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FACTOR MODELS FOR PORTFOLIO CREDIT RISK

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ABSTRACT. This paper gives a simple introduction to portfolio credit risk models of the factor model type. In factor models, the dependence between the individual defaults is driven by a small number of systematic factors. When conditioning on the realisation of these factors the defaults become independent. This allows to combine a large degree of analytical tractability in the model with a realistic dependency structure.

1. INTRODUCTION

In this paper we give an introduction to an important class of portfolio credit risk models: the factor models (also known as *conditionally independent credit risk models*). These models are among the few models that can replicate a realistic correlated default behaviour while still retaining a certain degree of analytical tractability. Many models that are used in practice are based upon this approach.

Default correlation is an important issue that arises in a large number of practical applications. Default correlation with respect to two obligors arises in the context of letter of credit backed debt, credit guarantees, counterparty risk (particularly in credit derivatives). Here the determination of the joint default probability of the two obligors is of particular interest. We will see below that this quantity depends very strongly on the default correlation between these obligors.

Default correlation with respect to multiple counterparties is highly relevant for the pricing and risk management of collateralised debt obligations, and for portfolio management of loan or bond portfolios and bank-wide risk management: Diversification of credit risk can only be successful if an adequate portfolio credit risk model is in place that can quantify the risks in the portfolio.

Dependence between the defaults of different obligors can be caused by several different fundamental factors. There may be direct links between the obligors (e.g. one obligor is a large creditor of the other, or one is the other's largest customer).

JEL Classification. G 13.

Key words and phrases. Default Risk, Portfolio Models.

Comments and suggestions are welcome, all errors are my own.

Alternatively the links are more indirect but still strong. Industrial firms may use the same input factors (and be exposed to the same price shocks here) or sell on the same markets (and thus depend on the demand there). The general state of an industry or the economic cycle in a region/country also strongly influence the credit quality of otherwise unrelated debtors.

Historically, defaults tend to cluster as the following examples from the USA show:

- Oil industry 22 companies defaulted 1982–1986
- Railroad Conglomerates: One default each year 1970–1977
- Airlines: 3 defaults 1970–1971, 5 defaults 1989–1990
- Thrifts: (Savings and Loan Crisis) 19 defaults 1989-1990
- Casinos / Hotel Chains: 10 defaults 1990
- Retailers: $>$20 defaults 1990–1992
- Construction / Real Estate: 4 defaults 1992.

The structure of the rest of the paper is as follows:

- First, we cover some basic modelling issues and notation.
- Then the distribution of default losses under independence is briefly analysed.
- In the next section we cover the case when defaults are driven by one common factor,
- and then the case when defaults are driven by several common factors.

2. Default Correlation Basics

2.1. **Terminology.** Although it has become common to talk about default *correlation*, the term *correlation* is misleading. The classical linear correlation coefficient that we know from the analysis of share prices, exchange rates and interest rates, is a very inadequate measure of d dependence between defaults in a portfolio. This is implicitly recognized by many portfolio managers and risk managers who use different concepts to measure the total credit risk in a portfolio of credit exposures. The conclusive measure of the risk of the portfolio is the full distribution of its returns which is also the quantity that we are mostly concerned with. Many risk-relevant features of this distribution can be summarised in a few key numbers like the expected loss and VaR numbers at various levels.

In this paper default *correlation* will be used as a generic term for interdependent defaults. Otherwise the specific term *linear correlation* will be used.

2.2. **Linear default correlation, conditional default probabilities, joint default probabilities.** As it is difficult to gain an intuitive feeling for the size and effect of the linear default correlation coefficient we set it into context with two more accessible quantities: the conditional default probability and the joint default probability.

We consider two obligors A and B and a fixed time horizon T. The probability of a default of A before T is denoted by p_A, the default probability of B by p_B. We assume that these probabilities are exogenously given.

Knowledge of these quantities is not yet sufficient to determine either

- the probability, that A *and* B default before T: the joint default probability p_{AB}
- i.e. the probability that A defaults before T, *given that* B *has defaulted before* T, or the probability that B defaults given that A has defaulted: the conditional default probabilities $p_{A|B}$ and $p_{B|A}$
- the linear correlation coefficient ϱ_{AB} between the default events $\mathbf{1}_{\{A\}}$ and $\mathbf{1}_{\{B\}}$. (The default indicator function equals one $\mathbf{1}_{\{A\}} = 1$ if A defaults before T, and $\mathbf{1}_{\{A\}} = 0$ if A does *not* default.)

We need at least one of the quantities above to calculate the others. The connection is given by Bayes' rule:

$$p_{A|B} = \frac{p_{AB}}{p_B}, \qquad p_{B|A} = \frac{p_{AB}}{p_A} \tag{1}$$

and by the definition of the lineare correlation coefficient

$$\varrho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}}. \tag{2}$$

2.3. **Why Correlations?** Default correlations are very important because default probabilities are very small. ϱ_{AB} can have a much larger effect on the than usual (e.g. for equities etc).

The joint default probability is given by

$$p_{AB} = p_A p_B + \varrho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)} \tag{3}$$

and the conditional default probabilities are:

$$p_{A|B} = p_A + \varrho_{AB}\sqrt{\frac{p_A}{p_B}(1-p_A)(1-p_B)} \tag{4}$$

To illustrate the significance of the size of the default correlation let us assume the following orders of magnitude: $\varrho_{AB} = \varrho = O(1)$ is not very small and $p_A = p_B = p \ll 1$ small, e.g. $\varrho_{AB} = 10\%$ and $p = 1\%$.

$$p_{AB} = 0.01 * 0.01 + 0.1 * 0.01 * 0.99 = 0.00109 \approx p^2 + \varrho p \approx \varrho p \tag{5}$$

$$p_{A|B} = 0.01 + 0.1 * 0.99 = 0.109 \approx \varrho. \tag{6}$$

In equations (5) and (6) the correlation coefficient **dominates** the joint default probabilities and the conditional default probabilities.

2.4. **The need for theoretical models of default correlations.** In this subsection it will be shown that for several reasons, structural theoretical models are indispensable for the successful assessment of the risk of default correlation. These models must be able to explain and predict default correlations from more fundamental variables.

The first of these reasons lies in the data upon which the assessment of default correlation is based. There are several possible data sources, none of which is perfect:

- Actual rating and default events:
  The obvious source of information on default correlation is the historical incidence of joint defaults of similar firms in a similar time frame. This data is objective and directly addresses the modelling problem. Unfortunately, because joint defaults are rare events, historical data on joint defaults is very sparse. To gain a statistically useful number of observations, long time ranges (several decades) have to be considered and the data must be aggregated across industries and countries. In many cases direct data will therefore not be available.
- Credit Spreads
  Credit spreads contain much information about the default risk of traded bonds, and changes in credit spreads reflect changes in the markets' assessment of the riskiness of these investments. If the credit spreads of two obligors are strongly correlated it is likely that the defaults of these obligors are also correlated. Credit spreads have the further advantage that they reflect market information (therefore they already contain risk premia) and that they can be observed far more frequently than defaults. Disadvantages are problems with data availability, data quality (liquidity), and the fact that there is no theoretical justification for the size and strength of the link between credit spread correlation and default correlation[1].
- Equity Correlations
  Equity price data is much more readily available and typically of better quality than credit spread data. The connection between equity prices and credit risk is not obvious, this link can only be established by using a theoretical model. Consequently a lot of pre-processing of the data is necessary until a statement about default correlations can be made.

Therefore the only viable data sources require theoretical models which predict default behaviour.

The second, and most important reason for the usage of default models is the fact that the specification of *full* joint default probabilities is simply too complex: While there are only four joint default events for two obligors (none defaults, A defaults, B defaults and both default), there are 2^N joint default events for N obligors. For a realistic number of obligors it is impossible to enumerate these probabilities.

[1]Theoretically, two obligors could exhibit independent credit spread dynamics up to default, but still have very high default correlation.

This situation is different from normally distributed random variables where the $N(N-1)/2$ elements of the correlation matrix are sufficient to describe the dependency structure.

3. INDEPENDENCE

Assume the following (very simplified) situation:

Assumption 1 (Homogeneous Portfolio, Independent Defaults)**.**

(i) *We consider default and survival of a portfolio until a fixed time-horizon of T. Interest-rates are set to zero*[2]*.*
(ii) *We have a portfolio of N exposures to N different obligors.*
(iii) *The exposures are of identical size L, and have identical recovery rates of c.*
(iv) *The defaults of the obligors happen independently of each other. Each obligor defaults with a probability of p before the time-horizon T.*

The assumptions in assumption 1 will be relaxed in coming sections.

We call X the number of defaults that actually occurred until time T. The loss in default is then

$$X(1-c)L, \tag{7}$$

the number of defaults multiplied with exposure size and one minus the recovery rate. Therefore, to assess the distribution of the default losses it is sufficient to know the distribution of the number X of defaults.

In this situation the distribution of defaults is given by the well-known *Binomial Distribution.* The probability of exactly $X = n$, (with $n \leq N$) defaults until time T is

$$\mathbf{P}\,[\,X = n\,] = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} =: b(n;\ N, p), \tag{8}$$

and the probability of up to N defaults is the sum over all potential numbers of defaults up to n

$$\mathbf{P}\,[\,X \leq n\,] = \sum_{m=0}^{n} \binom{N}{m} p^m (1-p)^{N-m} =: B(n;\ N, p). \tag{9}$$

Equations (8) and (9) define the Binomial Density function $b(n;\ N, p)$ and the Binomial Distribution function $B(n;\ N, p)$.

3.1. Properties of the Binomial Distribution Function.

An example for a typical Binomial density function is given in figure 1. The density has an extremely thin tail, the 99% VaR lies at 11 defaults, the 99.9% VaR at 13 and the 99.99% VaR at 15 defaults. As table 1 shows, these VaR numbers are not significantly changed when different individual default probabilities are used. When the number of obligors approaches infinity, the

[2]This assumption is not critical, we could also just consider the forward values of the exposures until T.

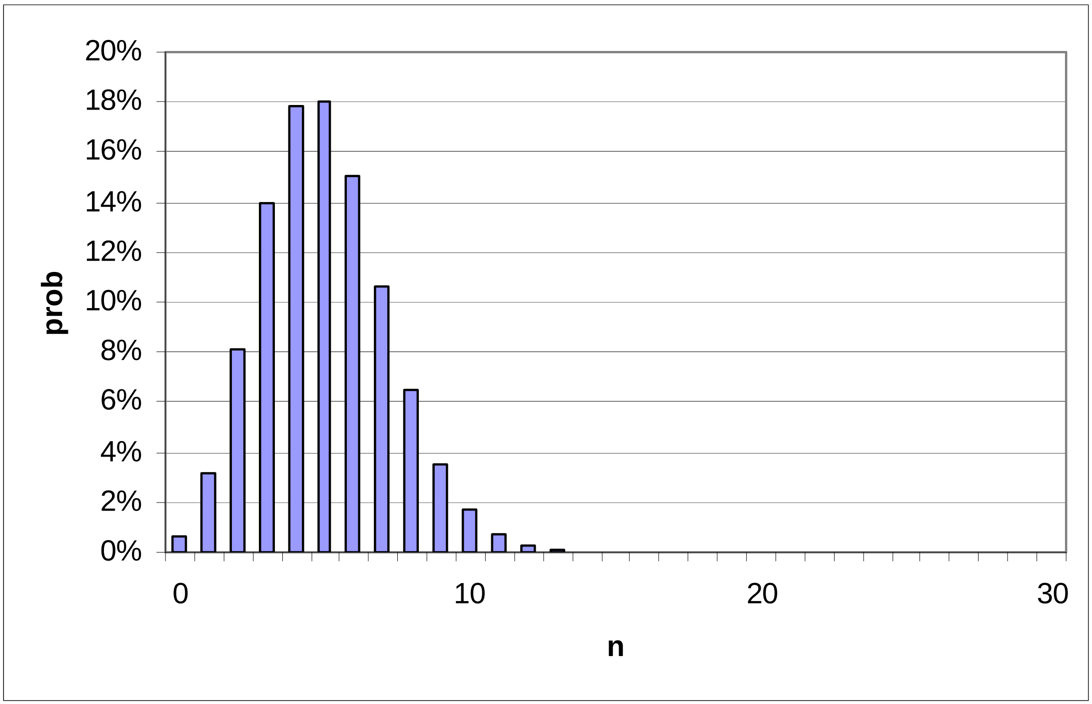

FIGURE 1. Default loss density function under independence. Parameters: Number of obligors $N = 100$, individual default probability $p = 5\%$

Default Probability	99.9% VaR Level
1%	5
2%	7
3%	9
4%	11
5%	13
6%	14
7%	16
8%	17
9%	19
10%	20

TABLE 1. VaR levels as a function of the individual default probability under independence of defaults.

central limit theorem ensures the convergence of the distribution function to a normal distribution function, hence the bell-shaped form of the default loss density function in figure 1.

3.2. **The other extreme: Perfectly dependent defaults.** The extreme case of default correlation is given when the defaults are perfectly dependent. In the situation of assumption 1 this means, that

- *either all* obligors default (with 5% probability)
- *or none* of the obligors defaults (this happens with 95% probability).

Note that this situation is still compatible with the previously made assumption that each obligor defaults with a probability of 5%. It is impossible to recover any information about the likelihood of joint defaults from the individual default probabilities in this situation.

When the individual default probabilities are not identical across the obligors it is possible to extract some information about the likelihood of joint defaults from the individual default probabilities, but not much. For example, the joint default probability of a certain set of obligors must be less than or equal to the smallest individual default probability of these obligors[3].

4. One- Factor Dependence

We need a model to represent correlated defaults. Here we use a very simplified form of the so-called firm's value models which is reportedly[4] due to Vasicek (1997). This approach is also used in Finger (1999) and in Belkin et al. (1998).

4.1. A simplified firm's value model.

Assumption 2 (Simplified Firm's Value Model)**.** *The default of each obligor is triggered by the change of the value of the assets of its firm. The value of the assets of the n'th obligor at time t is denoted by $V_n(t)$.*
We assume that $V_n(T)$ is normally distributed. Without loss of generality we set the initial asset values to zero $V_n(0) = 0$ and standardise their development such that $V_n(T) \sim \Phi(0, 1)$.
Obligor n defaults if its firm's value falls below a pre-specified barrier $V_n(T) \leq K_n$.
The asset values of different obligors are correlated with each other. The variance-covariance matrix[5] of the $V_1, \ldots, V_N$ is denoted by Σ.

This model is very similar in nature to the JPMorgan Credit Metrics model.

Note that in this setup we can still calibrate the model to reflect different individual default probabilities p_n over the time horizon by setting the barrier level to that level

[3] These bounds are a version of the so-called Frechet bounds.
[4] Although it is frequently cited, this working paper cannot be obtained from KMV or any of the usual working paper archives.
[5] The variance-covariance matrix coincides here with the correlation matrix.

p_2	Asset Corr.	p_{12}	$p_{2\|1}$	$p_{2\|1}/p_2$	Default Corr.
0.02	48	0.006	0.45	23	3.54
0.02	65	0.012	0.90	45	7.24
0.26	48	0.052	3.91	15	8.32
0.26	65	0.087	6.54	25	14.32

TABLE 2. Joint default probabilities p_{12}, conditional default probabilities $p_{2|1}$ and default correlations in a practical firm's value model. Source: Dt. Bank, Units: %

which replicates the given individual default probability. This level is

$$K_n = \Phi^{-1}(p_n). \tag{10}$$

The other calibration parameters are the $\frac{1}{2}N(N-1)$ elements of the covariance matrix Σ. These elements do not affect the individual default probabilities but only the joint default behaviour of the portfolio.

4.2. **Example Firm's Value Model.** Table 4.2 shows a practical numerical example with two Japanese banks as obligors. The indidvidual default probability of the first bank is $p_1 = 1,33\%$, the default probability of the second bank is either $p_2 = 0.02\%$ or $p_2 = 0.2\%$. The presence of asset value correlation has a significant effect on the joint and conditional default probabilities. This effect is larger for the high quality situation $p_2 = 0.02\%$ than for the low credit quality situation.

It should also be noted that the (linear) *asset* correlation coefficient is not equal to the (linear) *default* correlation coefficient. Typically, the default correlation coefficient is much smaller than the asset correlation.

4.3. A one-factor version of the firm's value model.

Assumption 3 (One-Factor Model)**.** *The values of the assets of the obligors are driven by a common, standard normally distributed factor Y component and an idiosyncratic standard normal noise component ϵ_n*

$$V_n(T) = \sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_n \qquad \forall\, n \leq N, \tag{11}$$

where Y and $\epsilon_n,\ n \leq N$ are i.i.d. standard normally $\Phi(0,1)$-distributed.

Using this approach the values of the assets of two obligors n and $m \neq n$ are correlated with linear correlation coefficient ϱ. The important point is that *conditional on the realisation of the systematic factor Y, the firm's values and the defaults are* ***independent****.*

4.4. **The distribution of the defaults.** In the following we assume that all obligors have the same default barrier $K_n = K$ and the same exposure $L_n = 1$.

By the law of iterated expectations, the probability of having exactly n defaults is the average of the conditional probabilities of n defaults, averaged over the possible realisations of Y and weighted with the probability density function $\phi(y)$:

$$\mathbf{P}\,[\,X = n\,] = \int_{-\infty}^{\infty} \mathbf{P}\,[\,X = n \mid Y = y\,]\,\phi(y)dy. \tag{12}$$

Conditional on $Y = y$, the probability of having n defaults is

$$\mathbf{P}\,[\,X = n \mid Y = y\,] = \binom{N}{n} (p(y))^n\,(1 - p(y))^{N-n}\,, \tag{13}$$

where we used the conditional independence of the defaults in the portfolio.

The individual conditional default probability $p(y)$ is the probability that the firm's value $V_n(T)$ is below the barrier K, given that the systematic factor Y takes the value y:

$$\begin{aligned} p(y) &= \mathbf{P}\,[\,V_n(T) < K \mid Y = y\,] \\ &= \mathbf{P}\left[\,\sqrt{\varrho}\,Y + \sqrt{1-\varrho}\,\epsilon_n < K \mid Y = y\,\right] \\ &= \mathbf{P}\left[\,\epsilon_n < \frac{K - \sqrt{\varrho}\,Y}{\sqrt{1-\varrho}} \mid Y = y\,\right] \\ &= \Phi(\frac{K - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}}). \end{aligned} \tag{14}$$

Substituting this into equation (12) yields

$$\mathbf{P}\,[\,X = n\,] = \int_{-\infty}^{\infty} \binom{N}{n} \left(\Phi(\frac{K - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}})\right)^n \left(1 - \Phi(\frac{K - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}})\right)^{N-n} \phi(y)dy. \tag{15}$$

Hence the distribution function of the defaults is

$$\mathbf{P}\,[\,X \le m\,] = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{\infty} \left(\Phi(\frac{K - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}})\right)^n \left(1 - \Phi(\frac{K - \sqrt{\varrho}\,y}{\sqrt{1-\varrho}})\right)^{N-n} \phi(y)dy. \tag{16}$$

Figure 2 shows the distribution of the default losses for the previous portfolio (100 obligors, 5% individual default probability) under different asset correlations. Increasing asset correlation (and thus default correlation) leads to a shift of the probability weight to the left ('good' events) and to the tail on the right. Very good events (no or very few defaults) become equally more likely as very bad events (many defaults). It should be noted that the deviation of the loss distribution function from the distribution under independence is already significant for very low values for the asset correlation (e.g. 10%).

The most significant effect for risk management is the increased mass of the loss distribution in its tails. Figure 3 uses a logarithmic scale for the probabilities to show this effect more clearly. While the probabilities decrease very quickly for independence and very low correlations, the probability of a joint default of 30 obligors is still above 10 bp for asset correlations of 30% or 50%.

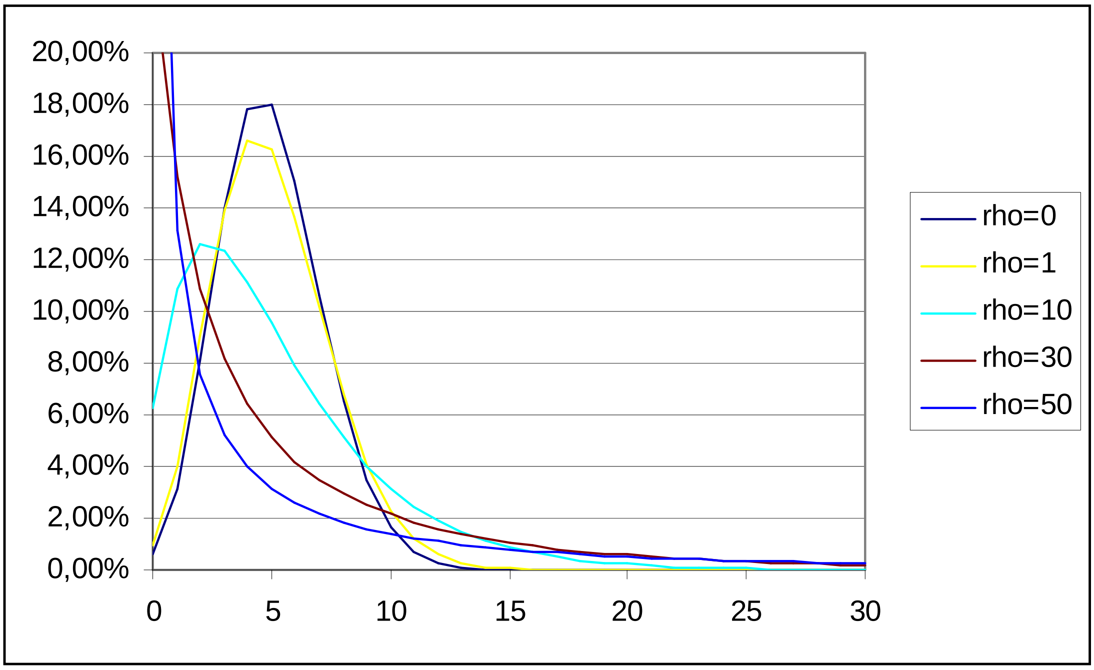

FIGURE 2. Default losses under correlation (one-factor model). Parameters: Number of obligors $N = 100$, individual default probability $p = 5\%$, asset correlation rho in percentage points: 0, 1, 10, 30, 50

Asset Correlation	99.9% VaR Level	99% VaR Level
0%	13	11
1%	14	12
10%	27	19
20%	41	27
30%	55	35
40%	68	44
50%	80	53

TABLE 3. 99.9% and 99% VaR levels as a function of the asset correlation in the one-factor model. Parameters: 100 obligors, 5% individual default probability

The 99.9% and 99% VaR levels shown in table 3 show the dramatically increased tail probabilities when correlation in the asset values is taken into account. The 99.9% VaR at a moderate asset correlation of 10% and an individual default probability of 5% is 27, which is worse than the 99.9% VaR of a portfolio of independent credits with a much higher individual default probability of 10% (the VaR is 20 in the latter case).

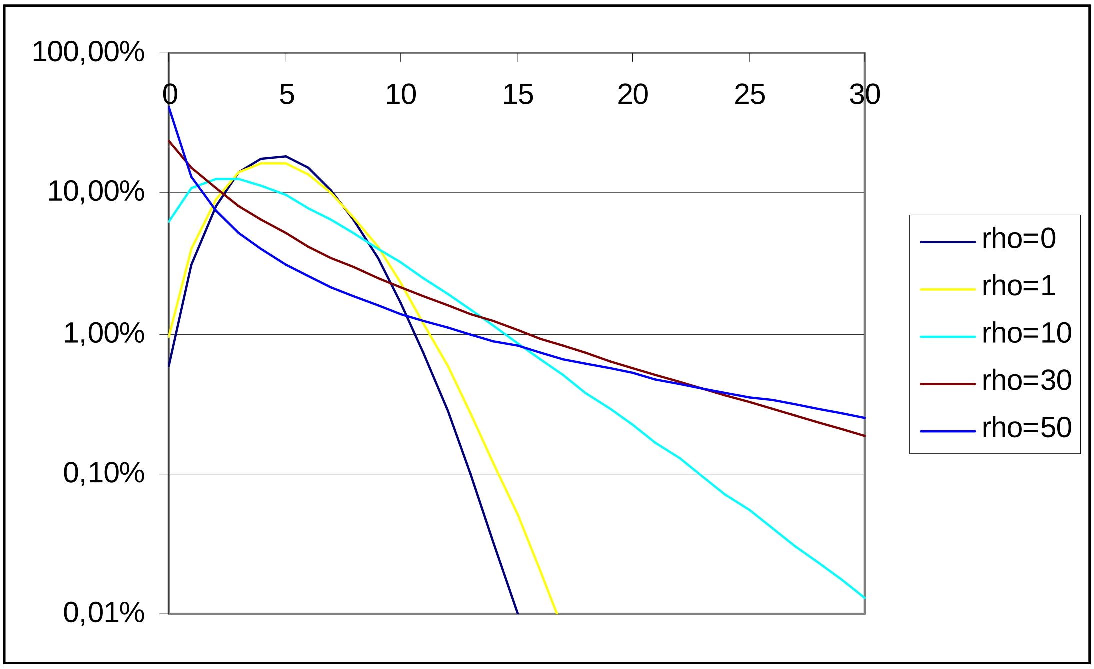

FIGURE 3. Default losses under correlation (one-factor model). Logarithmic scale. Parameters: Number of obligors $N = 100$, individual default probability $p = 5\%$, asset correlation rho in percentage points: 0, 1, 10, 30, 50

4.5. **The large portfolio approximation.** Even more tractability than in equations (15) and (16) can be reached if the number of obligors N tends to infinity.

Assumption 4 (Large Uniform Portfolio)**.** *The portfolio consists of a very large $N \to \infty$ number of credits of uniform size. Let X now denote the* fraction *of the defaulted securities in the portfolio.*

Individual defaults are still triggered by the simple firm's value model. Thus the individual default probability (conditional on the realisation y of the systematic factor Y) is given by equation (14)

$$p(y) = \Phi(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}).$$

Conditional on the realisation y of Y the individual defaults happen independently from each other. Therefore, in a very large portfolio, the law of large numbers ensures that the *fraction* of obligors that actually defaults is (almost surely) exactly equal to the individual default probability:

$$\mathbf{P}\,[\, X = p(y) \mid Y = y \,] = 1. \tag{17}$$

If we know Y, then we can predict the fraction of credits that will default with certainty.

Now we do not know the realisation of Y yet, but we can invoke iterated expectations to reach

$$\mathbf{P}[\, X \leq x \,] = \mathbf{E}[\, \mathbf{P}[\, X \leq x \mid Y \,]\,] \tag{18}$$

$$= \int_{-\infty}^{\infty} \mathbf{P}[\, X \leq x \mid Y = y \,]\, \phi(y) dy \tag{19}$$

using equation (17)

$$= \int_{-\infty}^{\infty} \mathbf{P}[\, X = p(y) \leq x \mid Y = y \,]\, \phi(y) dy \tag{20}$$

$$= \int_{-\infty}^{\infty} \mathbf{1}_{\{p(y) \leq x\}} \phi(y) dy = \int_{-y^*}^{\infty} \phi(y) dy = \Phi(y^*) \tag{21}$$

Here y^* is chosen such that $p(-y^*) = x$, and $p(y) \leq x$ for $y > -y^*$. (Remember that the individual default probability $p(y)$ *decreases* in y.) Thus y^* is

$$y^* = \frac{1}{\sqrt{\varrho}} \left(\sqrt{1-\varrho}\, \Phi^{-1}(x) - K \right). \tag{22}$$

and combining the results yields the distribution function of the loss fraction X:

$$F(x) := \mathbf{P}[\, X \leq x \,] = \Phi \left(\frac{1}{\sqrt{\varrho}} \left(\sqrt{1-\varrho}\, \Phi^{-1}(x) - \Phi^{-1}(p) \right) \right). \tag{23}$$

Taking the derivative of the distribution function with respect to x yields the corresponding probability density function $f(x)$:

$$f(x) = \sqrt{\frac{1-\varrho}{\varrho}} \exp \left\{ \frac{1}{2} (\Phi^{-1}(x)) - \frac{1}{2\varrho} \left(\Phi^{-1}(p) - \sqrt{1-\varrho}\, \Phi^{-1}(x) \right)^2 \right\}. \tag{24}$$

Obviously, infinitely large portfolios do not occur in practice, but the quality of the approximation is remarkable. In the present example (100 obligors, 5% individual default probability, 30% asset value correlation) the relative error in the tail of the distribution is around 0.1 – 0.2, i.e. the large-portfolio value of the default distribution deviates by a factor of up to ± 0.2 from the exact value. Given the uncertainty about the correct input for the asset correlation this error is negligible in many cases.

Figures 4 and 5 show the limit distribution of the default losses. For very low numbers of defaulted obligors (one or zero defaults) the approximation is incorrect because these events are discrete in reality, but must be represented as continuous events in the infinitely large portfolio. The approximation also has problems with very low (less than 1%) asset correlation coefficients, again here the discreteness of the real-world becomes noticeable: As the asset correlation coefficient tends to zero, the discrete distribution tends towards a Binomial $b(n;\ N, p)$ distribution, while the distribution of the infinitely large portfolio tends to a default fraction of exactly p. Apart from this, the quality of the approximation is remarkable.

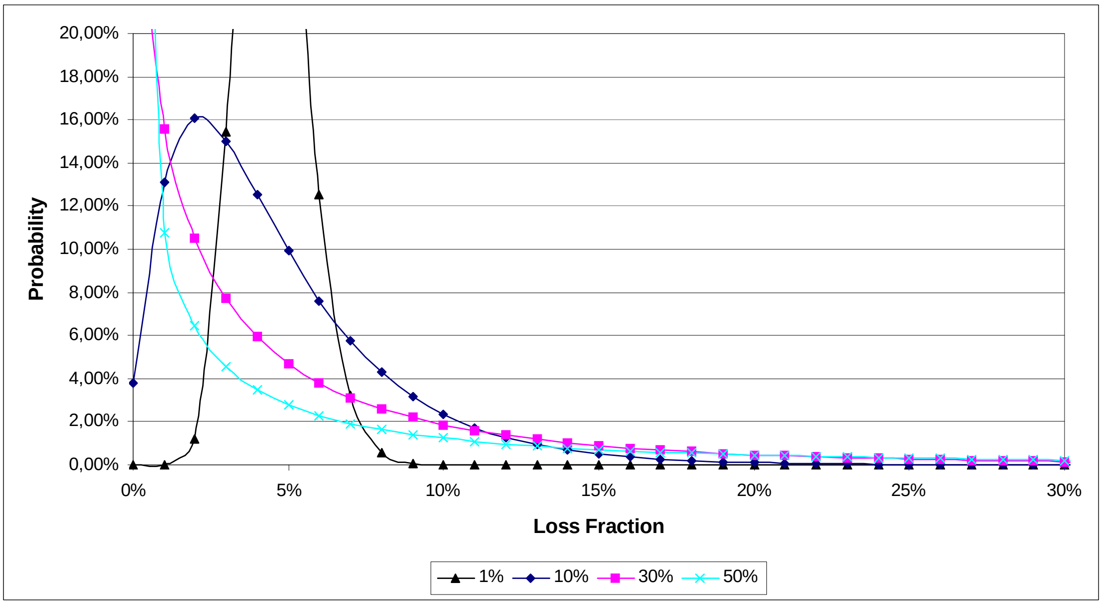

FIGURE 4. The limiting distribution of default losses for different asset value correlations.

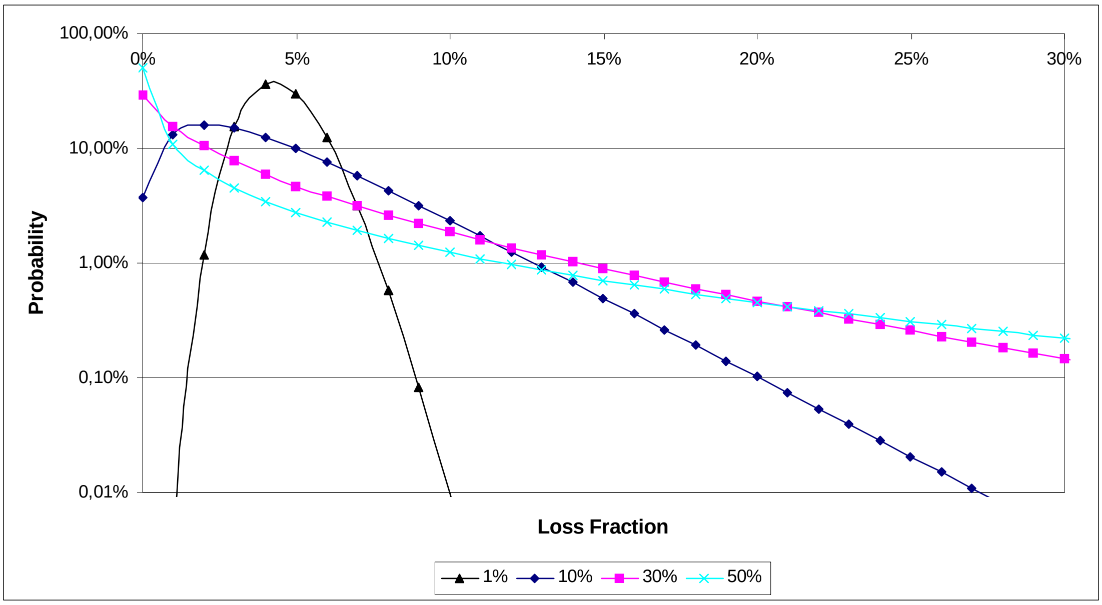

FIGURE 5. The limiting distribution of default losses for different asset value correlations, logarithmic scale.

5. GENERALISATIONS

The results of section 4 will now be generalised to show the influence of some, seemingly innocent assumptions, about the nature of the default process.

5.1. **Volatility-Uncertainty.** We replace the dynamics (11) with

$$V_n = \frac{1}{t}(\sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_n) \qquad \forall\, n \leq N, \tag{25}$$

where t is χ^2-distributed with ν degrees of freedom and independent of Y and the ϵ_n.

This setup changes the distribution of the firms' values V_n from a multivariate normal distribution to a multivariate t-distribution. The t-distribution arises whenever the variance of a normally distributed random variable must be estimated, it can be considered to represent stochastic volatility in the firms' values.

The expression for the conditional default risk (equation (14)) must be conditioned on t and Y, it takes the following form

$$p(y) = \mathbf{P}\left[\, \epsilon_n/t < \frac{K - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}} \mid Y = y \,\right] = H(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}). \tag{26}$$

The large-portfolio approximation now yields the generalisation of equation (23), a general loss distribution

$$F(x) := \mathbf{P}\,[\, X \leq x \,] = 1 - G\left(\frac{K}{\sqrt{\varrho}} - \sqrt{\frac{1-\varrho}{\varrho}}\, H^{-1}(x)\right). \tag{27}$$

5.2. **General Distribution Functions.** The careful reader will have observed that in the derivation of the results for the loss distributions in sections 4 we never explicitly used the functional form of the Normal Distribution. We therefore replace assumption 3 with 5.

Assumption 5 (Generalised One-Factor Model)**.** *The values of the assets of the obligors are driven by a common factor Y which has distribution function $G(y)$, and an idiosyncratic noise component ϵ_n which is distributed according to the distribution function $H(\epsilon)$*

$$V_n(T) = \sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_n \qquad \forall\, n \leq N, \tag{28}$$

where $Y \sim G$, and the $\epsilon_n,\ n \leq N$ are i.i.d. $H(\epsilon)$-distributed.
If the respective moments of Y and ϵ_n exist we assume w.l.o.g. that these random variables are centered and standardised.

Changing the distribution functions of the systematic factor and the noise term will *not* affect the linear correlation between the values V_n and V_m of two firms (provided the second moments exist), but it may have a large impact on the default risk of a portfolio of obligors.

The expression for the conditional default risk (equation (14)) now becomes

$$p(y) = \mathbf{P}\left[\, \epsilon_n < \frac{K - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}} \mid Y = y \,\right] = H(\frac{K - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}). \tag{29}$$

The large-portfolio approximation now yields the generalisation of equation (23), a general loss distribution

$$F(x) := \mathbf{P}[\, X \le x \,] = 1 - G\left(\frac{K}{\sqrt{\varrho}} - \sqrt{\frac{1-\varrho}{\varrho}}\, H^{-1}(x)\right). \tag{30}$$

6. MULTI-FACTOR DEPENDENCE

6.1. **Multifactor conditional models without rating transitions.** The results of the previous section can be extended to include more than one driving systematic factor for the development of the obligors' asset values. The model presented here is similar to Lucas et al. (1999).

Assumption 6 (Multifactor Firm's Value Model)**.** *The asset values of the firms are driven by a vector Y of J driving factors. Each factor influences the value of the n-th firm's assets with a weight of β_n^j. The weight vector of the n-th firm is called β_n. Thus*

$$V_n = \sum_{j=1}^{J} \beta_n^j Y_j + \epsilon_n \tag{31}$$

where ϵ_n is the idiosyncratic noise of firm n.
Firm n defaults if its firm's value is below the barrier for this firm: $V_n \le K_n$.
The factors and errors are normally distributed

- $Y \sim \Phi(0, \Omega_Y)$*: the driving factors in firm's values*
- $\epsilon_n \sim N(0, \omega_n^2)$*: idiosyncratic errors,*

and $(Y, \epsilon_1, \ldots, \epsilon_N)$ are independent.

Note that the assumption of uniform default probability across the portfolio has been given up.

Like in the one-factor case we need to derive the conditional default probabilities $p_n(y)$ of the different obligors to derive the distribution of the number X of defaults. Given a realisation y of the factor vector Y, the default probability of obligor n is

$$\begin{aligned} p_n(y) &= \mathbf{P}[\, V_n \le K_n \mid Y = y \,] \\ &= \mathbf{P}\left[\, \epsilon_n \le K_n - \sum_{j=1}^{J} \beta_n^j y_j \,\right] = \Phi\left(\frac{K_n - \sum_{j=1}^{J} \beta_n^j y_j}{\omega_n}\right). \end{aligned} \tag{32}$$

In general, the conditional default probabilities will be different for obligors with different factor loadings, even if they have the same default barrier K. For this reason assuming different unconditional default probabilities (and thus default barriers) does not complicate the model any further.

Using the conditional default probabilities from equation (32) we can now derive the conditional probability of having m defaults in the whole portfolio:

$$\mathbf{P}[\, X = m \mid Y = y \,] = \sum_{|M|=m} \left(\prod_{n \in M} p_n(y) \prod_{n' \notin M} (1 - p_{n'}(y)) \right), \tag{33}$$

where the sum is over all subsets $M \subset \{1, \ldots, N\}$ with exactly m elements: $|M| = m$. The unconditional probability of having m defaults is now

$$\mathbf{P}[\, X = m \,] = \int_{\mathbb{R}^J} \sum_{|M|=m} \left(\prod_{n \in M} p_n(y) \prod_{n' \notin M} (1 - p_{n'}(y)) \right) \phi(y|\, \Omega_Y)\, dy$$

$$= \sum_{|M|=m} \int_{\mathbb{R}^J} \left(\prod_{n \in M} p_n(y) \prod_{n' \notin M} (1 - p_{n'}(y)) \right) \phi(y|\, \Omega_Y)\, dy \tag{34}$$

Equations (33) and (34) are in closed-form but its numerical implementation is often impossible because of the large number of summation elements in the formula[6] Fortunately, in many practically relevant cases the number of summands can be reduced significantly.

6.2. **Portfolios of two asset classes.** As a simple example we consider the case where the portfolio of obligors can be decomposed into two classes of homogeneous obligor types.

Assumption 7 (Homogeneous Classes)**.** *Assume that the portfolio consists of two classes of obligors: C_1 and C_2. There are N_1 obligors of class C_1 and N_2 obligors of class C_2. Obligors of the same class have the same default barriers K_1 (or K_2) and factor loadings:*

$$V_{n_1} = Y_1 \beta_1^1 + Y_2 \beta_1^2 + \epsilon_{n_1} \qquad \text{for } n_1 \text{ in } C_1 \tag{35}$$

$$V_{n_2} = Y_1 \beta_2^1 + Y_2 \beta_2^2 + \epsilon_{n_2} \qquad \text{for } n_2 \text{ in } C_2. \tag{36}$$

The two factors Y_1 and Y_2 and the noises ϵ_{n_1} and ϵ_{n_2} are i.i.d. $\Phi(0,1)$ distributed.

In many practical credit risk modelling problems the information about the obligors is given in this form: The obligors are classified into different risk classes by criteria like industry, country, rating, but within the individual risk class no distinction is made between the obligors. Similar classifications are also made by the well-known Credit Metrics (JPMorgan & Co. Inc. (1997)) and Credit Risk+ (Credit Suisse First Boston (1997)) models.

The obligors' assets *within* one class are correlated with a correlation coefficient of ϱ_1 and ϱ_2 respectively, and the conditional survival probability in class 1 is given by

$$p_1(y) = \mathbf{P}[\, V_{n_1} \le K_1 \mid Y = y \,]$$

$$= \mathbf{P}\left[\, \epsilon_{n_1} \le K_1 - y_1 \beta_1^1 - y_2 \beta_1^2 \,\right] = \Phi\left(K_1 - y_1 \beta_1^1 - y_2 \beta_1^2 \right). \tag{37}$$

[6]For 10 defaults out of 100 obligors there are $\binom{100}{10}$ summation elements, this is a number with 19 digits.

The survival probability $p_2(y)$ for class 2 is determined analogously. Then the conditional probability of having m_1 defaults in class C_1 is given by the Binomial probability $b(m_1;\ N_1, p_1(y))$. Thus the total conditional probability of observing m defaults in the whole portfolio is

$$\mathbf{P}\,[\,X = m \mid Y = y\,] = \sum_{m_1=0}^{m} b(m_1;\ N_1, p_1(y))b(m - m_1;\ N_2, p_2(y)) \tag{38}$$

because, if m_1 defaults occur in the first class, we need $m - m_1$ defaults in the second class to reach a total of m defaults[7]. The unconditional probability of m defaults is reached by integrating over all possible realisations of the factors:

$$\mathbf{P}\,[\,X = m\,] = \sum_{m_1=0}^{m} \int_{\mathbb{R}^2} b(m_1;\ N_1, p_1(y))b(m - m_1;\ N_2, p_2(y))\phi(y)dy. \tag{39}$$

It is straightforward to generalise this approach to more than two classes of obligors.

6.3. **Rating Classes and Normal Approximation.** Another generalisation of the model is the introduction of rating classes, which enables us to model changes in the market values of the assets in the portfolio before a default occurs.

Assumption 8 (Rating Transitions)**.** *Similar to the credit-metrics approach we introduce rating transitions which are driven by changes in the asset values V_n when the firm's values exceed certain barriers*

$$c_{kl}.$$

Here c_{kl} is the barrier for a transition from rating class k to rating class l. The barriers are calibrated to exogenous rating transition probabilities.

If obligor n's rating changes from class k to class l there is an associated loss in value

$$\pi_{kl} L_n.$$

Here, closed-form solutions can also be written down, but their implementation generates even more problems than the implementation of equations (33) and (34). Therefore another way has to be found to reach the distribution of the default losses.

Again, we start from the conditional distribution of the values of the obligors' assets. The probability that the assets of obligor n are below the barrier c_{kl}, given that $Y = y$ is

$$\begin{aligned} p_n^{kl}(y) &= \mathbf{P}\,[\,V_n \le c_{kl} \mid Y = y\,] \\ &= \mathbf{P}\left[\,\epsilon_n \le c_{kl} - \sum_{j=1}^{J} \beta_n^j y_j\,\right] = \Phi\left(\frac{c_{kl} - \sum_{j=1}^{J} \beta_n^j y_j}{\omega_n}\right). \end{aligned} \tag{40}$$

[7]Here we also need to check that $N_1 \ge m_1$ and $N_2 \ge m - m_1$.

From the conditional rating transition probabilities and the associated price changes π_{kl} we can derive the conditional mean and variance of the value of the n-th asset:

$$\mu_n(y) := \sum_l L_n \pi_{kl} p_n^{kl}(y) \tag{41}$$

$$\sigma_n^2(y) := \sum_l (L_n \pi_{kl} - \mu_n(y))^2 p_n^{kl}(y). \tag{42}$$

The *normal approximation* now approximates the conditional distribution of the change in the value of the portfolio with a normal distribution with the same conditional mean and variance as the conditional mean and variance of the portfolio:

$$\mu(y) := \sum_{n=1}^N \mu_n(y) \tag{43}$$

$$\sigma^2(y) := \sum_{n=1}^N \sigma_n^2(y). \tag{44}$$

If the number of assets in the portfolio is large (and the individual default probabilities are not too small), this is a good approximation. In Lucas et al. (1999) there is a detailed numerical analysis of the quality of this approximation. Using this approximation the conditional distribution of the portfolio's change in value is the standard normal distribution

$$\mathbf{P}[\, X \leq x \mid Y = y \,] = \Phi\left(\frac{x - \mu(y)}{\sigma(y)}\right). \tag{45}$$

The unconditional distribution of the value changes of the portfolio is reached by integrating over the possible realisations of Y

$$\mathbf{P}[\, X \leq x \,] = \int_{\mathbb{R}^J} \Phi\left(\frac{x - \mu(y)}{\sigma(y)}\right) \phi(y|\, \Omega_Y)\, dy. \tag{46}$$

7. Some Remarks on Implementation

A few remarks on the implementation of conditionally independent credit risk models:

7.1. **Estimation with Probit Regression.** It is not a coincidence that the setup of the model in equation (31) looks very much like a linear regression model. The only difference to classical linear regression is the fact that the firm's value V_n cannot be observed, but only the default / survival event. This situation is known as a *hidden variables regression problem* in the econometric literature, and the best-known methods for problems of this kind are *LOGIT and PROBIT regression*. In the case of normally distributed residuals ϵ_n we are dealing with a PROBIT regression model. There is already a large literature on these models which can be directly used here.

There is one caveat: **Check the regression residuals for independence!**

The models crucially depend on the independence assumption for the ϵ_n. If these are still correlated, we cannot use the Binomial distribution and we have missed a portion of the correlation in the model. Usually this will result in an underestimation of the tail probabilities. Therefore, independence of the regression residuals is not just a esoteric exercise of statisticians but an important model requirement.

7.2. **Calibration to historical default experiences.** As an alternative to the Probit estimation of the fundamental asset value processes, one can calibrate the large portfolio distribution function (23) directly. Besides the average default rate p there is only one unknown parameter ϱ in equation (23). Given a historically observed distribution of default rates for a given industry / country / rating class one can imply these two parameters by fitting the distribution (23) to the historical observations. These parameters can then be used as asset value correlations and default probabilities in the models with fewer obligors.

This approach has the advantage of ensuring a realistic, historically confirmed shape for the distribution of default rates. Even if this approach is not taken, it should be used as a benchmark to check the results of the Probit regression.

8. Conclusion

The effect of default correlation can be very large, sometimes as large as the default probabilities themselves. Default correlations are especially important for worst cases and VaR. Therefore these effects must not be ignored in the management of portfolios of credit sensitive instruments. The structure of the modelling problem and the lack of available data makes structural models of default correlation indispensable.

In this paper a particularly simple and tractable class of default risk models is presented: the conditionally independent models. These models were built up from the most simple case of a homogeneous portfolio to the fully complex case of a multi-factor portfolio with different exposure sizes and possible value changes due to rating transitions.

Furthermore, the influence of the different parameters on key risk management numbers (VaR) was shown, and the quality of the various approximations was discussed.

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