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When Bidding More is Not Enough: All-Pay Auctions with Handicaps

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Abstract

We consider a standard two-player all-pay auction with private values, where the valuation for the object is private information to each bidder. The crucial feature is that one bidder is favored by the allocation rule in the sense that he need not bid as much as the other bidder to win the auction. Analogously, the other bidder is handicapped by the rule as overbidding the rival may not be enough to win the auction. Clearly, this has important implications on equilibrium behavior. We fully characterize the equilibrium strategies for this auction format and show that there exists a unique pure strategy Bayesian Nash Equilibrium.

Keywords: All-pay auction, contest, asymmetric allocation rule, rent-seeking, asymmetric information

JEL-Classification: D44, D88

1 Introduction

Motivation and results Auctions in which bidders compete for one unit of an indivisible good have been widely studied in recent years. Although the models differ along many dimensions, one common feature is that *the object is awarded to the bidder who submits the highest bid*. Contrary to that, our paper analyzes a two-player all-pay auction with incomplete information in which the highest bid does not necessarily win the auction. Instead, the allocation rule is asymmetric in the sense that one bidder is *favored* as he need not bid as much as the other bidder to win the auction. Analogously, the other bidder is *handicapped* by the rule as overbidding the rival may not be enough to win the auction.

The empirical significance of our setting comes from the well-known fact that all-pay auction are strategically equivalent to discriminatory contests. In discriminatory contests, each party exerts costly effort to compete with other parties for a prize, and the party who outbids all of their competitors wins the contest and receives the prize. In reality, the allocation rule in discriminatory contests is often asymmetric in the sense described above. For example, in German procurement auctions, although local authorities are obliged to choose the bidders with the lowest price, there is a clause according to which it can award the contract to a local bidder when this bidder's price is not more than 5 per cent higher than the lowest bidder's price. As a second example, consider the "in dubio pro reo"-rule in criminal law. According to this rule, a defendant will only be convicted if his lawyer presents considerably less quantity and quality of evidence than the prosecutor. Finally, assume that an enterprise hires a consulting firm, and suppose that firm A has a done some excellent in-house consulting before. Then, we often observe in reality that a potential entrant B is awarded the contract only if the quality of its proposal is considerably above the quality of A's proposal. Note that this can indeed be interpreted as an all-pay auction, because each firm (and not only the winner) has to exert effort to prepare a proposal.

To the best of our knowledge, these kind of asymmetric discriminatory contests have not yet been analyzed in a general framework.[1] In this paper, we fully characterize

[1] See the literature review below.

the equilibrium strategies for this two-player all pay auction with handicaps when each bidder's valuation is private information. We show that there exists a unique pure strategy Bayesian Nash Equilibrium. Two further results are also worth emphasizing: first, the revenue equivalence theorem does not apply in our setting, since the bidder with the higher valuation will not win the auction with certainty. Second, although it is generally possible that the handicapped player bids more than the favored bidder if the valuations are identical, we show that it is not possible that the handicapped player wins the auction when his valuation is lower than the favored bidder's valuation. Hence, an inefficient allocation of the object can only result when the favored bidder wins the auction although he has the lower valuation.

Literature There is a large recent literature analyzing the all-pay auction: Baye, Kovenock, and de Vries (1996) provide a complete analysis of the all-pay auction under complete information. With asymmetric information, Krishna and Morgan (1997) extend the classic model by Milgrom and Weber (1982) where signals are generally affiliated to also include the first and the second price all-pay auctions. Lizzeri and Persico (2000) analyze under which conditions there exist unique pure strategy equilibria in general auction games, including the all-pay auction.[2] Amann and Leininger (1996) and Maskin and Riley (2000) consider auctions in which bidders are asymmetric in the sense that the valuations for each bidder are drawn from different distributions. This also implies that the bidder with the highest valuation does no longer win the object with certainty. While Maskin and Riley (2000) confine attention to winner-pay auctions, our paper is more related to Amann and Leininger (1996) as they analyze the all-pay auction. Moreover, we adopt and extend their approach for determining the equilibrium bidding strategies from a system of differential equations. As stated above, in all these papers and contrary to our model, the winner of the auction is the high bidder.

In contrast to the auction literature, there are a few papers considering contests with handicaps. Konrad (2002) assumes that handicaps arise from the fact that incumbents need to spend less resources in order to win the discriminatory contest. However, he

[2] The issue of existence of pure-strategy equilibria in a more general class of simultanous games with asymmetric information is also extensively analyzed in Athey (2001).

restricts attention to complete information, so that only mixed strategy equilibria exist. In the context of bribery games, Clark and Riis (2000) consider an all-pay auction where two players compete for a government contract awarded by a corrupt official. In such a setting, the authors show that the auctioneer can increase his expected revenue by introducing asymmetry in our sense. However, they confine attention to the case where valuations for the contract are uniformly distributed. Bernardo, Talley, and Welch (2000) analyze a litigation game where the litigants' evidence is unequally weighed by the court. Since evidence production is costly, this leads in fact to a contest with handicaps. However, the game is a modelled as a Tullock contest, where each player wins the price with some probability depending on his effort (or bid).[3] This is different from our approach since the identity of the winner is stochastic even for given bids.

The remainder of the paper is organized as follows: In section 2 the basic model is presented. Section 3 derives the equilibrium strategies and contains our main results. In section 4 we discuss an example, while section 5 concludes.

2 The Model

Basic Setup We consider a private value all-pay auction where 2 risk-neutral bidders indexed $i = 1, 2$ compete for a single object to be sold. Each bidder has valuation $v_i \in [0, 1]$ drawn from a common distribution function $F(v) \in C^1$ satisfying $F(0) = 0$ where the density function $F'(v)$ is positive valued on $(0, 1)$. The realization of v_i (bidder i's "type") is private information to bidder i. We analyze equilibria in which the bidding strategy of bidder i is a function of his type, i.e. $b_i : [0, 1] \to \Re_0^+$.

The specific feature of this auction is the allocation rule: Denoting by $W \in \{1, 2\}$ the identity of the winner, we have

$$W = 1 \Leftrightarrow b_1 > t \cdot b_2 \text{ and } W = 2 \Leftrightarrow b_2 > \frac{1}{t} \cdot b_1 \tag{1}$$

where a coin is flipped in case that $b_1 = t \cdot b_2$ holds so that each bidder wins with

[3] To illustrate, in the simplest symmetric two-person Tullock contest, player i exerts effort e_i and wins with probability $\pi_i = \frac{e_i}{e_i + e_j}$. In the asymmetric version considered by Bernardo, Talley, and Welch (2000), the probability is $\pi_i = \frac{te_i}{te_i + e_j}$ where $t \neq 1$.

probability $\frac{1}{2}$. Thus, bidder 1 wins the auction only if he bids at least t-times as much as bidder 2, while bidder 2 wins if he bids at least $\frac{1}{t}$-times as much as bidder 1. Without loss of generality we confine attention to the case $t \geq 1$. Therefore, bidders 1 and 2 will be referred to as the "handicapped" and the "favored" bidder, respectively.[4] Clearly, for $t = 1$ this is simply the standard all-pay auction with private values. The value of t is commonly known.

Payoffs Following the setup of the model, for given bids b_1 and b_2, payoffs are

$$\pi_1(b_1, b_2, v_1; t) = \begin{cases} v_1 - b_1 & \text{if } b_1 > tb_2 \\ \frac{1}{2}v_1 - b_1 & \text{if } b_1 = tb_2 \\ -b_1 & \text{if } b_1 < tb_2 \end{cases} \tag{2}$$

and

$$\pi_2(b_1, b_2, v_2; t) = \begin{cases} v_2 - b_2 & \text{if } b_2 > \frac{1}{t}b_1 \\ \frac{1}{2}v_2 - b_2 & \text{if } b_2 = \frac{1}{t}b_1 \\ -b_2 & \text{if } b_2 < \frac{1}{t}b_1 \end{cases} . \tag{3}$$

Finally, *expected* payoffs are denoted by Π_i and given by

$$\Pi_1(b_1, b_2, v_1; t) = v_1 \Pr(b_1 > tb_2(v_2)) - b_1 \tag{4}$$

and

$$\Pi_2(b_1, b_2, v_2; t) = v_2 \Pr(b_2 > \frac{1}{t}b_1(v_1)) - b_2. \tag{5}$$

3 Equilibrium Analysis

Since this is a static game with incomplete information, the equilibrium concept used is Bayesian Nash Equilibrium (BNE). A vector of bids $(b_1^*(v_1), b_2^*(v_2))$ is a BNE if the

[4] Note that the asymmetry here refers to the allocation rule. This is different to "asymmetric auctions" in the sense of Amann and Leininger (1996) and Maskin and Riley (2000), where the valuations v_1 and v_2 are drawn from different distributions.

following set of conditions is satisfied:

$$\Pi_i(b_i^*(v_i), b_j^*(v_j); t) \geq \Pi_i(b_i, b_j^*(v_j); t) \text{ for all } b_i \in \Re_0^+ \text{ and } i, j \neq i = 1, 2. \tag{6}$$

In equilibrium, no bidder must be able to increase his expected payoff by choosing a bidding strategy other than $b_i^*(v_i)$, given that the opponent adheres to his equilibrium strategy. The following definition proves useful for further reference:

Definition 1 *Consider some function $x : A \to \Re$. Then define: $D_x := \{a \in A : x(a) \in \Re^+\}$.*

The restricted domain $D_{x(a)}$ contains only those elements a in A whose image $x(a)$ is positive. We can then state the following result concerning the properties of the equilibrium bidding strategies:

Lemma 1 (Equilibrium Bidding Strategies) *$b_i^* : D_{\mathsf{b}_\mathsf{i}} \to (0, b_i(1)]$ is a monotone increasing bijection on a non-empty set $D_{\mathsf{b}_\mathsf{i}} \subseteq [0,1]$ and differentiable almost everywhere.*

Proof. See Appendix 1. ■

Uniqueness of Equilibrium We first show that in this framework an equilibrium is unique whenever it exists. The issue of existence is addressed below. Note that Lemma 1 also ensures existence of the inverse mapping $\rho_i : [0, b_i^*(1)] \to D_{\mathsf{b}_\mathsf{i}}$, i.e. $\rho_i(b) \equiv b_i^{-1}(b)$ is the valuation bidder i must have in order to bid b. Equipped with this result we can now characterize the equilibrium bidding strategies in more detail. The maximization problem for bidder 1 when bidder 2 is playing some strategy $b_2(v_2)$ is given by

$$\max_{b_1} \; v_1 \cdot \Pr(b_1 > t \cdot b_2(v_2)) - b_1 = v_1 \cdot F(\rho_2(\frac{b_1}{t})) - b_1, \tag{7}$$

while for bidder 2, when bidder 1 is playing strategy $b_1(v_1)$ we have

$$\max_{b_2} \; v_2 \cdot \Pr(b_2 > \frac{1}{t} \cdot b_1(v_1)) - b_2 = v_2 \cdot F(\rho_1(t \cdot b_2)) - b_2. \tag{8}$$

The first order conditions to these maximization problems lead to the following system of ordinary first order differential equations which must be satisfied by solution candidates

$b_1^*(v_1)$ and $b_2^*(v_2)$:

$$v_1 \cdot F'(\rho_2(\frac{b_1(v_1)}{t})) \cdot \rho_2'(\frac{b_1(v_1)}{t}) \cdot \frac{1}{t} = 1. \quad (9)$$

$$v_2 \cdot F'(\rho_1(t \cdot b_2(v_2))) \cdot \rho_1'(t \cdot b_2(v_2)) \cdot t = 1. \quad (10)$$

For a given set of initial conditions, this system determines a unique trajectory of bidding strategies as it is Lipschitz continuous for $v_i > 0$. That there is only a single pair of initial conditions (such that a solution to Eqns. (9) and (10) is indeed unique) follows from the following results concerning the properties of the *equilibrium bid distributions* $G_{i=1,2} := F(\rho_i(b_i^*(v_i))) : D_{G_i} \to [0,1]$:[5]

Lemma 2 (Equilibrium Bid Distributions) *In any BNE, the bid distributions G_1 and G_2 have the following properties:*
(i) $D_{G_1} = (0, b_1^*(1)]$ and $D_{G_2} = (0, b_2^*(1)]$ where $b_1^*(1) = t \cdot b_2^*(1)$.
(ii) G_i is continuous and strictly monotone increasing $\forall i = 1, 2$.
(iii) If $G_i(0) > 0$, then $G_{j \neq i}(0) = 0$.
(iv) There is a single set of admissible initial conditions.

Proof. See Appendix 2. ■

Part (i) of the Lemma characterizes one main difference of an all-pay auction with handicaps compared to the standard model where $t = 1$ holds. Clearly, it can never be optimal for bidder 2 (the favored bidder) to submit bids larger than $\frac{1}{t}$-times the maximum bid of bidder 1 (the handicapped bidder) since he already wins with probability one when bidding $b_2 = \frac{1}{t} \cdot b_1(1)$. Part ii) establishes that, in equilibrium, bid distributions must ensure that no bidder can increase his expected profit by submitting a lower bid while leaving the probability of winning the auction unchanged which is due to the all-pay rule. Part iii) says that only one bidder's bid function can have an atom at zero. Intuitively, this follows from the fact that, given that one bidder's bid function has an atom at zero, the other bidder can always be better of by bidding some $x > 0$ whenever his valuation is positive. As one consequence, the coexistence of different sets of admissible initial conditions is ruled out as stated in part iv).

[5] Similar statements for the case $t = 1$ have for example been derived by Amann and Leininger (1996).

Existence of Equilibrium Rather than modifying equation system (9) and (10) directly, we extend the method adopted by Amann and Leininger (1996) who have analyzed the case $t = 1$ for valuations v_1 and v_2 drawn from different distributions. The advantage of this method is that it simplifies the problem of simultaneously solving a system of differential equations into a sequential procedure. This enables us to prove our main result:

Theorem 1 *There exists a unique pure-strategy Bayesian Nash-Equilibrium in which bidder* 1 *(the handicapped bidder) chooses*

$$b_1^*(v_1) = \int_{\max\{0,k^{-1}(0)\}}^{v_1} t \cdot k(V)F'(V)dV \tag{11}$$

and in which bidder 2 *(the favored bidder) chooses*

$$b_2^*(v_2) = \frac{b_1^*(k^{-1}(v_2))}{t} \tag{12}$$

where the bijection $k(v_1;t) : D_{b_1} \to D_{b_2}$ *is implicitly given by the differential equation*

$$\frac{dk(v_1;t)}{dv_1} = \frac{t \cdot k(v_1;t) \cdot F'(v_1)}{v_1 \cdot F'(k(v_1;t))}. \tag{13}$$

Proof. Using a bijection $k : D_{b_1} \to D_{b_2}$, the first order conditions (9) and (10) can be transformed into a set of differential equations expressed in a single variable v_1. Substituting $k(v_1)$ for v_2 in Eqn. (10) yields

$$v_1 \cdot F'(\rho_2(\frac{b_1(v_1)}{t})) \cdot \rho_2'(\frac{b_1(v_1)}{t}) \cdot \frac{1}{t} \quad = \quad 1 \tag{14}$$

$$k(v_1) \cdot F'(\rho_1(t \cdot b_2(k(v_1)))) \cdot \rho_1'(t \cdot b_2(k(v_1))) \cdot t \quad = \quad 1. \tag{15}$$

We can also make use of the identity of the two equations to yield

$$v_1 \cdot F'(\rho_2(\frac{b_1}{t})) \cdot \rho_2'(\frac{b_1}{t}) \cdot \frac{1}{t} = k(v_1) \cdot F'(\rho_1(t \cdot b_2(k(v_1))) \cdot \rho_1'(t \cdot b_2(k(v_1)) \cdot t. \tag{16}$$

Now consider the bijection

$$k(v_1;t) = \rho_2(\frac{b_1(v_1)}{t}) \tag{17}$$

with derivative

$$\frac{dk(v_1;t)}{dv_1} = \rho_2'(\frac{b_1(v_1)}{t}) \cdot \frac{db_1(v_1)}{dv_1} \cdot \frac{1}{t}. \tag{18}$$

Thus, $k(v)$ maps every type of bidder 1 onto that type of bidder 2 who bids *1/t- times as much as* bidder 1. Note that due to our previous results and together with the appropriate boundary condition $k(1) = 1$, Eqn. (17) defines indeed a bijection between the domains of the different strategies which is differentiable almost everywhere. Using Eqn. (17) allows us to rewrite Eqn. (16) as

$$\begin{aligned} v_1 \cdot F'(k(v_1;t)) \cdot \frac{dk(v_1;t)}{dv_1} \cdot \frac{1}{\frac{db_1(v_1)}{dv_1}} &= \\ k(v_1) \cdot F'(\rho_1(tb_2(\rho_2(\frac{b_1(v_1)}{t})))) \cdot \rho_1'(tb_2(\rho_2(\frac{b_1(v_1)}{t})) \cdot t \\ \Leftrightarrow v_1 \cdot F'(k(v_1;t)) \cdot \frac{dk(v_1;t)}{dv_1} \cdot \frac{1}{\frac{db_1(v_1)}{dv_1}} &= k(v_1) \cdot F'(\rho_1(b_1)) \cdot \rho_1'(b_1) \cdot t \\ \Leftrightarrow v_1 \cdot F'(k(v_1;t)) \cdot \frac{dk(v_1;t)}{dv_1} &= k(v_1) \cdot F'(v_1) \cdot \rho_1'(b_1) \cdot \frac{db_1(v_1)}{dv_1} \cdot t. \end{aligned} \tag{19}$$

Finally, as $\rho_1(b_1(v_1)) = v_1$, it follows that $\rho_1'(b_1) = \frac{dv_1}{db_1}$ which implies that $\rho_1'(b_1) \cdot \frac{db_1(v_1)}{dv_1} = 1$. Hence, we end up with the single ordinary differential equation

$$\frac{dk(v_1;t)}{dv_1} = \frac{t \cdot k(v_1;t) \cdot F'(v_1)}{v_1 \cdot F'(k(v_1;t))}. \tag{20}$$

The boundary condition $k(1;t) \equiv 1$ and the assumptions on $F(v)$ guarantee a unique solution for $k(v_1;t)$. Moreover, the equilibrium bidding strategy of bidder 1 must satisfy the differential equation

$$\frac{db_1(v_1)}{dv_1} = \frac{1}{\rho_1'(b_1)} = \frac{1}{\rho_1'(t \cdot b_2(k(v_1;t))} = t \cdot k(v_1;t) \cdot F'(v_1) \tag{21}$$

where the last step follows from Eqn. (15). Together with $b_1(k^{-1}(0)) = 0$ and the definition of $k(v_1;t)$, closed form solutions for the equilibrium bidding strategies are given by

$$b_1^*(v_1) = \int_{\max\{0,k^{-1}(0)\}}^{v_1} t \cdot k(V;t) \cdot dF(V) \tag{22}$$

$$b_2^*(v_2) = \frac{b_1^*(k^{-1}(v_2))}{t} \tag{23}$$

as stated in the Theorem. ■

Inefficient Allocation when $t > 1$ Clearly, the allocation of the object in our auction does not only depend on the two bidders valuations but also on the allocation rule expressed by t. Hence, we can not exclude that the object is awarded to a bidder whose valuation is lower than his competitor's valuation. Furthermore, without further assumptions on the distribution function $F(\cdot)$ (see the example below), we can not say if the favored or the handicapped player bids more for identical valuations. However, we can show that the handicapped player will never win the auction if his valuation is lower. This means that, even if he may bid more aggressively for particular distribution functions and for particular valuations, this can never outweigh his handicap. It follows that an inefficient allocation of the object can only result when bidder 2 (the favored bidder) wins the auction although he has a lower valuation. This is expressed in the following Proposition, where $W^* \in \{1,2\}$ denotes the identity of the winner in equilibrium:

Proposition 1 *i) In any BNE, there can only exist the case where $v_1 > v_2$ but $W^* = 2$, while the case where $v_2 > v_1$ but $W^* = 1$ does not occur with positive probability.*
ii) The expected equilibrium welfare loss due to inefficient allocation of the object is given by

$$L^* \equiv \begin{cases} \int_0^1 \int_{k(v_1;t)}^{v_1} (v_2 - v_1) F'(v_2) dv_2 F'(v_1) dv_1 < 0 & \textit{if } k(v_1;t) < v_1 \\ 0 & \textit{otherwise} \end{cases} . \tag{24}$$

iii) This expected equilibrium welfare loss is the greater for large values of t, i.e. $\frac{dL^*}{dt} < 0$.

Proof. See Appendix 3. ■

With respect to the welfare loss expressed in parts (ii) and (iii) of Proposition 1, we have simply calculated the conditional expectation of the difference in the valuations of player 2 and 1, given that $v_2 - v_1 < 0$, and that player 2 nevertheless wins the auction. Although this seems to be a natural definition of the welfare loss, one has to keep in mind that asymmetries are often introduced for welfare concerns not explicitly modelled here.[6]

4 An Example

To better understand the impact of the asymmetry generated by $t > 1$ on the bidders' behavior, we consider the special case where the v_i are uniformly distributed, i.e. $F(v) = v$. Differential equation (20) then becomes

$$k'(v_1; t) = \frac{t \cdot k(v_1; t)}{v_1}. \tag{25}$$

Using standard techniques, the solution has the form of some polynomial $k(v_1; t) = \alpha v_1^{\beta} + \gamma$ which leads to $k(v_1; t) = v_1^t$ as the unique solution satisfying $k(1; t) = 1$. Substituting in Eqn. (22) yields

$$b_1^*(v_1; t) = \int_0^{v_1} t \cdot V^t dV = \frac{t}{t+1} v_1^{t+1} \tag{26}$$

and, by definition of $k(v_1; t)$

$$b_2^*(v_2; t) = \frac{b_1(k^{-1}(v_2))}{t} = \frac{1}{t+1} v_2^{(t+1)/t}. \tag{27}$$

This leads to the bid functions $G_1 = ((\frac{1+t}{t})b_1^*)^{\frac{1}{t+1}}$ and $G_2 = ((1+t)b_2^*)^{\frac{t}{t+1}}$ which both satisfy $G_i(0) = 0$ (and hence are atomless) and $G_i(b^*(1)) = 1$. Clearly, $b_i^*(v_i; t)$ is increasing in v_i satisfying $b_i^*(0) = 0$. Moreover, the equilibrium bidding strategies satisfy the support constraint $b_1^*(1) = \frac{t}{t+1} = t \cdot b_2^*(1) = \frac{1}{1+t}$ as required by Lemma 2. For the comparative statics with respect to t, the results are not as clear-cut: The following figure shows

[6] For instance, in the consulting example described in the introduction, the enterprise introduces an asymmetry because it has a positive ex ante bias for one firm.

$b_2^*(v_2; t)$ as a function of t (where $t \geq 1$) for $v_2 = \frac{1}{3}$:[7]

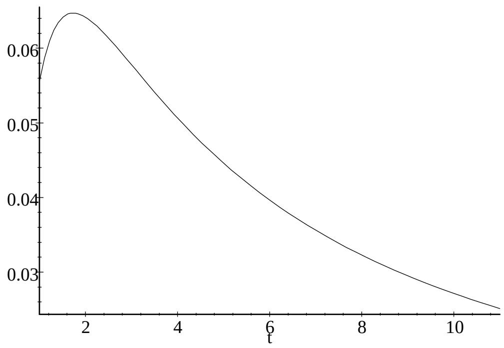

Figure 1: $b_2^*(v_2 = \frac{1}{3})$ as a function of t.

The intuition for this non-monotonicity result is best explained by looking at marginal costs and benefits from increasing b_i: Since marginal cost is always equal to 1 due to the all-pay rule, we can safely confine attention to the analysis of marginal benefit. Given bidder 1's equilibrium strategy $b_1^*(v_1; t)$, bidder 2's expected benefit (payoff net of cost) is $v_2 \cdot \Pr(b_2 > \frac{1}{t} b_1^*(v_1; t))$. When t increases by Δt, there are two effects: i) bidder 2 wins the auction not only in case that $b_2 > \frac{1}{t} b_1^*(\cdot)$ but already when $b_2 > \frac{1}{t+\Delta t} b_1^*(\cdot)$. This effect (the "direct effect") unambiguously increases the marginal benefit for bidder 2 and thus his equilibrium bid. ii) as t changes, also $b_1^*(v_1; t)$ changes by $\frac{d}{dt} b_1^*(v_1; \cdot) \cdot \Delta t$ and this also effects the probability of winning and thus the marginal benefit from increasing b_2 (the

[7] For a formal description of the comparative statics analysis with respect to t, define $t_i^{\max}(v_i) \in \arg\max_t b_i^*(v_i, t)$. One gets

$$t_1^{\max}(v_1) = \frac{1}{2 \ln v_1} \left(-\ln v_1 - \sqrt{(\ln^2 v_1 - 4 \ln v_1)} \right)$$

which is increasing in v_1 as

$$\frac{d}{dv_1} \left(\frac{1}{2 \ln v_1} \left(-\ln v_1 - \sqrt{(\ln^2 v_1 - 4 \ln v_1)} \right) \right) = -\frac{1}{(\ln v_1)\, v_1 \sqrt{((\ln v_1)(\ln v_1 - 4))}} > 0.$$

Moreover, $t_1^{\max}(v_1) = 1 \Leftrightarrow v_1 = e^{-\frac{1}{2}}$. Finally, $\lim_{v_1 \to 1} t_1^{\max}(v_1) = \infty$, so that $b_1^*(v_1; t)$ is monotone decreasing in t for $0 \leq v_1 \leq e^{-\frac{1}{2}}$, a concave function in t with an interior maximum at $t_1^{\max}(v_1)$ for $e^{-\frac{1}{2}} < v_1 < 1$ and strictly increasing in t if $v_1 = 1$.

Performing the same exercise for bidder 2 yields

$$t_2^{\max}(v_2) = -\frac{1}{2} \ln v_2 + \frac{1}{2} \sqrt{(\ln^2 v_2 - 4 \ln v_2)}$$

which is strictly decreasing in v_2. As $t_2^{\max}(v_2) = 1 \Leftrightarrow v_2 = e^{-\frac{1}{2}}$ and $\lim_{v_2 \to 0} t_2^{\max}(v_2) = \infty$, it follows that $b_2^*(v_2; t)$ is monotone increasing in t if $v_2 = 0$, a concave function in t with an interior maximum at $t_2^{\max}(v_2)$ for $0 < v_2 < e^{-\frac{1}{2}}$, and strictly decreasing in t if $v_2 \geq e^{-\frac{1}{2}}$.

”indirect effect”). When $\frac{d}{dt}b_1^*(v_1;t) > 0$, then competition gets tougher which increases the marginal benefit so that both effects go in the same direction (as is shown in footnote 4, a region where $\frac{d}{dt}b_1^*(v_1;t) > 0$ does not exist for all v_1).[8] As t becomes large, then eventually $\frac{d}{dt}b_1^*(v_1;t) < 0$, so that competition gets weaker and marginal benefit from increasing b_2 decreases. When this effect is so strong that it overcompensates the direct effect, then $b_2^*(v_2;t)$ also decreases.[9] An analogous argument holds for bidder 1, except that the direct effect always leads to lower marginal benefit as he does no longer win whenever $b_1 > t \cdot b_2^*(\cdot)$ but only when $b_1 > (t + \Delta t) \cdot b_2^*(\cdot)$.

5 Conclusion

In this paper, we have analyzed a two-player all-pay auction where one bidder is handicapped by the auction rule while the other is favored. The relevance of our analysis is due to the fact that all-pay auctions are strategically equivalent to discriminatory contests where these asymmetries are often observed in reality. We have shown that there exists a unique Bayesian Nash Equilibrium. Furthermore, it is impossible that the handicapped player wins the auction when he has a lower valuation. Whether the equilibrium bidding strategy of each bidder is increasing or decreasing in t depends on t itself, on bidder i's valuation v_i, and on $F(\cdot)$.

Coming back to the strategic equivalence of all-pay auctions and discriminatory contests, one can also interpret the bids as (socially useless) efforts undertaken to secure a rent. Then, it would be an interesting extension to compare the welfare loss from the allocation inefficiencies caused by the possibility of the favored party winning the contest even if it has the lower valuation to the welfare gain from the fact that total effort may be lower in such a contest with handicaps. However, one would then have also to take into account that handicaps are frequently introduced because the contest designer has a specific utility function. For instance, he explicitly wants to support local suppliers, or he

[8] The point is that increasing the bid becomes less attractive if the probability of winning is already high, since the probability is only increasing at a decreasing rate at least up from a certain point (this follows simply from the fact that the winning probability is bounded above by one).

[9] Thus, in the terminology of Bulow, Geanakoplos, and Klemperer (1985), bids are ”strategic complements”.

may believe that penalizing an innocent defendant is worse than acquitting a defendant who is guilty.

Appendix

1 Proof of Lemma 1

To prove the several characteristics of bidding strategies, we proceed in three steps. First, we show that the structure of the payoff function induces non-decreasing strategies. Together with continuity, this in turn implies strict monotonicity and therefore differentiability and bijectivity on the restricted domain D_{b_i}.

As a first step consider monotonicity. For any $v_i', v_i \in [0,1]$ and for $v_i' > v_i$ incentive compatibility requires

$$\begin{aligned} \Pi_i(b_i(v_i), v_i) &\geq \Pi_i(b_i(v_i'), v_i) \\ \Pi_i(b_i(v_i'), v_i') &\geq \Pi_i(b_i(v_i), v_i') \end{aligned}$$

Taking the sum of both conditions and reordering yields:

$$\Pi_i(b_i(v_i'), v_i') - \Pi_i(b_i(v_i'), v_i) \geq \Pi_i(b_i(v_i), v_i') - \Pi_i(b_i(v_i), v_i).$$

Using the explicit structure of the pay-off function, this leads to

$$\begin{aligned} (v_1' - v_1)\,\Pr(b_1(v_1') > t \cdot b_2) &\geq (v_1' - v_1)\,\Pr(b_1(v_1) > t \cdot b_2) \\ (v_2' - v_2)\,\Pr(b_2(v_2') > \frac{1}{t} \cdot b_1) &\geq (v_2' - v_2)\,\Pr(b_2(v_2) > \frac{1}{t} \cdot b_1) \end{aligned}$$

But this only holds if $b_i(v_i') \geq b_i(v_i)$ which proves monotonicity.

We will prove continuity by contradiction. Assume that b_1 is not continuous at $x \in (0, b_1(1))$. Stated differently $b_1(x) > \lim_{\epsilon \to 0} b_1(x - \epsilon) \equiv \underline{b}_1(x)$. This implies, that bidder 2 will not submit some bid $b_2 \in (\underline{b}_1(x)/t, b_1(x)/t)$ as he can always reduce costs while the probability to win the auction remains unchanged. Anticipating this, there is no reason for bidder 1 to increase bids from $\underline{b}_1(x)$ to $b_1(x)$. Hence, we end up with a contradiction.

Note, that the same result can be derived for the continuity of strategies of the favored player by a permutation of indices and the appropriate modification of probabilities to win the auction. Furthermore, as $F'(v) \neq 0 \ \forall v \neq 0$, this result holds for the entire interval of valuations.

Now assume that $b_i(v_i)$ is not strictly increasing on the restricted domain D_{b_i}. That means, there is an interval $I \subseteq (0,1]$ of finite length with $b_i(v_i) \equiv \underline{b} > 0 \forall v_i \in I$. Given such a strategy profile of bidder i, bidder j maximizes his expected payoff as given by Eqn. (5). To be specific, let $i = 1$ and $j = 2$. Now assume bidder 2 bids $(\underline{b} - \epsilon)/t$. Then his pay-off is

$$v_2 \Pr(\underline{b} - \epsilon > b_1) - (\underline{b} - \epsilon)/t$$

(with an appropriate valuation v_2). Now assume bidder 2 bids $(\underline{b} + \epsilon)/t$ instead. His expected pay-off function is then

$$v_2 \Pr(\underline{b} + \epsilon > b_1) - (\underline{b} + \epsilon)/t$$

Bidder 2 profits from such a deviation as can be seen when $\epsilon \longrightarrow 0$

$$\begin{aligned}
\lim_{\epsilon \longrightarrow 0} (v_2 \Pr(b_1 &> \underline{b} + \epsilon) - (\underline{b} + \epsilon)/t - (v_2 \Pr(b_1 > \underline{b} - \epsilon) - (\underline{b} - \epsilon)/t)) \\
&= \lim_{\epsilon \longrightarrow 0} (v_2(\Pr(b_1 \in [\underline{b} - \epsilon, \underline{b} + \epsilon])) - 2\frac{\epsilon}{t}) \\
&= v_2 \Pr(b_1 = \underline{b}) > 0
\end{aligned}$$

Therefore bidder 2 will always bid slightly above $\underline{b}/t$ instead of slightly below, but that contradicts continuity. Analogously, a gap in bidding strategies of bidder 1 can be deduced from a plateau in bidder 2's equilibrium strategies. This proves strict monotonicity on the restricted domain. Therefore bidding strategies are differentiable almost everywhere and a bijection from the restricted domain D_{b_i} onto $(0, b_i(1)]$. Finally, D_{b_i} has to be non-empty, as it can never be part of an equilibrium that both bidders or only one bidder send zero bids for the entire valuation space.

2 Proof of Lemma 2

Part i) Clearly, $b_i(0) = 0$ determines the lower bound of D_{G_i}. Moreover, denoting by $b_i^{\max}$ the maximum bid of bidder i, it follows from Lemma 1 that $\rho(b_i^{\max}) = \max\{v_i\} = 1$ must hold for bidder i. This implies that bidder 1 can never be better off by bidding too high, i.e. $b_1 \leq t \cdot b_2^{max}$ has to hold. Analogously, neither will bidder 2 bid more than necessary to win the auction with probability 1, i.e. $b_2 \leq \frac{1}{t} \cdot b_1^{max}$ has to hold. Of course, this must also be true for $b_1^{\max}$ and $b_2^{\max}$, respectively, i.e. $b_1^{\max} \leq t \cdot b_2^{max}$ and $b_2^{\max} \leq \frac{1}{t} \cdot b_1^{max}$ must hold. Rearranging yields

$$b_2^{max} \geq \frac{1}{t} \cdot b_1^{max} \leq b_2^{\max}$$

from which it follows that $b_2^{max} = \frac{1}{t} \cdot b_1^{max}$ or equivalently, $b_1^{max} = t \cdot b_2^{max}$ must hold. We refer to this as the *final condition.*

Part ii) Follows immediately from our assumptions on $F(v)$ and Lemma 1.

Part iii) Suppose, $G_j(0) = g > 0$. We show that, for all $v_i \in [0, 1]$, there is some positive bid $x > 0$ for bidder i such that he is strictly better off than with bidding $b_i = 0$: With $b_i = 0$, bidder i's loses whenever $b_j > 0$ (which happens with probability $(1 - g)$) wins with probability $\frac{1}{2}$ whenever $b_j = 0$ (which happens with probability g) so that his expected payoff is simply $v_i \cdot \frac{g}{2}$. When submitting a positive bid $x > 0$, he wins with certainty when $b_j = 0$ and, depending on x (and t), may even win when $b_j > 0$. Thus we have:

$$\Pi_i(x, \cdot) = v_i \cdot G_j(x) - x \geq v_i \cdot g - x > v_i \cdot \frac{g}{2} = \Pi_i(0, \cdot)$$

where the last inequality holds whenever $x < \frac{g}{2} \cdot v_i$, so that for all $v_i > 0$, there exist $x > 0$ which satisfies this condition.

Part iv) As the first order conditions consist of two ordinary first order differential equations which are Lipschitz continuous for $v_i > 0$, any set of initial conditions $(b_i(v_i) = c_i, i = 1, 2)$ determines unique trajectories $b_i(v_i)$. In the following, we show that part

(i) and part (iii) together with the so-called no-crossing property of equilibrium bids (see Lizzeri and Persico (2000)) implies, that there is only *one* admissible set of initial conditions.

First note, that the final condition in part (i) reduces the freedom to choose initial conditions by one, as for a given $b_i(1)$, $b_{j\neq i}(1)$ is fixed. On the other hand part (iii) requires that at least one bidder i sends finite bids for every positive valuation $b_i(v_i) > 0 \forall v_i > 0$.

Consequently, for two sets of initial conditions to co-exist, in at least one set one of the bidder's bid-distributions has to have an atom at zero. Furthermore, one of the two following properties of the corresponding equilibrium bid functions would have to hold.[10] (a) The atom of one bidder's bid distribution is smaller against a tougher strategy of his opponent. (b) At least one bidder bids the same for a given valuation against two distinct opponent's strategies. In the following we show that none of the two requirements can be fulfilled in equilibrium.

As to (a), consider the first order conditions for $\underline{v}_i$ given by $F(\underline{v}_i) \equiv G_i(0)$ and denoting bidder i's type who only just sends a zero bid and a second equilibrium denoted by $\widetilde{(\cdot)}$

$$\begin{aligned} \underline{v}_i \frac{d}{db} G_j(0) &= 1 \\ \widetilde{\underline{v}}_i \frac{d}{db} \widetilde{G}_j(0) &= 1 \end{aligned}$$

But (a) requires that $\frac{d}{db} G_j(0) > \frac{d}{db} \widetilde{G}_j(0)$ and $\underline{v}_i > \widetilde{\underline{v}}_i$ are satisfied simultaneously which is a contradiction to the structure of the first order conditions.

A similar argument contradicts (b). The first order conditions [11] for player 1 with valuation v_1 against two distinct strategies of player 2 (once more distinguished by $\widetilde{(\cdot)}$

$$\begin{aligned} v_1 \frac{d}{db_1} G_2(b_1/t) &= 1 \\ v_1 \frac{d}{db_1} \widetilde{G}_2(b_1/t) &= 1 \end{aligned} \tag{28}$$

can not be fulfilled simultaneously. Therefore co-existing sets of initial conditions are not

[10] To see this it suffices to plot ρ_i against b_i for $i = 1, 2$ as detailed in Lizzeri and Persico (2000).

[11] Once again we restrict ourselves to the favored bidder without loss of generality as the argument is independent of t.

feasible.

3 Proof of Proposition 1

Part i) As for the first case, in any BNE, bidder 1 loses the auction whenever $\Pr(b_1^*(v_1) < t \cdot b_2^*(v_2)) = \Pr(v_2 > k(v_1;t))$ which simply follows from the definition of $k(v_1;t)$: Since $k(v_1;t)$ gives that type of bidder 2 who bids $\frac{1}{t}-$times as much as bidder 1 (which would result in a tie), bidder 1 loses the auction whenever $v_2 > k(v_1;t)$. In order to violate Pareto efficiency, also $v_1 > v_2$ must hold. As we have seen for the symmetric case with $t = 1$, differential equation (20) leads to $k(v_1;1) = v_1$ so that we would have $\Pr(v_2 > v_1) = 0$ whenever $v_1 > v_2$. We now show that $k(v_1;t)$ as given by (20) will be decreasing in t, so that for all $t > 1$ there may exist v_1, v_2 such that $\Pr(v_2 > k(v_1;t)) > 0$ even when $v_1 > v_2$. It then follows from that the (unique) solution to Eqn. (20) satisfying the initial condition $k(1,t) = 1$ is implicitly given by the following integral equation:

$$0 = \int_{k(v_1;t)}^{1} \frac{\frac{\partial F(u)}{\partial u}}{tu} du - \int_{v_1}^{1} \frac{\frac{\partial F(u)}{\partial u}}{u} du \tag{29}$$

Clearly, this equation is continuous is k and t and its derivative with respect to k is non-zero, so that we can apply the implicit function theorem to Eqn. (29) to get

$$\frac{dk(v_1;t)}{dt} = -\frac{k(v_1;t)}{tF'(k(v_1;t))} \int_{k(v_1;t)}^{1} k(v_1;t) \frac{\frac{\partial F(u)}{\partial u}}{u} du \leq 0. \tag{30}$$

Contrary to that consider the second case: Bidder 2 loses whenever $\Pr(b_2^*(v_2) < \frac{1}{t} \cdot b_1^*(v_1)) = \Pr(k^{-1}(v_2;t) < v_1)$. Again, for this outcome not to be efficient, we must also have $v_2 > v_1$. Again, for $t = 1$ we get $k^{-1}(v_2;1) = v_2$ from which it follows that $\Pr(v_2 < v_1) = 0$ when $v_2 > v_1$. However, contrary to the first case, we can show that $k^{-1}(v_2;t)$ is increasing in t, so that this condition can never be satisfied for all $t > 1$ either. To be specific, we have

$$\frac{dk^{-1}(v_2;t)}{dt} = \frac{k^{-1}(v_2;t)}{tF'(k^{-1}(v_2;t))} \int_{k^{-1}(v_2;t)}^{1} \frac{\frac{\partial F(u)}{\partial u}}{u} du \geq 0. \tag{31}$$

Part ii) For any $v_1 \in [0,1]$, a welfare loss occurs whenever bidder 2 has the lower valuation, but wins the object. Thus, by definition of $k(v_1;t)$, bidder 1 loses whenever $v_2 > k(v_1;t)$ holds. It follows that for all values of v_2 satisfying $k(v_1;t) < v_2 < v_1$, we get a welfare loss $(v_2 - v_1) < 0$. Taking expectations over v_1 then yields Eqn. (24).

Part iii) For the case $k(v_1;t) - v_1 < 0$, taking the derivative of L^* with respect to t yields

$$-\int_0^1 (k(v_1;t) - v_1)dv_1 \cdot \frac{dk(\cdot)}{dt} < 0 \qquad (32)$$

as it was shown in Eqn. (30) that $k(v_1;t)$ is decreasing in t.

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