRB) approach of Basel II it is assumed that idiosyncratic risk has been fully diversified away. The impact of undiversified idiosyncratic risk on portfolio Value-at-Risk can be quantified via a *granularity adjustment* (GA). We provide an analytic formula for the GA in an extended single-factor CreditRisk$^+$ setting incorporating double default effects. It accounts for guarantees and their effect of reducing credit risk in the portfolio. Our general GA very well suits for application under Pillar 2 of Basel II as the data inputs are drawn from quantities already required for the calculation of IRB capital charges.

Key words: analytic approximation, Basel II, counterparty risk, double default, granularity adjustment, IRB approach, securitization

JEL Codes: G31, G28

1. Introduction

In the portfolio risk-factor frameworks that underpin both industry models of credit Value-at-Risk (VaR) and the Internal Ratings-Based (IRB) risk weights of Basel II, credit risk in a portfolio arises from two sources, systematic and idiosyncratic. Idiosyncratic risk represents the effects of risks that are particular to individual borrowers. Under the Asymptotic Single Risk Factor (ASRF) framework on which the IRB approach is based, it is assumed that bank portfolios are *perfectly* fine-grained in the sense that the largest individual exposures account for an infinitely small share of total portfolio exposure. In such a portfolio idiosyncratic risk is fully diversified away, so that economic capital depends only on systematic risk. Real-world portfolios are not, of course, perfectly fine-grained. The asymptotic assumption might be approximately valid for some of the largest bank portfolios, but clearly would be much less satisfactory for portfolios of smaller or more specialized institutions. When there are material name concentrations, there will be a residual of undiversified idiosyncratic risk in the portfolio. The IRB formula omits the contribution of this residual to the required economic capital.
The impact of undiversified idiosyncratic risk on portfolio VaR can be assessed via a methodology known as *granularity adjustment* (GA). It is derived as a first-order asymptotic approximation for the effect of diversification in large portfolios. The basic concepts and approximate form for the granularity adjustment were first introduced by Michael Gordy in 2000 for application in Basel II (see Gordy [2003]). It was then substantially refined and put on a more rigorous foundation by Wilde [2001] and Martin and Wilde [2003] using theoretical results from Gouriéroux et al. [2000]. Recently, Gordy and Lütkebohmert [2007] proposed and evaluated a granularity adjustment suitable for application under Pillar 2 of Basel II.[1]

All these methods, however, do not account for guarantees and general hedging instruments within a credit portfolio. This paper aims at filling this gap as the exclusion of hedging instruments represents, of course, a rather severe limitation since it is not at all rare that credit exposures are hedged in some way. For example, granting loans and transferring the risk afterwards is a typical business for a bank. The relevance of hedging instruments is also acknowledged by the Basel

[1] Two other earlier works on the GA are Emmer and Tasche [2005] and Pykthin and Dev [2002]. See Lütkebohmert [2009] for more information on the development of the GA and a discussion of the different methods. Note also that recently the GA methodology to quantify the effect of idiosyncratic risk has proved useful in quite different contexts. Gouriéroux and Monfort [2008] derive GAs for optimal portfolios. That is, they quantify the error in efficiency if one uses an optimal portfolio consisting of finitely many assets only in order to proxy the true, perfectly-diversified market portfolio. In Gagliardini and Gouriéroux [2009] the authors define and compute a GA within a derivative pricing model.

Committee as Basel II (Basel Committee on Banking Supervision [2006]) discusses extensively credit risk mitigation (CRM) techniques. These include, for example, ordinary guarantees, collateral securitization and credit derivatives such as credit default swaps. Today, credit derivatives might be the most common guarantee instrument. At least their market has grown rapidly over the last years. In the Mid-Year 2007 Market Survey Report of the International Swaps and Derivatives Association (ISDA), the notional amount of outstanding credit derivatives was estimated to be $45.46 trillion.[2]
It is reasonable that a financial institution should be able to decrease its capital requirements if it buys protection for its exposures. This is also important from a regulatory point of view, because it sets the incentive for banks to hedge their credit risk. Therefore, in 2005 the Basel Committee made an amendment to the 2003 New Basel Accord concerning the treatment of guarantees in the IRB approach (see Basel Committee on Banking Supervision [2005]).[3] In the New Basel Accord of 2003, banks were allowed to adopt a so-called *substitution approach* to hedged exposures. Roughly speaking, under this approach a bank can compute the risk-weighted assets for a hedged position as if the credit exposure was a direct exposure to the obligor's guarantor. Therefore, the bank may have only a small or even no benefit in terms of capital requirements from obtaining the protection. Since the 2005 amendment, for each hedged exposure the bank can choose between the substitution approach and the so-called *double default treatment.* The latter, inspired by Heitfield and Barger [2003], takes into account that the default of a hedged exposure only occurs if both the obligor and the guarantor default ("double default"). There are rather strict requirements on the obligor and the guarantor for application of the double default treatment. Moreover, the parameters chosen in calculating the double default probability are quite conservative. We refer to Grundke [2008] for a meta-study on this issue. It has been shown in Heitfield and Barger [2003]) that this double default treatment can lead to a significant decrease in capital requirements under the Advanced IRB approach.

Since the double default treatment in the IRB approach is also based on the assumption of an infinitely granular portfolio, it seems natural to investigate the impact of guarantees on possible adjustments for undiversified idiosyncratic risk as represented e.g. by the GA. In this paper we address this issue and derive a GA that takes into account double default effects. The GA is derived as a first-order asymptotic approximation for the effect of diversification in large portfolios

[2]See O'Kane [2008] for a comparison of several studies on the topic.

[3]Meanwhile the amendment also has been incorporated in a revised version of the 2003 Basel accord, Basel Committee on Banking Supervision [2006]. If not noted otherwise, this is the version we refer to as "Basel II".

within an extended version of the CreditRisk$^+$ model that allows for idiosyncratic recovery risk.[4] Note, however, that our methodology could in principle be applied to any model of portfolio credit risk that is based on a conditional independence framework. We derive an analytic solution for the granularity adjustment in a very general setting with several partially hedged positions where the guarantors can also act as obligors in the portfolio themselves. Moreover, we present some results on the performance of our new formula. In particular, we study the impact of guarantees and double default effects on the risk weighted assets of Basel II. Similar to the revised GA of Gordy and Lütkebohmert [2007] our generalization only requires data inputs which are already available when calculating IRB capital charges and reserve requirements. The fact that the GA is analytical allows for a fast computation and avoids the simulation of the rare double default events. Thus it very well suits for application under Pillar 2 of Basel II.

We start in Section 2 by introducing our basic notations and the CreditRisk$^+$ setting we apply. Moreover, in this section we provide a review of the GA methodology without guarantees. In Section 3 we provide some illustrative examples of our main result and discuss the main difficulties that occur when deriving a GA in the presence of guarantees. In particular, we discuss the various scenarios and interactions between obligors and guarantors that can occur in practice. Section 4 gives the main result for an arbitrary number of partially hedged positions in the portfolio and discusses multiple hedging of a single obligor. Here we also provide a numerical example on the performance of our novel GA. In Section 5 we conclude and discuss our assumptions and results. The Appendix A provides proofs of our results. Appendix B contains a comparison study of our model with the treatment of double default effects within the IRB approach.

2. Notations and Basic GA Methodology

Our model presents an extension of the granularity adjustment introduced in Gordy and Lütkebohmert [2007] which is based on the single factor CreditRisk$^+$ model allowing for idiosyncratic recovery risk. Note, however, that our general GA can in principle be applied to any risk-factor model of portfolio credit risk that is based on a conditional independence framework.

Let X denote the systematic risk factor which we assume to be unidimensional to achieve consistency with the ASRF framework of Basel II. Denote the probability density function of X by $h(X)$. In our specific setting we assume X to be Gamma

[4] CreditRisk$^+$ is a widely used industry model developed by Credit Suisse Financial Products [1997].

distributed with mean 1 and variance $1/\xi$ for some $\xi > 0$.[5] We consider a portfolio consisting of N obligors indexed by $n = 1, 2, \ldots, N$. Suppose that exposures of each obligor have been aggregated so that there is a unique position for each obligor in the portfolio. We refer to Gordy and Lütkebohmert [2007] for a discussion of this assumption. Assume that the first $K \geq 0$ positions are hedged by some guarantors who might or might not be part of the portfolio themselves. The remaining $N - K$ positions are unhedged.[6] Denote by EAD_n the exposure at default of obligor n and let $s_n = \mathrm{EAD}_n / \sum_{i=1}^N \mathrm{EAD}_i$ be its share on total exposure. Applying an actuarial definition of loss as in the CreditRisk$^+$ model we define the loss rate of obligor n as $U_n = \mathrm{LGD}_n \cdot \mathrm{D}_n$, where D_n is a default indicator equal to 1 if obligor n defaults and 0 otherwise. Here $\mathrm{LGD}_n \in [0, 1]$ denotes the loss given default rate of obligor n which is assumed to be random and independent of D_n with expectation ELGD_n and volatility VLGD_n. The systematic risk factor X generates correlation across obligor defaults by shifting the default probabilities. Conditional on $X = x$ the default probability of obligor n is

$$\mathrm{PD}_n(x) = \mathrm{PD}_n \cdot (1 - w_n + w_n \cdot x) \tag{2.1}$$

where PD_n is the unconditional default probability and w_n is a factor loading specifying the extent to which obligor n depends on the systematic factor X.

We denote the loss variable of a portfolio with K hedged positions and $N - K$ unhedged positions by $L_{K,N-K}$.[7] Note that in the situation without guarantees we have conditional independence between obligors in the portfolio and thus can express the portfolio loss as

$$L_{0,N} = \sum_{n=1}^{N} s_n U_n. \tag{2.2}$$

Denote the q^{th} percentile of the distribution of some random variable X by $\alpha_q(X)$ and for ease of notation we will sometimes use $x_q = \alpha_q(X)$ instead. When economic capital is measured as Value-at-Risk at the q^{th} percentile, we wish to estimate $\alpha_q(L_{K,N-K})$. The IRB formula, however, delivers the q^{th} percentile of the conditional expected loss $\alpha_q(\mathbb{E}[L_{K,N-K}|X])$. The difference

$$\alpha_q(L_{K,N-K}) - \alpha_q(\mathbb{E}[L_{K,N-K}|X]) \tag{2.3}$$

is the "exact" adjustment for the effect of undiversified idiosyncratic risk in the portfolio. This interpretation is justified in a conditional independence setting by the fact that $\alpha_q(\mathbb{E}[L_{K,N-K}|X])$ converges to $\alpha_q(L_{K,N-K})$ as the portfolio becomes

[5] For the calibration of the parameter ξ we refer to Gordy and Lütkebohmert [2007].

[6] In the following quantities with a subindex n refer to the single obligor n and are defined for arbitrary $n = 1, \ldots, N$.

[7] In general, when we use notations with two lower subindices, the first index gives the number of hedged positions and the second index gives the number of unhedged positions in the considered portfolio. This will be convenient when we derive the GA for portfolios with K hedged positions.

more and more fine-grained.[8] Such an exact adjustment cannot be obtained in analytical form, but we can construct a Taylor series approximation in orders of $1/N$. Therefore, we define the conditional expectation and conditional variance of obligor n's loss variable by $\mu_n(x) = \mathbb{E}[U_n|x]$ and $\sigma_n^2 = \mathbb{V}[U_n|x]$ and on portfolio level the quantities

$$\mu_{K,N-K}(x) = \mathbb{E}[L_{K,N-K}|x] \tag{2.4}$$

$$\sigma^2_{K,N-K}(x) = \mathbb{V}[L_{K,N-K}|x]. \tag{2.5}$$

Based on theoretical results of Gouriéroux et al. [2000] one can show that a first-order approximation of (2.3), which defines our granularity adjustment, can be obtained as

$$GA_{K,N-K} = \frac{-1}{2h(x_q)} \frac{d}{dx} \left(\frac{\sigma^2_{K,N-K}(x) h(x)}{\mu'_{K,N-K}(x)} \right) \Big|_{x=x_q}. \tag{2.6}$$

This result is independent of the question whether there are some hedged positions in the portfolio since only the quantities $\mu_{K,N-K}(x)$ and $\sigma_{K,N-K}(x)$ are sensitive to this decision. Gordy and Lütkebohmert [2007] reformulate this result within a CreditRisk$^+$ framework and derive a simple analytic formula for the GA in the case without guarantees which we will briefly review in the remainder of this section.

Assume a portfolio with N unhedged exposures. First, note that due to the conditional independence framework in the case without hedged positions the quantities in equations (2.4) and (2.5) can be expressed as

$$\mu_{0,N}(x) = \mathbb{E}[L_{0,N}|x] = \sum_{n=1}^{N} s_n \mu_n(x) \tag{2.7}$$

$$\sigma^2_{0,N}(x) = \mathbb{V}[L_{0,N}|x] = \sum_{n=1}^{N} s_n^2 \sigma_n^2(x). \tag{2.8}$$

In analogy to Gordy and Lütkebohmert [2007] we now reparameterize the inputs of the GA formula (2.6), i.e. the quantities $\mu_n(x)$ and $\sigma_n^2(x)$ for $n = 1, \dots, N$. Therefore, for every obligor n let $\mathcal{R}_n$ be the expected loss (EL) reserve requirement and $\mathcal{K}_n$ the unexpected loss (UL) capital requirement as a share of EAD_n. In the default-mode setting of CreditRisk$^+$ these quantities can be expressed as

$$\mathcal{R}_n = \mathbb{E}[U_n] = \mathrm{ELGD}_n \cdot \mathrm{PD}_n \tag{2.9}$$

$$\mathcal{K}_n = \mathbb{E}[U_n|x_q] - \mathbb{E}[U_n] = \mathrm{ELGD}_n \cdot \mathrm{PD}_n \cdot w_n \cdot (x_q - 1). \tag{2.10}$$

Further, let $\mathcal{K}_{0,N} = \sum_{n=1}^{N} s_n \mathcal{K}_n$ denote the required capital per unit exposure for the portfolio as a whole. Since the conditional default probability in a CreditRisk$^+$

[8] See Gordy [2003], Proposition 5, for assumptions and a proof of this result.

framework equals $\mathrm{PD}_n(x) = \mathrm{PD}_n \cdot (1 - w_n + w_n \cdot x)$ we obtain

$$
\begin{aligned}
\mu_n(x_q) &= \mathcal{K}_n + \mathcal{R}_n \\
\mu_n'(x_q) &= \mathcal{K}_n/(x_q - 1) \\
\mu_n''(x_q) &= 0.
\end{aligned}
\tag{2.11}
$$

Moreover, it can be shown that

$$
\sigma_n^2(x) = \mathcal{C}_n \mu_n(x) + \mu_n^2(x) \frac{\mathrm{VLGD}_n^2}{\mathrm{ELGD}_n^2}
\tag{2.12}
$$

and thus

$$
\frac{d}{dx} \sigma_n^2(x_q) = \mathcal{C}_n \mu_n'(x_q) + 2\mu_n'(x_q)\mu_n(x_q) \frac{\mathrm{VLGD}_n^2}{\mathrm{ELGD}_n^2}
\tag{2.13}
$$

with

$$
\mathcal{C}_n = \frac{\mathrm{ELGD}_n^2 + \mathrm{VLGD}_n^2}{\mathrm{ELGD}_n}.
$$

Noting that $\mu'(x_q) = \sum_{n=1}^N s_n \mathcal{K}_n/(x_q - 1) = \mathcal{K}_{0,N}/(x_q - 1)$ and exercising the differentiation operator, one can reformulate equation (2.6) in the case without hedging as

$$
\begin{aligned}
GA_{0,N} &= \frac{1}{2\mathcal{K}_{0,N}} \cdot \left(-(x_q - 1) \frac{h'(x_q)}{h(x_q)} \right) \sigma_{0,N}^2(x_q) \\
&\quad - \frac{1}{2} \cdot \frac{\left(\frac{d}{dx}\sigma_{0,N}^2(x_q)\right) \mu_{0,N}'(x_q) - \sigma_{0,N}^2(x_q)\mu_{0,N}''(x_q)}{(\mu_{0,N}'(x_q))^2}.
\end{aligned}
\tag{2.14}
$$

Defining

$$
\delta = -(x_q - 1) \frac{h'(x_q)}{h(x_q)}
$$

and using that $\mu_{0,N}''(x_q) = 0$ the previous equation simplifies to

$$
\begin{aligned}
GA_{0,N} &= \frac{1}{2\mathcal{K}_{0,N}} \cdot \delta \sigma_{0,N}^2(x_q) - \frac{1}{2} \cdot \frac{\frac{d}{dx}\sigma_{0,N}^2(x_q)}{\mu_{0,N}'(x_q)} \\
&= \frac{1}{2\mathcal{K}_{0,N}} \left(\delta \sigma_{0,N}^2(x_q) - (x_q - 1) \frac{d}{dx} \sigma_{0,N}^2(x_q) \right).
\end{aligned}
\tag{2.15}
$$

Inserting the CreditRisk$^+$ representations of the terms $\mu_{0,N}(x_q)$ and $\sigma_{0,N}^2(x_q)$ and their derivatives, Gordy and Lütkebohmert [2007] obtain

$$
\begin{aligned}
GA_{0,N} &= \frac{1}{2\mathcal{K}_{0,N}} \sum_{n=1}^N s_n^2 \left[\left(\delta \mathcal{C}_n (\mathcal{K}_n + \mathcal{R}_n) + \delta (\mathcal{K}_n + \mathcal{R}_n)^2 \cdot \frac{\mathrm{VLGD}_n^2}{\mathrm{ELGD}_n^2} \right) \right. \\
&\quad \left. - \mathcal{K}_n \left(\mathcal{C}_n + 2(\mathcal{K}_n + \mathcal{R}_n) \cdot \frac{\mathrm{VLGD}_n^2}{\mathrm{ELGD}_n^2} \right) \right].
\end{aligned}
\tag{2.16}
$$

It is the aim of this paper to extend this result to the situation with guarantees and to derive a simple closed-form GA that is able to account for double default effects and which is consistent with the ASRF model underlying Basel II.

3. Some Illustrative Examples and Discussion of the Methodology

In this section we provide some illustrative examples of our general GA formula given in Theorem 1. We start by discussing in some detail the main problems that occur in the presence of guarantees. Therefore it suffices for the beginning to study the case $K = 1$, i.e. we consider a portfolio consisting of an exposure to obligor 1, which is partially hedged by a guarantor g_1, and $N-1$ unhedged positions.[9] Note that partial hedging is of particular importance here as for the GA computation, exposures to a single obligor first have to be aggregated.[10] Thus if one exposure to an obligor is hedged and there are also some unhedged exposures to this obligor, we have to face the problem of partial hedging in the GA computation. For $0 \leq \lambda \leq 1$ denote by $(1-\lambda)\,\mathrm{EAD}_1$ the unhedged portion and by $\lambda\,\mathrm{EAD}_1$ the hedged portion of the exposure to obligor 1. All derivations in this paper will be given for the case where there is direct exposure to guarantors. That is, guarantors are themselves obligors in the portfolio. In the current case we thus let $g_1 = 2$ and s_2 is the exposure share of the guarantor, obligor 2. The situation where there is actually no direct exposure to the guarantor then simply is obtained as the special case where the exposure $s_2 = 0$.

In this situation the loss rates associated with the unhedged exposure to obligor 1, the direct exposure to the guarantor and the hedged exposure to obligor 1 can no longer be treated as conditionally independent. The IRB treatment of double default effects, however, ignores this issue by not specifying the relationships of the guarantors with the credit portfolio. Implicitly it is assumed that there are only perfect full hedges, that guarantors are not obligors in the portfolio themselves and that different obligors are hedged by different guarantors. To treat the possible interactions appropriately we construct a *composite instrument* with loss rate $\hat{U}_1$ and exposure share $\hat{s}_1 = \lambda s_1$ consisting of the hedged portion $\lambda\,\mathrm{EAD}_1$ of the exposure to obligor 1. Note that the loss rate of the unhedged portion $(1-\lambda)\,\mathrm{EAD}_1$ of the exposure to obligor 1 is given by U_1. In the following we will use the notation "hat" for a quantity referring to a *hedged* obligor and its guarantor. Thus, in general, such a quantity will depend on characteristics of both the hedged obligor and its guarantor. Note that, when obligor 1 defaults and the guarantor 2 survives, the latter will pay for the hedged exposure such that the exposure to obligor 1 is only lost in case when both obligor 1 and obligor 2 default. Therefore, let $\hat{U}_1 = U_1 U_2$.

[9] From now on we will think of ordinary guarantees as the hedging instruments although our results can be applied to all types of CRM techniques as indicated in the Introduction. For example, the "guarantor" could also be the protection seller within a credit default swap contract.

[10] For a detailed discussion of this problem we refer to Gordy and Lütkebohmert [2007].

We define the EL capital requirement for the composite instrument as

$$\begin{aligned}
\hat{\mathcal{R}}_1 &\equiv \mathbb{E}[\hat{U}_1] = \mathbb{E}[U_1 U_2] = \mathbb{E}[\mathbb{E}[U_1 U_2|X]] = \mathbb{E}\left[\mathbb{E}[U_1|X]\cdot\mathbb{E}[U_2|X]\right] \\
&= \mathrm{ELGD}_1\,\mathrm{ELGD}_2\cdot\mathbb{E}[\mathrm{PD}_1\cdot(1+w_1\cdot(X-1))\cdot\mathrm{PD}_2\cdot(1+w_2\cdot(X-1))] \\
&= \mathrm{ELGD}_1\,\mathrm{ELGD}_2\,\mathrm{PD}_1\,\mathrm{PD}_2\cdot(1+w_1w_2\cdot\mathbb{V}[X]) \\
&= \mathcal{R}_1\mathcal{R}_2 + \frac{\mathcal{K}_1\mathcal{K}_2}{(x_q-1)^2\xi},
\end{aligned}$$

which follows from the fact that the Bernoulli random variables D_1 and D_2 are independent conditional on the systematic risk factor X, which is Gamma distributed with mean 1 and variance $1/\xi$. Moreover, the UL capital contribution for the composite instrument is given by

$$\begin{aligned}
\hat{\mathcal{K}}_1 &\equiv \mathbb{E}[\hat{U}_1|x_q] - \mathbb{E}[\hat{U}_1] = \mathbb{E}[U_1U_2|x_q] - \mathbb{E}[U_1U_2] \\
&= \mathrm{ELGD}_1\,\mathrm{PD}_1\cdot(1+w_1(x_q-1))\cdot\mathrm{ELGD}_2\,\mathrm{PD}_2\cdot(1+w_2(x_q-1)) - \hat{\mathcal{R}}_1 \\
&= \mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_1\mathcal{R}_2 + \mathcal{R}_1\mathcal{K}_2 - \frac{\mathcal{K}_1\mathcal{K}_2}{(x_q-1)^2\xi}.
\end{aligned}$$

The portfolio loss $L_{1,N-1}$ in case of a single partial hedge can no longer be expressed by equation (2.2) but is given by

$$\begin{aligned}
L_{1,N-1} &= L_{0,N-1} + \lambda s_1\hat{U}_1 + (1-\lambda)s_1U_1 \\
&= L_{0,N-1} + s_1U_1\left(\lambda U_2 + (1-\lambda)\right).
\end{aligned} \tag{3.1}$$

Note that in the definition of $L_{0,N-1}$ the exposure shares are also defined as $\mathrm{EAD}_n\,/\sum_{i=1}^{N}\mathrm{EAD}_i$, i.e. with respect to the portfolio consisting of N positions.

Remark 1. We want to point out here that the loss rates in the above definition of the portfolio loss are no longer conditionally independent as the loss rates U_2 for the guarantor, U_1 for the unhedged exposure to obligor 1 and $\hat{U}_1$ for the composite instrument are conditionally dependent. However, it still makes sense to define the GA as the difference in terms of percentiles between the portfolio loss and its conditional expectation, equation (2.3), as long as the exposures that are hedged by internal guarantors are sufficiently small as shares of total portfolio exposure. Otherwise, the asymptotic result underlying the computation of portfolio VaR under the ASRF model breaks down.[11] Within the IRB treatment of double default effects this problem is more severe because of the additional correlation assumed in that setting.[12]

[11] See Gordy [2003], p.203, for further details.

[12] See Section 5 and the Appendix B for details.

To obtain the GA we must compute the conditional expectation $\mu_{1,N-1}(x)$ and the conditional variance $\sigma^2_{1,N-1}(x)$ referring to the above definition of loss, equation (3.1), and also derivatives of these expressions. Since in the current case no other obligor in the portfolio is hedged by guarantor 2, all of the $N-2$ ordinary obligors are independent of obligor 2 and the composite instrument conditional on the systematic risk factor X. Thus our approach will be to express $L_{1,N-1}$ as a deviation from $L_{0,N-2}$, $\mu_{1,N-1}(x)$ as a deviation from $\mu_{0,N-2}(x)$, and so on. We then show that these quantities also can be expressed as deviations from $L_{0,N-1}, \mu_{0,N-1}(x)$ and so on. This way the GA computation can partially be traced back to the one in Gordy and Lütkebohmert [2007] that was sketched in Section 2. This is the main idea for the proof of our first result which is summarized in the following Proposition. For the proof we refer to the Appendix A.

Proposition 1 (GA Formula in Case of a Single Partial Hedge)**.** *The granularity adjustment for the case where a portion λ of the exposure to obligor 1 is hedged by obligor 2 is given by*

(3.2)

$$\begin{aligned}\widetilde{GA}_{1,N-1} \;&=\; \frac{\mathcal{K}_{0,N-1}}{\mathcal{K}_{1,N-1}(\lambda)}\overline{GA}_{0,N} + \frac{s_1\lambda\mathcal{K}_1\mathcal{K}_2}{(\mathcal{K}_{1,N-1}(\lambda))^2}\sigma^2_{0,N-1}(x_q)\\ &\quad+\frac{\left(s_1^2\hat{\mathcal{C}}_1(\lambda)+2s_1s_2\lambda\mathcal{C}_2\right)}{2\mathcal{K}_{1,N-1}(\lambda)}\left(\delta(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1)-(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1))\right)\end{aligned}$$

where

$$\mathcal{K}_{1,N-1}(\lambda) := \mathcal{K}_{0,N-1} + s_1\left(\lambda\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right)+(1-\lambda)\mathcal{K}_1\right)$$

and

(3.3)

$$\begin{aligned}\overline{GA}_{0,N} \;&:=\; GA_{0,N-1} + \frac{s_1^2(1-\lambda)^2}{2\mathcal{K}_{0,N-1}}\left[\delta\left(\mathcal{C}_1(\mathcal{K}_1+\mathcal{R}_1)+(\mathcal{K}_1+\mathcal{R}_1)^2\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right)\right.\\ &\qquad\left.-\left(2\mathcal{K}_1(\mathcal{K}_1+\mathcal{R}_1)\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}+\mathcal{C}_1\mathcal{K}_1\right)\right].\end{aligned}$$

Here $GA_{0,N-1}$ is the GA formula for the portfolio with $N-1$ ordinary obligors, equation (2.16). Further, we used the notation

$$\hat{\mathcal{C}}_1(\lambda) := \lambda^2\mathcal{C}_1\mathcal{C}_2 + 2\lambda(1-\lambda)\mathcal{C}_1. \qquad (3.4)$$

The notation $\widetilde{GA}_{K,N-K}$ indicates that we simplified the expression for the GA by neglecting terms that are of order $\mathcal{O}(\frac{1}{N^2}\cdot\mathrm{PD}^3\cdot\mathrm{ELGD}^3)$ or even higher and thus would contribute little to the GA.[13]

[13]For more details on this argument see the proof and Remark 2.

The second term in equation (3.3) is the standard GA contribution of the non-hedged part $(1-\lambda)s_1$ of the exposure to obligor 1.[14] Thus in the first term of equation (3.2) we have summarized the contribution to the GA belonging to the unhedged part of the portfolio, i.e. to exposures $\mathrm{EAD}_2, \ldots, \mathrm{EAD}_N$ and to the unhedged portion $(1-\lambda)\,\mathrm{EAD}_1$ of the exposure to obligor 1. The third term of equation (3.2) depends only on the hedged obligor and its guarantor. It represents the contribution to the GA that is purely due to the hedged exposure to obligor 1. Note that this term also contains a part which vanishes when there is no direct exposure to the guarantor, i.e. when $s_2 = 0$, which leads to a reduction of the GA. The second term depends on all obligors in the portfolio. Hence, there is no additive decomposition of $\widetilde{GA}_{1,N-1}$ into the portfolio components belonging to the $N-1$ ordinary obligors and the hedged position and its guarantor. Note that for $\lambda = 1$ we have $\hat{\mathcal{C}}_1(\lambda) = \mathcal{C}_1\mathcal{C}_2$ and $\overline{GA}_{0,N} = GA_{0,N-1}$.

Remark 2. Studying equation (3.2) in more detail we will see that double default effects are second order effects $\mathcal{O}(1/N^2)$ in the GA. Therefore, we assume a homogeneous portfolio where each exposure share equals $s_n = 1/N$ and PDs and ELGDs are constant for all obligors. Assume that the exposure to obligor 1 is fully hedged by obligor 2, i.e. $\lambda = 1$. Recall that by the definition of $\mathcal{K}_{1,N-1}(\lambda)$ for such a portfolio we have

$$\begin{aligned}
\mathcal{K}_{1,N-1}(\lambda) &= \sum_{n=2}^{N} s_n\mathcal{K}_n + s_1(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2) + \mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)) \\
&= \frac{N-1}{N}\mathcal{K}_1 + \frac{1}{N}(2\mathcal{K}_1^2 + 2\mathcal{K}_1\mathcal{R}_1).
\end{aligned}$$

Thus for large N the terms $\mathcal{K}_n/\mathcal{K}_{1,N-1}(\lambda) = N/(N-1+2(\mathcal{K}_1+\mathcal{R}_1))$ are approximately equal to 1. Similarly one can show that $\mathcal{K}_{0,N-1}/\mathcal{K}_{1,N-1}(\lambda)$ is also approximately equal to 1. Moreover, one can easily show that for a homogeneous portfolio $GA_{0,N-1}$ is proportional to $1/N$. Thus the first term in equation (3.2) is of order $1/N$. Furthermore, for a homogeneous portfolio the quantity $\sigma^2_{0,N-1}(x_q) = \sum_{n=2}^{N} s_n^2\sigma_n^2(x_q) = \frac{1}{N^2}\sum_{n=2}^{N}\sigma_n^2(x_q)$ is proportional to $(N-1)/N^2$. Hence, for large N the second term in equation (3.2) is approximately proportional to $1/N^2$. Similarly we obtain that the third term is proportional to $1/N^2$. Hence the main contribution to the portfolio GA comes from the unhedged part of the portfolio while double default effects still contribute second order to the GA. Therefore, also in terms of the GA, a bank will be rewarded significantly with lower capital requirements when buying credit protections.

We now extend the previous model by allowing for several hedged positions in the portfolio. For the analysis it is sufficient to consider only two hedged positions

[14] Compare with formula (2.16).

as this illustrates all possible interactions between obligors and guarantors and the extension to more than two hedged positions will be straightforward. Let us first generalize the notations from the previous situation to the case with several guarantees. Therefore, consider a portfolio where the exposures to the first K obligors are partially hedged by some guarantors $g_1, \ldots, g_K \in \{K+1, \ldots, N\}$.[15] Denote the hedged fraction of the loan to obligor $n \in \{1, \ldots, K\}$ by $\lambda_n \in [0,1]$ and define the vector $\lambda = (\lambda_1, \ldots, \lambda_K) \in [0,1]^K$. We define composite instruments for all hedged obligors by $\hat{U}_n = U_n \cdot U_{g_n}$ for $n = 1, \ldots, K$. The portfolio loss is then given by

$$L_{K,N-K} = L_{0,N-K} + \sum_{n=1}^{K} s_n \left(\lambda_n \hat{U}_n + (1-\lambda_n) U_n \right). \tag{3.5}$$

Moreover, we generalize the definition for the EL and UL capital of the composite instruments for arbitrary n as follows

$$\begin{aligned} \hat{\mathcal{R}}_n &= \mathcal{R}_n \mathcal{R}_{g_n} + \frac{\mathcal{K}_n \mathcal{K}_{g_n}}{(x_q-1)^2 \xi} \\ \hat{\mathcal{K}}_n &= \mathcal{K}_n \mathcal{K}_{g_n} + \mathcal{K}_n \mathcal{R}_{g_n} + \mathcal{R}_n \mathcal{K}_{g_n} - \frac{\mathcal{K}_n \mathcal{K}_{g_n}}{(x_q-1)^2 \xi}. \end{aligned}$$

Furthermore, we also extend the definition of $\hat{\mathcal{C}}_1(\lambda)$ to

$$\hat{\mathcal{C}}_n(\lambda_n) = \lambda_n^2 \mathcal{C}_n \mathcal{C}_{g_n} + 2\lambda_n(1-\lambda_n)\mathcal{C}_n \tag{3.6}$$

and we generalize the notation $\mathcal{K}_{1,N-1}(\lambda)$ to the case of K partially hedged positions by setting

$$\mathcal{K}_{K,N-K}(\lambda) = \mathcal{K}_{0,N-K} + \sum_{k=1}^{K} s_k \left[\lambda_k (\mathcal{K}_k (\mathcal{K}_{g_k} + \mathcal{R}_{g_k}) + \mathcal{K}_{g_k}(\mathcal{K}_k + \mathcal{R}_k)) + (1-\lambda_k)\mathcal{K}_k \right].$$

Finally, we naturally extend the definition of $\overline{GA}_{0,N}$ to the case with K partially hedged loans.

In the case of two guarantees we have to distinguish two different scenarios. First it is possible that two different guarantors hedge two different obligors. Therefore, we consider a portfolio with two partially hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3, \ldots, N)$ where $g_1 \neq g_2$. The portfolio loss is then given by

$$L_{2,N-2} = L_{0,N-2} + s_1 \left(\lambda_1 \hat{U}_1 + (1-\lambda_1) U_1 \right) + s_2 \left(\lambda_2 \hat{U}_2 + (1-\lambda_2) U_2 \right). \tag{3.7}$$

Note that in the above equation, terms referring to the hedged obligor 1 are conditionally independent from those referring to the hedged obligor 2. This is why we can compute the conditional mean and conditional variance of the corresponding composite instruments for the hedged exposure to obligor 1 and obligor 2 separately

[15] We will discuss the case $g_n \in \{1, \ldots, K\}$ in Remark 3.

by applying the same methods as in the case of a single hedged position. For details see the Appendix A.

Another possible scenario with two guarantees is that one guarantor hedges two different obligors. Similarly to the previous case, we consider a portfolio with two hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3,4,\dots,N)$. However, the obligors now have the same guarantor $g_1 = g_2$. For ease of notation let $g_1 = g_2 = 3$. Then the portfolio loss is given by

$$\begin{aligned} L_{2,N-2} &= L_{0,N-3} + \Big(s_3U_3 + s_1\lambda_1U_1U_3 + s_2\lambda_2U_2U_3\Big) \\ &\quad + \Big(s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2\Big). \end{aligned} \tag{3.8}$$

Neglecting third and higher order terms in EL and UL capital contributions, one can show that the expressions for $\mu_{2,N-2}(x_q)$ and $\sigma^2_{2,N-2}(x_q)$ and their derivatives do not depend on whether both obligors have different guarantors or the same guarantor. Consequently the formula for the granularity adjustment also has to be the same as in the case with different guarantors. It is summarized in the following proposition. For the proof we refer to the Appendix A. It can be shown that the granularity adjustment in the case of the same guarantor is larger, but only in third order terms which are neglected in our simplified version.

Proposition 2 (GA Formula in Case of Two Partial Hedges)**.** *The GA in the case where a portion λ_1 of the exposure to obligor 1 is hedged by guarantor g_1 and a portion λ_2 of the exposure to obligor 2 is hedged by guarantor g_2 is given by*

$$\begin{aligned} \widetilde{GA}_{2,N-2} &= \frac{\mathcal{K}_{0,N-2}}{\mathcal{K}_{2,N-2}(\lambda)}\overline{GA}_{0,N} + \frac{s_1\lambda_1\mathcal{K}_1\mathcal{K}_{g_1} + s_2\lambda_2\mathcal{K}_2\mathcal{K}_{g_2}}{(\mathcal{K}_{2,N-2}(\lambda))^2}\sigma^2_{0,N-2}(x_q) \\ &+ \frac{\Big(s_1^2\hat{\mathcal{C}}_1(\lambda) + s_1 s_{g_1}\lambda_1\mathcal{C}_{g_1}(\lambda)\Big)}{2\mathcal{K}_{2,N-2}(\lambda)}\Big(\delta(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) - (\mathcal{K}_1(\mathcal{K}_{g_1} + \mathcal{R}_{g_1}) + \mathcal{K}_{g_1}(\mathcal{K}_1 + \mathcal{R}_1))\Big) \\ &+ \frac{\Big(s_2^2\hat{\mathcal{C}}_2(\lambda) + s_2 s_{g_2}\lambda_2\mathcal{C}_{g_2}(\lambda)\Big)}{2\mathcal{K}_{2,N-2}(\lambda)}\Big(\delta(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) - (\mathcal{K}_2(\mathcal{K}_{g_2} + \mathcal{R}_{g_2}) + \mathcal{K}_{g_2}(\mathcal{K}_2 + \mathcal{R}_2))\Big) \end{aligned} \tag{3.9}$$

where we again neglected terms that are of order $\mathcal{O}(\frac{1}{N^2}\cdot \mathrm{PD}^3\cdot \mathrm{ELGD}^3)$ or higher.

4. Granularity Adjustment for a Portfolio with Several Guarantees

In this section we provide a general formula for the GA of a portfolio with several guarantees. Here we not only extend the previous result from 2 to K hedged obligors in the portfolio[16], but we further allow for different parts of the exposure to the

[16]From the computations in the previous section this essentially is straightforward.

same obligor to be hedged by several distinct guarantors.[17] This generalization is necessary for several applications. Suppose, for example, there are three loans to obligor 1 (indexed by $1, 2$ and 3) in the portfolio. Loans 1 and 2 are guaranteed by two different guarantors $g_{1,1}$ and $g_{1,2}$, respectively, whereas loan 3 is unhedged.[18] For the computation of the GA all three loans first have to be aggregated into a single loan. Let $\lambda_{1,1}$ and $\lambda_{1,2}$ denote the fractions of the first and second loan to obligor 1, respectively, on the aggregated position. The fraction $1 - \lambda_{1,1} - \lambda_{1,2}$ of the aggregated position is the unhedged part. In this section we will derive the contribution of such a partially hedged obligor 1 to the GA.

More generally, suppose we have a portfolio with N obligors of which the first $K \leq N$ are hedged and the entries of the tuple $\lambda_n = (\lambda_{n,1}, \ldots, \lambda_{n,j_n})$ are the portions of the exposure EAD_n to obligor n $(n = 1, \ldots, K)$ which are hedged by guarantors $g_{n,1}, \ldots, g_{n,j_n}$, respectively. Denote by Λ the collection of all tuples $\lambda_1, \ldots, \lambda_K$. In this situation the portfolio loss can be written as

$$L_{K,N-K} = L_{0,N-K} + \sum_{n=1}^{K} \sum_{i=1}^{j_n} s_n \lambda_{n,i} U_n U_{g_{n,i}} + s_n \left(1 - \sum_{i=1}^{j_n} \lambda_{n,i}\right) U_n.$$

To write down the final version of the GA, we generalize the notations of Section 3. First we naturally generalize the notation $\mathcal{K}_{K,N-K}(\lambda)$ to the case of multiple hedges per obligor which we then denote by $\mathcal{K}_{K,N-K}(\Lambda)$ and we generalize the notation $\hat{\mathcal{C}}_n(\lambda_n)$ in the following way

$$\hat{\mathcal{C}}_n(\lambda_{n,i}) = \lambda_{n,i}^2 \mathcal{C}_n \mathcal{C}_{g_{n,i}} + 2\lambda_{n,i} \left(1 - \sum_{i=1}^{j_n} \lambda_{n,i}\right) \mathcal{C}_n.$$

Similarly the notation $\overline{GA}_{0,N}$ is adapted by replacing the terms $1 - \lambda_n$ by the quantities $1 - \sum_{i=1}^{j_n} \lambda_{n,i}$.

We can now formulate our main result, a single analytic formula for the granularity adjustment that applies to any of the afore mentioned hedging situations.[19]

Theorem 1 (General GA Formula)**.** *Consider a portfolio with an arbitrary number of hedged positions where every hedging instrument may be any type of credit risk mitigation technique. Exposures to the same obligor may be hedged by different guarantors and for every exposure only parts may be hedged. Guarantors may or may not be obligors in the portfolio themselves and they may hedge exposures of more than one obligor. The total exposure shares of the positions that are hedged by guarantors who are part of the portfolio themselves, however, has to be sufficiently small such that the asymptotic result underlying the ASRF model still holds. With*

[17] For the case when the same exposure is hedged by more than one guarantor, see Remark 3.

[18] For simplicity, think of full hedges, although the argument works as well for partial hedges.

[19] By treating the case of multiple hedging of the same exposure as proposed in Remark 3 the formula indeed applies to all possible hedging combinations.

the notations above the granularity adjustment of such a portfolio can be computed by means of the following analytic formula

(4.1)

$$\begin{aligned}\widetilde{GA}_{K,N-K} &= \frac{\mathcal{K}_{0,N-K}}{\mathcal{K}_{K,N-K}(\Lambda)}\overline{GA}_{0,N} + \frac{\sigma^2_{0,N-K}(x_q)}{(\mathcal{K}_{K,N-K}(\Lambda))^2}\sum_{n=1}^{K}\sum_{i=1}^{j_n} s_n\lambda_{n,i}\mathcal{K}_n\mathcal{K}_{g_{n,i}}\\ &\quad+\frac{1}{2\mathcal{K}_{K,N-K}(\Lambda)}\sum_{n=1}^{K}\sum_{i=1}^{j_n}\Big(s_n^2\hat{\mathcal{C}}_{n,i}(\lambda_{n,i})+2s_ns_{g_{n,i}}\lambda_{n,i}\mathcal{C}_{g_{n,i}}\Big)\\ &\qquad\cdot\Big(\delta(\hat{\mathcal{K}}_{n,i}+\hat{\mathcal{R}}_{n,i})-(\mathcal{K}_n(\mathcal{K}_{g_{n,i}}+\mathcal{R}_{g_{n,i}})+\mathcal{K}_{g_{n,i}}(\mathcal{K}_n+\mathcal{R}_n))\Big).\end{aligned}$$

The notation $\widetilde{GA}_{K,N-K}$ indicates that we simplified the expression for the GA by neglecting terms that are of order $\mathcal{O}(\frac{1}{N^2}\cdot \mathrm{PD}^3\cdot \mathrm{ELGD}^3)$ or even higher and thus would contribute little to the GA.

Remark 3. Note that by the previous derivations it is obvious that a loan which is hedged by several guarantors will contribute only third order to the GA. The same is true when a guarantor itself is hedged. In these cases, we suggest a substitution approach as applied in Basel Committee on Banking Supervision [2006]. That is, whenever there are multiple guarantors to a single loan, the risk manager can choose one guarantor whose characteristics (i.e. PD, ELGD, EL and UL capital contributions) enter the GA formula.

Before we begin with a discussion of our main result in Section 5 we provide a numerical example in order to study the impact of hedging on the GA.

Example 1. *Consider an artificial portfolio P which is the most concentrated portfolio that is admissible under the EU large exposure rules.*[20] *To this purpose we divide a total exposure of €6000 into one loan of size €45, 45 loans of size €47 and 32 loans of size €120. We assume a constant* PD *of 1% and constant* ELGD *of 45%. Now suppose that all 32 loans of size €120 are completely hedged by different guarantors who are not part of the portfolio themselves. For these guarantors we assume a constant* PD *of 0.1% and a constant* ELGD *of 45%. Moreover, we fix the effective maturity for all obligors and guarantors to $M = 2.5$ years.*

Our generalized GA formula (4.1) leads to an add-on for undiversified idiosyncratic risk of $\widetilde{GA}_{32,46} = 0.83\%$ of total exposure, i.e. €49.80.[21] *To study the impact of hedging on economic capital we computed the IRB capital for portfolio P using the double default treatment in the IRB approach.*[22] *Then the economic capital for portfolio P with 32 guarantees equals 4.71% or €282.41. Hence, our novel GA formula*

[20]See Directive 93/6/EEC of 15 March 1993 on the capital adequacy of investment firms and credit institutions.

[21]In our numerical results we always fix the variance parameter of the systematic risk factor as $\xi = 0.125$. Moreover we computed the variance of LGD as $\mathrm{VLGD}_n^2 = \frac{1}{4}ELGD_n \cdot (1 - ELGD_n)$.

[22]See the Appendix B for more details on this approach.

leads to an add-on on economic capital of 17.62%. We now compare this result with the analogous computations using the GA formula (2.16) that does not take into account the hedging relations. The latter formula would yield a granularity adjustment of GA = 1.68% *of total exposure, i.e. €100.80. Thus, if we had to ignore the hedging relationships in portfolio P in the GA computation the add-on would be 35.67%. Hence, accounting for guarantees within the computation of the GA can significantly reduce the capital requirement for undiversified idiosyncratic risk. In our example of portfolio P the reduction is by approximately 50%. Table 1 summarizes the results of our example.*

Table 1: Impact of Guarantees on GA and IRB Capital Requirements

Portfolio P	*GA*	*IRB capital*	*add-on for GA*
without guarantees	*1.68%*	*4.71%*	*35.67%*
with guarantees	*0.83%*	*4.71%*	*17.62%*

The EC is in both cases computed using the IRB treatment of double default effects and thereby accounting for the 32 hedged positions in Portfolio P. The add-on for GA on EC is defined as the quotient of the sum of GA and EC over the EC.

Remark 4. Note that for a homogeneous portfolio where all exposures have the same size and PDs and ELGDs are also identical for all obligors, hedging can also have the opposite effect and increase the GA. This is due to the fact that hedging can shift the exposure distribution of the portfolio to a more concentrated distribution. For such a homogeneous portfolio for example, the exposure distribution is uniform and the portfolio can be considered as almost perfectly diversified for large N. When we assume now that some of the exposures in the portfolio are guaranteed by some other obligors in the portfolio, the portfolio becomes more concentrated and thus the GA increases.

5. Discussion and Conclusion

In this paper we derived a granularity adjustment that accounts for credit risk mitigation techniques in a very general setting. The derivation of our main result, Theorem 1, is rather complex because it considers all possible interactions between obligors and guarantors that can occur in practice. However, it relies on a simple model of double default that allows for an analytical solution. Therefore, simulations of the very rare double default events can be avoided. Moreover, the GA is parsimonious with respect to data requirements as its inputs are needed for the computation of Pillar 1 economic capital under the IRB approach anyway. This is a very important quality since the data inputs can post the most serious obstacle for practical application. Thus, our general GA formula is very well suited for application under Pillar 2 of Basel II.

Let us now discuss the underlying assumptions of our main result, formula (4.1), in more detail. Here, we will focus only on the assumptions related to the treatment of double default effects in the GA. For a discussion of the general assumptions of the GA methodology we refer to Gordy and Lütkebohmert [2007] and Lütkebohmert [2009]. The latter also contains a comparison with related approaches.

Our model of double default effects is based on the assumption that the loss rate of the exposure to an obligor which is hedged by a guarantor is given by the product of the individual loss rates which are assumed to be independent conditional on the systematic risk factor. Thus we implicitly assume that the obligor's default (triggering the guarantee payment) must not be too much of a burden to the guarantor. The same problem arises in the IRB treatment of double default effects. To mitigate it, conditions on obligors and guarantors can be imposed in order to qualify for their hedging relationship to be accounted for. See Basel Committee on Banking Supervision [2005] and Grundke [2008] for a discussion of the conditions. The IRB treatment of double default effects further assumes some additional correlation since the obligor and its guarantor are correlated not only through the systematic risk factor but also through an additional factor. It should be noted, however, that correlation cannot capture the asymmetry in their relationship, i.e. the guarantor should suffer much more from the default of the obligor than vice versa. Therefore, we argue that assuming extra high correlation as is implied by the dependence on an additional factor in the IRB approach, is problematic, in particular, when there is direct exposure to the guarantor. Given the default of the guarantor this would imply a higher probability of default for the obligor which does not seem to be empirically justified. A better approach in our opinion would be to increase the guarantor's unconditional default probability appropriately as this also caputers the before mentioned asymmetry. Within a simple structural model of default, Grundke [2008] shows that the additional correlation of 0.7 fixed in the IRB treatment of double default effects approximately corresponds to an increase of 100% in the guarantors unconditional probability of default.

We further note that under the ASRF model that underpins Basel II one must be careful when introducing additional correlation between obligors in the portfolio. The exposure shares of obligors that are correlated through more than the common risk factor must be sufficiently small. This is because otherwise the asymptotic result underlying the computation of portfolio VaR under the ASRF model breaks down (see Gordy [2003], p. 209, for further details). This might be the case if e.g. several loans in the portfolio are guaranteed by a large insurance company and, in particular, if there is direct exposure to that guarantor. This problem is not addressed in Basel Committee on Banking Supervision [2005]. For a detailed comparison between the IRB treatment of double default effects and our approach within the GA we refer the reader to Appendix B.

As our GA formula is parameterized to achieve consistency with the IRB approach, one could also argue to compute the GA with double default effects in a *two-step approach,* where in a first step we compute the GA *without* considering double default effects (and obtain the result of Gordy and Lütkebohmert [2007], formula (2.16)). In a second step we could then compute the UL capital requirement $\mathcal{K}_n^{DD}$ for a hedged obligor n as in the IRB treatment of double default effects and insert this parameter instead of $\mathcal{K}_n$ in the GA formula. This two-step procedure, however, essentially ignores any interaction of the guarantor with the rest of the portfolio. That is, it even ignores the common dependence induced by the systematic risk factor. Hence, roughly speaking, under a two-step approach the computation of EL and UL for a given portfolio and the computation of the GA are solved separately (rather than jointly) and then are put together naively. This, of course, implies a fairly easy derivation, however, with the shortcoming of missing any mathematical justification.
In contrast to this procedure, the *bottom-up approach* we used to derive the GA given by formula (4.1) incorporates double default effects right in the beginning. More precisely, our treatment of double default effects enters the model setup (the portfolio loss distribution) rather than just the model's "solution", the final GA formula. Thus it avoids the inconsistencies and disadvantages involved with a two-step procedure. The drawback is that this fully rigorous derivation is much more complex. In the current case, however, we saw that the derivation is tractable and even leads to a rather simple (in terms of parameters) analytical solution which can easily be implemented. This solution correctly incorporates all the different interactions between the obligors and the guarantors that can occur. In the case of our Example 1 the two-step method would lead to a GA of 1.49% of total exposure, i.e €89.40. Thus the capital reducing effect of the guarantees would be much lower in this approach than in our rigorous model-based approach.

Appendix A. Proofs

Proof of Proposition 1. In the situation of a single partial hedge the portfolio loss $L_{1,N-1}$ is given by equation (3.1). The conditional expectation of the loss ratio of the composite instrument $\hat{U}_1$ is given by

$$\hat{\mu}_1(x) = \mathbb{E}[\hat{U}_1|x] = \mathrm{ELGD}_1\,\mathrm{ELGD}_2\,\mathrm{PD}_1(x)\,\mathrm{PD}_2(x) = \mu_1(x)\mu_2(x). \quad \text{(A.1)}$$

Equations (3.1) and (A.1) imply that the conditional mean of the portfolio loss is

$$\mu_{1,N-1}(x) = \mu_{0,N-1}(x) + \lambda s_1 \hat{\mu}_1(x) + (1-\lambda)s_1\mu_1(x). \tag{A.2}$$

Taking the derivative yields

$$\mu'_{1,N-1}(x) = \mu'_{0,N-1}(x) + \lambda s_1\left(\mu'_1(x)\mu_2(x) + \mu_1(x)\mu'_2(x)\right) + (1-\lambda)s_1\mu'_1(x)$$

and for the second derivative we obtain

$$\mu''_{1,N-1}(x) \quad = \quad 2\lambda s_1 \mu'_1(x)\mu'_2(x)$$

since the second derivative of $\mu_n(x)$ vanishes for any $n = 1,\ldots,N$. Using the CreditRisk$^+$ notation of Section 3, the conditional expectation of the portfolio loss ratio and its derivatives can be expressed as

$$\begin{aligned}
\mu_{1,N-1}(x_q) &= \mu_{0,N-1}(x_q) + s_1\lambda(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) + s_1(1-\lambda)(\mathcal{K}_1 + \mathcal{R}_1) \\
\mu'_{1,N-1}(x_q) &= \frac{\mathcal{K}_{1,N-1}(\lambda)}{x_q - 1} \\
\mu''_{1,N-1}(x_q) &= \frac{2s_1\lambda}{(x_q-1)^2}\mathcal{K}_1\mathcal{K}_2
\end{aligned} \tag{A.3}$$

where $\mathcal{K}_{1,N-1}(\lambda)$ is defined in Proposition 1. Hence, it remains to compute the conditional variance of the portfolio loss and its derivative. For the conditional variance of the portfolio loss ratio we obtain

(A.4)

$$\begin{aligned}
& \mathbb{V}[s_2U_2 + \lambda s_1\hat{U}_1 + (1-\lambda)s_1U_1|x] \\
= \; & \mathbb{V}[s_2U_2 + (1-\lambda)s_1U_1|x] + \mathbb{V}[\lambda s_1\hat{U}_1|x] + 2\,\mathrm{Cov}[s_2U_2 + (1-\lambda)s_1U_1, \lambda s_1\hat{U}_1|x]
\end{aligned}$$

and the last term can be written as

$$2s_1s_2\lambda\,\mathrm{Cov}[U_2, U_1U_2|x] + 2s_1^2\lambda(1-\lambda)\,\mathrm{Cov}[U_1, U_1U_2|x].$$

Since U_1 and U_2 are conditionally independent one can show that

$$\mathrm{Cov}[U_2, U_1U_2|x] = \mu_1(x)\sigma_2^2(x) \;\text{ and }\; \mathrm{Cov}[U_1, U_1U_2|x] = \mu_2(x)\sigma_1^2(x). \tag{A.5}$$

Recall that for independent random variables Y_1 and Y_2 the following relation holds

$$\mathbb{V}[Y_1Y_2] = \mathbb{V}[Y_1]\mathbb{V}[Y_2] + \mathbb{V}[Y_1]\mathbb{E}[Y_2]^2 + \mathbb{V}[Y_2]\mathbb{E}[Y_1]^2. \tag{A.6}$$

Using these results equation (A.4) can be written as

$$\begin{aligned}
& \mathbb{V}[s_2U_2 + s_1\lambda U_1U_2 + s_1(1-\lambda)U_1|x] \\
= \; & s_2\sigma_2^2(x) + s_1^2(1-\lambda)^2\sigma_1^2(x) + s_1^2\lambda^2\left(\sigma_1^2(x)\sigma_2^2(x) + \sigma_1^2(x)\mu_2^2(x) + \sigma_2^2(x)\mu_1^2(x)\right) \\
& +2s_1s_2\lambda\mu_1(x)\sigma_2^2(x) + 2s_1^2\lambda(1-\lambda)\mu_2(x)\sigma_1^2(x)
\end{aligned}$$

and therefore the conditional variance of the portfolio loss ratio is

$$
\begin{aligned}
\sigma^2_{1,N-1}(x) \quad &= \quad \sigma^2_{0,N-1}(x) + s_1^2(1-\lambda)^2\sigma_1^2(x) \\
&\quad +2s_1s_2\lambda\mu_1(x)\sigma_2^2(x) + 2s_1^2\lambda(1-\lambda)\mu_2(x)\sigma_1^2(x) \\
&\quad +s_1^2\lambda^2\left(\sigma_1^2(x)\sigma_2^2(x) + \sigma_1^2(x)\mu_2^2(x) + \sigma_2^2(x)\mu_1^2(x)\right).
\end{aligned} \tag{A.7}
$$

Evaluating at x_q and inserting equations (2.11) and (2.12) gives an expression in $\mathcal{K}_n$ and $\mathcal{R}_n$. These quantities are typically quite small so that products of these contribute little to the GA.[23] As double default effects will be second order effects, i.e. of order $\mathcal{O}(1/N^2)$ as discussed in Remark 2, we will throughout this paper neglect third and higher order terms in $\mathcal{K}_n$ and $\mathcal{R}_n$. For this argument note that due to relations (2.11) and (2.12) the terms $\mu_n(x_q)$ and $\sigma_n^2(x_q)$ and their derivatives are all of order 1 in $\mathcal{K}_n$ and $\mathcal{R}_n$. Moreover, if an expression for the conditional variance of the loss ratio involves a product of three or more of these terms it will also yield products of three or more of these terms in the derivative. Finally, when computing the GA using formula (2.6), third or higher order terms in $\mathcal{K}_n$ and $\mathcal{R}_n$ can never turn into more significant lower order terms. This is obvious from the following derivations. Therefore, in the following we will always compute the expressions for the conditional variance of the portfolio loss and its derivative without third or higher order terms in $\mathcal{K}_n$ and $\mathcal{R}_n$ since these terms are of order $\mathcal{O}(1/N^2 \cdot \mathrm{PD}^3 \cdot \mathrm{ELGD}^3)$ or even higher and thus would yield negligible terms anyway. Thus with these simplifications we obtain

$$
\begin{aligned}
\sigma^2_{1,N-1}(x_q) \quad &\approx \sigma^2_{0,N-1}(x_q) + s_1^2(1-\lambda)^2\left(\mathcal{C}_1(\mathcal{K}_1+\mathcal{R}_1) + (\mathcal{K}_1+\mathcal{R}_1)^2\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right) \\
&\quad +2s_1s_2\lambda\mathcal{C}_2(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + s_1^2\left[\lambda^2\mathcal{C}_1\mathcal{C}_2 + 2\lambda(1-\lambda)\mathcal{C}_1\right](\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) \\
\tfrac{d}{dx}\sigma^2_{1,N-1}(x_q) \;&\approx\; \frac{d}{dx}\sigma^2_{0,N-1}(x_q) + \frac{s_1^2(1-\lambda)^2}{x_q-1}\left(\mathcal{C}_1\mathcal{K}_1 + 2\mathcal{K}_1(\mathcal{K}_1+\mathcal{R}_1)\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right) \\
&\quad +\frac{2s_1s_2\lambda\mathcal{C}_2}{x_q-1}\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2) + \mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right) \\
&\quad +\frac{s_1^2\left[\lambda^2\mathcal{C}_1\mathcal{C}_2 + 2\lambda(1-\lambda)\mathcal{C}_1\right]}{x_q-1}\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2) + \mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right).
\end{aligned}
$$

We define the variance of the unhedged part of the portfolio as

$$
\bar{\sigma}^2_{0,N}(x_q) := \sigma^2_{0,N-1}(x_q) + s_1^2(1-\lambda)^2\left[\mathcal{C}_1(\mathcal{K}_1+\mathcal{R}_1) + (\mathcal{K}_1+\mathcal{R}_1)^2\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right]. \tag{A.8}
$$

[23] $\mathcal{K}_n$ and $\mathcal{R}_n$ are essentially products of $\mathrm{PD}_n \in [0,1]$ and $\mathrm{ELGD}_n \in [0,1]$.

Applying further the notation of $\hat{\mathcal{C}}_1(\lambda)$ in Proposition 1, we can reformulate the conditional variance of the portfolio loss and its derivative at x_q as

(A.9)

$$\sigma^2_{1,N-1}(x_q) \approx \bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda)(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + 2s_1s_2\lambda\mathcal{C}_2(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1)$$

$$\frac{d}{dx}\sigma^2_{1,N-1}(x_q) \approx \frac{d}{dx}\bar{\sigma}^2_{0,N}(x_q) + \frac{s_1^2\hat{\mathcal{C}}_1(\lambda)+2s_1s_2\lambda\mathcal{C}_2}{x_q-1}\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right).$$

We now use these representations to compute the GA in the case of one hedged position. Therefore, first note that the formula for the "full" GA, equation (2.6), can be reformulated as

$$\begin{aligned} GA_{1,N-1} &= \frac{1}{2\mathcal{K}_{1,N-1}(\lambda)}\Bigg(\delta\sigma^2_{1,N-1}(x_q) - (x_q-1)\frac{d}{dx}\sigma^2_{1,N-1}(x_q) \\ &\qquad + (x_q-1)\frac{\sigma^2_{1,N-1}(x_q)\mu''_{1,N-1}(x_q)}{\mu'_{1,N-1}(x_q)}\Bigg). \end{aligned} \tag{A.10}$$

Rearranging and using the simplified expressions for the conditional variance and its derivative, equation (A.9), this becomes

(A.11)

$$\begin{aligned} \widetilde{GA}_{1,N-1} &= \frac{1}{2\mathcal{K}_{1,N-1}(\lambda)}\left(\delta\bar{\sigma}^2_{0,N}(x_q) - (x_q-1)\frac{d}{dx}\bar{\sigma}^2_{0,N}(x_q)\right) \\ &+ \frac{\left(s_1^2\hat{\mathcal{C}}_1(\lambda)+2s_1s_2\lambda\mathcal{C}_2\right)}{2\mathcal{K}_{1,N-1}(\lambda)}\left(\delta(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) - \left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right)\right) \\ &+ \frac{1}{2\mathcal{K}_{1,N-1}(\lambda)}\left((x_q-1)\frac{\sigma^2_{1,N-1}(x_q)\mu''_{1,N-1}(x_q)}{\mu'_{1,N-1}(x_q)}\right). \end{aligned}$$

Unlike in the case without hedging the last summand of equation (A.11) does not vanish since $\mu''_{1,N-1}(x_q) = 2\lambda s_1\mu'_1(x_q)\mu'_2(x_q) = 2\lambda s_1\mathcal{K}_1\mathcal{K}_2/(x_q-1)^2$ is in general not zero. We have

$$\begin{aligned} &\frac{\sigma^2_{1,N-1}(x_q)\mu''_{1,N-1}(x_q)}{\mu'_{1,N-1}(x_q)} \\ &= \frac{2\lambda s_1\mathcal{K}_1\mathcal{K}_2}{\mathcal{K}_{1,N-1}(\lambda)(x_q-1)}\left(\bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda)(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + 2s_1s_2\lambda\mathcal{C}_2(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1)\right). \end{aligned}$$

The last two summands are very small and can be neglected.[24] Using this result, inserting the GA formula for the portfolio with $N-1$ ordinary obligors, equation (2.16), and using the notation $\overline{GA}_{0,N}$ we obtain from equation (A.11) the GA formula of Proposition 1. □

Proof of Proposition 2. We start with the situation where two different guarantors hedge two different obligors. Therefore, we consider a portfolio with

[24] The expression $\mathcal{K}_n/\mathcal{K}_{1,N-1}$ should be reasonably close to 1 so that the neglected terms are of order $\mathcal{O}(1/N^3)$.

two partially hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3,\dots,N)$ where $g_1 \neq g_2$. The portfolio loss is then given by equation (3.7). Similarly to equation (A.3) we obtain for the conditional expectation of the portfolio loss and its derivatives

$$
\begin{aligned}
\mu_{2,N-2}(x_q) &= \mu_{0,N-2}(x_q) + s_1\lambda_1(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + s_1(1-\lambda_1)(\mathcal{K}_1+\mathcal{R}_1) \\
&\qquad + s_2\lambda_2(\hat{\mathcal{K}}_2+\hat{\mathcal{R}}_2) + s_2(1-\lambda_2)(\mathcal{K}_2+\mathcal{R}_2) \\
\mu'_{2,N-2}(x_q) &= \frac{\mathcal{K}_{2,N-2}(\lambda)}{x_q-1} \\
\mu''_{2,N-2}(x_q) &= \frac{2}{(x_q-1)^2}\left(s_1\lambda_1\mathcal{K}_1\mathcal{K}_{g_1} + s_2\lambda_2\mathcal{K}_2\mathcal{K}_{g_2}\right).
\end{aligned}
\tag{A.12}
$$

Note that in the equation for the portfolio loss, terms referring to the hedged obligor 1 are conditionally independent to those referring to the hedged obligor 2. This is why we can compute the contributions to the variance of the portfolio loss separately for obligor 1 and obligor 2. Each component is obtained as in the proof of Proposition 1. Thus, for the conditional variance of the portfolio loss ratio and its derivative we obtain the natural extensions of equation (A.9), namely

$$
\begin{aligned}
\sigma^2_{2,N-2}(x_q) &\approx \bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda_1)(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + 2s_1s_{g_1}\lambda_1\mathcal{C}_{g_1}(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) \\
&\qquad + s_2^2\hat{\mathcal{C}}_2(\lambda_2)(\hat{\mathcal{K}}_2+\hat{\mathcal{R}}_2) + 2s_2s_{g_12}\lambda_2\mathcal{C}_{g_2}(\hat{\mathcal{K}}_2+\hat{\mathcal{R}}_2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\sigma^2_{2,N-2}(x_q) &\approx \frac{d}{dx}\bar{\sigma}^2_{0,N}(x_q) \\
&\quad + \frac{s_1^2\hat{\mathcal{C}}_1(\lambda_1) + 2s_1s_{g_1}\lambda_1\mathcal{C}_{g_1}}{x_q-1}(\mathcal{K}_1(\mathcal{K}_{g_1}+\mathcal{R}_{g_1}) + \mathcal{K}_{g_1}(\mathcal{K}_1+\mathcal{R}_1)) \\
&\quad + \frac{s_2^2\hat{\mathcal{C}}_2(\lambda_2) + 2s_2s_{g_2}\lambda_2\mathcal{C}_{g_2}}{x_q-1}(\mathcal{K}_2(\mathcal{K}_{g_2}+\mathcal{R}_{g_2}) + \mathcal{K}_{g_2}(\mathcal{K}_2+\mathcal{R}_2)).
\end{aligned}
$$

Here we naturally extended the definition (A.8) of $\bar{\sigma}^2_{0,N}(x)$ to the case with two guarantees. Thus, in case of two partially hedged positions the equivalent to equation (3.2) is given by equation (3.9), the result of Proposition 2.

Now consider the case where one guarantor hedges two different obligors. Similarly to the previous case we consider a portfolio with two hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3,4,\dots,N)$. However, the obligors now have the same guarantor $g_1 = g_2 = 3$. Then the portfolio loss is given by equation (3.8). It is obvious that the conditional expectation of the portfolio loss and its derivatives are also given by equation (A.12) where terms referring to the composite instrument of course have to be adjusted to the current situation. The conditional variance of the portfolio loss can be written as

(A.13)

$$
\begin{aligned}
&\mathbb{V}[L_{2,N-2}|x] \\
= \ &\mathbb{V}[L_{0,N-3}|x] + \mathbb{V}[s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x] \\
&+\mathbb{V}\left[s_{g_1}U_{g_1} + s_1\lambda_1 U_1 U_{g_1} + s_2\lambda_2 U_2 U_{g_1}|x\right] \\
&+2\,\mathrm{Cov}\Big[s_{g_1}U_{g_1} + s_1\lambda_1 U_1 U_{g_1} + s_2\lambda_2 U_2 U_{g_1}, s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x\Big].
\end{aligned}
$$

We can compute the individual terms further using the same technique as in the case of a single partial hedge. Applying formula (A.5) then reduces the covariance term to

$$
\begin{aligned}
&2\,\mathrm{Cov}\left[s_{g_1}U_{g_1} + s_1\lambda_1 U_1 U_{g_1} + s_2\lambda_2 U_2 U_{g_1}, s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x\right] \\
= \ &2s_1^2\lambda_1(1-\lambda_1)\sigma_1^2(x)\mu_{g_1}(x) + 2s_2^2\lambda_2(1-\lambda_2)\sigma_2^2(x)\mu_{g_1}(x)
\end{aligned}
$$

and the second variance term equals

$$
\mathbb{V}[s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x] = s_1^2(1-\lambda_1)^2\sigma_1^2(x) + s_2^2(1-\lambda_2)^2\sigma_2^2(x).
$$

The third variance in equation (A.13) can be computed using formula (A.6). Neglecting again higher order terms in capital contributions one can show that

$$
\begin{aligned}
&\mathbb{V}[U_{g_1}\left(s_{g_1} + s_1\lambda_1 U_1 + s_2\lambda_2 U_2\right)|x_q] \\
= \ &\sigma_{g_1}^2(x_q)\left(\lambda_1^2 s_1^2\sigma_1^2(x_q) + \lambda_2^2 s_2^2\sigma_2^2(x_q) + 2s_{g_1}\lambda_1 s_1\mu_1(x_q) + 2s_{g_1}\lambda_2 s_2\mu_1(x_q) + s_{g_1}^2\right).
\end{aligned}
$$

Then the conditional variance of the portfolio loss can be expressed as

$$
\begin{aligned}
\sigma_{2,N-2}^2(x_q) \ = \ &\bar{\sigma}_{0,N}^2(x_q) \\
&+\mu_{g_1}(x_q)\mu_1(x_q)\left(\lambda_1^2 s_1^2\mathcal{C}_1\mathcal{C}_{g_1} + 2s_{g_1}\lambda_1 s_1\mathcal{C}_{g_1} + 2s_1^2\lambda_1(1-\lambda_1)\mathcal{C}_1\right) \\
&+\mu_{g_1}(x_q)\mu_2(x_q)\left(\lambda_2^2 s_2^2\mathcal{C}_{g_1}\mathcal{C}_2 + 2s_{g_1}\lambda_2 s_2\mathcal{C}_{g_1} + 2s_2^2\lambda_2(1-\lambda_2)\mathcal{C}_2\right).
\end{aligned}
$$

Inserting the definition (3.6) for $\hat{\mathcal{C}}_n(\lambda_n)$ and for the EL and UL capital $\hat{\mathcal{R}}_n$ and $\hat{\mathcal{K}}_n$, respectively, we obtain

(A.14)

$$
\begin{aligned}
\sigma_{2,N-2}^2(x_q) \ = \ &\bar{\sigma}_{0,N}^2(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda_1)(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) + 2s_{g_1}s_1\lambda_1\mathcal{C}_{g_1}(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) \\
&+s_2^2\hat{\mathcal{C}}_2(\lambda_2)(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) + 2s_{g_1}s_2\lambda_2\mathcal{C}_{g_1}(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2)
\end{aligned}
$$

which coincides with the expression for $\sigma_{2,N-2}^2(x_q)$ in the previous case. That is, if higher order terms in EL and UL capital contributions are neglected, the expressions for $\mu_{2,N-2}(x_q)$ and $\sigma_{2,N-2}^2(x_q)$ and their derivatives do not depend on whether both obligors have different guarantors or the same guarantor. Obviously the formula for the granularity adjustment also has to be the same as in the case with different guarantors. Thus, it is given by equation (3.9). $\square$

Proof of Theorem 1. The generalization to the case of several guarantees uses the same techniques as the proof of Proposition 2 since no further interactions will appear. We omit the proof here because the computations become rather tedious and do not provide any additional insight. $\square$

Appendix B. Comparison with the Treatment of Double Default Effects within the IRB Approach

There are certain similarities between our approach to the treatment of double default effects within the GA and the way double default effects are accounted for in the IRB approach of Basel II. For a better comparison of both methods we will briefly review the derivation and final formulas for the latter. Within the IRB approach banks may choose between the simple substitution approach outlined in the Introduction and a double default approach where risk-weighted assets for exposures subject to double default are calculated as follows.[25] One first computes the UL capital requirement $\mathcal{K}_n$ for the hedged obligor in the same way as the UL capital requirement for an unhedged corporate exposure[26] with ELGD_n replaced by ELGD_{g_n} and in the computation of the maturity adjustment PD_n is replaced by the minimum of PD_n and PD_{g_n}. Then the capital requirement $\mathcal{K}_n^{DD}$ for the hedged exposure is calculated by multiplying $\mathcal{K}_n$ by an adjustment factor depending on the PD of the guarantor, namely

$$\mathcal{K}_n^{DD} = K_n \cdot (0.15 + 160 \cdot \mathrm{PD}_{g_n}).$$

Finally, the risk-weighted asset amount for the hedged exposure is computed in the same way as for unhedged exposures. Note that the multiplier $(0.15 + 160 \cdot \mathrm{PD}_{g_n})$ is derived as a linear approximation to the UL capital requirement for hedged exposures using the exact conditional expected loss function and the capital requirement for the unhedged exposure according to the usual IRB formula. Therefore, the ASRF framework, which also presents the basis for the computation of the risk weighted assets in the IRB approach, is used in an extended version. Specifically, it is assumed that the asset returns r_n (resp. r_{g_n}) of an obligor and its guarantor are no longer conditionally independent given the systematic risk factor X but also depend on an additional risk factor Z_{n,g_n} which only affects the obligor and its

[25]Compare Basel Committee on Banking Supervision [2006], paragraph 284.

[26]The latter is defined in paragraphs 272 and 273 of Basel Committee on Banking Supervision [2006].

guarantor. More precisely,

$$r_n = \sqrt{\rho_n} X + \sqrt{1-\rho_n} \left(\sqrt{\psi_{n,g_n}} Z_{n,g_n} + \sqrt{1-\psi_{n,g_n}} \epsilon_n \right),$$

where ρ_n is the asset correlation of obligor n, ψ_{n,g_n} is a weight specifying the sensitivity of obligor n to the factor Z_{n,g_n} and ϵ_n is the idiosyncratic risk factor of obligor n. By implicitly assuming that all hedges are perfect full hedges, guarantors are themselves not obligors in the portfolio and all guarantors are external, the joint default probability of the obligor and its guarantor can be computed explicitly as

$$\begin{aligned} & \mathbb{P}\left(\{\text{default of obligor n}\} \cap \{\text{default of guarantor}\, g_n\}\right) \\ = \; & \Phi_2\left(\Phi^{-1}(\mathrm{PD}_n), \Phi^{-1}(\mathrm{PD}_{g_n}); \rho_{n,g_n}\right), \end{aligned}$$

where ρ_{n,g_n} is the correlation between obligor n and its guarantor g_n. Here $\Phi_2(\cdot,\cdot;\rho)$ denotes the cumulative distribution function of the bivariate normal distribution with correlation ρ.

This setting translated into an actuarial definition of loss corresponds to our approach when $Z_{n,g_n} = 0$, i.e. when the obligor and the guarantor are conditionally independent given the systematic risk factor X and where X is assumed to be Gamma distributed. For the composite instrument, however, we have a direct dependence between obligor and guarantor such that its default probability is given by the joint default probability[27]

$$\mathbb{P}\left(D_n = 1, D_{g_n} = 1\right) = \mathrm{PD}_n \, \mathrm{PD}_{g_n} \cdot \left(1 + w_n w_{g_n} \frac{1}{\xi}\right).$$

We want to point out that in contrast to the IRB treatment of double defaults, our approach also holds when we have partial hedging and when several obligors are hedged by the same guarantor. To include internal guarantors in our model only a weak additional assumption is necessary in order to ensure the assumptions underlying the ASRF model.[28] Since the expected loss $\Phi_2\left(\Phi^{-1}(\mathrm{PD}_n), \Phi^{-1}(\mathrm{PD}_{g_n}); \rho_{n,g_n}\right) \cdot \mathrm{ELGD}_n \, \mathrm{ELGD}_{g_n}$ should in general be rather small, in Basel Committee on Banking Supervision [2005] this term is set equal to zero. In our case we could argue similarly and thus set $\hat{\mathcal{R}}_n = 0$ which implies that the UL-capital for the composite instrument equals $\hat{\mathcal{K}}_n = \mathcal{K}_n \mathcal{K}_{g_n} + \mathcal{K}_n \mathcal{R}_{g_n} + \mathcal{K}_n \mathcal{K}_{g_n}$. This would also simplify the expressions in our final GA formula slightly.

Further, we want to point out that in Basel Committee on Banking Supervision [2005] no double recovery effects are recognized within the double default treatment under Pillar 1.[29] Our GA formula is more flexible in this sense as it is given for arbitrary ELGD_{g_n}, although, of course, we could impose the same requirement by setting the loss given default of the guarantor equal to 100%.

---

[27] Compare this to the computation of $\hat{\mathcal{R}}_1$ in Section 3 and note that the probability of the Bernoulli event equals the expectation of the Bernoulli random variable.

[28] This cannot be expected under the assumptions of additional correlation.

[29] See Basel Committee on Banking Supervision [2005], paragraph 206, for their reasoning.

References

Basel Committee on Banking Supervision. *The application of Basel II to trading activities and the treatment of double default effects*. Bank for International Settlements, 2005.

Basel Committee on Banking Supervision. *Basel II: international convergence of capital measurement and capital standards: a revised framework*. Publication No. 128. Bank for International Settlements, 2006.

Credit Suisse Financial Products. *CreditRisk$^+$: a credit risk management framework*. Credit Suisse Financial Products, London, 1997.

S. Emmer and D. Tasche. Calculating credit risk capital charges with the one-factor model. *Journal of Risk*, 7(2):85–103, 2005.

P. Gagliardini and C. Gouriéroux. Approximate derivative pricing for large homogeneous assets with systematic risk. Working Paper, available from http://www.e-fern.org/public /files/file0025285.pdf, 2009.

M. Gordy. A risk-factor model foundation for ratings-based bank capital rules. *Journal of Financial Intermediation*, 12(3):199–232, 2003.

M. Gordy and E. Lütkebohmert. *A granularity adjustment for Basel II*. Deutsche Bundesbank, 2007. Discussion Paper "Banking and Financial Studies" 01/2007.

C. Gouriéroux and A. Monfort. Granularity adjustment for efficient portfolios. Working Paper, available from http://www.e-fern.org/public/files/file2845806.pdf, 2008.

C. Gouriéroux, J.P. Laurent, and O. Scaillet. Sensitivity analysis of values at risk. *Journal of Empirical Finance*, 7(3-4):225–245, 2000.

P. Grundke. Regulatory treatment of the double default effect under the new Basel accord: how conservative is it? *Review of Managerial Science*, 2:37–59, 2008.

E. Heitfield and N. Barger. Treatment of double-default and double-recovery effects for hedged exposures under Pillar I of the proposed new Basel accord. White Paper by the Staff of the Board of Governors of the Federal Reserve System, 2003.

E. Lütkebohmert. *Concentration risk in credit portfolios*. EAA Lecture Notes, Springer, 2009.

R. Martin and T. Wilde. Unsystematic credit risk. *Risk Magazine*, 15:123–128, 2003.

D. O'Kane. *Modelling single-name and multi-name credit derivatives*. John Wiley & Sons, 2008.

M. Pykthin and A. Dev. Analytical approach to credit risk modelling. *Risk Magazine*, 15:26–32, 2002.

T. Wilde. Probing granularity. *Risk Magazine*, 14:103–106, 2001.