

BONN ECON DISCUSSION PAPERS

Discussion Paper 14/2006

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by

Xia Su

June 2006



Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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Xia Su*

Bonn Graduate School of Economics
University of Bonn
Adenauerallee 24-26
53113 Bonn Germany
xia.su@uni-bonn.de

Revised in June 2006

*I am grateful for comments and suggestions to A. Mahayni, K. Sandmann, M. Suchaneki and to seminar participants at French Finance Association Annual Meeting (AFFI) 2006. The author remains responsible for all errors of omission and commission.

Hedging Basket Options by Using a Subset of Underlying Assets

Abstract

This paper proposes two-step static hedging strategies for European basket options by using only plain-vanilla options on a subset of underlying assets. The basic idea is stimulated from a static super-hedging strategy dependent on the whole basket. However, it would be too complicated to handle when there is a large number of assets in the basket. It becomes even worse when some of the underlying assets are illiquid or not available for trading. Meanwhile, this strategy completely neglects the correlation structure of the basket which has indeed a great effect on the basket option's price. To solve these problems, Principal Components Analysis is used to figure out the subset of dominant assets through a careful study on the modified covariance of the basket. On this basis, the optimal strikes of those significant assets' plain-vanilla options are obtained in the second step via optimization. The optimality criterion depends on the risk attitude of hedgers and is defined by a certain risk measure, e.g., super-replication, minimum expected shortfall given a constraint on the hedging cost. Through analyzing a numerical example, it is concluded that this static hedging portfolio captures a trade-off between reduced hedging costs and overall super-replication. Even without considering transaction costs, hedging by using only a subset of underlying assets performs well: only a reasonable small hedging error arises when investing the capital required by the super-hedging portfolio which is composed of plain-vanilla options on all underlying assets and hence is difficult to implement or even not available in the market.

Key words: Basket options, Principal Components Analysis, Super-replication, Expected shortfall.

JEL subject classification: G13.

1 Introduction

A basket option is an option whose final payoff is linked to a portfolio or “basket” of underlying assets. The basket can be any weighted sum of underlyings as long as the weights are all positive. Various types of basket options have emerged in the market and become increasingly popular as a tool for reducing risks since the early 1990s. As well acknowledged in the literature, the inherent challenge in pricing and hedging basket options stems primarily from the analytical intractability of the distribution of a weighted average of correlated lognormals. Another difficulty is due to the time-dependent correlation structure of the underlying basket which is not available in the market and has to be estimated from scarce option data or from historical time series. The current common practice is to assume it as a constant.

So far, several methods have been proposed for hedging European-style basket options. Engelmann and Schwendner (2001) [9] derive a perfect hedging strategy by running Monte Carlo simulations. Pellizzari (2004) [24] as well as Ashraff, Tarczon and Wu (1995) [1] develop a static partial hedge which minimizes the variance of the discrepancy between the final payoffs of the target basket option and the hedging portfolio. The least expensive upper bound or super-hedging portfolio is pursued by different methods: by solving an optimization problem in d’Aspremont and El-Ghaoui (2003) [2] and Laurence and Wang (2003) [18], via Fréchet bounds by Cherubini and Luciano (2002) [5] in a Copula framework and by solving Lagrange functions in Hobson, Laurence and Wang (2005) [14]. In the presence of multiple assets, the calculation of these methods is generally complicated and yields large hedging portfolios related to all the underlying assets. In this work, we develop new static hedging strategies consisting of plain vanilla options only on a subset of constituent assets.

The basic idea of this work is inspired by a static hedging strategy which dominates the final payoff of a basket option by using plain-vanilla options on all the underlying assets. However, it is indeed impractical to consider such a strategy based on all the underlying assets. As often observed in the market, most of the new contracts are related to a large number of underlying assets. In this case, hedging with all the underlying assets would be not only computationally expensive, but also would create high transaction costs which greatly reduce the hedging efficiency. Furthermore, hedging with a subset of assets becomes more practical and essential when some of the underlying assets are illiquid or not even available for trading¹. Thus, it is desirable to find a strategy to hedge a basket option by using only a subset of assets at a reasonable cost. To this end, Lamberton and Lapeyre (1992) [17] design a dynamic approximate hedging portfolio which consists of plain-vanilla options on the sub-basket identified by a multiple regression analysis. In practice, according to Nelken (1999) [20], the sub-hedge-basket is most often determined simply according to the liquidity or exposure of the underlying assets.

This article aims at introducing another approach, Principal Components Analysis (PCA), to select the hedging assets. PCA is one of the classical data mining tools to reduce dimensions in multivariate data by choosing the most effective orthogonal factors

¹This is possible when the underlying is a mutual fund.

to explain the original variables. This objective can be easily realized by decomposing the covariance matrix. Thus, this method is quite easy to implement with almost instantaneous calculation as well as reasonable accuracy. Meanwhile, it alleviates one inherent drawback of the static super-hedging strategy: the correlation effect is completely neglected such that the hedging portfolio is independent of the correlation structure of the underlying basket. It hence works well only for those basket options with strongly correlated underlying assets and the performance decreases greatly with the correlation. In contrast, by means of our new method, the correlation effect is well incorporated in the hedging strategy through the study of a modified covariance structure of the basket.

Through a proper combination of the super-hedging strategy and the newly-proposed asset selection technique, a two-step static hedging method is designed consisting of plain-vanilla options only on the dominant assets in the basket. In the first step, the appropriate set of hedging assets is figured out by means of PCA while taking the correlation structure of the basket into the consideration. Then, the optimal strikes of the options on the chosen sub-basket are calculated by solving an optimization problem. Surely, a subset could not perfectly track the original underlying basket and may leave some risk exposure uncovered. In this context, different optimality criteria can be designed to pursue super- or partial-replications. Basically, the criterion depends on the risk attitude of the hedger. They may favor a super-replication to eliminate all risks. Alternatively, with a constraint on the hedging cost at the initial date, optimal strikes are computed by minimizing a particular risk measure, e.g., the variance of the hedging error or the expected shortfall.

Considering that there is only a limited number of options traded in the market, we have to make a proper adjustment on the optimization problem: one condition has to be imposed such that the strikes are restricted in the given set. Generally, the hedging portfolio can be obtained by using a numerical searching algorithm. However, such a numerical optimization is computationally inefficient especially when the (sub-)basket is large and when a big number of strikes is available in the market for the chosen hedging assets. In this context, a simple calibration procedure, convexity correction method, is developed for super-hedging portfolios. Those optimal but unavailable options are approximated by a linear combination of two options with neighboring strikes. This calibration method maintains not only the super-replication property but also gives the cheapest portfolio attainable in the market.

Due to the lack of the distribution of the underlying basket, the hedging portfolios are obtained numerically through Monte Carlo simulations. A numerical study shows that the hedging error (measured by the expected shortfall) at the maturity date decreases with the optimal strikes and hence the hedging cost. As a result, the newly-proposed static hedging portfolio by a subset of underlying assets achieves a trade-off between reduced hedging costs and overall super-replication. It is also demonstrated that hedging with only several underlying assets gives a satisfactory performance: when the super-hedging portfolio composed of plain-vanilla options on all the underlying assets does not exist or is not easily implementable, hedging with several underlying assets generates only a reasonably small hedging error by investing the same capital as the hedging cost of the super-hedging portfolio. Furthermore, such a hedging portfolio creates far less

transaction costs than the super-hedging portfolio based on all the underlying assets if it is available. It enhances in turn the performance of the new hedging strategy by using only several underlying assets.

The remainder of the paper is organized as follows: Section 2 gives basic assumptions and points out three problems to deal with in this work after introducing a static super-hedging method. To cope with a large number of underlying assets and complicated correlation structure, Section 3 develops an asset selection technique based on PCA. On this basis, a two-step static hedging strategy is proposed in Section 4 by properly combining the asset selection technique and the static super-hedging strategy. Numerical results are reported in Section 5. Finally, Section 6 concludes the paper.

2 Assumptions and Problem Formulation

To develop a new hedging portfolio for European basket options, we give first basic assumptions and notations used throughout the paper. Then, this section moves on to formulating three problems in a static super-hedging portfolio dependent on all the underlying assets, which is a cornerstone of this work initiating the basic idea.

2.1 The Financial Market

Consider a financial market with a bank account B and N risky assets S_i , $i = 1, \dots, N$. The dynamics of the bank account, which is continuously compounded with a constant risk free interest rate $r \geq 0$, are given by

$$dB(t) = rB(t)dt.$$

In order to model the N risky assets, the standard N -dimensional Wiener process $W = (W_1(t), \dots, W_N(t))$ is defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{Q})$, where $\mathcal{Q}$ is the risk-neutral probability measure. As often assumed in the literature, the price process of each risky asset S_i , $i = 1, \dots, N$ follows a geometric Brownian motion and one-dimensional Brownian motions W_i , $i = 1, \dots, N$ are correlated with each other according to a constant parameter. More explicitly, under the risk-neutral probability measure $\mathcal{Q}$ we have

$$\begin{aligned} dS_i(t) &= (r - q_i)S_i(t)dt + \sigma_i S_i(t)dW_i(t) & i = 1, \dots, N \\ \rho_{ij}dt &= dW_i(t)dW_j(t) & i \neq j, \end{aligned}$$

where σ_i and q_i are the volatility and continuously compounded dividend yield of asset i , respectively and $\rho_{i,j} \in [-1, 1]$ denotes the constant correlation between assets i and j . In this way, the market is complete since sources of uncertainty have the same number as risky assets. Moreover, the determinant of the corresponding correlation structure is assumed to be strictly nonzero to ensure market completeness.

In addition to the above-mentioned primary assets, there are also T -contingent claims, such as plain-vanilla calls on each risky asset S_i with strike price $k \in \mathcal{K}^{(i)}$, the set of all strike prices traded in the market, and maturity date T

$$C_T^{(i)}(k) = (S_i(T) - k)^+, \quad i = 1, \dots, N.$$

In this financial market, we are going to design a hedging strategy for an European basket call on the N risky assets with maturity date T and strike price K

$$BC_T(K) = \left(\sum_{i=1}^N \omega_i S_i(T) - K \right)^+,$$

where each risky asset is weighted by a positive constant ω_i , $i = 1, \dots, N$. This hedging strategy for an European basket *call* can be easily translated into one for the corresponding European basket *put* based on the put-call parity result (see Laurence and Wang (2003) [18] and Deelstra, Liinev and Vanmaele (2004) [8])

$$\left(K - \sum_{i=1}^N \omega_i S_i(T) \right)^+ = \left(\sum_{i=1}^N \omega_i S_i(T) - K \right)^+ + \left(K - \sum_{i=1}^N \omega_i S_i(T) \right).$$

To measure the effectiveness of a hedging portfolio (HP), the hedging cost (HC) is defined as the price of the hedging portfolio at the initial date 0. Meanwhile, the hedging error at the maturity date T is simply denoted as HE , giving the difference between the payoffs of the basket option and the hedging portfolio, i.e., $BC_T(K) - HP_T$.

2.2 Problem Formulation

Theoretically, a perfect hedge is achievable for European basket options in a complete market. However, due to the lack of the distribution of the underlying basket and the knowledge of related parameters and most importantly a large number of the underlying assets, it is indeed impossible and impractical to hedge basket call options perfectly by all the constitute assets. In this context, a static hedging method is developed only related to a subset of underlying assets. This newly-designed hedging strategy is built up on the basis of a static super-hedging portfolio which consists of plain-vanilla call options on all the constituent assets with optimal strike prices. Hence, this subsection presents the super-hedging portfolio and points out three problems that remains to be fixed in the subsequent sections.

2.2.1 A Static Super-Hedging Strategy

This static hedging strategy aims at finding the least expensive portfolio whose final payoff always dominates that of a basket call. The idea is stimulated by Jensen's inequality for the final payoff of a basket call:

$$\begin{aligned} BC_T(K) &= \left(\sum_{i=1}^N \omega_i S_i(T) - K \right)^+ \\ &= \left[\sum_{i=1}^N \omega_i (S_i(T) - k_i) \right]^+ \\ &\leq \sum_{i=1}^N \omega_i (S_i(T) - k_i)^+ = \sum_{i=1}^N \omega_i C_T^{(i)}(k_i). \end{aligned}$$

First, ω_i is taken out of the bracket such that the equality holds if and only if $\sum_{i=1}^N \omega_i k_i = K$. The second transformation is due to Jensen's inequality. That is, the payoff of any portfolio consisting of N plain-vanilla calls is never lower than that of the corresponding basket call. Moreover, as a consequence of the no-arbitrage assumption, the price of a financial product is given by the discounted expected final payoff under the risk-neutral measure $\mathcal{Q}$. The corresponding relationship between the prices of a basket call and the hedging portfolio is then obtained as

$$e^{-rT} E_{\mathcal{Q}} \left[\left(\sum_{i=1}^N \omega_i S_i(T) - K \right)^+ \right] \leq \sum_{i=1}^N \omega_i e^{-rT} E_{\mathcal{Q}} [(S_i(T) - k_i)^+] . \quad (1)$$

For the purpose of hedging, one would like to look for a portfolio of plain-vanilla call options with the optimal strike prices such that it is the cheapest hedging strategy to dominate the final payoff of a basket call. As a result, a minimization problem has to be dealt with: minimize the price of a weighted portfolio of standard options with respect to k_i 's subject to the condition that the sum of $\omega_i k_i$'s is equal to K , i.e.,

$$\min_{k_i} \quad \sum_{i=1}^N \omega_i e^{-rT} E_{\mathcal{Q}} [(S_i(T) - k_i)^+] \quad (2)$$

$$\text{s.t.} \quad \sum_{i=1}^N \omega_i k_i = K. \quad (3)$$

The optimal sequence of strikes k_i^* is uniquely determined by the following proposition.

Proposition 2.1. *Suppose the underlying assets of a basket option follow geometric Brownian motions and the BS model is valid, then the optimal k_i^* 's satisfying*

$$BC_0(K) \leq \sum_{i=1}^N \omega_i e^{-rT} E_{\mathcal{Q}} [(S_i(T) - k_i^*)^+]$$

are uniquely obtained by solving a set of equations:

$$k_i = S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp \left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2} \sigma_1 \sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1} q_1 - q_i \right) \right] \right\} \quad (4)$$

$$\sum_{i=1}^N \omega_i k_i = K.$$

The rigorous proof is provided in the Appendix. The optimization problem is treated in the BS framework and solved by the corresponding Lagrange function. In this sense, it is similar to Hobson, Laurence and Wang (2005) [14]. Nevertheless, they analyze the problem in a model independent context thus with a focus on proving the existence of a super-replicating strategy but not giving the solution form.

2.2.2 Discussions

A Large Number of Underlying Assets The first concern of this work is on the number of underlying assets. Adopting this hedging strategy, one has to hold options on

all the underlying assets. It becomes almost impossible in practice when the basket option is contingent on a large number of assets. This is not only computationally expensive but also creates unfavorable high transaction costs. Moreover, the problem would be much worse when some of the underlying assets are illiquid or even not available for trading. Hence, it is essential and practical to find a strategy to hedge basket options at a reasonable cost but with only a subset of assets.

Correlation Effect This static hedging portfolio is an upper bound. In this way, all the risks are avoided, which is the second best for risk managers as the first best, perfect hedging, is almost impossible or complicated. The similar idea was once applied by Nielsen and Sandmann (2003) [21] to Asian options². It is well-known that Asian options and basket options are similar in structure: both of them are average options. The only difference lies in that Asian options relate to the prices of one unique asset during a prespecified time interval and basket options to the prices of several assets at the maturity date. This makes however a big difference in the performance of this super-hedging strategy, which is mainly due to the correlation effect. Following the same idea, a super-replicating portfolio for Asian options is obtained consisting of options on the same underlying asset but with different maturity dates. In this case, the correlations are in effect auto-correlations and indeed involved endogenously in calculation. While, the hedging portfolio above for basket options is composed of a portfolio of plain-vanilla options thus completely independent of the correlation structure between assets³. It is clearly an advantage of this method since it alleviates the difficulty of basket options hedging in controlling the correlation structure with scarce reliable data. However as easily observed, the upper bound performs well only when the underlying assets are strongly correlated, for example when all the constituent stocks belong to the same industry. The performance decreases in correlation with too much overhedge and high hedging cost. In this sense, correlation should not be totally neglected but be properly treated.

Discrete Set of Strikes Traded So far, this strategy is derived in an idealized situation where all the option prices on the constituent assets with a continuum of strikes are known. That is, $\mathcal{K}^{(i)}$, the set of all strike prices of options traded in the market on the underlying asset S_i , is a continuum interval. With this full information, the portfolio could be obtained by simply computing a Lagrangian function in the BS framework. However, options are traded only with a limited number of strikes in reality. Thus, the obtained portfolio has to be calibrated accordingly.

3 Significant Asset Selection

Given the multi-dimensional nature of basket options, the derived hedging strategy is often composed of all the underlying assets. In practice, the underlyings in the contract are differently weighted and sometimes some with pretty small weights. Thus, one can simply hedge such basket options by neglecting those assets. However, this is rather

²Such a super-hedging portfolio is also developed in Simon, Goovaerts and Dhaene (1999) [26] by using some results from risk theory on stop-loss order and comonotone risks.

³It becomes even clear when observing that the correlation does not appear in the calculation of the optimal k_i 's in (4).

arbitrary and lacks a theoretical foundation for the general case. This paper offers a criterion for asset selection by using Principal Components Analysis (PCA) approach. Moreover, it demonstrates that the correlation effect is well incorporated in the asset selection procedure.

3.1 Principal Components Analysis

PCA is a popular method for dimensionality reduction in multivariate data analysis. Thus, it is useful in visualizing multidimensional data, and most importantly, identifying the underlying principal factors of the original variables. PCA is originated by Pearson (1901) [22] and proposed later by Hotelling (1933) [15] for the specific adaptations to correlation structure analysis. Its idea has been well described, among others, in Harman (1967) [11], Härdle and Simar (2003) [12] and Srirastava and Khatri (1979) [27]. We follow here the lines of Härdle and Simar (2003).

The main objective of PCA is to reduce the dimensionality of a data set without a significant loss of information. This is achieved by decomposing the covariance matrix into a vector of eigenvalues ordered by importance and eigenvectors. To be precise, consider the asset prices vector $\mathcal{S} = (S_1, \dots, S_N)^T$ with

$$E(\mathcal{S}) = \mu \quad \text{and} \quad Var(\mathcal{S}) = \Sigma = E[(\mathcal{S} - \mu)(\mathcal{S} - \mu)^T].$$

PCA decomposes the covariance matrix into its eigenvalues and eigenvectors as

$$\Sigma = \Gamma \Lambda \Gamma^T, \tag{5}$$

where $\Lambda = diag(\lambda_1, \dots, \lambda_N)$ is the diagonal eigenvalue matrix with $\lambda_1 > \dots > \lambda_N$ and Γ the matrix of corresponding eigenvectors

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN} \end{pmatrix}$$

or simply $(\gamma_1, \dots, \gamma_N)$ given by the columns of the matrix. Principal Components (PCs) transformation is then defined as the product of the eigenvectors and the original matrix less the mean vector

$$P = \Gamma^T(\mathcal{S} - \mu). \tag{6}$$

That is, the PC transformation is a linear transformation of the underlying assets. Its elements $P_1, \dots, P_N$ are named i -th Principal Components since they can be considered as the underlying factors that influence the underlying assets with decreasing significance as measured by the size of the corresponding eigenvalues.

The ability of the first n ($n < N$) PCs to explain the variation in data is measured by the relative proportion of the cumulated sum of eigenvalues

$$\pi_n = \frac{\sum_{j=1}^n \lambda_j}{\sum_{j=1}^N \lambda_j}.$$

If a satisfactory percentage of the total variance has been accounted for by the first few components, the remaining PCs can be ignored as the assets are already well represented without significant loss of information. Usually, the first several n PCs are chosen such that over 75% of the variance are accounted for or simply the first three factors are selected ($n = 3$) for the convenience of visualizing the data.

The weighting of the PCs, or simply the element of each eigenvector, describes how the original variables are interpreted by the factors. This could be validated by considering the covariance between the PC vector P and the original vector $\mathcal{S}$

$$\begin{aligned} Cov(\mathcal{S}, P) &= E(\mathcal{S}P^T) - E\mathcal{S}E P^T \\ &= E(\mathcal{S}\mathcal{S}^T\Gamma) - \mu\mu^T\Gamma \\ &= \Sigma\Gamma \\ &= \Gamma\Lambda\Gamma^T\Gamma \\ &= \Gamma\Lambda. \end{aligned} \tag{7}$$

It implies that the correlation $r_{ij} = \rho_{S_i, P_j}$ between the variable S_i and the PC P_j is⁴

$$r_{ij} = \frac{\gamma_{ij}\lambda_j}{(\sigma_{S_i}^2\lambda_j)^{1/2}} = \gamma_{ij} \left(\frac{\lambda_j}{\sigma_i^2} \right)^{1/2}.$$

Clearly, γ_{ij} is proportional to the covariance of S_i and P_j . The higher it is, the more related is the i -th asset to the j -th PC. Hence, γ_{ij} are usually called factor loadings, characterizing the relationship between the original variables S_i , $i = 1, \dots, N$ and the derived factors, i.e., P_j 's, $j = 1, \dots, n$. Furthermore, one can easily find

$$\sum_{j=1}^N \lambda_j \gamma_{ij}^2 = \gamma_i^T \Lambda \gamma_i$$

is indeed the (i, i) -element of the matrix $\Gamma\Lambda\Gamma^T = \Sigma$. Summing up all r_{ij}^2 yields

$$\sum_{j=1}^N r_{ij}^2 = \frac{\sum_{j=1}^N \lambda_j \gamma_{ij}^2}{\sigma_i^2} = \frac{\sigma_i^2}{\sigma_i^2} = 1.$$

Thus, r_{ij}^2 is calculated in the standard practice measuring the proportion of variance of S_i explained by P_j .

3.2 Application to Basket Options Hedging

Now based on the principle of PCA, hedging asset selection could be completed in four steps as follows:

Step I: Find the covariance matrix of the underlying basket.

As assumed in Section 2, each underlying asset follows a geometric Brownian motion.

Hence, we have in matrix

$$d\mathcal{S}_t = d \begin{pmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_N(t) \end{pmatrix} = \begin{pmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_N(t) \end{pmatrix} r dt + \begin{pmatrix} \sigma_1 S_1(t) dW_1(t) \\ \sigma_2 S_2(t) dW_2(t) \\ \vdots \\ \sigma_N S_N(t) dW_N(t) \end{pmatrix}$$

⁴Note that $Var(P_j) = \lambda_j$. For the detailed derivation please check the referred books.

by assuming the dividend is zero for simplicity⁵. As the basket is composed of correlated lognormal distributed multivariates, its covariance turns out to be a complicated matrix whose elements are in exponentials. To ease the evaluation procedure, the following covariance is taken into consideration

$$Cov(dw\mathcal{S}) = \begin{pmatrix} \omega_1^2 \sigma_1^2 S_1^2(t) & \omega_1 \omega_2 \sigma_1 \sigma_2 \rho_{12} S_1(t) S_2(t) & \cdots & \omega_1 \omega_N \sigma_1 \sigma_N \rho_{1N} S_1(t) S_N(t) \\ \omega_1 \omega_2 \sigma_1 \sigma_2 \rho_{12} S_1(t) S_2(t) & \omega_2^2 \sigma_2^2 S_2^2(t) & \cdots & \omega_2 \omega_N \sigma_2 \sigma_N \rho_{2N} S_2(t) S_N(t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1 \omega_N \sigma_1 \sigma_N \rho_{1N} S_1(t) S_N(t) & \omega_2 \omega_N \sigma_2 \sigma_N \rho_{2N} S_2(t) S_N(t) & \cdots & \omega_N^2 \sigma_N^2 S_N^2(t) \end{pmatrix} dt,$$

which is the covariance of the change of the basket instead of the basket itself. In this way, the covariance structure is greatly simplified while all the effects of involved parameters are well maintained on the basket option price. However, what are those non-anticipating prices of $S_i(t)$, $i = 1, \dots, N$?

Another trick here is to consider the ratio of $\frac{d\omega\mathcal{S}}{\mathcal{S}}$ such that asset prices $S_i(t)$ are cancelled out in the covariance structure as below

$$Cov\left(\frac{d\omega\mathcal{S}}{\mathcal{S}}\right) = \begin{pmatrix} \omega_1^2 \sigma_1^2 & \omega_1 \omega_2 \sigma_1 \sigma_2 \rho_{12} & \cdots & \omega_1 \omega_N \sigma_1 \sigma_N \rho_{1N} \\ \omega_1 \omega_2 \sigma_1 \sigma_2 \rho_{12} & \omega_2^2 \sigma_2^2 & \cdots & \omega_2 \omega_N \sigma_2 \sigma_N \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1 \omega_N \sigma_1 \sigma_N \rho_{1N} & \omega_2 \omega_N \sigma_2 \sigma_N \rho_{2N} & \cdots & \omega_N^2 \sigma_N^2 \end{pmatrix} dt.$$

This covariance structure should work well except for the case in which spot prices of the underlying assets differ significantly from one another. Hence, one has to additionally take the effect of asset prices into consideration. In such an extreme case, those underlying assets with greatly high prices should be always chosen (even with a relatively low volatility) due to its absolute dominant effect on the basket option price.

Obviously, the correlation effect of the underlying basket is also examined via asset selection method. In contrast to the usual practice of decomposing the correlation matrix as recommended in PCA textbooks, the modified covariance is however used in this application. This is simply because weights and individual asset volatilities do have a great impact on the basket option price.

Step II: Decompose the covariance matrix into eigenvalues ordered in significance and the corresponding eigenvectors. This evaluation procedure could be easily done by many programs such as Matlab, Mathematica, C++ etc.

Step III: Choose the first several important principal components according to the cumulative proportion of the explained variance.

Step IV: Select $N_1 < N$ most dominant underlying assets by examining their cumulative r^2 with the chosen principal components. The selection can be done in two ways: First, if the number of hedging assets, N_1 , is beforehand determined, the list of least important assets is checked out after a comparison of cumulative r^2 . If there is no

⁵The decomposition procedure and result are actually the same for the case with a constant continuous dividend rate since dividends have no any effect on the covariance structure.

prior requirement on the number of assets, a more careful study of the cumulative r^2 has to be done to find the most effective assets.

Remark 3.1. *It is noted that some information of the original basket is lost in the fourth step by taking the underlying assets which are strongly dependent on the first several PCs. Meanwhile, it loses a quantitative measure of the explained variance. This can not be improved by PCA itself and may lead to lower performance while hedging basket options. In order to overcome this obstacle, we introduce an optimization problem aiming at reducing the discrepancy between final payoffs of the basket call and the hedging portfolio on the sub-basket. See the discussion in Section 4.2.*

4 New Static Hedging Strategies by Using a Subset of Assets

A number of two-step static hedging methods are proposed in this section by properly combining the static super-hedging portfolio and the asset selection technique. First, the sub-hedge-basket is determined by PCA. Then, the hedging portfolio is designed to be composed of call options written on these N_1 most important underlying assets with optimal strikes. They are chosen according to an optimality criterion defined by a particular risk measure. We furthermore demonstrate that the obtained hedging strategies capture a trade-off between reduced hedging costs and overall super-replication of basket options.

4.1 First Step: Hedging Asset Selection

As the first step of the newly-designed static hedging method, PCA is utilized to find the subset of important assets in the basket through a careful study of the modified covariance structure. In this way, all the underlying assets are newly indexed and regrouped into two subsets: one subset of N_1 assets of high significance S_j , where $j = 1, \dots, N_1$ and one with the other $N - N_1$ assets S_j , where $j = N_1 + 1, \dots, N$. Then the final payoff of the basket option can be rewritten as

$$\begin{aligned}
\left(\sum_{j=1}^N \omega_j S_j(T) - K \right)^+ &\stackrel{(A)}{=} \left(\sum_{j=1}^{N_1} \omega_j S_j(T) - K_1 + \sum_{j=N_1+1}^N \omega_j S_j(T) - K_2 \right)^+ \\
&\stackrel{(B)}{\leq} \underbrace{\left(\sum_{j=1}^{N_1} \omega_j S_j(T) - K_1 \right)^+}_I + \underbrace{\left(\sum_{j=N_1+1}^N \omega_j S_j(T) - K_2 \right)^+}_{II} \\
&\stackrel{(C)}{\leq} \underbrace{\sum_{j=1}^{N_1} \omega_j (S_j(T) - k_j)^+}_{I'} + \underbrace{\sum_{j=N_1+1}^N \omega_j (S_j(T) - k_j)^+}_{II'}, \tag{8}
\end{aligned}$$

where $K = K_1 + K_2$, $\sum_{j=1}^{N_1} \omega_j k_j = K_1$ and $\sum_{j=N_1+1}^N \omega_j k_j = K_2$.

That is, the final payoff if a basket call is always dominated by two portfolios of plain-vanilla call options denoted as I' and II' in (C). This result is achieved by applying two times Jensen's inequality in (B) and (C). Serving as a trick for the further derivation, the strike of the basket option K is in (A) split into K_1 and K_2 where $K_1, K_2 \in [0, K]$ such that the final payoff of the basket call is first dominated by two basket calls on the two disjoint subsets of the original underlying assets. Then following the same idea as in the previous section, one could find portfolios of plain-vanilla call options to further dominate the two basket options. Clearly, if $N_1 = N$ and $K_1 = K$ (or $N_1 = 0$ and $K_1 = 0$), the obtained hedging portfolio consists of all the underlying assets. Thus, hedging with all the assets discussed in Section 3.1 is one special case. With the assumption of no arbitrage, we can get the same relationship for the price at the initial date of the basket call and the portfolio of plain-vanilla call options, after taking expectations and discounting their final payoffs under the risk neutral risk measure $\mathcal{Q}$:

$$BC_0(K) \leq \underbrace{\sum_{j=1}^{N_1} \omega_j e^{-rT} E_{\mathcal{Q}} \left[(S_j(T) - k_j)^+ \right]}_{I'} + \underbrace{\sum_{j=N_1+1}^N \omega_j e^{-rT} E_{\mathcal{Q}} \left[(S_j(T) - k_j)^+ \right]}_{II'}. \quad (9)$$

4.2 Second Step: Optimal Strikes Computation

Since the new hedging portfolio is only related to the dominant assets, our concern here is simply on part I' . Thus, in the second step, we have to search for the optimal strike prices k_j^* 's of the plain-vanilla calls in the hedging portfolio to cover as well as possible the risks that basket options are exposed to.

As mentioned in Introduction, such hedging portfolios could not be a perfect replication due to impracticability and impossibility of trading all the underlying assets. It could be nevertheless a super or partial-hedge as required by hedgers. In any case, the designed hedging portfolios are derived through an optimization problem satisfying a certain optimality criterion.

4.2.1 Optimality Criteria

Basically, the optimality criteria depend on the risk attitude of hedgers and are defined by particular risk measures. For instance, the criteria considered in this paper are designed to achieve super-replication, the minimum variance of the hedging error or the minimum expected shortfall given a certain initial hedging cost.

Criterion 1: Super-Replicate the Basket Option

The first constraint is imposed on part I' in order to keep the the hedging portfolio's payoff at the maturity date always higher than that of the basket option. In this way, this hedging portfolio eliminates all the risks of holding a basket option. It is achieved by solving an

optimization problem with the constraint of no sub-replication. More explicitly,

$$\min_{K_1, k_j} \sum_{j=1}^{N_1} \omega_j e^{-rT} E_{\mathcal{Q}} [(S_j(T) - k_j)^+] \quad (10)$$

$$s.t. \quad \mathbb{P}_{\mathcal{Q}} \left[\sum_{j=1}^{N_1} \omega_j (S_j(T) - k_j)^+ \geq \left(\sum_{j=1}^N \omega_j S_j(T) - K \right)^+ \right] = 1 \quad (11)$$

$$\begin{aligned} \sum_{j=1}^{N_1} \omega_j k_j &= K_1 \\ 0 &\leq K_1 \leq K. \end{aligned}$$

One may notice that one additional parameter K_1 is involved in the optimization problem. Although it is not relevant to the hedging portfolio, it simplifies the computation algorithm. The argument is as follows: First, the optimal k_j^* can be determined uniquely by Proposition 3.1 to super-replicate the basket option on the dominant assets for any given strike price $K_1 \in [0, K]$. However, K_1 has to be lower enough such that the basket option is well dominated simply by part I' . Consequently, we only need to compute such dominating hedging portfolios with all possible K_1 and then find out the least expensive one. The final hedging portfolio is then composed of only plain-vanilla call options on significant assets with the optimal strikes such that the basket options are super-replicated. However from a practical point of view, this hedging portfolio may overhedge too much and hence require a high hedging cost. This is partly due to the property of super-hedging portfolio whose hedging costs have to be high enough to stay always on the safe side. In addition, since the hedging portfolio is related to the sub-basket, more capital has to be invested because some risks arise from neglecting those insignificant assets. As a result, partial hedging strategies may be considered to achieve the trade-off of reduced hedging costs and overall super-replication.

Criterion 2: Minimize the Variance of Hedging Errors Given a Constraint on the Hedging Cost (HC)

With restricted capital, one can only cover risks or minimize shortfall risks as well as possible. The shortfall risk is for instance in this case measured by the variance of hedging errors. A hedging portfolio is pursued to minimize the variance of hedging errors when the hedging cost is constrained to be lower than V_0 , the maximal capital that hedgers would like to invest to hedge the basket option. Formally, it is expressed as

$$\min_{k_j} E_{\mathcal{Q}} \left[\left(\left(\sum_{j=1}^N \omega_j S_j(T) - K \right)^+ - \sum_{j=1}^{N_1} \omega_j (S_j(T) - k_j)^+ \right)^2 \right] \quad (12)$$

$$s.t. \quad \sum_{j=1}^{N_1} \omega_j e^{-rT} E_{\mathcal{Q}} [(S_j(T) - k_j)^+] \leq V_0 \quad (13)$$

$$k_j \geq 0 \quad \forall \quad j = 1, \dots, N_1.$$

Criterion 3: Minimize the Expected Shortfall Given $HC \leq V_0$

One main drawback of the quadratic criterion is that it punishes both positive and negative differences between the payoffs of the hedging portfolio and the basket option. Actually,

for the purpose of hedging, only the negative difference is not favored. To avoid such a problem, some other effective risk measures could be considered. The expected shortfall (ES) is in the context of hedging basket options defined by $E[(BC_T - HP_T)^+]$. Obviously, it accounts for only the positive hedging error. Meanwhile as a risk measure, it takes into account not only the probability of exposed risks but also the size. Hence, it is often used in the literature recently as a risk indicator. In this case, the optimization problem becomes

$$\min_{k_j} E_{\mathcal{Q}} \left[\left(\left(\sum_{j=1}^N \omega_j S_j(T) - K \right)^+ - \sum_{j=1}^{N_1} \omega_j (S_j(T) - k_j)^+ \right)^+ \right] \quad (14)$$

$$s.t. \quad \sum_{j=1}^{N_1} \omega_j e^{-rT} E_{\mathcal{Q}} [(S_j(T) - k_j)^+] \leq V_0 \quad (15)$$

$$k_j \geq 0 \quad \forall \quad j = 1, \dots, N_1,$$

where V_0 is again the restriction on the hedging cost.

To summarize, our new hedging portfolio is composed of plain-vanilla call options only on the dominant underlying assets in the basket with optimal strikes. This hedging portfolio is achieved by first identifying the subset of hedging assets by means of PCA, and then figuring out the optimal strikes for the call options on these assets based on an optimality criterion, e.g., super-replication, minimum variance or minimum ES given a certain investment into the hedge. The chosen criterion depends on the risk attitude of hedgers. The more risk averse he is, the tighter the criterion on the hedging error is, and the more probable the hedging portfolio with subset assets super-hedges basket options. In this context, the static hedging strategy presented in this paper finds a compromise between reduced hedging costs and overall super-replication. It is worth mentioning that all the optimization problems above are solved numerically by running Monte Carlo simulations because of the lack of the distribution of the underlying basket.

4.3 Hedging with a Discrete Set of Strikes

The above optimization problems are constructed in an ideal situation where the optimal strikes are always available in the market. In reality, $\mathcal{K}^{(i)}$, the set of all strike prices of options traded in the market on the underlying asset S_i , is however never a continuum range or interval but a discrete set. It makes a direct impact on the hedging portfolio since the optimal hedging product may not exist. Hence, the optimization problems have to be modified when considering only discrete sets of strikes traded. Generally, they are solved by running numerical searching optimization, which searches numerically the cheapest portfolio confined in the given strike set and also satisfying the constraint.

More explicitly, the set of traded strikes for asset S_j ($j = 1, \dots, N_1$) entails $m + 1$ strikes in the increasing order, i.e., $\mathcal{K}^{(j)} = (k_0^{(j)}, k_1^{(j)}, \dots, k_m^{(j)})$ with $k_i^{(j)} < k_{i+1}^{(j)}$ for $i + 1 \leq m$ and $k_0^{(j)} = 0$, namely, the least possible strike is such that the call option is the asset itself. Take the super-hedging strategy as an example. By restricting the hedging instruments to be those available in the market, the optimization problem for the super-hedging portfolio

is modified as

$$\begin{aligned} \min_{k_j \in \mathcal{K}^{(j)}} \quad & \sum_{j=1}^{N_1} \omega_j e^{-rT} E_{\mathcal{Q}} [(S_j(T) - k_j)^+] \\ \text{s.t.} \quad & BC_T(K) \leq \sum_{j=1}^N \omega_j C_T^{(j)}(k_i). \end{aligned}$$

Although the idea is straightforward, this optimization problem is only solvable by using numerical methods which is computationally intractable for a large number of underlying assets and a wide choice in strikes. Suppose each component asset has m options traded, the numerical search has to be done among all the possible combinations of those options of the order m^{N_1} . It is in general rather large since m is about 10 in reality and N_1 possibly a big number. To gain computational efficiency, another simple calibration method, convexity correction, is developed for super-hedging portfolios via approximating the option's price with the optimal strike by two traded options with the neighboring strikes.

Recall the main property of a convex function: its value at a particular point is bounded from above by a linear interpolation of two neighboring values. This can be used to maintain the super-replication feature of the desired hedging portfolio since the BS call option price is well-known to be convex with respect to the strike price. Assume some optimal strikes $k_{opt}^{(j)}$'s obtained by solving the optimization problem (10) are not always traded in the market. For those assets whose call options with strike price $k_{opt}^{(j)}$ are not traded, one can replace them by a linear combination of two call options with the neighboring strikes $k_i^{(j)}$ and $k_{i+1}^{(j)}$ such that

$$C^{(j)}(k_{opt}^{(j)}) \leq \beta^* C^{(j)}(k_i^{(j)}) + (1 - \beta^*) C^{(j)}(k_{i+1}^{(j)}),$$

where $\beta^* = \frac{k_{i+1}^{(j)} - k_{opt}^{(j)}}{k_{i+1}^{(j)} - k_i^{(j)}}$. In this way, the upper bound for a basket call option can be generally expressed for $j = 1, \dots, N_1$

$$\sum_{k_{opt}^{(j)} \text{ traded}} \omega_j C^{(j)}(k_{opt}^{(j)}) + \sum_{k_{opt}^{(j)} \text{ non-traded}} \omega_j \left(\beta^* C^{(j)}(k_i^{(j)}) + (1 - \beta^*) C^{(j)}(k_{i+1}^{(j)}) \right).$$

Consequently by means of convexity correction, a super-hedging strategy is achieved consisting of one or two traded call options on each dominant asset. As shown by numerical results, the price of the convexity-corrected hedging portfolio is quite close to the original optimal portfolio. Moreover, this calibrated hedging portfolio is formally shown in Hobson, Laurence and Wang (2005) [14] to be always optimal (in the sense of the cheapest super-replication) in the model independent framework.

5 Numerical Results

In this section, we give some numerical results for the new two-step static hedging strategy. Here we use the example that is first presented in Milevsky and Posner (1998) [19].

Basically, it is an index-linked guaranteed investment certificate offered by Canada Trust Co., fusing a zero coupon bond with a basket option that is stuck at the spot rate of the underlying indices. Here we are interested in hedging the embedded basket option of a weighted average of the renormalized G-7 indices

$$BC_T = \left(\sum_{i=1}^7 \omega_i \frac{S_i(T)}{S_i(t)} - 1 \right)^+.$$

That is, effectively, a call option on the rates of return of 7 indices. The necessary pricing parameters are given in Table 1 and 2. In addition, the risk-free interest rate is assumed to be deterministic and equal to 6.3%⁶.

country	index	weight (in %)	volatility (in %)	dividend yield (in %)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	20.68	2.39
U.K.	FTSE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 1: G-7 Index-linked Guaranteed Investment Certificate

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.46	-0.22	0.32
Italy	0.04	0.50	0.70	0.46	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 2: Correlation Structure of G-7 Index-linked Guaranteed Investment Certificate

5.1 Asset Selection Through PCA

Given the data above, the modified covariance structure of the G-7 index-linked guaranteed investment certificate is easily calculated as the product of the weights, variance and correlation. An implementation of the decomposition on this modified covariance gives then the eigenvalue vector in the order of significance

$$\lambda = (0.0017316, 0.0012689, 0.00080498, 0.00036031, 0.00012665, 0.000054805, 0.000026991)^T,$$

⁶One important issue has to be mentioned for this illustrative example. Since the underlying assets are stock indices of different countries, exchange rate risks between different currencies will be involved in pricing and hedging the basket option. Here, in order to fully focus on the hedging issue, we neglect these risks by simply assuming that all the indices are traded in the market and are denominated in the same currency.

and the eigenvectors γ_j in columns of the matrix

$$\Gamma = \begin{pmatrix} 0.19059 & -0.10933 & 0.073686 & 0.12389 & 0.044403 & 0.85033 & 0.45377 \\ 0.19804 & 0.20187 & 0.28185 & 0.89761 & -0.012084 & -0.084268 & -0.16628 \\ 0.33086 & 0.6309 & 0.5035 & -0.38989 & -0.24896 & 0.098936 & -0.12331 \\ 0.18838 & 0.14423 & 0.081178 & -0.10171 & 0.95992 & -0.030681 & -0.066223 \\ 0.090682 & 0.14994 & 0.083136 & 0.035052 & 0.00055669 & -0.47195 & 0.85931 \\ -0.17192 & -0.56988 & 0.79228 & -0.091422 & 0.039491 & -0.089673 & -0.0046174 \\ 0.86123 & -0.42569 & -0.1429 & -0.08373 & -0.11355 & -0.16831 & -0.091729 \end{pmatrix}.$$

Based on the knowledge of the eigenvalues and eigenvectors, one can determine the most significant factors according to the (cumulative) proportions of explained variance. As the results in Table 3 show, the first PC already explains around 40% of the total variation. An additional 55% is captured by the next three PCs. The remaining three PCs explain a considerably small amount of total volatility. In all, the first four PCs together account for about 95% of the total variation associated with all 7 assets. It suggests that we can capture most of the variability in the data by choosing the first four principal components and neglecting the other three.

eigenvalue	proportion of variance	cumulated proportion
0.0017316	0.39586	0.39586
0.0012689	0.29008	0.68594
0.00080498	0.18403	0.86997
0.00036031	0.082374	0.95235
0.00012665	0.028954	0.98130
0.000054805	0.012529	0.99383
0.000026991	0.0061707	1

Table 3: Proportion of Variance Explained by PCs

The final step is to find the optimal subset of the underlying assets by checking the cumulative r^2 of each asset with the first four components. If two assets are planned to be used in the hedging portfolio, we need only find out the two most important assets from the basket. To achieve this result, the individual r and cumulative r^2 with the first four PCs are reported in Table 4. The assets are ordered in the significance: S_6 , S_7 , S_2 , S_3 , S_1 , S_5 and S_4 . Hence, the subset of optimal hedging assets is composed of S_6 (Japan Nikkei 225) and S_7 (U.S. S&P 500). If the restriction on the number of assets is relaxed, a careful check has to be made on the cumulative r^2 . An obvious cut-off can be found between S_3 and S_1 , as indicated by the large discrepancy of the cumulative r^2 (the difference between 99.09% and 65.94%). Therefore, we can finally determine the subset of assets for the purpose of hedging consisting of four assets of S_2 (Germany DAX), S_3 (France CAC 40), S_6 (Japan Nikkei 225) and S_7 (U.S. S&P 500).

5.2 Static Hedging with the Selected Four Dominant Assets

With the selected assets, the static hedging strategy could be achieved by figuring out the optimal strikes for the call options on these assets. In the following, only the numerical

	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$\sum_{j=1}^4 r_{ij}^2$
S_1	0.68666	-0.33719	0.18101	0.2036	0.65941
S_2	0.37811	0.32993	0.3669	0.78175	0.99757
S_3	0.44383	0.72448	0.46052	-0.23858	0.99086
S_4	0.53617	0.35141	0.15754	-0.13206	0.45323
S_5	0.4195	0.59378	0.26223	0.073969	0.60279
S_6	-0.22944	-0.65105	0.72093	-0.055656	0.99934
S_7	0.91422	-0.38683	-0.10343	-0.040545	0.99778

Table 4: Correlation Between the Original Variables and the PCs

results for the hedging portfolios with four assets are shown. Generally, a hedge with four assets works better than that with two assets due to the importance of S_2 and S_3 in the basket. Moreover, as the weights in the basket are not changed after asset selection, the hedging subset surely better duplicates the original basket when more assets are included in the hedging portfolio. Nevertheless, the proper number of assets should be chosen in practice by comparing the additional hedging cost and the reduced hedging error.

To give a hint on the performance of this new static hedging method, the hedging cost is compared to the basket options price. All the prices of basket options and the corresponding hedging portfolios are obtained numerically by Monte Carlo simulations with a number of simulated paths equal to 500,000. Such a simulation procedure guarantees that the basket option price of 100 contracts is relatively accurate to the second digit as shown in Table 5. In addition to the hedging cost, the expected value of the hedging error at the maturity date is reported for each hedging portfolio to account for hedging performance. Based on the definition in Section 2, negative hedging errors are favorable, suggesting that the basket option is well hedged with no risk exposure any more. Meanwhile, a special attention is paid to ES which plays a major role as a risk indicator to measure the hedging result. Especially, the strike of this basket option is varied with different values $K \in \{0.90, 0.95, 1.00, 1.05, 1.10\}$ and the maturity date $T \in \{1, 3, 5, 10\}$ years to gain an overall view of the hedging performance across maturities and strikes. Moreover, the set of strikes traded in the market for each asset is assumed to be $\mathcal{K}^{(i)} = \{0, 0.15, 0.30, 0.45, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00, 1.05, 1.10, 1.15, 1.20\}$ for all $i = 1, \dots, N$.

Table 6 presents the results of static super-hedging portfolio with only four assets based on the first criterion. Convexity correction technique is used when the optimal strikes are not available for trading. Consequently, two options have to be included in the portfolio for one asset as often observed in the table. Super-replication is not available for those options with long maturity of 10 years due to large volatility involved. Otherwise, this hedging strategy well dominates the basket option, as shown by negative expected hedging errors and zero shortfall probability required in the optimization. However, super-replication requires a rather low K_1 and hence a pretty high hedging cost which amounts to even almost 7 times the basket option price for the case $T = 1$ and $K = 1.10$. Especially, Figure 1 is designed to demonstrate how K_1 influences the hedging cost and the hedging error. Clearly, K_1 has two opposite effects on the hedging

Table 5: MC Simulated Basket Call Prices and Standard Errors (in Bracket)
for 100 Contracts with 500,000 Simulations

	$T = 1$	$T = 3$	$T = 5$	$T = 10$
$K = 1.10$	1.50 (0.0053)	7.61 (0.0177)	13.75 (0.0293)	26.25 (0.0604)
$K = 1.05$	3.19 (0.0077)	10.24 (0.0197)	16.47 (0.0310)	28.73 (0.0611)
$K = 1.00$	5.90 (0.0099)	13.33 (0.0215)	19.50 (0.0323)	31.14 (0.0618)
$K = 0.95$	9.56 (0.0115)	16.81 (0.0227)	22.74 (0.0333)	33.68 (0.0623)
$K = 0.90$	13.88 (0.0124)	20.60 (0.0236)	26.17 (0.0339)	36.24 (0.0626)

Table 6: Super-Hedging Portfolio with Four Dominant Assets

K	T	BC_0	K_1	HC	$E[HE]$	k_2	k_3	k_6	k_7
0.90	1	13.88	0.46	30.30	-17.51	0.65/0.70	0.45/0.60	0.60/0.65	0.60/0.65
	3	20.60	0.22	53.75	-40.01	0.30/0.45	0.15/0.30	0.30/0.45	0.30
	5	26.17	0.08	63.52	-51.18	0/0.15	0/0.15	0/0.15	0/0.15
0.95	1	9.56	0.53	23.71	-15.08	0.75	0.60/0.65	0.70/0.75	0.70/0.75
	3	16.81	0.24	51.89	-42.34	0.30/0.45	0.15/0.30	0.30/0.45	0.30/0.45
	5	22.74	0.13	59.76	-50.70	0.15/0.30	0/0.15	0.15/0.30	0.15/0.30
1.00	1	5.90	0.56	21.78	-16.91	0.75/0.80	0.65/0.70	0.75/0.80	0.75
	3	13.33	0.26	50.13	-44.46	0.30/0.45	0.15/0.30	0.30/0.45	0.30/0.45
	5	19.50	0.15	58.84	-53.91	0.15/0.30	0/0.15	0.15/0.30	0.15/0.30
1.05	1	3.19	0.64	14.49	-12.04	0.85/0.90	0.75/0.80	0.85/0.90	0.85/0.90
	3	10.24	0.38	40.17	-36.17	0.45/0.60	0.30/0.45	0.45/0.60	0.45/0.60
	5	16.47	0.18	56.58	-54.96	0.15/0.30	0/0.15	0.15/0.30	0.15/0.30
1.10	1	1.50	0.70	10.01	-9.06	0.95	0.85/0.90	0.90/0.95	0.90/0.95
	3	7.61	0.46	33.42	-31.17	0.65/0.70	0.45/0.60	0.65	0.60/0.65
	5	13.75	0.19	55.78	-57.64	0.30	0.15	0.15/0.30	0.15/0.30

performance: a reduction in K_1 decreases the expected shortfall and meanwhile increases the hedging cost. Thus, a higher hedging cost is unavoidable to achieve super-replication. Besides, it demonstrates that the hedging strategy proposed in this paper is exactly to achieve a trade-off between successful hedges and reduced hedging costs by varying strikes.

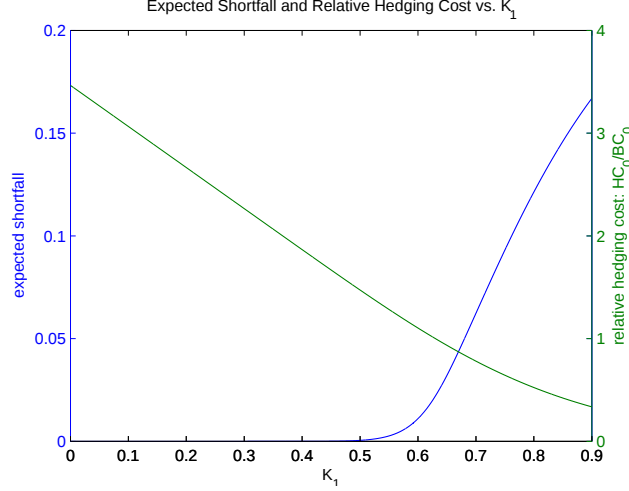


Figure 1: Expected Shortfall and Relative Hedging Cost vs. K_1 for the Basket Call with $T = 3$ and $K = 0.9$

When relaxing the strong requirement of super-replication, the hedging cost can be surely decreased, for instance, in the hedging portfolio obtained by taking the second criterion. As formulated in the model, the variance of the hedging error is minimized given a certain hedging cost V_0 . Here, two constraints are imposed on the hedging cost: BC_0 , the basket option price, and $HP(7)$, the hedging cost of the static super-hedging portfolio with all 7 underlying assets. First as shown in Table 7, both constraints lead to sub-replication, leaving some risks uncovered. Even for the case of $T = 1$ and $K = 1.10$, there is no such a portfolio when $V_0 = BC_0$ due to the limited number of strikes traded in the market. Given the strike set, all possible combinations of traded options have higher prices than the basket call. As the ES (the identifier of the hedging performance) decreases with the hedging cost, better results are achieved with the constraint of $HP(7)$: Opposite to the positive expected hedging error and high ES obtained in the case of $V_0 = BC$, the hedging error turns out to be negative on average and the ES decreases greatly to 4% – 9% across all maturities and strikes of the basket call. This result indicates that hedging with four assets gives a relatively satisfactory performance: only a reasonable low hedging error arises when investing the same capital as the hedging cost of the super-hedging portfolio composed of plain-vanilla options on all 7 assets which is supposed to be not available or at least not easily implementable. Moreover, such a hedging portfolio creates far less transaction costs than the super-hedging portfolio based on all the underlying assets if it exists. It enhances in turn the performance of the new hedging strategy by using only 4 underlying assets.

In addition, one can easily observe that the hedging portfolio given $V_0 = HP(7)$ performs better for short maturity. The relative ES and the variance of the hedging error of such portfolios always increase with T . Nevertheless, the variance of the hedging error

Table 7: Minimum-Variance Hedging Portfolio with Four Dominant Assets

K	T	BC_0	$V_0 = BC_0$										$HP(7)$		$V_0 = HP(7)$						
			HC	$E[HE]$	$Var[HE]$	$ES\%$	k_2	k_3	k_6	k_7	HC	$E[HE]$	$Var[HE]$	$ES\%$	k_2	k_3	k_6	k_7			
0.90	1	13.88	13.62	0.28	0.0008	9.48	0.85	0.80	0.95	0.85	15.05	14.93	-1.12	0.0007	4.80	0.85	0.70	0.95	0.85		
	3	20.60	20.57	0.03	0.0030	10.49	0.85	0.70	1.10	0.75	22.72	22.70	-2.54	0.0026	5.18	0.80	0.65	1.00	0.75		
	5	26.17	26.08	0.13	0.0054	11.25	0.80	0.60	1.10	0.75	28.43	28.41	-3.07	0.0051	6.22	0.75	0.60	1.00	0.70		
	10	36.24	36.10	0.26	0.0148	13.31	0.70	0.45	1.15	0.60	38.1	38.02	-3.36	0.0146	9.30	0.65	0.30	1.10	0.60		
0.95	1	9.56	9.29	0.29	0.0010	14.89	0.95	0.85	1.10	0.90	11.50	11.36	-1.91	0.0008	4.56	0.90	0.80	1.00	0.90		
	3	16.81	16.76	0.05	0.0031	13.39	0.90	0.80	1.15	0.85	19.66	19.63	-3.41	0.0026	5.01	0.85	0.70	1.10	0.80		
	5	22.74	22.61	0.18	0.0058	13.49	0.90	0.65	1.15	0.85	25.68	25.58	-3.90	0.0051	6.13	0.85	0.60	1.10	0.75		
	10	33.68	33.64	0.07	0.0153	14.24	0.90	0.45	1.20	0.65	35.99	35.85	-4.08	0.0146	9.31	0.65	0.45	1.15	0.65		
1.00	1	5.90	5.85	0.06	0.0010	21.03	1.00	0.95	1.15	1.00	8.48	8.26	-2.51	0.0007	4.45	0.95	0.90	1.05	0.95		
	3	13.33	12.97	0.45	0.0034	18.91	1.00	0.90	1.20	0.95	16.87	16.79	-4.17	0.0025	4.90	0.95	0.75	1.15	0.85		
	5	19.50	19.42	0.11	0.0061	15.97	0.95	0.75	1.20	0.95	23.11	23.07	-4.89	0.0050	5.76	0.85	0.65	1.15	0.85		
	10	31.14	31.11	0.06	0.0158	15.62	0.85	0.60	1.20	0.80	33.95	33.73	-4.86	0.0146	9.31	0.80	0.45	1.20	0.70		
1.05	1	3.19	3.14	0.06	0.0008	31.82	1.05	1.10	1.20	1.10	6.04	5.74	-2.71	0.0006	4.30	1.00	1.00	1.10	1.00		
	3	10.24	10.11	0.16	0.0035	22.91	1.10	1.00	1.20	1.05	14.36	14.31	-4.92	0.0025	4.48	0.95	0.80	1.20	0.95		
	5	16.47	16.42	0.06	0.0066	19.32	1.10	0.90	1.20	1.00	20.73	20.55	-5.60	0.0050	5.74	0.95	0.70	1.20	0.90		
	10	28.73	28.67	0.12	0.0166	17.47	1.00	0.65	1.20	0.90	32.01	31.99	-6.11	0.0148	8.90	0.90	0.60	1.20	0.70		
1.10	1	1.50	Given the strike set, all possible portfolios' price is above BC_0 .																		
	3	7.61	7.60	0.02	0.0037	29.07	1.20	1.15	1.20	1.15	4.16	2.63	-1.20	0.0005	16.52	1.10	1.15	1.20	1.10		
	5	13.75	13.70	0.08	0.0073	23.92	1.20	1.05	1.20	1.10	12.14	11.88	-5.15	0.0024	4.57	1.05	0.90	1.20	1.00		
	10	26.25	26.01	0.45	0.0180	20.46	1.10	0.80	1.20	1.00	18.53	18.53	-6.55	0.0049	5.29	1.05	0.85	1.20	0.90		
											30.15	30.05	-7.14	0.0148	8.72	0.95	0.65	1.20	0.80		

Note: $ES\%$ denotes the relative ES , namely the expected value measured in percentage of shortfalls divided by the corresponding basket option payoffs at the maturity date T .

and ES differ insignificantly across the strikes of the basket option. Unfortunately, such general rules can not be summarized in the case of $V_0 = BC$. Especially, the hedging performance surprisingly turns out to be poorest for the shortest maturity $T = 1$. As to the obtained optimal strikes of the hedging portfolio, they generally increase with the strike of the basket option in both cases. However, because S_6 is negatively correlated to the other three assets, k_6 rises with T opposed to the decreasing relation of k_2 , k_3 , and k_7 to the maturity.

The minimum-expected-shortfall hedging portfolios are demonstrated in Table 8 given the same two constraints on the hedging cost, BC_0 and $HP(7)$. With a restricted number of options, the obtained hedging portfolio sometimes coincides with the minimum-variance hedging portfolio. Nevertheless, the risk measure in this case is the expected shortfall, which is in effect a stricter criterion than the second one concerning only the positive difference between the prices of the basket option and the corresponding hedging portfolio. As a result, the hedging cost is generally higher than that of the minimum-variance hedging portfolio. Obviously, it leads to a better performance with lower hedging error and ES. To achieve a even smaller ES, the hedging cost constraint is further raised in Table 9 to the Value at Risk at the level 10% of the basket option discounted payoff. Due to the lack of the distribution of the underlying basket, this has to be obtained by running simulations. Under this construction, the hedging cost of the hedging portfolio becomes surely higher (about $VaR_{0.10}$). It then gives a quite promising result that the ES is greatly reduced and turns out to be almost zero, except those basket options with a long time to maturity.

As also observed in the results above, relatively lower hedging costs are required for in- and at-the-money basket options to achieve almost the same relative ES compared with out-of-the-money options. Consequently, if aiming at capturing the trade-off between reduced hedging costs and successful replications, the hedging portfolio performs better for in- and at-the-money basket options. To clearly show the regions of sub- and super-replication, the payoffs of the basket option ($T = 3$, $K = 0.9$) and its minimum-ES hedging portfolio given $HC_0 = HP(7)$ are simulated and plotted in Figure 2. It can be observed that the basket option is completely hedged if the realized value of the basket is below or around the strike. The possibility of sub-replication rises with the value of the basket being above 1.00. Nevertheless, the hedging error is rather small compared to the basket option.

5.3 Remarks

Sometimes, the hedging performance is not that satisfactory especially for out-of-the-money options. It is mainly due to the following two factors.

- First, the sub-hedge-basket is composed of simply dominant assets without reallocating weights. Therefore, the value of the subset is only part of the original basket. The only tool in the model to match the payoff of the basket option is varying the strikes of the hedging instruments. However, their power to match the distribution is fairly limited since they do not change the shape of the distribution of the

Table 8: Minimum-Expected-Shortfall Hedging Portfolios with Four Dominant Assets (I)

K	T	BC ₀	V ₀ = BC ₀										HP(7)	V ₀ = HP(7)						
			HC	E[HE]	Var[HE]	ES%	k ₂	k ₃	k ₆	k ₇	HC	E[HE]	Var[HE]	ES%	k ₂	k ₃	k ₆	k ₇		
0.90	1	13.88	13.82	0.07	0.0009	8.98	0.95	0.75	0.90	0.85	15.05	15.04	-1.23	0.0008	4.59	0.90	0.70	0.90	0.85	
	3	20.60	20.57	0.03	0.0030	10.49	0.85	0.70	1.10	0.75	22.72	22.70	-2.54	0.0011	5.18	0.80	0.65	1.00	0.75	
	5	26.17	26.17	0.01	0.0055	11.14	0.70	0.65	1.15	0.75	28.43	28.41	-3.07	0.0051	6.22	0.75	0.60	1.00	0.70	
	10	36.24	36.24	-0.01	0.0149	12.98	0.65	0.45	1.10	0.65	38.10	38.10	-3.50	0.0147	9.20	0.75	0.30	1.00	0.60	
0.95	1	9.56	9.55	0.01	0.0011	13.60	0.90	0.85	1.15	0.90	11.50	11.48	-2.04	0.0009	4.39	0.90	0.75	1.05	0.90	
	3	16.81	16.78	0.04	0.0032	13.35	0.95	0.75	1.15	0.85	19.66	19.63	-3.41	0.0026	5.01	0.85	0.70	1.10	0.80	
	5	22.74	22.69	0.08	0.0059	13.32	0.90	0.60	1.20	0.85	25.68	25.64	-3.97	0.0052	6.07	0.75	0.65	1.15	0.75	
	10	33.68	33.68	-0.02	0.0155	14.19	0.85	0.60	1.10	0.65	35.99	35.95	-4.26	0.0148	9.19	0.80	0.45	1.00	0.65	
1.00	1	5.90	5.85	0.06	0.0010	21.03	1.00	0.95	1.15	1.00	8.48	8.40	-2.66	0.0008	4.14	0.95	0.85	1.10	0.95	
	3	13.33	13.31	0.03	0.0035	17.46	1.05	0.95	1.15	0.90	16.87	16.85	-4.24	0.0026	4.78	0.90	0.80	1.05	0.90	
	5	19.50	19.46	0.06	0.0062	15.90	1.00	0.70	1.20	0.95	23.11	23.10	-4.92	0.0050	5.75	0.90	0.60	1.15	0.85	
	10	31.14	31.14	-0.01	0.0159	15.58	0.85	0.65	1.15	0.80	33.95	33.94	-5.26	0.0149	9.02	0.85	0.45	1.05	0.75	
1.05	1	3.19	3.14	0.06	0.0008	31.82	1.05	1.10	1.20	1.10	6.04	6.01	-3.00	0.0007	3.68	1.00	0.90	1.10	1.05	
	3	10.24	10.17	0.09	0.0036	22.67	1.15	0.95	1.20	1.05	14.36	14.31	-4.92	0.0025	4.48	0.95	0.80	1.20	0.95	
	5	16.47	16.42	0.06	0.0066	19.32	1.10	0.90	1.20	1.00	20.73	20.71	-5.81	0.0051	5.55	0.90	0.70	1.15	0.95	
	10	28.73	28.73	-0.06	0.0169	17.36	1.10	0.60	1.15	0.90	32.01	32.01	-6.21	0.0149	8.89	0.80	0.70	1.20	0.70	
1.10	1	1.50	Given the strike set, all possible portfolios' price is above BC ₀ .																	
	3	7.61	7.60	0.02	0.0037	29.07	1.20	1.15	1.20	1.15	12.14	12.14	-5.47	0.0025	4.22	1.05	0.80	1.20	1.05	
	5	13.75	13.78	-0.03	0.0074	23.73	1.10	1.15	1.20	1.10	18.53	18.53	-6.55	0.0049	5.29	1.05	0.85	1.20	0.90	
	10	26.25	26.21	-0.08	0.0181	19.85	1.20	0.85	1.20	0.90	30.15	30.12	-7.27	0.0151	8.70	1.05	0.60	1.15	0.80	

K	T	BC_0	$V_0 = VaR_{10\%}$							
			HC	$E[HE]$	$Var[HE]$	$ES\%$	k_2	k_3	k_6	k_7
0.90	1	13.88	24.57	-11.38	0.0007	0.0009	0.45	0.65	0.95	0.70
	3	20.60	38.13	-21.17	0.0023	0.0061	0.15	0.30	0.90	0.65
	5	26.17	48.12	-30.07	0.0045	0.0141	0.15	0.45	0.85	0.15
	10	36.24	60.15	-44.90	0.0143	0.1148	0	0.30	0	0.15
0.95	1	9.56	19.91	-11.02	0.0006	0.0012	0.75	0.65	0.90	0.75
	3	16.81	34.19	-21.00	0.0021	0.0065	0.70	0.30	1.00	0.45
	5	22.74	44.49	-29.81	0.0047	0.0200	0.30	0.30	1.20	0.15
	10	33.68	58.95	-47.46	0.0140	0.0950	0.15	0.30	0	0.15
1.00	1	5.90	15.21	-9.92	0.0007	0.0039	0.90	0.75	1.00	0.75
	3	13.33	30.29	-20.48	0.0021	0.0072	0.65	0.30	0.95	0.70
	5	19.50	41.10	-29.59	0.0042	0.0164	0.65	0.30	1.15	0.15
	10	31.14	58.54	-51.44	0.0136	0.0713	0.15	0.15	0.15	0.15
1.05	1	3.19	10.63	-7.92	0.0008	0.0111	0.80	0.85	1.05	0.95
	3	10.24	26.28	-19.37	0.0023	0.0179	0.75	0.65	1.00	0.60
	5	16.47	37.55	-28.88	0.0041	0.0195	0.15	0.30	1.05	0.70
	10	28.73	56.58	-52.30	0.0134	0.0617	0.15	0.15	0.15	0.30
1.10	1	1.50	6.01	-4.80	0.0009	0.1631	1.00	0.90	1.10	1.05
	3	7.61	22.27	-17.71	0.0027	0.0130	0.80	0.60	1.10	0.75
	5	13.75	34.30	-28.16	0.0043	0.0207	0.45	0.30	1.05	0.70
	10	26.25	53.75	-51.62	0.0130	0.0702	0.15	0.15	0.60	0.15

Table 9: Minimum-Expected-Shortfall Hedging Portfolios with Four Dominant Assets (II)

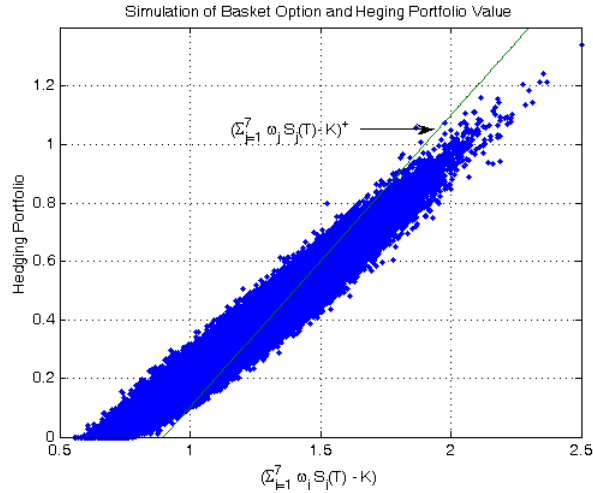


Figure 2: Simulations of the Basket Option ($T=3, K=0.9$) and the Minimum-Expected-Shortfall Hedge Portfolio with Constraint $V_0 = VaR_{0.10}$

sub-hedge-basket, but only shift the distribution closer to the original basket. This can be easily observed in Figure 3. By neglecting those insignificant underlying assets, the sub-basket experiences less extreme cases. However, since it is part of the original basket, it is located on the left of the original basket. Therefore, the function of varying strikes is to relocate the distribution of the hedging portfolio to the proper position near the basket option. As shown in the figure, the tighter the hedging criterion is, the further the distribution is shifted to the right.

- In addition, all the hedging portfolios designed in this paper are static. Hence, more capital may be required to well hedge the basket option. However, the model is restricted to be static under the construction of hedging with plain-vanilla options on the significant underlying assets. As the control variables in this model are strikes of these options, frequent trading on options with different strikes would cause great loss and additional transaction costs.

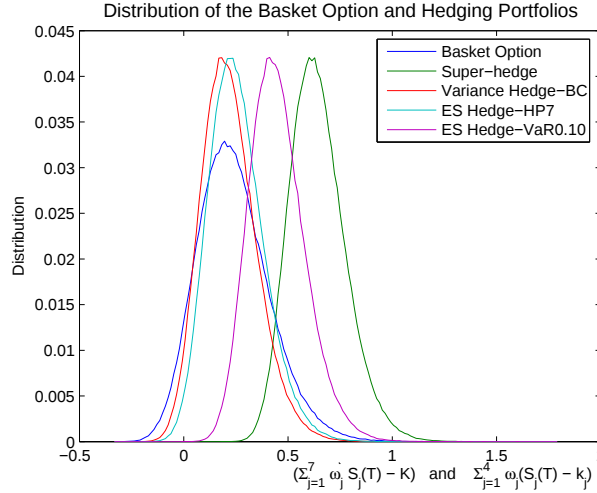


Figure 3: Distribution of the Underlying Basket ($T=3$, $K=0.9$) and the Hedging Portfolios

As a result, other control variables have to be considered to improve the hedging effect. One possible instrument is reallocating the weights of the hedging basket such that the new sub-hedge-basket can better match the distribution of the original basket. On this basis, dynamic hedging would also be possible by duplicating the basket option with the hedging assets. This would be an extension to be considered in future works.

6 Conclusion

In summary, Principal Components Analysis, a popular multivariate statistical method for dimension reduction, is applied to basket options hedging to select only a subset of underlying assets. The selection procedure is completed mainly by decomposing the modified covariance structure of the underlying basket into eigenvalues in the order of significance and eigenvectors. Hedging basket options based on only several assets can not only reduce transaction costs if combined with other hedging strategies, but also become practical and essential when some of the underlying assets are illiquid or not even available

for trading. Following this idea, a new two-step static hedging strategy is developed in this paper. It consists of plain-vanilla options on $N_1 < N$ dominant assets with optimal strike prices. The strikes are optimally chosen by numerically solving an optimization problem where the optimality criterion depends on the risk attitude of hedgers. As given in the paper, the first objective is to eliminate all the risks that the basket option is exposed to. Alternatively, optimal strikes are obtained by minimizing a particular risk measure, e.g., the variance of the hedging error or the expected shortfall, given a constraint on the hedging cost. Meanwhile considering the limited number of strikes traded in the market, complicated numerical optimizations have to be done by imposing another constraint that the optimal strikes are in the set of traded assets. Particularly, a simple and computationally efficient calibration procedure, convexity correction, is designed when achieving the super-replication hedging portfolio. As observed from the numerical results, the static hedging method achieves the trade-off between reduced hedging cost and overall super-replication. It demonstrates also that hedging with only a subset of assets works quite well even without considering reduced transaction costs, generating a reasonably small hedging error by investing the same capital as the super-hedging portfolio on all the underlying assets which is difficult to construct or is even not available in the market. Actually, its performance will become more satisfactory if the underlying basket is large and illiquid. Since the hedging performance is sensitive to the subset of the selected assets, it is recommended to examine hedging costs, involved transaction costs as well as hedging errors of several subsets. To achieve a better performance, hedging basket options with a subset of assets could be improved by reallocating weights of the sub-hedge-basket to approximately match the distribution of the original basket. This could be the extension for future research.

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Appendix: Proof of Proposition 1

Proof. First, the Lagrange function for the minimization problem is formed

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^N \omega_i e^{-rT} E[(S_i(T) - k_i)^+] + \lambda \left(\sum_{i=1}^N \omega_i k_i - K \right) \\ &= \sum_{i=1}^N \omega_i e^{-rT} \int_{\max(k_i, 0)}^{\infty} (x_i - k_i) f_i(x_i) dx_i + \lambda \left(\sum_{i=1}^N \omega_i k_i - K \right)\end{aligned}$$

where $f_i(x_i)$ is the lognormal density function under the risk-neutral martingale measure for the stock S_i . A necessary and sufficient condition for the sequence k_i to minimize the Lagrange function is found through the first order conditions:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial k_i} &= -\omega_i e^{-rT} \frac{\partial \max(k_i, 0)}{\partial k_i} \{ \max(k_i, 0) - k_i \} f_i[\max(k_i, 0)] \\ &\quad - \omega_i e^{-rT} \int_{\max(k_i, 0)}^{\infty} f_i(x_i) dx_i + \lambda \omega_i \\ &= 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \sum_{i=1}^N \omega_i k_i - K = 0.\end{aligned}$$

These conditions can be further simplified to

$$\frac{\partial \mathcal{L}}{\partial k_i} = -\omega_i \left(e^{-rT} \int_{\max(k_i, 0)}^{\infty} f_i(x_i) dx_i - \lambda \right) = 0 \quad \forall i = 1, \dots, N \quad (16)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^N \omega_i k_i - K = 0 \quad (17)$$

since the term of the first condition $\omega_i e^{-rT} \frac{\partial \max(k_i, 0)}{\partial k_i} \{ \max(k_i, 0) - k_i \} f_i[\max(k_i, 0)]$ is always equal to zero no matter which value $\max(k_i, 0)$ is going to take.

With these conditions, one can first prove that $k_i \in [0, K] \forall i = 1, \dots, N$ is always satisfied. Assume any specific $\bar{i}$ we have $k_{\bar{i}} < 0$. This implies that

$$\frac{\partial \mathcal{L}}{\partial k_i} \Big|_{k_i=k_{\bar{i}}} = -\omega_i (e^{-rT} - \lambda) = 0.$$

In this case, the first order condition (16) can be reduced to

$$\int_{\max(k_i, 0)}^{\infty} f_i(x_i) dx_i = 1 \quad \forall i = 1, \dots, N,$$

which implies the result that $k_i \leq 0 \quad \forall i = 1, \dots, N$. This contradicts however the second first order condition (17). Therefore, k_i 's are always positive and lie in the interval $[0, K]$.

Then, given $k_i \in [0, K]$, $\forall i = 1, \dots, N$, the first order condition (16) can be stated as

$$\Phi(d_2(S_i, k_i)) = \Phi(d_2(S_j, k_j)) \quad \forall i, j$$

where $d_2(S_i, k_i) = \frac{\ln\left(\frac{S_i(0)}{k_i}\right) + (r - q_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}$ as defined in the BS formula, and Φ again denotes the standard normal cumulative distribution function.

Furthermore, $\Phi(x)$ is bijective, the first condition (16) can be reduced to

$$d_2(S_i, k_i) = d_2(S_j, k_j) \quad \forall i = 1, \dots, N.$$

Then k_i can be all expressed in k_1 as

$$k_i = S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp \left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1\sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i \right) \right] \right\} \quad (18)$$

In summary, the optimal k_i 's are all positive and determined by solving the system of equations

$$\begin{aligned} k_i &= S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp \left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1\sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i \right) \right] \right\} \\ \sum_{i=1}^N \omega_i k_i &= K. \end{aligned}$$

The existing problem is whether there is always a solution and whether the solution is unique. This is shown in the following way:

First, k_i is a strictly increasing function of k_1 since the first derivative of k_i with respect to k_1

$$k'_i = \frac{S_i\sigma_i}{S_1\sigma_1} \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}-1} \exp \left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1\sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i \right) \right] \right\}$$

is always larger than zero.

Then the sum of the k_i 's as a function of k_1 given by

$$g(k_1) = \sum_{i=1}^N k_i = \sum_{i=1}^N S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp \left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1\sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i \right) \right] \right\}$$

is also continuous and increasing in k_1 , which could be proven again by checking its first derivative. Moreover,

$$g(k_1 = 0) = 0,$$

and

$$\begin{aligned} g(k_1 = K) &= \sum_{i=1}^n S_i \left(\frac{K}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp \left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1\sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i \right) \right] \right\} \\ &= K + S_2 \left(\frac{K}{S_1} \right)^{\frac{\sigma_2}{\sigma_1}} \exp \left\{ T \left[\left(1 - \frac{\sigma_2}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1\sigma_2 \right) + \left(\frac{\sigma_2}{\sigma_1}q_1 - q_2 \right) \right] \right\} + \dots \\ &\geq K. \end{aligned}$$

As a consequence, there is always a unique solution $k_i \in [0, K]$. $\square$