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Imitation Equilibrium

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Abstract

The paper presents the concept of an "imitation equilibrium" and explores it in the context of some simple oligopoly models. The concept applies to normal form games enriched by a "reference structure" specifying a "reference group" for every player. The reference group is a set of other players, whom the player may consider to imitate. Some of these players may not be suitable for imitation for various reasons. Only one of the most successful of the remaining members of the reference group is imitated. Imitation is the adoption of the imitated player's strategy.

Imitation equilibrium does not only mean absence of imitation opportunities but also stability against exploratory deviations of "success leaders", i. e. players most successful in their reference groups. Exploration declenches a process of imitation which either leads back to imitation equilibrium directly or by a "return path" after an unsuccessful deviation.

The imitation equilibrium concept is motivated by the experimental literature which suggests that under appropriate conditions imitation of the most successful relevant other is an important behavioral force. The concept may be useful for the evaluation of experimental data and for the planning of future experiments.

Keywords

Imitation equilibrium, oligopoly, normal form games, experimental economics

JEL Classification Codes

C72, C91, C92, L13

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1. Introduction

Cournot's oligopoly theory (1838) predicts convergence to the Cournot equilibrium in repeated play of his quantity variation model. He envisions a dynamic adjustment process driven by short run profit maximization against the expectation of unchanging competitors' quantities. The idea of convergence to Cournot equilibrium finds some support in the older literature on oligopoly experiments (Sauermann and Selten 1959, Stern 1967).

Deviations went in the direction of cooperative quantity restraint. Especially symmetric duopoly with common knowledge of demand and costs was conducive to joint profit maximization (Fouraker and Siegel 1963). The availability of verbal communication possibilities is another factor which enhances cooperation (Friedman 1972, Selten and Berg 1970). A tendency towards Cournot equilibrium arose under conditions without communication, with asymmetric costs and with little information about competitors' profits.

Surprisingly new oligopoly experiments with the quantity variation model show average quantities higher than those in Cournot equilibrium (Huck, Normann and Oechssler 1999). Imitation of the more successful, i. e. of other players with higher profits, is offered as an explanation. In symmetric Cournot oligopoly with constant average costs those who supply more have higher profits as long as price is above average costs. In this situation imitation of the more successful can be expected to result in a tendency towards competitive equilibrium. This has been pointed out in the economic literature on evolutionary game theory (Weibull 1995, Vega-Redondo 1999).

In the old experiments subjects usually were supplied with profit tables which made it easy to determine best replies. Obviously this facilitates short run profit maximization against the expected joint supply of the competitors. In the experiments by Huck, Normann and Oechssler, subjects did not have access to easy means of finding best replies, but they received feedback on the competitors' profits. Therefore their experimental situation may be more conducive to imitation of the more successful.

The older literature conveyed the impression that symmetry, communication possibilities and information are crucial influences on behavior in oligopoly situations. It seems to be necessary to add the information processing background as a fourth factor to this list. By this we mean tools like profit tables without which the available information cannot be easily exploited.

The oligopoly experiments mentioned up to now are based on simple models with only one action parameter. Experiments on more complex oligopoly situations can also be found in the older literature. A study of this kind (Todt 1970, 1972, 1975) presents experimental evidence for imitation of the more successful as an important feature of observed behavior. The oligopoly situation explored by Horst Todt involved two resort towns with three hotels in each of them. The hotels could choose between two categories (upper and lower quality), and they had to determine capacity, price and advertising. The oligopoly situation had the character of a complex dynamic game.

Todt's subjects did not indiscriminately imitate other more successful players, but only those who were most similar to themselves. These "nearest relatives" were hotels in the same category and, if possible, in the same resort. (A hotel which is the only one in its category has no nearest relatives.) The behavioral tendency observed by Todt is not just imitation of the more successful, but rather "imitation of the similar more successful".

In the interpretation of his results, Horst Todt combined his description of imitative behavior with an idea of local exploration. He proposed that players who are at least as successful as similar others may make small random changes of their action parameters, presumably in order to explore the possibilities for payoff improvement. However, he did not indicate how a player would evaluate the success of such exploratory behavior and how a player would respond to exploratory success or failure.

The work of Horst Todt is of special significance for this paper. Our concept of "imitation equilibrium" grew out of an attempt to capture the essence of his ideas on imitation of the more successful and exploratory local deviations by a formal behavioral equilibrium notion. A first definition was proposed in a reappraisal of Todt's work motivated by the occasion of his 70th birthday (Ostmann, Selten and Tietz 2000). The definitions presented here are a little different and more suitable for application, but the essential features remain unchanged.

It is necessary to distinguish between local and global imitation equilibria. The words "local" and "global" refer to the sets of exploratory deviations taken into account. Global imitation equilibrium requires stability against any exploratory deviation whereas the stability of local imitation equilibrium is restricted to sufficiently small exploratory deviations.

Our definitions apply to normal form games complemented by a "reference structure" which assigns a set of other players to each player. This set is called the "reference group" of the player under consideration. The players in the reference group are those who are sufficiently similar to be imitated if they are more successful. The reference structure is thought of as exogenously given. No attempt will be made to discuss the question how reference groups arise.

We speak of a "universal reference structure" if the reference set of a player is always the set of all other players. In many cases a non-universal reference structure suggests itself. Thus we may for example look at a market involving producers and spatially dispersed retailers. The reference group of a producer may be modeled as the set of all other producers and the reference group of a retailer may be the set of all neighboring retailers.

The imitation equilibrium concept will be applied to several oligopoly models. As we will see, the symmetric Cournot model with constant average costs complemented by the universal reference structure has a uniquely determined local imitation equilibrium. At this equilibrium all players supply the same amount and price equals average costs, as in the competitive equilibrium.

The second example to be explored is the asymmetric Cournot duopoly with unequal constant average costs, again complemented by the universal reference structure. In this case we also find a uniquely determined local imitation equilibrium. In this equilibrium both players supply the same amount, half of the monopoly supply of the low cost supplier if he were alone in the market. Here imitation of the more successful does not drive the price down to the competitive price. On the contrary, imitation combined with exploration moves the price up to a quasi-monopoly level.

Unlike in the case of the symmetric oligopoly the uniquely determined local imitation equilibrium of the asymmetric duopoly fails to be a global imitation equilibrium in a part of the parameter space where cost differences are relatively small.

The third oligopoly situation explored is mill price competition on a circle. The players are n firms located equidistantly on a circle. Each firm sets a mill price. Transport costs are carried by the customers who buy where it is cheapest including transport costs. The demand of an individual customer is completely inelastic below an upper limit for price plus transport costs, but zero above this limit. Customers are evenly distributed on the circle. Up to unimportant minor variations this is a standard model of location theory (Beckmann 1968). In the literature the model is usually not presented as a fully specified normal form game. We prove that this game has a uniquely determined pure equilibrium, referred to as the Cournot equilibrium of the model.

The reference group of a player is formed by its left and right neighbor. Attention is restricted to symmetric imitation equilibria at which all firms take the same price. For $n = 2$ and $n = 3$ there are uniquely determined local symmetric imitation equilibria and for $n > 3$ there is a whole range of such equilibria. It turns out that for all n the local symmetric imitation equilibria are also global ones. Interestingly, competition at imitation equilibrium is more intense than at Cournot equilibrium for $n = 2$ and $n = 3$, but not necessarily for $n > 3$, where the Cournot equilibrium is also an imitation equilibrium.

2. The concept of imitation equilibrium

2.1 Imitation models

As has been explained in the introduction the concept of a local or global imitation equilibrium refers to a normal form game complemented by a reference structure. Such a pair will be called an "imitation model". We restrict our attention to normal form games in which strategies can be varied continuously. Accordingly we introduce the following definitions.

An imitation model

$M = (G, R)$

has two constituents:

1) An n-player *normal form game* $G = (S_1, ..., S_n; H)$ where S_i is player *i's strategy set* and H is the *payoff function* which assigns the *payoff vector*

$$H(s) = \left(H_1(s),...,H_n(s)\right)$$

to every *strategy combination* $s = (s_1,...,s_n)$ with $s_i \in S_i$ for i = 1, ..., n. The set of all strategy combinations is denoted by S. For i = 1, ..., n the strategy set S_i is a non-empty convex subset of a Euclidean space. An element $s_i \in S_i$ is a *strategy of player i*. For $s \in S$ and i = 1, ..., n *player i's payoff* $H_i(s)$ *for s* is a real number.

2) A *reference structure R* which assigns a subset $R(i)$ of the *player set N* = {1, ..., n} to every $i \in N$, such that i does not belong to $R(i)$. The set $R(i)$ is called player i's reference group. The case that $R(i)$ is empty is not excluded.

Comment: Since we do not make use of mixed strategies, the word "strategy" always refers to a pure strategy. This is reflected by the above definition.

We will refer to a player by the pronoun "it". This seems to be justified, since we look at players as organizations rather than individuals. Firms are neither male nor female.

Experimental games often have discrete strategy sets. Thus in a Cournot oligopoly supplies may be restricted to integer multiples of a smallest money unit. Here we will not discuss the question how our concepts could be adjusted to such cases, even if this problem may need to be addressed in the evaluation of experiments. The restriction of our attention to continuously varying strategies permits us to concentrate on essential features of conceptual issues and theoretical results.

2.2 Informal conceptual preview

In the following we will informally explain some concepts. More precise definitions will be given later.

In this paper imitation is understood as a change of strategy, a replacement of one's current strategy by the strategy of a more successful member of the reference group. Obviously nobody can be imitated who uses the same strategy. A player together with those members of its reference group using the same strategy as it does form the group of "costrategists" of this player. Other members of the reference group of the player may be "incomparable" in the sense that they play strategies not in the player's strategy set. The remaining members of the reference group and the player itself are "comparable". Obviously it makes no sense to imitate a comparable player unless its payoff surpasses that of all costrategists. Among these players (if there are any) only those with the highest payoff are "success examples" of the player. It is assumed that only success examples are imitated. The strategy of a success example is an "imitation opportunity".

Two success examples may achieve the same payoff with different strategies. Therefore it cannot be excluded that a player has more then one imitation opportunity. Our definitions must take this into account, even if the possibility may rarely arise in particular cases.

The concept of imitation equilibrium is based on the idea that imitation goes on as long as there are players with imitation opportunities. It is assumed that at a strategy combination at which this is the case, all these players immediately imitate a success example. A new strategy combination is reached in this way and new imitation opportunities may present themselves there. In this way an “imitation process” takes its course which goes on until a strategy combination without imitation opportunities is reached.

We think of the imitation process as a journey through the space of strategy combinations. Inspired by this image we refer to strategy combinations with imitation possibilities as “way stations” and to strategy combinations without imitation opportunities as “destinations”.

At a way station each player with imitation possibilities immediately takes one of them. They all act simultaneously. A new strategy combination to which the imitation process can move in this way is called a “successor station” of the way station under consideration. Since a player may have several imitation opportunities, a way station may have several successor stations. An “imitation path” is a string of strategy combinations, such that each way station on the string is followed by one of its successors. A finite imitation path ends with a destination, but the possibility of an infinite imitation path is not excluded. Such a path would proceed from way station to way station and never reach a destination.

In his theory of economic development, Schumpeter (1939) portrays business cycles as driven by innovation and imitation. Innovation only occurs when imitation has run its course. Similarly the concept of imitation equilibrium is based on the assumption that exploration does not happen as long as the imitation process is going on. Imitation responses are thought of as quick and exploration will be considered only after things have settled down and a destination has been reached.

Following Horst Todt (1970, 1972, 1975) it is also assumed that exploration activities are restricted to those who are “success leaders” with respect to their reference group, in the sense that the player’s profit is at least as high as he highest in its reference group.

An imitation equilibrium must be a destination, but this is not enough. Additional stability properties are required. A global imitation equilibrium must be stable against any exploratory deviation of a success leader and a local one only against sufficiently small ones.

It will now be explained what stability against an exploratory deviation means. Consider a strategy combination which is a candidate for imitation equilibrium and a deviation of a success leader from it. The deviation leads from the candidate to a new strategy combination, referred to as the “deviation start”. In order to examine whether the candidate is stable or not we have to look at all imitation paths beginning with the deviation start. We call these paths the “deviation paths”. A first stability requirement is that *no deviation path is infinite.* Assume that this is the case. Then each deviation path ends with a destination to which we refer as the “deviation destination” of this path.

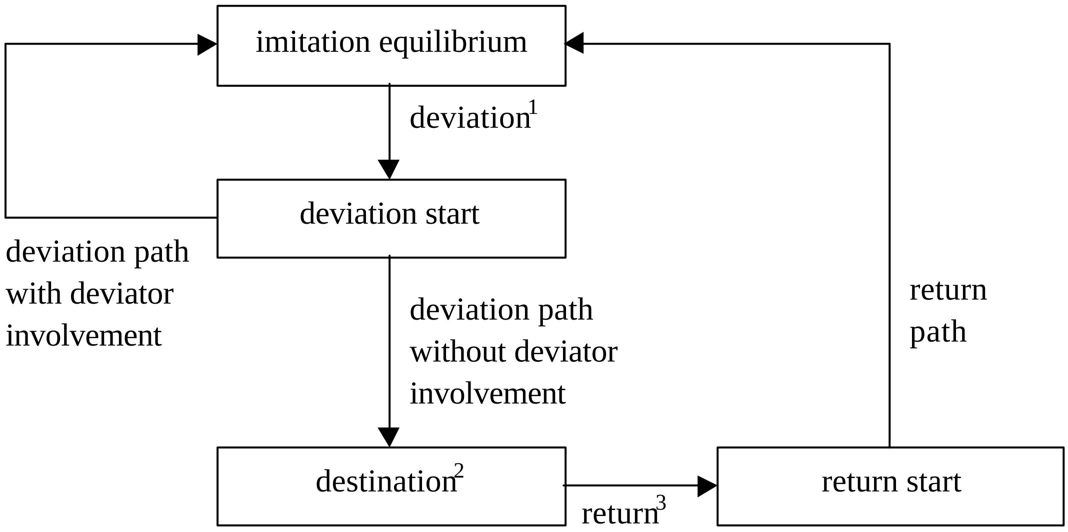
We speak of a deviation path "without deviator involvement" if at the way stations on the path the deviator never has an imitation opportunity. On a deviation path "with deviator involvement" the deviator takes an imitation opportunity at least once.

The concept of an imitation equilibrium is based on the idea that under appropriate circumstances a deviation will be abolished in favor of a return to the old strategy. In the case of a deviation path with deviator involvement the deviation is abolished in favor of an imitation opportunity, and as we see it, the question of a return to the old strategy does not arise any more after this has happened. We may say that imitation supercedes exploration. Therefore a second stability requirement is that *the destination reached by a deviation path with deviator involvement must be the imitation equilibrium.*

Imagine that a destination has been reached on a deviation path without deviator involvement. Suppose that at this destination the deviator's payoff is at least as high as at the imitation equilibrium candidate. Then the deviator is not dissatisfied with the results of its exploratory deviation. It is assumed that in this situation the deviator sticks to its deviation. This has the consequence that the imitation equilibrium candidate is not reached again. Therefore a third stability requirement is that *at every destination reached by a deviation path without deviator involvement the deviator's payoff is lower than at the imitation equilibrium.*

Now suppose that at the end of a deviation path without deviator involvement a destination is reached at which the deviator's payoff is lower than at the imitation equilibrium candidate. It is assumed that in this situation the deviator is dissatisfied with the result of its exploratory deviation and therefore returns to its old strategy in the hope to get its old payoff back. Thereby the destination is changed to a new strategy combination which we call a "return start". If we speak of a "return start" we mean a strategy combination arising in this way from a destination of a deviation path without deviator involvement and with lower payoffs at the destination than at the imitation equilibrium candidate. At a return start the imitation process is set in motion again. An imitation path beginning with a return start is called a "return path". A fourth stability requirement is that *every return path is finite and reaches the imitation equilibrium as its destination.*

The four stability requirements together define stability of an imitation equilibrium with respect to a given deviation. They may be looked upon as conditions on a dynamic process declenched by the deviation. This process generates a sequence of strategy combinations which, however, is not uniquely determined in general. In this sense the process is indeterminate. Figure 1 schematically represents the possibilities which can arise in the case of stability. In this case the process must come back to the imitation equilibrium eventually, either directly by a deviation path or indirectly at the end of a return path.



1) Only deviations of success leaders are considered.
2) The deviator's payoff at the destination must be lower than at the imitation equilibrium.
3) The deviator returns to his old strategy in the imitation equilibrium.

Figure 1: Schematic representation of the possibilities for the dynamic process declenched by a deviation in the case of stability

In order to provide a better overview we now repeat the four stability requirements with convenient names attached to them:

1. *Finiteness requirement:* No deviation path is infinite.

2. *Involvement requirement:* The destination reached by a deviation path with deviator involvement must be the imitation equilibrium.

3. *Payoff requirement:* At every destination reached by a deviation path without deviator involvement the deviator's payoff is lower than at the imitation equilibrium.

4. *Return requirement:* Every return path is finite and reaches the imitation equilibrium as its destination.

An imitation equilibrium must be a destination. This is one of the defining properties of both global and local imitation equilibrium. A global imitation equilibrium is a destination which satisfies the four stability requirements for all possible deviations of success leaders. In the case of a local imitation equilibrium the four requirements are not imposed on all these deviations but only on those which are within an arbitrary chosen small positive distance from the equilibrium.

2.3 Definitions and notation

All definitions refer to a fixed but arbitrary imitation model $M = (G,R)$ as described in 2.1. The *extended reference group* $\bar{R}(i)$ of player i is the union of $R(i)$ and i. For every strategy combination $s = (s_1,...,s_n)$ we distinguish three types of members of the extended reference group. The *costrategists* of i are all players $k \in \bar{R}(i)$ with $s_k = s_i$. The players *comparable* to i are all players $k \in \bar{R}(i)$ with $s_k \in S_i$. Players in $\bar{R}(i)$ which are neither costrategists of nor comparable to i are called *incomparable* to i. The set of all costrategists of i at s is denoted by $C_i(s)$ and the set off all players comparable to i is denoted by $R_i(s)$. A *success example for i at s* is a player $j \in R(i)$ with

$$H_j(s) = \max_{k \in \bar{R}_i(s)} H_k(s) > \max_{k \in C_i(s)} H_k(s).$$

A strategy of a success example j is called an *imitation opportunity of i at s.* The set of all imitation opportunities of i at s is denoted by $I_i(s)$ and is referred to as the *imitation opportunity set of i at s.*

A strategy combination $s = (s_1,...,s_n)$ is called a *way station* if $I_i(s)$ is non-empty for at least one player i and a *destination* if $I_i(s)$ is empty for all players i. If s is a way station, then a *successor station* of s is a strategy combination $u = (u_1,...,u_n)$ such that the following two conditions hold:

$$u_i = s_i \qquad \text{for } I_i(s) = \varnothing,$$

$$u_i \in I_i(s) \qquad \text{for } I_i(s) \neq \varnothing.$$

A *finite imitation path* is a sequence $s^1, ... , s^m$ of strategy combinations such that for $j = 2, ... , m$ the strategy combination s^j is a successor station of s^{j-1} and s^m is a destination. An *infinite imitation path* is an infinite sequence $s^1, s^2, ...$ of strategy combinations such that for $j = 2, 3, ...$ the strategy combination s^j is a successor station of s^{j-1}. An *imitation path* is either a finite or an infinite imitation path. The definition does not exclude the special case of a sequence s^1 starting and ending with a destination.

A *success leader* at a strategy combination s is either a player j whose reference group $R(j)$ is empty or a player j with

$$H_j(s) \geq \max_{k \in R(j)} H_k(s)$$

in the case that $R(j)$ is non-empty. Obviously a player j whose payoff at s is maximal among all payoffs at s must be a success leader. It is also clear that the imitation opportunity set $I_j(s)$ of a success leader at s must be empty, since a success leader cannot have a success example.

Let $s = (s_1,...,s_n)$ be a strategy combination and let (j, t_j) be a pair in which j is a player and t_j is one of j's strategies. The strategy combination resulting from s by replacing its j-th component s_j by t_j and leaving all other components unchanged is denoted by $s/(j, t_j)$:

$s/(j,t_j)=(s_1,\ldots,s_{j-1},t_j,s_{j+1},\ldots,s_n).$

The pair (j, t_j) is a *deviation from s* if t_j is different from s_j. In this case $s/(j, t_j)$ is called the *deviation start* generated by the deviation (j, t_j) from s and an imitation path beginning with $s/(j, t_j)$ is called a *deviation path generated by* the deviation (j, t_j) from s.

The shorter notation s/t_j instead of $s/(j, t_j)$ is often found in the game-theoretic literature. However, in our context it is better to identify the player whose strategy is replaced, since overlaps among strategy sets are of crucial importance.

Let $s^1, \ldots, s^m$ or $s^1, s^2, \ldots$ be a deviation path generated by a deviation (j, t_j) from s. We speak of a deviation path *without deviator involvement* if for all s^k on the path the j-th component is t_j and of a deviation path *with deviator involvement* if this is not the case for at least one s^k on the path.

If $s^1, \ldots, s^m$ is a finite deviation path generated by the deviation (j, t_j) from s, then the destination s^m is called *reached by* (j, t_j) *from s*. We say that s^m is reached from s *with* or *without deviator involvement* if $s^1, \ldots, s^m$ is a deviation path with or without deviator involvement, respectively. The set of all destinations reached by (j, t_j) from s with deviator involvement is denoted by $D_j(s, t_j)$. Similarly, $D_{-j}(s, t_j)$ stands for the set of all destinations reached by (j, t_j) from s without deviator involvement. $D(j, t_j)$ is the set of all destinations reached by (j, t_j) from s. For $u \in D_{-j}(s, t_j)$ the strategy combination $u/(j, s_j)$ is called the *return start after u* and an imitation path generated by $u/(j, s_j)$ is called a *return path after u*.

We now have formally introduced all the auxiliary definitions needed for a formal restatement of the four stability requirements loosely explained in the previous section. In the following it should be kept in mind that only deviations of success leaders are considered, even if the definition of stability against a deviation is more general.

A destination $s=(s_1,\ldots,s_n)$ is *stable against* the deviation (j, t_j) from s if the following four *stability requirements* are satisfied:

Finiteness requirement: Every deviation path generated by the deviation (j, t_j) from s is finite.

Involvement requirement: $D_j(s, t_j) \subseteq \{s\}$.

Payoff requirement: $H_j(u) < H_j(s)$ *for every* $u \in D_{-j}(s, t_j)$.

Return requirement: For every $u \in D_{-j}(s, t_j)$ the return path after u is finite and has s as its destination.

A strategy combination $s=(s_1,\ldots,s_n)$ is a *global imitation equilibrium* if it is a destination which for every success leader j at s is stable against all deviations (j, t_j) from s.

A strategy combination $s=(s_1,\ldots,s_n)$ is a *local imitation equilibrium* if it is a destination and if a positive number $\varepsilon > 0$ exists such that for every success leader j at s the destination s is stable against all deviations (j, t_j) with $|t_j - s_j| < \bar{e}$, where $|t_j - s_j|$ denotes the Euclidean distance between s_j and t_j.

3. Application to the symmetric linear Cournot model

3.1 The model

The symmetric Cournot oligopoly has the structure of an n-person game with the oligopolists i = 1, ..., n as players and profits as payoffs. The strategy set S_i of player i is the set of all real numbers x_i with x_i 0. We will use the following symbols:

x_i	supply of oligopolist i,
x	total supply,
p	price,
c	constant unit costs,
H_i	profits.

The variables are related to each other as follows:

$$x = x_1 + \ldots + x_n,$$

$$p = \begin{cases} b - ax & \text{for } x \leq b/a \\ 0 & \text{else,} \end{cases}$$

$$H_i = (p - c)x_i \quad \text{for } i = 1,\ldots,n.$$

The parameters a, b and c are positive constants with

$$b > c.$$

It can be seen immediately that this *profitability condition* is necessary for the possibility of positive profits.

We investigate an imitation model (G, R) which combines the Cournot oligopoly G with the universal reference structure R. As has been explained in the introduction the *universal reference structure* assigns the set of all other players to each player.

3.2 Cournot equilibrium

In Cournot equilibrium, quantities, prices and profits are as follows:

$$x_i = \frac{b-c}{a(n+1)} \text{ for } i = 1,\ldots, n,$$

$$p = c + \frac{b-c}{n+1},$$

$$H_i = \frac{1}{a}\left(\frac{b-c}{n+1}\right)^2.$$

The derivation of these formulas is elementary and will not be presented here.

3.3 The imitation equilibrium

It will be shown that the symmetric linear Cournot oligopoly has a uniquely determined local imitation equilibrium, namely the strategy combination

$$s^* = (x_0, \ldots, x_0)$$

in which every oligopolist offers the same quantity

$$x_0 = \frac{b-c}{an}.$$

At this strategy combination price equals unit costs and all profits are zero. The uniquely determined local imitation equilibrium is also a global imitation equilibrium.

Lemma 1*:* s^* is a global imitation equilibrium.

Proof: Obviously s^* is a destination. All players are success leaders. Assume that player j deviates to a quantity $x_+ > x_0$. This leads to a deviation start $s^*/(j, x_+)$ with a price smaller than c. There j's profit is smaller than that of all other players, since j supplies more than they do and unit profits are negative. Player j is induced to imitate one of them and the imitation path immediately leads back to s^*.

Now assume that player j deviates to a quantity $x_- < x_0$. This leads to a price greater than c. At the deviation start $s^*/(j, x_-)$ player j's profit is smaller than that of the other players since j supplies less than they do and unit profits are positive. Here, too, the deviation path immediately leads back to s^*.

Lemma 2: s^* is the only local imitation equilibrium.

Proof: We distinguish three possible cases concerning the relationship of the price p to the unit cost c.

(1) $p > c$,

(2) $p < c$,

(3) $p = c$.

Consider a strategy combination $s = (x_1,..., x_n)$ with $p > c$ or $p < c$. It is clear that s is a destination if and only if all x_i are equal. In this case all profits are equal and no player has any imitation opportunities, whereas otherwise profits are unequal and at least one player has an imitation opportunity. This is different in case (3) in which players with different quantities have the same profit zero.

Case (1): In this case a local imitation equilibrium must have the form

$$s = (y,..., y)$$

in which every oligopolist supplies the same amount y with

$$0 \leq y < \frac{b-c}{an} \quad .$$

Obviously all players are success leaders at s. Arbitrarily near to y a number $y_+ > y$ can be found such that

$$y_+ + (n-1)y < \frac{b-c}{a}$$

holds. If s is a local imitation equilibrium then it must be stable against a deviation (j, y_+) with y_+ sufficiently near to y. At $s/(j, y_+)$ player j's profit is greater than that of the other players. He is imitated by all of them and this leads to the new destination $s_+ = (y_+,..., y_+)$ reached without deviator involvement. There j earns less than at s and therefore returns to y. At the return start $s_+/(j, y)$ player j's profit is lower than that of the others. He imitates one of them. Thereby s_+ is reached. The return path does not end in s but in s_+. Therefore s is not stable against (j, y_+). Consequently s fails to be a local imitation equilibrium.

Case (2): In this case a local imitation equilibrium must have the form

$$s = (y,..., y)$$

with

$$y > \frac{b-c}{an} \quad .$$

All players are success leaders at s. Arbitrarily near to y a number $y_- < y$ can be found such that

$$y_- + (n-1)y > \frac{b-c}{a}$$

holds. If s is a local imitation equilibrium then it must be stable against a deviation (j, y_-) with y_- sufficiently close to y. At $s/(j, y_-)$ player j's profit is greater than that of all other players, since unit profits are negative and j supplies less than the others. All other players imitate j and thereby the deviation path ends at $s_- = (y_-,..., y_-)$. There player *j* earns more than at *s*. The payoff requirement is violated. It follows that *s* is not a local imitation equilibrium.

Case (3): In this case a local imitation equilibrium *s* different from s^* must be a strategy combination

$$s = (x_1,..., x_n)$$

with different supplies for at least two players. Let $\overline{x}$ be the maximal and $\underline{x}$ be the minimal supply in *s*. In view of $p = c$ all players have the same profit zero and all of them are success leaders. Let *j* be a player with $x_j = \overline{x}$. Consider a deviation (j, x_+) with $x_+ > 0$ from *s*. If *s* is a local imitation equilibrium, then *s* must be stable against all deviations of this kind with x_+ sufficiently near to $\overline{x}$. At $s/(j, x_+)$ the price is lower than *c*. Therefore the players *k* with $x_k = \underline{x}$ have the highest profit there. For all other players $\underline{x}$ is an imitation opportunity. The deviation path leads to the new destination $\underline{s} = (\underline{x},..., \underline{x})$ with deviator involvement. Contrary to the involvement requirement the destination *s* is not reached. Consequently *s* cannot be a local imitation equilibrium.

Theorem 1: The symmetric linear Cournot oligopoly (as described in this section) combined with the universal reference structure has a uniquely determined local imitation equilibrium s^*. At s^* price equals average costs and each oligopolist has the same supply $(b-c)/an$. Moreover s^* is also a global imitation equilibrium.

Proof: The assertion is an immediate consequence of lemma 1 and lemma 2.

Comment: One may think that the interaction of imitation and exploration modeled in this paper generally drives price down to the level of perfect competition. However, as we will see, this is not the case. The result obtained in this section crucially depends on the symmetry of the situation.

4. The linear Cournot duopoly with different costs

4.1The model

The asymmetric linear Cournot duopoly is similar to the model treated in the previous section. There are only two competitors but their constant unit costs are different. They are *c* for duopolist 1 and $c + h$ for duopolist 2 where *h* is a positive constant. It is convenient to set up the model in terms of player 1's unit profits $g = p - c$ instead of price. This can be done by inserting $g + c$ for *p* in the demand equation. We then use the freedom to fix the quantity unit and the money unit in such a way that the negative slope and the intercept become 1. With these normalizations the linear Cournot duopoly takes the following form:

$x = x_1 + x_2,$

$$g = \begin{cases} 1 - x & \text{for } 1 - x \geq -c \\ 0 & \text{else,} \end{cases}$$

$H_1 = gx_1$

$H_2 = (g - h)x_2$

$0 < h < \frac{1}{2}$

Symbols:

x_i supply of duopolist *i*,
x total supply,
g duopolist 1's unit profit,
h cost difference.

The inequality $1 - x \geq -c$ replaces $p \geq 0$. It can be seen immediately that $g - h$ is duopolist 2's unit profit. h is constrained to the interval $0 < h < \frac{1}{2}$, since this leads to a situation with an internal Cournot equilibrium (see below). We assume the universal reference structure.

4.2 The Cournot equilibrium

It can be seen without difficulty that the Cournot equilibrium of the model is as follows:

$$x_1 = \frac{1+h}{3},$$

$$x_2 = \frac{1-2h}{3},$$

$$x = \frac{2-h}{3},$$

$$g = \frac{1+h}{3},$$

$$H_1 = \left(\frac{1+h}{3}\right)^2,$$

$$H_2 = \left(\frac{1-2h}{3}\right)^2.$$

Obviously H_1 increases and H_2 decreases with h in the interval $0 \leq h \leq \frac{1}{2}$. The formulas are not valid for $h > \frac{1}{2}$. There we have $x_1 = \frac{1}{2}$ and $x_2 = 0$. It can be seen that $h < \frac{1}{2}$ is necessary and sufficient for the existence of an internal Cournot equilibrium.

4.3 Local imitation equilibrium

Lemma 3: The strategy combination $\left(\frac{1}{4},\frac{1}{4}\right)$ is a local imitation equilibrium.

Proof: Consider a strategy combination $(x_0\,,x_0)$ with $0 \le x_0 \le \frac{1}{2}$. Obviously $(x_0\,,x_0)$ is a destination. 1's profit at $(x_0\,,x_0)$ is

$$H_1 = f(x_0) = x_0(1-2x_0).$$

In view of

$$\frac{df(x_0)}{dx_0} = 1-4x_0$$

the function $f(x_0)$ has a maximum at $x_0 = \frac{1}{4}$. At $\left(\frac{1}{4},\frac{1}{4}\right)$ player 1's profit is greater than that of player 2. Therefore 1 is a success leader there and 2 is not. For every x_0 sufficiently near to $\frac{1}{4}$ we still have $H_1 > H_2$ at $\left(x_0,\frac{1}{4}\right)$. Therefore, at this strategy combination player 2 has an imitation opportunity. The deviation path starting with $\left(x_0,\frac{1}{4}\right)$ immediately reaches $(x_0\,,x_0)$ without deviator involvement. Player 1's payoff at $(x_0\,,x_0)$ is lower than at $\left(\frac{1}{4},\frac{1}{4}\right)$, since $f(x_0)$ has its maximum at $x_0 = \frac{1}{4}$. Therefore player 1, the deviator, returns to $x_1 = \frac{1}{4}$. At $\left(\frac{1}{4},x_0\right)$ player 1's payoff is greater than that of player 2, since x_0 is smaller than $\frac{1}{4}$. Therefore player 2 has an imitation opportunity at $\left(\frac{1}{4},x_0\right)$ and the return path immediately leads back to $\left(\frac{1}{4},\frac{1}{4}\right)$. This shows that $\left(\frac{1}{4},\frac{1}{4}\right)$ is a local imitation equilibrium.

Comment: We want to show that $\left(\frac{1}{4},\frac{1}{4}\right)$ is the only local imitation equilibrium. Obviously every strategy combination $(x_0\,,x_0)$ is a destination. However, there are other strategy combinations with this property, namely those with $H_1 = H_2$. The following lemma serves to exclude the possibility that one of them is a local imitation equilibrium.

Lemma 4: Let (x_1,x_2) be a strategy combination with

$$H_1(x_1,x_2) = H_2(x_1,x_2).$$

Then (x_1,x_2) is not a local imitation equilibrium.

We first look at the special case $g = 0$. In this case we have $H_1 - H_2 = hx_2 = 0$ and therefore $x_2 = 0.$ In view of $g = 1-x$ and $x = x_1$ this yields $x_1 = 1.$ We now show that (1,0) is not a local imitation equilibrium. Player 1's payoff at (1,0) is zero. Suppose that player 1 deviates to a supply $x_1 > 1$ arbitrarily near to 1. At $(x_1,0)$ player 1's payoff is negative. However, player 2 has zero profits and therefore is a success example for player 1. 1 imitates 2 and the deviation path reaches (0,0) with deviator involvement. This shows that (1,0) is not a local imitation equilibrium.

Now assume $g \ne 0$. We first look at he special case $x_1 = x_2 = 0$. At the destination (0,0) both players have zero profits. Player 1 may deviate to an arbitrarily small e. For sufficiently small e player 1's profit at $(e,0)$ is positive whereas that of player 2 is zero. Therefore player 2 imitates player 1 at $(e,0)$ and the new destination (e,e) is reached without deviator involvement. There both profits are positive, contrary to the payoff requirement. This shows that (0,0) is not an imitation equilibrium.

We now assume $g>0$ and $x_1>0$. Obviously $H_1=H_2$ implies $g>h$ and $x_2>x_1$. Suppose that player 1 deviates to a supply $x_+>x_1$ with $2x_+<x_1+x_2$. Arbitrarily near to x_1 such an x_+ can be found. In view of

$$\frac{\partial(H_1-H_2)}{\partial x_1}=g-x_1+x_2>0$$

player 1's profit at (x_+,x_2) is greater than that of player 2 for x_+ sufficiently near to x_1. Player 2 imitates player 1 at this deviation start and the new destination (x_+,x_+) is reached without deviator involvement. In view of $2x_+<x_1+x_2$ unit profits are greater than at (x_1,x_2), contrary to the payoff requirement. It follows that (x_1,x_2) is not a local imitation equilibrium.

It remains to examine the case $g<0$ and $x_1>0$. Let (x_1,x_2) be a strategy combination with these properties and with $H_1=H_2$. Obviously we must have $x_2>0$ and $x_1>x_2$. Suppose that player 1 deviates to an x_+ with $x_+>x_1$. In view of the fact that $\partial(H_1-H_2)/\partial x_1$ is negative, player 1's profit is lower than that of player 2 at (x_+,x_2) if x_+ is sufficiently near to x_1. At this deviation start player 1 imitates player 2 and the new destination (x_2,x_2) is reached with deviator involvement. Contrary to the involvement requirement the process does not lead back to (x_1,x_2). Therefore (x_1,x_2) is not a local imitation equilibrium. This completes the proof.

Lemma 5: Let $x_0\geq 0$ be a number with $x_0\neq\frac{1}{4}$. Then the strategy combination (x_0,x_0) is not a local imitation equilibrium.

Proof: As we have seen in the proof of lemma 3 player 1's profit $H_1(x_0,x_0)$ has its maximum at $x_0=\frac{1}{4}$. Moreover, the derivative of $H_1(x_0,x_0)$ with respect to x_0 is positive for $x_0<\frac{1}{4}$ and negative for $x_0>\frac{1}{4}$. In addition to this we have

$$H_1(x_0,x_0)-H_2(x_0,x_0)=hx_0.$$

By lemma 3 the strategy combination (0,0) is not a local imitation equilibrium, since both profits are equal there. We can assume $x_0>0$. For $x_0>0$ player 1's payoff is always greater than that of player 2 in a sufficiently small neighborhood of (x_0,x_0). Therefore at (x_0,x_0) player 1 may deviate to some x_1 between x_0 and $\frac{1}{4}$, but sufficiently near to x_0. At (x_1,x_0) player 1's payoff is higher than that of player 2 and player 2 imitates player 1. Thereby the new destination (x_1,x_1) is reached with deviator involvement. At (x_1,x_1) player 1 has a higher payoff than at (x_0,x_0). Therefore (x_0,x_0) is not a local imitation equilibrium.

Theorem 2: The asymmetric linear Cournot oligopoly as described in this section has one and only one local imitation equilibrium, namely the strategy combination $\left(\frac{1}{4},\frac{1}{4}\right)$.

Proof: A destination must either be of the form (x_0,x_0) or it must have the property $H_1=H_2$. Therefore the theorem follows by lemmata 3, 4, and 5.

Comment: At the uniquely determined local imitation equilibrium the price, unit costs plus unit profits, is the same one as the monopoly price player 1 would take if player 1 were alone in the market. The Cournot equilibrium unit profit $(1+h)/3$ is smaller than $\frac{1}{2}$ in the interval $0<c<\frac{1}{2}$. We may say that in the case of the asymmetric linear Cournot oligopoly, local imitation equilibrium does not drive prices down to average costs like in the symmetric case, but rather up to a quasi-monopoly level. The fact that this holds for even very small cost differences means that there is a sharp discontinuity with respect to this parameter.

We will now turn our attention to the question under which circumstances the uniquely determined local imitation equilibrium is also a global one. As we will see this is not generally true. A global imitation equilibrium does not exist if the cost difference is too small. The following theorem shows that the dividing line between existence and non-existence of global imitation equilibrium is at the cost difference $h=1-\sqrt{3/4}$. This is approximately equal to .134.

Theorem 3: The uniquely determined local imitation equilibrium of the asymmetric linear Cournot oligopoly as described in this section is also a global imitation equilibrium if the cost difference parameter *h* satisfies the inequality

$$h>1-\sqrt{\tfrac{3}{4}}.$$

Otherwise no global imitation equilibrium exists.

Proof: We first show that the local imitation equilibrium is a global one if the condition is satisfied. Suppose player 1 deviates from $\left(\frac{1}{4},\frac{1}{4}\right)$ to a supply x_1 with $x_1 \neq \frac{1}{4}$. We distinguish three cases:

(1) $H_1\left(x_1,\frac{1}{4}\right)<H_2\left(x_1,\frac{1}{4}\right)$

(2) $H_1\left(x_1,\frac{1}{4}\right)=H_2\left(x_1,\frac{1}{4}\right)$

(3) $H_1\left(x_1,\frac{1}{4}\right)>H_2\left(x_1,\frac{1}{4}\right)$

In case (1) player 1 imitates player 2 and thereby the local imitation equilibrium is reached again. Obviously case (1) does not pose any difficulties. We now turn our attention to case (2). Here a destination is reached by the deviation. We have to show that at this destination player 1 has a lower payoff than at the local imitation equilibrium. An easy calculation shows

$$H_2\left(x_1,\tfrac{1}{4}\right)-H_1\left(x_1,\tfrac{1}{4}\right)=x_1^{\,2}-x_1+\tfrac{3}{16}-\tfrac{1}{4}h.$$

If the expression on the right hand side vanishes we have

$$x_1^{\,2}-x_1=-\tfrac{3}{16}+\tfrac{1}{4}h$$

and therefore

$$H_1 = x_1\left(1 - x_1 - \tfrac{1}{4}\right) = -\tfrac{3}{16} + \tfrac{1}{4}h - \tfrac{1}{4}x_1 .$$

In view of $h < \frac{1}{2}$ this shows that H_1 must be negative. At the local imitation equilibrium player 1's payoff is $\frac{1}{8}$. Therefore player 1 returns to the local imitation equilibrium. The local imitation equilibrium is stable against deviations of this kind.

Now consider case (3). The expression for $H_2\left(x_1, \frac{1}{4}\right) - H_1\left(x_1, \frac{1}{4}\right)$ derived above shows that a deviation $x_1 \neq \frac{1}{4}$ fitting this case must have the property

$$x_1^{\,2} - x_1 + \tfrac{3}{16} - \tfrac{1}{4}h < 0 .$$

In this case player 2 imitates player 1. This leads to the destination (x_1, x_1). As we have seen in the proof of lemma 3, player 1's maximal payoff for destination (x_0, x_0) is reached at $x_0 = \frac{1}{4}$. Therefore at (x_1, x_1) player 1's payoff is lower than at $\left(\frac{1}{4}, \frac{1}{4}\right)$. Therefore player 1 returns to its strategy $\frac{1}{4}$ in the local equilibrium. It is now important what happens at $\left(\frac{1}{4}, x_1\right)$. Player 2 will imitate player 1 if we have

$$H_1\left(\tfrac{1}{4}, x_1\right) - H_2\left(\tfrac{1}{4}, x_1\right) > 0 .$$

This is equivalent to

$$x_1^{\,2} - (1-h)x_1 + \tfrac{3}{16} > 0 .$$

The expression on the left hand side has its minimum at

$$x_1 = \frac{1-h}{2} .$$

For this deviation the expression has the value

$$\left(\frac{1-h}{2}\right)^2 - (1-h)\frac{1-h}{2} + \frac{3}{16} = -\frac{(1-h)^2}{4} + \frac{3}{16} .$$

Player 2 always imitates player 1 at $\left(\frac{1}{4}, x_1\right)$ if this value is positive. This is the case for

$$h > 1 - \sqrt{\tfrac{3}{4}} .$$

If *h* satisfies the above inequality, then player 2 imitates player 1. Thereby $\left(\frac{1}{4}, \frac{1}{4}\right)$ is reached again. It is now clear that the local imitation equilibrium is also a global one if the condition on *h* is satisfied.

It remains to show that for

$$0 < h \leq 1 - \sqrt{\tfrac{3}{4}}$$

the local imitation equilibrium fails to be a global one. Assume that this inequality holds and suppose that player 1 deviates to

$$x_1 = \frac{1-h}{2} \ .$$

In this case we obtain

$$H_2\left(x_1, \tfrac{1}{4}\right) - H_1\left(x_1, \tfrac{1}{4}\right) = \left(\frac{1-h}{2}\right)^2 - \frac{1-h}{2} + \frac{3}{16} - \frac{h}{4} = \frac{h^2 - h}{4} - \frac{1}{16} \ .$$

In view of $h < \frac{1}{2}$ the last expression is negative. This means that at $\left(x_1, \tfrac{1}{4}\right)$ player 1 earns more than player 2, and player 2 imitates player 1. Again, at (x_1, x_1) player 1's payoff is lower than at $\left(\tfrac{1}{4}, \tfrac{1}{4}\right)$. Therefore player 1 returns to its strategy $\frac{1}{4}$, but now we have

$$H_1\left(\tfrac{1}{4}, x_1\right) - H_2\left(\tfrac{1}{4}, x_1\right) = -\frac{(1-h)^2}{4} + \frac{3}{16} \le 0 \ .$$

This means that either $\left(\tfrac{1}{4}, x_1\right)$ is a destination or player 1 imitates player 2 and (x_1, x_1) is reached with deviator involvement. In both cases the local imitation equilibrium is not reached again. This shows that $\left(\tfrac{1}{4}, \tfrac{1}{4}\right)$ is not stable against the deviation $x_1 = (1-h)/2$. Consequently the local imitation equilibrium is not a global one if h is not greater than the bound $1-\sqrt{3/4}$. Since every global equilibrium must be a local one, there cannot be any other global equilibrium. This completes the proof.

5. Mill price competition on the circle

5.1 The model

Imagine a circular island settled only along he coast line with an insurmountable mountain in the middle. There are *n* producers $1,...,n$ equidistantly located on a circular coastal road $(n \ge 2)$. The distance unit is chosen in such a way that the distance between two adjacent suppliers is 1. All suppliers have the same unit cost *c*. There are constant unit transport costs *t*.

For the sake of simplicity demand is assumed to be completely inelastic below a maximum price $\bar{p}$. This willingness to pay includes transport costs. All transport costs are carried by the customers. A customer buys as cheaply as possible including transport costs, provided this can be done without surpassing the willingness to pay $\bar{p}$. Demand is evenly distributed along the circular road.

It is convenient to set up the model in terms of unit profits instead of prices. We do not permit prices below costs. Each supplier *i* chooses a unit profit $g_i \ge 0$. This means that the strategy set of a player *i* is the set of all non-negative numbers.

Symbols

g_i	player i's unit profit, $i=1,...,n$.
c	unit costs.
v	road coordinate $0 \leq v \leq n$. For $i=1,..., n$ player i is located at $v=i$.
$g(v)$	local price (minus c).
$\overline{g}$	maximum local price (minus c).
t	transport costs.
M_{im}	set of all v served by i and $m-1$ others, $i=1,...,n$, $m=1,...,n$.
L_{im}	total length of the road segments in M_{im}, $i=1,...,n$, $m=1,...,n$.
L_i	total demand of i's product, $i=1,...,n$.
x_i	road segment on the right of i served by i alone, $i=1,...,n$.
y_i	road segment on the left of i served by i alone, $i=1,...,n$.

5.1.1 Local price, demand and payoff

The distance $|v-i|$ is to be understood as the distance on the road, not necessarily as the absolute value of $v-i$ if travelling from i to v to the left is shorter. The local price is the price including transport costs paid by a customer at location v if he or she buys anything at all. Since our model

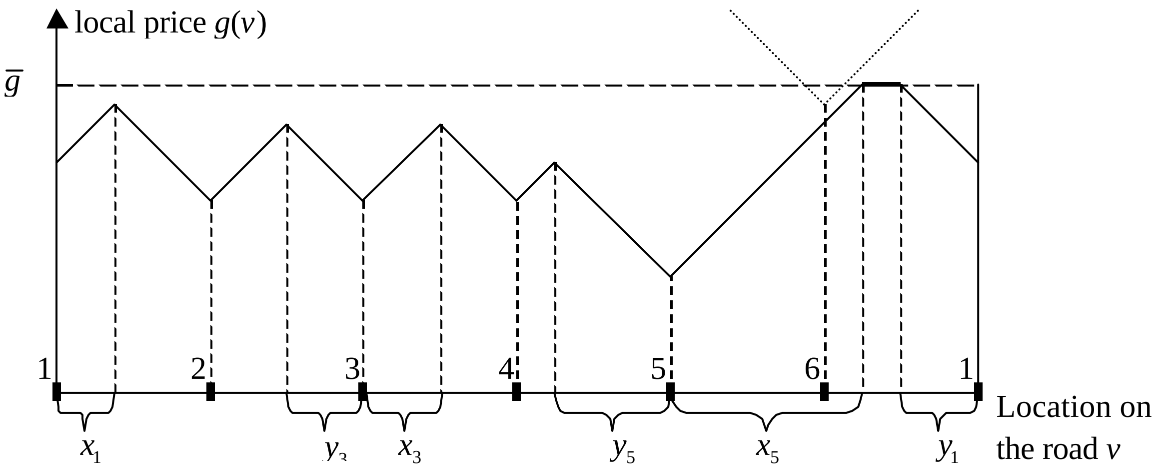


Figure 2: The determination of local price minus c

is set up in terms of unit profits rather than price, we define local price minus c as follows:

$$g(v)=\min\left[\overline{g}, \min_{i=1,...,n} g_i+|v-i|t\right].$$

In the following we will simply speak of *local price*. The qualification “minus c” will be omitted for the sake of brevity. We say that a player i *serves* v if we have

$$g_i + |v-i|t = g(v)\ .$$

M_{im} is the set of all points served by i together with $m-1$ other players. It can be seen without difficulty that M_{im} is a finite collection of line segments. Let L_{im} be the total length of line segments in M_{im}. For $i=1,...,m$ the *demand for i’s product* is defined as follows:

$$L_i = \sum_{m=1}^{n} \frac{1}{m} L_{im} \quad \text{for } i=1,...,n\ .$$

Player i’s payoff is

$$H_i = g_i L_i \qquad \text{for } i=1,...,n\ .$$

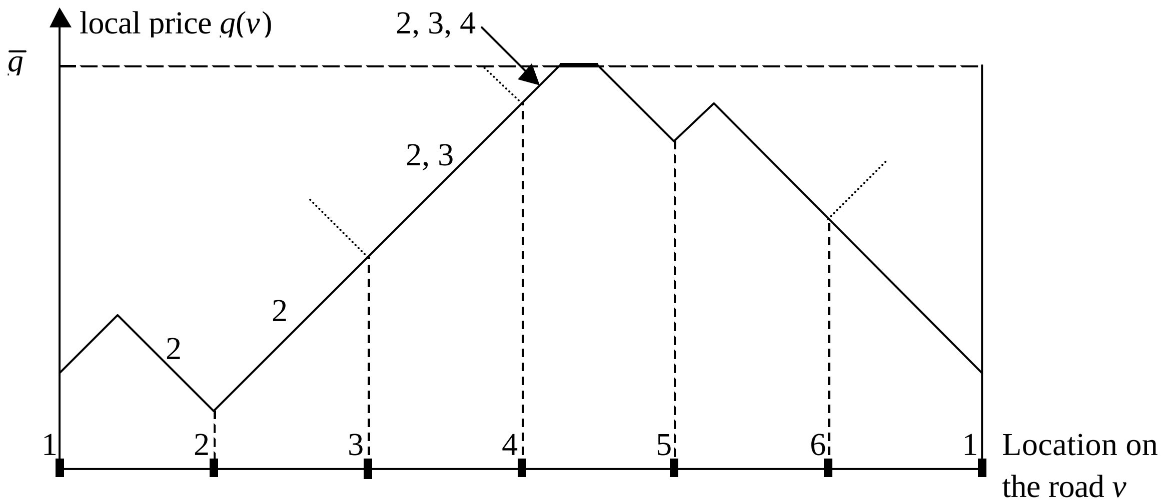


Figure 3: A special situation.

The determination of the local price and the demand for i’s product is illustrated by Figure 2. Usually there are only finitely many points served by more than one player and their total length is zero. However, special situations may arise like in Figure 3 where player 2’s payoff must be computed as follows:

$L_{21} = \frac{5}{3}$

$L_{22} = 1$

$L_{23} = \frac{1}{4}$

and therefore

$L_2 = \frac{5}{3} + \frac{1}{2} + \frac{1}{12} = 2\frac{1}{4}$.

5.1.2 A condition on the maximum local price

It will be assumed that

$\bar{g} > 2\frac{1}{3}t$

holds for the maximal local price. As we will see, this condition makes $\bar{g}$ high enough not to matter in equilibrium.

5.1.3 The reference structure

We will consider two reference structures, a *narrow* reference structure R_1 and a *wider* reference structure R_2. For $n \geq 3$ the narrow reference group $R_1(i)$ consists of player *i*'s left and right neighbor and the wider reference group $R_2(i)$ consists of player *i*'s two left and two right neighbors. For $n = 2$ both reference structures collapse to the universal one. This is true also for $n = 3$ and in the case of R_2 even for $n = 4$ and $n = 5$. Only for $n > 5$ both reference structures are different from the universal one.

The results are the same for R_1 and R_2, but the proofs are slightly different.

5.1.4 Payoffs for regular and semi-regular strategy combinations

Player indices involving addition and subtraction like $i+1$ and $i-1$ are to be understood modulo *n*. This means that for $i = n$ the index $i+1$ is interpreted as 1 and for $i = 1$ the index $i-1$ as *n*.

A strategy combination

$s = (g_1, ..., g_n)$

is called *semi-regular* if we have

$|g_{i+1} - g_i| < t \quad$ for $i = 1, ..., n$

and *regular* if in addition to this the following is true:

$g(v) < \bar{g} \quad$ for $0 < v \leq n$.

The inequality defining semi-regularity has the consequence that no player serves the customers at the location of another. If there is a region with $g(v) = \overline{g}$, then no customer inside this region is served by anyone.

Assume that $s = (g_1, ..., g_n)$ is regular. In this case each player *i* serves a road segment of length x_i to the right of *i* and another one of length y_i to the left of *i*. Apart from the *n* points at which the regions served by two neighbors touch, there are no regions served by at least two players. Therefore we have

$$L_i = x_i + y_i$$

and accordingly

$$H_i = g_i (x_i + y_i)$$

for $i = 1, ..., n$. The numbers x_i and y_i are determined as follows:

$$g_i + tx_i = g_{i+1} + t(1 - x_i)$$

$$g_i + ty_i = g_{i-1} + t(1 - y_i).$$

This is equivalent to

$$2tx_i = t + g_{i+1} - g_i$$

$$2ty_i = t + g_{i-1} - g_i \ .$$

We obtain

$$x_i = \frac{1}{2} + \frac{1}{2t}(g_{i+1} - g_i)$$

$$y_i = \frac{1}{2} + \frac{1}{2t}(g_{i-1} - g_i).$$

In view of $|g_i - g_i| < t$ it is clear that x_i and y_i are positive numbers smaller than 1. Addition of both equations yields:

$$x_i + y_i = 1 + \frac{1}{t}\left(\frac{g_{i+1} + g_{i-1}}{2} - g_i\right)$$

and therefore

$$H_i = g_i\left[1 + \frac{1}{t}\left(\frac{g_{i+1} + g_{i-1}}{2} - g_i\right)\right]$$

for $i = 1, ..., n$. This payoff formula holds for regular strategy combinations, but not necessarily for other ones.

Now suppose that $s=(g_1,\ldots,g_n)$ is semi-regular, but not regular. In this case there are still no regions of positive length served by at least two players, but there may be regions served by nobody. It may happen that the region served by i to the right of i or to the left of i is determined by the condition that the cost of buying at i should not be higher than $\bar{g}$. This means that x_i and y_i cannot surpass the numbers determined by

$$g_i + tx_i = \bar{g}$$

$$g_i + ty_i = \bar{g}.$$

This is equivalent to

$$x_i = \frac{1}{t}(\bar{g} - g_i)$$

$$y_i = \frac{1}{t}(\bar{g} - g_i).$$

We obtain x_i as the minimum of this value and the one determined earlier. The same is true for y_i:

$$x_i = \min\left[\frac{1}{2} + \frac{1}{2t}(g_{i+1} - g_i),\ \frac{1}{t}(\bar{g} - g_i)\right]$$

$$y_i = \min\left[\frac{1}{2} + \frac{1}{2t}(g_{i-1} - g_i),\ \frac{1}{t}(\bar{g} - g_i)\right].$$

As before we have

$$H_i = g_i(x_i + y_i)$$

for $i=1,\ldots,n$. If we apply the payoff formulas for the regular case to a semi-regular strategy combination we may obtain a payoff that is too high, but never one that is too low. This fact will be important later.

5.2 Cournot equilibrium

If we talk of *Cournot equilibrium,* we mean a Nash equilibrium in pure strategies of an oligopoly model which is a normal form game. In this sense the mill price competition oligopoly as described above has a uniquely determined Cournot equilibrium $s^* = (t,\ldots,t)$ at which every player i chooses $g_i = t$. Even though this is well known in the literature (e. g. Beckmann 1968) it may nevertheless be useful to provide a more rigorous proof. The problem is not as easy as it may seem to be if one restricts one's attention to regular strategy combinations.

Lemma 6: Let $s=(g_1,\ldots,g_n)$ be a strategy combination which is not semi-regular. Then s is not a Cournot equilibrium.

Proof: It can be seen without difficulty that we must have $L_{j1} = 0$ for at least one player *j*. (Examples are players 6 in Figure 2 and players 3 and 4 in Figure 3.) Suppose that we have $L_j = 0$. The local price $g(j)$ at *j* is at least *t*, since the neighbors of *j* have non-negative unit profits. Therefore player *j* can deviate to a strategy $t' < t$ and thereby obtain a positive payoff in $s/(j, t')$. Obviously *s* cannot be a Cournot equilibrium in this case.

Now assume $L_j > 0$ and $L_{j1} = 0$. Then we have $L_{jm} > 0$ for some *m* with $m > 1$. At the points in the regions M_{jm} with $L_{jm} > 0$ player *j* shares its customers with other players. Suppose that player *j* deviates to a strategy $s'_j = s_j - e$ where e is a small positive number. Then in $s/(j, s'_j)$ all these customers will be served by *j* alone. This means that L_j at $s/(j, s'_j)$ is at least twice as high as at *s*. Since e can be arbitrarily small it follows that player *j* can improve its payoff in this way. Therefore *s* is not a Cournot equilibrium.

Lemma 7: Let $s = (g_1, ..., g_n)$ be a strategy combination which is not regular. Then *s* is not a Cournot equilibrium.

Proof: In view of lemma 6 we can restrict our attention to semi-regular strategy combinations. Let *s* be a semi-regular one which is not regular. This means that in the formulas for x_i and y_i at the end of the section on payoffs for regular and semi-regular strategy combinations the second term after the minimum operator is equal to x_i or y_i in at least one case. In other words, for some player *j* either x_i or y_i is equal to $(\bar{g} - g_i)/t$ since one border point of the region served by *i* is determined by the maximal local price $\bar{g}$ rather than the competition of *i*'s neighbor. In view of the semi-regularity of *s* we must have

$$(\bar{g} - g_i)/t < 1.$$

Together with $\bar{g} < 3t$ this yields

$$g_j > \bar{g} - t > 2t \,.$$

We will show that a player *j* of this kind can improve its payoff by a small decrease of g_j and that therefore *s* cannot be a Cournot equilibrium. For this purpose we look at the left partial derivative of H_j with respect to g_j:

$$\left.\frac{\partial H_j}{\partial g_j}\right|_{-} = x_j + y_j + g_j \left(\left.\frac{\partial x_j}{\partial g_j}\right|_{-} + \left.\frac{\partial y_j}{\partial g_j}\right|_{-} \right).$$

It can be seen that the left partial derivative of x_i or y_i is at least $-1/2t$, the derivative of the first term after the minimum operator. Moreover it is a consequence of the semi-regularity of *s* that we have

$$x_j + y_j < 2.$$

Therefore we have

$$\left.\frac{\partial H_j}{\partial g_j}\right|_{-} < 2-\frac{1}{t}2t = 0 .$$

This completes the proof of the lemma.

Theorem 4: For every $n = 2,3,...$ the mill price competition oligopoly as described in this section has exactly one Cournot equilibrium, namely the strategy combination $s^* = (t,\dots,t)$ at which for $i = 1,...,n$ player i chooses $g_i = t$.

Proof: We first show that s^* is the only candidate for a Cournot equilibrium. In view of lemma 7 we can restrict our attention to regular strategy combinations. Assume that $s = (g_1,..., g_n)$ is regular. Player i's payoff function for regular combinations

$$H_i = g_i\left[1+\frac{1}{t}\left(\frac{g_{i+1}+g_{i-1}}{2}\right)-g_i\right]$$

is concave in g_j . We have

$$\frac{\partial H_i}{\partial g_i} = 1+\frac{1}{t}\left(\frac{g_{i+1}+g_{i-1}}{2}\right)-2g_i .$$

At $g_j = 0$ this derivative is positive. For all regular strategy combinations s with $g_j > 0$ for $i = 1,...,n$ a whole ε-neighborhood of s consists of regular strategy combinations. Therefore we must have

$$\frac{\partial H_i}{\partial g_i} = 0 \text{ for } i = 1,...,n$$

at a Cournot equilibrium. These conditions form a linear equation system for the unit profit:

$$2g_i - \frac{g_{i+1}}{2} - \frac{g_{i-1}}{2} = t \text{ for } i = 1,...,n .$$

Obviously the matrix of the system has a dominant diagonal and therefore a non-vanishing determinant. Therefore the system has a unique solution. It can be seen immediately that this solution is

$$g_i = t \text{ for } i = 1,...,n .$$

Accordingly, $s^* = (t,\dots,t)$ is the uniquely determined Cournot equilibrium.

We now show that s^* is a Cournot equilibrium. Let $s^1 = s/(j, t+\varepsilon)$ with $\varepsilon \geq -t$ be a regular strategy combination. Then we have

$$H_j\left(s^1\right) = (t+\varepsilon)\left(1-\frac{\varepsilon}{t}\right) = t-\frac{\varepsilon^2}{t}$$

if s^1 is regular. This follows by the payoff formulas for regular combinations in 5.1.4. In view of $\bar{g} > 3t$ the combination s^1 is regular for $-t < \boldsymbol{e} < t$. However, the formula for $H_j\left(s^1\right)$ above holds for $\boldsymbol{e} = -t$ and $\boldsymbol{e} = +t$, too, since in both cases $H_j\left(s^1\right)$ is equal to zero. It is also zero for $\boldsymbol{e} > t$. Obviously $H_j\left(s^1\right)$ is smaller than $H_j\left(s\right)$ for every $\boldsymbol{e} \neq 0$ with $\boldsymbol{e} \geq -t$. This completes the proof.

5.3 Imitation equilibrium

5.3.1 Preview

A strategy combination $g = \left(g_1,..., g_n\right)$ for the mill price competition oligopoly is called *symmetric* if we have

$$g_i = g_0 \qquad \text{for } i = 1,..., n.$$

A local and global imitation equilibrium is called *symmetric* if it has this property. In this section we restrict our attention to symmetric imitation equilibria. The question whether other ones exist and what they look like seems to be difficult and no attempt will be made to answer it here.

As we will see in 5.3.6, the mill price competition model has a uniquely determined symmetric local imitation equilibrium for $n = 2$ and $n = 3$, namely

$$\left(\frac{t}{2}, \frac{t}{2}\right) \qquad \text{for } n = 2$$

and

$$\left(\frac{2t}{3}, \frac{2t}{3}, \frac{2t}{3}\right) \qquad \text{for } n = 3.$$

For $n > 3$ there is a whole range of symmetric local equilibria, namely

$$\left(g_0,..., g_0\right) \qquad \text{with } \frac{2t}{3} \leq g_0 \leq t \qquad \text{for } n = 4,5,... \ .$$

The range has the imitation equilibrium unit profit as its lower border and the Cournot equilibrium unit profit as its upper border. Surprisingly, competition is more intense than at Cournot equilibrium for $n = 2$ and $n = 3$, whereas for $n > 3$ this is not necessarily the case. There are no other symmetric local imitation equilibria. As we will see in 5.3.6, all of them are also global imitation equilibria.

5.3.2 Payoffs for regular binary strategy combinations

In the following we will often have to look at the question what happens if our player, any player j deviates from a symmetric strategy combination $s = \left(g_0,..., g_0\right)$ to a strategy $g_0 + \boldsymbol{e}$ where $\boldsymbol{e}$ is a not necessarily very small positive or negative number. Such a deviation leads to a finite or infinite deviation sequence entirely made up of strategy combinations in which only the strategies g_0 and $g_0 + e$ occur, since starting from such a combination, no new strategy can come in by imitation. We call such combinations *binary*. It is useful to get an overview over payoffs of binary strategy combinations. We will assume that all strategy combinations considered are regular.

Obviously the payoff of a player *i* at a regular strategy combination depends on the strategies of $i-1$, *i*, and $i+1$ only. Let $s = (g_1,..., g_n)$ be a regular binary strategy combination with $g_i = g_0$ or $g_i = g_0 + e$ for $i = 1,..., n$. We introduce the following notation:

$A_k(g_0, e)$ is the payoff of a player *i* at *s* who plays g_0 and has exactly *k* neighbors who play $g_0 + e$ at *s*. The payoff $A_k(g_0, e)$ is defined for $k = 0,1,2,...$.

$B_m(g_0, e)$ is the payoff of a player *i* at *s* who plays $g_0 + e$ and has exactly *m* neighbors who play $g_0 + e$ at *s*. The payoff $B_m(g_0, e)$ is defined for $m = 0,1,2,...$.

In view of the formula for payoffs at regular strategy combinations derived in 5.1 we have

$$A_k(g_0, e) = g_0\left(1 + \frac{k}{2t}e\right)$$

for $k = 0,1,2,...$ and

$$B_m(g_0, e) = (g_0 + e)\left(1 + \frac{m-2}{2t}e\right)$$

for $m = 0,1,2,...$. The expressions for A_k and B_m can be rewritten as follows:

$$A_k(g_0, e) = g_0 + g_0\frac{k}{2t}e\ ,$$

$$B_m(g_0, e) = g_0 + \left(1 + \frac{(g_0 + e)(m-2)}{2t}\right)e\ .$$

It will often be necessary to look at payoff differences between neighbors playing different strategies. Therefore we introduce the following definition:

$$D_{mk}(g_0, e) = B_m(g_0, e) - A_k(g_0, e)$$

for $k = 0,1,2,...$ and $m = 0,1,2,...$. Obviously D_{mk} can be explicitly expressed as follows:

$$D_{mk}(g_0, e) = \left(1 - \frac{(2-\mathrm{m})(g_0 + e) + k\,g_0}{2t}\right)e\ .$$

5.3.3 Payoffs for deviation starts

Let $s=(g_0,...,g_0)$ be a symmetric strategy combination with $0\le g_0\le t$ and let

$$s^1=s/(j,g_0+\boldsymbol{e})$$

with $\boldsymbol{e}\ge -g_0$ be the deviation start after a deviation of *j* to $g_0+\boldsymbol{e}$. We will look at the payoffs of all players at s^1. It is necessary to distinguish the cases $n=2, n=3,$ and $n>3$. Consider the case $n=2$. Here player *j* is a "double neighbor" of the other player *i* with $i\ne j$, i. e. we have $j-1=j+1=i$. Therefore we have

$$H_j(s^1)=B_0(g_0,\boldsymbol{e})$$

$$H_i(s^1)=A_2(g_0,\boldsymbol{e})$$

if s^1 is regular. It can be seen immediately that in view of $\bar{g}>3t$ these formulas hold also for $\boldsymbol{e}=t$. There player *j* receives zero. Nothing happens if player *j* increases $\boldsymbol{e}$ from *t* to a value above *t*. Payoffs for $\boldsymbol{e}>t$ are the same as for $\boldsymbol{e}=t$. Negative $\boldsymbol{e}$ lead to regular deviation starts s^1 for $g_o<t$ or $\boldsymbol{e}>-t$. In this case both payoffs are zero. These are also the values of B_0 and A_2 for $g_0=t$ and $\boldsymbol{e}=-t$. Therefore the following result holds.

Result for *n* = 2:

$$H_j(s^1)=\begin{cases}B_0(g_0,\boldsymbol{e}) & \text{for } -g_0\le\boldsymbol{e}\le t\\ B_0(g_0,t) & \text{for } \boldsymbol{e}>t\end{cases}$$

$$H_i(s^1)=\begin{cases}A_2(g_0,\boldsymbol{e}) & \text{for } -g_0\le\boldsymbol{e}\le t\\ A_2(g_0,t) & \text{for } \boldsymbol{e}>t\end{cases}$$

with $i\ne j$.

Now consider the case $n=3$. Here players *j* and $j+1$ have only one neighbor playing $g_0+\boldsymbol{e}$. Therefore we have

$$H_j(s^1)=B_0(g_0,\boldsymbol{e})$$

$$H_{j+1}(s^1)=H_{j-1}(s^1)=A_1(g_0,\boldsymbol{e})$$

if s^1 is regular. Everything else is similar to the case $n=2$. These formulas hold for $\boldsymbol{e}=t$, too, and for $\boldsymbol{e}>t$ the payoffs are the same as for $\boldsymbol{e}=t$. At $g_0=-t$ and $\boldsymbol{e}=t$ player *j* receives zero in view of $g_j=0$. However, the road segment between $j+1$ and $j-1$ is still served by player $j+1$, but together with player *j*. The situation of $j-1$ is analogous. Each of the two receives $t/4$ (see Figure 4).

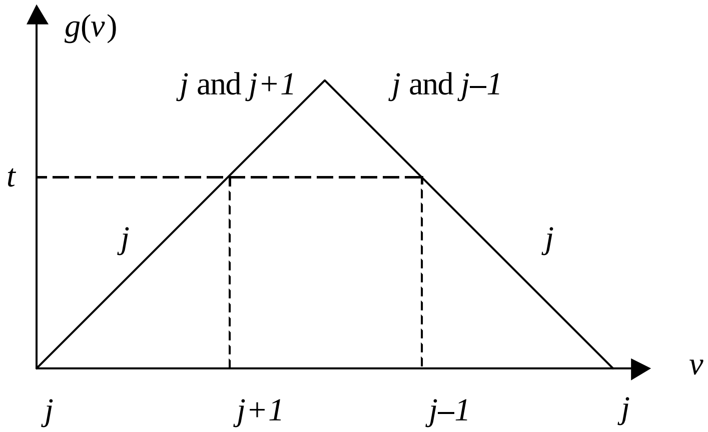


Figure 4: The situation for $g_j = t$ and $\boldsymbol{e} = -t$

Result for *n* = 3:

The payoffs for player *j* are the same as for $n = 2$. The neighbors of *j* have the following payoffs:

$$H_{j+1}\left(s^1\right) = H_{j-1}\left(s^1\right) = \begin{cases} A_1\left(g_0, \boldsymbol{e}\right) & \text{for } g_0 < t \text{ and } -g_0 \le \boldsymbol{e} \le t \\ A_1\left(g_0, \boldsymbol{e}\right) & \text{for } g_0 = t \text{ and } -t < \boldsymbol{e} \le t \\ A_1\left(g_0, t\right) & \text{for } \boldsymbol{e} > t \\ \dfrac{t}{4} & \text{for } g_0 = t \text{ and } \boldsymbol{e} = -t. \end{cases}$$

We now consider the case $n > 3$. It can be seen without difficulty that here the situation for players $j-1$, *j*, and $j+1$ is the same as in the case $n = 3$. However, now there are other players. These other players remain unaffected by the deviation of *j*, since the greatest possible deviation of *j* is $\boldsymbol{e} = -t$.

Result for *n* > 3:

The payoff for *j* is the same as for $n = 2$. The payoffs for *j*'s neighbors $j-1$ and $j+1$ are the same as for $n = 3$. Other players have the same payoff as at *s:*

$$H_i\left(s^1\right) = H_i\left(s\right) = g_0 \quad \text{for } i \notin R(j).$$

5.3.4 Global imitation equilibria

Lemma 8: Let $s = \left(g_0, \ldots, g_0\right)$ be a symmetric strategy combination. *s* is a global imitation equilibrium if we have

$g_0 = \frac{t}{2}$ for $n = 2$

$g_0 = \frac{2t}{3}$ for $n = 3$

$\frac{2t}{3} \le g_0 \le t$ for $n = 4,5,\ldots$.

Proof: Let $s^1 = s/(j, g_0 + e)$ be the deviation start after a deviation of a player *j*. Consider the case $n = 2$. We have

$$H_j\left(s^1\right) - H_{j+1}\left(s^1\right) = \begin{cases} D_{02}\left(\frac{t}{2}, e\right) & \text{for } -g_0 \le e \le t \\ D_{02}\left(\frac{t}{2}, t\right) & \text{for } e > t. \end{cases}$$

The last equation in 5.3.2 yields

$$D_{02}\left(\frac{t}{2}, e\right) = \left(1 - \frac{2t + 2e}{2t}\right)e = -\frac{e^2}{t}.$$

It is clear that therefore any deviation of player *j* leads to a deviation start at which *j*'s payoff is smaller than that of $j+1$. It follows that *j* imitates $j+1$ and the deviation path immediately comes back to *s*. This shows that *s* is a global imitation equilibrium.

Now consider the case $n = 3$. Here we obtain

$$H_j\left(s^1\right) - H_{j+1}\left(s^1\right) = \begin{cases} D_{01}\left(\frac{2t}{3}, e\right) & \text{for } g_0 < t \text{ and } -g_0 \le e \le t \\ D_{01}\left(\frac{2t}{3}, e\right) & \text{for } g_0 = t \text{ and } -t < e \le t \\ D_{01}\left(\frac{2t}{3}, t\right) & \text{for } e > t \\ -\frac{t^2}{4} & \text{for } g_0 = t \text{ and } e = -t. \end{cases}$$

The payoff difference $H_j\left(s^1\right) - H_{j-1}\left(s^1\right)$ has the same value. We have

$$D_{01}\left(\frac{2t}{3}, e\right) = \left(1 - \frac{2t + 2e}{2t}\right)e = -\frac{e^2}{t}.$$

as in the case $n=2$. This shows that $H_j\left(s^1\right)$ is smaller than $H_{j+1}\left(s^1\right)$ and $H_{j-1}\left(s^1\right)$ for every deviation to g_0+e with $e\neq 0$. As in the case $n=2$ we can conclude that *j* imitates one of *j*'s neighbors. Thereby s is reached again. s is a global imitation equilibrium.

In the remainder of the proof we assume $n>3$. Consider a deviation of player *j* to a strategy g_0+e with $e>0$. At s^1 the payoff of *j* may or may not be higher then *j*'s payoff g_0 at *s*, but *j*'s neighbors will always profit from an increase of *j*'s unit profit. Their payoffs will be greater at s^1 than at *s*. They have no reason to imitate $j+2$ or $j-2$ who still receive the same payoff g_0 as at *s*. We have

$$D_{01}\left(g_0,e\right)=\left(1-\frac{3g_0+2e}{2t}\right)e\,.$$

For $g_0\geq 2t/3$ and $e>0$ this is always negative. Therefore at s^1 player *j* imitates one of *j*'s neighbors and thereby the deviation path immediately leads back to *s*.

Now consider a deviation of player *j* to a strategy g_0+e with $e<0$. Here the deviation will decrease the payoffs of *j*'s neighbors and $D_{01}\left(g_0,e\right)$ may be positive. However, player *j* cannot be a success example for *j*'s neighbors unless the payoff $H_i\left(s^1\right)=g_0$ of players *i* other than $j-1,\ j,$ and $j+1$ are lower than $H_j\left(s^1\right)$. We therefore look at

$$H_j\left(s^1\right)-g_0=B_0\left(g_0,e\right)-g_0=\left(1-\frac{g_0+e}{t}\right)e\,.$$

In view of $g_0\leq t$ this is negative for $e<0$. Hence *j*'s payoff at s^1 is smaller than the payoff g_0 of $j+2$ and $j-2$. Accordingly, $j+1$ and $j-1$ do not imitate *j* even if *j*'s payoff is higher, since their other neighbors have a still higher payoff with the strategy g_0. In such cases s^1 is a destination. Player *j* then returns to *j*'s original strategy g_0, since *j* earns less at s^1 than at *s*. Thereby the return path immediately leads back to *s*. It follows that *s* is a global imitation equilibrium.

5.3.5 Exclusion of other local imitation equilibria

A global imitation equilibrium always is a local one, too. We can exclude all further symmetric local or global imitation equilibria by proving that there are no other symmetric local imitation equilibria than those described by lemma 8. We first prove a lemma about the local properties of the payoff differences D_{mk}. Then we will apply this result in order to exclude the possibility of symmetric local imitation equilibria $\left(g_0,\ldots,g_0\right)$ with $g_0=t$. Then we will look at the cases $n=2,\ n=3,$ and $n>3$ separately.

Lemma 9: For $m=0,1$ and $k=0,1,2$ an $e_0>0$ exists for every $g_0\geq 0$ such that for $0<e<e_0$ we have

$$D_{mk}\left(g_0,\boldsymbol{e}\right)>0 \qquad \text{for } g_0<\frac{2t}{2-m+k}$$

$$D_{mk}\left(g_0,\boldsymbol{e}\right)<0 \qquad \text{for } g_0\geq\frac{2t}{2-m+k}$$

and for every $\boldsymbol{e}$ with $-\boldsymbol{e}_0<\boldsymbol{e}<0$ we have

$$D_{mk}\left(g_0,\boldsymbol{e}\right)<0 \qquad \text{for } g_0\leq\frac{2t}{2-m+k}$$

$$D_{mk}\left(g_0,\boldsymbol{e}\right)>0 \qquad \text{for } g_0>\frac{2t}{2-m+k}.$$

Proof: In view of $m\leq 1$ and $k\geq 0$ the denominator $2-m+k$ of the *critical level* $2t/\left(2-m+k\right)$ is always positive. We first look at the special case that g_0 is exactly equal to this critical level:

$$g_0=\frac{2t}{2-m+k}.$$

In this case we have

$$D_{mk}\left(g_0,\boldsymbol{e}\right)=\left(1-\frac{2-m+k}{2t}g_0-\frac{2-m}{2t}\boldsymbol{e}\right)\boldsymbol{e}$$

$$D_{mk}\left(g_0,\boldsymbol{e}\right)=-\frac{2-m}{2t}\boldsymbol{e}^2.$$

In view of $m\leq 1$ in this case $D_{mk}\left(g_0,\boldsymbol{e}\right)$ is always negative in agreement with the lemma regardless of whether $\boldsymbol{e}$ is positive or negative. Suppose that g_0 is greater than the critical level. Then $\boldsymbol{e}_0$ can be chosen in such a way that $g_0-\boldsymbol{e}_0$ is still greater than this level. This has the consequence

$$1-\frac{\left(2-m\right)\left(g_0+\boldsymbol{e}\right)+kg_0}{2t}<0$$

for all $\boldsymbol{e}$ with $0<\left|\boldsymbol{e}\right|<\boldsymbol{e}_0$. Therefore in this case $D_{mk}\left(g_0,\boldsymbol{e}\right)$ is negative for $0<\boldsymbol{e}<\boldsymbol{e}_0$ and positive for $\boldsymbol{e}_0<\boldsymbol{e}<0$.

Now suppose that g_0 is smaller than the critical level. Then $\boldsymbol{e}_0$ can be chosen in such a way that $g_0-\boldsymbol{e}_0$ is still smaller than the critical level. This has the consequence

$$1-\frac{\left(2-m\right)\left(g_0+\boldsymbol{e}\right)+k}{2t}>0$$

for all $\boldsymbol{e}$ with $0<\left|\boldsymbol{e}\right|<\boldsymbol{e}_0$. Therefore in in this case $D_{mk}\left(g_0,\boldsymbol{e}\right)$ is positive for $\boldsymbol{e}>0$ and negative for $\boldsymbol{e}<0$. This completes the proof of the lemma.

Lemma 10: Let $s=\left(g_0,...,g_0\right)$ be a symmetric strategy combination with $g_0>t$. Then *s* is not a local imitation equilibrium.

Proof: It is necessary to distinguish the following three cases:

Case 1: $g_0 \geq \overline{g}$.

Case 2: $\overline{g}>g_0>\overline{g}-\dfrac{t}{2}$.

Case 3: $\overline{g}-\dfrac{t}{2}\geq g_0>t$.

In case 1 all payoffs at *s* are zero and this is not changed by a deviation to g_0+e with $e>0$. A new destination is reached in this way. Therefore *s* is not a local imitation equilibrium in case 1.

In case 2 some customers remain unserved. This is not changed by a deviation of a player *j* to a strategy g_0-e with $e>0$ and $g_0-e>\overline{g}-(t/2)$. At $s^1=s/\left(j,g_0-e\right)$ player *j*'s payoff is higher than that of $j-1$ and $j+1$. They imitate it. Thereby the deviation path ends with a new destination for $n=2$ and $n=3$. For $n>3$ then $j-1$ and $j+1$ are imitated by their left and right neighbor and similar steps follow until finally the destination $\left(g_0-e,...,g_0-e\right)$ is reached. There *j*'s payoff is higher than at *s*. Therefore *s* is not a local imitation equilibrium.

In case 3 the payoff formulas for regular strategy combinations apply to all binary strategy combinations with $g_i=g_0$ or $g_i=g_0+e$ for $i=1,...,n$ for any sufficiently small negative e. Suppose that a player *j* deviates from *s* to g_0+e with $e<0$. In view of lemma 9 and $g_0>t$ we have

$$H_j\left(s^1\right)-H_{j+1}\left(s^1\right)=D_{0k}\left(g_0,e\right)>0$$

with $k=2$ for $n=2$, and $k=1$ for $n>2$. This inequality also holds with $H_{j-1}\left(s^1\right)$ instead of $H_{j+1}\left(s^1\right)$. Therefore at the deviation start s^1 each of *j*'s neighbors imitates it. Thereby a combination s^2 is reached. If s^2 is not yet the destination $\left(g_0+e,...,g_0+e\right)$, then there are one or two players playing g_0 with neighbors playing g_0+e. In the case of two such players each of them receives $A_1\left(g_0,e\right)$ whereas the neighbor playing g_0+e receives $B_1\left(g_0,e\right)$. It follows by lemma 9 that we have

$$D_{11}\left(g_0,e\right)<0$$

since in this case the critical level is *t*. If $n>2$ is even, then just before the end of the deviation path only one player playing g_0 is left over. This player receives $A_2\left(g_0,e\right)$ whereas his neighbors receive $B_1\left(g_0,e\right)$. In view of lemma 9 and $g_0>t$ we have

$$D_{12}\left(g_0,e\right)<0.$$

Therefore the deviation path always ends with the destination $(g_0+e,\ldots,g_0+e)$. Obviously s is not stable against such deviations. It follows that s is not a local imitation equilibrium.

Lemma 11: The global imitation equilibria described by lemma 8 are the only symmetric local imitation equilibria for $n=2$ and $n=3$.

Proof: Let $s=(g_0,g_0)$ or $s=(g_0,g_0,g_0)$ be a symmetric strategy combination with $0\le g_0\le t$. It is clear that for positive or negative e of sufficiently small absolute value all binary combinations with $g_i=g_0$ or $g_i=g_0+e$ are regular. In the following we will always assume that the absolute value $|e|$ of e is sufficiently small in this sense and also sufficiently small in the sense of lemma 9. In view of lemma 10 it is not necessary to look at strategy combinations with $g_0>t$. We have to distinguish four cases:

(i) $n=2$ and $0\le g_0<\frac{t}{2}$,

(ii) $n=2$ and $\frac{t}{2}<g_0\le t$,

(iii) $n=3$ and $0\le g_0<\frac{2t}{3}$,

(iv) $n=3$ and $\frac{2t}{3}<g_0\le t$.

We first look at case (i). Suppose that player 2 deviates to a strategy g_0+e with $e>0$. At $s^1=(g_0,g_0+e)$ player 2 receives $B_0(g_0,e)$ and 1 receives $A_2(g_0,e)$. According to lemma 9 we have

$$D_{02}(g_0,e)=B_0(g_0,e)-A_2(g_0,e)>0$$

for $g_0<t/2$. Therefore at s^1 player 2 receives more than player 1. Player 1 imitates player 2 and thereby the destination $s^2=(g_0+e,g_0+e)$ is reached. Player 2 receives $B_2(g_0,e)$ at s^2 and 1 receives $A_0(g_0,e)$ at s^2.

$$D_{20}(g_0,e)=B_2(g_0,e)-A_0(g_0,e)>0.$$

Therefore player 2 does not return to g_0. consequently s is not a local imitation equilibrium in case (i).

We now look at case (ii). Suppose that player 2 deviates to g_0+e with $e<0$. Again player 2 receives $B_0(g_0,e)$ and 1 receives $A_2(g_0,e)$. it follows by lemma 9 that $D_{02}(g_0,e)$ is positive for $g_0>t/2$. Therefore at $s^1=(g_0,g_0+e)$ player 1 imitates 2 and the new destination $s^2=(g_0+e,g_0+e)$ is reached. 2's payoff is $B_2(g_0,e)$ at s^2 and $A_0(g_0,e)$ at s. In view of $D_{20}(g_0,e)=e$ we have $B_2(g_0,e)<A_2(g_0,e)$. Therefore player 2 returns to g_0. At the return

start $s^3 = (g_0 + e, g_0)$ player 2 receives $B_0(g_0, e)$ and 1 receives $A_2(g_0, e)$. In view of $D_{20}(g_0, e) > 0$ player 2 imitates player 1. In this way the return path leads to the destination s^2 and not to *s*. It follows that in case (ii) *s* is not a local imitation equilibrium.

Now consider case (iii). Suppose that player 2 deviates to $g_0 + e$ with $e > 0$. At $s^1 = s/(2, g_0 + e)$ player 2 receives B_0 and player 1 receives A_1. In view of lemma 8 we have

$$D_{01}(g_0, e) = B_0(g_0, e) - A_1(g_0, e) > 0$$

for $g_0 < 2t/3$. Therefore players 1 and 3 imitate player 2 at s^1. This leads to the destination $s^2 = (g_0 + e, g_0 + e, g_0 + e)$. Player 2 receives $B_2(g_0, e)$ at s^2 and $A_0(g_0, e)$ at *s*. In view of $D_{02}(g_0, e) = e$ we can conclude that 2 does not return to g_0 and that therefore *s* is not a local imitation equilibrium in case (iii).

Finally we look at case (iv). Suppose that player 2 deviates to $g_0 + e$ with $e < 0$. It follows by lemma 9 that $D_{01} > 0$ holds. Therefore player 2 has a higher payoff than player 1 at $s^1 = s/(2, g_0 + e)$. Player 2's neighbors 1 and 3 imitate 2. In view of $D_{02}(g_0, e) = e$ player 2's payoff $B_2(g_0, e)$ at $s^2 = (g_0 + e, g_0 + e, g_0 + e)$ is smaller than 2's payoff A_0 at (g_0, g_0, g_0). Therefore 2 returns to g_0. By lemma 9 we have $D_{20}(g_0, e) > 0$ for $g_0 > 2t/3$. Therefore 2 has a lower payoff than 1 at the return start $s^3 = (g_0 + e, g_0, g_0 + e)$. Player 2 imitates 1 or 3 and thereby the destination s^2 is reached by the return path, and not *s*. In case (iv) *s* is not a local imitation equilibrium, either.

Lemma 12: For $n > 3$ the global equilibria described by lemma 8 are the only local imitation equilibria.

Proof: Let $s = (g_0, ..., g_0)$ be a symmetric strategy combination with

$$0 \le g_0 < \frac{2t}{3}.$$

We have to show that *s* is not a local imitation equilibrium. By lemma 10 we know that *s* is not a local imitation equilibrium for $g_0 > t$. As in the proof of lemma 11 we will assume that e is sufficiently small in the sense that the binary strategy combinations involving g_0 and $g_0 + e$ are regular and that lemma 9 can be applied.

Suppose that a player *j* deviates to $g_0 + e$ with $e > 0$. Then at $s^1 = s/(j, g_0 + e)$ player *j* has the payoff $B_0(g_0, e)$. In view of lemma 9 we have

$$D_{00}(g_0, e) > 0 \qquad \text{for } g_0 < t.$$

Therefore $B_0(g_0, e)$ is greater than $B_i(g_0, e) = g_0$ for players *i* other than $j-1$, *j*, and $j+1$. Moreover, we have

$D_{01}(g_0, e) > 0 \qquad\qquad \text{for } g_0 < \frac{2t}{3}$

This has the consequence that player *j*'s payoff at s^1 is greater than that of *j*'s neighbors and that of their other neighbors $j-2$ and $j+2$. Therefore *j*'s neighbors imitate *j* and we receive a binary strategy combination in which $j-1, j,$ and $j+1$ play $g_0 + e$ and all other players use g_0. Now the payoff difference for neighbors with different strategies is

$D_{11}(g_0, e) > 0 \qquad\qquad \text{for } g_0 < t$.

The imitation of neighbors playing $g_0 + e$ by neighbors using g_0 goes on until no player or only one player with strategy g_0 is left over. In the latter case the relevant payoff difference is

$D_{12}(g_0, e) \qquad\qquad \text{for } g_0 < \frac{2t}{3}$.

Also in this case finally the symmetric strategy combination $s^2 = (g_0 + e, ..., g_0 + e)$ is reached as the final destination of the deviation path. At s^2 player *j* receives the payoff $g_0 + e$ which is higher than *j*'s payoff at *s*. Consequently *j* does not return to *s*. It is clear that *s* is not a local imitation equilibrium.

5.3.6 Symmetric imitation equilibria of the mill price competition model

The following theorem presents a complete overview over all symmetric local and global imitation equilibria of the mill price competition model.

Theorem 5: The mill price competition model as described in this section has the following symmetric local and global imitation equilibria:

$\left(\frac{t}{2}, \frac{t}{2}\right) \qquad\qquad \text{for } n = 2,$

$\left(\frac{2t}{3}, \frac{2t}{3}, \frac{2t}{3}\right) \qquad\qquad \text{for } n = 3,$

$(g_0, ..., g_0) \text{ with } \frac{2t}{3} \le g_0 \le t \qquad \text{for } n = 4, 5, ...$.

These are the only symmetric local imitation equilibria of this model.

Proof: The theorem is an immediate consequence of lemma 8 together with lemmata 10, 11, and 12.

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