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by

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Optimal Seedings in Elimination Tournaments Revisited*

Matthias Kräkel†

Abstract

The paper addresses the problem of optimally matching heterogeneous players in a two-stage two-type Lazear-Rosen tournament in which the semifinal losers are eliminated. The organizer of the tournament can either choose two homogeneous semifinals – one between two strong players and the other one between two weak players – or two heterogeneous semifinals, each between one strong and one weak player. I identify conditions under which the organizer is strictly better off from two homogeneous semifinals if he wants to maximize total expected effort and the strong players' win probability. This finding is contrary to both the typical procedure used in real sporting contests and previous results based on all-pay auctions and the Tullock contest. Hence, my findings point out that the optimal design of elimination tournaments crucially depends on the underlying contest-success technology.

Key Words: Lazear-Rosen tournament, heterogeneous match, homogeneous match.

JEL Classification: D44, D72.

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1 Introduction

There exist many situations from different fields that can be best described by a winner-take-all competition. For example, in sporting contests individual athletes or teams compete for championship. In internal labor markets, employees participate in job-promotion tournaments for a vacant position at a higher tier of the corporate hierarchy. In addition, political contests, litigation contests, rent-seeking contests, R&D races, the race for technological or fashion leadership in certain markets, military conflicts, and song, beauty and cooking contests also have the typical characteristics of a winner-take-all competition.

In these situations, the contestants do not necessarily compete against each other in a grand contest. Instead, they are often initially assigned to several sub-contests whose winners are allowed to continue in a higher-order tournament, whereas the sub-contest losers are eliminated. There are several examples for such elimination tournaments:[1] Sporting contests are frequently divided into a group phase and a subsequent knock-out phase, where the group winners meet in the round of sixteen. Thereafter, the respective winners are matched in the quarter final and so on. At the end, the winner of the final receives the overall winner prize. Another example is given by job-promotion tournaments. Here, managers typically have to win subordinate tournaments (e.g., for becoming section or division head) before they get the opportunity to compete for a top position at their corporation's headquarters (e.g., the CEO position).

In this paper, I analyze how an organizer should optimally design the structure of such elimination tournaments. In particular, the organizer has to decide on the matching of different player types in the single sub-contests, which is called *seeding* (Groh et al., forthcoming). The answer to the question on what is the optimal seeding may crucially depend on the objective function of the tournament organizer. As a somehow natural objective func-

[1] For these and additional examples, see the introduction of Groh et al. (forthcoming) and Moldovanu and Sela (2006), p. 71.

tion, the tournament organizer can be interested in maximizing players' total expected effort for a given winner prize.[2] In the case of a sporting contest, the local organizer of the sports event may be free to choose the structure of the tournament, but the winner prize is determined by an exogenous sponsor or the respective sporting association. In that case, maximization of players' total effort can be interpreted as maximizing the entertainment of the audience. In the case of a job-promotion tournament within internal labor markets, the firm owner may be primarily interested in maximizing workers' overall effort for given wages being attached to jobs.

The examples of sporting contests and job-promotion tournaments indicate a second possible objective of the tournament organizer, namely the maximization of the top players' win probability.[3] A national sporting association can be very interested in this aim, if the winner of a national contest does not only obtain a winner prize but is also qualified for an international tournament. In a job-promotion tournament, the owner of a firm usually prefers the most able employee to climb the ladder to the top. Suppose that the design of the tournament which maximizes total expected effort, being thus optimal from an incentive perspective, is not optimal with respect to selecting high-ability individuals for top positions. In that case, the firm owner may be tempted to breach the rules of the tournament by declaring a more able employee the tournament winner although another employee has performed best. If the employees anticipate that such opportunistic behavior of the owner will happen with a high probability, incentives from participating in a job-promotion tournament will be erased. Altogether, if the incentive aim and the selection aim lead to different solutions to the design problem, the organizer must credibly commit either to single out one of the two aims or to decide according to an optimal weighing of the two aims.

I consider a four-player two-type elimination tournament and address

[2] See, e.g., Gradstein and Konrad (1999), Moldovanu and Sela (2006), Cohen and Sela (2007, 2008), Aoyagi (2010), Moldovanu et al. (2007, forthcoming), Groh et al. (forthcoming), Franke et al. (forthcoming) and Fu and Lu (forthcoming).

[3] See also Höchtl. et al. (2011) and Groh et al. (forthcoming).

both possible objective functions – maximization of total expected effort and maximization of the top players' win probability. In this setting, there are two semifinals whose winners compete in the subsequent final for a winner-take-all prize, whereas the semifinal losers are eliminated and obtain nothing. Two players have a high type (i.e., a high valuation of winning) and two players have a low type (i.e., a low valuation of winning), which implies two possible seedings for the semifinals: The tournament organizer can either choose a homogeneous seeding or a heterogeneous one. In the former case (*hom*), the organizer matches the two high types in one semifinal and the two low types in the other semifinal. In the latter case (*het*), the organizer chooses two identical semifinals each between one high type and one low type. I use a Lazear-Rosen (1981) type of tournament for modeling the winner-take-all competition.[4] My results show that it can be optimal for the tournament organizer *not* to match low types ("underdogs") and high types ("favorites") together in the same semifinal, which is the typical procedure used in real sporting contests. Particularly, if noise in the tournament is uniformly distributed – as in the setting of Meyer (1991) – the design *hom* dominates the design *het* irrespective of whether the organizer wants to maximize total expected effort or the top players' win probability. Thus, there is no trade-off between the two aims.

The intuition for my finding is the following: Design *hom* leads to balanced competition in both semifinals, which is beneficial for the organizer since players' equilibrium efforts are higher the more balanced the competition.[5] Moreover, all semifinalists anticipate unbalanced competition in the final for sure, implying low effort costs in the final and, hence, a rather large expected utility from participating in the final. This second effect also boosts overall incentives in the semifinals. In addition, the two high-type or top players already know in the semifinal that, in case of winning, they will be matched with a weaker opponent in the final. Thus, their win probability

[4] The underlying win technology has also been referred to as difference-form contest-success function; see, e.g., Dixit (1987), Baik (1998) and Che and Gale (2000).

[5] See, e.g., Rosen (1986), pp. 707-710.

in the final will be quite large, so that especially the high-type players have a very large incentive to win the semifinal. Note that the design *hom* has two drawbacks: First, since it is clear that one high-type and one low-type player will reach the final, total effort from the final will be rather small due to unbalanced competition. Second, the low-type players in the semifinal anticipate that they will have a rather low win probability in the final. However, if the incentive advantages from the two homogeneous semifinals dominate these incentive disadvantages from a surely heterogeneous final, design *hom* will be optimal for maximizing total expected effort. Design *hom* is also beneficial for maximizing the win probability of a high-type player: In contrast to *het*, design *hom* guarantees that one high-type player reaches the final for sure. In the final, the high-type player has a quite large win probability since he competes against a low-type opponent.

My finding sharply contrasts with the main result of Groh et al. (forthcoming) who analyze the same design problem but use an all-pay auction with complete information instead of a Lazear-Rosen tournament (or difference-form contest-success function [csf]). They show that design *het* is optimal for maximizing both total expected effort and the top players' win probability. My contrary findings point out that the optimal design of elimination tournaments crucially depends on the underlying contest-success technology.

Note that the all-pay auction with complete information can be seen as an extreme case of the Lazear-Rosen tournament. In fact, by assuming that the noise distribution is degenerate, the Lazear-Rosen tournament immediately turns into an all-pay auction. Suppose that the tournament organizer can choose between two different monitoring technologies. He may either choose a perfect monitoring technology leading to an all-pay auction and high monitoring costs, or a less precise monitoring technology implying a Lazear-Rosen tournament and lower monitoring costs. Under either contest type, the organizer has the advantage that one design is optimal in both dimensions – maximizing total expected effort (incentive dimension) and the top players' win probability (selection dimension). However, I can show that for uniformly distributed noise the optimally designed all-pay auction (using design *het*) dominates any optimally designed Lazear-Rosen tournament (using design *hom*) in both the incentive and the selection dimensions. The decision problem which kind of contest to choose then boils down to solving the trade-off between better performance (in both dimensions) but higher monitoring costs of an all-pay auction compared to a Lazear-Rosen tournament.

My paper is organized as follows. The next section gives a brief overview about related work. Sections 3 and 4 introduce the model and present the solution to the tournament game. Building on the findings of Section 4, the next two sections compare the two possible seedings *hom* and *het* with respect to total expected effort (Section 5) and the top players' win probability (Section 6). Section 7 concludes

2 Related Literature

Previous work on elimination tournaments typically builds on either the all-pay auction or the Tullock csf.[6] As has been stressed above, my paper is closest to the recent work by Groh et al. (forthcoming), who also consider seedings of heterogeneous contestants in two semifinals. However, contrary to my paper, Groh et al. apply an all-pay auction with complete information, which leads to the converse outcome – *het* dominates *hom*. Moldovanu and Sela (2006) consider an all-pay auction where cost functions are private information of the contestants. They analyze whether a contest organizer should design a single grand contest among all $n \geq 2$ players or split the players into sub-contests with sub-contest losers being eliminated and winners competing against each other in the final. If the organizer wants to maximize total expected effort, the grand contest dominates splitting given that cost functions are linear, but splitting can be advantageous if cost functions are convex. This finding shows that the optimal design of tournaments does not only depend on the csf but also on the shape of the players' cost functions.

[6] See Konrad (2009), pp. 164–169, for an overview.

The vast majority of the papers on elimination tournaments uses the Tullock csf. Rosen (1986) is the first paper at all that addresses elimination tournaments. There, the n players compete in pairwise elimination matches where in each round the losers receive certain prizes and are not allowed to continue, but the match winners are assigned to the next stage in which each one competes against another winner to further climb the ladder. Rosen shows that the prize spread for winning the final should be very large to induce sufficient effort levels to all players during the complete multistage elimination tournament.

Gradstein and Konrad (1999) investigate whether the contest organizer is better off from choosing a single grand contest or a sequence of pairwise elimination tournaments to maximize total expected effort. Using a Tullock contest, they show that the grand contest is optimal if the csf is sufficiently discriminatory, but otherwise a sequence of elimination tournaments would be advantageous.

Amegashie (1999, 2000) considers a multi-stage elimination tournament between a given number of homogeneous players, who compete against each other according to a Tullock csf. In an initial period, the total number of contestants is divided into sub-contests of equal size. The sub-contest winners are allowed to participate in the final whereas the losers are eliminated. Amegashie shows that an elimination tournament with sub-contests is better for maximizing overall effort than a single grand contest if the csf's discriminatory power in the sub-contests is sufficiently large compared to the discriminatory power in the final.

Harbaugh and Klumpp (2005) analyze a four-player two-stage elimination tournament with a Tullock csf. Players have a fixed budget for their effort choices. In the beginning, they have to share their effort budget into the amount that is invested in the semifinal and the remainder which is exerted in the final given that the respective player reaches the final. In equilibrium, the weaker player exerts more effort than his stronger opponent in each semifinal. However, these asymmetric effort choices cannot completely equalize the semifinalists' win probabilities. Due to the fixed budget, weak players have a quite low win probability when meeting a strong player in the final.

Similar to my paper, Höchtl et al. (2011) address the question of optimal seeding in elimination tournaments.[7] However, they use a Tullock contest with linear impact function, and assume costs to be linear and each contestant's valuation of the winner prize to be uniform. Höchtl et al. show that *hom* is better than *het* if the tournament organizer wants to maximize total expected effort, but *het* is better than *hom* for maximizing the top players' win probability. Thus, given a Tullock csf, the incentive and selection aims lead to a strict trade-off, contrary to the Lazear-Rosen tournament and the all-pay auction.

Fu and Lu (forthcoming) analyze a multi-stage elimination tournament between n homogeneous contestants. The tournament organizer chooses the structure of the tournament (allocation of prize money, number of stages, number of winners for each stage) to maximize total expected effort. Fu and Lu show that, for a Tullock csf with concave impact function, the organizer should give the entire prize money to the player that wins the final. In addition, it is optimal to maximize the number of elimination stages.

With the exception of Höchtl et a. (2011) all papers on elimination tournaments with Tullock csf do not address the problem of optimal seeding.

3 The Model

In analogy to Groh et al. (forthcoming), I consider a two-stage elimination tournament between four players. In stage 1, pairs of two players are matched within two semifinals that take place simultaneously. Matching is deterministic, that is, individual players are seeded by the organizer of the tournament, called O. After the semifinals, the game ends for the losers, who obtain zero. However, the winners of the semifinals enter stage 2 where

[7] See Rosen (1986), pp. 709-710, on a related numerical example on seeding versus random matching.

they meet one another in the final and compete for a given prize. As in the semifinals, the loser obtains zero.

Players only differ in their individual valuations of the prize for winning the final. I assume that there are two players whose value of the prize is given by v_H, whereas the two other players' winner prize amounts to $v_L \in (0, v_H)$.[8] Contrary to Groh et al. (forthcoming), I do not consider an all-pay auction with complete information, but a tournament with difference-form csf (see, among many others, Lazear and Rosen 1981, Dixit 1987, Baik 1998, Che and Gale 2000). In particular, I assume that in a match between players i and j, who choose non-negative efforts e_i and e_j, organizer O observes the noisy signal s with

$$s = \begin{cases} s_i & \text{if} \quad e_i - e_j > \varepsilon \\ s_j & \text{if} \quad e_i - e_j < \varepsilon, \end{cases} \tag{1}$$

where s_i (s_j) indicates that player i (j) has outperformed player j (i). Thus, the realization of the signal depends on the players' efforts and the variable ε that describes an unobservable, exogenous random term (e.g., measurement error or noise) with density $g(\varepsilon)$ and cdf $G(\varepsilon)$. I follow the usual assumption that $g(\varepsilon)$ is symmetric about zero and weakly unimodal.[9]

Effort e_i entails costs on player i that are described by the function $c(e_i) = \kappa e_i^2/2$ with $\kappa > 0$ being sufficiently large to ensure positive expected utilities in equilibrium (see, e.g., (4) below). Similarly to Schöttner (2008), I assume that

$$v_H \cdot \sup_{\Delta e} |g'(\Delta e)| < \inf_{e>0} c''(e) = \kappa \tag{2}$$

to guarantee the existence of pure-strategy equilibria. Thus, the second-order conditions are assumed to hold in each possible final and semifinal. This condition is clearly satisfied in the example given below with uniformly distributed noise.

[8] Groh et al. (forthcoming) assume that the four individual prizes of the players satisfy $v_1 \geq v_2 \geq v_3 \geq v_4 > 0$. In Subsection 3.1.1, they consider the two-type case analyzed in this paper.

[9] Many tournament papers even assume strict unimodality with a unique mode at zero (e.g., Dixit 1987, Drago et al. 1996, Hvide 2002, Chen 2003).

Each contestant maximizes the expected prize of winning the tournament minus his effort costs. Following Groh et al. (forthcoming), I consider two possible objective functions that organizer O wants to maximize for exogenously given prizes v_H and v_L:[10] On the one hand, organizer O can be interested in maximizing the sum of expected efforts of all players over all rounds. Given that players' efforts are productive (e.g., entertainment), this objective function seems quite natural. On the other hand, O may be interested in maximizing the probability that a v_H-player wins in the final. As Groh et al. (forthcoming) point out, this objective plays an important role in the statistical literature. Under either objective function, O can choose between two different seedings – one that constitutes two homogeneous semifinals (called design *hom*), and one that constitutes two heterogeneous semifinals (design *het*).

4 Solution to the Tournament Game

4.1 Final

I start with the solution of the final between players i and j with valuations $v_i, v_j \in \{v_L, v_H\}$, choosing efforts e_i^F and e_j^F, respectively. Player i maximizes

$$EU_i^F\left(e_i^F, e_j^F\right) = v_i \cdot G\left(e_i^F - e_j^F\right) - c\left(e_i^F\right),$$

whereas player j maximizes

$$EU_j^F\left(e_j^F, e_i^F\right) = v_j \cdot \left[1 - G\left(e_i^F - e_j^F\right)\right] - c\left(e_j^F\right).$$

[10] Groh et al. (forthcoming) also consider two additional criteria, whose application is not interesting in the given setting.

Given condition (2), the equilibrium $\left(e_i^{F*}, e_j^{F*}\right)$ can be described by the first-order conditions

$$g\left(e_i^{F*} - e_j^{F*}\right) = \frac{c'\left(e_i^{F*}\right)}{v_i} = \frac{c'\left(e_j^{F*}\right)}{v_j},$$

which – using the specific cost function – yields

$$\left(e_i^{F*}, e_j^{F*}\right) = \begin{cases} \left(\frac{vg(0)}{\kappa}, \frac{vg(0)}{\kappa}\right) & \text{if} \quad v_i = v_j = v \\ \left(\frac{v_i g\left(\Delta e^{F*}\right)}{\kappa}, \frac{v_j g\left(\Delta e^{F*}\right)}{\kappa}\right) & \text{if} \quad v_i \neq v_j \end{cases} \tag{3}$$

with $\Delta e^{F*} := e_i^{F*} - e_j^{F*}$. Note that $g(0) \geq g\left(\Delta e^{F*}\right)$ since g is symmetric about zero and weakly unimodal. Hence, each player chooses at least as much effort in a homogeneous match as in a heterogeneous one. Expected utilities in equilibrium read as

$$EU_{ij}^{F*} = \begin{cases} \frac{v}{2} - \frac{v^2 g(0)^2}{2\kappa} & \text{if} \quad v_i = v_j = v \\ v_i G\left(\Delta e^{F*}\right) - \frac{v_i^2 g\left(\Delta e^{F*}\right)^2}{2\kappa} & \text{if} \quad v_i \neq v_j \end{cases} \tag{4}$$

with $EU_{ij}^{F*} := EU_i^F\left(e_i^{F*}, e_j^{F*}\right)$ and $G\left(\Delta e^{F*}\right) \gtrless \frac{1}{2}$ if $\Delta e^{F*} \gtrless 0$. Not surprisingly, each player's equilibrium effort strictly increases in his valuation of the winner prize. However, the same is not true for the expected utility since the convex effort costs also increase in v_i. Recall that, by assumption, κ is sufficiently large so that $EU_{ij}^{F*} > 0$.

4.2 Semifinals

We have to differentiate between two possible constellations for the semifinals. Suppose that O decides to organize two *homogeneous semifinals* (design *hom*) – one semifinal between the two v_H-players and the other one between the two v_L-players. In the v_H-semifinal, each participant knows that, in case of winning, he will be matched with a v_L-opponent in the final. Let H_1 and H_2 denote the two v_H-players, $e_{H_i}^S$ $(i = 1, 2)$ their respective effort levels, and

$EU^S_{H_iH_j}$ $(i,j=1,2;i\neq j)$ their expected utilities from participating in the v_H-semifinal. Then the players' objective functions can be written as follows:

$$\begin{aligned} EU^S_{H_1H_2} &= G\left(e^S_{H_1}-e^S_{H_2}\right)EU^{F*}_{HL}-c\left(e^S_{H_1}\right) \\ EU^S_{H_2H_1} &= \left[1-G\left(e^S_{H_1}-e^S_{H_2}\right)\right]EU^{F*}_{HL}-c\left(e^S_{H_2}\right). \end{aligned}$$

Since existence of pure-strategy equilibria is guaranteed by (2), the first-order conditions

$$g\left(e^S_{H_1}-e^S_{H_2}\right)EU^{F*}_{HL}=c'\left(e^S_{H_1}\right)=c'\left(e^S_{H_2}\right)$$

show that the v_H-semifinal has a unique equilibrium which is symmetric: $\left(e^{S*}_{H_1},e^{S*}_{H_2}\right)=\left(e^{S*}_{H,hom},e^{S*}_{H,hom}\right)$ with $e^{S*}_{H,hom}=\left[g\left(0\right)EU^{F*}_{HL}\right]/\kappa$. Analogously, in the v_L-semifinal each player, who anticipates to be matched with a v_H-player in the final when winning the semifinal, chooses effort $e^{S*}_{L,hom}=\left[g\left(0\right)EU^{F*}_{LH}\right]/\kappa$ in equilibrium. Depending on the value difference v_H-v_L and the magnitude of the cost parameter κ, the v_H-players may choose more effort than the v_L-players in the semifinals (i.e., if $EU^{F*}_{HL}>EU^{F*}_{LH}$) or less (if $EU^{F*}_{HL}<EU^{F*}_{LH}$).

Now suppose that O decides to organize two *heterogeneous semifinals* each with one v_H-player and one v_L-player (design *het*). Let $e^{S_n}_H$ $(e^{S_n}_L)$ denote the effort level chosen by the v_H-player (v_L-player) in semifinal $n=1,2$, and $EU^{S_n}_{HL}$ $(EU^{S_n}_{LH})$ the player's respective expected utility from participating in that semifinal. Thus, for the first semifinal, we obtain

$$\begin{aligned} EU^{S_1}_{HL} &= G\left(\Delta e^{S_1}\right)\left(G\left(\Delta e^{S_2}\right)EU^{F*}_{HH}+\left[1-G\left(\Delta e^{S_2}\right)\right]EU^{F*}_{HL}\right)-c\left(e^{S_1}_H\right) \\ EU^{S_1}_{LH} &= \left[1-G\left(\Delta e^{S_1}\right)\right]\left(G\left(\Delta e^{S_2}\right)EU^{F*}_{LH}+\left[1-G\left(\Delta e^{S_2}\right)\right]EU^{F*}_{LL}\right)-c\left(e^{S_1}_L\right) \end{aligned}$$

with $\Delta e^{S_n}:=e^{S_n}_H-e^{S_n}_L$ $(n=1,2)$. Given (2), equilibria can be described by the first-order conditions. I restrict the analysis to the most plausible case of symmetric equilibria where players of the same type behave identically (i.e., $e^{S_1}_H=e^{S_2}_H$ and $e^{S_1}_L=e^{S_2}_L$). The existence of additional asymmetric equilibria cannot be ruled out. However, in the example with uniformly distributed noise considered below there exists a unique equilibrium which is symmetric.

First-order conditions together with symmetry yield that v_H-players choose $e^{S*}_{H,het}$ in equilibrium and v_L-players $e^{S*}_{L,het}$ with

$$\begin{aligned} e^{S*}_{H,het} &= \frac{g\left(\Delta e^{S*}_{het}\right)}{\kappa}\left(EU^{F*}_{HL} - \left(EU^{F*}_{HL} - EU^{F*}_{HH}\right)G\left(\Delta e^{S*}_{het}\right)\right) \\ e^{S*}_{L,het} &= \frac{g\left(\Delta e^{S*}_{het}\right)}{\kappa}\left(EU^{F*}_{LL} - \left(EU^{F*}_{LL} - EU^{F*}_{LH}\right)G\left(\Delta e^{S*}_{het}\right)\right) \end{aligned}$$

and $\Delta e^{S*}_{het} := e^{S*}_{H,het} - e^{S*}_{L,het}$. At first sight, it is not clear which type of player chooses more effort in the heterogeneous semifinals. However, I can state under which conditions the stronger player exerts more effort in equilibrium:[11]

Lemma 1 *If $EU^{F*}_{HH} > EU^{F*}_{LH}$, then $e^{S*}_{H,het} > e^{S*}_{L,het}$.*

The lemma is based on the assumption that it is better to be a strong player than a weak one when meeting another strong player in the final (i.e., $EU^{F*}_{HH} > EU^{F*}_{LH}$).[12] Under this condition, v_H-players choose higher efforts than v_L-players in the heterogeneous semifinals. Note that $EU^{F*}_{HH} > EU^{F*}_{LH}$ is only a sufficient condition for $e^{S*}_{H,het} > e^{S*}_{L,het}$ and may be much too strong in many cases.

[11] All proofs are relegated to the appendix.

[12] Condition $EU^{F*}_{HH} > EU^{F*}_{LH}$ can be verified if the distribution of ε is further specified. If, e.g., ε is uniformly distributed over $[-\bar{\varepsilon}, \bar{\varepsilon}]$, the condition boils down to $4\bar{\varepsilon}^2 > v_H - v_L$, which is always satisfied under an interior solution. If player heterogeneity is additive (see, e.g., O'Keeffe et al. (1984); Bull et al. (1987); Akerlof and Holden (forthcoming)), the relative-performance signal is given by

$$s = \begin{cases} s_i & \text{if} \quad e_i + a_i - e_j - a_j > \varepsilon \\ s_j & \text{if} \quad e_i + a_i - e_j - a_j < \varepsilon, \end{cases}$$

with a_i and a_j as the players' abilities. Assuming a uniform tournament prize v, two high-ability players ($a_i = a_H > 0$) and two low-ability ones ($a_i = a_L \in (0, a_H)$) would allow to write $EU^{F*}_{HH} > EU^{F*}_{LH}$ as a pure condition of the primitives of the model (see the additional pages for the referees)

$$v < 2\kappa \frac{\frac{1}{2} - [1 - G(\Delta a)]}{g(0)^2 - g(\Delta a)^2}.$$

5 Total Expected Effort

We can use the results for the equilibrium efforts in the final and the two semifinals under the two different seedings to answer the question whether organizer O prefers the design *hom* or the design *het* if he wants to maximize total effort. Under the design *hom*, total effort from both semifinals is given by

$$\Sigma_{hom}^{S} = 2e_{H,hom}^{S*} + 2e_{L,hom}^{S*} = \frac{2g(0)}{\kappa}\left(EU_{HL}^{F*} + EU_{LH}^{F*}\right) \tag{5}$$

and total (expected) effort from the subsequent final by

$$\Sigma_{hom}^{F} = \frac{g\left(\Delta\hat{e}^{F*}\right)}{\kappa}(v_H + v_L) \quad \text{with } \Delta\hat{e}^{F*} := e_H^{F*} - e_L^{F*} > 0. \tag{6}$$

Under the design *het*, the four equilibrium efforts from both heterogeneous semifinals sum up to

$$\begin{aligned}\Sigma_{het}^{S} &= 2e_{H,het}^{S*} + 2e_{L,het}^{S*} \\ &= 2\frac{g\left(\Delta e_{het}^{S*}\right)}{\kappa}\left[\left(1 - G\left(\Delta e_{het}^{S*}\right)\right)\left(EU_{HL}^{F*} + EU_{LL}^{F*}\right)\right. \\ &\qquad \left. + \left(EU_{HH}^{F*} + EU_{LH}^{F*}\right)G\left(\Delta e_{het}^{S*}\right)\right].\end{aligned} \tag{7}$$

Expected equilibrium efforts from the subsequent final amount to

$$\begin{aligned}\Sigma_{het}^{F} &= 2\left(\frac{g(0)}{\kappa}\left(v_H G\left(\Delta e_{het}^{S*}\right)^2 + v_L\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]^2\right)\right. \\ &\qquad \left. + \frac{g\left(\Delta\hat{e}^{F*}\right)}{\kappa}(v_H + v_L)G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]\right).\end{aligned} \tag{8}$$

Comparing the effort choices under the two different designs leads to the following results:

Proposition 1 *(i) If $EU_{HH}^{F*} > EU_{LH}^{F*}$, then $\Sigma_{hom}^{S} > \Sigma_{het}^{S}$. (ii) $\Sigma_{hom}^{F} \leq \Sigma_{het}^{F}$.*

The proposition shows that expected efforts in the final are always greater or equal under design *het* compared to *hom*. The intuition for this result is based on the fact that equilibrium efforts in the final are particularly high if (a) a player's winner prize is v_H instead of v_L, and if (b) there is a homogeneous instead of a heterogeneous match. Under design *hom*, we unambiguously have a heterogenous final between one v_H-player and one v_L-player, so that total effort is quite low. However, under design *het*, there are four possible constellations for the final. In three constellations at least one v_H-player enters the final and in two constellations there is a homogeneous match, so that expected efforts are relatively high.

The two arguments (a) and (b) can be similarly applied to the semifinals. Concerning argument (a), at first sight, both *hom* and *het* seem to be identical because in either case there are two v_H-players and two v_L-players. However, from the viewpoint of a semifinalist, not the values v_H and v_L are decisive for the creation of incentives, but the corresponding expected utilities EU_{ij}^{F*}, which also depend on the type of the other finalist. There are four possible values – EU_{HL}^{F*}, EU_{HH}^{F*}, EU_{LL}^{F*}, and EU_{LH}^{F*} – and the relation between these values is not clear in most of the cases since it depends on the interplay of the win probability, the winner prize and the effort costs. It is only clear that $EU_{HL}^{F*} > EU_{HH}^{F*}$.

However, note that the disadvantage of a surely heterogeneous final under *hom* turns into a strong advantage concerning the *hom*-semifinals. In the *hom*-semifinals, all four players anticipate that, due to heterogeneous competition, they will have rather low effort costs if they enter the final. This fact implies rather large expected utilities of participating in the final and, hence, strong incentives in the *hom*-semifinals. Especially the two players in the v_H-semifinal have very strong incentives because they anticipate that they have a large win probability $G\left(\Delta\hat{e}^{F*}\right) > 1/2$ and relatively small effort costs when being matched with a v_L-player in the final for sure.

Applying argument (b) to the semifinals shows that the designs *hom* and *het* strictly differ. In the two homogeneous semifinals, equilibrium efforts are quite high as indicated by $g(0)$ in expression (5), whereas efforts are rather low in the two heterogeneous semifinals, indicated by $g\left(\Delta e_{het}^{S*}\right)$ in

(7). Altogether, as Proposition 1 shows, the incentive advantages of the two homogeneous semifinals can be so strong that total effort from the two *hom*-semifinals exceeds that from the two *het*-semifinals.

According to Proposition 1, it is not straightforward which seeding maximizes total expected effort from the whole tournament. However, further specifying the distribution of ε can lead to a clear-cut result. For analytical tractability, I follow Meyer (1991, Section 5.2) and assume that ε is uniformly distributed over $[-\bar{\varepsilon}, \bar{\varepsilon}]$ with $\bar{\varepsilon}$ being sufficiently large to guarantee interior solutions to the tournament game. Furthermore, I assume that $\kappa = 1$ to save notation. For this specification, the following result can be obtained:

Proposition 2 *If ε is uniformly distributed, then $\Sigma^S_{hom} + \Sigma^F_{hom} > \Sigma^S_{het} + \Sigma^F_{het}$.*

Proposition 2 shows that, under uniformly distributed noise, the tournament organizer O should prefer the design *hom* to the design *het* if he wants to maximize overall expected effort for a given tournament prize. This finding is in sharp contrast to the result of Groh et al. (forthcoming), who consider an all-pay auction with complete information, but qualitatively in line with the finding of Höchtl et al (2011), using a Tullock csf. Interestingly, my result holds although two of the main arguments in favor of *hom* are eliminated by the use of the uniform distribution. In the discussion of Proposition 1 above, I have argued that incentives in the *hom*-semifinals are quite large – compared to *het* – since (1) all semifinalists face homogeneous instead of heterogeneous competition, and since (2) the surely heterogeneous final unambiguously leads to low effort costs and, hence, a rather high expected utility from participating in the final. These two arguments are absent under Proposition 2, because the uniform distribution has a constant density, implying $g(0) = g\left(\Delta e^{S*}_{het}\right)$. Nevertheless, the design *hom* dominates the design *het*.

6 The Top Players' Win Probability

In this section, the two designs *hom* and *het* will be compared with respect to the second possible objective of the tournament organizer O – maximizing the probability that one of the v_H-players wins the tournament. This objective can be motivated as follows. Consider, for example, the case of a job-promotion tournament where the winner of the final is promoted to an important management position. Then not only incentives and, therefore, total effort should play a role, but also the type of the promoted individual. Let, for the moment, v_H and v_L also indicate the suitability of the players for the management position with v_H-players being more suited for the job.[13] In that situation, O should be interested in how the top players' win probabilities differ under the designs *hom* and *het*.

Under design *hom*, the final must be a match between a v_H- and a v_L-player, so that the top players' win probability is simply given by $G\left(\Delta\hat{e}^{F*}\right)$ with $\Delta\hat{e}^{F*} > 0$ (see (6)). Under design *het*, one of the v_H-players will win the tournament if the final is between two v_H-players or if there is a heterogeneous final but the v_H-player wins. Thus, the top players' win probability is higher under *hom* than under *het* iff

$$G\left(\Delta\hat{e}^{F*}\right) > G\left(\Delta e_{het}^{S*}\right) G\left(\Delta e_{het}^{S*}\right) + 2G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right] G\left(\Delta\hat{e}^{F*}\right)$$

$$\Leftrightarrow G\left(\Delta\hat{e}^{F*}\right) > \frac{G\left(\Delta e_{het}^{S*}\right)^2}{G\left(\Delta e_{het}^{S*}\right)^2 + \left[1 - G\left(\Delta e_{het}^{S*}\right)\right]^2}. \tag{9}$$

The left-hand side of (9) is strictly larger than $1/2$ since $\Delta\hat{e}^{F*} > 0$. The right-hand side describes a function that is monotonically increasing in $G\left(\Delta e_{het}^{S*}\right)$, taking value 0 at $G\left(\Delta e_{het}^{S*}\right) = 0$ and value 1 at $G\left(\Delta e_{het}^{S*}\right) = 1$ (i.e., technically, it has the characteristics of a cdf of $G\left(\Delta e_{het}^{S*}\right)$). The numerator describes the probability that, under *het*, both semifinals are won by the v_H-players, whereas the denominator describes the probability that the final

[13] On the additional pages for the referees, I explicitly model player heterogeneity via the players' individual abilities.

is homogeneous. Hence, the right-hand side of (9) is the probability of a beneficial homogeneous final relative to the probability that the final will be a homogeneous one under *het*. If this relation is sufficiently small, O will prefer *hom*. Condition (9) is clearly satisfied for $\Delta e_{het}^{S*} < 0$, otherwise it depends on the relation between $\Delta \hat{e}^{F*}$ and Δe_{het}^{S*}.

Clear-cut results can only be derived when further specifying the underlying distribution G. If ε is again uniformly distributed over $[-\bar{\varepsilon}, \bar{\varepsilon}]$, the top players' win probability under *hom* is given by $G\left(\Delta \hat{e}^{F*}\right) = \left(2\bar{\varepsilon}^2 + v_H - v_L\right) / \left(4\bar{\varepsilon}^2\right)$,[14] but under *het* it amounts to

$$G\left(\Delta e_{het}^{S*}\right) G\left(\Delta e_{het}^{S*}\right) + 2G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right] G\left(\Delta \hat{e}^{F*}\right)$$

$$= \frac{4\bar{\varepsilon}^2\left(16\bar{\varepsilon}^4 + \left(v_H - v_L\right)^2\right)\left(v_H - v_L + 4\bar{\varepsilon}^2\right)^2 + \left(v_H - v_L\right)^5}{8\bar{\varepsilon}^2\left(16\bar{\varepsilon}^4 + \left(v_H - v_L\right)^2\right)^2},$$

which is strictly smaller than $\left(2\bar{\varepsilon}^2 + v_H - v_L\right) / \left(4\bar{\varepsilon}^2\right)$. These findings are summarized in the following proposition:

Proposition 3 *The top players' win probability is higher under hom than under het, iff* $G\left(\Delta \hat{e}^{F*}\right) > G\left(\Delta e_{het}^{S*}\right)^2 / \left(G\left(\Delta e_{het}^{S*}\right)^2 + \left[1 - G\left(\Delta e_{het}^{S*}\right)\right]^2\right)$. *If* ε *is uniformly distributed, this condition is satisfied.*

Propositions 2 and 3 together show that, for the case of uniformly distributed noise, design *hom* strictly dominates design *het* in (A) maximizing total effort as well as (B) maximizing the top players' win probability. This finding is in sharp contrast to that of Groh et al. (forthcoming) for the all-pay auction with complete information, where *het* dominates *hom* in both dimensions (A) and (B). My result also strictly differs from the finding of Höchtl et al. (2011) on the Tullock contest, where O faces a trade-off concerning objectives (A) and (B): *hom* is better than *het* for maximizing total effort, but the top players' win probability is maximized under *het*. Altogether, this comparison demonstrates that the optimal design of elimination tournaments crucially depends on the kind of underlying csf.

[14] Again, $\kappa = 1$ is assumed to save notation.

In principle, the all-pay auction, considered by Groh et al. (forthcoming), and the Lazear-Rosen tournament (or difference-form csf), analyzed in this paper, belong to the same csf-class and can, therefore, be directly compared with each other. If we assume that $\varepsilon = 0$ in (1), the Lazear-Rosen tournament immediately turns into an all-pay auction with complete information. Moreover, we can imagine that the organizer O has the possibility to choose between different monitoring technologies that differ in precision and costs. For simplicity, I consider the case of uniformly distributed noise. Then, the relative performance signal s has the precision $1/Var(\varepsilon) = 3/\bar{\varepsilon}^2$ and we can characterize a monitoring technology with noise $\varepsilon \sim U(-\bar{\varepsilon}, \bar{\varepsilon})$ by the monitoring-cost function $C(\bar{\varepsilon}) > 0$ with $C'(\bar{\varepsilon}) < 0$ (i.e., the less precise the performance signal the less costly will be the underlying monitoring technology). In this case, the use of an all-pay auction (i.e., a tournament with perfect monitoring) is associated with costs $C(0)$, whereas the Lazear-Rosen tournament with noise $\varepsilon_0 \sim U(-\bar{\varepsilon}_0, \bar{\varepsilon}_0)$ ($\bar{\varepsilon}_0 > 0$) leads to monitoring cost $C(\bar{\varepsilon}_0) < C(0)$.

Let Σ^*_{tour} denote total effort generated by an optimally designed Lazear-Rosen tournament (i.e., design *hom*) with noise $\varepsilon \sim U(-\bar{\varepsilon}, \bar{\varepsilon})$, and $\Sigma^*_{all-pay}$ total effort generated by an optimally designed all-pay auction (i.e., design *het*) given the same effort-cost function as in the tournament, $c(e) = e^2/2$. Then, it can be shown that $\Sigma^*_{all-pay} > \Sigma^*_{tour}$ (see the appendix). Furthermore, the all-pay auction also dominates the tournament in the dimension (B): In the all-pay auction with optimal design *het*, the two v_L-players in the semifinals anticipate that possible participation in the final yields zero expected payoffs for them. This fact minimizes their incentives in the semifinals, so that the two v_H-players almost surely reach the final. Hence, the probability that a top player wins the final is approximately 1 in the all-pay auction. In the Lazear-Rosen tournament with optimal design *hom*, however, the final will be between a v_L-player and a v_H-player, so that the top players' win probability is strictly smaller than 1. Altogether, we obtain the following result:

Proposition 4 *If O can choose between an all-pay auction and a tournament with noise $\varepsilon_0 \sim U(-\bar{\varepsilon}_0, \bar{\varepsilon}_0)$ ($\bar{\varepsilon}_0 > 0$), he will prefer the all-pay auction given that monitoring does not lead to costs. If, however, monitoring is costly and described by the monitoring-cost function C, the organizer O will prefer the tournament given that $C(0) - C(\bar{\varepsilon})$ is sufficiently large.*

Neither the all-pay auction nor the tournament leads to a trade-off between the objectives (A) maximizing total expected effort and (B) maximizing the top players' win probability when O chooses the optimal design. Since the all-pay auction dominates the tournament in both dimensions (A) and (B), O will unambiguously prefer the all-pay auction if monitoring is free. However, if the all-pay auction under complete information requires monitoring costs $C(0)$, the organizer's preference will change if $C(0)$ is prohibitively large relative to $C(\bar{\varepsilon}_0)$ under a tournament with imprecise monitoring.

7 Conclusion

Multi-stage elimination tournaments are often combined with explicit seeding rules that determine the initial pairings of players. In sporting contests, we can frequently observe seeding rules that generate matches between underdogs and favorites (see Groh et al., forthcoming). Such seedings should prevent that favorites meet each other in the tournament very early, which could destroy suspense about the outcome of the whole tournament. However, it is not clear whether such seedings indeed maximize entertainment. Heterogeneous seedings can lead to rather boring matches at the beginning of a tournament so that the audience stays away and waits for more balanced matches in later rounds. The results of this paper show that, indeed, homogeneous seedings may be optimal to maximize entertainment if spectators do not have a too strong preference for a homogeneous final.

Moreover, unbalanced competition and, hence, low overall effort levels in the beginning of an elimination tournament can lead to outcomes that are highly determined by luck. As a consequence, there is a substantial probability that favorites lose against underdogs in early rounds. Intuitively, we can imagine that underdogs only choose low efforts in heterogeneous seedings since they are the presumable losers anyway, so that the best they can do is saving effort costs. However, the best response of the favorites then is to withhold effort as well. At the end, the outcome of early rounds can be mainly determined by luck. The findings of this paper for the setting with uniformly distributed noise clearly supports this fear: The probability that an underdog wins the tournament is strictly higher under the heterogeneous seeding than under the homogeneous one.

Appendix

Proof of Lemma 1:

The result can be shown by contradiction. Suppose that $EU_{HH}^{F*} > EU_{LH}^{F*}$ but $e_{H,het}^{S*} < e_{L,het}^{S*}$, which implies

$$\begin{aligned}
& EU_{HL}^{F*} - \left(EU_{HL}^{F*} - EU_{HH}^{F*}\right) G\left(\Delta e_{het}^{S*}\right) \\
& < EU_{LL}^{F*} - \left(EU_{LL}^{F*} - EU_{LH}^{F*}\right) G\left(\Delta e_{het}^{S*}\right) \\
\Leftrightarrow\ & \frac{EU_{HL}^{F*} - EU_{LL}^{F*}}{\left(EU_{HL}^{F*} - EU_{LL}^{F*}\right) - \left(EU_{HH}^{F*} - EU_{LH}^{F*}\right)} < G\left(\Delta e_{het}^{S*}\right) \qquad (10)
\end{aligned}$$

with

$$\begin{aligned}
& \left(EU_{HL}^{F*} - EU_{LL}^{F*}\right) - \left(EU_{HH}^{F*} - EU_{LH}^{F*}\right) \\
= \ & \left(v_H - v_L\right)\left[G\left(\Delta \hat{e}^{F*}\right) - \frac{1}{2}\right] + \left(v_H^2 + v_L^2\right)\frac{g\left(0\right)^2 - g\left(\Delta \hat{e}^{F*}\right)^2}{2\kappa} > 0 \qquad (11)
\end{aligned}$$

Condition (10) can only be satisfied if

$$\frac{EU_{HL}^{F*} - EU_{LL}^{F*}}{\left(EU_{HL}^{F*} - EU_{LL}^{F*}\right) - \left(EU_{HH}^{F*} - EU_{LH}^{F*}\right)} < 1 \Leftrightarrow EU_{HH}^{F*} < EU_{LH}^{F*},$$

a contradiction.

Proof of Proposition 1:

(i) $\Sigma_{hom}^{S} > \Sigma_{het}^{S}$ can be rewritten as

$$\begin{aligned}
& g\left(0\right)\left(EU_{HL}^{F*} + EU_{LH}^{F*}\right) > \\
& g\left(\Delta e_{het}^{S*}\right)\left[\left(1 - G\left(\Delta e_{het}^{S*}\right)\right)\left(EU_{HL}^{F*} + EU_{LL}^{F*}\right) + \left(EU_{HH}^{F*} + EU_{LH}^{F*}\right) G\left(\Delta e_{het}^{S*}\right)\right],
\end{aligned}$$

which is true if

$$EU_{HL}^{F*} + EU_{LH}^{F*} > \left(1 - G\left(\Delta e_{het}^{S*}\right)\right)\left(EU_{HL}^{F*} + EU_{LL}^{F*}\right) + \left(EU_{HH}^{F*} + EU_{LH}^{F*}\right) G\left(\Delta e_{het}^{S*}\right)$$

$$\Leftrightarrow \left(EU_{HL}^{F*} - EU_{HH}^{F*}\right) G\left(\Delta e_{het}^{S*}\right) > \left(EU_{LL}^{F*} - EU_{LH}^{F*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]. \quad (12)$$

Since, according to Lemma 1, $\Delta e_{het}^{S*} > 0 \Rightarrow G\left(\Delta e_{het}^{S*}\right) > 1/2$ under $EU_{HH}^{F*} > EU_{LH}^{F*}$, inequality (12) is satisfied because $EU_{HL}^{F*} - EU_{HH}^{F*} > EU_{LL}^{F*} - EU_{LH}^{F*}$ (see (11)).

Result (ii) is proven by contradiction. Suppose that $\Sigma_{hom}^{F} > \Sigma_{het}^{F}$, which can be rewritten as

$$\begin{aligned} & v_L\left[g\left(\Delta\hat{e}^{F*}\right)\left(1 - 2G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]\right) - 2g\left(0\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]^2\right] \\ > \; & v_H\left[2g\left(0\right)G\left(\Delta e_{het}^{S*}\right)^2 - g\left(\Delta\hat{e}^{F*}\right)\left(1 - 2G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]\right)\right]. \end{aligned}$$

This condition can only be satisfied if

$$\begin{aligned} & g\left(\Delta\hat{e}^{F*}\right)\left(1 - 2G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]\right) - 2g\left(0\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]^2 \\ > \; & 2g\left(0\right)G\left(\Delta e_{het}^{S*}\right)^2 - g\left(\Delta\hat{e}^{F*}\right)\left(1 - 2G\left(\Delta e_{het}^{S*}\right)\left[1 - G\left(\Delta e_{het}^{S*}\right)\right]\right) \\ \Leftrightarrow \; & g\left(\Delta\hat{e}^{F*}\right) > g\left(0\right), \end{aligned}$$

a contradiction.

Proof of Proposition 2:

Let players i and j with valuations $v_i, v_j \in \{v_L, v_H\}$ compete in the final. Using the quadratic cost function $c\left(e_i\right) = e_i^2/2$ and the uniform distribution $G\left(x\right) = \left(x + \bar{\varepsilon}\right)/\left(2\bar{\varepsilon}\right)$ immediately shows that both players' objective functions are strictly concave so that the equilibrium efforts and the corresponding expected utilities from participating in the final are

$$e_i^{F*} = \frac{v_i}{2\bar{\varepsilon}} \quad \text{and} \quad EU_{ij}^{F*} = \frac{v_i\left(4\bar{\varepsilon}^2 + v_i - 2v_j\right)}{8\bar{\varepsilon}^2} \quad \text{with } i,j \in \{L,H\}.$$

In order to guarantee that $e_H^{F*} - e_L^{F*} \in \left(-\bar{\varepsilon}, \bar{\varepsilon}\right)$ and $EU_{ij}^{F*} > 0$, I impose the technical restriction

$$\bar{\varepsilon}^2 > \frac{v_H}{2} - \frac{v_L}{4}. \quad (13)$$

Consider first design *hom*. In the v_H-semifinal, the players' objective functions can be written as follows:

$$
\begin{aligned}
EU^S_{H_1H_2} &= \frac{e^S_{H_1} - e^S_{H_2} + \bar{\varepsilon}}{2\bar{\varepsilon}} EU^{F*}_{HL} - \frac{\left(e^S_{H_1}\right)^2}{2} \\
EU^S_{H_2H_1} &= \left[1 - \frac{e^S_{H_1} - e^S_{H_2} + \bar{\varepsilon}}{2\bar{\varepsilon}}\right] EU^{F*}_{HL} - \frac{\left(e^S_{H_2}\right)^2}{2}.
\end{aligned}
$$

Thus, in equilibrium, each v_H-player chooses effort $e^{S*}_{H,hom} = EU^{F*}_{HL}/\left(2\bar{\varepsilon}\right)$ in the semifinal. Analogously, equilibrium effort of each participant in the v_L-semifinal is given by $e^{S*}_{L,hom} = EU^{F*}_{LH}/\left(2\bar{\varepsilon}\right)$, so that we obtain

$$
\begin{aligned}
\Sigma^S_{hom} + \Sigma^F_{hom} &= \frac{2\left(EU^{F*}_{HL} + EU^{F*}_{LH}\right) + v_H + v_L}{2\bar{\varepsilon}} \\
&= \frac{8\bar{\varepsilon}^2\left(v_H + v_L\right) + v_H^2 - 4v_Hv_L + v_L^2}{8\bar{\varepsilon}^3}. \qquad (14)
\end{aligned}
$$

Now we consider design *het*. The participants of the first semifinal maximize

$$
\begin{aligned}
EU^{S_1}_{HL} = {} & \frac{e^{S_1}_H - e^{S_1}_L + \bar{\varepsilon}}{2\bar{\varepsilon}} \left(\frac{e^{S_2}_H - e^{S_2}_L + \bar{\varepsilon}}{2\bar{\varepsilon}} EU^{F*}_{HH}\right. \\
& \left. + \left[1 - \frac{e^{S_2}_H - e^{S_2}_L + \bar{\varepsilon}}{2\bar{\varepsilon}}\right] EU^{F*}_{HL}\right) - \frac{\left(e^{S_1}_H\right)^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
EU^{S_1}_{LH} = {} & \left[1 - \frac{e^{S_1}_H - e^{S_1}_L + \bar{\varepsilon}}{2\bar{\varepsilon}}\right] \left(\frac{e^{S_2}_H - e^{S_2}_L + \bar{\varepsilon}}{2\bar{\varepsilon}} EU^{F*}_{LH}\right. \\
& \left. + \left[1 - \frac{e^{S_2}_H - e^{S_2}_L + \bar{\varepsilon}}{2\bar{\varepsilon}}\right] EU^{F*}_{LL}\right) - \frac{\left(e^{S_1}_L\right)^2}{2}
\end{aligned}
$$

and the participants of the second semifinal

$$\begin{aligned}
EU_{HL}^{S_2} &= \frac{e_H^{S_2} - e_L^{S_2} + \bar{\varepsilon}}{2\bar{\varepsilon}} \left(\frac{e_H^{S_1} - e_L^{S_1} + \bar{\varepsilon}}{2\bar{\varepsilon}} EU_{HH}^{F*} \right. \\
&\qquad \left. + \left[1 - \frac{e_H^{S_1} - e_L^{S_1} + \bar{\varepsilon}}{2\bar{\varepsilon}} \right] EU_{HL}^{F*} \right) - \frac{\left(e_H^{S_2}\right)^2}{2} \\
EU_{LH}^{S_2} &= \left[1 - \frac{e_H^{S_2} - e_L^{S_2} + \bar{\varepsilon}}{2\bar{\varepsilon}} \right] \left(\frac{e_H^{S_1} - e_L^{S_1} + \bar{\varepsilon}}{2\bar{\varepsilon}} EU_{LH}^{F*} \right. \\
&\qquad \left. + \left[1 - \frac{e_H^{S_1} - e_L^{S_1} + \bar{\varepsilon}}{2\bar{\varepsilon}} \right] EU_{LL}^{F*} \right) - \frac{\left(e_L^{S_2}\right)^2}{2}.
\end{aligned}$$

Since all objective functions are strictly concave, the equilibrium efforts are described by the four first-order conditions, which yield[15]

$$\begin{aligned}
e_H^{S_1*} &= e_H^{S_2*} = \frac{EU_{HL}^{F*} EU_{LH}^{F*} - EU_{HH}^{F*} EU_{LL}^{F*} + 2\bar{\varepsilon}^2 \left(EU_{HH}^{F*} + EU_{HL}^{F*} \right)}{8\bar{\varepsilon}^3 - 2\bar{\varepsilon} \left(EU_{HH}^{F*} + EU_{LL}^{F*} - EU_{HL}^{F*} - EU_{LH}^{F*} \right)} \\
&= \frac{16\bar{\varepsilon}^4 \left(4\bar{\varepsilon}^2 - v_L \right) - v_L \left(v_H - v_L \right)^2}{16\bar{\varepsilon}^3 \left(16\bar{\varepsilon}^4 + \left(v_H - v_L \right)^2 \right)} v_H =: e_{H,het}^{S*} \\
e_L^{S_1*} &= e_L^{S_2*} = \frac{EU_{HL}^{F*} EU_{LH}^{F*} - EU_{HH}^{F*} EU_{LL}^{F*} + 2\bar{\varepsilon}^2 \left(EU_{LL}^{F*} + EU_{LH}^{F*} \right)}{8\bar{\varepsilon}^3 - 2\bar{\varepsilon} \left(EU_{HH}^{F*} + EU_{LL}^{F*} - EU_{HL}^{F*} - EU_{LH}^{F*} \right)} \\
&= \frac{16\bar{\varepsilon}^4 \left(4\bar{\varepsilon}^2 - v_H \right) - v_H \left(v_H - v_L \right)^2}{16\bar{\varepsilon}^3 \left(16\bar{\varepsilon}^4 + \left(v_H - v_L \right)^2 \right)} v_L =: e_{L,het}^{S*}.
\end{aligned}$$

Altogether, expected total effort from seeding het amounts to

$$\Sigma_{het}^S + \Sigma_{het}^F =$$

$$\begin{aligned}
&2e_{H,het}^{S*} + 2e_{L,het}^{S*} + 2e_H^{F*} G \left(e_{H,het}^{S*} - e_{L,het}^{S*} \right)^2 + 2e_L^{F*} \left[1 - G \left(e_{H,het}^{S*} - e_{L,het}^{S*} \right) \right]^2 \\
&\qquad + \left(e_H^{F*} + e_L^{F*} \right) 2G \left(e_{H,het}^{S*} - e_{L,het}^{S*} \right) \left[1 - G \left(e_{H,het}^{S*} - e_{L,het}^{S*} \right) \right] =
\end{aligned}$$

$$\frac{64\bar{\varepsilon}^6 \left(v_H + v_L \right) + 8\bar{\varepsilon}^4 \left(v_H^2 + v_L^2 - 4v_H v_L \right) + \left(2\bar{\varepsilon}^2 \left(v_H + v_L \right) - v_H v_L \right) \left(v_H - v_L \right)^2}{4\bar{\varepsilon}^3 \left(16\bar{\varepsilon}^4 + \left(v_H - v_L \right)^2 \right)}.$$

[15] Condition (13) ensures that $e_{H,het}^{S*} - e_{L,het}^{S*} \in (-\bar{\varepsilon}, \bar{\varepsilon})$.

The direct comparison of total expected efforts shows that

$$\Sigma^S_{hom} + \Sigma^F_{hom} > \Sigma^S_{het} + \Sigma^F_{het} \Leftrightarrow$$
$$(v_H - v_L)^2 \left(4\bar{\varepsilon}^2 (v_H + v_L) + (v_H - v_L)^2\right) > 0$$

is true.

Proof of Proposition 4:

Groh et al. (forthcoming) show that it is optimal for the tournament organizer to choose design *het* to maximize total expected effort. Under this design, the two v_L-players anticipate in the semifinals that they will have a zero expected payoff from participating in the final, irrespective of the type of their opponent. Consequently, they choose minimal efforts in the semifinals. The two v_H-players anticipate that they will meet each other in the final almost surely and win their respective semifinal with very low efforts. Total expected effort under *het* is thus approximately given by the two v_H-players' equilibrium effort choices in the final.

In a symmetric all-pay auction with complete information in which players can choose non-negative efforts without restriction, only mixed-strategy equilibria exist (see, e.g., Baye et al. 1996; Konrad 2009, Section 2.1). In case of a convex cost function, $c(e)$, and two homogeneous contestants with winner prize v_H the equilibrium strategy of each v_H-player in the final is given by the cdf $F(e) = c(e)/v_H$ with $e \in [0, c^{-1}(v_H)]$ (Kaplan et al. 2003; Konrad 2009, p. 27). Hence, total expected effort from the all-pay auction amounts to

$$\Sigma^*_{all-pay} = 2 \cdot \int_0^{c^{-1}(v_H)} e \cdot \frac{c'(e)}{v_H} de = \frac{2^{\frac{5}{2}} v_H^{\frac{1}{2}}}{3},$$

where the last equality follows from the specific cost function $c(e) = e^2/2$. Recall that in the tournament setting considered in this paper, given uniformly distributed ε, design *hom* is optimal for the organizer O to maximize total expected effort. This effort $\Sigma^*_{tour} = \Sigma^S_{hom} + \Sigma^F_{hom}$ is described by (14).

The claim $\Sigma^*_{tour} < \Sigma^*_{all-pay}$ can be shown by contradiction. Suppose that

$$\begin{aligned}
\Sigma^*_{tour} &> \Sigma^*_{all-pay} \Leftrightarrow \\
v_H^2 - 4v_H v_L + v_L^2 &> 8\bar{\varepsilon}^2 \left(\frac{32\sqrt{2v_H}}{3} \bar{\varepsilon} - (v_H + v_L) \right). \qquad (15)
\end{aligned}$$

The right-hand side is positive since

$$\bar{\varepsilon} > \frac{3\,(v_H + v_L)}{32\sqrt{2v_H}}$$

is true due to $\bar{\varepsilon} \overset{(13)}{>} \sqrt{\frac{v_H}{2} - \frac{v_L}{4}} > 3\,(v_H + v_L)\,/[32\sqrt{2v_H}]$. Inserting for the best possible case, $\bar{\varepsilon} = \sqrt{\frac{v_H}{2} - \frac{v_L}{4}}$, in (15) and rewriting yields

$$5v_H^2 - v_L^2 - 2v_H v_L - \frac{32}{3}\sqrt{2v_H}\,(2v_H - v_L)^{\frac{3}{2}} > 0.$$

Replacing v_H by av_L with $a > 1$, leads to

$$5a^2 - 2a - \frac{32}{3}\sqrt{2a}\,(2a - 1)^{\frac{3}{2}} - 1 > 0,$$

which is false.

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Additional pages for the referees

Condition $EU_{HH}^{F*} > EU_{LH}^{F*}$ under additive heterogeneity:

Assume again that the tournament organizer O makes use of a relative-performance signal which either indicates that player i ($s = s_i$) or player j ($s = s_j$) has performed better. Let this signal be given by

$$s = \begin{cases} s_i & \text{if} \quad e_i + a_i - e_j - a_j > \varepsilon \\ s_j & \text{if} \quad e_i + a_i - e_j - a_j < \varepsilon, \end{cases}$$

with a_i and a_j as the players' abilities. I assume that players have a uniform tournament prize v and that there are two high-ability players ($a_i = a_H > 0$) and two low-ability ones ($a_i = a_L \in (0, a_H)$).

In the final, the two participants i and j maximize their objective functions

$$\begin{aligned} EU_i^F\left(e_i^F, e_j^F\right) &= v \cdot G\left(e_i^F + a_i - e_j^F - a_j\right) - c\left(e_i^F\right) \\ EU_j^F\left(e_j^F, e_i^F\right) &= v \cdot \left[1 - G\left(e_i^F + a_i - e_j^F - a_j\right)\right] - c\left(e_j^F\right). \end{aligned}$$

Let, again, an equilibrium in pure strategies exist. In that case, this equilibrium $\left(e_i^{F*}, e_j^{F*}\right)$ can be described by the first-order conditions

$$vg\left(e_i^{F*} + a_i - e_j^{F*} - a_j\right) = c'\left(e_i^{F*}\right) = c'\left(e_j^{F*}\right).$$

Hence, in equilibrium, players always choose identical efforts in the final:

$$\left(e_i^{F*}, e_j^{F*}\right) = \begin{cases} \left(\frac{vg(0)}{\kappa}, \frac{vg(0)}{\kappa}\right) & \text{if} \quad a_i = a_j \\ \left(\frac{vg(\Delta a)}{\kappa}, \frac{vg(\Delta a)}{\kappa}\right) & \text{if} \quad a_i \neq a_j \end{cases}$$

with $\Delta a := a_H - a_L$. The corresponding expected utilities read as follows:[16]

$$\begin{aligned}
EU_{HH}^{F*} &= EU_{LL}^{F*} = \frac{v}{2} - \frac{v^2 g(0)^2}{2\kappa} \\
EU_{HL}^{F*} &= vG(\Delta a) - \frac{v^2 g(\Delta a)^2}{2\kappa} \\
EU_{LH}^{F*} &= v[1 - G(\Delta a)] - \frac{v^2 g(\Delta a)^2}{2\kappa}
\end{aligned}$$

with

$$EU_{HL}^{F*} > EU_{HH}^{F*} = EU_{LL}^{F*} \quad \text{and} \quad EU_{HL}^{F*} > EU_{LH}^{F*}.$$

Under the design *hom*, the expected utilities of the two high-ability players in the semifinal are given by

$$\begin{aligned}
EU_{H_1H_2}^{S} &= G\left(e_{H_1}^S - e_{H_2}^S\right) EU_{HL}^{F*} - c\left(e_{H_1}^S\right) \\
EU_{H_2H_1}^{S} &= \left[1 - G\left(e_{H_1}^S - e_{H_2}^S\right)\right] EU_{HL}^{F*} - c\left(e_{H_2}^S\right),
\end{aligned}$$

yielding equilibrium efforts $e_{H_1}^{S*} = e_{H_2}^{S*} = e_{H,hom}^{S*} := \left[g(0)\, EU_{HL}^{F*}\right]/\kappa$. Analogously, in the semifinal between the two low-ability players, each participant chooses $e_{L,hom}^{S*} = \left[g(0)\, EU_{LH}^{F*}\right]/\kappa$ in equilibrium. Total efforts in the semifinal, Σ_{hom}^S, and in the final, Σ_{hom}^F, amount to[17]

$$\begin{aligned}
\Sigma_{hom}^S &= \frac{2g(0)}{\kappa}\left(EU_{HL}^{F*} + EU_{LH}^{F*}\right) = \frac{2g(0)}{\kappa}\left(v - \frac{v^2 g(\Delta a)^2}{\kappa}\right) \\
\text{and} \quad \Sigma_{hom}^F &= 2\frac{vg(\Delta a)}{\kappa}.
\end{aligned}$$

Under the design *het*, the players' objective functions in semifinal 1 are

[16] Let v be sufficiently small to guarantee strictly positive values.

[17] Let, again, v be sufficiently small to guarantee strictly positive values.

given by

$$\begin{aligned} EU_{HL}^{S_1} &= G\left(\Delta e^{S_1}+\Delta a\right)\left(G\left(\Delta e^{S_2}+\Delta a\right)EU_{HH}^{F*}\right. \\ &\quad \left.+\left[1-G\left(\Delta e^{S_2}+\Delta a\right)\right]EU_{HL}^{F*}\right)-c\left(e_H^{S_1}\right) \qquad \text{and} \\ EU_{LH}^{S_1} &= \left[1-G\left(\Delta e^{S_1}+\Delta a\right)\right]\left(EU_{LH}^{F*}G\left(\Delta e^{S_2}+\Delta a\right)\right. \\ &\quad \left.+\left[1-G\left(\Delta e^{S_2}+\Delta a\right)\right]EU_{LL}^{F*}\right)-c\left(e_L^{S_1}\right). \end{aligned}$$

The first-order conditions show that, if players of the same type behave identically in equilibrium, efforts in this symmetric equilibrium are described by

$$\begin{aligned} e_{H,het}^{S*} &= \frac{g\left(\Delta e_{het}^{S*}+\Delta a\right)}{\kappa}\left(EU_{HL}^{F*}-\left(EU_{HL}^{F*}-EU_{HH}^{F*}\right)G\left(\Delta e_{het}^{S*}+\Delta a\right)\right) \\ e_{L,het}^{S*} &= \frac{g\left(\Delta e_{het}^{S*}+\Delta a\right)}{\kappa}\left(EU_{LL}^{F*}-\left(EU_{LL}^{F*}-EU_{LH}^{F*}\right)G\left(\Delta e_{het}^{S*}+\Delta a\right)\right) \end{aligned}$$

with $\Delta e_{het}^{S*} := e_{H,het}^{S*}-e_{L,het}^{S*}$. We have

$$\begin{gathered} e_{H,het}^{S*} > e_{L,het}^{S*} \Leftrightarrow EU_{HL}^{F*}-EU_{LL}^{F*} > \\ [\left(EU_{HL}^{F*}-EU_{LL}^{F*}\right)-\left(EU_{HH}^{F*}-EU_{LH}^{F*}\right)]G\left(\Delta e_{het}^{S*}+\Delta a\right) \Leftrightarrow \\ \frac{EU_{HL}^{F*}-EU_{LL}^{F*}}{(EU_{HL}^{F*}-EU_{LL}^{F*})-(EU_{HH}^{F*}-EU_{LH}^{F*})} > G\left(\Delta e_{het}^{S*}+\Delta a\right) \end{gathered}$$

with

$$\left(EU_{HL}^{F*}-EU_{LL}^{F*}\right)-\left(EU_{HH}^{F*}-EU_{LH}^{F*}\right)=\frac{v^2}{\kappa}\left(g\left(0\right)^2-g\left(\Delta a\right)^2\right)>0$$

Hence, a sufficient condition for $e_{H,het}^{S*} > e_{L,het}^{S*}$ is given by

$$\begin{gathered} \frac{EU_{HL}^{F*}-EU_{LL}^{F*}}{(EU_{HL}^{F*}-EU_{LL}^{F*})-(EU_{HH}^{F*}-EU_{LH}^{F*})} > 1 \Leftrightarrow \\ EU_{HH}^{F*} > EU_{LH}^{F*} \Leftrightarrow \\ 2\kappa\frac{\frac{1}{2}-\left[1-G\left(\Delta a\right)\right]}{g\left(0\right)^2-g\left(\Delta a\right)^2} > v, \end{gathered}$$

the condition used in the footnote.

All equilibrium efforts from both heterogeneous semifinals sum up to

$$\begin{aligned}
\Sigma_{het}^{S} &= 2\frac{g\left(\Delta e_{het}^{S*}+\Delta a\right)}{\kappa}[\left(EU_{HL}^{F*}+EU_{LL}^{F*}\right)\left(1-G\left(\Delta e_{het}^{S*}+\Delta a\right)\right) \\
&\quad +\left(EU_{HH}^{F*}+EU_{LH}^{F*}\right)G\left(\Delta e_{het}^{S*}+\Delta a\right)] \\
&= 2\frac{g\left(\Delta e_{het}^{S*}+\Delta a\right)}{\kappa}[\left(EU_{HL}^{F*}+EU_{LL}^{F*}\right) \\
&\quad +\left(EU_{HH}^{F*}+EU_{LH}^{F*}-EU_{HL}^{F*}-EU_{LL}^{F*}\right)G\left(\Delta e_{het}^{S*}+\Delta a\right)].
\end{aligned}$$

The expected efforts from the subsequent final amount to

$$\begin{aligned}
\Sigma_{het}^{F} &= G\left(\Delta e_{het}^{S*}\right)^{2}2\frac{vg\left(0\right)}{\kappa} \\
&\quad +\left[1-G\left(\Delta e_{het}^{S*}\right)\right]^{2}2\frac{vg\left(0\right)}{\kappa} \\
&\quad +2G\left(\Delta e_{het}^{S*}\right)\left[1-G\left(\Delta e_{het}^{S*}\right)\right]2\frac{vg\left(\Delta a\right)}{\kappa} \\
&= 2\frac{vg\left(0\right)}{\kappa}\left(G\left(\Delta e_{het}^{S*}\right)^{2}+\left[1-G\left(\Delta e_{het}^{S*}\right)\right]^{2}\right) \\
&\quad +2G\left(\Delta e_{het}^{S*}\right)\left[1-G\left(\Delta e_{het}^{S*}\right)\right]2\frac{vg\left(\Delta a\right)}{\kappa}.
\end{aligned}$$

Players choose lower or equal efforts in the *hom*-final than in the *het*-final if

$$\begin{gathered}
\Sigma_{hom}^{F}=2\frac{vg\left(\Delta a\right)}{\kappa}\leq 2\frac{vg\left(0\right)}{\kappa}\left(G\left(\Delta e_{het}^{S*}\right)^{2}+\left[1-G\left(\Delta e_{het}^{S*}\right)\right]^{2}\right) \\
+2G\left(\Delta e_{het}^{S*}\right)\left[1-G\left(\Delta e_{het}^{S*}\right)\right]2\frac{vg\left(\Delta a\right)}{\kappa}=\Sigma_{het}^{F}\Leftrightarrow \\
g\left(\Delta a\right)\leq g\left(0\right),
\end{gathered}$$

which is true. Comparison of total efforts from the semifinal yields

$$\begin{gathered}
\Sigma_{hom}^{S}>\Sigma_{het}^{S}\Leftrightarrow g\left(0\right)\left(EU_{HL}^{F*}+EU_{LH}^{F*}\right)> \\
g\left(\Delta e_{het}^{S*}+\Delta a\right)[\left(EU_{HL}^{F*}+EU_{LL}^{F*}\right)\left(1-G\left(\Delta e_{het}^{S*}+\Delta a\right)\right) \\
+\left(EU_{HH}^{F*}+EU_{LH}^{F*}\right)G\left(\Delta e_{het}^{S*}+\Delta a\right)],
\end{gathered}$$

which is true if

$$\left(EU_{HL}^{F*} - EU_{HH}^{F*}\right) G\left(\Delta e_{het}^{S*} + \Delta a\right) > \left(EU_{LL}^{F*} - EU_{LH}^{F*}\right)\left[1 - G\left(\Delta e_{het}^{S*} + \Delta a\right)\right].$$

Recall that under $EU_{HH}^{F*} > EU_{LH}^{F*} \Leftrightarrow v < 2\kappa \frac{\frac{1}{2} - [1 - G(\Delta a)]}{g(0)^2 - g(\Delta a)^2}$ we have $\Delta e_{het}^{S*} > 0$, so that $G\left(\Delta e_{het}^{S*} + \Delta a\right) > 1/2$ and, thus, the last inequality is satisfied because $EU_{HL}^{F*} - EU_{HH}^{F*} > EU_{LL}^{F*} - EU_{LH}^{F*}$ (see above).

Altogether, the setting with additive heterogeneity leads to exactly the same results as Proposition 1 above.