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# The Uncertain Mortality Intensity Framework: Pricing and Hedging Unit-Linked Life Insurance Contracts

by

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# The Uncertain Mortality Intensity Framework: Pricing and Hedging Unit-Linked Life Insurance Contracts

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**Abstract**

We study the valuation and hedging of unit-linked life insurance contracts in a setting where mortality intensity is governed by a stochastic process. We focus on model risk arising from different specifications for the mortality intensity. To do so we assume that the mortality intensity is almost surely bounded under the statistical measure. Further, we restrict the equivalent martingale measures and apply the same bounds to the mortality intensity under these measures. For this setting we derive upper and lower price bounds for unit-linked life insurance contracts using stochastic control techniques. We also show that the induced hedging strategies indeed produce a dynamic superhedge and subhedge under the statistical measure in the limit when the number of contracts increases. This justifies the bounds for the mortality intensity under the pricing measures. We provide numerical examples investigating fixed-term, endowment insurance contracts and their combinations including various guarantee features. The pricing partial differential equation for the upper and lower price bounds is solved by finite difference methods. For our contracts and choice of parameters the pricing and hedging is fairly robust with respect to misspecification of the mortality intensity. The model risk resulting from the uncertain mortality intensity is of minor importance.

*Keywords:* unit-linked life insurance contracts, mortality model risk, price bounds, stochastic control

*JEL:* G13, G22, C61

## 1. Introduction

Mortality is a major risk factor for life insurance companies and pension funds that needs to be modeled properly. In recent years, it has been widely accepted that mortality changes over time in an unpredictable way and stochastic models have been developed to adequately capture the systematic mortality risk. For stochastic models valuing of mortality-linked liabilities and determining the required market reserves, see for instance Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), and Dahl and Møller (2006). Stochastic models with an emphasis on securitizing mortality risk by introducing survivor bonds as hedging instruments are discussed by, e.g., Blake et al. (2006) and Cairns et al. (2006). Each mortality model is a possible description of the mortality risk. Melnikov and Romaniuk (2006) show that different mortality models perform differently in the risk management of a unit-linked pure endowment contract and warns us to be careful when choosing one mortality model against another. In this paper we provide a framework for assessing the mortality model risk embedded in unit-linked life insurance contracts arising from different specifications for the mortality intensity.

Unit-linked life insurance contracts are popular and widely used on the insurance market.[1] They provide either death benefit or maturity benefit or both. The benefits are linked to an underlying asset with or without certain guarantees so that the policyholders have the opportunity to participate in the financial market and (eventually) be protected from the downside development of the financial market. Many unit-linked life insurance contracts

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[1]In the 1990s, united-linked life insurance was very popular in household financial planning. The share of unit-linked premiums increased from 20% in 1997 to 36% in 2001 of the total life insurance premiums which accounted for over 10% of the GDP of western Europe, see Swiss Re (2003). Although the enthusiasm in unit-linked insurance from the policyholder side declined during financial market crashes, e.g., at the end of 2001 and between 2007 and 2010, this business is expected to boom again when the capital market recovers from the depression. According to Swiss Re (2003), a simple regression analysis shows that a 10% rise in the stock market led to a 15% increase in single-premium unit-linked sales.

also embed options in them, e.g., the surrender option allowing the policyholders to terminate the contracts prematurely and the guaranteed annuity option giving the policyholders the right to convert a lump sum payment at the maturity into annuities at a predetermined rate. Depending on the payoff structures of the contracts, the effect of the mortality model risk may also be different. By investigating the effect of the mortality model risk we are able to know whether its importance is under or over-emphasized for different contract types.

In our paper instead of inputting different mortality models into the same pricing and hedging problem and comparing their performances as Melnikov and Romaniuk (2006), we set up a more flexible framework saying that we do not know the exact process of the mortality intensity but are able to figure out its upper and lower bound under the statistical measure. Further, we restrict the set of equivalent martingale measures such that the same bounds apply to the mortality intensity under these measures. This setup allows us to study various contract types more efficiently and we call it the uncertain mortality intensity framework, see Avellaneda et al. (1995) for a related framework for pricing stock options when the volatility process is unknown but bounded.

Within our framework we do not intend to find the fair value of a contract but its price bounds. The price bounds are solutions to the partial differential equations associated to a stochastic control problem. The upper price bound is found by choosing the worst-case mortality intensity at any time during the life time of the contract so that the contract value is maximized. Whereas the lower price bound is found by setting the mortality intensity to the best-case value in the sense that the contract value would be minimized. The effect of our approach is quite similar to that of the practice in traditional life insurance like pure endowment insurance and term insurance. An insurance company usually puts itself on the safe side by adjusting the premium by a loading factor defined as a percentage markup from the actuarially fair value of insurance. This is equivalent to assuming lower mortality intensity for pure endowment insurance and higher mortality intensity for term insurance. However, since our approach chooses the worst (or best) possible mortality intensity dynamically, we are able to deal with more complex contract structures where the safest mortality intensity at any time also depends on the price of the underlying asset. As a result, the higher the difference between the upper and the lower price bounds, the greater impact would the mortality model risk have on the contracts considered. In this way we are able to identify whether model risk is potentially deteriorating the fair evaluation of the contracts.

Further we examine hedging strategies induced by the price bounds. We show that the induced hedging strategies indeed produce a dynamic superhedge and subhedge under the statistical measure when the unsystematic mortality risk is eliminated by pooling together an increasing number of similar contracts. This justifies the bounds for the mortality intensity under the pricing measures. We provide numerical examples investigating fixed-term, endowment insurance contracts and their combinations including various guarantee features. The pricing partial differential equation for the upper and lower price bounds is solved by finite difference methods. For our contracts and choice of parameters pricing and hedging is fairly robust with respect to misspecification of the mortality intensity, with at most a mispricing of 4% for single premium contracts and at most 2% for periodic premium payment. We conclude that model risk resulting from the uncertain mortality intensity is of minor importance.

The structure of the paper is as follows. In Section 2 we describe both the financial market and the insurance market. In Section 3 we formalize the uncertain mortality intensity framework. Based on the model setup, we introduce in Section 4 the optimal control rule of the mortality intensity within its upper and lower bounds so that the price bounds are found. This enables us to build in mean superhedging strategies which are discussed in Section 5. Section 6 illustrates the theoretical results by providing a numerical analysis for different types of unit-linked life insurance contracts. Section 7 concludes.

## 2. Setup

The model for the financial market and the insurance market is developed subsequently. Both markets are jointly specified on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The probability $\mathbb{P}$ is called the real world measure and is sometime also referred to as statistical measure. We assume that the probability space is large enough to support an $n$-dimensional Wiener process $W = [W^1, W^2, \ldots, W^n]$ and a random time $\tau$. The time horizon is denoted by $T$.

The financial market consists of a risky asset with price process $S$ and a riskless money market account with price process $B$. The latter is given by $B_t = \exp\{\int_0^t r(u)\,\mathrm{d}u\}$, $0 \le t \le T$, where the risk-free interest rate $r$ is a deterministic and continuous function. The risky asset price process $S$ is governed by the stochastic differential equation:

$$\mathrm{d}S_t = \alpha(t, S_t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}W_t^1, \quad 0 \le t \le T. \tag{1}$$

where $\alpha$ is the local mean rate of return and $\sigma$ is the volatility. The dividend structure $D$ is given by

$$\mathrm{d}D_t = q(t, S_t) S_t \mathrm{d}t, \quad 0 \le t \le T. \tag{2}$$

where $q$ is a continuous deterministic function.[2] The financial market modeled in this way is complete and arbitrage free and is called $\mathbb{F}^S$ market. Here, $\mathbb{F}^S = \left(\mathcal{F}_t^S\right)_{0\le t\le T}$ is the augmented natural filtration generated by the stock price process $S$. Since $\sigma > 0$ it follows that the augmented natural filtration generated by the first component $W^1$ of the Wiener process $\mathbb{F}^1 = (\mathcal{F}_t^{W^1})_{0\le t\le T}$ coincides with the market filtration $\mathbb{F}^S$.

The insurance market is modeled by the random time $\tau$ denoting the death time of an individual aged $x$ at the starting time 0.[3] For simplicity of notation we will omit the age variable $x$ in the subsequent discussion of mortality related variables. The filtration generated by the right-continuous indicator process $H_t = 1_{\{\tau \le t\}}$, for $t \in [0, T]$, is denoted $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le T}$. The mortality is potentially influenced by an $m$-dimensional environment process $X = [X_1, \ldots, X_m]$ with dynamics

$$\mathrm{d}X_t = \alpha_X(t, X_t)\, \mathrm{d}t + \Sigma_X(t, X_t)\, \mathrm{d}W_t, \quad 0 \le t \le T, \tag{3}$$

[2] We assume that the coefficients $\alpha$ and $\sigma$ are regular enough to ensure the existence of a solution to the SDE (1), see for instance Protter (2004), Ch. V, Sec. 3. Additionally, we assume that $\alpha$, $\sigma$ and $q$ are uniformly bounded and $\sigma$ is bounded away from zero to ensure the integrability of $S$, related portfolio value processes, and to ensure the existence of the measure change from $\mathbb{P}$ to an equivalent martingale measure $\mathbb{Q}$.

[3] In Section 5 we consider the case of a family of random times $(\tau_i)_{i\ge1}$ and the corresponding contracts.

where $\alpha_X$ is a $\mathbb{R}^m$-valued function and $\Sigma_X$ is a $\mathbb{R}^{n\times m}$-valued function, both regular enough to ensure the existence of a solution to the SDE. By definition it is clear that $X$ is adapted to the filtration generated by the Wiener process $W$, say, $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. Note that $\mathbb{F}^S \subseteq \mathbb{F}$, and further denote the joint filtration by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. The financial market model for unit-linked life insurance contracts is then called the $\mathbb{G}$ market.

*2.1. Dependence of Financial Market and Insurance Market*

The probabilistic connection between $W$ and $\tau$ is now formalized. In broad terms we assume that we are in a setting frequently used in the credit risk literature, see Bielecki and Rutkowski (2001), part II, for a detailed treatment. In particular, we assume that on $(\Omega, \mathcal{G}, \mathbb{P})$ there exists a unit exponentially distributed random variable $E_1$ that is independent of $W$ and further that there exists a nonnegative $\mathbb{F}$-adapted process $\nu$ such that $\tau$ can be represented by

$$\tau = \inf\left\{ t \geq 0 : \int_0^t \nu_s \, \mathrm{d}s \geq E_1 \right\} , \quad a.s.,$$

with the usual convention that the infimum over the empty set is $\infty$, and the integrability condition $\int_0^t \nu_s \, \mathrm{d}s < 0$ holds almost surely, for all $t \geq 0$.

We then have the following representation

$$M_t \;=\; H_t - \int_0^{t\wedge\tau} \nu_s \mathrm{d}s \,, \quad 0 \leq t \leq T,$$

where $M$ is a $(\mathbb{P}, \mathbb{G})$-martingale, see Bielecki and Rutkowski (2001), p.153, Prop. 5.1.3. In our context the intensity $\nu$ is known as mortality intensity.

By specification of $\tau$ through $E_1$ and $\nu$ and the assumed independence, the $\sigma$-fields $\mathcal{F}_T$ and $\mathcal{H}_t$ are independent given $\mathcal{F}_t$ under the real world probability measure $\mathbb{P}$. Although we may perceive the death probability of the individual, we do not know when the death event really happens. Hence, $\tau$ is an inaccessible $\mathbb{G}$ stopping time but not an $\mathbb{F}$ stopping time. On the other hand, the financial market is not influenced by the introduction of $\tau$. Accordingly, the $\mathbb{G}$ market for unit-linked life insurance contracts is free of arbitrage.[4] However, given

[4] In particular any square integrable $(\mathbb{F}, \mathbb{P})$-martingale is also a square integrable $(\mathbb{G}, \mathbb{P})$-martingale. This is also known as hypotheses (H), see Jeulin and Yor (1979).

that there are no products to hedge against the mortality risk (that is the fluctuation of $\nu$ and the mortality event indicated by $H$), the $\mathbb{G}$ market is incomplete, and hence, there should be infinitely many equivalent martingale measures.

*2.2. Equivalent Martingale Measures*

The set of equivalent martingale measures is studied. Given a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}, \mathbb{G})$, the Radon-Nikodym density process $\eta$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ is

$$\eta_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(Y|\mathcal{G}_t), \quad \mathbb{P} - a.s., \tag{4}$$

for some $\mathcal{G}_T$-measurable random variable $Y$ with $\mathbb{P}(Y > 0) = 1$ and $\mathbb{E}_{\mathbb{P}}(Y) = 1$.

Now, we characterize the set of equivalent measure for our setting and also the set of equivalent martingale measures. The set of equivalent measures is given by Prop. 5.1.3 in Bielecki and Rutkowski (2001), p.162. Let $\mathbb{Q}$ be a probability measure equivalent to the real world probability measure $\mathbb{P}$ with the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ defined by (4). Then we can write

$$\eta_t = 1 + \int_0^t \eta_{u-}(\varphi_u \mathrm{d}W_u + \phi_u\, \mathrm{d}M_u), \quad 0 \le t \le T\,, \tag{5}$$

where $\varphi$ and $\phi$ are $\mathbb{G}$-predictable stochastic processes. The change of measure affects the martingales $W$ and $M$ as follows. Define the processes $W^Q$ and $M^Q$ by

$$W_t^Q = W_t - \int_0^t \varphi_s\, \mathrm{d}s\,, \qquad \text{and} \qquad M_t^Q = H_t - \int_0^{t\wedge\tau} (1+\phi_u)\,\nu_u\, \mathrm{d}u\,, \quad 0 \le t \le T\,. \tag{6}$$

Then $W^Q$ is a $(\mathbb{G}, \mathbb{Q})$-Wiener process and $M^Q$ is an $(\mathbb{G}, \mathbb{Q})$-martingale, and $\mu = (1+\phi)\,\nu$ is the $\mathbb{Q}$-intensity of $\tau$. Moreover, $\mu$ can be chosen to be $\mathbb{F}$-adapted, see Remark following Corollary 5.3.1 in Bielecki and Rutkowski (2001), p.164.

**Proposition 1.** *If $\mathbb{Q}$ is an equivalent martingale measure, i.e. $\mathbb{Q} \sim \mathbb{P}$ and $S/B$ is a $(\mathbb{G}, \mathbb{Q})$-martingale, then $\{W^{1Q}\}_{t\in[0,T]}$ is uniquely determined by the market price of risk*

$$\varphi_t^1 = -\frac{\alpha(t, S_t) - r(t) + q(t, S_t)}{\sigma(t, S_t)}, \quad 0 \le t \le T\,. \tag{7}$$

*Proof of Proposition 1.* This follows from the Second Fundamental Theorem, confer Björk (2009), p.151, Theorem 10.17 and p.204, Sec.14.6. □

Proposition 1 indicates that when we restrict to the $\mathbb{F}^S$ market, there is a unique martingale measure, which we denote as $\mathbb{Q}^{\mathbb{F}^S}$. Under any equivalent martingale measure $\mathbb{Q}$, the dynamics of the stock price is

$$\mathrm{d}S_t \;=\; (r(t) - q(t, S_t))\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}W_t^{1Q}, \quad 0 \le t \le T\,. \tag{8}$$

However, when we observe the extended market with both the financial and the mortality risks, we cannot find a riskless benchmark security. Hence, $\phi$, or equivalently, the risk-neutral mortality intensity is not uniquely defined. Theoretically, among a whole class of equivalent martingale measures, the insurance companies can choose any one depending on their risk attitude. We denote the set of equivalent martingale measures by $\mathcal{Q}$, i.e.

$$\mathcal{Q} \;=\; \left\{ \mathbb{Q} \sim \mathbb{P} : \varphi_t^1 = -\frac{\alpha(t, S_t) - r(t) + q(t, S_t)}{\sigma(t, S_t)} \right\}\,. \tag{9}$$

**Remark 1.** The fair valuation of an insurance liability is carried out under a specific risk-neutral measure. Choosing the valuation measure in an incomplete market is a difficult task. Alternatively, the model can be completed by adding asset that cover the entire risk factors. Biffis and Millossovich (2006) assumes that there is a liquid secondary market where the insurers can continuously trade their books of policies making it possible to access both short and long positions. This results into a complete market situation where the valuation measure is unique. Another possibility to uniquely determine the risk-neutral measure is to introduce standardized mortality linked products such as longevity bonds which are liquidly traded on the market, see Blake et al. (2006). However, a fully developed secondary insurance market does not exist yet and the securitization of mortality risk is still at its infancy stage with most of the mortality linked securities only being tailored to the customers. Hence, in this paper, we still assume that there is not a unique market price of mortality risk and that there are infinitely many martingale measures $\mathbb{Q}$ equivalent to $\mathbb{P}$, under which the prices of the insurance contracts do not allow arbitrage.

*2.3. Examples*

The setup described so far accommodates a large class of models discussed in the literature. We illustrate the use of the environment process $X$ by some prominent examples.

The mean reverting Brownian Gompertz approach of Milevsky and Promislow Milevsky and Promislow (2001) is given by

$$\begin{aligned} \nu_t &= \nu_0 \, e^{gt+\sigma X_t}, \quad 0 \le t \le T, \\ \mathrm{d}X_t &= -b \, X_t \mathrm{d}t + \mathrm{d}W_t^2, \quad 0 \le t \le T, \quad X_0 = 0, \end{aligned} \tag{10}$$

where $g, \sigma, \nu_0 > 0$ and $b \ge 0$.

Dahl Dahl (2004) and Dahl and Møller Dahl and Møller (2006) use the extended CIR model, i.e. $\nu = X$ with

$$\mathrm{d}X_t = (\beta_t - \gamma_t \, X_t) \, \mathrm{d}t + \sigma_t \sqrt{X_t} \mathrm{d}W_t^2, \quad 0 \le t \le T, \tag{11}$$

where $\beta_t$, $\gamma_t$ and $\sigma_t$ are positive bounded functions satisfying $2\,\beta_t \ge \sigma_t^2$, $0 \le t \le T$.

Biffis (2005) studies affine processes of the form

$$\begin{aligned} \mathrm{d}X_t^1 &= \gamma_1 (X_t^2 - X_t^1) \mathrm{d}t + \sigma_1 \sqrt{X_t^1} \, \mathrm{d}W_t^2, \quad 0 \le t \le T, \\ \mathrm{d}X_t^2 &= \gamma_2 (m(t) - X_t^2) \, \mathrm{d}t + \sigma_2 \sqrt{X_t^2 - m^*(t)} \, \mathrm{d}W_t^3, \quad 0 \le t \le T, \end{aligned} \tag{12}$$

where $\gamma_i, \sigma_i > 0$, $i = 1, 2$, and the mortality intensity is $\nu = X^1$, the stochastic mean reversion level is $\bar{\nu} = X^2$, $m(t)$ is a suitable demographic basis, and $m^*(t)$ is a time varying lower boundary for the stochastic drift $X^2$.

The mortality intensity is typically modeled under the statistical measure $\mathbb{P}$. The life tables are calculated on the basis of real world data. When going to a pricing measure $\mathbb{Q}$, often structure preserving transformations are allowed for, e.g., Dahl and Møller Dahl and Møller (2006) relate the $\mathbb{P}$-mortality intensity $\nu$ to the $\mathbb{Q}$-mortality intensity $\mu$ by assuming $\mu_t = (1 + g(t)) \, \nu_t$, where $g$ is a deterministic continuously differentiable function.

## 3. Uncertain Mortality Intensity

Our setup is a generically incomplete market model and we cannot obtain a unique price for a unit-linked life insurance contract. However, we are able to find its price bounds under certain assumptions. In this paper, we admit that we cannot perceive the dynamics of the mortality intensity exactly. Instead of applying a specific mortality model, as is done for example in Biffis (2005), Dahl (2004), Dahl and Møller (2006), and Milevsky and Promislow (2001), we assume less stringently that we know the upper and lower bounds of the mortality intensity. As is shown in our numerical section below, this assumption can be motivated by a statistical analysis of survival data and the confidence bounds for the estimated mortality intensity arising there, see, e.g., Lee and Carter (1992). The concept of an uncertain input parameter to a pricing model is related to Avellaneda et al. (1995). They discus the pricing and hedging of derivative securities in an incomplete market where the incompleteness is attributed to the uncertainty of the future volatility of the underlying asset. As suggested by them, we will use stochastic optimal control techniques to identify the best-case scenario and the worst-case scenario of the mortality intensity dynamics, to derive the upper and lower price bounds of the unit-linked life insurance contracts.

The above assumption on the boundedness of the mortality intensity is now made precise.

**Assumption 1** ($\mathbb{P}$-Bounds for Mortality Intensity). *The mortality intensity $\nu$ is an $\mathbb{F}$-adapted stochastic process satisfying*

$$\underline{\mu}(t) \leq \nu_t \leq \bar{\mu}(t)\,, \textit{almost surely}\,, \quad 0 \leq t \leq T\,, \tag{13}$$

*where $0 < \underline{\mu} \leq \bar{\mu} < \infty$ are continuous functions on $[0,T]$.*

**Remark 2.** The bounds $\underline{\mu}$ and $\bar{\mu}$ for $\nu$ are assumed to hold almost surely whereas the examples in 2.3 allow $\nu$ to take values in $\mathbb{R}^+$. Accordingly, these examples are not included in our setup once Assumption 1 is invoked. However, these models satisfy the boundedness condition typically with a high probability when assuming that both, parameters for a model in 2.3 and the bounds $\underline{\mu}$ and $\bar{\mu}$, here in terms of confidence bounds, are calculated from the same data set. In fact, by increasing the confidence level for the calculation of $\underline{\mu}$ and $\bar{\mu}$ the probability is increased of a stochastic model for $\nu$ also fulfilling the boundedness condition. In Section 6 numerical results are obtained when working with a 99.9% confidence bound.

For pricing derivatives the dynamics of the risk factors under an equivalent martingale measure is relevant. Our market model is incomplete and we can choose between an infinite range of equivalent martingale measures, see Proposition 1 and the discussion thereafter. An incomplete financial market is typically completed by adding assets to the market model such that all risk factors are traded. However, for insurance risk we can use the diversification rationale as an alternative. The diversification applies in our setting to the life insurance risk given by the time of death $\tau$ parameterized by the mortality intensity $\nu$. Diversification is driven by the strong law of large numbers and thus tied to the statistical measure $\mathbb{P}$. Based on this rationale, pricing must take all the possible scenarios of death events into account. Consequently, the most suitable equivalent martingale measure should be chosen among all the possible ones so that the diversification works to eliminate the mortality risk. Since the possible scenarios of death events do not change under any pricing measure although their probability distributions are different[5], we impose the boundedness assumption under $\mathbb{P}$ also to any pricing measure $\mathbb{Q} \in \mathcal{Q}$, defined in Eq. (9).

**Assumption 2** ($\mathbb{Q}$-Bounds for Mortality Intensity)**.** *Let $\mathbb{Q}$ be an equivalent martingale measure given in Eq.* (9) *with $\mathbb{F}$-adapted mortality intensity $\mu$. Assume that*

$$\underline{\mu}(t) \leq \mu_t \leq \bar{\mu}(t)\,, \quad 0 \leq t \leq T\,, \tag{14}$$

*where $\underline{\mu}$ and $\bar{\mu}$ are the same functions as in Assumption 1.*

The restricted set of martingale measures satisfying Assumption 2 is now specified.

**Definition 1.** *Given that Assumption 1 holds, denote by $Q^b$ the set of equivalent martingale measures $\mathbb{Q} \in \mathcal{Q}$ that satisfy Assumption 2, i.e.*

$$\mathcal{Q}^b \quad = \quad \left\{ \mathbb{Q} \in \mathcal{Q} : \underline{\mu} \leq \mu \leq \bar{\mu}, \ \textit{where } \mu \textit{ is the } \mathbb{Q}\textit{-intensity of } \tau \right\}\,. \tag{15}$$

In Section 4 we establish upper and lower price bounds for specific unit-linked life insurance contracts. Subsequently, in Section 5 we show that the upper price bound indeed

[5]This indicates that given the same the confidence bound, the confidence level under the equivalent martingale measure $\mathbb{Q}$ is not identical with the confidence level under the real world measure $\mathbb{P}$. We assume that the difference in the confidence level is not significantly big.

leads to the cheapest superhedge once diversification is applied such that the biometric risk is eliminated. The respective results for the lower price bound and the most expensive subhedge follows analogously.

## 4. Pricing Unit-Linked Life Insurance Contracts

### *4.1. Payoff Structures*

Now we introduce a unit-linked life insurance contract with Markovian payoff structures to the $\mathbb{G}$ market. The contract has the life time of $T$ years. It may be obtained by the policyholders upon upfront single payment or a continuous flow of premiums[6]. When the policyholder dies at $\tau < T$, the contract pays $\Psi(\tau, S_\tau)$ immediately. When he survives time $T$, the payment is $\Phi(S_T)$. The policyholder is entitled to this payoff structure if he pays the premium required. We assume that the cumulated premium payment at time $t$ is $A_t = A_0 + \int_0^t \Gamma(u, S_u)\,\mathrm{d}u$ where $\Gamma$ refers to the instantaneous premium payment rate on the annual basis. Through a concrete definition of $\Psi$, $\Phi$ and $A$, we obtain different types of contracts. Some examples are:

- Term insurance: $\Psi(\tau, S_\tau) > 0$, for $\tau \leq T$, and $\Phi(S_T) = 0$, for $\tau > T$.
- Pure endowment insurance: $\Psi(\tau, S_\tau) = 0$, for $\tau \leq T$, and $\Phi(S_T) > 0$, for $\tau > T$.
- Endowment insurance: $\Psi(\tau, S_\tau) > 0$, for $\tau \leq T$, and $\Phi(S_T) > 0$ for $\tau > T$.
- Single premium: $A_t = A_0 =$ constant.
- Periodic premium: $A_t$ is increasing in $t$.

The contract cash flows specified by the functions $\Phi$, $\Psi$, and $\Gamma$ have to satisfy certain integrability conditions. These are summarized below.

[6] In reality, periodic premiums are paid monthly or yearly. We assume the continuous flow of premiums just for illustration simplicity.

**Assumption 3.** *The functions* $\Phi$*,* $\Psi$*, and* $\Gamma$ *satisfy the following integrability conditions*

$$\mathbb{E}^{\mathbb{Q}}\left[|\Phi(S_T)| + \int_0^T |\Psi(t, S_t)| + |\Gamma(t, S_t)| \,\mathrm{d}t\right] < \infty\,,$$

*where* $\mathbb{Q}$ *is an equivalent martingale measure.*

Note that if the condition holds for a specific $\mathbb{Q} \in \mathcal{Q}$ then it holds for any other equivalent martingale measure. The reason is that all equivalent martingale measures coincide on $\mathcal{F}_T^S$ and the random variable where the expectation is taken is $\mathcal{F}_T^S$-measurable.

Unit-linked life insurance contracts can also have exotic features not covered by our setup, e.g., a surrender guarantee or a guaranteed annuity option. In case of a surrender guarantee the policyholder can surrender the contract and receives the surrender payment replacing all payments afterward originally specified by $\Phi$, $\Psi$ and $\Gamma$. The surrender payment may or may not be linked to the underlying asset. If the contract specifies a guaranteed annuity rate $a$ at which the policyholder has the right to convert the terminal payment into annuities at time $T$, then the terminal value of the contract becomes the original terminal value times a call option on the annuity rate.[7] Unit-linked life insurance contracts with exotic features are important contract types. They can be discussed when extending our framework. However, in this paper we work with unit-linked life insurance contracts with rather simple payoff structures as was specified at the beginning of this section. By this we can explain the method we adopt to analyze the risk management of unit-linked life insurance contracts under mortality model risk.

### *4.2. Arbitrage Free Prices and Price Bounds*

Fix an equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}$ with mortality intensity $\mu$. An arbitrage free price of the contract $(\Phi, \Psi, \Gamma)$ is given by the conditional expectation of the discounted cashflow under $\mathbb{Q}$, see, e.g., Björk (2009). Decomposing the contract into its components the arbitrage free prices of the death benefit $V^{\mu,\Psi}$, the survival benefit $V^{\mu,\Phi}$, and the premium

[7] The payoff is $\Phi(S_T) \cdot \max(1, a\,\mathbb{E}_{\mathbb{Q}}[\int_T^\infty \exp\{-\int_T^u (r + \mu_s)du\}])$.

$V^{\mu,\Gamma}$ are:

$$\begin{aligned} V_t^{\mu,\Psi} &= B_t \mathbb{E}^{\mathbb{Q}}\left[B_\tau^{-1} 1_{\{t<\tau\le T\}} \Psi(\tau, S_\tau) | \mathcal{G}_t\right], \quad V_t^{\mu,\Phi} = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} 1_{\{\tau>T\}} \Phi(S_T) | \mathcal{G}_t\right], \\ V_t^{\mu,\Gamma} &= B_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T B_u^{-1} 1_{\{u<\tau\le T\}} \Gamma(u, S_u)\, \mathrm{d}u \,\middle|\, \mathcal{G}_t\right], \end{aligned}$$

and the arbitrage free price of the aggregate contract $V^\mu$ is then

$$V_t^\mu = V_t^{\mu,\Psi} + V_t^{\mu,\Phi} - V_t^{\mu,\Gamma}, \quad 0 \le t \le T.$$

The arbitrage free prices for a life insurance contract and its components under a specific equivalent martingale measure $\mathbb{Q}$ can be given in a more explicit form. Duffie et al. (1996) have shown that

$$\begin{aligned} V_t^{\mu,\Psi} &= 1_{\{\tau>t\}} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1} \Psi(u, S_u) \mu_u \,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right], V_t^{\mu,\Phi} = 1_{\{\tau>t\}} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1} \Phi(S_T) \,\middle|\, \mathcal{F}_t\right], \\ V_t^{\mu,\Gamma} &= 1_{\{\tau>t\}} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1} \Gamma(u, S_u)\, \mathrm{d}u \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T, \end{aligned} \tag{16}$$

where $\hat{B}_\cdot = \exp\{\int_0^\cdot (r(s) + \mu_s)\, \mathrm{d}s\}$ represents a mortality risk adjusted money market account that also depends on the choice of $\mathbb{Q}$ via $\mu$.

According to the so-called reduced forms above we can consider the contract price as the discounted value of a fictitious security whose dividend payment at $t$ is $\Psi(t, S_t)\, \mu_t - \Gamma(t, S_t)$ and final payment is $\Phi(S_T)$. The fictitious discount factor is $\hat{B}$. In this fictitious world, we can ignore the mortality risk in the form of $\tau$ and consider the insurance contract merely as a contingent claim on the fictitious financial market.

From now on, we discuss the pricing problem within the class of equivalent martingale measures $\mathcal{Q}^b$ where the mortality intensity is bounded from below and from above, i.e. we suppose that Assumptions 1 and 2 hold. The worst case that may happen to the insurance company with regard to the death benefit, the survival benefit, the premium and the contract price, respectively, is

$$\bar{V}_t^\Psi = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu,\Psi}, \quad \bar{V}_t^\Phi = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu,\Phi}, \quad \underline{V}_t^\Gamma = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu,\Gamma},$$

and

$$\bar{V}_t = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^\mu, \quad 0 \le t \le T.$$

In view of the results of Duffie et al. (1996) presented above, we can transfer the problem of choosing the best or worst equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}^b$ to the problem of choosing the best or worst mortality intensity $\mu \in \mathcal{U}_t$, where $\mathcal{U}_t$ is the set of $\mathbb{F}$-predictable processes $\mu$ on $[t, T]$ such that $\underline{\mu}(s) \leq \mu_s \leq \bar{\mu}(s)$, for $t \leq s \leq T$. In particular, we may write

$$\bar{V}_t^{\Psi} = 1_{\{\tau>t\}} \operatorname*{ess\,sup}_{\mu \in \mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T \hat{B}_u^{-1} \Psi(u, S_u) \mu_u \, \mathrm{d}u \middle| \mathcal{F}_t \right], \tag{17}$$

$$\bar{V}_t^{\Phi} = 1_{\{\tau>t\}} \operatorname*{ess\,sup}_{\mu \in \mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}} \left[ \hat{B}_T^{-1} \Phi(S_T) \middle| \mathcal{F}_t \right], \tag{18}$$

$$\underline{V}_t^{\Gamma} = 1_{\{\tau>t\}} \operatorname*{ess\,inf}_{\mu \in \mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T \hat{B}_u^{-1} \Gamma(u, S_u) \, \mathrm{d}u \middle| \mathcal{F}_t \right], \tag{19}$$

and

$$\bar{V}_t = 1_{\{\tau>t\}} \operatorname*{ess\,sup}_{\mu \in \mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T \hat{B}_u^{-1} \Psi(u, S_u) \mu_u \, \mathrm{d}u + \hat{B}_T^{-1} \Phi(S_T) - \int_t^T \hat{B}_u^{-1} \Gamma(u, S_u) \, \mathrm{d}u \middle| \mathcal{F}_t \right]. \tag{20}$$

By specifying the stock price dynamics with (8), we have actually fixed the equivalent martingale measure on $\mathbb{F}^S$. Instead of looking for the optimal martingale measure on the $\mathbb{G}$ market, we convert the problem into looking for the $\mathbb{F}$-adapted process $\mu$. The expressions in (17-20) are stochastic control problems with control process $\mu$. The prices $V^{\mu,\Phi}$ and $V^{\mu,\Gamma}$ depend on $\mu$ monotonously. When considering $\mathbb{Q} \in \mathcal{Q}^b$, the highest arbitrage free price for the death benefit $\bar{V}_t^{\Phi}$ is obtained for $\mu = \underline{\mu}$, and the lowest value for the premium income $\underline{V}^{\Gamma}$ is obtained for $\mu = \bar{\mu}$. With regard to $\bar{V}^{\Psi}$ and $\bar{V}$, we apply stochastic control techniques to obtain the respective solutions.

The stock price process $S$ is Markovian and the payoff functions are simple in the sense that they depend on time and the current value of the stock price. This suggest the standard setup of a stochastic control problem with state variable $(t, s)$, feedback control $\mu \in \mathcal{U}(t, s)$, and maximization problem

$$\bar{v}(t, s) = \sup_{\mu \in \mathcal{U}(t,s)} \mathbb{E}^{t,s} \left[ \int_t^T \frac{\hat{B}_t}{\hat{B}_u} \Psi(u, S_u) \mu_u \, \mathrm{d}u + \frac{\hat{B}_t}{\hat{B}_T} \Phi(S_T) - \int_t^T \frac{\hat{B}_t}{\hat{B}_u} \Gamma(u, S_u) \, \mathrm{d}u \right], \tag{21}$$

where $\mathcal{U}(t, s) = \{\mu : [t, T] \times \mathbb{R}^+ \mapsto \mathbb{R}^+ : \underline{\mu}(u) \leq \mu(u, x) \leq \bar{\mu}(u) \,, \text{ all } t \leq u \leq T, x \in \mathbb{R}^+\}$ and $\mathbb{E}^{t,s}$ denotes the expectation conditional on $S(t) = s$ under the measure $\mathbb{Q}^{\mathbb{F}^S}$. Recall, that

$\hat{B}$ also depends on $\mu$ and in particular $\frac{\hat{B}_t}{\hat{B}_u} = \exp\{-\int_t^u r(s) + \mu(s, S_s)\,\mathrm{d}s\}$. Observe that the term inside the conditional expectation is $\mathcal{F}_T^S$-measurable.

According to the theorem of the Hamilton-Jacobi-Bellman equation (Confer Yong (1997) as well as Yong and Zhou (1999)), $\bar{v}$ is the solution to:

$$0 = \sup_{\mu \in \mathcal{U}(t,s)} \{\mathcal{L}\bar{v}(u,s) + \Psi(u,s)\,\mu(u,s) - \Gamma(u,s) - \bar{v}(u,s)[\mu(u,s) + r(u)]\} \,, \quad (22)$$

$$\Phi(s) = \bar{v}(T,s)\,. \quad (23)$$

where

$$\mathcal{L}f(u,s) = \frac{\partial f}{\partial u}(u,s) + (r(u) - q(u,s))\, s\, \frac{\partial f}{\partial s}(u,s) + \frac{1}{2}\sigma^2(u,s)\, s^2\, \frac{\partial^2 f}{\partial s^2}(u,s). \quad (24)$$

The part of (22) that is depending on the control $\mu$ is given by $[\Psi(u,s) - \bar{v}(u,s)]\,\mu(u,s)$ and is linear in $\mu$. Hence, we obtain pointwise

$$\sup_{\mu \in \mathcal{U}(t,s)} [\Psi(u,s) - \bar{v}(u,s)]\mu(u,s) = \begin{cases} [\Psi(u,s) - \bar{v}(u,s)]\bar{\mu}(u), & \text{if } \Psi(u,s) \geq \bar{v}(u,s), \\ [\Psi(u,s) - \bar{v}(u,s)]\underline{\mu}(u), & \text{if } \Psi(u,s) < \bar{v}(u,s). \end{cases}$$

The maximizer $\mu^\star$ is thus

$$\mu^\star(t,s) = \begin{cases} \bar{\mu}(t), & \text{if } \Psi(t,s) \geq \bar{v}(t,s), \\ \underline{\mu}(t), & \text{if } \Psi(t,s) < \bar{v}(t,s). \end{cases} \quad (25)$$

Plugging the pointwise maximizer in (22) gives

$$0 = \mathcal{L}\bar{v}(u,s) + \Psi(u,s)\,\mu^\star(u,s) - \Gamma(u,s) - \bar{v}(u,s)[\mu^\star(u,s) + r(u)]\,, \quad (26)$$

$$\Phi(s) = \bar{v}(T,s)\,. \quad (27)$$

In fact, the calculation above produces a candidate $\bar{v}$ for solution of the maximization problem (21). Moreover, we want this candidate $\bar{v}$ to solve the more general problem (20) in the sense that $1_{\{\tau > t\}}\,\bar{v}(t, S_t) = \bar{V}_t = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}^b} V_t^\mu$. We will show this in Theorem 1 below. To do so we require the following integrability condition.

**Assumption 4.** *Denote* $\overline{v} \in C^{1,2}$ *the solution to the partial differential equation in (26-27). Assume the following integrability condition holds*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left(\frac{\partial \overline{v}}{\partial s}(t,S_t)\,\sigma(t,S_t)\,S_t\right)^2 \mathrm{d}t\right] < \infty\,.$$

**Theorem 1.** *Given the setup in Section 2, suppose Assumptions 1-3 hold. Denote* $\bar{v} \in C^{1,2}$ *the solution to the boundary value problem in* (26) *with terminal condition* (27) *and suppose that Assumption 4 holds. Then*

$$1_{\{\tau>t\}}\,\overline{v}(t,S_t) \;=\; \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu}\,, \quad 0 \le t \le T\,.$$

*In particular, the mortality intensity* $\mu^\star$ *that maximizes the contract value is given by* (25).

*Proof of Theorem 1.* First, we have to establish that indeed $1_{\{\tau>t\}}\,\overline{v}(t,S_t) = V_t^{\mu^\star}$ where $\mu^\star$ is the optimal control given in (25). Ito's lemma gives

$$\begin{aligned}\mathrm{d}\overline{v}(t,S_t) \;&=\; \left(\frac{\partial \overline{v}}{\partial t}(t,S_t) + (r(t)-q(t,S_t))\,S_t\frac{\partial \overline{v}}{\partial s}(t,S_t) + \frac{1}{2}\,\sigma(t,S_t)^2\,S_t^2\,\frac{\partial^2 \overline{v}}{\partial s^2}(t,S_t)\right)\mathrm{d}t\\ &\quad+\sigma(t,S_t)\,S_t\,\frac{\partial \overline{v}}{\partial s}(t,S_t)\,\mathrm{d}W_t^{1Q}\,.\end{aligned}$$

Now, $\overline{v}$ satisfies (26) by assumption and thus can be written as

$$\begin{aligned}\mathrm{d}\overline{v}(t,S_t) \;&=\; \left(r(t)\,\overline{v}(t,S_t) - (\Psi(t,S_t) - \overline{v}(t,S_t))\,\mu^\star(t,S_t) + \Gamma(t,S_t)\right)\mathrm{d}t\\ &\quad+\sigma(t,S_t)\,S_t\,\frac{\partial \overline{v}}{\partial s}(t,S_t)\,\mathrm{d}W_t^{1Q}\,.\end{aligned}$$

The differential is a linear stochastic differential equation with formal solution

$$\begin{aligned}\overline{v}(u,S_u) \;&=\; e^{\int_t^u r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\Big(\overline{v}(t,S_t) - \int_t^u e^{-\int_t^w r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\sigma(w,S_w)\,S_w\,\frac{\partial \overline{v}}{\partial s}(w,S_w)\,\mathrm{d}W_w^{1Q}\\ &\quad-\int_t^u e^{-\int_t^w r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\,(\Psi(w,S_w)\,\mu^\star(w,S_w) - \Gamma(w,S_w))\,\mathrm{d}w\Big)\,.\end{aligned}$$

Set $u=T$ and recall that $\overline{v}(T,S_T) = \Phi(S_T)$. Then

$$\begin{aligned}e^{-\int_t^T r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\Phi(S_T) \;&=\; \overline{v}(t,S_t) - \int_t^T e^{-\int_t^u r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\sigma(u,S_u)\,S_u\,\frac{\partial \overline{v}}{\partial s}(u,S_u)\,\mathrm{d}W_u^{1Q}\\ &\quad-\int_t^T e^{-\int_t^u r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\,(\Psi(u,S_u)\,\mu^\star(u,S_u) - \Gamma(u,S_u))\,\mathrm{d}u\,.\end{aligned}$$

Solving for $\overline{v}(t,S_t)$ and taking the expectation with respect to $\mathcal{F}_t$ we obtain

$$\begin{aligned}\overline{v}(t,S_t) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\Phi(S_T)\middle|\,\mathcal{F}_t\right]\\ &+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(s)+\mu^\star(s,S_s)\,\mathrm{d}s}\left(\Psi(u,S_u)\,\mu^\star(u,S_u)-\Gamma(u,S_u)\right)\,\mathrm{d}u\middle|\,\mathcal{F}_t\right],\end{aligned}$$

where the martingale part vanishes because of Assumption 4. Recalling the reduced form representation of $V^\mu = V^{\mu,\Psi}+V^{\mu,\Phi}-V^{\mu,\Gamma}$ in (16) we see that the candidate $\overline{v}$ is indeed a value function, i.e. $\mathbf{1}_{\{\tau>t\}}\,\overline{v}(t,S_t)=V_t^{\mu^\star}$, for $0\le t\le T$.

Next, the optimality of $\mu^\star$ and the corresponding value function $\overline{v}$ is established. We fix a measure $\mathbb{Q}\in\mathcal{Q}^b$ and as usual denote by $\mu$ the mortality intensity under $\mathbb{Q}$. Define the $\mathbb{F}$-adapted process $U^\mu$ by

$$U_t^\mu = \hat{B}_t\mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1}\Phi(S_T)+\int_t^T \hat{B}_u^{-1}\Psi(u,S_u)\mu_u\,\mathrm{d}u-\int_t^T \hat{B}_u^{-1}\Gamma(u,S_u)\,\mathrm{d}u\middle|\mathcal{F}_t\right],\quad 0\le t\le T,$$

such that $V_t^\mu=\mathbf{1}_{\{\tau>t\}}\,U_t^\mu$, $0\le t\le T$. Define the accompanying martingale $M^\mu$ by

$$M_t^\mu = \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1}\Phi(S_T)+\int_0^T \hat{B}_u^{-1}\Psi(u,S_u)\mu_u\,\mathrm{d}u-\int_0^T \hat{B}_u^{-1}\Gamma(u,S_u)\,\mathrm{d}u\middle|\mathcal{F}_t\right],\quad 0\le t\le T.$$

Verify that $\mathbb{E}^{\mathbb{Q}}|M_T|<\infty$ by Assumptions 2 and 3 and $M^\mu$ is indeed an $(\mathbb{Q},\mathbb{F})$-martingale. Now, $\mathbb{Q}$ coincides with $\mathbb{Q}^{\mathbb{F}^S}$ on $\mathcal{F}_T^S$ and for the process $U^{\mu^\star}$ and $M^{\mu^\star}$ defined by

$$\begin{aligned}U_t^{\mu^\star} &= \hat{B}_t^\star\,\mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{\star-1}\Phi(S_T)+\int_t^T \hat{B}_u^{\star-1}\Psi(u,S_u)\mu^\star(u,S_u)\,\mathrm{d}u-\int_t^T \hat{B}^{\star-1}\Gamma(u,S_u)\,\mathrm{d}u\middle|\mathcal{F}_t\right],\\ M_t^{\mu^\star} &= \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{\star-1}\Phi(S_T)+\int_0^T \hat{B}_u^{\star-1}\Psi(u,S_u)\mu^\star(u,S_u)\,\mathrm{d}u-\int_0^T \hat{B}_u^{\star-1}\Gamma(u,S_u)\,\mathrm{d}u\middle|\mathcal{F}_t\right],\end{aligned}$$

where $\hat{B}_\cdot^\star=\exp\{\int_0^\cdot(r(s)+\mu^\star(s,S_s))\,\mathrm{d}s\}$, for $0\le t\le T$, it holds that $U_t^{\mu^\star}=\bar{v}(t,S_t)$ and $M^{\mu^\star}$ is an $\mathbb{F}^S$-adapted $(\mathbb{Q},\mathbb{F})$-martingale. For $\eta\in\{\mu,\mu^\star\}$ we can connect $U^\eta$ and $M^\eta$ via

$$M_t^\eta = e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,U_t^\eta+\int_0^t e^{-\int_0^u(r(s)+\eta_s)\,\mathrm{d}s}\,\Psi(u,S_u)\,\eta_u\,\mathrm{d}u-\int_0^t e^{-\int_0^u(r(s)+\eta_s\,\mathrm{d}s}\,\Gamma(u,S_u)\,\mathrm{d}u,$$

or, alternatively, in the form of the stochastic differential

$$\begin{aligned}\mathrm{d}M_t^\eta &= e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,\mathrm{d}U_t^\eta-(r(t)+\eta_t)\,e^{-\int_0^t(r(s)+\eta_s\,\mathrm{d}s}\,U_t^\eta\,\mathrm{d}t\\ &+e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,\Psi(t,S_t)\,\eta_t\,\mathrm{d}t-e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,\Gamma(t,S_t)\,\mathrm{d}t,\quad 0\le t\le T.\end{aligned}$$

Solving for $\mathrm{d}U_t^\eta$ yields

$$\mathrm{d}U_t^\eta = (r(t) + \eta_t)\, U_t^\eta\, \mathrm{d}t - \Psi(t, S_t)\, \eta_t\, \mathrm{d}t + \Gamma(t, S_t)\, \mathrm{d}t + \mathrm{d}\hat{M}_t^\eta\,, \quad 0 \le t \le T\,,$$

where $\mathrm{d}\hat{M}_t^\eta = e^{\int_0^t (r(s)+\eta_s)\,\mathrm{d}s}\,\mathrm{d}M_t^\eta$ is a $(\mathbb{Q}, \mathbb{F})$-martingale since $r$ and $\eta$ are uniformly bounded by a deterministic constant. The next step is to define the difference process

$$X_t^\mu = U_t^{\mu^\star} - U_t^\mu\,, \quad 0 \le t \le T\,.$$

The terminal value of $X^\mu$ is $X_T^\mu = U_T^{\mu^\star} - U_T^\mu = \Phi(S_T) - \Phi(S_T) = 0$. The stochastic differential of $X^\mu$ is given by

$$\begin{aligned} \mathrm{d}X_t^\mu &= \mathrm{d}U_t^{\mu^\star} - \mathrm{d}U_t^\mu \\ &= \Big[(r(t) + \mu_t)\, X_t^\mu + \Big(U_t^{\mu^\star} - \Psi(t, S_t)\Big)\, (\mu^\star(t, S_t) - \mu_t)\Big]\, \mathrm{d}t + \mathrm{d}\hat{M}_t^{\mu^\star} - \mathrm{d}\hat{M}_t^\mu\,. \end{aligned}$$

The above differential can be interpreted as a linear stochastic differential with formal solution

$$\begin{aligned} X_u^\mu &= e^{\int_t^u r(s)+\mu_s\,\mathrm{d}s}\Big(X_t^\mu + \int_t^u e^{-\int_t^w (r(s)+\mu_s)\,\mathrm{d}s}(\mathrm{d}\hat{M}_w^{\mu^\star} - \mathrm{d}\hat{M}_w^\mu) \\ &\quad - \int_t^u e^{-\int_t^w r(s)+\mu_s\,\mathrm{d}s}\, \big[U_w^{\mu^\star} - \Psi(w, S_w)\big]\, \left[\mu^\star(w, S_w) - \mu_w\right]\, \mathrm{d}w\Big)\,, \quad 0 \le t \le u \le T\,. \end{aligned}$$

Set $u = T$, solve for $X_t^\mu$ and recall $X_T^\mu = 0$. Taking the expectation conditioned on $\mathcal{F}_t$ eliminates the $(\mathbb{Q}, \mathbb{F})$-martingale and

$$X_t^\mu \;=\; \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^w r(s)+\mu_s\,\mathrm{d}s}\, \big[U_w^{\mu^\star} - \Psi(w, S_w)\big]\, \left[\mu^\star(w, S_w) - \mu_w\right]\, \mathrm{d}w \,\middle|\, \mathcal{F}_t\right]$$

Recall that $U_t^{\mu^\star} = \bar{v}(t, S_t)$ and $\mu \in \mathcal{Q}^b$, i.e., $\underline{\mu}(t) \le \mu_t \le \bar{\mu}(t)$, for $0 \le t \le T$. Then by the definition of $\mu^\star$ in (25) we have that the integrand inside the conditional expectation is nonnegative. This implies that $X_t^\mu \ge 0$ or, equivalently, $\bar{v}(t, S_t) \ge U_t^\mu$, for $0 \le t \le T$. Multiplying on the indicator process $\mathbf{1}_{\{t<\tau\}}$ we obtain

$$\mathbf{1}_{\{t<\tau\}}\, \bar{v}(t, S_t) \;\ge\; V_t^\mu\,, \quad 0 \le t \le T\,,$$

for all $\mathbb{Q} \in \mathcal{Q}^b$ and corresponding $\mathbb{Q}$-mortality rate $\mu$, establishing the optimality. □

**Remark 3.** The lower bound of the arbitrage free prices $\underline{V}_t = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu}$ can be obtained analogously. The minimizer $\mu_\star$ is obtained by swapping $\underline{\mu}$ and $\bar{\mu}$ in (25). The solution to the partial differential equation (26) where $\mu_\star$ is replaced by $\mu^\star$ with terminal condition (27) is denoted $\underline{v}$. Then $\mathbf{1}_{\{t<\tau\}}\,\underline{v}(t,S_t) = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu}$.

**Remark 4.** The value maximizing mortality $\mu^\star$ in(25) is $\mathbb{F}^S$-adapted. However, the set of admissible controls is much larger allowing for $\mathbb{F}$-adapted control processes. Obviously, the information generated by the environment process $X$ is not needed for finding the maximal arbitrage free price. This result can be explained by properties of our model. The risk introduced by $X$ cannot be hedged since no liquidly traded assets are available for trading and potentially eliminating the associated risk. Further, the bounds $\underline{\mu}$ and $\bar{\mu}$ in Assumptions 1 and 2 are almost sure bounds and do not depend on the environment process $X$. Both properties together explain that the optimal control process $\mu^\star$ can be determined based on $\mathbb{F}^S$, the information generated by the traded asset with price process $S$.

**Remark 5.** We can summarize the optimal control rule concerning the death benefit, the survival benefit, the premium and the whole contract as follows so as to obtain their worst-case and best-case values.

| | *worst case* | *best case* |
|---|---|---|
| *death benefit* | $\bar{\mu}$ *if* $\Psi \geq \bar{v}^{\Psi}$ | $\underline{\mu}$ *if* $\Psi \geq \underline{v}^{\Psi}$ |
| | $\underline{\mu}$ *if* $\Psi < \bar{v}^{\Psi}$ | $\bar{\mu}$ *if* $\Psi < \underline{v}^{\Psi}$ |
| *survival benefit* | $\underline{\mu}$ | $\bar{\mu}$ |
| *premium* | $\bar{\mu}$ | $\underline{\mu}$ |
| *whole contract* | $\bar{\mu}$ *if* $\Psi \geq \bar{v}$ | $\underline{\mu}$ *if* $\Psi \geq \underline{v}$ |
| | $\underline{\mu}$ *if* $\Psi < \bar{v}$ | $\bar{\mu}$ *if* $\Psi < \underline{v}$ |

The worst-case value of the contract is its upper price bound and the best-case value is the lower price bound of the contract.

Theorem 1 indicates that the price bound of an insurance contract usually cannot be obtained by keeping $\mu$ to its lower or upper bound. The simple rule of keeping $\mu$ to its lower or upper bound is only possible for some special cases. Here are two examples:

*Pure endowment insurance with single premium.* In this case, we have $\Psi = 0$ and $\Gamma = 0$, and hence the value of $\mu$ is irrelevant for the death benefit part and the premium part. The maximal value for the survival benefit and is obtained by setting $\mu = \underline{\mu}$ and then is $\bar{v} = \bar{v}^{\Phi}$, on $(0, T]$. Similarly, the minimal value for the survival benefit is obtained by setting $\mu = \bar{\mu}$ and then $\underline{v} = \underline{v}^{\Phi}$, on $[0, T)$.

*Term insurance with single premium or periodic premiums.* The death benefit takes the form $\Psi(t, s) = Ke^{gt}$ with $g \leq r$ or $\Psi(t, s) = S_t$. In the former case, we have

$$\begin{aligned} v^{\mu,\Psi}(t, s) &= K\,\mathbb{E}^{t,s}\left[\int_t^T \exp\left(-\int_t^u (r(s) - g)s\,\mathrm{d}s\right)\exp\left(-\int_t^u \mu_s\,\mathrm{d}s\right)\mu_u\,\mathrm{d}u\right] \\ &\leq K\,\mathbb{E}^{t,s}\left[\int_t^T \exp\left(-\int_t^u \mu_s\,\mathrm{d}s\right)\mu_u\,\mathrm{d}u\right] \\ &\leq K\left(1 - \mathbb{E}^{t,s}\left[\exp\left(-\int_t^T \mu_u\,\mathrm{d}u\right)\right]\right) \;\leq\; K \;\leq\; Ke^{gt} = \Psi(t, s), \end{aligned}$$

and in the latter case, there is

$$\begin{aligned} v^{\mu,\Psi}(t, s) &= \mathbb{E}^{t,s}\left[\exp\left(-\int_t^\tau r(s)ds\right)\,S_\tau\,1_{\{\tau\leq T\}}\middle|\,\tau > t\right] \\ &\leq \mathbb{E}^{t,s}\left[\exp\left(-\int_t^\tau r(s)ds\right)\,S_\tau\middle|\,\tau > t\right] \;=\; s \;=\; \Psi(t, s). \end{aligned}$$

In both cases we know that $v^{\mu} = v^{\mu,\Psi} - v^{\mu,\Gamma} \leq v^{\mu,\Psi} \leq \Psi$ and therefore the maximum (minimum) value is obtained when $\mu$ is set to $\bar{\mu}$ ($\underline{\mu}$). In the single premium case we have $\bar{v} = \bar{v}^{\Psi}$ and $\underline{v} = \underline{v}^{\Psi}$, on $[0, T)$. In the periodic premium case we have $\bar{v} = \bar{v}^{\Psi} - \underline{v}^{\Gamma}$ and $\underline{v} = \underline{v}^{\Psi} - \bar{v}^{\Gamma}$, on $[0, T)$.

### *4.3. Connection to American-style Financial Contracts*

We return to the partial differential equation for a given $\mu$:

$$0 \;=\; \mathcal{L}v(u, s) + \Psi(u, s)\,\mu(u, s) - \Gamma(u, s) - v(u, s)\,[\mu(u, s) + r(u)], \quad t \leq u \leq T, \tag{28}$$

with terminal condition $v(T, s) = \Phi(s)$. If we allow $\mu$ to move between $[0, \infty)$ which is beyond its original bounds, by setting

$$\mu(u, s) = \begin{cases} 0, & \text{if } \; v(u, s) > \Psi(u, s) \\ \infty, & \text{if } \; v(u, s) \leq \Psi(u, s) \end{cases} \tag{29}$$

for $t \leq u < T$, we force the contract to stop immediately when the contract value reaches the death benefit from above so that $v(u,s) \geq \Psi(u,s)$ is always satisfied. This is equivalent to an optimal stopping problem, whose linear complementarity formulation is

$$\begin{aligned} [\mathcal{L}v(u,s) - \Gamma(u,s) - r(u)\,v(u,s)]\;[v(u,s) - \Psi(u,s)] \;&=\; 0, \\ \mathcal{L}v(u,s) - \Gamma(u,s) - r(u)\,v(u,s) \;\leq\; 0, \quad \text{and} \quad v(u,s) - \Psi(u,s) \;&\geq\; 0, \end{aligned}$$

for $t \leq u < T$, with adjusted terminal condition $v(T,s) = \max(\Phi(s), \Psi(T,s))$. Similar to Dai et al. (2007) where the prepayment of mortgage loans is discussed, (28) together with (29) can be visualized as the penalty approximation to the linear complementarity formulation following the theory of variational inequalities of free boundary problems.

**Remark 6.** Equation (29) specifies the optimal control of $\mu$ within a broader bound which corresponds to the larger set of martingale measure $\mathcal{Q}$ defined by (9). Setting $\mu = \mu^\star$ in (28), where $\mu^\star$ is given in (25), can be viewed as using a suboptimal stopping strategy which does not necessarily yield $v(u,s) \geq \Psi(u,s)$. The value function of the optimal stopping problem will produce a superhedge. In contrast the upper price bound $\bar{v}$ based on $\mathbb{Q}^b$ cannot produce a superhedge in general. However, a superhedge will arise when diversification of the unsystematic mortality risk is taken into account.

## 5. Hedging Unit-Linked Life Insurance Contracts

The upper and lower price bounds for unit-linked life insurance contracts in Theorem 1 and Remark 3 suggest the implementation of hedging strategies related to these bounds. The financial risk driven by $S$ can be eliminated by these strategies since the risk is represented by a traded asset. In contrast, the mortality risk cannot be eliminated in general. The trading strategies corresponding to the upper (lower) price bound cannot produce a superhedge (subhedge) for a specific single contract. However, mortality risk can be diversified by considering a sufficiently large number of policyholders, and then a superhedge (subhedge) can be produced in the limit.

We consider a community of policyholders of size $N$ where the contracts for each individual are identical and given by $(\Phi, \Psi, \Gamma)$. Further, we assume that the death times of the policyholders $(\tau_i)_{i=1,\dots,N}$ are independent given $\mathcal{F}_T$.[8] The number of individuals that have died until $t$ is denoted by $X_t^N$ and the number of policyholders that are still alive at time $t$ is denoted by $\bar{X}_t^N$, respectively, i.e.

$$X_t^N = \sum_{i=1}^N \mathbf{1}_{\{\tau_i \le t\}} \,, \qquad \text{and} \qquad \bar{X}_t^N = N - X_t^N = \sum_{i=1}^N \mathbf{1}_{\{t<\tau_i\}} \,, \quad 0 \le t \le T \,. \tag{30}$$

Fix as input parameter the potentially misspecified mortality intensity $\tilde{\mu} = (\tilde{\mu}(t, S_t))_{0\le t\le T}$ that is Markovian with state vector $(t, S_t)$. Compute the price of the contract of a policyholder who is alive at time $t$ as solution $\tilde{v}$ to

$$\mathcal{L}\tilde{v}(t,s) - \Gamma(t,s) + \tilde{\mu}(t,s)\,[\Psi(t,s) - \tilde{v}(t,s)] - r(t)\tilde{v}(t,s) \;=\; 0, \qquad \text{and} \qquad \tilde{v}(T,s) = \Phi(s), \tag{31}$$

where $\mathcal{L}$ is given in (24).

The potentially misspecified value $\tilde{V}^N$ of the outstanding contracts is thus given by the number of living policyholders $\bar{X}^N$ times the corresponding value $\tilde{v}$, i.e.

$$\tilde{V}_t^N = \bar{X}_t^N \, \tilde{v}(t, S_t) \,, \quad 0 \le t \le T \,. \tag{32}$$

In the formulation of $\tilde{V}^N$ in (32) we have two potential sources of error. Firstly, the individual contract may be incorrectly priced by $\tilde{v}$ and secondly, the mortality risk cannot be hedged in our setup and thus the jumps of $\bar{X}^N$ will introduce a further error. In the following we analyze the error when setting up a hedging portfolio based on (32).

The hedging strategy and resulting portfolio processes are now specified. Let the left-continuous and adapted process $h^N$ denote the holdings in the risky asset $S$. (We do not have to specify the holding in the money market account since we are considering self-financing strategies and can use the budget constraint.) The portfolio process $V^N$ with initial value $V_0^N$ is defined by the stochastic differential equation

$$\mathrm{d}V_t^N = (V_t^N - h_t^N S_t)\, r(t)\, \mathrm{d}t + h_t^N\, (\mathrm{d}S_t + \mathrm{d}D_t) + \bar{X}_t^N\, \Gamma(t, S_t)\, \mathrm{d}t - \Psi(t, S_t)\, \mathrm{d}X_t^N \,. \tag{33}$$

[8] The underlying model in Section 2.1 is extended canonically, i.e., take an i.i.d. family of r.v.s $(E_n)_{n=1,\dots,N}$ that are unit exponentially distributed and independent of $W$. Then define the death times $(\tau_n)_{n=1,\dots,N}$ by $\tau_n = \operatorname{ess\,inf}\{t \ge 0 : \int_0^t \nu_s \,\mathrm{d}s \ge E_n\}$, $n = 1, \dots, N$.

The portfolio process is self-financing given the inflow of premium payment rate $\Gamma$ of the active contracts $\bar{X}^N$ and the discrete time outflows of the death benefits $\Psi$ at times the individuals pass away given by $X^N$. The hedge implied by (32) is given by

$$h_t^N = \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t, S_t), \quad 0 \leq t \leq T. \tag{34}$$

It is clear that the price $\tilde{v}$ and the corresponding hedge ratio $h$ are both determined by the assumed mortality intensity $\tilde{\mu}$. For a specific choice of $\tilde{\mu}$ the resulting hedging error is analyzed under the real world measure $\mathbb{P}$. The error has two additive components: a jump-martingale component capturing unsystematic mortality risk, and a predictable finite variation component which is determined by the systematic mortality risk.

**Theorem 2.** *Fix $\tilde{\mu} = \tilde{\mu}(t, s)$ and determine $\tilde{v}$ as the solution to* (31)*. Fix the size of the community of policyholders $N$ and then define $h^N$ by* (34)*. For the corresponding portfolio process $V^N$ in* (33) *the hedging error $E^N$ relative to $\tilde{V}^N$ given in* (32) *is defined by $E^N = V^N - \tilde{V}^N$. Then the hedging error and has the following $\mathbb{P}$-dynamics:*

$$\mathrm{d}E_t^N = E_t^N\, r(t)\,\mathrm{d}t + \bar{X}_t^N\,[\Psi(t,S_t) - \tilde{v}(t,S_t)]\,[\tilde{\mu}(t,S_t) - \nu_t]\,\mathrm{d}t + [\tilde{v}(t,S_t) - \Psi(t,S_t)]\,\mathrm{d}M_t^N\,,$$

*with initial value $E_0^N = V_0^N - N\,\tilde{v}(0, S_0)$. Moreover, $M^N = X^N - \int_0^\cdot \bar{X}_t^N\,\nu_t\,\mathrm{d}t$ is a $\mathbb{P}$-martingale.*

*Proof of Theorem 2.* Write the stochastic differential of the portfolio value process using the definitions of the strategy $h^N$ in (34) and the risky assets dynamics in (1) and (2):

$$\begin{aligned}
\mathrm{d}V_t^N &= \left[V_t^N - \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,S_t\right] r(t)\,\mathrm{d}t + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,[\alpha(t,S_t) + q(t,S_t)]\,S_t\,\mathrm{d}t \\
&\quad + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,\sigma(t,S_t)\,S_t\,\mathrm{d}W_t + \bar{X}_t^N\,\Gamma(t,S_t)\,\mathrm{d}t - \Psi(t,S_t)\,\mathrm{d}X_t^N \\
&= V_t^N\,r(t)\,\mathrm{d}t + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,\left(\alpha(t,S_t)\,S_t\,\mathrm{d}t + \sigma(t,S_t)\,S_t\,\mathrm{d}W_t\right) - \Psi(t,S_t)\,\mathrm{d}M_t^N \\
&\quad - \bar{X}_t^N \left[\frac{\partial \tilde{v}}{\partial s}(t,S_t)\,[r(t) - q(t,S_t)]\,S_t - \Gamma(t,S_t) + \Psi(t,S_t)\,\nu_t\right] \mathrm{d}t\,.
\end{aligned}$$

To the last line we apply the partial differential equation (31) and then

$$\begin{aligned}
\mathrm{d}V_t^N &= V_t^N\, r(t)\,\mathrm{d}t + \bar{X}_t^N\, \frac{\partial \tilde{v}}{\partial s}(t,S_t)\ [\alpha(t,S_t)\, S_t\, \mathrm{d}t + \sigma(t,S_t)\, S_t\, \mathrm{d}W_t] - \Psi(t,S_t)\,\mathrm{d}M_t^N \\
&\quad + \bar{X}_t^N \Big[\frac{\partial \tilde{v}}{\partial t}(t,S_t) + \frac{1}{2}\,\sigma^2(t,S_t)\,\frac{\partial^2 \tilde{v}}{\partial s^2}(t,S_t) - r(t)\,\tilde{v}(t,S_t) \\
&\qquad + [\tilde{\mu}(t,S_t) - \nu_t]\,[\Psi(t,S_t) - \tilde{v}(t,S_t)] - \tilde{v}(t,S_t)\,\nu_t\Big]\,\mathrm{d}t\,.
\end{aligned}$$

Using the product rule we obtain the stochastic differential of $\tilde{V}^N = \bar{X}^N\, \tilde{v}(\cdot, S)$:

$$\begin{aligned}
\mathrm{d}\tilde{V}_t^N &= \bar{X}_t^N \left[\frac{\partial \tilde{v}}{\partial t}(t,s) + \alpha(t,S_t)\, S_t\, \frac{\partial \tilde{v}}{\partial s}(t,s) + \frac{1}{2}\sigma(t,S_t)^2\, S_t^2\, \frac{\partial^2 \tilde{v}}{\partial s^2}(t,S_t) - \tilde{v}(t,S_t)\,\nu_t\right]\mathrm{d}t \\
&\quad + \bar{X}_t^N\, \sigma(t,S_t)\, S_t\, \frac{\partial \tilde{v}}{\partial s}(t,s)\,\mathrm{d}W_t - \tilde{v}(t,S_t)\,\mathrm{d}M_t^N\,.
\end{aligned}$$

Now collect the terms from the two equations above to compute stochastic differential of the hedging error $E^N = V^N - \tilde{V}^N$:

$$\begin{aligned}
\mathrm{d}E_t^N &= E^N\, r(t)\,\mathrm{d}t + \bar{X}_t^N\,[\tilde{\mu}(t,S_t) - \nu_t]\,[\Psi(t,S_t) - \tilde{v}(t,S_t)]\,\mathrm{d}t \\
&\quad + [\tilde{v}(t,S_t) - \Psi(t,S_t)]\,\mathrm{d}M_t^N\,.
\end{aligned}$$

Finally, to verify that $M^N$ is a $\mathbb{P}$-martingale see Bielecki and Rutkowski (2001), Proposition 5.1.3., p. 153. □

**Remark 7.** The stochastic differential equation for hedge error $E^N$ in Theorem 2 has the straight-forward solution

$$\begin{aligned}
E_t^N &= e^{\int_0^t r(s)\,\mathrm{d}s}\,\big(V_0^N - N\,\tilde{v}(0,S_0)\big) + \int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,\bar{X}_u^N\,[\Psi(u,S_u) - \tilde{v}(t,S_u)]\,[\tilde{\mu}(t,S_u) - \nu_u]\,\mathrm{d}u \\
&\quad + \int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,[\tilde{v}(u,S_u) - \Psi(u,S_u)]\,\mathrm{d}M_u^N\,.
\end{aligned}$$

Suppose that the community of policyholders is very large, then the following corollary gives a limit result.

**Corollary 1.** *In the setting of Theorem 2 define the scaled hedging error* $\bar{E}_t^N$ *by*

$$\bar{E}_t^N = \frac{1}{N}\,E_t^N\,, \quad 0 \le t \le T\,.$$

*Assume that the following limit exists* $\lim_{N\to\infty} V_0^N/N =: v$*, then*

$$\sup_{0\le t\le T} \left|\bar{E}_t^N - \bar{E}_t^\infty\right| \xrightarrow{\mathbb{P}} 0\,,$$

*where*

$$\begin{aligned}\bar{E}_t^\infty &= e^{\int_0^t r(s)\,\mathrm{d}s}\,(v - \tilde{v}(0,S_0)) \\ &\quad + \int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\, e^{-\int_0^u \nu_s\,\mathrm{d}s}\,[\Psi(u,S_u) - \tilde{v}(u,S_u)]\,[\tilde{\mu}(u,S_u) - \nu_u]\,\mathrm{d}u\,.\end{aligned}$$

*Proof of Corollary 1.* See Appendix A. □

**Remark 8.** Corollary 1 can be applied to the upper price bound $\bar{v}$ with mortality intensity $\mu^\star$ defined in (25), see Theorem 1. The normalized hedge error $\bar{E}^N$ converges uniformly in probability to $\bar{E}^\infty$ by Corollary 1. And the $\mathbb{P}$-dynamics of $\bar{E}^\infty$ are given by

$$\mathrm{d}E_t^\infty = E_t^\infty\, r(t)\,\mathrm{d}t + e^{-\int_0^t \nu_u\,\mathrm{d}u}\,[\Psi(t,S_t) - \bar{v}(t,S_t)]\,[\mu^\star(t,S_t) - \nu_t]\,\mathrm{d}t\,, \quad 0\le t\le T\,.$$

Observe that $\bar{E}^\infty$ is of finite variation and, assuming a nonnegative initial value, $E_0^\infty \ge 0$, is nondecreasing. To see this, recall that by Assumption 1 the realized mortality rate $\nu$ is bounded, i.e. $\underline{\mu}(t) \le \nu_t \le \bar{\mu}(t)$, $0 \le t \le T$. The value function $\bar{v}$ and mortality intensity $\mu^\star$ are specified such that $[\Psi(t,S_t) - \bar{v}(t,S_t)]\,[\mu^\star(t,S_t) - \nu_t]$ is nonnegative. Accordingly, the upper price bound $\bar{v}$ indeed produces a superhedge when we are allowed to diversify the unsystematic mortality risk by the law of large numbers.

## 6. Numerical Results

In this section we analyze several unit-linked life insurance contracts in the uncertain mortality intensity framework. In addition to providing price bounds we also produce the optimally controlled regions of the mortality intensity.

As the underlying asset we take the S&P 500 index in the USA with the starting price $S_0 = 1073$. The dividend rate is assumed to be 0. We adopt the assumption that the volatility is constant over the time with $\sigma = 0.1833$. With regard to the interest rate, we assume that it is constant over the time with $r = 0.03$. The life insurance contracts that we

| | payoff $\Psi(\tau, S_\tau)$ at $\tau$ (death) | payoff $\Phi(S_t)$ at $T$ (survival) |
|---|---|---|
| I | $S_\tau$ | $\max(S_0 e^{g_1 T}, S_T)$ |
| II | $S_0 e^{g_1 \tau}$ | $\max(S_0 e^{g_1 T}, S_T)$ |
| III | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $S_T$ |
| IV | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $\max(S_0 e^{g_1 T}, S_T)$ |
| V | $\min(S_0 e^{g_2 \tau}, S_\tau)$ | $\min(S_0 e^{g_2 T}, S_T)$ |
| VI | $\min(\max(S_0 e^{g_1 \tau}, S_\tau), S_0 e^{g_2 \tau})$ | $\min(\max(S_0 e^{g_1 T}, S_T), S_0 e^{g_2 T})$ |

Table 1: Payoff structures for different life insurance contracts.

study are described in Table 1 where $g_1 = 0.02$ refers to the minimum guarantee rate and $g_2 = 0.06$ refers to capped rate.

The policyholders are supposed to be 40 years old at the beginning and the contract lasts 30 years. In Figure 1, we display the forecast of the mortality intensity $\hat{\nu}$ over the next 30 years based on the model and results of Lee and Carter (1992). Additionally, upper and lower bounds are included corresponding to a pointwise 99.9% confidence interval, see Appendix C for details. In the following, these bounds are assumed to be the mortality bounds $\underline{\mu}$ and $\bar{\mu}$ for Assumptions 1 and 2.

We consider both the single premium and the periodic premium cases. In the single premium case, we calculate the lump sum amount a policyholder needs to pay if the mortality intensity moves in the most adverse way from the viewpoint of the insurance company such that the discounted benefit payment is maximized in expectation. In the periodic premium case, we assume that the policyholder pays continuously a prespecified cash flow until he dies. The prespecified cash flow is defined to be a fixed amount which is obtained by meeting the following fair premium principle.

**Definition 2.** *A unit-linked life insurance contract is fair if and only if the expected payment to the policyholder equals the expected premium paid by the policyholder at the initial date under the measure $\mathbb{Q} \in \mathcal{Q}^b$ with $\mathbb{Q}$-mortality intensity $\hat{\nu}$.*

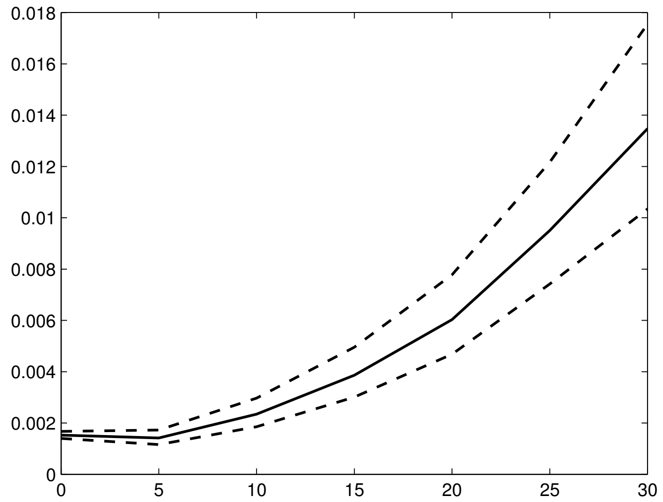


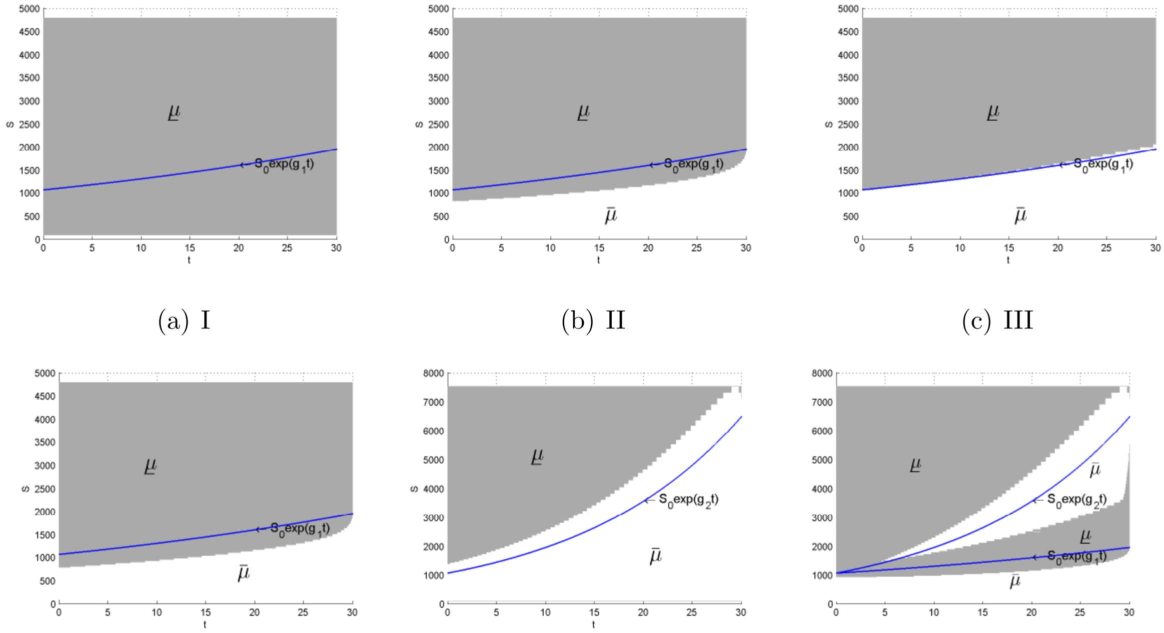

Figure 1: The forecast of the mortality intensity including bounds based on pointwise 99.9% confidence level.

*6.1. Single Premium Case*

We first analyze the single premium case. In Figure 2 we present the optimally controlled regions where $\mu$ should be set to its lower bound $\underline{\mu}$ or to its upper bound $\bar{\mu}$ so that the upper price bounds of the contracts are obtained.

For contract type I, $\mu$ needs merely to be set to its lower bound $\underline{\mu}$ during the whole life time of the contract. By looking more closely at its payoff structure, we see that the reason is self-evident. The present value of the survival payoff exceeds the present value of the death benefit. Therefore, the contract value $v(t, S_t)$ is always greater than the death benefit $\Psi(t, S_t)$. According to Theorem 1, we will obtain the upper price bound by always setting $\mu$ to $\underline{\mu}$. However, in most cases, we cannot follow the simple rule of restricting to one bound of $\mu$. The dynamic control scheme recommended in Theorem 1 should be implemented to get the price bounds.

For contract types II-IV, the upper bound and the lower bound regions of the mortality intensity are divided slightly below or at the minimum guarantee curve $S_0 e^{g_1 t}$, $0 \leq t \leq T$. When the asset price remains at a much lower level than the minimum guarantee curve,




(a) I (b) II (c) III

(d) IV (e) V (f) VI

Figure 2: Optimally controlled regions for $\mu$ (single premium)

the chance for the policyholder to participate in the earnings of the risky asset is very low. The policyholder has higher possibility to obtain the minimum guaranteed amount which, however, increases at a lower rate than the investment in the riskless money market amount. Hence, it is optimal if the policyholder is able to quit the contract for the better investment alternative. The mortality intensity $\mu$ should be set to $\bar{\mu}$. On the other hand, if the asset price is high enough, the policyholder can benefit more from the risky asset which is even protected by the minimum guarantee, and hence, a higher mortality intensity becomes optimal.

For contract type V, we also see that the upper-left part of the figure is the $\underline{\mu}$-region and the lower-right part is the $\bar{\mu}$-region. The two regions are divided near the curve of the capped amount $S_0\, e^{g_2 t}$, $0 \leq t \leq T$. When the asset price is well above this curve it is optimal to keep the mortality intensity to its minimum $\underline{\mu}$. The policyholder has a high possibility of obtaining the capped amount growing at rate $g_2$ which is higher than risk-free rate of $r$. On the contrary, when the asset price is lower than the capped amount, the immediate death benefit is higher than any later proceeds from the contract. The mortality intensity should be set to its upper bound $\bar{\mu}$. The payoff structure of the contract type VI is the mixture the contract types IV and V, and hence, the optimally controlled regions of $\mu$ shown in Figure 2 (f) is also the mixture of Figure 2 (d) and 2 (e).

In Table 2 we present the pricing results of the single premium case by inserting different scenarios of the mortality intensity to the examples we have presented in Table 1.

| | $\Psi$ | $\Phi$ | $\mu = \mu^f$ | $\mu = \underline{\mu}$ | $\mu = \bar{\mu}$ | $\mu \in [\underline{\mu}, \bar{\mu}]$ | | $\mu \in [0, \infty)$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | lower | upper | lower | upper |
| I | $S_\tau$ | $\max(S_0 e^{g_1 T}, S_T)$ | 1267.4 | 1275.2 | 1257.8 | 1257.8 | 1275.2 | 1073.0 | 1307.5 |
| II | $S_0 e^{g_1 \tau}$ | $\max(S_0 e^{g_1 T}, S_T)$ | 1228.4 | 1242.7 | 1211.0 | 1203.8 | 1248.0 | 795.1 | 1357.3 |
| III | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $S_T$ | 1109.6 | 1102.4 | 1118.4 | 1102.2 | 1118.7 | 1073.0 | 1357.2 |
| IV | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $\max(S_0 e^{g_1 T}, S_T)$ | 1303.9 | 1304.6 | 1303.2 | 1301.2 | 1306.4 | 1075.3 | 1357.3 |
| V | $\min(S_0 e^{g_2 \tau}, S_\tau)$ | $\min(S_0 e^{g_2 T}, S_T)$ | 916.4 | 916.2 | 916.8 | 914.3 | 918.7 | 855.6 | 1071.0 |
| VI | $\min(\max(S_0 e^{g_1 \tau}, S_\tau)$ $, S_0 e^{g_2 \tau})$ | $\min(\max(S_0 e^{g_1 T}, S_T)$ $, S_0 e^{g_2 T})$ | 1147.3 | 1147.6 | 1146.9 | 1143.8 | 1150.7 | 1010.4 | 1252.9 |

Table 2: Contract prices with different scenarios of the mortality intensity (single premium)

In the fourth to the sixth columns the mortality intensity is not dynamically controlled but only set to the forecasted value, its lower bound and its upper bound respectively. The seventh and eighth columns show the results when the mortality intensity $\mu$ lies in $[\underline{\mu}, \bar{\mu}]$. When $\mu$ is controlled least optimally, we obtain the lower price bound in the seventh column; while the upper price bound in the eighth column is obtained when $\mu$ is controlled most optimally. Furthermore, we present the case when $\mu$ lies in $[0, \infty)$. This is an unrealistic representation of the mortality risk, which corresponds to the choice of the optimal equivalent martingale measure within the whole class of $\mathcal{Q}$ so that the price is minimized (column 9) or maximized (column 10). As we have discussed in Section 4.3, the maximized price is equal to the price of a pure American-style financial contract. The upper price is the initial wealth required for setting up a superhedge when diversification of the unsystematic mortality risk is ignored, see Remark 6.

As we have analyzed previously, for contract type I we follow the simple rule of keeping to $\underline{\mu}$ ($\bar{\mu}$) in order to obtain the upper (lower) price bound. This is can be seen once again in Table 2. While for the other contract types, the prices obtained by simply inserting $\underline{\mu}$

and $\bar{\mu}$ as well as $\mu^f$ over the time interval $[0, T]$ all lie within the lower and the upper price bounds. However, when we observe the differences between the two price bounds, we find they are not significantly big.[9] Even for contract type II where we see the largest gap, it has not exceeded 4% of the contract price. Hence, for the pricing purpose, it does not matter too much which scenario of the mortality intensity we implement into the pricing problem as long as it is a reasonable scenario within its confidence interval. This result indicates that mortality model risk does not have a huge effect on the risk management of unit-linked life insurance contracts we focus on, see Table 1. This argument is also valid when we increase the confidence level of the mortality intensity. In Table 3 we present the lower and upper price bounds at the different confidence levels of 99.9%, 99.99% and 99.999%. We notice that the differences between the lower and the upper price bounds are although wider but have not varied too much over the different confidence levels.

| | $\Psi$ | $\Phi$ | 99.9% | | 99.99% | | 99.999% | |
|---|---|---|---|---|---|---|---|---|
| | | | lower | upper | lower | upper | lower | upper |
| I | $S_\tau$ | $\max(S_0 e^{g_1 T}, S_T)$ | 1257.8 | 1275.2 | 1254.8 | 1277.2 | 1252.2 | 1278.7 |
| II | $S_0 e^{g_1 \tau}$ | $\max(S_0 e^{g_1 T}, S_T)$ | 1203.8 | 1248.0 | 1198.7 | 1251.1 | 1194.0 | 1253.7 |
| III | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $S_T$ | 1102.2 | 1118.7 | 1100.4 | 1121.5 | 1099.0 | 1124.0 |
| IV | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $\max(S_0 e^{g_1 T}, S_T)$ | 1301.2 | 1306.4 | 1300.7 | 1306.9 | 1300.2 | 1307.3 |
| V | $\min(S_0 e^{g_2 \tau}, S_\tau)$ | $\min(S_0 e^{g_2 T}, S_T)$ | 914.3 | 918.7 | 913.9 | 919.1 | 913.5 | 919.5 |
| VI | $\min(\max(S_0 e^{g_1 \tau}, S_\tau), S_0 e^{g_2 \tau})$ | $\min(\max(S_0 e^{g_1 T}, S_T), S_0 e^{g_2 T})$ | 1143.8 | 1150.7 | 1143.1 | 1151.3 | 1142.5 | 1151.9 |

Table 3: Price bounds for different confidence levels (single premium)

Melnikov and Romaniuk (2006) show that different mortality models display different risk management performances for unit-linked pure endowment contracts. These are not contradictory results but give us the hint that mortality model risk can be alleviated by contract design. For contract type I and contract types III-VI, both the death benefit and

[9]To present this issue into more detail we show the differences between the upper and lower price bounds over the life time of the contracts depending on the price of the underlying in Figure B.4 in Appendix B. Although the maximal difference for contract type II in Figure B.4 (b) is near 400 when the underlying asset price is near 6000, it is not crucial when we take it into consideration that the possibility for the underlying asset price to rise to such a high level from the starting value of 1073 is quite small.

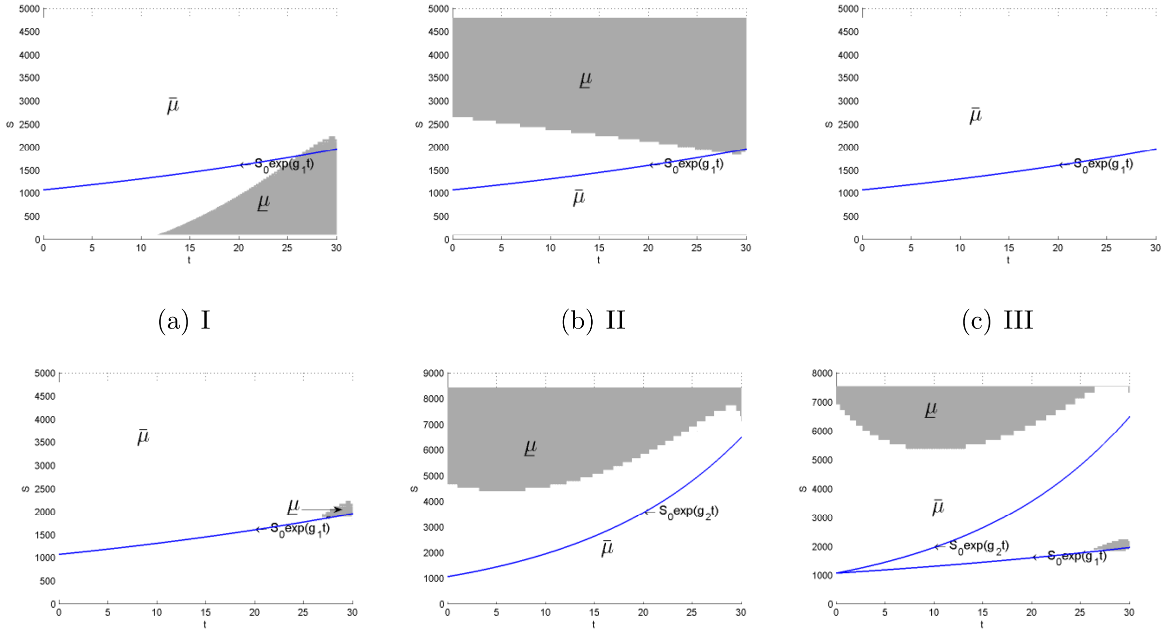
the terminal payment are strongly associated with the performance of the underlying asset. Hence the contracts are more a financial product than an insurance product. For contract type II as well as the unit-linked pure endowment insurance in Melnikov and Romaniuk (2006), the death benefit is either a deterministic amount independent of the index performance or is zero. The risk profiles of the death benefit before time $T$ and the survival benefit at time $T$ are quite different which makes it crucial to know whether the death event may take place earlier or later.

*6.2. Periodic Premium Case*

Now we come to the periodic premium case. We consider the same payoff structures as before, see Table 1. However, the policyholder does not need to pay the premium at the beginning but pays it in arrears during the life time of the contract but maximally till his death time. For simplicity, we assume that the premium is paid continuously at a constant instantaneous rate $\Gamma$ which is determined according to the fair contract principle given in Definition 2. If the mortality intensity develops as forecasted, the contract price should be 0 at the beginning. Since it is usually not the case, a different scenario of the mortality intensity will ex post lead to the situation that the premiums are either overpaid or underpaid by the policyholders on average. In the former case, the insurance company earns on average a surplus due to the misspecification of the mortality risk. However, in the latter case, it will find itself losing money. Because ex ante the insurance company has no complete information about the future, it is safe for it to be pessimistic and to assume the worst-case scenario for the pricing purpose.

In Figure 3 we see how $\mu$ should be optimally controlled so that the contract prices obtained enable the insurance company to stay on the safe side. Due to the introduction of the periodic premium payment, the insurance company bears higher risk that earlier death of the policyholder stops it from collecting the initial investment but does not reduces its obligation of benefit payment. Hence, we see that the optimally controlled regions look totally different in comparison with Figure 2.

For contract type I, it is not optimal to keep to the lower bound of $\mu$ any more. During




(a) I (b) II (c) III

(d) IV (e) V (f) VI

Figure 3: Optimally controlled regions of $\mu$ (periodic premium)

the early life time of the contract, the policyholder has not paid too much premium. This indicates that once he dies prematurely, he has the right to get the value of the underlying asset only at trivial costs. Hence, it is optimal if $\mu$ takes the upper bounded value. As the policyholder survives most part of the contract's life, he should have already paid a great part of the premium. At this time, the optimal $\mu$ depends on the spot price of the underlying asset again. When $S$ is very high, it is still profitable to stop the contract as immediate as possible, that is, to set $\mu$ to $\bar{\mu}$, because to go on paying the premium will not bring more benefit in expectation. On the contrary, if $S$ is very low, the policyholder would prefer to pay the premium so that he gets the chance to receive a higher survival benefit. In this case, $\mu$ should optimally be set to $\underline{\mu}$.

For contract type II, the higher the underlying asset price is, the higher is the possibility that the policyholder will obtain a higher survival benefit, whose advantage outweighs the premium to be paid, and hence, $\underline{\mu}$ is optimal. In contrast, the lower the underlying asset price is, the higher is the possibility that the policyholder can only receive the guaranteed amount at the increasing rate of $g_1$. Since $g_1 < r$, further premium payment is not worthwhile for the policyholder and a higher mortality intensity, namely, $\bar{\mu}$ would be better. We also see that the critical asset price that divides the two regions decreases with time. This is due to the fact that the premium that has already been paid are sunk costs and the choice of the optimal scenario only depends on the balance between the future benefit and premium payment. Hence, as time moves on, the advantage of the higher survival benefit over premium payment already reveals at a lower level of $S$ in comparison with the previous stage.

For contract type III, the mortality intensity should always be set to $\bar{\mu}$, meaning that it is always optimal to stop the contract as soon as possible. The present value of the death benefit is always great or equal than the present value of the survival benefit. Hence, an optimal mortality intensity should also be $\bar{\mu}$.

Contract type IV has similar payoff structure as contract type III except that the survival benefit also provides a minimum guarantee. The minimum guarantee for the terminal date $T$ has trivial effect during most life time of the contract where the optimally controlled region is identical to the region in contract type III. Only when the contract is close to the maturity date does the minimum guarantee for the survival benefit matter for the optimal choice of $\mu$. When the asset price is very close to the minimum guarantee, the death benefit is close to the spot asset price and is lower than the continuation value even when the periodic premium payment is taken into account. In this case, $\mu$ should be set to its lower bound $\underline{\mu}$.

Contract type V sets a limit to both the death benefit and the survival benefit. When the asset price during the early life time of the contract is high enough, the return rate of the insurance contract is $g_2 > r$, and the policyholder would be willing to keep paying the premium so that the contract keeps alive and that he earns more than a pure investment in the financial market. In this case, $\mu$ is optimally set to $\underline{\mu}$ to count for this worst-case scenario from the perspective of the insurance company. Moreover, we see the non-monotonicity in the critical asset price. This is due to two effects. On the one hand, as the cap increases with the time, the critical value of $S$ should also increase with the time. On the other hand, when the contract is farther away from the maturity date, the policyholder has more premium to pay. His incentive of continuing the contract is big only when he knows the possibility that the return of the contract keeps at a higher level. When the asset price is not high enough, due to the premium payment, it is usually optimal if the contract stops as soon as possible and $\mu$ should be set to $\bar{\mu}$.

Contract type VI is a mixture of contract type IV and contract type V. Therefore, the optimally controlled regions are a combination of Figure 3 (d) and (e).

| | | | | Profit and Loss of Insurance Company | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | True Mortality Intensity | | | | |
| | $\Psi$ | $\Phi$ | $\Gamma$ | $\mu^f$ | $\mu = \underline{\mu}$ | $\mu = \bar{\mu}$ | $\mu \in [\underline{\mu}, \bar{\mu}]$ | |
| | | | | | | | lower | upper |
| I | $S_\tau$ | $\max(S_0 e^{g_1 T}, S_T)$ | 67.02 | 0.00 | 3.36 | -3.40 | 10.84 | -10.67 |
| II | $S_0 e^{g_1 \tau}$ | $\max(S_0 e^{g_1 T}, S_T)$ | 65.04 | 0.00 | -1.67 | -0.33 | 22.95 | -17.85 |
| III | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $S_T$ | 58.32 | 0.00 | 10.36 | -26.80 | 10.36 | -26.80 |
| IV | $\max(S_0 e^{g_1 \tau}, S_\tau)$ | $\max(S_0 e^{g_1 T}, S_T)$ | 68.94 | 0.00 | 10.63 | -12.91 | 10.63 | -12.91 |
| V | $\min(S_0 e^{g_2 \tau}, S_\tau)$ | $\min(S_0 e^{g_2 T}, S_T)$ | 48.46 | 0.00 | 8.32 | -9.83 | 8.81 | -10.32 |
| VI | $\min(\max(S_0 e^{g_1 \tau}, S_\tau)$ $, S_0 e^{g_2 \tau})$ | $\min(\max(S_0 e^{g_1 T}, S_T)$ $, S_0 e^{g_2 T})$ | 60.66 | 0.00 | 9.56 | -11.62 | 9.95 | -12.00 |

Table 4: Contract prices for different scenarios of the mortality intensity (periodic premium).

As we have shown in Section 5, the hedging strategies based on the upper price bound will ensure the insurance company to build a superhedging position if enough policyholders are pooled together. In Table 4 we present the average profit and loss for the insurance company on a single contract for different scenarios of the realized mortality intensity. In column 4 the constant instantaneous premium rate $\Gamma$ is given which enables the contract price to be zero under the assumption that the mortality intensity moves as is forecasted in the future, see also column 5 for zero profit and loss. However, in reality, mortality risk cannot be forecasted with certainty. In column 6 and 7 we show what the contract price would be if the mortality intensity keeps to its lower bound (the extreme case of the longevity risk) and to its upper bound respectively. Column 8 and 9 display the lower price bound when the mortality intensity develops in the most favorable way for the insurance company and the upper price bound when it develops the most unfavorably. The results are qualitatively similar for all contracts. Exemplarily we discuss contract type I. We see that a lower mortality intensity than the forecast is of no risk to the insurance company. The benefit to be paid is less than the premium to be collected on average. The insurance company also does not suffer from the model risk if the mortality intensity develops in the most favorable way as is indicated in column 8. If the mortality intensity is higher than forecasted (column 7) or changes its value in the most unfavorable way (column 9) to the insurance company, the insurance company will find itself not being able to fulfill its obligation totally with the premium collected. All the prices under different scenarios of the mortality intensity lie within the lower price bound and the upper price bound that have been found dynamically according to Theorem 1. The upper price bound theoretically enables the insurance company to manage the financial risk dynamically under the model risk concerning the mortality intensity. However, when we compare it to the premium payment rate, we see it counts for about 2 months' premium, or equivalently, about 1% of the whole amount of premium expected to be collected, which for the insurance company may not be a large amount. Similar results can be found in the other contract types which indicates once again that mortality model risk has little price impact for contracts considered here.

## 7. Conclusion

We have investigated the influence of mortality model risk on unit-linked life insurance contracts. This investigation is undertaken within an uncertain mortality intensity framework where we assume reasonable bounds for the unknown mortality intensity. The magnitude of the mortality model risk can be easily identified by carrying out a stochastic control analysis and establishing upper and lower price bounds of unit-linked life insurance contracts, see Theorem 1. In addition, superhedging strategies are suggested under mortality model risk when assuming that the number of policyholders is large, see Theorem 2 and Corollary 1.

We show that when the risk profiles of the death benefit and the survival benefit are not significantly different, the effect of the mortality model risk may not be very large indeed. The contract prices in our examples have little sensitivity with respect to changes in the mortality intensity. For the single premium version the overall contract price differences were well below 4%. In the periodic premium case the deviation from the fair price was in the same range, and was not exceeding a six month premium income. In this case, other risk sources such as interest rate risk and equity risk deserve more attention than mortality model risk.

Our framework can be extended in many useful directions. The setup can be directly extended to include an American feature where the policyholder has the right to quit the contract for a pre-specified payoff, the surrender guarantee. Further, other risk factors such as interest rate risk and other facets of equity risk such as volatility risk can be included in the setup. The so extended framework can then be analyzed for model risk by similar methods as used here.

## Appendix A. Proof of Corollary 1

*Proof of Corollary 1.* Take the integral representation of the hedge error $E^N$ in Remark 7 and divide this by $N$ to obtain $\bar{E}^N$. Using the triangular inequality we can study each of the three expressions separately and establish uniform convergence in probability to the corresponding counterpart in $\bar{E}^\infty$. The first expression yields

$$\frac{1}{N}\, e^{\int_0^t r(s)\,\mathrm{d}s}\, \left(V_0^N - N\,\tilde{v}(0,S_0)\right) \to e^{\int_0^t r(s)\,\mathrm{d}s}\, \left(v - \tilde{v}(0,S_0)\right) \quad \text{for } N \to \infty\,,$$

by the assumption $v = \lim_{N\to\infty} V_0^N/N$. The convergence is uniform in $t$ since $r$ is deterministic and the integral $\int_0^t r(s)\,\mathrm{d}s$ is a deterministic and continuous function. Accordingly, the expression $e^{\int_0^t r(s)\,\mathrm{d}s}$ is uniformly bounded by a constant on the compact $[0,T]$.

The error in the second expression is then

$$R_t^N = \int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\, \left(\frac{\bar{X}_u^N}{N} - e^{-\int_0^t \nu_s\,\mathrm{d}s}\right) \left[\Psi(u,S_u) - \tilde{v}(u,S_u)\right] \left[\tilde{\mu}(u,S_u) - \nu_u\right] \mathrm{d}u\,,$$

where we have deducted the integral part of $\bar{E}^\infty$. Then we can establish the following uniform bound

$$\sup_{0\le t\le T} |R_t^N| \;\le\; \sup_{0\le t\le T} \left|\frac{\bar{X}_t^N}{N} - e^{-\int_0^t \nu_s\,\mathrm{d}s}\right| \int_0^T e^{\int_t^T r(s)\,\mathrm{d}s}\, |\Psi(t,S_t) - \tilde{v}(t,S_t)|\, |\tilde{\mu}(t,S_t) - \nu_t|\,\mathrm{d}t\,.$$

The bound depends on $N$ only in the first component. The second component is almost surely finite, and we are left to show that the first component vanishes in probability. To do so, consider

$$e^{-\int_0^t \nu_s\,\mathrm{d}s}\,Y_t^N = \frac{\bar{X}_t^N}{N} - e^{-\int_0^t \nu_s\,\mathrm{d}s}\,, \quad \text{or, equivalently, } Y_t^N = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{t<\tau_i\}}\,e^{\int_0^t \nu_s\,\mathrm{d}s} - 1\,.$$

Then $Y^N$ is a local $\mathbb{P}$-martingale, see Bielecki and Rutkowski (2001), Lemma 5.1.7., p. 152. Moreover, $Y^N$ is square integrable since we assume that $\nu$ is uniformly bounded in $t$, and therefore

$$[Y^N, Y^N]_t = \frac{1}{N^2}\sum_{\{\tau_i\le t\}} e^{2\int_0^{\tau_i}\nu_s\,\mathrm{d}s}\,,$$

is uniformly bounded in $t$ by a deterministic constant on the compact $[0,T]$. Doob's maximal quadratic inequality then gives

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_t^N|^2\right) \le 4\,\mathbb{E}\left([Y^N,Y^N]_T\right) \le \frac{4}{N}\,e^{2\int_0^T \bar{\mu}(s)\,\mathrm{d}s} = O(1/N)\,,$$

Accordingly, $\sup_{0\le t\le T}|Y_t^N|$ tends to zero in $L^2(\mathbb{P})$ and hence in probability. This establishes the uniform convergence of $R^N$ to zero in probability.

Finally, consider the third expression

$$Z_t^N = \frac{1}{N}\int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,[\tilde{v}(u,S_u) - \Psi(u,S_u)]\,\mathrm{d}M_u^N\,.$$

The process $Z^N$ is a $\mathbb{P}$-martingale with quadratic variation

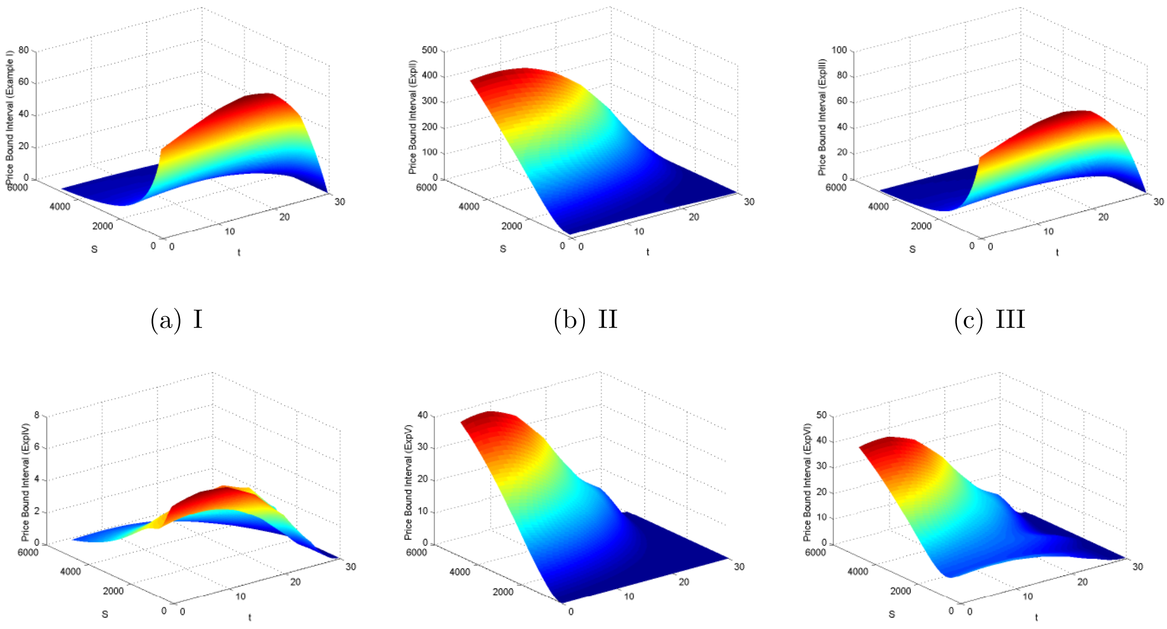
$$[Z^N,Z^N]_t = \frac{1}{N^2}\int_0^t e^{2\int_u^t r(s)\,\mathrm{d}s}\,[\tilde{v}(u,S_u) - \Psi(u,S_u)]^2\,\mathrm{d}X_u^N\,.$$

Note that we can find a localizing sequence of stopping times $(\sigma_n)_{n\ge 1}$ such that the stopped process $(\tilde{v}(\cdot,S) - \Psi(\cdot,S))^{\sigma_n}$ is uniformly bounded by a deterministic constant, say $C_n$. Thus, without loss of generality we may assume that $(\tilde{v}(\cdot,S) - \Psi(\cdot,S))$ is indeed bounded by real number, say by $C>0$. Then $Z^N$ is square integrable and by Doob's maximal quadratic inequality we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|Z_t^N|^2\right) &\le 4\,\mathbb{E}\left([Z^N,Z^N]_T\right)\\ &= \frac{4}{N}\,\mathbb{E}\left[\int_0^T e^{2\int_t^T r(s)\,\mathrm{d}s}\,[\tilde v(t,S_t)-\Psi(t,S_t)]^2\,e^{-\int_0^t \nu_s\,\mathrm{d}s}\,\nu_t\,\mathrm{d}t\right]\\ &\le \frac{4}{N}\,e^{2\int_0^T |r(s)|\,\mathrm{d}s}\,C^2\int_0^T \bar\mu(t)\,\mathrm{d}t = O(1/N)\,.\end{aligned}$$

This implies the uniform convergence of $Z^N$ to zero in $L^2(\mathbb{P})$ and hence in probability. The localizing sequence $(\sigma_n)_{n\ge 1}$ will give in general the result of the uniform convergence of $Z^N$ to zero in probability. □

## Appendix B. Figure B.4



(a) I (b) II (c) III

(d) IV (e) V (f) VI

Figure B.4: Differences between upper and lower price bounds (single premium).

## Appendix C. The Bounds of the Mortality Intensity

The bounds of the mortality intensity are obtained through the Lee-Carter model. The Lee-Carter model in Lee and Carter (1992) and its various extensions, see, e.g., Lee (2000), have been used successfully to forecast the death rates of the population in many developed countries, such as USA, Canada, Japan, Chile, Belgium, Austria and Australia.

Lee and Carter (1992) describe the logs of the age-specific death rates $m(t,x)$ by a linear function of an unobserved period-specific intensity index $k_t$ with age-specific parameters $a_x$ and $b_x$.[10] After the estimation of $a_x$ and $b_x$ as well as $k_t$ in the past, a time series model is applied to describe the dynamics of $k$ so as to forecast its future development, based on which the central death rate is forecasted and its confidence interval was found. The link between the central death rate and the mortality intensity is $m_t = \int_t^{t+1} \nu_u \, du$.[11] Assuming that there would be no extreme change with the mortality intensity within one year, the same confidence interval that bounds the death rate should also be a suitable bound for the mortality intensity. The age-specific parameters we use in this paper are presented in Table C.5.

Concerning the mortality index $k$, we take the ARIMA time series model estimated by Lee and Carter but without the dummy term, namely, $k_t = k_{t-1} - 0.365 + e_t$. The standard error of the estimation (see) is assumed to be 0.651 and $k_0$ is equal to $-18$. Also for the sake of simplicity, we make the assumption that there are no estimation errors with the age-specific parameters. The uncertainty of the mortality forecast is supposed to be only attributed to the random behavior of the mortality index $k$. Under this assumption, we obtain that the confidence interval of the mortality intensity at confidence level $p$ by

$$\begin{aligned} \mu_{40+t} \quad \in \quad & \Big[\exp\Big(a_{40+\lfloor t\rfloor} + b_{40+\lfloor t\rfloor}\,(k_0 - \lfloor t\rfloor\, 0.365) - q\, b_{40+\lfloor t\rfloor}\,\text{see}\,\sqrt{\lfloor t\rfloor}\Big)\,, \\ & \exp\Big(a_{40+\lfloor t\rfloor} + b_{40+\lfloor t\rfloor}\,(k_0 - \lfloor t\rfloor\, 0.365) + q\, b_{40+\lfloor tu\rfloor}\,\text{see}\,\sqrt{\lfloor t\rfloor}\Big)\Big]\,, \end{aligned}$$

for $0 \le t \le 30$, and $q = \Phi^{-1}((1+p)/2)$.

[10] The constraint serves to normalize the solutions, see Lee and Carter (1992). It should be pointed out that here $x$ denotes the age of an individual at time $t$ instead at time 0 as we have referred to in the previous part. We allow this abuse of notation to keep consistent with the literature on the Lee-Carter model. In the later part, we always refer $x$ to the age of an individual at time 0.

[11] Focusing only on the policyholders who are aged $x$ at time 0, we omit $x$ in the index and denote the central death rate at time $t$ as $m_t$.

Table C.5: The age-specific parameters $a_x$ and $b_x$

| $x$ | $a_x$ | $b_x$ |
|---|---|---|
| 40-44 | -5.51323 | 0.05279 |
| 45-49 | -5.09024 | 0.04458 |
| 50-54 | -4.65680 | 0.03830 |
| 55-59 | -4.25497 | 0.03382 |
| 60-64 | -3.85608 | 0.02949 |
| 65-69 | -3.47313 | 0.02880 |
| 70-74 | -3.06117 | 0.02908 |
| 75-79 | -2.63023 | 0.03240 |
| 80 | -2.20498 | 0.03091 |

## References


Avellaneda, M., Levy, A., Paras, A., 1995. Pricing and hedging derivative securities in markets with uncertain volatilities. Applied Mathematical Finance 2, 73–88.

Bielecki, T., Rutkowski, M., 2001. Credit Risk: Modeling, Valuation and Hedging. Springer, Berlin.

Biffis, E., 2005. Affine processes for dynamic mortality and actuarial valuations. Insurance: Mathematics and Economics 37, 443–468.

Biffis, E., Millossovich, P., 2006. The fair value of guaranteed annuity options. Scandinavian Actuarial Journal 2006, 23–41.

Björk, T., 2009. Arbitrage Theory in Continuous Time. Oxford University Press Inc.

Blake, D., Cairns, A., Dowd, E., 2006. Pricing death: Frameworks for the valuation and securitization of mortality risk. ASTIN Bulletin 36, 79–120.

Cairns, A., Blake, D., Dowd, K., 2006. A two-factor model for stochastic mortality with parameter uncertainty. Journal of Risk and Insurance 73, 687–718.

Dahl, M., 2004. Stochastic mortality in life insurance: Market reserves and mortality-linked insurance contracts. Insurance: Mathematics and Economics 35, 113–136.

Dahl, M., Møller, T., 2006. Valuation and hedging of life insurance liabilities with systematic mortality risk. Insurance: Mathematics and Economics 39, 193–217.

Dai, M., Kwok, Y., You, H., 2007. Intensity-based framework and penalty formulation of optimal stopping problems. Journal of Economic Dynamics & Control 31, 3860–3880.

Duffie, D., Schroder, M., Skiadas, C., 1996. Recursive valuation of defautable securities and timing of resolution of uncertainty. The Annals of Applied Probability 6, 1075–1090.

Jeulin, T., Yor, M., 1979. Inégalité de hardy, semimartingales, et faux-amis, in: Dellacherie, C., Meyer, P., Weil, M. (Eds.), Séminaire de Probabilités XIII (Lecture Notes in Mathematics). Springer. volume 721, pp. 332–359.

Lee, R., 2000. The lee-carter method for forecasting mortality, with various extensions and applications. North American Actuarial Journal 4, 80–91.

Lee, R., Carter, L., 1992. Modeling and forecasting u. s. mortality. Journal of the American Statistical Association 87, 659–671.

Melnikov, A., Romaniuk, Y., 2006. Evaluating the performance of gompertz, makeham and lee-carter mortality models for risk management with unit-linked contracts. Insurance: Mathematics and Economics 39, 310–329.

Milevsky, M., Promislow, S., 2001. Mortality derivatives and the option to annuitise. Insurance: Mathematics and Economics 29, 299–318.

Protter, P., 2004. Stochastic Integration and Differential Equations. Springer, Heidelberg.

Swiss Re, 2003. Unit-linked life insurance in western europe: regaining momentum? Sigma .

Yong, J., 1997. Relations among odes, pdes, fsdes, bsdes, and fbsdes, in: Proceedings of the 36the Conference on Decision & Control, San Diego, California USA.

Yong, J., Zhou, X., 1999. Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer, New York.