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The Effect of Secondary Markets on Equity-Linked Life Insurance with Surrender Guarantees

by

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Abstract

We study the effect of secondary markets on equity-linked life insurance contracts with surrender guarantees. The policyholders are assumed to be boundedly rational in giving up their contracts, and a proportion of policyholders will access the secondary markets instead of surrendering the contracts to the insurance company. We formulate the valuation problems from both the insurance company's and the policyholders' perspectives and characterize the contract values by deriving the respective pricing PDEs. Comparative statics are derived indicating the effect of the level of the policyholder's rationality and secondary market characteristics such as accessibility and competition on the contract values. The pricing PDEs are solved numerically via the Crank-Nicolson scheme to study the implication of the inclusion of a secondary market. We show that a secondary market generally increases the risk borne by the insurance company and the policyholders profit from the secondary market only when the secondary market is sufficiently competitive. Furthermore, we derive the necessary condition for the existence of a fair contract in this context and study the effect of the secondary market on fair contract design.

Keywords: equity-linked life insurance contracts, surrender guarantee, bounded rationality, fair contract analysis, secondary market

JEL: G13, G22, C65

1 Introduction

The world market for life insurance contracts is huge with a premium volume of 2,332 billion US dollars in 2009.[1] Statistics show that roughly 50% of the contracts in most developed countries are terminated (surrendered) early.[2] As the surrender guarantees offered by the primal insurers are usually far less than the contract values themselves, secondary markets have been developed, which allow the policyholders to sell their policies to third parties at relatively better prices. In the US and the UK, which are the world's largest and the world's third largest life insurance markets respectively, these secondary markets have a long history,[3] and have been growing in the last few decades.[4] In other countries like Japan or Germany, secondary markets for life insurance contracts have been established recently and a substantial increase in the trading volume on these markets could be observed.[5]

In this paper, we analyze the effect secondary markets have on the valuation of equity-linked life insurance contracts with surrender guarantees. On the valuation of such contracts without secondary markets there exists a large literature. The key problem of valuing these contracts is to model the surrender behavior of the policyholders. Most literature assumes surrender to be induced purely by endogenous reasons and considers the premature contract termination as an optimal stopping problem. The contract valuation is hence conducted within the American-style contingent claim framework. Prominent examples are Grosen and Jorgensen [21, 22], Bacinello [2, 3], Bacinello et. al. [4]. Recently, there is also the argument that policyholders surrender contracts for both endogenous and exogenous reasons and surrender behavior should be modeled with both of these surrender reasons in mind. Contract valuation in this spirit can be found for example in Albizzati and Geman [1], DeGiovanni [20] and Li and Szimayer [26].

While the literature on equity-linked life insurance contracts with surrender guarantees is large, the impact of secondary markets on these contracts is rarely examined. In the

[1] SwissRe [30].

[2] Gatzert [17], Bundesverband Vermögensverwalter im Zweitmarkt Lebensversicherung (BVZL) e.V. [9]

[3] For the UK it can be traced back to 1844, for the US to 1911. See BVZL e.V. [9].

[4] For the US, BVZL gives a volume of 2 million US dollar in 1990, 12 billion US dollar in 2008, and estimates a traded volume of 30 billion dollars for 2017. On the UK secondary market, 20,000 contracts with a price volume of 200 million GBP have been traded in 1996, which increased to 200,000 contracts with a price volume of 500 million GBP in 2003. See Gatzert [17].

[5] For example, the price volume of traded policies in Germany raised from €50 million in 2000 to €1.4 billion in 2007. The total volume of terminated contracts increased from €8.2 billion (2000) to €13.8 billion (2009). See BVZL e.V. [9].

literature we only find a few articles on secondary markets either in a specialized setting or for other contract types. Gatzert [17] compares the major secondary markets, evaluates the market potential and points out possible effects on these markets, but does not address the quantification of these effects. Gatzert, Hoermann, and Schmeiser [18] simulate surrender profits in a model of heterogeneous insurance holders, and analyze the effect of asymmetric surrender behavior on the secondary market. Giacolone [19] reviews the secondary market for life insurance contracts with viatical transactions[6] as do Doherty and Singer [15], who focus on welfare aspects of secondary markets in this case. Equity-linked life insurance contracts are also popular contract types on secondary markets, and we aim with our study to shed some light on this issue.

We address the valuation of equity-linked life insurance contracts with surrender guarantees in the presence of secondary markets in two steps. First, we formulate a financial market augmented for mortality risk. In this market we consider an equity-linked life insurance contract with a surrender option, following Li and Szimayer [26], see also Stanton [29], Dai et al. [13], and DeGiovanni [20]. As the second step we extend the setup and add as a new feature the secondary market. For doing so, the surrender strategy of the policyholder is adapted to also allow for the sale of the contract on the secondary market.

More precisely, we formulate the financial market model consisting of a riskless asset and a risky asset that is the reference fund for the contract. The financial market is extended to also include mortality risk, that is the death time of the insured individual, which we assume to admit a deterministic mortality intensity. Consequently, the mortality risk is unsystematic and can be diversified for a large pool of similar contracts. Then the surrender options and the secondary market are added. The policyholder can now walk away from the contract either by exercising the surrender option, i.e. by giving back the contract to the insurance company, or, by selling the contract to a third party on the secondary market. To model this we need to specify two objects, the time when the contract is given up and the mode of giving up the contract. The time when the contract is given up by the policyholder is defined by a random time with a stochastic intensity that is bounded from below and from above. The lower bound represents giving up the contract for exogenous reasons due to the

[6] Viaticors are policyholders with sharply reduced life expectancy due to severe illness. The first secondary markets for life insurance products have been established in the US for people with drastically reduced life expectancy, in particular persons inflicted by HIV (See Giacolone [19]).

policyholder being exposed to financial constraints, e.g., liquidity needs caused by financial distress. The intensity increases to its upper bound whenever the policyholder is financially better off from ending the contract compared to continuing the contract. Therefore, the upper bound limits the optimal timing for giving up the contract and the setup is referred to as bounded rationality, see, e.g., Stanton [29].

The two ways of giving up the contract, either by exercising the surrender option or by selling the contract on the secondary market, are assigned to the representative policyholder by randomization. The probability that the representative policyholder can access the secondary market is therefore a parameter capturing the representative policyholder's awareness of the secondary market. Once the secondary market is accessed and the contract is sold to a third party we assume that the contract buyer is a finance professional exercising the surrender option financially optimal. Then the contract value is given by the price of the corresponding American claim in the presence of diversifiable mortality risk. The price for selling the contract on the secondary market thus cannot exceed the price of the corresponding American option. As well, the price cannot drop below the surrender value, otherwise the policyholder would rather exercise the surrender option than selling the contract on the secondary market. The bargaining power of the both parties determines how the profit that arises from the policyholder's access to the secondary market is shared. In the end the policyholder can compute the expected early termination value of the contract and compares this value to the continuation value. Based on the specification of the behavior of the policyholder and the respective payoffs the pricing PDEs is derived for the contract value from the perspective of the representative policyholder. Further, the stochastic representation of the contract value is provided using Feynman-Kac. The profit sharing between the policyholder and the potential buyer of the contract on the secondary market leads to a value differential between the policyholder and the insurer. The value from the perspective of the insurance company has to account for all costs incurred by the contract. Using this fact and the behavior of the representative policyholder as input the pricing PDE for the contract value from the perspective of the insurance company is derived. Again, the stochastic representation of the contract value is provided using Feynman-Kac. The contract values from both perspectives, the policyholder's and insurance company's, are then analyzed for their sensitivities when changing relevant parameters. This is then followed by a numerical analysis where the pricing PDEs are solved by the Crank-Nicolson scheme. The analysis highlights the effect of the inclusion of the secondary market on the surrender behavior of the policyholder. By varying the parameters describing the secondary market and also the rationality parameters we provide a risk analysis for the insurer. This is followed by an investigation of the welfare for the policyholder and a fair contract analysis. Overall, we find that the policyholder may only profit partly from the secondary market. Although the introduction of the secondary market may increase the payout to the policyholder, it is not necessarily beneficial for him if the welfare increase is associated with the increase of the premium. Generalizing from the representative policyholder to two groups of policyholders, one uninformed and one informed of the existence of the secondary market, we demonstrate that the secondary market is profitable for no policyholders when the policyholders have no sufficient bargaining power on the secondary market. If, on the contrary, the secondary market is competitive enough, the informed policyholders profit from the secondary market while the uninformed policyholders bear the costs incurred by it. For the insurance company the secondary market brings a challenge in regard to the risk management of the contracts. If the secondary market is introduced at a sudden, the premiums charged before may not be adequate to support the hedging strategies of the insurance companies. Further, the projected cashflows from the contracts may alter due to the surrender options being exercised optimally by the contract buyers on the secondary market, ultimately affecting the insurer's liquidity management.

The paper is organized as follows. The model is presented in Section 2. In Section 3 the pricing of the insurance contract is carried out. Comparative statics are presented in Section 4. Numerical results, in particular, the risk analysis for the insurance company, the welfare analysis for the policyholder and the fair contract analysis are provided in Section 5. Section 6 concludes.

2 Model

In order to price equity-linked life insurance contracts, a model for both the financial and the insurance market is necessary. The model presented in this section is based on Li and Szimayer [26], but extends their market model by a secondary market on which policyholders may sell their life insurance contracts. The decision behavior of the policyholder of choosing between holding the contract, exercising the surrender option or to sell the contract on the secondary market is a further key component. The model for the decision process is motivated and formalized for a representative policyholder.

2.1 Financial and Insurance Market

The financial market is defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and consists of a risk-less money market account with price process B and risky asset with price process S. The risky asset is assumed not to pay any dividends and plays the role of the reference fund for the equity-linked life insurance contract studied in what follows. We fix a time horizon $T > 0$ and define the dynamics of the price processes by

$$\mathrm{d}B_t = r(t)\, B_t \,\mathrm{d}t\,, \quad \text{for } 0 \le t \le T \text{ and } B_0 = 1\,, \tag{1}$$

$$\mathrm{d}S_t = a(t, S_t)\, S_t \,\mathrm{d}t + \sigma(t, S_t)\, S_t \,\mathrm{d}W_t\,, \quad \text{for } 0 \le t \le T\,, \text{ and } S_0 = s > 0\,. \tag{2}$$

The deterministic function r denotes the short rate, the functions a and $\sigma > 0$ are drift and volatility of the risky asset, and W is a Wiener process under the real world measure $\mathbb{P}$. Both price processes are assumed to be Markovian, i.e. they do not depend on past events, only on the present state. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ of the financial market is generated by the Wiener process, i.e. the Wiener process reflects all the information available on the financial market.

By definition the financial market is arbitrage free and complete. In other words, there exists a unique equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ under which the dynamics of the risky asset satisfy

$$\mathrm{d}S_t = r(t)\, S_t \,\mathrm{d}t + \sigma(t, S_t)\, S_t \,\mathrm{d}\hat{W}_t, \tag{3}$$

where $\hat{W}$ is a Wiener process under $\mathbb{Q}$ with $\mathrm{d}\hat{W}_t = \mathrm{d}W_t \;+\; \frac{a-r}{\sigma}\mathrm{d}t$, for $0 \le t \le T$.

The insurance market is modeled by two random times τ and λ potentially ending the financial contract. The time τ refers to the death time of an individual aged y at time $t = 0$ when the contract is signed. The time λ refers to the time when the policyholder decides to give up the contract either by exercising the surrender option or by selling the contract on the secondary market.[7] The jump process associated with τ is H with $H_t = 1_{\{\tau \leq t\}}$, for $0 \leq t \leq T$, and generates the filtration $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$. The hazard rate of the random time τ (or the mortality intensity) is denoted by μ and is assumed to be a deterministic function. Under this assumption, the mortality risk can be diversified over a large pool of policyholders. The jump process associated with λ is J with $J_t = 1_{\{\lambda \leq t\}}$, for $0 \leq t \leq T$. It generates the filtration $\mathbb{J} = (\mathcal{J}_t)_{0 \leq t \leq T}$. The hazard rate of the random time λ is denoted by γ, and is also called the surrender intensity.[8] By introducing the random time λ, and correspondingly, the surrender intensity γ, we can actually represent a large family of insurance contracts. For the degenerate case where $\gamma = \infty$, the insurance contracts are European style. When γ is allowed to take finite values, the policyholder can walk away from the contract. In contrast to the mortality intensity μ, the surrender intensity γ is not deterministic but depends on the monetary rationality of the policyholder in making surrender decisions by comparing the contract value and the surrender value. Since the contract value and eventually also the surrender value are linked to the risky asset S, γ is assumed to be $\mathbb{F}$-measurable. The exact form of γ will be specified in Section 2.2.

The nature of equity-linked life insurance policies is that they are linking the financial market and the insurance market. To model the information on the linked market, the filtrations $\mathbb{F}$, $\mathbb{H}$ and $\mathbb{J}$ need to be combined. Bielecki and Rutkowski [6] give an account on the technicalities to combine these filtrations, see Section 7, pp.197. We give a brief summary of their key results relevant to our situation.

Starting under the original probability space $(\Omega, \mathcal{G}, \mathbb{P})$ we first specify the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ carrying all the relevant information by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \vee \mathcal{J}_t$, for $0 \leq t \leq T$. Recalling that $\mathbb{F}$ is the filtration generated by the Wiener process W we assume that W remains a Wiener process for the enlarged filtration $\mathbb{G}$. The processes H and J both admit intensities μ and λ that are $\mathbb{F}$-adapted. Now, we additionally assume that μ and λ are the respective $\mathbb{G}$-intensities, i.e. the processes $\hat{M}^H = (\hat{M}_t^H)_{0 \leq t \leq T} = (H_t - \int_0^{t \wedge \tau} \mu(u)\, \mathrm{d}u)_{0 \leq t \leq T}$ and $\hat{M}^J = (\hat{M}_t^J)_{0 \leq t \leq T} = (J_t - \int_0^{t \wedge \lambda} \gamma_u\, \mathrm{d}u)_{0 \leq t \leq T}$ are both $\mathbb{G}$-martingales, and that joint

[7] In case the contract is sold on the secondary market the contract is still alive. However, the policyholder is no longer holding the contract.

[8] Surrender is here understood as both ways the policy holder can walk away from the contract, i.e. by exercising the surrender option and by selling it on the secondary market.

jumps of H and J occur with zero probability, i.e. $\mathbb{P}(\tau = \lambda) = 0$.

The Radon-Nikodym density process for the measure change $\mathbb{P} \to \mathbb{Q}$ is defined as

$$\eta_t = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = \mathbb{E}[Y|\mathcal{G}_t] \quad \mathbb{P} - a.s., \tag{4}$$

where Y is a $\mathcal{G}_T$-measurable random variable with $\mathbb{P}(Y > 0) = 1$ and $\mathbb{E}^{\mathbb{P}}[Y] = 1$. According to Bielecki and Rutkowski [6], Proposition 7.1.3, p. 201, it has the following integral representation

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-}(\varphi_u \mathrm{d}\hat{W}_u + \xi_u^H \mathrm{d}\hat{M}_u^H + \xi_u^J \mathrm{d}\hat{M}_u^J), \tag{5}$$

where φ, ξ^H and ξ^J are $\mathbb{G}$-predictable processes.

Set $\varphi = -\frac{a-r}{\sigma}$ and $\xi^H = \xi^J = 1$, then by Proposition 7.2.1. in Bielecki and Rutkowski [6], $\hat{W}$ in the risk-neutral dynamics of the risky asset in (3) is also $\mathbb{Q}$-Brownian motion on the enlarged filtration $\mathbb{G}$. Further, μ and γ are the intensities of τ and λ under the equivalent martingale measure $\mathbb{Q}$ and filtration $\mathbb{G}$. Thus, valuation under the risk-neutral measure $\mathbb{Q}$ and on the extended filtration $\mathbb{G}$ is possible and carried out in Section 3. However, prior to carrying out the valuation we have to model the yet unspecified intensity γ governing the likelihood of the policyholder walking away from the contract.

2.2 Decision Behavior of the Representative Policyholder

In our setup the policyholder of an equity-linked life insurance contract with surrender guarantees can choose to continue the contract or to end the contract either be exercising the surrender option or by selling the contract on the secondary market. The related valuation problem could be addressed as a standard contingent-claim pricing problem assuming a rational agent in a perfect financial market, however, we follow a different approach.

We develop the decision process of a representative policyholder facing financial constraints, e.g., liquidity needs caused by financial distress. Our representative agent is not a finance expert and may not always realize that he is better off from giving up the contract and instead continues holding the contract. In case he opts to walk away from the contract he can exercise the surrender option and give back the contract to the insurance company for the prespecified surrender value or he can sell the contract to a third party on the secondary market. Here, we also assume that the representative policyholder is not fully rational in his decision. The two ways of ending the contract are assigned by randomization. The probability that the representative policyholder accesses the secondary market is therefore a parameter capturing the representative policyholder's awareness of the secondary market.

Once the secondary market is accessed and the contract is sold to a third party we assume that the contract buyer is a finance professional exercising the surrender option financially optimal. Then the contract value is given by the price of the corresponding American claim in the presence of diversifiable mortality risk. The price for selling the contract on the secondary market thus cannot exceed the price of the corresponding American option. As well, the price cannot drop below the surrender value, otherwise the policyholder would rather exercise the surrender option than selling the contract on the secondary market. The bargaining power of the both parties determines how the profit that arises from the policyholder's access to the secondary market is shared. In the end the policyholder can compute the expected early termination value of the contract and compares this value to the continuation value.

The decision process is now formalized. We are given an equity-linked life insurance contract written on S with maturity T. Denote by V^C the contract value from the perspective of the representative policyholder and by L the surrender benefit. The third party on the secondary market potentially buying the contract is assumed to be an agent in a perfect financial market with no frictions and access to all relevant information. Consequently, the buyer on the secondary market exercises the surrender option financially optimal and the contract value is then the price of the corresponding American-style option in the presence of mortality risk and is denoted by V^{Am}. The probability that the representative agent can access the financial market is $p \in [0,1]$, both under $\mathbb{P}$ and $\mathbb{Q}$.[9] This is captured by the random variable X that is independent of $\mathcal{G}_T$ and is taking the value 1 with probability p (access to secondary market) and the value 0 with probability $1-p$ (no access to secondary market).[10] The bargaining power of the representative policyholder on the secondary mar-

[9]This invariance of the probability is based on the underlying assumption that the associated risk is unsystematic and hence diversifiable.

[10]Formally, we have to enlarge the filtration $\mathbb{G}$ to also include the information generated by the process $(X\,1_{\{\lambda\leq t\}})_{0\leq t\leq T}$ revealing X at λ.

ket is given by $\kappa \in [0, 1]$ and quantifies the fraction of the additional value created by the secondary market that remains with the policyholder.

Suppose that at time t the representative policyholder gives up the contract, i.e. $\lambda = t$, and suppose further he is able to access the secondary market, i.e. $X = 1$. Since the representative policyholder could give the contract back to the insurance company, he requires at the least the surrender benefit L to sell the contract on the secondary market. The contract value for a potential buyer on the secondary market is the price of the corresponding American-style claim in the presence of mortality risk with price process V^{Am}, with $V^{Am} \geq L$. This is the highest possible price offered on the secondary market. The added value created by the secondary market is thus $V^{Am} - L$. Now, using the bargaining power parameter $\kappa \in [0, 1]$ the price offered for the contract on the secondary market is $(1 - \kappa) L + \kappa V^{Am} = L + \kappa (V^{Am} - L)$. Taking into account the randomization given by X, the expected contract value when giving up the contract at t is then

$$(1 - p) L(t) + p (L(t) + \kappa [V_t^{Am} - L(t)]) = L(t) + p \kappa (V_t^{Am} - L(t)) . \tag{6}$$

Now, we specify the surrender intensity γ of the representative policyholder for ending the contract following the approach Li and Szimayer [26] that is dating back to Stanton [29]. The representative policyholder tends to terminate the contract for exogenous reasons, e.g., caused by financial constraints, at rate $\underline{\rho} \geq 0$. In case the expected proceeds from ending the contract in (6) exceed the continuation value of the contract V^C then the contract termination occurs at a higher intensity $\overline{\rho}$, with $\overline{\rho} \geq \underline{\rho}$, i.e.

$$\gamma_t = \begin{cases} \underline{\rho}\,, & \text{for } L(t) + p\kappa \left[V_t^{Am} - L(t)\right] < V_t^C\,, \\ \overline{\rho}\,, & \text{for } L(t) + p\kappa \left[V_t^{Am} - L(t)\right] \geq V_t^C\,. \end{cases} \tag{7}$$

The intensity difference $\overline{\rho} - \underline{\rho}$ can be interpreted as a the level of rationality of the representative policyholder.

3 Contract Valuation

Given the market model and the policyholder's decision behavior, the pricing of the insurance contract is now possible. This is carried out first from the perspective of the representative policyholder to obtain V^C. Then the value from the perspective of the insurance company V^I is derived using as input the behavior of the representative policyholder captured by γ. The difference in the contract values is the premium (or cost) for introducing the secondary market.

We consider the case of single premium contracts, the results can be extended to the case of continuous premiums without much difficulty. The payoff structure of the insurance contract is divided into three parts: the benefit at maturity, denoted $\Phi(S_T)$, the benefit at death $\Psi(\tau, S_\tau)$ and the benefit if the contract is terminated early, given by $L(\lambda)$.[11] The payoff functions considered here are

$$\Phi(T, S_T) = P \max\left(\alpha\,(1+g)^T, \left(\frac{S_T}{S_0}\right)^k\right), \tag{8}$$

$$\Psi(\tau, S_\tau) = P \max\left(\alpha\,(1+g_d)^\tau, \left(\frac{S_\tau}{S_0}\right)^{k_d}\right), \tag{9}$$

$$L(\lambda) = (1-\beta_\lambda)\,P\,(1+h)^\lambda\,. \tag{10}$$

Here, α is the fraction of the premium guaranteed to yield the minimum rate g, usually smaller than the risk free rate,[12] P the single premium and k the participation coefficient specifying the degree to which the policyholder participates in gains of the risky asset underlying the insurance contract. Mostly, $g = g_d$ and $k = k_d$, i.e. death is not penalized by the insurer.[13] Surrendering the contract early is penalized however, captured by the time dependent function β. In most cases, β is a function decreasing in time, aiming to punish early surrender over continuation of the contract. Bernard and Lemieux [5] state that for example in Canada, by law the cash surrender value cannot be lower than the guaranteed minimum rate, i.e. $g \leq h$.[14]

[11] The payoff structure is taken from Bernard and Lemieux [5] and also employed by Li and Szimayer [26].

[12] See [5], p. 446. Note that by $g < r$ early surrender may be attractive if the stock market does not perform well, so that investing in the money market account will provide a higher return that is risk-less.

[13] See [5], p. 446.

[14] See Bernard and Lemieux [5], p. 447.

Remark 1. *The value of an American option in the presence of mortality risk then describes the maximal possible value of the contract and assumes that the contract holder is facing no financial constraints in an efficient market. On* $\{t < \tau \wedge \lambda \wedge T\}$ *the value of the American option is* $V_t^{Am} = v^{Am}(t, S_t)$ *where* $v^{Am} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ *satisfies the free boundary value problem given by the PDE*

$$
\begin{aligned}
0 =& \frac{\partial v^{Am}}{\partial t}(t,s) + r(t)s\frac{\partial v^{Am}}{\partial s}(t,s) + \frac{1}{2}\sigma^2(t,s)s^2\frac{\partial^2 v^{Am}}{\partial s^2}(t,s) + \mu(t)\Psi(t,s) \\
& - (r(t) + \mu(t))\, v^{Am}(t,s)\,,
\end{aligned}
$$

with constraint $v^{Am}(t,s) \geq L(t)$ *on* $[0,T) \times \mathbb{R}_+$ *and terminal condition* $v^{Am}(T,s) = \Phi(s)$*, for* $s \in \mathbb{R}_+$*.*

3.1 Representative Policyholder's Contract Value

In the following, we derive the pricing PDE for the contract value from the perspective of the policyholder V^C. Our derivation extends Li and Szimayer [26] to allow for a secondary market. The basis of the derivation remains the balance law as stated by Dai et. al. [13]. The expected return of the contingent claim specified by the insurance contract has to equal the risk free rate under the risk-neutral measure as this is a no-arbitrage condition. Contracts that have not terminated for any reason, i.e. on $\{t < \lambda \wedge \tau \wedge T\}$ with $x \wedge y := \min(x, y)$, satisfy

$$
r(t)V_t^C\,\mathrm{d}t = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}V_t^C|\mathcal{G}_t\right]\,. \tag{11}
$$

On the above set, the following possible cases may happen:

1. The conditional probability that death occurs over $(t, t + \mathrm{d}t)$ while surrender does not is $\mu(t)\,\mathrm{d}t\,(1 - \gamma_t p\,\mathrm{d}t - \gamma_t(1 - p)\,\mathrm{d}t) = \mu(t)\,\mathrm{d}t$. Note that the decision to access a secondary market is irrelevant here.

2. The conditional probability that death does not occur over $(t, t + \mathrm{d}t)$ while surrender does and the secondary market is used is $\gamma_t p\,\mathrm{d}t(1 - \mu(t)\,\mathrm{d}t) = \gamma_t p\,\mathrm{d}t$.

3. The conditional probability that death does not occur over $(t, t + \mathrm{d}t)$ while surrender

does and the secondary market is not used is $\gamma_t(1-p)\,\mathrm{d}t(1-\mu\,\mathrm{d}t) = \gamma_t(1-p)\,\mathrm{d}t$.

4. The conditional probability that both surrender and death happen over $(t, t+\mathrm{d}t)$ is 0 as in Li and Szimayer [26]. Again, the decision to enter a secondary market is irrelevant, as the probability to get to a point where this decision matters is zero anyway.

Analogous to Li and Szimayer [26] the contract value at time $t \leq \lambda \wedge \tau \wedge T$ is assumed to take the form

$$\begin{aligned} V_t^C =& \mathbb{1}_{\{t<\tau\wedge\lambda\}} v(t, S_t) + \mathbb{1}_{\{t=\tau\leq\lambda\}} \Psi(\tau, S_\tau) + \mathbb{1}_{\{t=\lambda<\tau,\, X=0\}} L(\lambda) \\ &+ \mathbb{1}_{\{t=\lambda<\tau,\, X=1\}} \left[L(\lambda) + \kappa\big(V_\lambda^{Am} - L(\lambda)\big)\right], \end{aligned} \tag{12}$$

where v is a suitably differential function $v : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$. By (7) we see that γ depends on the current continuation value V^C and the expected termination payoff driven by V^{Am}. V^{Am} can be expressed as functions of time t and price of the risky asset s, see Remark 1, and therefore γ can be written as a function of (t, s, v), i.e. $\gamma : [0,T]\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}_+$, and set $\gamma_t = \gamma(t, S_t, v(t, S_t))$. As given by Li and Szimayer [26] the occurrence of the event of death changes the contracts payoff $\Psi(t, s)$ leaving a change in payment that is $\Psi(t, s) - v(t, s)$. Basically, the claim to receive v is lost and replaced by the payment of Ψ. Similarly, the payment liability is effected by the surrender action. The secondary market changes the payoff to be dependent on the state of the decision variable X. The change in payment remains to be $L(t) - v(t, S_t)$ if $X = 0$, but if the secondary market is accessed then $X = 1$ and the payoff change is $L(t)+\kappa[v^{Am}(t, S_t)-L(t)]-v(t, S_t)$. The representative policyholder does not know in advance whether the secondary market will be accessed or not.

Using the above changes in payment liabilities, the balance law (11) can be written as[15]

$$\begin{aligned} r(t)v(t, S_t)\mathrm{d}t =& \mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}v(t, S_t)\,|\mathcal{F}_t\right] + \left[L(t) + \kappa\left(V_t^{Am} - L(t)\right) - v(t, s)\right] p\gamma_t\mathrm{d}t \\ &+ \left[L(t) - v(t, S_t)\right](1-p)\gamma_t\mathrm{d}t + \left[\Psi(t, S_t) - v(t, S_t)\right]\mu(t)\mathrm{d}t. \end{aligned} \tag{13}$$

Equation (13) carries economic interpretation: The change in the contract's value can be split up in the changes in value due to the different surrender and death events and the

[15] See Li and Szimayer [26].

change in value originating in the continuation value. All these components have to equal the risk free rate in total, since the pricing takes place under the risk neutral measure. The expected change in the continuation value is based on filtration $\mathbb{F}$ as, in economic terms, the death risk process does not influence the stock prices. Expanding the increment of the continuation value by Itó's Lemma gives

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}v(t,S_t)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathcal{L}v(t,s)\,\mathrm{d}t + \sigma(t,S_t)\frac{\partial v}{\partial s}(t,S_t)\mathrm{d}W_t\middle|\,\mathcal{F}_t\right] = \mathcal{L}v(t,S_t)\mathrm{d}t, \tag{14}$$

where the differential operator $\mathcal{L}$ is given by

$$\mathcal{L}f(t,s) = \frac{\partial f}{\partial t}(t,s) + r(t)s\frac{\partial f}{\partial s}(t,s) + \frac{1}{2}\sigma^2(t,s)s^2\frac{\partial^2 f}{\partial s^2}(t,s).$$

Thus, the balance law produces

$$\begin{aligned} 0 =& \mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma(t,s)(1-p)L(t) \\ &+ \gamma(t,s,v(t,s))p\left[L(t) + \kappa\left(v^{Am}(t,s) - L(t)\right)\right] - \left(r(t) + \mu(t) + \gamma(t,s,v(t,s))\right)v(t,s). \end{aligned}$$

A further no-arbitrage condition is $v(T,s) = \Phi(s)$, for all $s > 0$, i.e. the value of the contract that has survived up to maturity will be the same as the value of the payout specified for this case. This completes the derivation of the following proposition:

Proposition 1 (Pricing PDE, Representative Policyholder)**.** *For the contract value V^C given by* (12) *the price function v is the solution of the partial differential equation*

$$\begin{aligned} 0 =& \mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma(t,s,v(t,s))\left[L(t) + p\kappa\left(v^{Am}(t,s) - L(t)\right)\right] \\ &- [r(t) + \mu(t) + \gamma(t,s,v(t,s))]v(t,s)\,, \end{aligned} \tag{15}$$

for $(t,s) \in [0,T) \times \mathbb{R}_+$ with terminal condition $v(T,s) = \Phi(s)$, for $s \in \mathbb{R}_+$. The solution of (15) *together with Remark 1 and equation* (7) *then characterize the intensity γ.*

In the following corollary, a stochastic representation formula of the Feynman-Kac type is obtained:

Corollary 1 (Stochastic Representation Formula, Representative Policyholder)**.** *The value* *of the contract V^C can be represented on $\{t < \tau \wedge \lambda \wedge T\}$ by*

$$\begin{aligned} V_t^C =&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(y)+\mu(y)+\gamma_y \mathrm{d}y} \Phi(S_T) \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma_y \mathrm{d}y} \mu(u)\Psi(u, S_u)\mathrm{d}u \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma_y \mathrm{d}y} \gamma_u \left[L(u) + p\kappa \left(V_u^{Am} - L(u) \right) \right] \mathrm{d}u \middle| \mathcal{F}_t \right]. \end{aligned} \tag{16}$$

3.2 Insurance Company's Contract Value

The derivation of the contract value for the insurance company is broadly similar to that for the representative policyholder. However, there are some distinct differences. The contact value from the perspective of the insurance company V^I depends on the behavior of the representative policyholder as described by γ. Thus γ and indirectly also v serve here as an input parameter. Further, in case the contract is sold on the secondary market the insurance company has to account for the full costs.

In the spirit of (12) we express the contract value V^I by

$$V_t^I = \mathbb{1}_{\{t<\tau\wedge\lambda\}} u(t, S_t) + \mathbb{1}_{\{t=\tau\leq\lambda\}} \Psi(\tau, S_\tau) + \mathbb{1}_{\{t=\lambda<\tau,\, X=0\}} L(\lambda) + \mathbb{1}_{\{t=\lambda<\tau,\, X=1\}} V_\lambda^{Am}, \tag{17}$$

where $u : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ is the related value function.

Proposition 2 (Pricing PDE, Insurance Company)**.** *Suppose that the contract value for the representative policyholder is given by v and the intensity is given by γ, respectively, both according to Proposition 1. For the contract value V^I given by* (17) *the price function u is the solution of the partial differential equation*

$$\begin{aligned} 0 =&\mathcal{L}u(t, s) + \mu(t)\Psi(t, s) + \gamma(t, s, v(t, s))(1 - p)L(t) + \gamma(t, s)pv^{Am}(t, s) \\ &- [r(t) + \mu(t) + \gamma(t, s, v(t, s))]u(t, s) \end{aligned} \tag{18}$$

for $(t, s) \in [0, T) \times \mathbb{R}_+$ with terminal condition $u(T, s) = \Phi(s)$ for $s \in \mathbb{R}_+$.

Again, we have an immediate corollary giving the stochastic representation of the price function.

Corollary 2 (Stochastic Representation Formula, Insurance Company)**.** *The value of the contract V^I can be represented on $\{t < \tau \wedge \lambda \wedge T\}$ by*

$$\begin{aligned} V_t^I =& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(y)+\mu(y)+\gamma_y \mathrm{d}y} \Phi(S_T) \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma_y \mathrm{d}y} \mu(u)\Psi(u, S_u)\mathrm{d}u \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma_y \mathrm{d}y} \gamma_u \left[(1-p)L(u) + pV_u^{Am}\right]\mathrm{d}u \middle| \mathcal{F}_t \right]. \end{aligned} \tag{19}$$

The relationship between the insurance company's value and the representative policyholder's value is that, as expected, the insurance company's value is greater than that of the representative policyholder. This is made precise below and follows directly from Corollary 1 and Corollary 2

Corollary 3. *The value difference of the contract from the perspective of the insurance company and from the perspective of the representative policyholder, respectively, can be represented on $\{t < \tau \wedge \lambda \wedge T\}$ by*

$$V_t^I - V_t^C = p\,(1-\kappa)\,\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma_y \mathrm{d}y} \gamma_u \left[V_u^{Am} - L(u)\right]\mathrm{d}u \middle| \mathcal{F}_t \right], \tag{20}$$

and is non-negative.

Remark 2. *The pricing PDE and the stochastic representation formula can be extended to incorporate continuous premiums. Further, the constant surrender parameters $\underline{\rho}$ and $\overline{\rho}$ and can be allowed to be functions of the time and the price of the risky asset. Then the results are still valid under the extended setup.*

In the traditional sense, an equity-linked life insurance is fair if and only if the expected payment to the policyholder equals the premium paid by the policyholder at the initial date. Such a fair contract does not necessarily exist if the insurer charges V^I but the policyholder is paid only V^C in expectation.

Proposition 3. *If a fair equity-linked life insurance contract exists on the insurance market with a secondary market, then one of the following conditions must be satisfied*

1) $p = 0$, i.e., there is no possibility to access the secondary market;

2) $\kappa = 1$, i.e., the secondary market is completely competitive;

3) $(\underline{\rho}, \bar{\rho}) = (0, \infty)$, i.e., the policyholder faces no financial constraints and acts monetarily rational.

Proof. From Corollary 3 we see that $V^I = V^C$ if one of the above conditions is met. The contract parameters can then be specified so that the expected payment to the policyholder equals the expected premium paid by the policyholder at the initial date. $\square$

4 Comparative Statics

In the following, the effects of changes in the model parameters are analyzed. The analysis is restricted to the single premium case, but can be extended to the continuous premium case, as the premium will cancel immediately when studying value differentials.

For the representative policyholder we can derive a comprehensive set of comparative statics. The more rational the representative policyholder is, or, the less financial constraints he is facing, the higher is the contract value. Thus the contract value is increasing for decreasing likelihood of exogenous surrender ($\underline{\rho}$) and for increasing rationality ($\overline{\rho}$). The impact of the secondary market parameters is that the increasing probability of access to the secondary market (p) and increasing bargaining power of the representative policyholder (κ) result in an increasing contract value.

Proposition 4 (Comparative Statics for Representative Policyholder)**.** *For $0 \leq \underline{\rho} \leq \overline{\rho}$ and $0 \leq p, \kappa \leq 1$ denote by v the representative policyholder's value function, and for the set of parameters $0 \leq \underline{\rho}' \leq \overline{\rho}'$ and $0 \leq p', \kappa' \leq 1$, denote the respective value function by v', both as given in Proposition 1. Suppose that $\underline{\rho}' \leq \underline{\rho}$, $\overline{\rho}' \geq \overline{\rho}$, and $p'\,\kappa' \geq p\,\kappa$, then $v'(t,s) \geq v(t,s)$, for all $(t,s) \in [0,T] \times \mathbb{R}_+$.*

Proof. The pairs (v, γ) and (v', γ') are solutions to the PDE (15) with respective parameters, see Proposition 1. Consider the difference $z = v' - v$. First, the boundary condition of z is computed, i.e. $z(T,s) = v'(T,s) - v(T,s) = \Phi(s) - \Phi(s) = 0$, for all $s \in \mathbb{R}_+$. By taking the difference of (15) for v' and v we obtain the PDE describing z on $[0,T)\times\mathbb{R}_+$, i.e.

$$\begin{aligned}0 =&\mathcal{L}z(t,s) + [\gamma'(t,s,v'(t,s)) - \gamma(t,s,v(t,s))]\left[\left(L(t) + p'\,\kappa'(v^{Am}(t,s) - L(t))\right) - v'(t,s)\right]\\ &+ \gamma(t,s,v(t,s))(p'\,\kappa' - p\,\kappa)[v^{Am}(t,s) - L(t)] - [r(t) + \mu(t) + \gamma(t,s,v(t,s))]z(t,s)\,.\end{aligned}$$

We see that the sign of z depends on the sign of A given by

$$\begin{aligned}A(t,s) =&(\gamma'(t,s,v'(t,s)) - \gamma(t,s,v(t,s)))\left(\left[L(t) + p'\,\kappa'(v^{Am}(t,s) - L(t))\right] - v'(t,s)\right)\\ &+ \gamma(t,s,v(t,s))(p'\,\kappa' - p\,\kappa)[v^{Am}(t,s) - L(t)]\,,\end{aligned}$$

what can be inferred, e.g., from the Feynman-Kac stochastic representation formula, i.e.

$$z(t,s) = \mathbb{E}^{\mathbb{Q}}_{t,s}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma(y,S_y,v(y,S_y))\mathrm{d}y} A(u,S_u)\,\mathrm{d}u\right]\,.$$

Now, $A \geq 0$ implies exactly what we want to show, i.e. $z \geq 0$, or, equivalently $v' \geq v$. To establish this we analyze the first component of A. For the case $v' > L + p'\,\kappa'(v^{Am} - L)$ we have $\gamma' = \underline{\rho}'$ by (7). Thus $(\gamma' - \gamma) \leq \underline{\rho}' - \underline{\rho} \leq 0$ by assumption. Both factors constituting the first component of A are non-positive, hence their product is non-negative. For the case $v' \leq L + p'\,\kappa'(v^{Am} - L)$ we have $\gamma' = \overline{\rho}'$ by (7). And then $(\gamma' - \gamma) \geq \overline{\rho}' - \overline{\rho} \geq 0$ by assumption. Now, both factors are non-negative and so is there product. It remains to investigate the second component of A. Note that $\gamma \geq 0$, $p'\,\kappa' \geq p\,\kappa$ by assumption, and $v^{Am} \geq L$, to see that also the second component of A is non-negative. This finishes the proof. □

In economic terms, the above proposition states that a contract is more valuable, if the option to surrender the contract is used "more rationally". This means that such an insurance contract has higher value compared to a second one if the number of surrenders during periods in which it is rational to lapse the contract, is higher (i.e. surrender takes place after a shorter period of waiting). We have a further natural interpretation of the above results relating to the secondary market. Given all else remains constant, an increased willingness to access a secondary market for the contract raises the value of this life insurance. Further, a raise in the policyholder's share of the profits made through optimal exercise increases the contracts value also.

For the insurance company's contract value the dependence on the parameters is complex since the policyholder's contract value also has an impact via the policyholder's behavior (γ). We can provide the following result.

Proposition 5 (Comparative Statics for Insurance Company)**.** *For* $0 \le \underline{\rho} \le \overline{\rho}$ *and* $0 \le p, \kappa \le 1$ *denote by* u *the insurance company's value function, and for the set of parameters* $0 \le \underline{\rho}' \le \overline{\rho}'$ *and* $0 \le p', \kappa' \le 1$*, denote the respective value function by* u'*, both as given in Proposition 2. Suppose that* $\underline{\rho}' = \underline{\rho}$*,* $\overline{\rho}' = \overline{\rho}$*,* $p'\kappa' = p\,\kappa$*, and* $\kappa' \le \kappa$*, then* $u'(t,s) \ge u(t,s)$*, for all* $(t,s) \in [0,T] \times \mathbb{R}_+$*.*

Proof. First note that the corresponding contract values from the perspective of the representative policyholder we have $v' = v$ by Proposition 1 and assumption $\underline{\rho}' = \underline{\rho}$, $\overline{\rho}' = \overline{\rho}$, $p'\kappa' = p\,\kappa$. Consequently, we have that $\gamma' = \gamma$, where we have also used (7). Now, we can apply Corollary 2 for u' and u. Taking the difference we see that the first two summands cancel out and we obtain

$$u'(t,s) - u(t,s) = (p' - p)\,\mathbb{E}^{\mathbb{Q}}_{t,s}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma_y \mathrm{d}y}\left(v^{Am}(u,S_u) - L(u)\right)\gamma_u\,\mathrm{d}u\right].$$

Observe that $\kappa' \le \kappa$ implies $p' \ge p$ by the assumed constraint $p'\kappa' = p\,\kappa$. Finally, it follows that $u' \ge u$. □

5 Numerical Analysis

5.1 Numerical Methodology

Four major steps are undertaken to solve the pricing PDEs (15) and (18) via finite differences. In the beginning of the analysis, boundary conditions for the PDE are derived to solve the problem with and without secondary markets. Then the optimal stopping values needed in the PDEs are computed numerically via the Crank-Nicolson scheme that is set up in the second step. Especially, when solving the pricing PDE for the policyholder the surrender intensity has to be determined simultaneously. The methodology used to deal with this issue is presented in step three. This allows the solution of the generalized PDE including secondary markets in the final step.

First, we address the boundary conditions for the PDE given in (15). Because a grid with the two dimensions time and price is used, three boundary conditions have to be found. In a similar context, DeGiovanni [20] derives boundary conditions that are adapted to fit the problem under consideration. The boundary condition for maturity is already given by Φ, i.e. an alive contract will pay out the value at maturity specified by the insurance contract. The two other conditions are more demanding, however.

For $s = 0$, a suitable boundary condition of the PDE is obtained numerically. At $s = 0$, the PDE given in (15) satisfies

$$\begin{aligned}0 =& \mathcal{L}v(t,0) + \mu(t)\Psi(t,0) - [r(t) + \mu(t) + \gamma(t,0,v(t,0))]v(t,0) \\ &+ \gamma(t,0,v(t,0))(1-p)L(t) + \gamma(t,0,v(t,0))p\left[L(t) + \kappa(v^{Am}(t,0) - L(t))\right]\end{aligned}$$

This is equivalent to

$$0 = \frac{\partial v}{\partial t}(t,0) + \mu(t)\Psi(t,0) + \overline{\rho}(1-p)L(t) + \gamma(t,0,v(t,0))p\left[L(t) + \kappa(v^{Am}(t,0) - L(t))\right].$$

In addition, the PDE for the pure American-style contract given in Remark 1 satisfies for $s = 0$

$$\frac{\partial v^{Am}}{\partial t}(t,0) + \mu(t)\Psi(t,0) = 0\,,$$

with constraint $v^{Am}(\cdot,0) \geq L$. These two ODEs are of the Ricatti type and can be solved numerically by a straight-forward finite difference scheme that is omitted here.

The third boundary condition required is the one for the case $s \to \infty$, where infinity has to be replaced by a suitable value s_{max} in the numerical procedure. This condition is based on a discrete approximation of the derivative of v with respect to s. For large values of s, surrender will only happen for exogenous reasons. This boundary does not require the operator splitting method suggested by DeGiovanni [20], as there is no interaction of an interest rate model and the insurance value in contrast to the problem studied by DeGiovanni. However, one insight of this paper is that the lapse rate has to be $\underline{\rho}$, as there are no incentive to surrender the contract for endogenous reasons. This means that the discretized version of the contract value for s_{max} is given as

$$v(t, s_{max}) = \mu(t)\Delta t\Psi(t, s_{max})e^{-r\Delta t} + (1 - \mu(t)\Delta t)e^{-r\Delta t}\big[\underline{\rho}\Delta L(t) + (1 - \underline{\rho}\Delta t)v(s_{max}, t + \Delta t)\big],$$

where Δt is a small but not infinitesimal time step.

These three boundary conditions set up the boundaries of the discretized grid on which the contract value can be established using the Crank-Nicolson finite difference scheme.

5.2 Analysis of Financial Implications

This section studies some examples of the contracts analysed in Section 3. In particular, the setting of Li and Szimayer [26] is used to get numerical results that capture the effect of secondary markets for equity-linked life insurance contracts of the above type. The following issues are to be investigated. Firstly, we study the effect of the secondary market on policyholders' surrender behavior, which at the same time depends on the policyholders' monetary rationality degree. Secondly, we compare the contract values with and without a secondary market for both the insurance company and the policyholders and investigate the impact of the secondary market for both parties. Thirdly, we conduct the fair contract analysis from the insurance company's perspective and call for more care in contract design when the existence of a secondary market is not negligible.

The parametrization is specified to be identical to the one used in Li and Szimayer [26] for comparability: The risky asset has a volatility of $\sigma = 0.2$ and $S_0 = 1000$ as a starting value. The interest rate is taken to be constant and equal to $r = 4\%$. The single premium is $P = 100$, and the time to maturity is $T = 10$ years. The percentage of the premium covered by the guarantee is $\alpha = 0.85$, while the guaranteed rates for both the final payment as well as for premature termination, whether due to death or surrender, is $g = g_d = h = 2\%$. The participation coefficient for gains of the underlying asset is $k = k_d = 0.9$. The policyholder is assumed to be forty years old when he enters the contract. The penalty function for early surrender, β, is assumed to have the penalty rates $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$ and $\beta_t = 0$ for $t \geq 5$. The deterministic mortality intensity takes the form $\mu(t) = A + Bc^{y+t}$, where $A = 5.0758 \times 10^{-4}$, $B = 3.9342 \times 10^{-5}$, and $c = 1.1029$.

5.2.1 Effect of Secondary Market on Surrender Behavior

Li and Szimayer [26] have studied policyholders' surrender behavior by developing a separating boundary between the high and the low surrender intensities. Policyholders are more likely to surrender the contracts (represented by the high surrender intensity $\bar{\rho}$) when the underlying asset has a relatively low value. While surrender events take place less likely (represented by the low surrender intensity $\underline{\rho}$) when the underlying asset value is relatively high. The interest rate effect, the time effect and the penalty effect were analyzed to explain the non-smooth increase of the separating boundary. In this section we analyze how the separating boundary, and correspondingly the surrender behavior, is affected by the introduction of the secondary market.

As we have addressed above, the secondary market is featured by the competitiveness of the secondary market κ and the policyholder's access probability p. The surrender behavior is assumed to be affected by the product of p and κ in our model. Assuming that we increase $p \times \kappa$, the average surrender benefit is increased, say, to $\tilde{L}$, see (6), and hence the contract value V^C increases. If the contribution of $\tilde{L}$ to the contract value is very high, so that $V^C \geq \tilde{L}$ is still satisfied even when S is lower, we would expect the separating boundary to move downwards. On the contrary, if the contribution of $\tilde{L}$ is not high enough, the separating boundary may stay unchanged or move upwards. The exact effect of $p \times \kappa$ could only be investigated numerically, which we present in Figure 1. $p = \kappa$ takes the value of 0, 0.2, 0.5 and 0.8 respectively. The surrender intensity is $\underline{\rho}$ above the separating boundary and $\bar{\rho}$ below the separating boundary.

It is clear that when $p \times \kappa = 0$, we are back to the model of Li and Szimayer [26] where a secondary market is not accessible to the policyholders. For $p \times \kappa > 0$, the secondary market comes into play. The surrender intensity in this case indicates the policyholder's surrender behavior either to the insurer or to the secondary market, instead of solely to the insurer when there is no secondary market. We see from Figure 1 that compared to the case with $p \times \kappa = 0$, the $\bar{\rho}$ region is enlarged while the $\underline{\rho}$ region shrinks for all the cases with $p \times \kappa > 0$. In the $\bar{\rho}$ region, we would expect more policyholders to be more likely to give up their contracts when it is monetarily better to do so. Since a fixed proportion (indicated by p) among these policyholders go to the secondary market, we would expect more contracts to be surrendered monetarily optimally in the future. At the same time, the

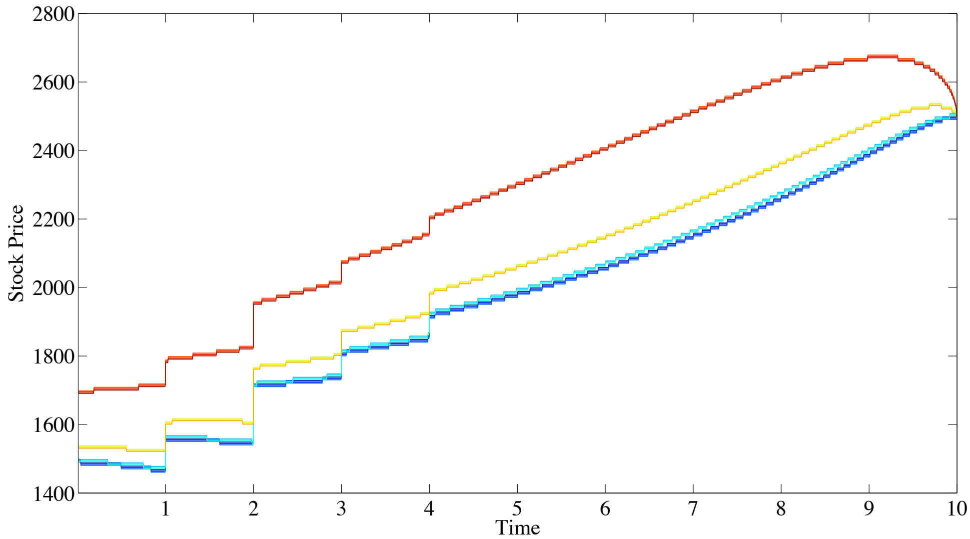

Figure 1: Separating Boundary for $p = \kappa = 0$ (blue), $p \times \kappa = 0.04$ (turquoise), $p \times \kappa = 0.25$ (yellow), and $p \times \kappa = 0.64$ (red).

shrink of the $\underline{\rho}$ region indicates that fewer policyholders will give up the contracts when it is monetarily disadvantageous to do so. Moreover, a proportional amount among them go to the secondary market which triggers the optimal surrender later on. Both of these aspects indicate that the insurer bears more risk when a secondary market emerges. As mentioned in Li and Szimayer [26], the kinks displayed in all the graphs are due to the discontinuous levels of penalties for early surrender. If early surrender is not penalized, e.g. by $\beta_t = 0$ for all $t \geq 0$, the graphs turn out to be smooth. This case does not alter the economic interpretation offered above in any way, however.

5.2.2 Risk Analysis for the Insurer

The separating boundary presented in Figure 1 has indicated that the emergence and the growth of the secondary market increases the risk borne by the insurer. In this section, we quantify the magnitude of the risk by comparing the contract values when there is and when there is not a secondary market. Besides, we study the interaction of policyholder's monetary rationality with the secondary market on the contract values by calculating the true contract values when $\underline{\rho}$ and $\bar{\rho}$ vary.

In Table 1 we present the contract values for $\underline{\rho} \in \{0, 0.03, 0.3\}$, $\bar{\rho} \in \{0, 0.03, 0.3, \infty\}$, $p \in$

$\{0.0, 0.2, 0.5, 0.8, 1.0\}$ and $\kappa \in \{0.0, 02, 0.5, 0.8, 1.0\}$. The first row with $(p, \kappa) = (0.0, 0.0)$ displays the contract values when there is no secondary market. We use it as the benchmark to investigate the risk for the insurer caused by the secondary market. For $(\underline{\rho}, \bar{\rho}) = (0, 0)$, the contract is actually of the European type and is hence not influenced by the secondary market. For $(\underline{\rho}, \bar{\rho}) = (0, \infty)$, both the policyholder and the secondary market are supposed to be able to exercise the surrender option monetarily optimally. Hence, it does not matter who is actually to exercise it, and the secondary market does not play a significant role in this case either. For the other cases we always observe the increase of the contract values whenever the secondary market is introduced.

For given (p, κ) combinations, we always observe that the contract value increases monotonically with the decrease of $\underline{\rho}$ and the increase of $\bar{\rho}$. For the insurance company, the lower $\underline{\rho}$ is, the lower would be the probability that the policyholder surrenders the contract suboptimally which increases the contract value. On the other hand, the lower would be the probability that the contract is sold to the secondary market and hence the lower is the chance that the optimal surrender is triggered. This aspect tends to decrease the contract value for given (p, κ). From the table we infer that the first effect dominates. With regard to $\bar{\rho}$, the higher $\bar{\rho}$ is, the higher is the probability that the contract is surrendered optimally and meanwhile the higher is the probability that the contract is sold to the secondary market. Hence, overall, a higher $\bar{\rho}$ indicates a higher contract value.

Now we further study the impact of p and κ separately. The insurer does not care about how the profits are shared between the secondary market and the original policyholder but only the total amount of extra money which is to flow out of the company due to the existence of the secondary market. Since the access probability p determines directly the profits generated by the secondary market, we observe the monotonic increase of the contract value with p for given κ. On the contrary, the competitiveness index κ only matters through the decision rule of the policyholder. The behavior of the contract value with κ can be distinguished in three cases. First, when $(\underline{\rho}, \bar{\rho}) = (0.03, 0.03), (0.3, 0.3)$, the policyholder's surrender decision is exogenously determined which is independent of κ. Hence, we do not see the change of contract value with κ. Second, when $(\underline{\rho}, \bar{\rho}) = (0.03, \infty), (0.3, \infty)$, the contract value is not influenced by κ either. Analogous to our analysis in Section 5.2.1, the increase of $\tilde{L}$ due to the increase of κ is not high enough to change the relationship

(p,κ)	$(\underline{\rho},\bar{\rho})$								
	$(0,0)$	$(0,0.03)$	$(0,0.3)$	$(0,\infty)$	$(0.03,0.03)$	$(0.03,0.3)$	$(0.03,\infty)$	$(0.3,0.3)$	$(0.3,\infty)$
(0.0,0.0)	101.4769	102.7011	107.2225	112.6733	98.4722	102.7793	106.7845	92.6242	95.0194
(0.2,0.0)	101.4769	102.8649	107.7913	112.6733	99.6142	104.308	107.9235	96.4809	98.5502
(0.5,0.0)	101.4769	103.1107	108.6445	112.6733	101.3273	106.6011	109.6320	102.266	103.8464
(0.8,0.0)	101.4769	103.3565	109.4976	112.6733	103.0403	108.8941	111.3406	108.0511	109.1426
(1.0,0.0)	101.4769	103.5204	110.0664	112.6733	104.1824	110.4228	112.4796	111.9078	112.6734
(0.2,0.2)	101.4769	102.8691	107.7986	112.6733	99.6142	104.3136	107.9235	96.4809	98.5502
(0.5,0.2)	101.4769	103.1391	108.6922	112.6733	101.3273	106.6357	109.6320	102.266	103.8463
(0.8,0.2)	101.4769	103.4340	109.6270	112.6733	103.0403	108.9903	111.3406	108.0511	109.1429
(1.0,0.2)	101.4769	103.6474	110.2771	112.6733	104.1824	110.5760	112.4796	111.9078	112.6734
(0.2,0.5)	101.4769	102.8742	107.8068	112.6733	99.6142	104.3194	107.9235	96.4809	98.5501
(0.5,0.5)	101.4769	103.1797	108.7567	112.6733	101.3273	106.6841	109.6320	102.266	103.8463
(0.8,0.5)	101.4769	103.5741	109.8370	112.6733	103.0403	109.1472	111.3407	108.0511	109.1425
(1.0,0.5)	101.4769	103.9224	110.6742	112.6733	104.1824	110.8796	112.4797	111.9078	112.6737
(0.2,0.8)	101.4769	102.8773	107.8117	112.6733	99.6142	104.3232	107.9235	96.4809	98.5501
(0.5,0.8)	101.4769	103.2125	108.8047	112.6733	101.3273	106.7204	109.6322	102.266	103.8463
(0.8,0.8)	101.4769	103.7236	110.0638	112.6733	103.0403	109.3226	111.3407	108.0511	109.1425
(1.0,0.8)	101.4769	104.2366	111.3373	112.6733	104.1824	111.4217	112.4798	111.9078	112.6733
(0.2,1.0)	101.4769	102.8779	107.8126	112.6733	99.6142	104.3238	107.9235	96.4809	98.5501
(0.5,1.0)	101.4769	103.2217	108.8170	112.6733	101.3273	106.7298	109.6321	102.2660	103.8463
(0.8,1.0)	101.4769	103.7669	110.1648	112.6733	103.0403	109.4063	111.3382	108.0511	109.1425
(1.0,1.0)	101.4769	104.3767	112.1153	112.6733	104.1824	112.0841	112.6754	111.9078	112.6745

Table 1: Contract values from the insurer's perspective for $\underline{\rho} \in \{0, 0.03, 0.3\}$, $\bar{\rho} \in \{0, 0.03, 0.3, \infty\}$, $p \in \{0.0, 0.2, 0.5, 0.8, 1.0\}$ and $\kappa \in \{0.0, 02, 0.5, 0.8, 1.0\}$.

between $\tilde{L}$ and V^C, so that the decision rule does not change with κ. This leads to the fact that the contract value does not vary with κ when the secondary market exists. Third, when $(\underline{\rho}, \bar{\rho}) = (0, 0.03), (0, 0.3), (0.03, 0.3)$, the contract value increases slightly with κ. The reason can be interpreted from Figure 1. Since the $\bar{\rho}$ region increases with κ for given p, the policyholder is more inclined to give up the contract when it is advantageous to do so. With a certain probability, the contract would be sold to the secondary market. Moreover, the shrink of the $\underline{\rho}$ region also contributes to the increase of the contract value. Thus, on the whole, the contract value increases with κ.

To study the interaction of the policyholder's monetary rationality with the secondary market, we present in Table 2, for policyholders with different monetary rationality degrees $(\underline{\rho}, \bar{\rho})$, the relative deviation of the contract values when there is a secondary market from the contract values when a secondary market does not exist. Comparing the columns for $(\underline{\rho}, \bar{\rho}) = (0.03, 0.03), (0.03, 0.3), (0.03, \infty)$, we see that the impact of the secondary market first increases with the rise of the endogenous surrender intensity and then decreases with it for given (p, κ) combinations. Since the relative deviation is 0 when $(\underline{\rho}, \bar{\rho}) = (0, \infty)$, which we have not displayed in the table, the same pattern can also be observed for $(\underline{\rho}, \bar{\rho}) = (0, 0.03), (0, 0.3), (0, \infty)$. This pattern is the joint work of the endogenous surrender intensity $\bar{\rho}$ and the margin from the secondary market $(V^{Am} - V)$ where V refers to the contract value without the secondary market. Although the increase of $\bar{\rho}$ indicates that the policyholder is more likely to surrender the contract to the secondary market when it is monetarily rational to do so, the margin decreases when the policyholder is more capable of surrendering the contract optimally by themselves. In which degree the contract value increases due to the introduction of the secondary market depends on the change of $\bar{\rho}(V^{Am} - V)$ with $\bar{\rho}$ at any time when the contract is likely to be surrendered endogenously by the policyholder to either the insurer or the secondary market. With the increase of $\bar{\rho}$, at the beginning, its increase dominates the decrease of the margin $(V^{Am} - V)$ so that the relative deviation increases. When $\bar{\rho}$ further increases, its increase is dominated by the decrease of the margin. This causes the decrease of the relative deviation. Differently, we observe the monotonic increase of the relative deviation with the exogenous surrender intensity $\underline{\rho}$ when we compare the columns with $(\underline{\rho}, \bar{\rho}) = (0, 0.3), (0.03, 0.3), (0.3, 0.3)$. A higher $\underline{\rho}$ indicates the lower monetary rationality degree of the policyholder. Hence, the margin from the secondary market increases. The double effect, i.e., $\underline{\rho}(V^{Am} - V)$, leads to the increase of the relative deviation of the contract value with the secondary market from the contract value without the secondary market.

	$(\underline{\rho}, \bar{\rho})$						
(p, κ)	$(0, 0.03)$	$(0, 0.3)$	$(0.03, 0.03)$	$(0.03, 0.3)$	$(0.03, \infty)$	$(0.3, 0.3)$	$(0.3, \infty)$
(0.2,0.0)	0.1595	0.5305	1.1597	1.4874	1.0666	4.1638	3.7159
(0.5,0.0)	0.3988	1.3262	2.8994	3.7185	2.6666	10.4096	9.2897
(0.8,0.0)	0.6382	2.1218	4.6390	5.9494	4.2666	16.6554	14.8635
(1.0,0.0)	0.7978	2.6523	5.7988	7.4368	5.3333	20.8192	18.5794
(0.2,0.2)	0.1636	0.5373	1.1597	1.4928	1.0666	4.1638	3.7159
(0.5,0.2)	0.4265	1.3707	2.8994	3.7521	2.6666	10.4096	9.2896
(0.8,0.2)	0.7136	2.2425	4.6390	6.0430	4.2666	16.6554	14.8638
(1.0,0.2)	0.9214	2.8488	5.7988	7.5859	5.3333	20.8192	18.5794
(0.2,0.5)	0.1685	0.5449	1.1597	1.4985	1.0666	4.1638	3.7158
(0.5,0.5)	0.4660	1.4309	2.8994	3.7992	2.6666	10.4096	9.2896
(0.8,0.5)	0.8500	2.4384	4.6390	6.1957	4.2667	16.6554	14.8634
(1.0,0.5)	1.1892	3.2192	5.7988	7.8813	5.3334	20.8192	18.5797
(0.2,0.8)	0.1716	0.5495	1.1597	1.5022	1.0666	4.1638	3.7158
(0.5,0.8)	0.4979	1.4756	2.8994	3.8345	2.6668	10.4096	9.2896
(0.8,0.8)	0.9956	2.6499	4.6390	6.3664	4.2667	16.6554	14.8634
(1.0,0.8)	1.4951	3.8376	5.7988	8.4087	5.3335	20.8192	18.5793
(0.2,1.0)	0.1722	0.5504	1.1597	1.5027	1.0666	4.1638	3.7158
(0.5,1.0)	0.5069	1.4871	2.8994	3.8437	2.6667	10.4096	9.2896
(0.8,1.0)	1.0378	2.7441	4.6390	6.4478	4.2644	16.6554	14.8634
(1.0,1.0)	1.6315	4.5632	5.7988	9.0532	5.5166	20.8192	18.5805

Table 2: Relative deviation (in %) of contract values with secondary market from contract values without secondary market for $(\underline{\rho}, \bar{\rho}) \in \{(0, 0.03), (0, 0.3), (0.03, 0.03), (0.03, 0.3), (0.03, \infty), (0.3, 0.3), (0.3, \infty)\}$ from insurer's perspective.

5.2.3 Welfare Analysis for the Policyholder

In this section we study the effect of secondary market for the representative policyholder. In Table 3 we present the contract values from the policyholder's perspective. We compare the values when the secondary market exists and when it does not. Similar to Table 1, we do not observe its effect when $(\underline{\rho}, \bar{\rho}) = (0, 0), (0, \infty)$. When $\kappa = 0$, all the profits are transferred to the contract buyer. The policyholder is in principle indifferent, whether to sell the contract back to the insurer or to the contract buyer on the secondary market. His welfare increases when both p and κ are different from 0. As is indicated in Proposition 1, it is $p \times \kappa$ that determines the contract value for the policyholder. Hence, we see in Table 3 that the contract values are identical for the (p, κ) combinations with the same $p \times \kappa$ value and the same degree of monetary rationality. The higher $p \times \kappa$ is, the higher is the increase in the welfare of the policyholder which is brought by the secondary market.

Furthermore, to study the interaction of policyholder's monetary rationality with the (p, κ) combinations and its effect on the welfare of the representative policyholder, we present in Table 4 the relative deviation of the contract values when there is secondary market from the values when there is no secondary market. We observe the same pattern as is demonstrated in Table 2, i.e., the relative deviation increases first with $\bar{\rho}$ and then decreases with it for $\bar{\rho} \in \{0.03, 0.3, \infty\}$, and the relative deviation increases monotonically with $\underline{\rho}$. The reason for this phenomenon is the same as is analyzed in Section 5.2.2.

Now we compare Tables 1 and 3. When $\kappa = 1.0$, the policyholder obtains all the profits generated by the secondary market. There is no difference between the contract values from the insurer's and the policyholder's perspectives. Moreover, since the secondary market has no effect for $(\underline{\rho}, \bar{\rho}) = (0, 0), (0, \infty)$, we do not see the difference in these cases either. In the other cases, we observe that the true contract values for the policyholder are always lower than the values for the insurer, because the profits generated by the secondary market are shared with contract buyer. Besides, the difference between them are higher for higher p. This is because higher p indicates that the policyholder is more likely to go to the secondary market and more profits are to be generated by the secondary market due to its competence to exercise the surrender option optimally. On the contrary, the difference between the two values shrinks with the increase of the κ since the benefits obtained by the policyholder is closer to premium charged by the insurer for higher κ. Through this comparison, we see that the introduction of the secondary market is not necessarily profitable to the policyholder if the increase of the welfare is associated with the increase of the premium. If the insurer takes the secondary market into account when calculating the premium, then the secondary market is only desirable for the representative policyholder when it is very competitive.

We know that the policyholder actually represents a large pool of policyholders. The surrender behavior of the representative policyholder summarizes the average behavior of the pool of policyholders. Within this pool, some policyholders are more informed of the existence of the secondary market than the others. Even when the secondary market is not competitive enough and the insurer charges a higher premium, those policyholders who are better informed may benefit from the secondary market and those who are less informed may bear the costs caused by the secondary market. Now we study which policyholders are really profiting from the secondary market.

We assume there are two types of policyholders. 50% of the policyholders are of type 1 and they have no access to the secondary market. The other 50% are of type 2, who have full access to the secondary market. This indicates that $p = 0.5$. Moreover, we assume that the policyholders of the two types have on average the same degree of monetary rationality, namely, $(\underline{\rho}, \bar{\rho}) = (0.03, 0.3)$. We look at the secondary markets with different degrees of competitiveness, $\kappa = 0, 0.2, 0.5, 1.0$. Since the secondary market is irrelevant for the type 1 policyholders, the contract value for this policyholder type is the same for different κ values, namely, 102.7793. When $\kappa = 0$, the contract value for the type 2 policyholders is 102.7793. However, if the insurer takes it into account that some policyholders will go to the secondary market, the premium could be 106.6011, see the $\{(p, \kappa), (\underline{\rho}, \bar{\rho})\} = \{(0.5, 0.0), (0.03, 0.3)\}$ entry in Table 1. The secondary market harms both types of policyholders. When $\kappa = 0.2$, the contract value for the type 2 policyholders is 104.3238, while the insurer may calculate the premium as 106.6357. Thus, on the secondary market with little competitiveness, no policyholders benefit from it either. When $\kappa = 0.5$, the type 2 policyholders benefit from the secondary market. Not only does their welfare increases from 102.7793 to 106.7298, but also they may pay lower premium than the welfare they have gained, i.e., 106.6841. The type 1 policyholders bear the costs incurred by the secondary market. At last, when $\kappa = 1.0$, it is still the type 1 policyholders who pay more premium to eliminate the higher risk faced by the insurer, while the type 2 policyholders profit from the existence of the secondary market.

		$(\underline{\rho}, \bar{\rho})$								
		$(0,0)$	$(0,0.03)$	$(0,0.3)$	$(0,\infty)$	$(0.03,0.03)$	$(0.03,0.3)$	$(0.03,\infty)$	$(0.3,0.3)$	$(0.3,\infty)$
(p,κ)	(0.0,0.0)	101.4769	102.7011	107.2225	112.6733	98.4722	102.7793	106.7845	92.6242	95.0194
	(0.2,0.0)	101.4769	102.7011	107.2225	112.6733	98.4722	102.7793	106.7845	92.6242	95.0194
	(0.5,0.0)	101.4769	102.7011	107.2225	112.6733	98.4722	102.7793	106.7845	92.6242	95.0194
	(0.8,0.0)	101.4769	102.7011	107.2225	112.6733	98.4722	102.7793	106.7845	92.6242	95.0194
	(1.0,0.0)	101.4769	102.7011	107.2225	112.6733	98.4722	102.7793	106.7845	92.6242	95.0194
	(0.2,0.2)	101.4769	102.7343	107.3369	112.6733	98.7006	103.0855	107.0123	93.3955	95.7255
	(0.5,0.2)	101.4769	102.7859	107.5117	112.6733	99.0432	103.5472	107.3540	94.5526	96.7847
	(0.8,0.2)	101.4769	102.8402	107.6907	112.6733	99.3858	104.0120	107.6957	95.7096	97.844
	(1.0,0.2)	101.4769	102.8779	107.8126	112.6733	99.6142	104.3238	107.9235	96.4809	98.5501
	(0.2,0.5)	101.4769	102.7859	107.5117	112.6733	99.0432	103.5472	107.3540	94.5526	96.7847
	(0.5,0.5)	101.4769	102.9271	107.9681	112.6733	99.8997	104.7159	108.2082	97.4451	99.4328
	(0.8,0.5)	101.4769	103.0919	108.4606	112.6733	100.7563	105.9115	109.0627	100.3376	102.0809
	(1.0,0.5)	101.4769	103.2217	108.8170	112.6733	101.3273	106.7298	109.6321	102.266	103.8463
	(0.2,0.8)	101.4769	102.8402	107.6907	112.6733	99.3858	104.0120	107.6957	95.7096	97.8440
	(0.5,0.8)	101.4769	103.0919	108.4606	112.6733	100.7563	105.9115	109.0627	100.3376	102.0809
	(0.8,0.8)	101.4769	103.4418	109.3737	112.6733	102.1267	107.9202	110.4294	104.9657	106.3179
	(1.0,0.8)	101.4769	103.7669	110.1648	112.6733	103.0403	109.4063	111.3382	108.0511	109.1425
	(0.2,1.0)	101.4769	102.8779	107.8126	112.6733	99.6142	104.3238	107.9235	96.4809	98.5501
	(0.5,1.0)	101.4769	103.2217	108.8170	112.6733	101.3273	106.7298	109.6321	102.266	103.8463
	(0.8,1.0)	101.4769	103.7669	110.1648	112.6733	103.0403	109.4063	111.3382	108.0511	109.1425
	(1.0,1.0)	101.4769	104.3767	112.1153	112.6733	104.1824	112.0841	112.6724	111.9078	112.6715

Table 3: Contract values for $\underline{\rho} \in \{0, 0.03, 0.3\}$, $\bar{\rho} \in \{0, 0.03, 0.3, \infty\}$, $p \in \{0.0, 0.2, 0.5, 0.8, 1.0\}$ and $\kappa \in \{0.0, 02, 0.5, 0.8, 1.0\}$ from the policyholder's perspective

5.2.4 Fair Contract Analysis

In Proposition 3, we have presented the necessary condition for the existence of a fair contract. When $p = 0$, it is actually equivalent to the absence of a secondary market. When $(\underline{\rho}, \bar{\rho}) = (0, \infty)$, the policyholder is assumed to be fully monetarily rational, which is hardly true in reality. The remaining necessary condition requires that the secondary market develops to a completely competitive market, i.e., $\kappa = 1$, so that a fair contract exists.

In this section, we conduct a brief fair contract analysis under the assumption that the secondary market is completely competitive. Furthermore, the policyholder's monetary rationality is supposed to satisfy $(\underline{\rho}, \bar{\rho}) = (0.03, 0.3)$, and the access possibility to the secondary market is $p = 0.5$. We study how the participation coefficient k should be modified, so that the contract issued is fair. Alternatively, other parameters could be adjusted in the similar way when keeping k constant and are hence not analyzed here.

In the case without a secondary market for the analyzed insurance contract, the contract is at par for $k = 0.8196$. When a secondary market does exist and it is completely

	$(\underline{\rho}, \bar{\rho})$						
(p, κ)	$(0, 0.03)$	$(0, 0.3)$	$(0.03, 0.03)$	$(0.03, 0.3)$	$(0.03, \infty)$	$(0.3, 0.3)$	$(0.3, \infty)$
(0.2,0.2)	0.0323	0.1067	0.2319	0.2979	0.2133	0.8327	0.7431
(0.5,0.2)	0.0826	0.2697	0.5799	0.7471	0.5333	2.0820	1.8578
(0.8,0.2)	0.1354	0.4367	0.9278	1.1994	0.8533	3.3311	2.9727
(1.0,0.2)	0.1722	0.5504	1.1597	1.5027	1.0666	4.1638	3.7158
(0.2,0.5)	0.0826	0.2697	0.5799	0.7471	0.5333	2.0820	1.8578
(0.5,0.5)	0.2201	0.6954	1.4496	1.8842	1.3332	5.2048	4.6447
(0.8,0.5)	0.3805	1.1547	2.3195	3.0475	2.1335	8.3276	7.4316
(1.0,0.5)	0.5069	1.4871	2.8994	3.8437	2.6667	10.4096	9.2896
(0.2,0.8)	0.1354	0.4367	0.9278	1.1994	0.8533	3.3311	2.9727
(0.5,0.8)	0.3805	1.1547	2.3195	3.0475	2.1335	8.3276	7.4316
(0.8,0.8)	0.7212	2.0063	3.7112	5.0019	3.4133	13.3243	11.8907
(1.0,0.8)	1.0378	2.7441	4.6390	6.4478	4.2644	16.6554	14.8634
(0.2,1.0)	0.1722	0.5504	1.1597	1.5027	1.0666	4.1638	3.7158
(0.5,1.0)	0.5069	1.4871	2.8994	3.8437	2.6667	10.4096	9.2896
(0.8,1.0)	1.0378	2.7441	4.6390	6.4478	4.2644	16.6554	14.8634
(1.0,1.0)	1.6315	4.5632	5.7988	9.0532	5.5166	20.8192	18.5805

Table 4: Relative deviation (in %) of contract values with secondary market from contract values without secondary market for $(\underline{\rho}, \bar{\rho}) \in \{(0, 0.03), (0, 0.3), (0.03, 0.03), (0.03, 0.3), (0.03, \infty), (0.3, 0.3), (0.3, \infty)\}$ from policyholder's perspective.

competitive, the contract is at par for $k = 0.72552$ which is about 11.5% lower than the k value in the former case. The lower participation coefficient helps to offset the profits generated from the secondary market and earned by the policyholder. Similar to our analysis in section 5.2.3, we now generalize from the representative policyholder to two groups of policyholders and see which group is hurt by the secondary market. Policyholders of group 1 have no information about the secondary market and policyholders of group 2 have knowledge about it. The fair participation coefficient k for group 1 is $k = 0.8196$ and for group 2 is $k = 0.6348$. Thus, group 1 is worse off for not being able to participate fairly in the return of the underlying assets. While group 2 is better off for profiting not only from the secondary market but also from the policyholders of group 1.

In this paper we have focused more on the effect of the secondary market on the policyholders with average monetary rationality. If we further generalize from the two groups of policyholders with different information about the secondary market to more groups of policyholders with different degrees of monetary rationality as well, we would find that the policyholders with the highest rationality and the full information about the secondary market profit the most from the secondary market. A comprehensive fair contract analysis concerning different degrees of monetary rationality is provided by Li and Szimayer [26]. Their analysis can be easily extended to our setting with secondary market and is hence omitted here.

6 Conclusion

To our knowledge this is the first paper which includes a secondary market in the valuation of equity-linked life insurance contracts with surrender guarantees. We have analyzed the effect of a secondary market within an intensity based framework where the surrender intensity of the policyholder is assumed to be bounded from below and from above. The access to the secondary market is modeled by randomization.

We have shown that the surrender behavior of the representative policyholder is affected by the secondary market. The existence of a secondary market increases the likelihood that the policyholder sells his contract either back to the insurer or to the secondary market when it is financially optimal to do so. Besides, the likelihood of surrendering in disadvantageous situations decreases, where we need to keep it in mind that with a certain possibility the secondary market will take over the contract. On the whole, the risk borne by the insurer increases due to the introduction of the secondary market and it increases further with the growth of the secondary market.

With the existence of a secondary market, the contract values from the insurer's perspective are usually not identical with the values from a representative policyholder's point of view. We have derived two pricing PDEs and Feynman-Kac type stochastic representations to characterize the contract values from the two perspectives. Comparative statics as well as our numerical analysis have shown that the contract value for the representative policyholder increases with the product of the access probability to the secondary market p and the competitiveness indicator κ. For the insurer, the contract value increases monotonically with the access probability p but not always with κ. This is because the insurer takes into account during the contract valuation that the contract buyers on the secondary market have the expertise to exercise the surrender option financially rational and κ influences the contract value from the insurer's perspective only through the surrender behavior modeled in equation (7).

We have also investigated the interaction of the policyholder's monetary rationality with the secondary market. Numerical results have shown that higher endogenous surrender intensity decreases the margin generated by the secondary market and thus does not necessarily increase the impact of the secondary market on contract values. On the contrary, higher exogenous surrender intensity increases the margin and hence enhances the effect of the secondary market.

Since the policyholder may only profit partly from the high rationality of the contract buyers on the secondary market, the contract values from the policyholder's perspective are usually lower than the values from the policyholder's perspective. Hence, although the introduction of the secondary market may increase the payout to the policyholder, it is not necessarily beneficial for him if the welfare increase is associated with the increase of the premium. Generalizing from the representative policyholder to two groups of policyholders, one uninformed and one informed of the existence of the secondary market, we have demonstrated that the secondary market is profitable for no policyholders when the policyholders have not bargaining power on the secondary market. If, on the contrary, the secondary market is competitive enough, the informed policyholders profit from the secondary market while the uninformed policyholders bear the costs incurred by it.

Only in special cases would the contract values from the two perspectives be identical. One of the special cases is $\kappa = 1$, i.e., the secondary market is completely competitive. Under this assumption, we have conducted the fair contract analysis. We have compared the contract parameters when the the secondary market exists and when it does not, and hence highlighted the influence of a secondary market on contract design.

The secondary market brings a challenge to the insurance companies in regard to the risk management of the contracts. Insurance contracts usually have long maturity time and may already have existed before the insurance companies' awareness about the emergence of the secondary market. If the secondary market is introduced at a sudden, the premiums charged before may not be adequate to support the hedging strategies of the insurance companies. In particular, although the policyholders may not rush to surrender their contracts simultaneously, the contract buyers on the secondary market may be able to do so due to their expertise in managing the contracts. Hence, once the secondary market comes into play, the insurance companies have to deal with the potential liquidity problem caused by the simultaneous surrender of the contracts.

A Proof of Corollary 1

Proof. Suppose V satisfies the PDE (15) with the given boundary condition $v(T, S_T) = \Phi(S_T)$, and define the process X via

$$\mathrm{d}X_s = a(t,s)\, X(s)\mathrm{d}t + \sigma(t,s)\, X(s)\mathrm{d}W(s).$$

Consider the function $Z(s)$, defined as

$$Z(s, X(s)) = e^{\int_t^s r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} v(s, X(s)).$$

Expanding the function Z by Itó's Lemma yields

$$\begin{aligned}
Z(T, X(T)) =& z(t,x) + \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \\
& \big(v_u(u, X(u)) - (r(u) + \mu(u) + \gamma(u, X(u)))v(u, X(u))\big)\mathrm{d}u \\
& + \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} r(u) x v_x(u, X(u))\mathrm{d}u \\
& + \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \sigma x v_x(u, X(u))\mathrm{d}W_u \\
& + \frac{1}{2}\int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \sigma^2 x^2 v_{xx}(u, X(u))\mathrm{d}u.
\end{aligned}$$

The left hand side is expanded by

$$\pm \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \mu(u)\Psi(u, X(u)) + \gamma(u, X(u))L(u) + p\kappa(V^{Am}(u, X(u)) - L(u)),$$

which can be rewritten as

$$
\begin{aligned}
Z(T, X(T)) =& z(t,x) + \int_t^T \Big[e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \\
& \big[v_u(u, X(u)) - (r(u) + \mu(u) + \gamma(u, X(u)))v(u, X(u))\big] \\
& + r(u)xv_x(u, X(u)) + \frac{1}{2}\sigma^2 x^2 v_{xx}(u, X(u)) \\
& + \mu(u)\Psi(u, X(u)) + \gamma(u, X(u))L(u) + p\kappa(V^{Am}(u, X(u)) - L(u))\Big]\mathrm{d}u \\
& + \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \sigma x v_x(u, X(u))\mathrm{d}W_u \\
& - \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \mu(u)\Psi(u, X(u)) + \gamma(u, X(u))L(u) \\
& + p\kappa\big(V^{Am}(u, X(u)) - L(u)\big)\mathrm{d}u.
\end{aligned}
$$

Recognising that v satisfies the PDE (15), meaning that the first integral vanishes, and rearranging results in

$$
\begin{aligned}
& Z(T, X(T)) + \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \\
& \mu(u)\Psi(u, X(u)) + \gamma(u, X(u))L(u) + p\kappa\big(V^{Am}(u, X(u)) - L(u)\big)du \\
& \qquad = z(t,x) + \int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \sigma x v_x(u, X(u))\mathrm{d}W_u.
\end{aligned}
$$

Applying expectations to this equation finally yields, assuming sufficient integrability,

$$
\begin{aligned}
& \mathbb{E}\Big[\int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \Phi(X(T))\Big] + \mathbb{E}\Big[\int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \mu(u)\Psi(u, X(u))\mathrm{d}u\Big] \\
& \qquad + \mathbb{E}\Big[\int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} \gamma(u, X(u))L(u)\mathrm{d}u\Big] \\
& \qquad + \mathbb{E}\Big[\int_t^T e^{\int_t^u r(y)+\mu(y)+\gamma(y,X(y))\mathrm{d}y} p\kappa\big(V^{Am}(u, X(u)) - L(u)\big)\mathrm{d}u\Big] \\
& = z(t,x),
\end{aligned}
$$

which is the required stochastic representation formula of the Feynman-Kac type and thus the proof is completed. □

B A Martingale Approach to Corollary 1

This appendix proves Corollary 1 using a martingale approach. Based on the model setup provided in Section 2, the time t value of an alive contract, i.e., $t < \lambda \wedge \tau \wedge T$, from the policyholder's perspective can be expressed as

$$
\begin{aligned}
V_t^C &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)\mathrm{d}u}\mathbb{1}_{\{\tau>T,\lambda>T\}}\Phi(S_T)\middle|\mathcal{G}_t\right] && \text{(Part 1)}\\
&+ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^\tau r(u)\mathrm{d}u}\mathbb{1}_{\{t<\tau\leq T,\tau<\lambda\}}\Psi(\tau,S_\tau)\middle|\mathcal{G}_t\right] && \text{(Part 2)}\\
&+ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^\lambda r(u)\mathrm{d}u}\mathbb{1}_{\{t<\lambda\leq T,\lambda<\tau\}}L(\lambda)\middle|\mathcal{G}_t\right] && \text{(Part 3)}\\
&+ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^\lambda r(u)\mathrm{d}u}\mathbb{1}_{\{t<\lambda\leq T,\lambda<\tau\}}p\kappa(V^{Am}(\lambda,s)-L(\lambda))\middle|\mathcal{G}_t\right] && \text{(Part 4)}
\end{aligned}
$$

Proposition 6. *Suppose the setup detailed in Section 2 and Section 3, the value process V_t^C has the following representation on $\{t < \lambda \wedge \tau \wedge T\}$*

$$
\begin{aligned}
V_t^C =&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(y)+\mu(y)+\gamma(y,s)dy}\Phi(S_T)\middle|\mathcal{F}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma(y,s)\mathrm{d}y}\mu(u)\Psi(u,S_u)\mathrm{d}u\middle|\mathcal{F}_t\right]\\
&+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma(y,s)\mathrm{d}y}\gamma(u,S_u)L(u)\mathrm{d}u\middle|\mathcal{F}_t\right]\\
&+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma(y,s)\mathrm{d}y}\gamma(u,S_u)p\kappa\big(V^{Am}(u,S_u)-L(u)\big)\mathrm{d}u\middle|\mathcal{F}_t\right]. \qquad (21)
\end{aligned}
$$

Proof. For the first three terms, Proposition 3 in Li and Szimayer [26] is applicable. Thus, the only remaining term that requires proof is term four. For that, we have precisely the same steps as for part 3 in Li and Szimayer, but with a modified payment. This is due to the nature of the decision to access the secondary market. It only influences the payment received once the contract is surrendered, but it does not affect the probabilities to get to time where this is relevant. Neither the flow of information modeled by the σ-algebras nor the probabilities are affected here. We have

$$
\begin{aligned}
\text{Part 4} =&\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{t<\lambda\leq T\}}\mathbb{1}_{\{\lambda<\tau\}}p\kappa\big(V^{Am}(\lambda,s)-L(\lambda)\big)e^{-\int_t^\lambda r(y)\mathrm{d}y}\middle|\mathcal{F}_t\wedge\mathcal{H}_t\wedge\mathcal{J}_t\right]\\
=&\mathbb{1}_{\{\lambda>t\}}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{t<\lambda\leq T\}}\mathbb{1}_{\{\lambda<\tau\}}e^{\int_0^t \gamma(y,S(y))\mathrm{d}y}p\kappa\big(V^{Am}(\lambda,s)-L(\lambda)\big)e^{-\int_t^\lambda r(y)\mathrm{d}y}\middle|\mathcal{F}_t\wedge\mathcal{H}_t\right].
\end{aligned}
$$

Here, Corollary 5.1.1 of Bielecki and Rutkowski [6], equation (5.13) has been applied. We now rewrite the expectation as an integral w.r.t $\mathbb{Q}$ in order to remove the indicator function, which gives

$$\mathbb{1}_{\{\lambda>t\}}\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^T \mathbb{1}_{\{\tau>u\}}e^{\int_0^t \gamma(y,S(y))\mathrm{d}y}p\kappa\big(V^{Am}(\lambda,s)-L(\lambda)\big)e^{-\int_t^u r(y)\mathrm{d}y}\mathrm{d}\mathbb{Q}(\lambda\leq u\big|\mathcal{F}_T)\bigg|\mathcal{F}_t\wedge\mathcal{H}_t\bigg].$$

Changing the order of integration and rewriting the integral as an integral w.r.t Lebesque measure produces

$$\mathbb{1}_{\{\lambda>t\}}\int_t^T \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\{\tau>u\}}e^{-\int_0^t \gamma(y,S(y))\mathrm{d}y}p\kappa\big(V^{Am}(\lambda,s)-L(\lambda)\big)$$
$$e^{-\int_t^u r(y)\mathrm{d}y}\gamma(u,S(u))e^{-\int_0^u \gamma(y,S(y))\mathrm{d}y}\bigg|\mathcal{F}_t\wedge\mathcal{H}_t\bigg]\mathrm{d}u.$$

Collecting terms and applying Corollary 5.1.1 to pull out the remaining indicator for mortality gives

$$\mathbb{1}_{\{\lambda>t\}}\int_t^T \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\bigg[e^{-\int_t^u r(y)+\mu(y)+\gamma(y,S(y))\mathrm{d}y}p\kappa\big(V^{Am}(\lambda,s)-L(\lambda)\big)\bigg|\mathcal{F}_t\bigg]du.$$

Changing the order of integration again (Fubini's theorem) results in

$$\mathbb{1}_{\{\lambda>t\}}\mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\bigg[\int_t^T e^{-\int_t^u r(y)+\mu(y)+\gamma(y,S(y))\mathrm{d}y}p\kappa\big(V^{Am}(\lambda,s)-L(\lambda)\big)\mathrm{d}u\bigg|\mathcal{F}_t\bigg].$$

This proofs the result, as both indicator functions are always equal to one because the function is only analyzed on $\{t<\lambda\wedge\tau\wedge T\}$. □

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