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On Irreversible Investment

by

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Abstract

This paper develops a general theory of irreversible investment of a single firm that chooses a dynamic capacity expansion plan in an uncertain environment. The model is set up free of any distributional or any parametric assumptions and hence encompasses all the existing models. As the first contribution, a general existence and uniqueness result is provided for the optimal investment policy. Based upon an alternative approach developed previously to dynamic programming problems, we derive the optimal *base capacity policy* such that the firm always keeps the capacity at or above the *base capacity*. The critical *base capacity* is explicitly constructed and characterized via a stochastic backward equation. This method allows qualitative insights into the nature of the optimal investment under irreversibility. (It is demonstrated that the marginal profit is indeed equal to the user cost of capital in free intervals where investment occurs in an absolutely continuous way at strictly positive rates. However, the equality is maintained only in expectation *on average* in blocked intervals where no investment occurs. Whenever the uncertainty is generated by a diffusion, the investment is singular with respect to Lebesgue measure. In contrast to the deterministic and Brownian motion case where lump sum investment takes place only at time zero, the firm responses in general more frequently in jumps to shocks. Nevertheless, lump sum investments are shown to be possible only at information surprises which is defined as unpredictable stopping time or unanticipated information jump even at the predictable time.) Furthermore, general monotone comparative statics results are derived for the relevant ingredients of the model. Finally, explicit solutions are derived for infinite time horizon, a separable operating profit function of Cobb–Douglas type and an exponential Lévy process modelled economic shock.

Key words and phrases: Sequential Irreversible Investment, Capacity Expansion, Singular Control Problem, Lévy Processes.

JEL subject classification: C61, D81, E22, G11

1 Introduction

Consumption and investment are fundamental economic activities. The act of investing of these two, by creating value and pushing the economy forward, is *the* distinguishing human activity that separates the fate of different societies. Investment is hindered by many frictions, though. It is thus of obvious importance to develop a theory of rational investment under frictions. One particularly significant class of frictions is due to (complete or partial) irreversibility as in many cases, for instance, entrepreneurs are unable to recover the capital invested due to sunk costs, adverse selection problems or institutional arrangements.

Irreversible investment problem is studied by an extensive literature. In the pioneering work, Arrow (1968) deals with the problem of irreversibility under perfect foresight; Pindyck (1988) and Bertola (1998) analyze the benchmark problem of a firm with Cobb–Douglas profit function and stochastic shocks modelled by a geometric Brownian motion. Their work has recently been extended to Markov processes with independent identically distributed increments (Boyarchenko (2004)) and regime shifts (Guo, Miao, and Morellec (2005)). However so far, little work has been done beyond specific classes of models. The present paper develops a general theory of irreversible investment under uncertainty which is free of any distributional or parametric assumptions. Based upon an alternative method to dynamic programming originally developed for utility maximization problems in Bank and Riedel (2001b), we develop a qualitative theory of irreversible investment that allows characterization of the investment behavior for any type of profit function and general stochastic processes. Furthermore, general monotone comparative statics is established for the relevant ingredients of the model.

We are now going to explain our contributions in more detail. A profit-maximizing single firm is considered who chooses a dynamic capacity expansion plan in a risky environment. The operating profit function depends on the current capacity of the firm and a stochastic process that models the uncertainty. In this way, the model covers not only all the previously studied models in economics but also the standard finance model where the uncertainty is usually specified by a semimartingale process.

To have a sound foundation for our theory, a general existence and uniqueness theorem is first developed, which is not available in the literature yet. Uniqueness of the optimal policy is trivial as usual, given a maximization problem of a strict concave functional. For the proof of existence, the optimal investment policy under perfect reversibility is taken as a benchmark case. As is well known, a firm in this case equates the marginal operating profit with the user cost of capital at all times. It is reasonable to assume that the problem under reversibility is finite, which in turn guarantees the well-posedness of the irreversible investment problem. On this basis, the existence result is obtained by additionally assuming that the running maximum of the optimal frictionless policy is integrable. This assumption is required to show that all sensible investment policies are bounded by the running maximum of the optimal frictionless policy. With this integrable upper bound, Komlos' Theorem can be used as a substitute for the lack of compactness in the infinite–dimensional space to identify a candidate optimal policy. Generally, it is impossible to relax our assumptions as the constructed model includes the setup where the optimal policies under reversibility and irreversibility coincide.

Moving on, we study the explicit construction of the optimal investment policy. As the starting point, the first–order condition is derived as done in Bertola (1998). In contrast to the frictionless model where only the immediate marginal operating profit comes into effect, all the changes in future marginal operating profits due to the current investment have to be taken into account. Consequently in case of irreversibility, the marginal operating profit from the current investment is given by the properly discounted expected present value of future marginal operating profits. The firm aims then to keep it below the cost of current investment at all times. In Bertola's explicit model, it is sufficient to verify the first–order condition with a guess of the optimal policy. Nevertheless due to irreversibility, the first–order condition is not binding frequently and hence can not be used to obtain solutions in general. To further proceed, we borrow an approach which is well known in inventory theory and make the ansatz that the optimal policy is going to be a so–called *base capacity policy*: there exists a base capacity (l_t), a stochastic process indicating the optimal capacity level the firm would like to have if it started with zero capacity at that point in time. The optimal policy is then to expand firm's capacity to the base capacity level if the current capacity is lower, or otherwise to maintain the current level.

Such a policy can be characterized as a function of the running maximum of the base capacity. Endowed with this tool, we rewrite the first–order condition in terms of the base capacity. In this way, the first–order condition is obtained as an *equality* by leaving past capacity aside as if the firm started at any point in time from scratch, although it is as noted above an inequality frequently due to the possibility of excess capacity. This leads to our key equation, a stochastic backward equation, that has been studied in other contexts before (Bank and Riedel (2001a) in the framework of intertemporal utility functions with memory, Bank and Föllmer (2003) and El Karoui and Karatzas (1994) for American options; and a general study of the mathematical properties of this equation is in Bank and ElKaroui (2004)). As this backward equation can always be solved numerically via backward induction, the irreversible investment problem is in principle completely solved.

In addition to the backward equation, we show that the base capacity can also be characterized via a family of optimal stopping problems. This formalizes in a rigorous way the approach taken by Pindyck (1988) who solves the irreversible investment problem by considering a continuum of American options for the next marginal investment. Starting from the first–order condition, we construct auxiliary levels L_t^τ. These numbers would be the optimal capacity level if it was optimal to invest at time t, wait until the next (possible)investment time τ. It is easy to see that these levels are chosen such that the discounted expected marginal operating profit equals the user cost of capital. It is then shown that the optimal base capacity l_t is the lower envelope of all these auxiliary levels L_t^τ. The firm thus solves at any point in time an optimal stopping problem that determines the next time of investment.

The auxiliary levels L_t^τ are very useful because one can infer properties of the optimal investment policy from those of the auxiliary levels. The auxiliary levels solve a simple equation and hence can be handled easily. As a first application, they are used to give a general qualitative characterization of the optimal policy. Following Arrow (1968), we distinguish free and blocked intervals. In a free interval, the firm invests in an absolutely continuous way at strictly positive rates[1]. It is shown that in free intervals the firm always equates the marginal operating profit with the user cost of capital. In this sense, it generalizes Arrow's result for the benchmark case of the frictionless world to the stochastic model. During a blocked interval, no investment occurs as the firm has excess capacity from the past. Using our construction of the auxiliary levels, it follows immediately that the marginal operating profit is equal to the user cost of capital only in expectation on average over

[1]Note that diffusion models that are usually studied do not have such free intervals. However, it is perfectly natural to consider other stochastic processes, such as compound Poisson processes. In this case, free intervals exist as shown by the specific examples in Section 6.

time.

Whenever uncertainty is generated by a diffusion, the optimal policy is going to be related to the running maximum of another diffusion. Therefore, investment will be generally singular with respect to Lebesgue measure. This means that positive investment occurs on a set of Lebesgue measure zero. Peculiar as this might seem, it is the well–known standard case in a Brownian model. Diffusions oscillate in such an irregular way that highly irregular action patterns have to be taken to keep them below a certain boundary. Under perfect foresight, Arrow shows that the firm usually invests in lump at time zero to boost the firm's capacity to a good level. Then, no lump sum investments occur afterwards as the firm anticipates the future changes and adjusts capacity in a smooth way. With stochastic jumps, lump sum investments may be the optimal response to shocks. Nevertheless, it is demonstrated that only *information surprises* lead to lump sum investments. Anticipated shocks are, as in Arrow, bolstered by continuous smooth adjustments. The concept of an information surprise is as defined in Hindy and Huang (1993) either an unpredictable stopping time or a predictable time but with a discontinuous information flow. For Markov processes with i.i.d. increments, jumps of a Poisson process is the only sources of shocks which occurs with surprise. Generally, discontinuity of the information flow is also possible. A paradigmatic example might be completely random decisions by the Federal reserve which occur, however, at fixed thus predictable times. In addition, we show that whenever a lump sum investment happens as a reaction to an information surprise, the capacity never jumps to "excess" capacity with respect to the operating profit. Thus, the firm remains cautious in the sense that it usually invests less than it would in a frictionless environment.

The auxiliary levels L_t^τ form the building block for our general monotone comparative statics results which are determined by applying implicit differentiation of the simple equation of the auxiliary levels. Following the methods and ideas from Topkis (1978) and Milgrom and Shannon (1994), we establish that the base capacity is monotonically increasing in the exogenous shock process when the operating profit function is supermodular, or equivalently, exhibits increasing differences in capacity and exogenous economic shock. To our knowledge, this is the first result in monotone comparative statics which takes a whole stochastic process as a parameter. Another two significant parameters of the model are depreciation and interest rate. In general, no monotone comparative statics hold true for any one of them alone. Instead, their sum, the user cost of capital, is the right quantity to study and we demonstrate that investment is decreasing in the user cost of capital.

Generally, numerical methods have to be used to identify the base capacity according to the algorithm given in the paper. Nevertheless, closed-form solutions of the optimal investment policy are possible for an infinite time horizon, separable operating profit functions of Cobb–Douglas type and shocks specified by an exponential of a Markov process with i.i.d. increments, namely, an exponential Lévy process. We show how to recover the results of Pindyck (1988), Bertola (1998), and Boyarchenko (2004) with our method. Under this construction, the base capacity is given by the exogenous economic shock multiplied by a constant factor expressed in terms of expectation. In this way, the marginal profit under the optimal investment plan is always kept below the user cost of capital times a markup factor.

The remainder of this paper is organized as follows. Section 2 presents the general model and derives the uniqueness and existence theorem. The heuristics and explicit construction of the auxiliary levels and the base capacity are provided in Section 3. Section 4 characterizes the optimal policy and Section 5 gives general comparative statics results for the irreversible investment problem. Explicit solutions are derived in Section 6 for the case that the firm is facing a Cobb–Douglas operating profit function and is subject to a multiplicative economic shock modelled by an exponential Lévy process. Finally, Section 7 concludes the paper with a short summary and remark.

2 Irreversible Investment Model and Uniqueness and Existence of Optimal Policies

To develop the sequential irreversible investment theory, a general model is first constructed where a single profit–maximizing firm chooses a dynamic capacity expansion plan in an uncertain environment. This setup encompasses all existing models in the literature. Then, this section moves on to the investigation of existence and uniqueness of the optimal investment strategy.

2.1 Irreversible Investment Model Setup

Consider a firm that chooses a dynamic capacity expansion plan over a time horizon $\hat{T} \leq \infty$ which can be finite or infinite. The operating profit flow of the firm is assumed to be summarized by a function $\pi(X_t, C_t)$ of current capacity C_t and some exogenous *state variable* X_t with values in some complete metric space[2] E. X_t can be regarded as an economic shock, reflecting the changes in, e.g., technologically feasible output, demand and macroeconomic conditions and so on, which have direct or indirect effects on the firm's profit. The stochastic process $(X_t)_{t\in[0,\hat{T}]}$ is formally defined on some underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t,\ 0 \le t \le \hat{T}), \mathbb{P})$ with an information filtration $(\mathcal{F}_t)_{0\le t\le \hat{T}}$ satisfying the usual conditions of completeness, i.e., $\mathcal{F}_0$ contains all the $\mathbb{P}$-null set of $\mathcal{F}$ and $\mathbb{F}$ is right continuous. In addition, X_t is known at time t, or formally, the process $(X_t)_{t\in[0,\hat{T}]}$ is progressively measurable w.r.t. $\mathcal{F}_t$. Suppose in addition that $\pi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and strictly concave in capacity C with derivative $\pi_c(x, c)$ that satisfies the Inada conditions

$$\lim_{c\to 0} \pi_c(x, c) = \infty$$

and

$$\lim_{c\to\infty} \pi_c(x, c) = 0$$

for all $x \in \mathbb{R}$. Moreover, there are no costs as long as no investment has been made, namely, $\pi(0) = 0$. As in Arrow (1968), the price of capital goods used to build up capacity is taken as the numéraire. Thus, the cost of investment is always 1 and the short–term interest rate at time t, r_t, is expressed in terms of capital goods not money[3]. Formally, $(r_t)_{t\in[0,\hat{T}]}$ is a nonnegative bounded optional process.

Given the operating profit, the firm chooses a plan $I = (I_t)_{t\in[0,\hat{T}]}$ of cumulative investments, a right–continuous adapted process. The initial investment $I_0 > 0$ indicates the size of the lump sum investment at time 0. As investment is irreversible, I has to be nondecreasing. The investment plan leads to a capacity $C^I = (C^I_t)_{t\in[0,\hat{T}]}$ that starts in $C^I_{0-} = 0$[4] and evolves according to the differential equation

$$dC^I_t = dI_t - \delta_t C^I_t dt \tag{1}$$

[2]Such an assumption is made for a microeconomic foundation, see e.g. Bertola (1998).

[3]This assumption is not restrictive. Indeed, it can be always achieved by a change of numéraire if, e.g., the price of capital goods is a bounded semimartingale.

[4]It is assumed here that the firm does not come into being with past capacity. All results hold true, however, with some initial capacity $C_{0-} > 0$. We distinguish between time $0-$ and 0 because a lump sum investment usually occurs at time 0 of size I_0 and brings the capacity $C_0 = C_{0-} + I_0$ at time 0.

for some *depreciation rate* $(\delta_t)_{t\in[0,\hat{T}]} \geq 0$, a nonnegative bounded optional process. An investment plan I is *admissible* if its net present value is finite, i.e.,

$$\mathbb{E}\left[\int_0^{\hat{T}} e^{-\int_0^t r_s ds} dI_t\right] < \infty\,.$$

In this context, the firm maximizes the expected present value of the future overall net profits

$$\Pi(I) = \mathbb{E}\left[\int_0^{\hat{T}} e^{-\int_0^t r_s ds}\Big(\pi\left(X_t, C_t^I\right)\, dt - dI_t\Big)\right] \qquad (2)$$

over all admissible plans I. Note that the net profit $\Pi(I)$ is well defined for all admissible plans but potentially infinite. In the next subsection, some conditions are given that ensure finiteness.

Before solving the sequential irreversible investment decision problem, we show that all models studied so far in the literature are included in our setup.

Example 2.1 *The general setup includes the deterministic case with an arbitrary deterministic interest rate r and the operating profit $\pi(t, C_t)$ (Here, time is the state variable, i.e., $X_t = t$). This case has been fully analyzed by Arrow (1968) in complete generality by using the calculus of variations, in particular Pontryagin's principle.*

Example 2.2 *The best studied special case under uncertainty has a separable operating profit function $\pi(x, c) = e^x \pi(c)$ and an infinite time horizon. Bertola (1998) and Pindyck (1988) take X as a Brownian motion with drift,*

$$X_t = \mu t + \sigma W_t, \qquad (3)$$

where W_t is the standard Brownian motion, μ and σ are the constant drift and volatility, respectively. Moreover, they assume a constant interest rate $r_t = r$, $\forall\, t \in [0, \infty)$, and a Cobb–Douglas operating profit function $\pi(c) = \frac{1}{1-\alpha} c^{1-\alpha}$ for some $0 < \alpha < 1$. Boyarchenko (2004) allows X to be a Lévy process, a Markov process with independent identically distributed increments. An interesting extension concerns regime shifts where the parameters of the Brownian motion switch between different states according to a continuous–time Markov chain, see Guo, Miao, and Morellec (2005). Kobila (1993) presents the general dynamic programming approach for nonseparable operating profit and diffusion state variables.

2.2 Existence and Uniqueness

Although a number of explicit case studies have been carried out, no general existence and uniqueness theorem is available in the literature. The present subsection provides sufficient conditions that ensure existence and uniqueness of a solution for the case of a finite horizon. Those for an infinite horizon are given in Appendix A.

Take an auxiliary function as the starting point. Due to the assumptions of the operating profit function π, the *indirect profit function*

$$\pi^*(x, r, \delta) = \max_{c \geq 0} \; \pi(x, c) - (r + \delta)c \tag{4}$$

exists for fixed parameters $x, r, \delta \in \mathbb{R}$. The unique maximizer denoted by $c^*(x, r, \delta)$ solves the first–order condition

$$\pi_c(x, c) = r + \delta \,.$$

Remark 2.3 *This auxiliary function describes the optimal investment under perfect reversibility, the so–called myopic decision rule. In this case, the marginal operating profit has to be equal to the* user cost of capital[5] *which is given in this paper by the sum of interest and depreciation rate, $r + \delta$. Compare this result with the discussion on the optimal capacity with perfect reversibility in Section 3.*

The following two conditions are imposed for existence and uniqueness of the optimal policies. We assume that the reversible investment problem has a finite value and that the overall maximum of optimal reversible capacity is integrable.

Assumption 2.4 *(i)* $\mathbb{E}\Big[\pi^*(X_t, r_t, \delta_t)\Big] < \infty, \quad \forall\, t \in [0,\ \hat{T}]$;

(ii) $K \triangleq \mathbb{E}\Big[\sup_{t \leq \hat{T}} c^*(X_t, r_t, \delta_t)\Big] < \infty\,.$

[5] As first defined by Jorgenson (1963), the user cost of capital is the opportunity cost of holding one unit of capital for a period in the standard neoclassical economics . It consists of three components: the financial cost of the capital measured by the interest rate r, the depreciation cost δ and the lost gain in the value of that unit of capital $\frac{E[dP_t]}{P_t}$ where P_t denotes the purchasing price of the capital.

The example below shows that these assumptions hold true generally for the widely studied separable operating profit function and Lévy processes with bounded positive jumps.

Example 2.5 *The benchmark example in the literature has the operating profit function* $\pi(x,c) = e^x \frac{c^{1-\alpha}}{1-\alpha}$ *with a constant parameter* $\alpha > 0$*. Assumption 2.4 is satisfied, if* X *is a Brownian motion as defined in* (3)*. More generally, Assumption 2.4 holds true for Lévy processes with bounded positive jumps.*

Proof: The maximizer of (4) is obtained as

$$c^*(x,r,\delta) = \frac{1}{(r+\delta)^{\frac{1}{\alpha}}} e^{\frac{x}{\alpha}}.$$

It gives then the optimal indirect operating profit

$$\pi^*(x,r,\delta) = \frac{1}{1-\alpha}(r+\delta)^{\frac{\alpha-1}{\alpha}} e^{\frac{x}{\alpha}}.$$

Thus, it is enough to show that $Z_t = \sup_{t \le \hat{T}} X_t$ satisfies $\mathbb{E}\left[e^{\lambda Z_t}\right] < \infty$ for all positive $\lambda > 0$. This always holds true for Brownian motion and more generally for Lévy processes with bounded positive jumps (see, e.g., Bertoin (1996), Chapter VII). □

Given these two assumptions, we are now ready to state and discuss the general existence and uniqueness theorem.

Theorem 2.6 *Under Assumption 2.4, there always exists a unique optimal irreversible investment plan* I^**.*

The proof of Theorem 2.6 is given in full detail in Appendix A.1. To briefly sketch the idea: We have a maximization problem of a concave functional (2). In this case, uniqueness is trivial due to the strict concavity of the objective function; and existence is usually achieved by continuity and some compactness (subsequence) principle. Given our assumptions, continuity of the profit functional is obtained through dominated convergence. We then restrict our attention only to the investment policies whose corresponding capacity stays below the overall maximal capacity under perfect reversibility. Nevertheless, this restriction does not exclude any promising policies as the firm in general would like to invest less under irreversibility. By Assumption 2.4 (*ii*), an integrable upper bound is achieved for all investment policies.

It allows one to use *Komlos' theorem* which seems not to be widely used in Economics although it is as Taylor–made for many optimization problems. Komlos' Theorem states that a sequence of random variables $(Z_n)_{n\in\mathbb{N}}$ which is upper–bounded in expectation has a subsequence (ζ_n) converging in the sense of the Law of Large Numbers to some random variable ξ. Here, we use a version of Komlos' theorem for increasing processes which is proven in Kabanov (1999) and Balder (1989). The limit identified by Komlos' Theorem turns out to be an optimal policy.

Remark 2.7 *In general, Assumption 2.4 (i) and (ii) are necessary for existence. To see this, consider the case in which the irreversibility constraint is never binding and the firm invests all the time. It is clearly the extreme case where the capacity of the firm comes to the maximum as if the investment is perfectly reversible. In this case, the optimal policy under reversibility also solves the investment problem under irreversibility. Thus, the irreversible problem is well–posed whenever the reversible case is. When there is no depreciation, the overall maximum of capacity is equal to the total investment* $I_{\hat{T}}$. *Then, this policy is admissible if and only if Assumption 2.4 (ii) is satisfied. It is therefore concluded that Assumption 2.4 cannot be weakened in general.*

The impossibility of relaxing these two assumptions can also be verified through explicit examples as follows. Suppose $\pi(x,c) = 2x\sqrt{c}$ *with a constant interest rate* $r > 0$ *and zero depreciation rate, i.e.,* $\delta = 0$. *Moreover,* X_t *is modelled such that* X_t *is a strictly increasing stochastic process (e.g.* $X_t = e^{N_t}$ *where* N_t *is a compound Poisson process with positive jumps). Here, the optimal policy in the reversible case is given by (compare our discussion in Section 3)*

$$X_t \frac{1}{\sqrt{C_t}} = r\,,$$

or

$$I_t = C_t = e^{-2rt} X_t^2 / r^2\,.$$

Hence, the optimal investment policy under reversible investment is strictly increasing.

3 Optimal Irreversible Investment Policies

Having established the existence and uniqueness of optimal policies, we are now going to find their explicit construction. For comparison purposes, the optimal investment rule is first briefly introduced when investment is perfectly reversible. Then, we develop the base capacity rule for the investment problem with complete irreversibility and show how to characterize the *base capacity* from the derived first–order condition. Basically, the optimal policy keeps the actual capacity above the *base capacity* in a minimal way. This defined *base capacity* is finally identified as the unique solution to a backward equation that can be numerically solved by backward induction for any general model setup.

Reversible Investment Policies If investment is perfectly reversible, the firm can adjust capacity by selling and purchasing the capital goods freely at every point of time. As is well known (Jorgenson (1963), see also Arrow (1968)), the optimal investment criterion is to equate the marginal operating profit with the user cost of capital, i.e.,

$$\pi_c\left(X_t, C_t^I\right) = r + \delta\,. \tag{5}$$

The optimal investment plan has a special "myopic" property in the sense that future expected marginal profits do not appear in Equation (5). The firm equates only the immediate marginal operating profit from capacity with the cost of renting a further marginal unit for an infinitesimal period. This cost is given by the interest rate augmented by the cost of replacing the depreciated amount of capacity. However, this does not mean that the firm is myopic. The optimal plan does not consider future marginal profits since the firm can resize its capacity in any desired way by purchasing or selling the capital.

Once investment is irreversible, the optimal investment plan is no longer of myopic nature as today's investment cannot be abandoned later on. Marginal operating profit from investment is therefore going to be a functional of all future marginal operating profits created by today's investment. To keep the notation simple, the interest and discount rate are assumed from now on to be constant. Nevertheless, the argument is valid for stochastic interest and discount rates as well.

Necessary Optimality Conditions under Irreversibility Before constructing the optimal policy, it is worth to note that some degree of integrability has to be imposed on the process X. Meanwhile, the following inequality is assumed to be true for all values $L > 0$

$$\mathbb{E}\int_0^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, Le^{-\delta s}\right) ds < \infty\,.$$

At any time, installation of any infinitesimal unit of capital will create a stream of marginal profits. At optimum, the marginal operating profit from investing has to be lower than or equal to the cost of investing. The investment cost at t discounted back to the initial time is e^{-rt}. Denote the marginal operating profit at time t following the investment plan I after discounting by $MO_t(I)$. Then, the necessary optimality conditions are given by

$$MO_t(I) \leq e^{-rt} \qquad \text{for all times } t \leq \hat{T} \tag{6}$$

and

$$MO_t(I) = e^{-rt} \qquad \text{whenever } dI_t > 0. \tag{7}$$

Conditions (6) and (7) can also be interpreted as the Kuhn-Tucker conditions for the optimality problem (2) with an inequality constraint $dI_t \geq 0$.

Marginal Operating Profit The marginal investment at time t first induces an immediate marginal gain of $\pi_c\left(X_t, C_t^I\right)$. As capital accumulation is irreversible, all future profits are increased marginally by

$$\pi_c\left(X_s, C_s^I\right) e^{-\delta(s-t)} \quad \forall\, s \in [t, \hat{T}]\,,$$

where the discount factor $e^{-\delta(s-t)}$ is due to the depreciation of current capital stocks[6]. This marginal gain has to be discounted by the interest rate as well to the initial date 0. Overall, the expected marginal operating profit conditional on the information at time t is given by

$$\begin{aligned} MO_t(I) &= \mathbb{E}\left[\int_t^{\hat{T}} e^{-rs}\pi_c\left(X_s, C_s^I\right) e^{-\delta(s-t)} ds \,\middle|\, \mathcal{F}_t\right] \\ &= e^{\delta t}\mathbb{E}\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{8}$$

[6]In the case of reversible investment, there is no such effect on future profits because earlier investments can be withdrawn at any time. Thus, it is sufficient to consider the marginal gain at present time t only.

Remark 3.1 *1.* (8) *is used by Bertola (1998) to check the optimality of certain policies. The heuristics that lead to* (8) *can also be made rigorous, see e.g. Duffie and Skiadas (1994) or Bank and Riedel (2001b) in the context of intertemporal utility maximization.*

2. Assume for the moment that the firm is infinitely lived with $\hat{T} = \infty$. The first–order condition can be reformulated as

$$\mathbb{E}\left[\int_t^\infty e^{-(r+\delta)s}\big[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\big]ds \,\middle|\, \mathcal{F}_t\right] \leq 0\,,$$

after multiplying $e^{-\delta t}$ at the both sides of Inequality (6) *and rewriting $e^{-(r+\delta)t} = \int_t^\infty (r+\delta)e^{-(r+\delta)s}ds$. In the reversible case, the integrand at the left–hand side is always equal to zero, as the marginal operating profit is always equal to the user cost of capital, $r+\delta$. In the irreversible case, the firm however aims to achieve the equality of the marginal operating profit and the user cost of capital only* in expectation on average in time*. The inequality becomes strict when capacity is excess at time t.*

The Base Capacity Generally, the first–order condition is not that helpful for finding the solution as it is not binding so frequently. Nevertheless, it is of great use for constructing a *base capacity* $(l_t)_{t\in[0,\hat{T}]}$, the capacity level that a firm would choose if it were about to start operating at time t regardless of the past capacity. In the following, we aim to show that the optimal policy is to keep the capacity above the base capacity in a minimal way. As a result, the firm does not invest if current capacity is above the base capacity; and does invest up to the base capacity level if current capacity is below the base capacity.[7]

Suppose that the firm follows the optimal investment plan: invest at some (random stopping) time τ_0, wait for a while till $\tau_1 > \tau_0$ and invest again. In this case, the first–order condition is binding at both times, namely,

$$MO_{\tau_0}(I) = e^{-r\tau_0} \text{ and } MO_{\tau_1}(I) = e^{-r\tau_1}\,.$$

[7]Such a policy is well known in operations research, especially inventory theory, see Porteus (1990) for instance.

Multiplying both equations with $e^{-\delta\tau_i}$, $i = 0, 1$, respectively and subtracting them from each another yields

$$\mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = \mathbb{E}\left[e^{-(r+\delta)\tau_0} - e^{-(r+\delta)\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right] ,$$

where the conditional expectation is taken with respect to the information available at time τ_0. The conditional expectation appears also at the right–hand side because τ_1 is generally random. Upon realizing that the difference on the right–hand side can be written as $\int_{\tau_0}^{\tau_1}(r+\delta)e^{-(r+\delta)s}ds$, one arrives at

$$\mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = 0\,.$$

As no investment occurs between τ_0 and τ_1, the capacity starts at some level L at time τ_0 and depreciates at the rate δ, i.e.,

$$C_s^I = Le^{-\delta(s-\tau_0)}$$

for $s \in (\tau_0, \tau_1)$. By plugging this back into the equation above one arrives at

$$\mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\left[\pi_c\left(X_s, Le^{-\delta(s-\tau_0)}\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = 0\,. \tag{9}$$

This equation has a unique solution $L_{\tau_0}^{\tau_1}$, a $\mathcal{F}_{\tau_0}$–measurable random variable[8].

The level $L_{\tau_0}^{\tau_1}$ will be the optimal capacity at time τ_0 if a *blocked interval*[9] starts at time τ_0. In general, the firm asks at time τ_0: when and how much should be invested (marginally or in lumps) next time? Taking the whole variety of possible levels $\left(L_{\tau_0}^{\tau_1}\right)_{\tau_1>\tau_0}$ and the irreversibility constraint into consideration, the *lowest* level is chosen as it is most favorable in the sense that it gives the maximal flexibility for future decisions. As a result,

$$l_{\tau_0} = \operatorname*{ess\,inf}_{\tau_1>\tau_0} L_{\tau_0}^{\tau_1} \tag{10}$$

is defined as the *base capacity*.

[8]The derivation given here is heuristic. Thus, the proof is not provided for the uniqueness of the solution to this implicit equation. This argument can be made rigorous however by considering that the marginal operating profit π_c is strictly decreasing to 0 in capacity.

[9]Please refer to the full discussion in Section 4.

Remark 3.2 *One might wonder why the firm would like to take the smallest of all auxiliary levels L_t^τ. The reasoning is given in the following way. Suppose that current capacity exceeds some $L_{\tau_0}^{\tau_1}$ and assume $\delta = 0$ for simplicity. From irreversibility, it is clear that $C_s > L_{\tau_0}^{\tau_1}$ for all times $s \in (\tau_0, \tau_1)$. By the definition of $L_{\tau_0}^{\tau_1}$, one obtains*

$$\begin{aligned}\mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] &< \mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-rs}\pi_c\left(X_s, L_{\tau_0}^{\tau_1}\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] \\ &= \mathbb{E}\left[e^{-r\tau_0} - e^{-r\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right] .\end{aligned}$$

It follows that

$$\begin{aligned}MO_{\tau_0}(I) &= \mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] + \mathbb{E}\left[\int_{\tau_1}^{\hat{T}} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] \\ &< \mathbb{E}\left[e^{-r\tau_0} - e^{-r\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right] + \mathbb{E}\left[\int_{\tau_1}^{\hat{T}} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] \\ &= \mathbb{E}\left[e^{-r\tau_0} - e^{-r\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right] + MO_{\tau_1}(I) \leq e^{-r\tau_0} ,\end{aligned}$$

where the first–order constraint is used in the last line. Thus, the necessary condition for investment at time τ_0 is that the current capacity has to be always less than or equal to all levels $L_{\tau_0}^{\tau_1}$ for $\tau_1 > \tau_0$, justifying the infimum in our definition of the base capacity.

Characterization of the Optimal Investment Policy: Tracking the Base Capacity Generally, the base capacity l is a widely fluctuating stochastic process. Irreversibility prevents the firm from exactly matching the base capacity at all times, e.g., when downward jumps occur or when the base capacity decreases at a higher rate than δ or when the base capacity decreases in a non–differentiable way as is typical for diffusion models. Therefore, a feasible capacity process C_t has to be found out that tracks the base capacity as closely as possible. According to the base capacity policy, $C_t \geq l_t$ has to hold in a minimal way at all times. Consequently, the correct means is to look for the smallest feasible capacity that dominates the base capacity.

If there is no depreciation, i.e., $\delta = 0$, C must be a nondecreasing process. That is, $C_t \geq C_s$ for $t > s$. Meanwhile, in accordance with the requirement

$C_s \geq l_s$, $C_t \geq l_s$ always holds for $s \leq t$, or equivalently,

$$C_t \geq \sup_{s \leq t} l_s \,.$$

Being the running maximum of the base capacities, $\sup_{s\leq t} l_s$ *is* surely a nondecreasing process, and hence can be a feasible capacity. Therefore, the running maximum

$$C_t = \sup_{s \leq t} l_s$$

is the smallest feasible capacity that dominates the base capacity. For the general case ($\delta > 0$), it is better to study the nondecreasing process $A_t = C_t e^{\delta t}$. By the same reasoning as above, one shows that A has to satisfy the relationship

$$A_t = \sup_{s \leq t} \left(l_s e^{\delta s} \right) \,. \tag{11}$$

The feasible capacity becomes then

$$C_t = e^{-\delta t} \sup_{s \leq t} \left(l_s e^{\delta s} \right) \,.$$

In the case of no depreciation, the corresponding investment plan is trivially obtained as $C^I = I$. In general, one can derive the investment plan from Equation (1), namely, $dI_t = dC_t^I + \delta C_t^I dt$. All these findings are summarized in the following definition.

Definition 3.3 *For a given optional process l and depreciation rate $\delta \geq 0$,*

$$C_t^{l,\delta} = e^{-\delta t} \sup_{s \leq t} \left(l_s e^{\delta s} \right) \tag{12}$$

is the capacity that tracks l at depreciation rate δ. *The investment plan that finances $C^{l,\delta}$ is denoted by $I^{l,\delta}$ and satisfies*

$$I_0^{l,\delta} = l_0 \quad \text{and} \quad dI_t^{l,\delta} = dC_t^{l,\delta} + \delta C_t^{l,\delta} dt \,.$$

If l is the base capacity as defined in Equation (10), *we call $I^{l,\delta}$ the* base capacity policy with depreciation rate δ.

Remark 3.4 *Note that the capacity that tracks the base capacity satisfies*

$$dC_t^{l,\delta} = -\delta C_t^{l,\delta} dt + e^{-\delta t} dA_t^{l,\delta} \,,$$

where $A^{l,\delta}$ is given by (11). *It follows that*

$$dI_t^{l,\delta} = e^{-\delta t} dA_t^{l,\delta} \,. \tag{13}$$

As a result, investment takes place if and only if the process $A^{l,\delta}$ increases; this in turn happens whenever the process $(l_s e^{\delta s})$ reaches a new all time high.

Stochastic Backward Equation and Optimality of the Base Capacity Policy It remains to be shown that the constructed base capacity policy is indeed optimal. To this end, an equation is achieved to determine the base capacity via backward induction. This equation is very similar to the first–order condition, but has the advantage of being an equality at all times almost surely. It is thus extremely useful for explicit computations and for qualitative assessments in subsequent sections.

The capacity $C_s^{l,\delta}$ at time $s > \tau$ created by the base capacity policy can be rewritten as

$$C_s^{l,\delta} = e^{-\delta s} \sup_{0 \le u \le s} l_u e^{\delta u} = e^{-\delta s} \max \left\{ \sup_{0 \le u \le \tau} l_u e^{\delta u}, \sup_{\tau \le u \le s} l_u e^{\delta u} \right\} .$$

Plugging it into the first–order inequality yields then

$$\mathbb{E}\left[\int_\tau^{\hat{T}} e^{-(r+\delta)s} \pi_c \left(X_s, e^{-\delta s} \max \left\{ \sup_{0 \le u \le \tau} l_u e^{\delta u}, \sup_{\tau \le u \le s} l_u e^{\delta u} \right\} \right) ds \,\middle|\, \mathcal{F}_\tau \right] \le e^{-(r+\delta)\tau} .$$

It is a strict inequality if we have excess capacity from the past. However, whenever the past capacity is ignored which is expressed exactly by the term $\sup_{0 \le u \le \tau} l_u e^{\delta u}$, it turns out to be an equality. Indeed, this equation is the first–order condition of a firm that starts at time τ with zero capacity.

Theorem 3.5 (Optimal Investment) *The base capacity policy $I^{l,\delta}$ as defined in Definition 3.3 is optimal. Furthermore, the base capacity l which is defined in* (10) *is obtained as the* unique *solution of the following* modified first–order condition*: for all stopping times $\tau < \hat{T}$*

$$\mathbb{E}\left[\int_\tau^{\hat{T}} e^{-(r+\delta)s} \pi_c \left(X_s, e^{-\delta s} \sup_{\tau \le u \le s} l_u e^{\delta u} \right) ds \,\middle|\, \mathcal{F}_\tau \right] = e^{-(r+\delta)\tau} . \qquad (14)$$

The rigorous proof is given in the Appendix. Clearly, it demonstrates that the base capacity policy is optimal. Meanwhile, it provides also a very useful tool to calculate the optimal policy. As the *unique* solution of the backward equation (14), the base capacity can be calculated numerically via backward induction and hence the optimal investment policy that tracks the running supremum of the base capacity. In this way, the optimal policy is fully characterized by the stochastic backward equation (14).

4 Qualitative Properties of Irreversible Investments

In the analysis of the deterministic case, Arrow (1968) distinguishes between *free* and *blocked* intervals. In free intervals, the irreversibility constraint is not binding and investment occurs at some rate, i.e., we have $dI_t = i_t dt$ for some investment rate $i_t > 0$. In blocked intervals, the firm would like to disinvest in blocked intervals, namely, $dI_t = 0$. Under uncertainty, the diffusion case studied by Bertola (1998), Pindyck (1988) has such blocked intervals as well. However, due to the special nature of diffusions, there exist no free intervals in the sense of Arrow (1968). Whenever investment occurs, it happens in a *singular* way: the set of time points at which investment occurs is of Lebesgue measure zero; hence there is no rate of investment. In general, all three types of investment can occur. In order to fully characterize the qualitative properties of the optimal investment plan, this section carries out a thorough analysis on the irreversible investment under uncertainty and compares the implications to those in Arrow (1968).

Given the general model discussed in the present paper, there exist in all three phenomena in irreversible investment: Every investment plan I can be decomposed into three parts,

$$I = I^a + I^j + I^{\perp},$$

where $I_t^a = \int_0^t i_u^a du$ with $i_u^a > 0$, $\forall\, t \in [0, \hat{T}]$ is the smooth investment with an absolutely continuous plan, $I_t^j = \sum_{n:\tau_n \leq t} \Delta_n$, $\forall\, t \in [0, \hat{T}]$ consists of lump sum investments Δ_n that take place at stopping times $(\tau_n)_{n\geq 0}$, and $I^{\perp}$ describes the singular part of the investment plan.

Free Intervals A random (optional) time interval $[\tau_0, \tau_1]$ is defined as a free interval when smooth investment appears. Throughout the free interval, investment occurs at a strictly positive rate, i.e.,

$$i_u^a > 0, \quad \forall\, u \in (\tau_0, \tau_1).$$

The following theorem generalizes the result of Arrow (1968) to the case of investment under uncertainty: The investment rate in free intervals corresponds to reversible case in the sense of the following theorem.

Theorem 4.1 *In free intervals* $[\tau_0, \tau_1]$*, the marginal operating profit is equal to the user cost of capital, i.e.,*

$$\pi_c\left(X_t, C_t^I\right) = r + \delta \qquad a.s.$$

for all $t \in (\tau_0, \tau_1)$*.*

PROOF: In a free interval where investment occurs continuously, the irreversibility constraint always binds for all $t \in (\tau_0, \tau_1)$ as

$$\mathbb{E}\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds \,\middle|\, \mathcal{F}_t\right] = e^{-(r+\delta)t}.$$

Define

$$H = \int_0^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds$$

and its conditional expectation given the information at time t as the martingale

$$M_t = \mathbb{E}\left[H \,|\, \mathcal{F}_t\right] .$$

We can then rewrite the first–order condition in the free interval as

$$M_t = \int_0^t e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds + e^{-(r+\delta)t} .$$

It follows that M is a martingale with an absolutely continuous sample path. As is well known, such martingales are constant (cf. Protter (1995), Chapter *II*, pp. $64-65$)[10]. Taking derivatives on both sides of the equation yields then

$$\pi_c\left(X_t, C_t^I\right) = r + \delta$$

as desired. □

[10] If a martingale has an absolutely continuous sample path, it must have finite variation. As stated in Protter (1995), a continuous martingale with paths of finite variation is constant.

Blocked Intervals In blocked intervals where no investment occurs, we have initially excess capacity in comparison with the benchmark reversible case. From the derivation of the base capacity, it is obvious that in blocked intervals the firm tries to equate the marginal operating profit and the user cost of capital in expectation on average over time:

Theorem 4.2 *In blocked intervals, the marginal operating profit equals the user cost of capital on average in expectation, formally*

$$\mathbb{E}\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right)-(r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_0}\right]=0\,.$$

The Set of Singular Investment If uncertainty is generated by a diffusion, singular investment will be generally encountered. Let $S=\{(\omega,t): dI_t^{\perp}(\omega)>0\}$ be the support of the random measure $I^{\perp}$. As noted above, this set has, by definition of the singular part $I^{\perp}$, Lebesgue measure zero. The following theorem is a direct consequence of the first–order condition.

Theorem 4.3 *On the support of the singular investment part $I^{\perp}$, the first–order condition is binding as*

$$\mathbb{E}\left[\int_{t}^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds \,\middle|\, \mathcal{F}_t\right]=e^{-(r+\delta)t} \quad dI^{\perp}-a.s.$$

Remark 4.4 *The great difference between singular and smooth investments lies in the fact that singular investment occurs not in an absolute way at a measurable rate. This is due to the nature of Brownian motions and diffusions. Generally speaking, if one wants to keep a Brownian motion below some boundary, actions have to be taken at very irregular steps.*

Lumpy Investment: Stopping Times and Information Surprises Arrow has shown in the deterministic model that lump sum investments do not occur except at time zero. The same holds true in the Brownian motion case analyzed by Bertola (1998) and Pindyck (1988). In general, jumps are however quite possible, e.g. when a Poisson–like jump occurs. To fully describe the jump effect of the investment, we assume that in the remainder of this section X is a (right–continuous) semimartingale and introduce the concept of information surprises, following in spirit Hindy and Huang (1992).

An information surprise occurs at the stopping time τ if τ is either an unpredictable stopping time or a predictable time at which the shock process X has an unpredictable jump, i.e. $\Delta X_\tau = X_\tau - X_{\tau-} \neq 0$ is not $\mathcal{F}_{\tau-}$–measurable. That is, there is some information shock even if the time of a certain event is known or predictable.

First, in contrast to the deterministic case, it is typically *not* the case that the firm equates the marginal operating profit with the user cost of capital when it invests in a lumpy fashion. Instead, uncertainty usually leads to a lower capacity as shown in the following theorem and example (Please compare it with Example 6.5).

Theorem 4.5 *Suppose that the optimal investment plan has a jump at the stopping time τ. Then we have*

$$\pi_c\left(X_\tau, C^I_\tau\right) \geq r + \delta\,.$$

In words: the capacity never jumps to an excess capacity (where the excess capacity is defined with respect to the operating profit).

Proof: Let τ be a stopping time with $\Delta I_\tau > 0$. For shorter notation, denote the difference of the marginal operating profit and the user cost of capital by $\zeta_t = \pi_c\left(X_t, C^I_t\right) - (r+\delta)$. In this way, it is only necessary to show $\zeta_\tau \geq 0$. Fix $\varepsilon \geq 0$. Let $\rho = \inf\{t \geq \tau : \zeta_t \geq -\varepsilon\}$ be the first time when ζ is greater than or equal to $-\varepsilon$. The first–order conditions, $MO_\tau = e^{-r\tau}$ and $MO_\rho \leq e^{-r\rho}$, are equivalent to

$$\mathbb{E}\left[\int_\tau^{\hat{T}} e^{-(r+\delta)s}\zeta_s ds \,\middle|\, \mathcal{F}_\tau\right] = e^{-(r+\delta)\hat{T}}\,,$$

$$\mathbb{E}\left[\int_\rho^{\hat{T}} e^{-(r+\delta)s}\zeta_s ds \,\middle|\, \mathcal{F}_\rho\right] \leq e^{-(r+\delta)\hat{T}}\,.$$

We obtain by taking the conditional expectation at time τ of their differences

$$0 \leq \mathbb{E}\left[\int_\tau^{\rho} e^{-(r+\delta)s}\zeta_s ds \,\middle|\, \mathcal{F}_\tau\right]\,.$$

By the definition of ρ, it follows that

$$0 \leq -\varepsilon\,\mathbb{E}\left[\int_\tau^{\rho} e^{-(r+\delta)s} ds \,\middle|\, \mathcal{F}_\tau\right]\,.$$

This is only possible when $\rho = \tau$ almost surely as $\rho \geq \tau$ as defined. Therefore, we have (from right–continuity of X and C^I) $\zeta_\tau \geq -\varepsilon$. As ε is arbitrary, $\zeta_\tau \geq 0$ follows. □

Example 4.6 *Consider a simple infinite horizon model in which there is only one shock taking place at time* 1*. Formally,* $X_t = 1$ *for* $0 \leq t < 1$*. At time* 1*, the shock jumps to either a good or a bad state with the same probability, i.e.,* $\mathbb{P}[X_1 = \xi] = \mathbb{P}[X_1 = \eta] = 1/2$ *for* $\xi > 1 > \eta > 0$*. After time* 1*,* X *stays constant, that is,* $X_t = X_1$ *for* $t > 1$*. Let* $(\mathcal{F}_t)_{t\in[0,\infty)}$ *be the filtration generated by* X*. The profit function is separable in the form of* $\pi(x, c) = x\pi(c)$ *for some nice function* π*. In addition, there is no depreciation,* $\delta = 0$*.*

It is easy to check that the following investment policy is optimal. The optimal base capacity at time 1 *satisfies* $X_1\pi'(l_1) = r$ *and stays constant afterwards, namely,* $l_t = l_1$ *for* $t \geq 1$*. Let* a *and* b *be the optimal base capacities after the good and bad shock, respectively. Then, we have* $\xi\pi'(a) = r$ *and* $\eta\pi'(b) = r$ *with* $a > b$*. Between time* 0 *and* 1*,* l *stays again constant after time* 0*,* $l_t = l_0 \ \forall\, t \in [0, 1)$*. Intuitively, it is due to the fact that no new information is released during that interval.*

At time 0*, the optimal level* l_0 *lies between* a *and* b *and gives the first–order condition in equality*

$$1 = \mathbb{E}\int_0^\infty e^{-rs} X_s \pi'(\sup_{u\leq s} l_u)\, ds\,.$$

After time 1*, the capacity stays constant at* l_0 *all the time afterwards with probability* $1/2$*. Otherwise, it jumps to* a *at time* 1 *when a good shock occurs. It gives then*

$$1 = \frac{1}{r}\pi'(l_0)(1 - e^{-r}) + \frac{1}{2r}\left[\xi\pi'(a) + \eta\pi'(l_0)\right] e^{-r}$$

or equivalently,

$$\pi'(l_0) = r\frac{1 - \frac{1}{2}e^{-r}}{1 - (1 - \frac{1}{2}\eta)e^{-r}}\,.$$

As $0 < \eta < 1$*, it is clear that* $\frac{1-\frac{1}{2}e^{-r}}{1-(1-\frac{1}{2}\eta)e^{-r}} > 1$ *and hence*

$$\pi'(l_0) > r\,.$$

The next theorem shows that the occurrence of lumpy investment is closely related to information surprises.

Theorem 4.7 *After the initial investment, jumps in investment occur only at information surprises.*

PROOF: Here we will show that it is never optimal to have a jump in the investment except at information surprises. Let $\mathcal{F}$ be the σ-field generated by X. Suppose that there is no information surprise at the optimal stopping time τ_0. That is, τ_0 is predictable and hence there exists a sequence of optional times $\{\tau_n\}$ such that $\tau_n < \tau_{n+1} < \tau_0$ and $\tau_n \nearrow \tau_0$ a.s. In addition, the filtration is continuous at τ_0, i.e., $\mathcal{F}_{\tau_0} = \vee_n \mathcal{F}_{\tau_n}$.

Furthermore as defined in Section 2, $(I_t)_{t\in[0,\hat{T}]}$ is a right–continuous and nondecreasing adapted process. Now assume that I_t has a jump at the stopping time $\tau_0 \in [0, \hat{T}]$. This implies that

$$I_{\tau_n} < I_{\tau_0} \qquad \forall\, n = 1, 2 \cdots ,$$

and that the corresponding capacity levels satisfy the following inequality

$$C^I_{\tau_n} < C^I_{\tau_0} \qquad \text{for large n.}$$

Since investment occurs at time τ_0, the irreversibility condition is binding at that time point

$$\mathbb{E}\left[\int_{\tau_0}^{\hat{T}} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C^I_s\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = 0\,. \tag{15}$$

Meanwhile, it is always valid that for any time τ_n

$$\mathbb{E}\left[\int_{\tau_n}^{\hat{T}} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C^I_s\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_n}\right] \le 0\,. \tag{16}$$

However, a contradictory result is obtained after reformulating the first–

order condition at τ_n

$$\begin{aligned} & \mathbb{E}\left[\int_{\tau_n}^{\hat{T}} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_n}\right] \\ = \;& \mathbb{E}\left[\int_{\tau_n}^{\tau_0} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_n}\right] \\ & + \mathbb{E}\left[\int_{\tau_0}^{\hat{T}} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_n}\right] \\ = \;& \mathbb{E}\left[\int_{\tau_n}^{\tau_0} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_n}\right], \end{aligned}$$

where the last equality is obtained as the second term is shown to be zero by taking conditional expectation of (15) given the information available up to τ_n.

Based on the fact that $\pi_c(x, c)$ is decreasing in c, one can easily get $\pi_c(X_{\tau_0}, C_{\tau_0}^I) < \pi_c(X_{\tau_n}, C_{\tau_n}^I)$ for large n if $X_{\tau_n} \approx X_{\tau_0}$ a.s. which holds due to the fact that there is no (X-generated) information surprise at τ_0[11]. It hence leads to

$$(r+\delta) \leq \pi_c(X_{\tau_0}, C_{\tau_0}^I) < \pi_c(X_{\tau_n}, C_{\tau_n}^I)$$

for large n by combining the statement in Theorem 4.5. Clearly, it gives the result

$$\mathbb{E}\left[\int_{\tau_n}^{\hat{T}} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_n}\right] > 0,$$

which contradicts the assertion (16). Thus, jumps in investment appear only at information surprises after the initial time. □

As one special case, the irreversible investment under certainty studied in Arrow (1968) possesses complete information set during the whole investment plan. As a result, lumpy investment takes place only at the initial time:

[11] This condition implies that X is continuous at τ_0. Intuitively, no (X-generated) information surprise means there is no "sudden" and "unexpected" change in X at τ_0. Hence, even if there is a discontinuity (or jump) at that time, one has already full knowledge before and will take forward-looking actions. Suppose for instance that there is an anticipated increase in demand. In this case, investors will be well prepared by investing at τ_n which converges to τ_0 almost surely by taking $X_{\tau_n} = X_{\tau_0}$.

Corollary 4.8 *In Arrows model, jumps occur only at time* 0*.*

In addition, it is worthwhile to point out that there is no information surprise when the information filtration is generated by Brownian motions. Consequently, lumpy investment appears in this case only once at the initial date.

Corollary 4.9 *If the information flow is continuous (as if generated by some Brownian motions), jumps in investment occur only at time* 0*.*

PROOF: First, when the information filtration is generated by Brownian motions, it is well–known that stopping times under this construction are always predictable. Obviously, there is no information surprise in this case since the information flow is always continuous. It proves then the corollary since jumps in investment as stated in Theorem 4.5 occur only at the initial time and at information surprises. □

5 Comparative Statics

An advantage of our approach to irreversible investment is that it easily leads to general monotone comparative statics. We are going to illustrate it in this section with two comparative statics results.

First, it is shown that the base capacity is monotonically increasing in shocks X whenever the operating profit function has increasing differences in shocks and capacity. A function is supermodular or equivalently exhibits increasing differences (see Topkis (1978)) if and only if the twice differentiable function satisfies

$$\frac{\partial^2}{\partial x \partial c}\pi(x,c) \geq 0\,.$$

The general theory by Topkis (1978) and Milgrom and Shannon (1994) suggests that this property is necessary to have monotone comparative statics. As there is no general theory of stochastic dominance for stochastic processes, we order the set of all stochastic processes by the partial order of being greater or equal almost surely everywhere.

Theorem 5.1 *Let X and Y be two progressively measurable stochastic processes with $X_t \geq Y_t$ for all $t \in [0,\hat{T}]$ almost surely. Denote the optimal* *base capacity under X (resp. Y) by l^X (resp. l^Y). Assume that the operating profit function is supermodular. Then the base capacity is monotonically increasing in X, i.e. $l_t^X \geq l_t^Y$ for all $t \in [0, \hat{T}]$ a.s.*

PROOF: As the base capacity level is the essential infimum of all candidates L_t^τ in (9), it is enough to show that the L_t^τ corresponding to the exogenous stock X is larger than that to Y, or equivalently, $L_t^{X,\tau} > L_t^{Y,\tau}$.

L_t^τ is by definition the unique solution of the first–order condition (9). Thus, we get an equality of the two conditions subject to different shocks X and Y as follows:

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c\left(X_s, L_t^{X,\tau}e^{-\delta(s-t)}\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c\left(Y_s, L_t^{Y,\tau}e^{-\delta(s-t)}\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{17}$$

Define a function

$$f(L) = \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\pi_c\left(X_s, Le^{-\delta(s-t)}\right) ds \,\middle|\, \mathcal{F}_t\right]$$

with values in the set of all $\mathcal{F}_t$–measurable random variables. As X dominates Y almost surely and supermodularity of π implies that π_c is increasing in x, one can easily obtain

$$\mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\pi_c\left(X_s, Le^{-\delta(s-t)}\right) ds \,\middle|\, \mathcal{F}_t\right] \geq \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\pi_c\left(Y_s, Le^{-\delta(s-t)}\right) ds \,\middle|\, \mathcal{F}_t\right]$$

for any a stopping time $\tau > t$. Clearly, (17) is valid if and only if $L_t^{X,\tau} > L_t^{Y,\tau}$ since π_c ia decreasing in c and hence the function $f(L)$ in L.

□

Remark 5.2 *An alternative proof via Topkis (1978) is also possible. Moreover, Theorem* 10 *in Milgrom and Shannon (1994) suggests that supermodularity is necessary for this type of monotone comparative statics.*

Remark 5.3 *Having considered a monotone shift in the shock, one would also like to ask what happens if the exogenous shock process becomes more risky. Unfortunately, there is no general theory of second order stochastic dominance for stochastic processes. A natural definition inspired by the* *well–known fact of second order stochastic dominance would be an unanimity principle for expected utility maximizers. Fix a certain discount rate ρ and consider arbitrary risk–averse expected utility maximizers that live from time t to some stopping time $\tau > t$. If all these rational agents would rather consume Y than X, then we call X riskier than Y. Formally, let X and Y be two progressively measurable stochastic processes. X is riskier than Y if*

$$\mathbb{E}\left[\int_t^\tau e^{-\rho s} u(X_s) ds \,\middle|\, \mathcal{F}_t\right] \leq \mathbb{E}\left[\int_t^\tau e^{-\rho s} u(Y_s) ds \,\middle|\, \mathcal{F}_t\right]$$

for all monotone and increasing functions $u : \mathbb{R} \to \mathbb{R}$ and all times $\tau > t$. However, one can show that such a definition boils down to the condition that X dominating Y in an almost sure sense which is exactly the monotonicity condition considered in Theorem 5.1.

We conclude this section by establishing a plausible result that the firm size is decreasing in the user cost of capital.

Theorem 5.4 *The base capacity is decreasing in the user cost of capital $r+\delta$.*

The complete proof is provided in Appendix C. Basically, one applies the implicit function theorem to the definition of the auxiliary variables L_t^τ. However, it has to be used carefully with some change in our case since the function at hand is a conditional expectation.

6 Explicit Solutions for a Lévy Shock and a Cobb-Douglas Operating Profit Function

Generally, numerical methods have to be adopted to identify the solutions. Nevertheless, a closed-form solution can be obtained for an infinitely-lived firm ($\hat{T} = \infty$) when the multiplicative economic shock is characterized by an exponential Lévy process and the firm is endowed with the operating profit function of the form

$$\pi(X_t, C_t) = \frac{1}{1-\alpha} X_t^\alpha C_t^{1-\alpha}, \quad 0 < \alpha < 1. \tag{18}$$

This construction is consistent with a competitive firm who produces at decreasing returns to scale or with a monopolist firm facing with a constant elasticity demand function and constant returns to scale production (as shown by Abel and Eberly (1996) and Morellec (2001)). Clearly, this function is concave with the first derivative $\pi_C = X_t^\alpha C_t^{-\alpha}$. In particular, the economic shock X_t is modelled by

$$X_t = x_0 e^{Y_t},$$

where x_0 is the initial value at $t = 0$ and Y_t is a Lévy process with zero initial value. Moreover, the interest and discount rate are assumed to be constant over time.

Computation of the Base Capacity As introduced in Section 3, the irreversible investment decision problem is solved by calculating the first–order condition and solving the achieved backward equation (14). Here, it is reduced to

$$\mathbb{E}\left[\int_\tau^\infty e^{-(r+\delta)s} X_s^\alpha \left(e^{-\delta s} \sup_{\tau \le u \le s} l_u e^{\delta u}\right)^{-\alpha} ds \,\middle|\, \mathcal{F}_\tau\right] = e^{-(r+\delta)\tau}, \qquad (19)$$

which can be explicitly solved by means of the strong Markov property and time homogeneity of Lévy processes, as given in the following theorem.

Theorem 6.1 *When the production function is of form* (18) *and the exogenous economic shock is characterized by an exponential Lévy process, the base capacity is identified as* $l_t = \kappa X_t$ *with*

$$\kappa = \left(\frac{1}{r+\delta}\mathbb{E}\left[e^{\alpha \underline{G}_{\tau(r+\delta)}}\right]\right)^{\frac{1}{\alpha}}, \qquad (20)$$

where $G_t = Y_t + \delta t$, $\underline{G}_t$ *is defined as* $\underline{G}_t = \inf_{0 \le u \le t} G_u$ *and* $\tau(r+\delta)$ *is an independent exponential distributed time with parameter* $r + \delta$*.*

PROOF: To prove it, construct first the base capacity in form of $l_u = \kappa X_u$. Then the left–hand side of Equation (19) can be reduced into

$$
\begin{aligned}
& \mathbb{E}\left[\int_\tau^\infty e^{-(r+\delta)s} X_s^\alpha \left(e^{-\delta s}\sup_{\tau\le u\le s} l_u e^{\delta u}\right)^{-\alpha} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= & \mathbb{E}\left[\int_\tau^\infty e^{-(r+\delta)s} X_s^\alpha \inf_{\tau\le u\le s} (\kappa X_u)^{-\alpha} e^{-\alpha\delta(u-s)} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= & \mathbb{E}\left[\int_\tau^\infty e^{-(r+\delta)s} \kappa^{-\alpha} \inf_{\tau\le u\le s} \left(\frac{x_0 e^{Y_s}}{x_0 e^{Y_u}}\right)^\alpha e^{-\alpha\delta(u-s)} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= & \mathbb{E}\left[\int_\tau^\infty e^{-(r+\delta)s} \kappa^{-\alpha} \inf_{\tau\le u\le s} e^{\alpha[(Y_s-Y_u)+\delta(s-u)]} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= & \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)(t+\tau)} \kappa^{-\alpha} \inf_{0\le u\le t} e^{\alpha[(Y_{t+\tau}-Y_{u+\tau})+\delta(t-u)]} dt \,\middle|\, \mathcal{F}_\tau\right],
\end{aligned}
$$

where the last step is obtained by assuming $t = s - \tau$.

It can be further simplified by the property of Lévy processes that $Y_t - Y_u$ has the same distribution as Y_{t-u} and is independent of the $\sigma-$Field $\mathcal{F}_u$.

$$
\begin{aligned}
& E\left[\int_0^\infty e^{-(r+\delta)(t+\tau)} \kappa^{-\alpha} \inf_{0\le u\le t} e^{\alpha[(Y_t-Y_u)+\delta(t-u)]} dt\right] \\
= & e^{-(r+\delta)\tau} \kappa^{-\alpha} E\left[\int_0^\infty e^{-(r+\delta)t} \inf_{0\le u\le t} e^{\alpha(G_t-G_u)} dt\right],
\end{aligned}
$$

where $G_t = Y_t + \delta t$ is clearly also a Lévy process.

In this way, κ is obtained as follows by defining $\overline{G}_t = \sup_{0\le u\le t} G_u$ and $\underline{G}_t = \inf_{0\le u\le t} G_u$

$$
\begin{aligned}
\kappa & = \left(\mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t} \inf_{0\le u\le t} e^{\alpha(G_t-G_u)} dt\right]\right)^{\frac{1}{\alpha}} \\
& = \left(\mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t} e^{\alpha(G_t-\overline{G}_t)} dt\right]\right)^{\frac{1}{\alpha}} \\
& = \left(\frac{1}{r+\delta}\mathbb{E}\left[e^{\alpha(G_{\tau(r+\delta)}-\overline{G}_{\tau(r+\delta)})}\right]\right)^{\frac{1}{\alpha}} \\
& = \left(\frac{1}{r+\delta}\mathbb{E}\left[e^{\alpha\underline{G}_{\tau(r+\delta)}}\right]\right)^{\frac{1}{\alpha}},
\end{aligned}
$$

where $\tau(r+\delta)$ is an independent exponential distributed time with parameter $r+\delta$ and the last equality is achieved by the duality theorem that $G_t - \overline{G}_t$ has the same distribution as $\underline{G}_t$ (see Bertoin (1996)). □

Remark 6.2 *According to the optimal investment policy, it is always maintained that $C_t \geq l_t$ at all time $t \in [0, \hat{T}]$. With the derived solution $l_t = \kappa X_t$, one can easily obtain*

$$\pi_c \;=\; X_t^{\alpha} C_t^{-\alpha} \leq X_t^{\alpha} (\kappa X_t)^{-\alpha} = \kappa^{-\alpha},$$

where $\kappa^{-\alpha} = (r+\delta)/\mathbb{E}\left[e^{\alpha \underline{G}_{\tau(r+\delta)}}\right]$. Obviously, the expectation term is valued only in $(0,1]$. It follows thus that the marginal operating profit under the optimal investment plan is always kept below the user cost of capital times a markup factor.

Computation of the Firm's Overall Profit and Well-Posedness of the Problem The preceding theorem obtains a solution of the stochastic backward equation for all Lévy processes. Taking it as a candidate for the optimal policy, we then have to check for optimality that it gives an admissible investment and that the resulted firm's value is finite. In infinite horizon models, this usually requires a constraint on the interest rate and on the growth rate of X. Here, only one condition is already sufficient as stated in the theorem below.

Theorem 6.3 *Assume that $r + \delta > \Psi(1)$ where $\Psi(1)$ is the Lévy–Laplace exponent of G defined by $\Psi(1) = \log E[e^{G_1}]$. Then the base capacity policy that keeps the capacity just above the base capacity $l_t = \kappa X_t$ is optimal. The overall profit of the firm is given by*

$$\Pi(I^*) = \frac{\alpha}{1-\alpha} \frac{x_0}{(r+\delta)^{\frac{1}{\alpha}}} \mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] \left(\mathbb{E}\left[e^{\alpha \underline{G}_{\tau(r+\delta)}}\right]\right)^{\frac{1}{\alpha}} .$$

Proof: Given the base capacity $l_t = \kappa X_t$, the optimal capacity follows then

$$C_t^{l,\delta} = e^{-\delta t} \sup_{s \leq t} l_s e^{\delta s} = x_0 \kappa e^{-\delta t} \sup_{s \leq t} e^{G_s} .$$

$I^{l,\delta}$ is admissible if and only if $\mathbb{E}\left[\int_0^\infty e^{-rt} dI_t^{l,\delta}\right] < \infty$. Expanding it by $dI_t^{l,\delta} = dC_t^{l,\delta} + \delta C_t^{l,\delta} dt$ and taking integration by parts yields

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-rt} dI_t^{l,\delta}\right] &= \mathbb{E}\left[\int_0^\infty e^{-rt}\left(dC_t^{l,\delta} + \delta C_t^{l,\delta} dt\right)\right] \\
&= \mathbb{E}\left[e^{-r\infty} C_\infty^{l,\delta}\right] + \mathbb{E}\left[\int_0^\infty e^{-rt}(r+\delta) C_t^{l,\delta} dt\right] \\
&= x_0\kappa\mathbb{E}\left[e^{-(r+\delta)\infty} \sup_{s\le\infty} e^{G_s}\right] + x_0\kappa\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] \\
&< \infty .
\end{aligned}$$

Hence, the admissibility is guaranteed if

$$\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] < \infty . \tag{21}$$

It is a sufficient condition since it implies $\mathbb{E}\left[e^{-(r+\delta)\infty} \sup_{s\le\infty} e^{G_s}\right] = 0$ a.s.

The Wiener–Hopf factorization tells that (21) holds true if and only if

$$\mathbb{E}\int_0^\infty e^{-(r+\delta)s + G_s} ds = \int_0^\infty e^{[\Psi(1)-(r+\delta)]s} ds < \infty$$

and hence

$$r + \delta > \Psi(1) , \tag{22}$$

where $\Psi(1)$ is the Lévy–Laplace exponent of G defined by $E[e^{G_t}] = e^{t\Psi(1)}$.

In this case, $I^{l,\delta}$ is the optimal investment plan with expected discounted cost at

$$\mathbb{E}\left[\int_0^\infty e^{-rt} dI_t^{l,\delta}\right] = x_0\kappa\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] .$$

Meanwhile, the optimal investment policy generates the overall profit

$$\begin{aligned}
\Pi(I^*) &= \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\frac{1}{1-\alpha} X_t^\alpha C_t^{*\,1-\alpha} dt - dI_t^*\right)\right] \\
&= \frac{x_0\kappa^{1-\alpha}}{(1-\alpha)(r+\delta)}\mathbb{E}\left[e^{\alpha G_{\tau(r+\delta)} + (1-\alpha)\overline{G}_{\tau(r+\delta)}}\right] - \kappa x_0\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] \\
&= \frac{x_0\kappa^{1-\alpha}}{(1-\alpha)(r+\delta)}\mathbb{E}\left[e^{\alpha(G_{\tau(r+\delta)} - \overline{G}_{\tau(r+\delta)})}\right]\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] - \kappa x_0\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] ,
\end{aligned}$$

where the last equality is obtained since $\overline{G}_t$ and $G_t - \overline{G}_t$ are independent by Theorem $VI.5(i)$ in Bertoin (1996).

It can be further simplified due to $\kappa^{-\alpha} = (r+\delta)\left(\mathbb{E}\left[e^{\underline{G}_{\tau(r+\delta)}}\right]\right)^{-1}$ and duality theorem

$$\begin{aligned}\Pi(I^*) &= \kappa\frac{x_0}{1-\alpha}\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] - \kappa x_0 \mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right] \\ &= \kappa x_0 \frac{\alpha}{1-\alpha}\mathbb{E}\left[e^{\overline{G}_{\tau(r+\delta)}}\right].\end{aligned}$$

It is worth to note that (22) is also necessary to achieve the well-posedness of our profit maximization problem. □

Remark 6.4 *For geometric Brownian motions, the irreversible investment problem is well-posed whenever* $r > \mu + \frac{1}{2}\sigma^2$ *where* μ *and* σ *are the constant drift and variance of the geometric Brownian motion* X*. This basically coincides those results in Pindyck (1988) and Bertola (1998). Boyarchenko (2004) derives the result for exponential Lévy processes under the additional restriction that the capacity remains bounded. This assumption is not required in this paper.*

Specific Examples In order to well illustrate this method and the derived *base capacity policy*, two examples are provided based on the specific model setup as follows:

Example 6.5 *As mostly often assumed in the literature,* X *is a geometric Brownian motion, that is,*

$$Y_t = \sigma W_t\,,$$

where W_t *is the standard Brownian motion and the constant volatility* $\sigma = 0.20$*. Additionally, the production parameter is given as* $\alpha = 0.80$*. The constant interest and discount rate are* $r = 8\%$*,* $\delta = 2\%$*, respectively.*

As shown in Figure 1, the base capacity evolves according to a geometric Brownian motion with a continuous path but in nowhere differentiable fashion. Investment is undertaken if and only if the current capacity is discounted or becomes lower than the base capacity. In any case, the optimal capacity is maintained to be equal or higher than the base capacity, although sometimes the firm would like to disinvest which is impossible due to irreversibility of the investment. Consequently, the investment plan in this case consists only of

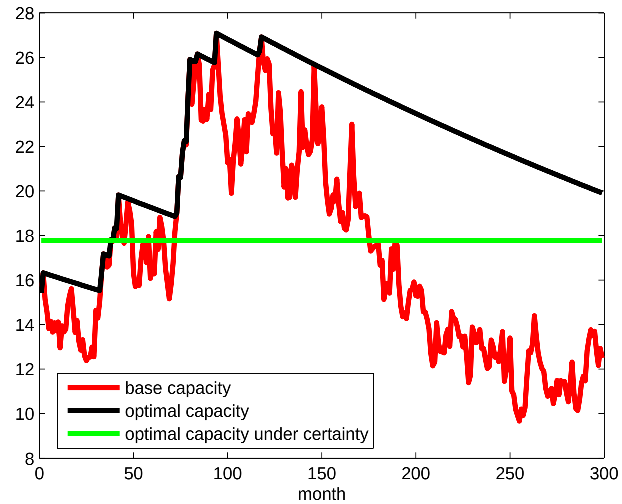

Figure 1: Optimal Capacity Level under Certainty and Uncertainty with Geometric Brownian Motion Modelled Shocks

singular investment and no investment. Jump in investment appears only at the initial time. Moreover, the initial jump is below the optimal capacity level under certainty that equals the marginal operating profit with the user cost of capital. Clearly, it coincides with Theorem 4.5 that irreversibility leads to underinvestment.

Example 6.6 *The next example models the economic shock by a Compound Poisson process*

$$Y_t = \mu t + \sum_{n=1}^{N_t} J_n \,,$$

where the drift term is constant with $\mu = 0.05$, $(N_t)_{t\geq 0}$ *is a Poisson process of intensity* $\lambda = 0.05$ *and* $J = (J_n)_{n\in\mathbb{N}}$ *a sequence of independent identically distributed random variables with density*

$$f(j) = \begin{cases} pc^+ e^{-c^+ j} & j \geq 0\,, \\ (1-p)c^- e^{c^- j} & j < 0\,. \end{cases}$$

with $c^+ = 0.10$, $c^- = 0.45$ *and* $p = 0.70$. *Under this assumption, the economic shock at time* t *has in all* N_t *possible upward and downward jumps which occur with probability* 70% *and* 30%, *respectively. Each positive/negative jump is exponentially distributed with parameter* c^+/c^-. *Keep* *all the other model parameters constant as given in Example 6.6. In this way, κ and hence the base capacity can be identified and plotted in Figure* (2)*.*

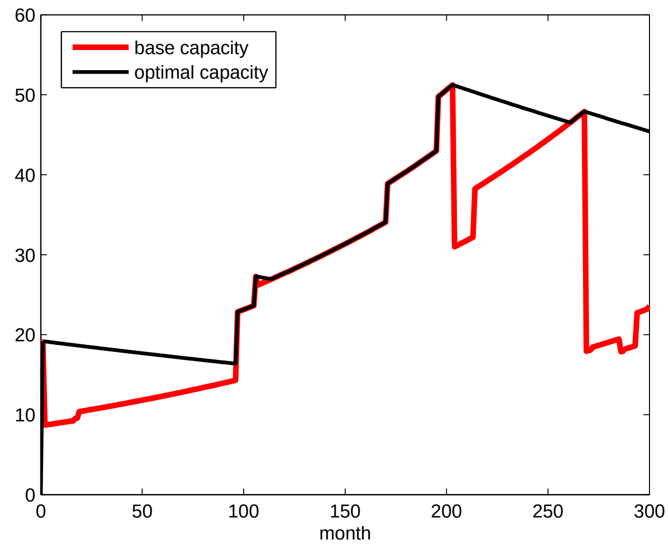

Figure 2: Optimal Capacity Level under Uncertainty with Compound Poisson Process Modelled Shocks

Obviously in this case where there is no Brownian motion term, there exist lump sum and also smooth investment, but no singular investment. Consequently, the whole investment plan can be easily divided into free and blocked intervals after jumps. Meanwhile as well observed, all the jumps in the optimal investment occur only at information surprises, i.e., when X jumps upward.

7 Conclusion

This paper studies sequential irreversible investment decision problems under uncertainty. The same problem is solved in Pindyck (1988) by the standard real options approach. The dynamic capacity choice problem is treated as a sequence of optimal stopping problems. Instead of focusing on how much to invest at each time, he starts from when the infinitesimal stock of capital should be invested. This is exactly the starting point of our method, which is based on Bank and Riedel (2001b) and first applied in this paper to the real options theory, to concern the marginal effect of investment at any given time. Similarly, Bertola (1998) solves the maximization problem (1) by identifying the optimality condition in the sense of marginal effect. On this basis, different techniques are applied to achieve the the optimal threshold investment level. Pindyck (1988) obtains the optimal trigger level of the investment by solving Hamilton-Jacobi-Bellman equations. Sticking to the marginal effect, Bertola (1998) identifies the marginal profit of the investment and solves its stochastic differential equation after assuming that there is a control barrier on the marginal profit. While, the method of this paper considers the marginal investment problem as a singular control problem and characterize the optimal investment policy by constructing and tracking a base capacity and solving our key stochastic backward equation.

This method is advantageous mainly in the following four aspects. First, it applies well to a general model which is free of any distributional and parametric assumptions. General existence and uniqueness theorem is derived for both finite and infinite horizons, which is to our knowledge the first result in the literature. Second, this method incorporates an economic interpretation in the derivation, enabling one to better understand the irreversible investment problem. More importantly, it allows for a general qualitative characterization of the optimal investment. Generally, the investment plan can be characterized by three different phenomena: smooth continuous investment, lump sum investment and singular investment. The marginal operating profit is equal to the user cost of capital only in free intervals where smooth investment occurs at positive rates. While in blocked intervals during which there is no investment, the equality of the marginal profit and the user cost of capital is maintained only in expectation on average over time. Lumpy sum investment is possible only with information surprises. Singular investment takes place in a nowhere differentiable fashion whenever the uncertainty is (partly) modelled diffusions. In addition, this method gives some monotone comparative statics results: When the operating profit function is supermodular, the base capacity increases monotonically with the exogenous shock; and the firm size always declines with the user cost of capital. Finally, explicit solutions is obtained for an infinitely-lived firm where he is endowed with the operating profit function of Cobb–Douglas type and the multiplicative economic shock is modelled by an exponential Lévy process. In this context, the base capacity is identified as the exogenous shock multiplied by a factor κ, which recovers the well-known result in the literature.

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A Proof of the Existence and Uniqueness

A.1 Finite Horizon

Theorem A.1 *Under Assumption 2.4, there always exists a unique optimal investment plan I^*.*

Proof: For simplicity, assume in the proof that the interest and depreciation rate r and δ are positive constants. The argument goes through also in the case of bounded, nonnegative processes with the corresponding and obvious changes, which is easily done without any difficulties but in terms of clumsier formula.

First, uniqueness follows directly from strict concavity as usual. Hence, it is not necessary to be more addressed here. The existence proof is not that trivial and consists of three steps. First, Assumption 2.4 (i) is shown to guarantee the finiteness of $\Pi(I)$. Step 2 demonstrates that one can restrict attention to those investment plans I which lead to the capacities that satisfy $\mathbb{E}\big[C^I_{\hat{T}}\big] \le K$, where the constant K is as defined in Assumption 2.4 (ii). In the third step, a suitable variant of Kilos' Theorem (Komlós (1967), see also Balder (1989) and Kabanov (1999)) is applied to obtain a sequence of investment plans (I^n) that converges in the Cesaro sense almost surely to some investment plan I^*. Concavity of the profit functional ensures the optimality of I^*.

Step 1. From Equation (1), one can write $dI_t = dC^I_t + \delta C^I_t dt$. This yields

$$\int_0^{\hat{T}} e^{-rt} dI_t = \int_0^{\hat{T}} e^{-rt} dC^I_t + \int_0^{\hat{T}} \delta e^{-rt} C^I_t dt\,.$$

Integration by parts gives

$$\int_0^{\hat{T}} e^{-rt} dC^I_t = e^{-r\hat{T}} C^I_{\hat{T}} + \int_0^{\hat{T}} r e^{-rt} C^I_t dt\,,$$

and hence

$$\int_0^{\hat{T}} e^{-rt} dI_t = e^{-r\hat{T}} C^I_{\hat{T}} + \int_0^{\hat{T}} (r+\delta) e^{-rt} C^I_t dt\,.$$

It follows then

$$\begin{aligned}\int_0^{\hat{T}} e^{-rt}\Big(\pi\left(X_t, C_t^I\right)\,dt - dI_t\Big) &\le \int_0^{\hat{T}} e^{-rt}\Big(\pi\left(X_t, C_t^I\right) - (r+\delta)C_t^I\Big)\,dt \\ &\le \int_0^{\hat{T}} e^{-rt}\pi^*(X_t, r, \delta)\,dt\,.\end{aligned}$$

This implies consequently

$$\Pi(I) \le \mathbb{E}\int_0^{\hat{T}} e^{-rt}\pi^*(X(t), r, \delta)dt < \infty\,,$$

and the problem has always a finite value $v^\star = \sup_I \Pi(I) < \infty$.

Step 2. In this step, an investment plan $\hat{I}$ with the corresponding capacity $\hat{C}$ is constructed such that it gives an upper bound for all reasonable plans in the sense that it is not worthwhile to have a higher capacity than $\hat{C}$. The basic idea is that it does not make sense to have a capacity higher than that one would have in the reversible case, c^*. A complication arises from the fact that $c^*\left(X_s, r, \delta\right)$ will generally be a process of unbounded variation and thus may not be a feasible capacity.

The trick here is to construct the investment plan that leads to a capacity $\hat{C} \ge c^*$ in a minimal way. Set

$$\hat{C}_t = e^{-\delta t}\sup_{s\le t}\left(c_s^* e^{\delta s}\right)\,, \tag{23}$$

where the notation is slightly abused by writing

$$c_s^* = c^*\left(X_s, r, \delta\right)\,.$$

Because of Assumption 2.4 and $\delta \ge 0$, $\hat{C}_{\hat{T}}$ is integrable as

$$\mathbb{E}\Big[\hat{C}_{\hat{T}}\Big] = \mathbb{E}\Big[\sup_{s\le\hat{T}} c_s^* e^{-\delta(\hat{T}-s)}\Big] \le \mathbb{E}\Big[\sup_{s\le\hat{T}} c_s^*\Big] < \infty\,,$$

where $\mathbb{E}\Big[\sup_{s\le\hat{T}} c_s^*\Big]$ is obviously equal to K specified in Assumption 2.4 with the deterministic r and δ.

The investment plan

$$\hat{I}_t = \hat{C}_t + \int_0^t \delta \hat{C}_s ds \tag{24}$$

is the feasible plan that leads to the capacity $\hat{C}$.

The claim to be demonstrated is that one can restrict attention to plans I with capacity $C^I \leq \hat{C}$. Let I be given and write $C = C^I$ for shorter notation. Define $\bar{C}_t = \min\left\{C_t, \hat{C}_t\right\}$ and $\bar{A}_t = e^{\delta t}\bar{C}_t$. Note that $(A_t)_{t\in[0,\hat{T}]}$ is also nondecreasing as $(C_t)_{t\in[0,\hat{T}]}$. The corresponding investment plan with capacity $C^{\bar{I}} = \bar{C}$ is denoted as $\bar{I}_t = \int_0^t e^{\delta s} d\bar{A}_s$.

Under this construction, the claim is valid if $\bar{I}$ is shown to be at least as good as I. Taking integration by parts again yields

$$\begin{aligned}
\Pi(\bar{I}) - \Pi(I) &= \mathbb{E}\Big[\int_0^{\hat{T}} e^{-rt}\left(\pi\left(X_t, \bar{C}_t\right) - (r+\delta)\bar{C}_t\right) dt\Big] - \mathbb{E}\Big[e^{-r\hat{T}}\bar{C}_{\hat{T}}\Big] \\
&\quad -\mathbb{E}\Big[\int_0^{\hat{T}} e^{-rt}\left(\pi\left(X_t, C_t\right) - (r+\delta)C_t\right) dt\Big] + \mathbb{E}\Big[e^{-r\hat{T}}C_{\hat{T}}\Big] \\
&= \mathbb{E}\Big[\int_0^{\hat{T}} e^{-rt}\left[\left(\pi\left(X_t, \bar{C}_t\right) - (r+\delta)\bar{C}_t\right) - \left(\pi\left(X_t, C_t\right) - (r+\delta)C_t\right)\right] dt\Big] \\
&\quad -\mathbb{E}\Big[e^{-r\hat{T}}\left(\bar{C}_{\hat{T}} - C_{\hat{T}}\right)\Big].
\end{aligned}$$

The last term is nonnegative because $\bar{C} \leq C$. The integrand in the first term is either zero when $\bar{C} = C$; or nonnegative when $\bar{C} < C$. In the second case of $\bar{C} < C$, it is clear that $C_t > \bar{C}_t \geq c_t^*$. As C_t is located at the right of the maximum c^* and the function $c \mapsto \pi(x,c) - (r+\delta)c$ is concave, one can find out that

$$\pi\left(X_t, \bar{C}_t\right) - (r+\delta)\bar{C}_t > \pi\left(X_t, C_t\right) - (r+\delta)C_t\,.$$

These arguments altogether lead to $\Pi(\bar{I}) \geq \Pi(I)$ as desired.

Step 3. In this way, the auxiliary problem

$$\sup_{I\colon C^I \leq \hat{C}} \Pi(I) = v^*$$

has the same value as the original problem. Choose an optimal sequence (I^n) for this auxiliary problem. Its value at time $\hat{T}$ has the following property:

$$I^n_{\hat{T}} = C^{I^n}_{\hat{T}} + \delta \int_0^{\hat{T}} C^{I^n}_s ds \leq (1+\delta\hat{T})C^{I^n}_{\hat{T}} \leq (1+\delta\hat{T})\hat{C}_{\hat{T}}.$$

This suggests that

$$\sup_n \mathbb{E}\big[\, I^n_{\hat{T}}\big] < \infty \,.$$

With this condition, Kilos Theorem (in the variant of Kabanov (1999)) can be thus applied here: assume without loss of generality that (I^n) converges in the Cesaro sense almost surely to some I^*, that is

$$J^n_t \triangleq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I^k_t = I^*_t \qquad a.s.$$

Through linearity, the corresponding capacities C^k converge also in the Cesaro sense almost surely. Moreover, the concavity of the profit function in capacity yields the final result that

$$\Pi\big(I^*\big) \geq \limsup \Pi\left(J^n\right) = v^* \,.$$

Therefore, I^* is the optimal investment plan that maximizes the firm's net profit. □

A.2 Existence for the Infinite Horizon Case

Of course, the naive generalization of Assumption 2.4 with

$$\mathbb{E}\big[\sup_{t<\infty} c^*_t\big] < \infty$$

is sufficient (by repeating the proof above for the finite horizon case). However, it is too strong in the infinite horizon case because the running maximum up to infinity will be in many contexts infinity. Indeed, the following weaker version of Assumption 2.4 is sufficient to guarantee the existence of the optimal sequential investment plan with the infinite horizon.

Assumption A.2 *(i)* $\mathbb{E}\big[\int_0^\infty e^{-\int_0^t r_s ds} \pi^*\left(X_t, r_t, \delta_t\right) dt\big] < \infty \ \forall\, t \in [0, \hat{T}]$.

(ii) $K \triangleq \mathbb{E}\big[\int_0^\infty e^{-\int_0^t r_s ds} d\hat{I}_t\big] < \infty$ *for* $\hat{I}$ *as given by (24).*

With this assumption, one can repeat almost verbatim the proof for the finite horizon case.

Theorem A.3 *Under Assumption A.2, there always exists one unique optimal investment plan* I^* *for the infinite–horizon sequential irreversible investment problem.*

B Proof of Theorem 3.5

Proof: Bank and ElKaroui (2004) perform a detailed analysis of the adjusted first–order equation (14). In particular, they show that the base capacity is the unique progressively measurable process that solves (14) (Theorem 1 and 3 therein). Given that the base capacity solves (14), we check now the first–order conditions (6) and (7). Let I^* denote the investment plan that finances $C^{l,\delta}$. From (8), the gradient at time t is given by

$$MO_t\left(I^*\right) = e^{\delta t}\,\mathbb{E}\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^{l,\delta}\right)ds\,\middle|\,\mathcal{F}_t\right].$$

As $C^{l,\delta}$ tracks the level l, the marginal profit of investment can be written as

$$MO_t\left(I^*\right) = e^{\delta t}\,\mathbb{E}\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{u\le s} l_s e^{\delta s}\right)ds\,\middle|\,\mathcal{F}_t\right].$$

Trivially, we have $\sup_{u\le s} l_s e^{\delta s} \ge \sup_{t\le u\le s} l_s e^{\delta s}$ and as marginal profit is decreasing in c, it follows with the help of the backward equation that

$$MO_t\left(I^*\right) \le e^{\delta t}\,\mathbb{E}\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{t\le u\le s} l_s e^{\delta s}\right)ds\,\middle|\,\mathcal{F}_t\right] = e^{-rt}.$$

This proves (6). When $dI_t^* > 0$, the process $\left(l_s e^{\delta s}\right)_{s\in[0,\hat{T}]}$ reaches a new running maximum at time t, that is,

$$l_t e^{\delta t} > l_u e^{\delta u} \quad \text{for all } u < t\,.$$

In this case, we have

$$\sup_{u\le s} l_s e^{\delta s} = \sup_{t\le u\le s} l_s e^{\delta s}\,,$$

which leads to

$$MO_t\left(I^*\right) = e^{\delta t}\,\mathbb{E}\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{t\le u\le s}\left(l_s e^{\delta s}\right)\right)ds\,\middle|\,\mathcal{F}_t\right] = e^{-rt},$$

and (7) is also satisfied by I^*. □

C Proof of Theorem 5.4

Proof: On the basis of (10), it is sufficient to check the relationship of the user cost of capital and L_t^τ. Let $A \in \mathcal{F}_t$ and set

$$\begin{aligned} f(L, r+\delta, X_t) &= \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c\left(X_s, Le^{-\delta(s-t)}\right) - (r+\delta)\right] ds\, \mathbb{1}_A\right] \\ &= \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\pi_c\left(X_s, Le^{-\delta(s-t)}\right) ds\, \mathbb{1}_A\right] + \mathbb{E}\left[\left(e^{-(r+\delta)\tau} - e^{-(r+\delta)t}\right) \mathbb{1}_A\right] . \end{aligned}$$

As L_t^τ is the unique solution of the first–order condition, we have always $f(L_t^\tau, r+\delta, X_t) = 0$. Hence, an implicit function exists to characterize L_t^τ as a function of the user cost of capital, $L_t^\tau = L(r+\delta)$, such that

$$f\left(L(r+\delta), r+\delta, X_t\right) = 0\,.$$

Taking the derivative of the function w.r.t. $r+\delta$ yields then

$$\frac{\partial L(r+\delta)}{\partial(r+\delta)} = -\frac{\partial f\left(L(r+\delta), r+\delta, X_t\right)}{\partial(r+\delta)} \Big/ \frac{\partial f\left(L(r+\delta), r+\delta, X_t\right)}{\partial L}\,.$$

The denominator at the left–hand side is easily obtained as

$$\frac{\partial f\left(L(r+\delta), r+\delta, X_t\right)}{\partial L} = \mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\pi_{cc}\left(X_s, L(r+\delta)e^{-\delta(s-t)}\right) e^{-\delta(s-t)} ds\, \mathbb{1}_A\right],$$

which is always negative since π is concave in capacity.

Meanwhile, the nominator is calculated and further reduced by using the condition $f(L(r+\delta), r+\delta, X_t) = 0$:

$$\begin{aligned} &\frac{\partial f\left(L(r+\delta), r+\delta, X_t\right)}{\partial(r+\delta)} \\ =\; &\mathbb{E}\left[\int_t^\tau -se^{-(r+\delta)s}\pi_c\left(X_s, L_t^\tau e^{-\delta(s-t)}\right) ds\, \mathbb{1}_A\right] - \mathbb{E}\left[\left(\tau e^{-(r+\delta)\tau} - te^{-(r+\delta)t}\right) \mathbb{1}_A\right] \\ <\; &\mathbb{E}\left[\int_t^\tau -se^{-(r+\delta)s}\pi_c\left(X_s, L_t^\tau e^{-\delta(s-t)}\right) ds\, \mathbb{1}_A\right] - t\mathbb{E}\left[\left(e^{-(r+\delta)\tau} - e^{-(r+\delta)t}\right) \mathbb{1}_A\right] \\ =\; &-\mathbb{E}\left[\int_t^\tau (s-t)e^{-(r+\delta)s}\pi_c\left(X_s, L_t^\tau e^{-\delta(s-t)}\right) ds\, \mathbb{1}_A\right] < 0\,. \end{aligned}$$

It follows that $\frac{\partial L(r+\delta)}{\partial(r+\delta)} < 0$.

□