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by

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Abstract

It is commonly assumed in private value auctions that bidders have no information about the realization of the other bidders' valuations. Nevertheless, an informative public signal about the realization may be released by a bidder while he learns his own valuation. Using a simple discrete asymmetric first-price auction setting, we show that a bidder may indeed benefit from the presence of an informative signal about his own valuation. We characterize the optimal signal and show that a signal is not beneficial if it is too precise. The latter result carries over to a general continuous asymmetric first-price auction model. Finally, we use a specific signaling structure with uniform distributions to show that signaling need not be beneficial for any precision of the signal.

JEL: D44, D82
Keywords: asymmetric auction, first-price auction, signaling

1 Introduction

Can it be beneficial to reveal some information about one's own valuation to another bidder in a first-price auction with private values? On the first glance, the answer seems to be an obvious *no*: one bidder receives additional information while the revealing bidder's information level stays the same. In principle, the informed bidder should be able to use this information to his own advantage and take away part of the profit of the revealing bidder. On the second glance however, things are not so clear: a bidder wants to appear weak in the eyes of his opponent, such that the opponent tries to profit from this weakness by reducing his bid. This increases the chances of winning for the bidder who reveals to be weak. Of course, there is also an opposing effect if a bidder appears strong. It is the goal of this paper to characterize circumstances under which it is profitable (or not profitable) to release an informative public signal while learning one's valuation.

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A typical situation where an informative signal could emerge can be found in the context of procurement auctions. Consider a manufacturer who wants to compete in a first-price procurement auction[1] to sell a new product. Before he takes part in the auction he has to acquire information about his production costs and about the quality of the new product. Costs and quality depend on the production technology and the costs for buying the necessary components. If the competitors in the auction are able to observe which components the manufacturer buys for which price, they update their beliefs about quality and price of the manufacturer's product. Nevertheless, the manufacturer is the only one who knows his production technology, while the competitors observe only an informative signal. How a signal is perceived by the competitors and how their updating works depends very much on the context, the possible production technologies and the competitor's beliefs about these things. If the manufacturer buys the components secretly, no signal is released. Usually, the manufacturer has the power to decide whether he uses a secret buying process or whether he makes its results public. For example, if he uses a request for quotation to acquire the components, the manufacturer provides public information about the specifications of the components he intends to use. Alternatively, he would be free to secretly approach possible suppliers and get their offers without revealing any public information.
In our model, two bidders take part in a first-price auction with private values. One of the two bidders has the option to release a signal about his valuation while he learns it. Thus, he has to commit to releasing the signal before he knows his valuation. In case a signal is released, the receiving bidder updates his beliefs about the valuation of the sending bidder. As a consequence, the two bidders bid as in an asymmetric auction. Furthermore, for each signal realization the resulting beliefs differ and thus do the distributions of the players' valuations in the auction. This is the major difficulty of this paper: to derive the expected profit of using these signals, an expectation over the bidders' payoffs of different asymmetric auctions has to be calculated. A closed-form solution for the bidders' equilibrium strategies is necessary to do this explicitly. Unfortunately, a general closed-form solution for asymmetric first-price auctions is not known.
A crucial element for the success of signaling is the structure of the signals. The results of this paper show that a very precise signal is not favorable from the sender's point of view. Nevertheless, we provide a signaling structure for which signaling is favorable: such a structure contains some information about the valuation, but is not too precise. However, in general a signaling precision guaranteeing the success of signaling does not need to exist: for a different structure, we show that signaling is never favorable for the sender, no matter what the precision is. In particular, one setting where signaling may be favorable is a simple discrete first-price auction setting. Each bidder's valuation and

[1]We think of a multi-attribute auction where bids are price-quality combinations evaluated by a scoring rule. This auction is essentially strategically equivalent to a standard first-price auction (see Asker and Cantillon (2008)). It is thus safe to transfer the results of this paper, which are obtained for standard first-price auctions, to procurement auctions.

the signal may be either high, medium, or low. The signal is informative in the sense that it will take the true value with a larger probability than the other two values, and the remaining two values are taken with equal probability. We show that releasing such a signal is beneficial for a bidder, as long as the signal is not too precise (the probability of revealing the true valuation is not close to one). Additionally, we derive the optimal probability of revealing the true valuation from the sending bidder's perspective.

Our other results are obtained in a continuous environment: the valuations of the two bidders are drawn from the same interval. Signals may realize in an interval around the true valuation. This interval is shifted for different realizations. The signal precision is given by the length of this interval – the shorter the interval, the more precise the signal. Using only mild assumptions on the signal distributions we give an explicit length of the interval such that signaling is not beneficial if the signals stem from an interval at most as long as this length. In our final setting, we assume all distributions to be uniform. This is the only continuous environment where a general explicit solution is known (Kaplan and Zamir (2007)) – in particular, a solution is needed that allows for different supports of the distributions of the bidders' valuations. With this signaling structure it is not beneficial for a bidder to release a signal about his realized valuation, irrespective of the signal precision.

This problem has not been addressed in the literature so far. The most related paper is Hoerner and Sahuguet (2007). They explain bluffing and jump bidding in a model with two bidders and an initial stage. In this initial stage, one of the two bidders makes an opening bid and the other bidder has to match it to start the actual auction following this stage. A similar feature to our model is the fact that the beliefs of the bidders change depending on the opening bid and thus an asymmetric auction is played afterward. However, the opening bid has to be paid in any case. Thus, the signaling happening in the initial stage has a direct influence on the payoff. Hoerner and Sahuguet (2007) concentrate mostly on an all-pay auction for the second stage, but also briefly discuss a discrete first-price auction related to the one we look at in parts of this paper. In a similar framework, Ye (2007) looks at the concept of indicative bidding. Potential bidders submit non-binding bids in a stage before the actual auction starts, which is related to the signals in our model. However, these bids are used to select the participants for the auction and thus have a direct influence on the payoffs. Furthermore, bidders only learn the highest rejected non-binding bid, such that the following auction is a symmetric one – and not asymmetric, as in our case.

Another related line of research is dealing with information acquisition in auctions. Bergemann and Valimaki (2002) study efficiency in a general mechanism design problem where agents do not know their type but may acquire a signal about it. More precise signals are more expensive. In contrast to our model, agents do not learn anything about the other agents, but only about themselves. Persico (2000) shows that agents acquire more information about their types in a first-price auction compared to a second-price auction.

Compte and Jehiel (2007) compare sealed-bid and dynamic formats, where some bidders are informed and others are uninformed. In their model, more information is acquired in the dynamic format, which goes along with a higher revenue for the seller.

Furthermore, our paper is connected to the literature on information disclosure by the seller. Milgrom and Weber (1982) show that a seller wants to disclose public information which is affiliated with the buyers' types. Eso and Szentes (2007) give a similar result when information is given to the bidders privately by the seller. Board (2009) studies the English auction where the seller may be worse off in some cases when releasing information. Looking for the optimal auction, in Bergemann and Pesendorfer (2007) the seller has full control how the buyers learn their types. Finally, Kaplan and Zamir (2000) explore the role of commitment.

The main difficulty of this paper lies in solving an asymmetric auction. We use the explicit solution for two bidders with uniform distributions and a general support by Kaplan and Zamir (2007). Plum (1992) provides the differential equations characterizing a general solution when the support of both bidders' distributions has the same lower bound. He also provides an explicit solution for power distributions. Numerical solutions are provided by Gayle and Richard (2008) and the general questions of uniqueness and existence are examined by Maskin and Riley (2000a, 2000b, 2003) and Lebrun (1999, 2006).

This paper is organized as follows: in Section 2 we introduce signaling in a discrete first-price auction. The general model with continuous typespaces is studied in Section 3 and a special case of this model with uniform distributions is given in Section 4. We conclude in Section 5. We derive the equilibrium for a discrete asymmetric auction in Appendix A and proofs are given in Appendix B.

2 Signaling in a Discrete Environment

We consider a first-price auction with two bidders, $i = 1, 2$, and discrete valuations $v_i \in V := \{0, 1, 2\}$. The valuations are independently distributed and private information of the bidders. $f_i(v_i)$ is the probability that valuation v_i is realized for bidder i. Bidder 1 may send a signal $s \in S := \{0, 1, 2\} = V$ about his realized valuation. The signal is common knowledge to both agents. The decision whether to send a signal or not is made before he knows his valuation. For a given $v_1 \in V$, we denote the probability of sending a signal value of s by $h(s|v_1)$. As the signaling should reveal some information about the true realization, we assume that $h(v_1|v_1) > f_1(v_1)$ and for $s \neq v_1$ we assume $h(s|v_1) < f_1(v_1)$. Consequently, bidder 2 updates his beliefs about bidder 1's true valuation to the posteriors $g(v_1|s)$ according to

$$g(v_1|s) = \frac{h(s|v_1) \cdot f_1(v_1)}{\sum_{j=0}^{2} h(s|j) \cdot f_1(j)}. \tag{1}$$

As a result, an asymmetric auction is played. To be able to study the consequences of signaling in a first-price auction, we need to know some properties of the equilibrium in this asymmetric auction. By Proposition 2 in Maskin and Riley (2000b) we know that a monotonic equilibrium exists in this setting if a *Vickrey tie-breaking rule* is used. According to this rule, ties are broken by performing a Vickrey auction among the bidders with the same bid. The resulting payment of the Vickrey tie-breaking auction has to be paid on top of the winning bid of the actual first-price auction. Ties in the Vickrey auction are broken by randomizing with equal probability. This kind of tie-breaking rule ensures that in equilibrium a bidder with a higher valuation may submit the same bid as another bidder with a lower valuation and still win the auction with probability one (while two bidders with the same valuation and the same bid win with equal probability). We assume a Vickrey tie-breaking rule in the following and concentrate on monotonic equilibria. The detailed derivation of the equilibrium, which is in mixed strategies, is given in Appendix A.

For concreteness, when studying signaling we assume that the a priori-distribution of both bidders' valuations is uniform, $f_i(v_i) = \frac{1}{3}$ for $i = 1, 2$ and $v_i \in V$. Furthermore, we assume that signaling is of the following form: both signal realizations not meeting the true valuation are equally likely, $h(s|v_1) = h(s'|v_1) < h(v_1|v_1)$ for $s \neq s' \neq v_1 \neq s$. Additionally, the probability of sending a signal containing the true valuation, the *signal precision* r, is assumed to be the same irrespective of the valuation. Hence, for all $v_1, v_1' \in V$ it holds that $r := h(v_1|v_1) = h(v_1'|v_1')$. Consequently, the posterior in (1) becomes $g(v_1|s) = h(s|v_1)$, as $\sum_{j=0}^{2} h(s|j) = 1$.

With the help of Proposition 16 in Appendix A we are able to calculate the expected revenue of using signals with precision r, $\pi_1^s(r)$. For each possible signal realization, different posteriors arise, and hence essentially a different asymmetric auction is played. The detailed profit of the bidders is derived in Appendix B, the overall profit is summarized in the following lemma.

Lemma 1 *Bidder 1's expected profit in this auction setting when he uses signals with precision r is given by*

$$\pi_1^s(r) = \frac{7}{36} + \frac{1}{6}r + \frac{1}{18}r\sqrt{13-12r} - \frac{1}{12}r^2 + \frac{1-r}{12} \cdot \frac{3+32r-3r^2+(1+r)\sqrt{9+78r+9r^2}}{3-3r+\sqrt{9+78r+9r^2}}.$$

Next, we derive the optimal signal precision r from bidder 1's perspective. This is done by maximizing bidder 1's expected profit as given in Lemma 1. We use the short notation $a := \sqrt{9+78r+9r^2}$ and $b := \sqrt{13-12r}$. Then, the first order condition amounts to

$$\frac{(54+18b)r^4 + (375-6ab+105b-18a)r^3 - (9ab+47a+713+519b)r^2}{ab(-3+3r-a)^2} + \frac{(\frac{76}{3}a+107b-12ab+245)r + 13a + 11ab + 33b + 39}{ab(-3+3r-a)^2} = 0 \qquad (2)$$

and we can state the following theorem:

Theorem 2 *The optimal signaling precision r^* in the discrete auction model is given by the solution to (2), with $r^* \approx 0.5462$. Signaling is beneficial for all r fulfilling $\frac{1}{3} < r < r'$ with r' being the larger solution of $\pi_1^s(r') - \frac{4}{9} = 0$. This yields $r' \approx 0.7572$.*

Proof $r^* \approx 0.5462$ is the unique solution to the first order condition (2). We furthermore need to show that it is in fact associated with a maximum: by continuity of the left hand side of (2) the uniqueness of the solution yields that a local maximum is a global maximum as well. Furthermore, a numerical calculation as in Figure 1 shows that there are r-values above and below r^* leading to a lower profit than r^*. Because of the continuity this is sufficient to show that r^* is a local maximum, and hence a global maximum.
To show the second part of the theorem, we note that the profit of using no signals (or signaling with a precision of $r = \frac{1}{3}$) yields an expected profit of $\frac{4}{9}$ for bidder 1. By our analysis of the first order condition we have essentially seen that $\pi_1^s(r)$ is monotonically increasing on $\left(\frac{1}{3}, r^*\right)$ and monotonically decreasing on $(r^*, 1)$. Hence, the zeros of $\pi_1^s(r) - \frac{4}{9}$ describe the boundaries of the interval for which signaling is beneficial. $\pi_1^s(r) - \frac{4}{9}$ has two zeros, the lower one being $\frac{1}{3}$ and the larger one being $r' \approx 0.7572$. $\square$

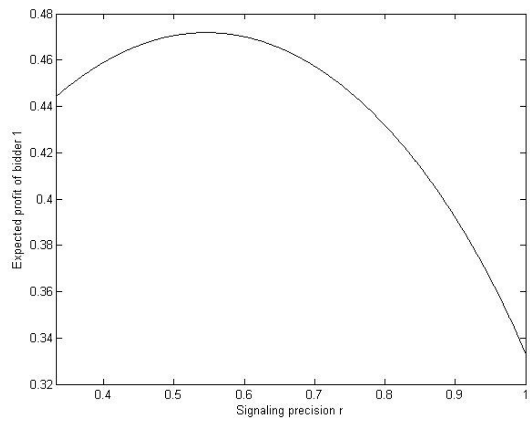

Figure 1: Expected profit of bidder 1 depending on the signaling precision.

As illustrated by Figure 1, the expected revenue of the signaling bidder is increasing as soon as informative signaling is introduced. There is a unique optimal signaling precision given the signaling structure we use. Furthermore, a general pattern of signaling is already visible here: if signaling gets too precise, it is not beneficial any more. Particularly, if the precision is very high, the revenue decrease is substantial. Nevertheless, as shown in Theorem 2, signaling is beneficial for quite a wide range of parameters.
If we increase the number of possible valuations in the set V, this basic insight does not change. In principle, the same analysis can be repeated for any number of valuations. In the natural extension of our example, the ex ante distribution of types is uniform, the average value stays the same and the signaling structure does not change: the signal takes the true value with a high probability and the remaining values with a smaller

probability, which is equal across all remaining types. However, a general statement is difficult to make, as we do not have an explicit general characterization of the equilibrium with n discrete types. We thus limit the explicit analysis to this small example and omit the detailed characterization of signaling with other numbers of types. Qualitatively, a basic analysis shows that the revenue without signaling is decreasing in the number of types, and it suggests that the interval of precisions for which signaling is profitable gets shorter in absolute and relative terms. The same is true for the maximum gain of signaling, which is achieved by using the optimal signaling precision. However, it is not clear how the profitability of signaling will develop in the limit for a large number of discrete types. Note that the shape of the signaling distribution becomes flatter with an increasing number of types – it is likely that a more peak-shaped form of the signals, like in the original three-type example, keeps up the profitability of signaling. Nevertheless, for the reasons mentioned above, we cannot prove this type of general statements for larger numbers of types.

3 Signaling in a Continuous Environment

We now introduce signaling when the agents have continuous type spaces. Valuations v_i are independently drawn from an interval $V = [\underline{v}, \bar{v}]$ and are private information of the bidders. $F_i(v_i)$ is the cumulative distribution function of bidder i's valuation with associated strictly positive density $f_i(v_i)$. Bidder 1 may send a signal $s \in S = [v_1 - d, v_1 + d]$ about his realized valuation, with $d \in \mathbb{R}_+$. The signal is common knowledge to both agents. We call d the *precision* of the signal. As in the discrete case, the decision whether to send a signal or not is made before the bidder learns his valuation. Given that a valuation v_1 is realized, the conditional distribution of the signal s with precision d is denoted by $H_d(s|v_1)$ and the corresponding density by $h_d(s|v_1)$. Note that the signals may be up to d higher (respectively lower) than the actual maximal (minimal) possible valuation.
After receiving s, bidder 2 correctly updates his beliefs that bidder 1's valuation is distributed on $[\max\{\underline{v}, s-d\}, \min\{\bar{v}, s+d\}] =: [\underline{s}(s,d), \bar{s}(s,d)]$ according to a cumulative posterior distribution function $G_d(v_1|s)$ with strictly positive density $g_d(v_1|s)$. We write $\underline{s}$ and $\bar{s}$ in short for $\underline{s}(s,d)$ and $\bar{s}(s,d)$ where the reference to s and d is clear. The overall expected profit of using signals is denoted by π^s, if no signals are used the expected profit is π. The expected profit of bidder 1, when he has valuation v_1 and a signal s has realized, is denoted by $\pi_d(v_1|s)$. As lower signal realizations lead to lower beliefs of bidder 2 and thus lower equilibrium bids with a higher profit of bidder 1, we concentrate on signaling structures fulfilling the following assumption, which is true for example for the uniform signaling presented in Section 4 (see Proposition 11).

Assumption 1 *Lower signal realizations increase the profit: $\pi_d(v_1|s)$ is weakly decreasing in s given fixed values of v_1 and d.*

Note that in the current section we do not further restrict the signal to take a specific form. Its informativeness comes from the fact that the true valuation of bidder 1 is determined by the signal with a precision of d.

Maskin and Riley (2000b) showed that in such a setting a pure-strategy equilibrium of the first-price auction with monotonic bid functions exists. We denote the monotonic equilibrium bidding strategy of agent i in case no signal is revealed by $\beta_i(v_i)$. In case the signal realization is s and the signal precision is d, we denote the strategy of agent i by $\beta_i(v_i|s,d)$. We focus on undominated equilibrium strategies and thus make use of the following assumption, similar to Maskin and Riley (2003):

Assumption 2 *Bidder i never bids more than his type v_i in equilibrium.*

Adapting a lemma of Maskin and Riley (2003) to our context, we can characterize the bid of the lowest possible type of bidder 1. Note that this lowest possible type depends on the signal realization.

Lemma 3 *If Assumption 2 holds, for any $d \in \mathbb{R}$ and any possible signal realization s, the lowest possible type $\underline{s}(s,d)$ of bidder 1 has an equilibrium bid of*

$$b_*(\underline{s}(s,d)) = \beta_1(\underline{s}(s,d)|s,d) = \max \arg\max_b F_2(b)(\underline{s}(s,d) - b).$$

Note that in case $\underline{s}(s,d) = \underline{v}$, $b_*(\underline{s}(s,d)) = \underline{v}$ holds. The following simple lemma shows that the highest possible type of bidder 1 always wins the auction:

Lemma 4 *The highest type $\bar{s}$ wins the auction with probability* 1 *in equilibrium.*

We now come to our main result of this section:

Theorem 5 *Assume*

$$d \leq \frac{1}{2} \cdot \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(b_*(v_1)\right)\right) - F_2\left(b_*(v_1)\right)\right] \cdot \left(v_1 - b_*(v_1)\right) f_1(v_1)\, dv_1,$$

then it is more profitable for bidder 1 not to reveal additional information about his valuation than revealing a signal s with precision d.

Proof Consider the lowest possible valuation of bidder 1, $\underline{s}$, with an equilibrium profit of $\pi_d(\underline{s}|s)$. Furthermore, recall from Lemma 4 that the highest type wins the auction with probability 1. In equilibrium, it is not profitable for $\underline{s}$ to imitate the bidding behavior of the highest type. Hence, it holds that

$$\pi_d(\underline{s}|s) \geq \underline{s} - \beta_1(\bar{s}|s,d).$$

We now compare the profit of the lowest and the highest type:

$$\begin{aligned} \pi_d(\bar{s}|s) - \pi_d(\underline{s}|s) &\leq (s + d - \beta_1(\bar{s}|s,d)) - (s - d - \beta_1(\bar{s}|s,d)) \\ &= 2d. \end{aligned} \tag{3}$$

For a signal s, any type $v_1 \in [\underline{s}, \bar{s}]$ makes a profit

$$\pi_d(v_1|s) \leq \pi_d(v_1|v_1 - d) \tag{4}$$
$$\leq \pi_d(\underline{s}(v_1 - d, d)|v_1 - d) + 2d \tag{5}$$
$$\leq \pi_d(v_1|v_1 + d) + 2d. \tag{6}$$

Here, (4) holds by Assumption 1 and (5) holds by using (3) as $v_1 = \bar{s}(v_1 - d, d)$. Finally, (6) follows directly from Lemma 3: the profit of the lowest type given there is obviously increasing in the value of the lowest type. Clearly, this increase in profit applies here as $v_1 = \underline{s}(v_1 + d, d) \geq \underline{s}(v_1 - d, d)$.

As a consequence, we can derive a bound on the expected profit bidder 1 makes in case the signal is sent. We can write the expected profit in case bidder 1 uses signals in the following way:

$$\begin{aligned}
\pi^s &= \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}-d}^{\bar{v}+d} \pi_d(v_1|s) h_d(s|v_1) f_1(v_1) \,\mathrm{d}s \,\mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \int_{\underline{v}-d}^{\bar{v}+d} \pi_d(v_1|s) h_d(s|v_1) \,\mathrm{d}s \,\mathrm{d}v_1 \\
&\leq \int_{\underline{v}}^{\bar{v}} f_1(v_1) \int_{\underline{v}-d}^{\bar{v}+d} \left(\pi_d(v_1|v_1 + d) + 2d\right) h_d(s|v_1) \,\mathrm{d}s \,\mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \left(\pi_d(v_1|v_1 + d) + 2d\right) \int_{\underline{v}-d}^{\bar{v}+d} h_d(s|v_1) \,\mathrm{d}s \,\mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \left(\pi_d(v_1|v_1 + d) + 2d\right) \cdot 1 \,\mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) \pi_d(v_1|v_1 + d) \,\mathrm{d}v_1 + 2d \\
&= \int_{\underline{v}}^{\bar{v}} f_1(v_1) F_2(b_*(v_1))(v_1 - b_*(v_1)) \,\mathrm{d}v_1 + 2d.
\end{aligned} \tag{7}$$

The last line holds by Lemma 3, as v_1 is the lowest possible type given a signal $v_1 + d$ and wins exactly against all opponent's types that are lower than his bid.

Now suppose to the contrary that revealing a signal s with precision d is more profitable than not revealing such a signal. Given that no signal is revealed, consider the following strategy β_1^+ of bidder 1: if his type realization is $v_1 \in [\underline{v}, \bar{v}]$, he plays as if a signal $v_1 + d$ was realized such that v_1 is the lowest possible type given this signal. By Lemma 3 we therefore get $\beta_1^+(v_1) = b_*(v_1)$. Our proof now proceeds as follows: we show that β_1^+ would be a profitable deviation for bidder 1 in comparison to his equilibrium strategy without signal realization, β_1.

We first calculate the profit π^+ of deviating to β_1^+:

$$\begin{aligned}
\pi^+ &= \int_{\underline{v}}^{\bar{v}} F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right)\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1) \,\mathrm{d}v_1 \\
&= \int_{\underline{v}}^{\bar{v}} \left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right) - F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1 - \beta_1^+(v_1)\right) f_1(v_1) \,\mathrm{d}v_1
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\underline{v}}^{\bar{v}} f_1(v_1)F_2\left(\beta_1^+(v_1)\right)\left(v_1-\beta_1^+(v_1)\right)\mathrm{d}v_1\\
&\overset{(7)}{\geq}\int_{\underline{v}}^{\bar{v}}\left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right)-F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1-\beta_1^+(v_1)\right)f_1(v_1)\,\mathrm{d}v_1+\pi^s-2d\\
&\geq\int_{\underline{v}}^{\bar{v}}\left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right)-F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1-\beta_1^+(v_1)\right)f_1(v_1)\,\mathrm{d}v_1+\pi-2d. \qquad (8)
\end{aligned}
$$

The last line holds because by assumption, the expected profit given no signaling takes place, π, is smaller than the expected profit with signaling, $\pi^s \geq \pi$. If we rearrange (8) we get the following:

$$
\pi^+-\pi\geq\int_{\underline{v}}^{\bar{v}}\left[F_2\left(\beta_2^{-1}\left(\beta_1^+(v_1)\right)\right)-F_2\left(\beta_1^+(v_1)\right)\right]\left(v_1-\beta_1^+(v_1)\right)f_1(v_1)\,\mathrm{d}v_1-2d.
$$

Here, as $\beta_1^+(v_1)=b_*(v_1)$, we can see that the deviation to β_1^+ is profitable if

$$
\int_{\underline{v}}^{\bar{v}}\left[F_2\left(\beta_2^{-1}\left(b_*(v_1)\right)\right)-F_2\left(b_*(v_1)\right)\right]\left(v_1-b_*(v_1)\right)f_1(v_1)\,\mathrm{d}v_1\geq 2d, \qquad (9)
$$

leading to $\pi^+-\pi\geq 0$. In these cases, we get a contradiction to the fact that β_1 is an equilibrium strategy. Thus, our initial assumption that revealing a signal with a precision d as in (9) must have been false and bidder 1 prefers not to reveal a signal. □

The theorem shows that a bidder never likes to use a signal that is too precise in the sense of d being very small. This bound on d we derived is independent of the precise distribution used for signaling (as long as Assumption 1 is fulfilled). However, it depends on the original distributions of the bidder's valuations. Note that the result does not say whether signaling is profitable or not for higher values of d. In the following example, we calculate the size of the bound for a uniform distribution.

Example 6 Suppose valuations are drawn from a uniform distribution on $[\underline{v},\bar{v}]=[0,1]$, hence $F_i(v_i)=v_i$ and $f_i(v_i)=1$. It is commonly known that equilibrium bids in a first-price auction are then given by $\beta_i(v_i)=\frac{v_i}{2}$. Furthermore, by Lemma 3 we know $b_*(v_1)=\max\arg\max_b F_2(b)(v_1-b)=b(v_1-b)=\frac{v_1}{2}$. This fixes the bound on the precision d according to Theorem 5:

$$
d\leq\frac{1}{2}\cdot\int_0^1\left[2\cdot\frac{v_1}{2}-\frac{v_1}{2}\right]\cdot\left(v_1-\frac{v_1}{2}\right)\cdot 1\,\mathrm{d}v_1=\frac{1}{2}\cdot\int_0^1\frac{v_1^2}{4}=\frac{1}{24}.
$$

Thus, for a signaling interval length smaller than $2d=\frac{1}{12}$ it is not profitable to make use of the signals.

4 Signaling via Uniform Distributions

We now consider the only class of distributions for which a complete characterization of equilibrium strategies in the asymmetric auction exists: the uniform distribution. This is a special case of the general continuous environment in Section 3. The aim of this section is to analyze the profitability of signaling for all possible signal precisions d. Ex ante, the valuations for both bidders are identically and independently distributed according to a uniform distribution on $[\underline{v}, \bar{v}]$. Accordingly, the cumulative distribution function F is given by $F(v) = \frac{v-\underline{v}}{\bar{v}-\underline{v}}$ and its density by $f(v) = \frac{1}{\bar{v}-\underline{v}}$. Bidder 1 has the option to ex ante commit to sending a signal s with a precision d after his valuation v_1 is realized. Specifically, the signal s is distributed uniformly on $[v_1 - d, v_1 + d]$. The corresponding cumulative distribution function is given by $H_d(s|v_1) = \frac{s-(v_1-d)}{2d}$ and its density by $h_d(s|v_1) = \frac{1}{2d}$. Hence, from an ex ante-perspective, we can derive the density $h_d(s)$ for a realization of signal s by the law of total probability:

$$h_d(s) = \int_{\underline{s}(s,d)}^{\bar{s}(s,d)} f(v_1) h_d(s|v_1)\, \mathrm{d}v_1 = \int_{\underline{s}(s,d)}^{\bar{s}(s,d)} \frac{1}{\bar{v}-\underline{v}} \cdot \frac{1}{2d}\, \mathrm{d}v_1 = \frac{\bar{s}(s,d) - \underline{s}(s,d)}{(\bar{v}-\underline{v})2d} \tag{10}$$

After observing a signal s, bidder 2 updates his belief to the posterior probability distribution $G_d(v_1|s)$ with density $g_d(v_1|s)$, which can be derived as follows, using Bayes' law:

$$g_d(v_1|s) = \frac{h_d(s|v_1) f(v_1)}{h_d(s)} = \frac{\frac{1}{2d} \cdot \frac{1}{\bar{v}-\underline{v}}}{\frac{\bar{s}(s,d)-\underline{s}(s,d)}{(\bar{v}-\underline{v})2d}} = \frac{1}{\bar{s}(s,d) - \underline{s}(s,d)}.$$

Thus, the posterior is distributed uniformly on $[\underline{s}(s,d), \bar{s}(s,d)]$. Given a signal realization s, the two bidders face the situation of an asymmetric auction with uniform distributions. The two bidders play as if bidder 1's value had been drawn uniformly from $[\underline{s}(s,d), \bar{s}(s,d)]$ and bidder 2's value from $[\underline{v}, \bar{v}]$. We denote the expected profit of bidder 1 in this auction by $\pi_1(s,d)$. The general inverse bidding strategies for this asymmetric auction have been derived by Kaplan and Zamir (2007) and can be found in Appendix B. Again, we denote the bidding strategy of bidder i by $\beta_i(v_i|s,d)$ as the bid depends on the realized valuation v_i, the realized signal s and the precision of the signal d. For notational simplicity, we write $\beta_i(v_i)$ whenever s and d are fixed. The expected profit is given as follows, using the substitution $(\beta_1)^{-1}(b) = v_1$ with boundaries $\underline{b}(s,d) = \beta_1(\underline{s}(s,d))$ and $\bar{b}(s,d) = \beta_1(\bar{s}(s,d))$:

$$\begin{aligned}
\pi_1(s,d) &= \int_{\underline{s}(s,d)}^{\bar{s}(s,d)} (v_1 - \beta_1(v_1)) \cdot F\left(\beta_2^{-1}(\beta_1(v_1))\right) g_d(v_1|s)\, \mathrm{d}v_1 \\
&= \int_{\underline{b}(s,d)}^{\bar{b}(s,d)} (\beta_1^{-1}(b) - b) \cdot F\left(\beta_2^{-1}(b)\right) \left(\beta_1^{-1}\right)'(b) \frac{1}{\bar{s}(s,d) - \underline{s}(s,d)}\, \mathrm{d}b \\
&= \int_{\underline{b}(s,d)}^{\bar{b}(s,d)} (\beta_1^{-1}(b) - b) \cdot \frac{\beta_2^{-1}(b) - \underline{v}}{\bar{v}-\underline{v}} \cdot \frac{\left(\beta_1^{-1}\right)'(b)}{\bar{s}(s,d) - \underline{s}(s,d)}\, \mathrm{d}b.
\end{aligned} \tag{11}$$

The ex ante expected profit of bidder 1 from using signals with a precision d, π_1^s, can be expressed as

$$\pi_1^s(\underline{v}, \bar{v}, d) = \int_{\underline{v}-d}^{\bar{v}+d} h_d(s) \pi_1(s,d)\, \mathrm{d}s. \tag{12}$$

Our main goal is to analyze whether signaling is profitable. To simplify the analysis, we first formulate a series of lemmas enabling us to restrict attention on F being uniform on

$[0,1]$. We formulate these lemmas in the general framework with bidders having valuations distributed on $[\underline{v}_i, \bar{v}_i]$. The proofs for all lemmas are given in Appendix B.

Lemma 7 *Suppose the supports of the valuations $[\underline{v}_i, \bar{v}_i]$ are transformed to $[\underline{v}_i^+, \bar{v}_i^+] = [\alpha\underline{v}_i + k, \alpha\bar{v}_i + k]$ with $\alpha, k \in \mathbb{R}_+$. Then, the inverse bidding strategies are transformed accordingly: $\underline{b}^+ = \alpha\underline{b} + k$, $\bar{b}^+ = \alpha\bar{b} + k$ and for all $\alpha b + k =: b^+ \in [\underline{b}^+, \bar{b}^+]$ it holds that $\left(\beta_i^+\right)^{-1}(b^+) = \alpha\beta_i^{-1}(b) + k$.*

Making use of this result, we can make a statement about a bidder's payoffs depending on the distribution parameters. Denote bidder i's payoff by $\pi_i(\underline{v}_1, \bar{v}_1, \underline{v}_2, \bar{v}_2)$.

Lemma 8 *Given the situation of Lemma 7, the expected profit changes according to*

$$\pi_i(\underline{v}_1^+, \bar{v}_1^+, \underline{v}_2^+, \bar{v}_2^+) = \pi_i(\alpha\underline{v}_1 + k, \alpha\bar{v}_1 + k, \alpha\underline{v}_2 + k, \alpha\bar{v}_2 + k) = \alpha\pi_i(\underline{v}_1, \bar{v}_1, \underline{v}_2, \bar{v}_2).$$

Transforming $[\underline{v}, \bar{v}]$ to $[\underline{v}^+, \bar{v}^+] := [\alpha\underline{v}+k, \alpha\bar{v}+k]$ and the signal precision d to $d^+ := \alpha d$, it is immediate to see that the bounds for valuations possibly generating a signal $s^+ = \alpha s + k$ change according to $\underline{s}^+(s^+, d^+) = \alpha\underline{s}(s,d) + k$ and $\bar{s}^+(s^+, d^+) = \alpha\bar{s} + k$. We can apply this to get the last lemma:

Lemma 9 *The expected profit from using signals changes according to*

$$\pi_1^s(\underline{v}^+, \bar{v}^+, d^+) = \pi_1^s(\alpha\underline{v} + k, \alpha\bar{v} + k, \alpha d) = \alpha\pi_1^s(\underline{v}, \bar{v}, d).$$

We can summarize our findings to state the following proposition:

Proposition 10 *Signaling is not profitable for valuations drawn from $[\underline{v}, \bar{v}]$ if and only if it is not profitable for valuations drawn from $[0,1]$,*

$$\pi_1^s(\underline{v}, \bar{v}, (\bar{v} - \underline{v})d) < \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v}) \iff \pi_1^s(0,1,d) < \pi_1(0,1,0,1).$$

Proof We use Lemmas 8 and 9 with $\alpha = \bar{v} - \underline{v}$ and $k = \underline{v}$ to conclude

$$\begin{aligned} & \pi_1^s(0,1,d) < \pi_1(0,1,0,1) \\ \iff \quad & (\bar{v} - \underline{v})\pi_1^s(0,1,d) < (\bar{v} - \underline{v})\pi_1(0,1,0,1) \\ \iff \quad & \pi_1^s(\underline{v}, \bar{v}, (\bar{v} - \underline{v})d) < \pi_1(\underline{v}, \bar{v}, \underline{v}, \bar{v}). \end{aligned}$$ $\square$

The following proposition shows that a better (lower) signal realization leads to higher profits and thus Assumption 1 made in Section 3 holds in this signaling structure.

Proposition 11 *Suppose bidder 1 has a valuation v_1 drawn from a uniform distribution on the support $[\underline{v}_1, \bar{v}_1]$ with $\underline{v}_1 \geq 0$ and bidder 2's valuation is drawn uniformly from $[0,1]$. Then, the profit of bidder 1 with valuation v_1 is weakly lower if v_1 is a realization from a uniform distribution on $[\underline{v}_1^+, \bar{v}_1^+]$ with $\underline{v}_1^+ \geq \underline{v}_1$ and $\bar{v}_1^+ \geq \bar{v}_1$ with one of the two inequalities being strict.*

The following theorem leads to the main result of this section.

Theorem 12 *For $d \geq \frac{\bar{v}-\underline{v}}{2}$, the expected profit $\pi_1^s(\underline{v},\bar{v},d)$ is monotonically increasing in d with*

$$\lim_{d\to\infty} \pi_1^s(\underline{v},\bar{v},d) = \pi_1(\underline{v},\bar{v},\underline{v},\bar{v}).$$

Proof By Proposition 10 it is sufficient to show the results for $[\underline{v},\bar{v}] = [0,1]$. The expected profit of signaling for $d \geq 0.5$ is given as follows:

$$\begin{aligned}
\pi_1^s(0,1,d) &\overset{(12)}{=} \int_{-d}^{1+d} h_d(s)\pi_1(s,d)\,\mathrm{d}s \\
&\overset{(10)}{=} \int_{-d}^{1+d} \frac{\bar{s}(s,d)-\underline{s}(s,d)}{2d}\pi_1(\underline{s}(s,d),\bar{s}(s,d),0,1)\,\mathrm{d}s \\
&= \int_{-d}^{1-d} \frac{s+d}{2d}\pi_1(0,s+d,0,1)\,\mathrm{d}s + \int_{1-d}^{d} \frac{1}{2d}\pi_1(0,1,0,1)\,\mathrm{d}s \\
&\qquad + \int_{d}^{1+d} \frac{1-(s-d)}{2d}\pi_1(s-d,1,0,1)\,\mathrm{d}s \\
&= \frac{1}{2d}\left(\int_0^1 t\pi_1(0,t,0,1)\,\mathrm{d}t + (d-(1-d))\pi_1(0,1,0,1) + \int_0^1 (1-t)\pi_1(t,1,0,1)\,\mathrm{d}t\right) \\
&= \pi_1(0,1,0,1) + \frac{1}{2d}\underbrace{\left(\int_0^1 t\pi_1(0,t,0,1)\,\mathrm{d}t - \pi_1(0,1,0,1) + \int_0^1 (1-t)\pi_1(t,1,0,1)\,\mathrm{d}t\right)}_{=:\tilde{c}}
\end{aligned}$$

$\tilde{c}$ is constant, and thus $\lim_{d\to\infty} \pi_1^s(\underline{v},\bar{v},d) = \pi_1(\underline{v},\bar{v},\underline{v},\bar{v})$. A calculation of $\tilde{c}$ shows $\tilde{c} \approx -0.03 < 0$. Hence, $\pi_1^s(0,1,d)$ is increasing. □

This theorem already proofs part of our main result:

Result 13 *For any precision of signals $d > 0$, signaling is less profitable:*

$$\pi_1^s(\underline{v},\bar{v},d) < \pi_1(\underline{v},\bar{v},\underline{v},\bar{v}).$$

This result is a generalization of Theorem 12 (for the case $d \geq \frac{\bar{v}-\underline{v}}{2}$) and Example 6 as an application of Theorem 5 (for the case $d \leq \frac{\bar{v}-\underline{v}}{24}$). For the remaining parameter values, we give a proof in Appendix B. The proof uses the assertion that an increase in the upper or lower end point of the support of bidder 1's uniform distribution also increases his expected profit. We do not provide a formal proof of this assertion. Nevertheless, a numerical calculation shows directly that the profit is increasing in d for all values in d and the result thus holds.

5 Conclusion

We showed that a bidder in a first-price auction might voluntarily commit to revealing an informative signal about his valuation. However, whether he does so or not depends on

several parameters, particularly the distribution and precision of the signals. As a general pattern, bidders have no incentive to reveal an informative signal if it is very precise. In a setting with only three possible valuations – high, medium or low – we derived the optimal signal and the range of precision for which signaling is beneficial. The analysis relies on a closed-form solution of the equilibrium strategies. Such an analysis is in principle feasible for other discrete sets of valuations and other shapes of signaling distributions as well. However, general statements for higher numbers of valuations are difficult to make without an explicit general characterization of discrete asymmetric equilibria. Nevertheless, the key insight can already be gleaned from the small example with three valuations: the voluntary release of an informative signal about one's own valuation can be beneficial.
It is likely that a similar shaped distribution of signals as in the discrete case would also make signaling profitable in the continuous setting. The distributions in such a family should be single-peaked on the same interval, differing in the position of the peak. Unfortunately, the explicit equilibrium strategies for a family of signals having that peaked shape is not known so far – and without knowledge of the explicit strategies it is difficult to estimate the expected revenue, as the auctions played differ with each signal realization. Hence, we chose to introduce informativeness of the signals by altering the support of the possible signals depending on the realized valuation. This enables us to get both, a result for a general class of distributions on a restricted set of signal precisions and a result for all signal precisions using uniform distributions. In these settings, signaling is not profitable for a bidder.

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A Appendix: Equilibrium of a Discrete Asymmetric Auction

We derive the necessary equilibrium properties of the asymmetric auction used in Section 2: a first-price auction with two bidders, $i = 1, 2$, with private values $v_1, v_2 \in V = \{0, 1, 2\}$, independently drawn according to the probabilities p^{v_1} and q^{v_2} respectively, using a Vickrey tie-breaking rule. Note that compared to Section 2, we change the notation of the probabilities. This is to avoid confusion: depending on the specific probabilities in the asymmetric auction, bidder 1 in Section 2 may take the role of either bidder 1 or bidder 2 in this appendix and a different notation minimizes the risk of mixing them up.

To start with the equilibrium analysis, first note that the equilibrium is in mixed strategies:

Lemma 14 *In this discrete first-price auction with Vickrey tie-breaking rule no pure-strategy equilibrium exists.*

Proof Consider two bidders with valuation 2 and suppose there is a (monotonic) pure-strategy equilibrium in which they bid differently. Then, the bidder submitting the strictly higher bid has an incentive to undercut his own bid such that he decreases his payment but still wins for sure. This cannot happen in equilibrium. In the same way, if both bidders submit the same highest bid, both of them have an incentive to slightly overcut the other bidder – the additional payment can be made arbitrary low, while the winning probability will make a fixed jump upwards (with the new bid, the bidder will always win the auction while he lost with positive probability before). □

A mixed equilibrium has the following structure:

Lemma 15 *In a mixed equilibrium of this discrete first-price auction with Vickrey tie-breaking rule*

1. *both bidders submit the same maximum bid b^*;*
2. *there cannot be an interval (b', b'') with $0 < b' < b'' < b^*$ in which any of the two bidders does not submit a bid;*
3. *bidders do not use atoms in $(0, b^*]$.*

Proof To prove the first part, we use the fact that for a given valuation bidders have to be indifferent between all bids they possibly submit. Hence, the maximum bid b^* has to be the same for both bidders – otherwise, the bidder with the higher one could profitably deviate from his maximum bid by slightly undercutting. For the second part, suppose that such an interval (b', b'') in which bidder i does not submit a bid would exist. Then, bidder j would not submit bids on this interval either and hence no bids at all would be submitted on this interval. Suppose bidder j does not place an atom on b''. Then, bidder i had a profitable deviation from his bid b'' by deviating to a bid in the interval (b', b''), lowering the price to pay in case of winning without losing any winning probability. If bidder j has an atom on b'', then either bidder i has no atom, and the argument above applies for bidder j – or bidder i has an atom as well. In this case, both bidders necessarily have a positive winning probability with their bid b'' and make profit using it[2] (otherwise, they would have a profitable deviation in the interval (b', b'')). However, as a consequence they have a profitable deviation by slightly increasing their bid, making a jump upwards in the winning probability on the expense of an arbitrarily low increase in payment. This cannot be the case in equilibrium. Finally, we note that bidders do not use atoms in $(0, b^*]$:

[2]Note that there is a positive mass of bidders with valuation 0 who always submit a bid of 0.

as already shown above, it is not possible that both bidders place an atom on the same bid in equilibrium. Similarly, if only bidder i places an atom on some $b' \in (0, b^*]$, bidder j has an incentive to bid slightly above b' instead of bidding in an interval $(b' - \varepsilon, b')$ for ε small enough. This increases the winning probability by at least the mass of the atom, while the payment is only increased by at most ε. Consequently, this atom cannot be part of an equilibrium in case $v_j > b'$. However, $v_j = b'$ cannot be part of an equilibrium as well, as bidder j would earn a profit arbitrarily close to 0 with the bid $b' - \varepsilon$, while he could get a fixed positive amount by simply bidding 0. We can thus conclude that bidders possibly only use atoms when bidding 0. □

Hence, we look for equilibria with bids on the whole interval $[0, b^*]$. Additionally, we assume w.l.o.g. that bidder 1 has a higher probability of having valuation 2, $p^2 \geq q^2$. The lowest possible equilibrium bid of bidder i with valuation v_i is denoted by $b_i(v_i)$, and bidder 1's winning probability with his bid $b_1(2)$ is q'. Similarly, his winning probability with a bid $b_1(1) = 0$ is given by $q'' \geq q^0$. $p'' \geq p^0$ is the respective probability for bidder 2. We are now ready to derive some necessary equilibrium conditions. A bidder with valuation 2 has the opportunity to win against all others for sure by submitting a bid of b^*. Then, he makes a profit of $2 - b^*$. All other bids submitted with valuation 2 have to generate the same profit. Consequently, in equilibrium both bidders mix symmetrically on $[\max\{b_1(2), b_2(2)\}, b^*]$. Hence, as we assumed that $p^2 \geq q^2$, it holds that $b_1(2) \leq b_2(2)$ and the first equilibrium condition is given by

$$2 - b^* = (1 - q^2)(2 - b_2(2)), \tag{13}$$

as bidder 1 wins the auction with a bid of $b_2(2)$ exactly against all bidder 2 types with a valuation of 1 or 0. If p^2 is strictly larger than q^2, bidder 1's lowest bid fulfills $b_1(2) < b_2(2)$, and bidder 1 with valuation 2 sometimes loses against bidder 2 who has valuation 1. We get a second condition involving bidder 1's winning probability with his bid $b_1(2)$, q':

$$2 - b^* = q'(2 - b_1(2)). \tag{14}$$

Similarly, bidder 2 with valuation 1 has to be indifferent between submitting a bid of $b_2(2)$ and $b_1(2)$ according to

$$(1 - q^2)(1 - b_2(2)) = (1 - p^2)(1 - b_1(2)). \tag{15}$$

Additionally, he gets the same profit by submitting a bid of $b_2(1) = 0$, having a winning probability of $p'' \geq p^0$:

$$(1 - p^2)(1 - b_1(2)) = p''(1 - 0). \tag{16}$$

Bidder 1 with valuation 1 is indifferent between submitting a bid of $b_1(2)$ or $b_1(1) = 0$, winning with probability $q'' \geq q^0$ in the latter case:

$$q'(1 - b_1(2)) = q''(1 - 0). \tag{17}$$

Note that due to the Vickrey tie-breaking rule, a bidder with valuation 1 wins against all opponents with valuation 0 in case he submits a bid of 0. Furthermore, at least one of $p'' = p^0$ and $q'' = q^0$ is always true: it cannot be the case that both bidders bid 0 with a positive probability when having valuation 1 – facing a bidder with the same valuation, tie-breaking will let them win only in half of the cases. Increasing the bid slightly would hence be a profitable deviation. Given these additional conditions, we have a linear equation system with five equations and five unknowns, pinning down the bidding intervals for the different valuations as stated in the following proposition:

Proposition 16 *In this discrete asymmetric auction setting with $p^2 \geq q^2$, bidder i's equilibrium bids have the following properties:*

1. *With valuation 2, bidder i mixes his bids on $[b_i(2), b^*]$;*
2. *with valuation 1, bidder i mixes his bids on $[0, b_i(2)]$, with possibly a mass point on 0;*
3. *with valuation 0, bidder i bids 0.*

The boundaries of the bidding intervals and the probability of bidding 0 are given as follows:

1. *In case $p'' = p^0$*

$$\begin{aligned} b^* &= 1 - p^0 + q^2 \\ b_1(2) &= 1 - \frac{p^0}{1-p^2} \\ b_2(2) &= 1 - \frac{p^0}{1-q^2} \\ q'' &= p^0 \cdot \frac{1+p^0-q^2}{1-p^2+p^0}. \end{aligned}$$

2. *In case $q'' = q^0$*

$$\begin{aligned} b^* &= 2 - q^0 - q' \\ b_1(2) &= 1 - \frac{q^0}{q'} \\ b_2(2) &= 2 - \frac{q^0+q'}{1-q^2} \\ p'' &= (1-p^2) \cdot \frac{q^0}{q'} \end{aligned}$$

with $q' = \frac{q^1}{2} + \sqrt{\left(\frac{q^1}{2}\right)^2 + (1-p^2)q^0}$.

Proof We start with the case $p'' = p^0$. It follows directly from (16) that

$$b_1(2) = 1 - \frac{p^0}{1-p^2}.$$

Plugging this into (15), we get

$$b_2(2) = 1 - \frac{p^0}{1-q^2}.$$

Then, (13) yields

$$b^* = 2 - (1-q^2)\left(1 + \frac{p^0}{1-q^2}\right) = 1 - p^0 + q^2.$$

The winning probabilities follow from (14) (for q') and (17) (for q''):

$$\begin{aligned} q' &= \frac{1+p^0-q^2}{1+\frac{p^0}{1-p^2}} = \frac{(1+p^0-q^2)(1-p^2)}{1-p^2+p^0} \\ q'' &= p^0 \cdot \frac{1+p^0-q^2}{1-p^2+p^0}. \end{aligned}$$

Next, we focus on the case $q'' = q^0$. Starting with (17), we get

$$b_1(2) = 1 - \frac{q^0}{q'}. \tag{18}$$

Combining (13) and (14) we can write

$$b_2(2) = 2 - \frac{q'}{1-q^2} \cdot \left(1 + \frac{q^0}{q'}\right) = 2 - \frac{q'+q^0}{1-q^2}. \tag{19}$$

The probability q' can be calculated by plugging (18) and (19) into (15):

$$\begin{aligned} & (1-q^2)\left(\frac{q'+q^0}{1-q^2} - 1\right) = (1-p^2) \cdot \frac{q^0}{q'} \\ \iff \quad & (q')^2 - q^1 \cdot q' - (1-p^2)q^0 = 0 \\ \implies \quad & q' = \frac{q^1}{2} + \sqrt{\left(\frac{q^1}{2}\right)^2 + (1-p^2)q^0}. \end{aligned}$$

Plugging q' in (18) and (19) yields the expressions stated in the proposition. (13) fixes b^* according to

$$b^* = 2 - (1-q^2) \cdot \frac{q'+q^0}{1-q^2} = 2 - q^0 - q'.$$

Finally, according to (16) we get

$$p'' = (1-p^2) \cdot \frac{q^0}{q'}. \qquad \square$$

Particularly, the proposition allows us to pin down the equilibrium profit of the bidders, which is all we need for calculating the profit of signaling. Hence, there is no need for a full characterization of equilibrium strategies in this place.
Finally, we give a characterization which of the cases $p'' = p^0$ or $q'' = q^0$ in Proposition 16 is the relevant one for some specific probability distributions. This lemma will be useful in the next section.

Lemma 17 *In Proposition 16, the case $p'' = p^0$ is relevant if $p^0 > q^0$. Furthermore, the case $q'' = q^0$ is relevant if either $p^2 > 1/3$, $p^1 = p^0 = \frac{1-p^2}{2}$, $q^2 = q^1 = q^0 = \frac{1}{3}$ or if $p^2 = p^1 = p^0 = \frac{1}{3}$, $q^0 > \frac{1}{3}$, $q^1 = q^2 = \frac{1-q^0}{2}$.*

Proof First note that $1 - p^2 \leq q'$: substituting the left-hand side of (14) with the right-hand side of (13) and dividing (15) by the resulting equation yields

$$\frac{1-p^2}{q'} \cdot \frac{1-b_1(2)}{2-b_1(2)} = \frac{1-b_2(2)}{2-b_2(2)} \qquad \Longleftrightarrow \qquad \frac{1-p^2}{q'} = \frac{2-b_1(2)-2b_2(2)+b_1(2)b_2(2)}{2-2b_1(2)-b_2(2)+b_1(2)b_2(2)}.$$

As

$$2-b_1(2)-2b_2(2)+b_1(2)b_2(2) \leq 2-2b_1(2)-b_2(2)+b_1(2)b_2(2) \qquad \Longleftrightarrow \qquad b_1(2) \leq b_2(2),$$

we know that $1 - p^2 \leq q' \iff b_1(2) \leq b_2(2)$, while the latter is true by our initial assumption $p^2 \geq q^2$. Consequently, by comparing (16) and (17) we get the general condition $p'' \leq q''$. Hence, if $p^0 > q^0$ is fulfilled, it can never be the case that $q'' = q^0$ because it would yield the contradiction $q'' = q^0 < p^0 \leq p''$.
In the case $p^2 > 1/3$, $p^1 = p^0 = \frac{1-p^2}{2}$, $q^2 = q^1 = q^0 = \frac{1}{3}$ the above argumentation cannot be applied as $q^0 > p^0$. We thus take a different approach and show that if $p'' = p^0$ were true, $q'' \geq q^0 = \frac{1}{3}$ would be violated. According to Proposition 16, q'' is given by

$$q'' = \frac{1-p^2}{2} \cdot \frac{1 + \frac{1-p^2}{2} - \frac{1}{3}}{1 - p^2 + \frac{1-p^2}{2}} = \frac{1}{3} \cdot \left(\frac{3}{2} - \frac{p^2}{2} - \frac{1}{3}\right).$$

We get the contradiction $q'' < q^0 = \frac{1}{3}$ in case

$$\frac{1}{3} \cdot \left(\frac{3}{2} - \frac{p^2}{2} - \frac{1}{3}\right) < \frac{1}{3} \qquad \Longleftrightarrow \qquad \frac{1}{3} < p^2,$$

which is true by our assumption.
Similarly, the case $p^2 = p^1 = p^0 = \frac{1}{3}$, $q^0 > \frac{1}{3}$, $q^1 = q^2 = \frac{1-q^0}{2}$ can be analyzed. Here, we have

$$q'' = \frac{1}{3} \cdot \frac{1 + \frac{1}{3} - \frac{1-q^0}{2}}{1 - \frac{1}{3} + \frac{1}{3}} = \frac{5}{18} + \frac{q^0}{6}.$$

Again, the contradiction $q'' < q^0$ is given iff

$$\frac{5}{18} + \frac{q^0}{6} < q^0 \qquad \Longleftrightarrow \qquad \frac{1}{3} < q^0,$$

which is true in the case we are analyzing. □

B Appendix: Proofs

Proof of Lemma 1

First note that each signal realizes with probability $\frac{1}{3}$. We will thus proceed by calculating the expected profit given a signal realization $s \in S$, denoted by $\pi_1(s,r)$, and then take the average of these profits. Suppose that a signal $s = 2$ is received. Then, $g(2|2) = r > \frac{1}{3} = f_2(2)$ and bidder 1 is associated with the p-probabilities in Proposition 16, while bidder 2 is associated with the q's. Hence, the two bidders are playing an asymmetric auction with posterior probabilities $p^2 = r$, $p^1 = p^0 = \frac{1-r}{2}$ and $q^0 = q^1 = q^2 = \frac{1}{3}$. As we assumed $r > \frac{1}{3}$, Lemma 17 tells us that $q'' = q^0$ has to hold in Proposition 16. The expected profit can be calculated according to

$$\begin{aligned}\pi_1(2,r) = p^2\left(2-b^*\right) + p^1 q^0 \left(1-0\right) &= r\left(\frac{1}{3}+\frac{1}{6}+\sqrt{\left(\frac{1}{6}\right)^2+\frac{1}{3}(1-r)}\right)+\frac{1-r}{2}\cdot\frac{1}{3}\\ &= \frac{1}{3}r+\frac{1}{6}r\sqrt{13-12r}+\frac{1}{6}.\end{aligned}$$

If the signal realizes to $s = 1$, posteriors are given by $g(2|1) = g(0|1) = \frac{1-r}{2}$, $g(1|1) = r$ and $f_2(0) = f_2(1) = f_2(2) = \frac{1}{3}$. Hence, $g(2|1) < f_2(2)$ and in the language of Proposition 16 bidder 1 and bidder 2 switch roles. Consequently, to get $\pi_1(1,r)$ we have to calculate the profit of the bidder 2-role in Proposition 16 in an asymmetric auction with $p^0 = p^1 = p^2 = \frac{1}{3}$ and $q^2 = q^0 = \frac{1-r}{2}$, $q^1 = r$. As $q^0 = \frac{1-r}{2} < \frac{1}{3} = p^0$, by Lemma 17 $p'' = p^0$ holds in Proposition 16. Thus, we get

$$\begin{aligned}\pi_1(1,r) = q^2\left(2-b^*\right)+q^1p^0 &= \frac{1-r}{2}\left(1+\frac{1}{3}-\frac{1-r}{2}\right)+\frac{1}{3}r\\ &= \frac{1}{6}r+\frac{5}{12}-\frac{1}{4}r^2.\end{aligned}$$

The last possible signal realization is $s = 0$. Then, posteriors are $g(2|0) = g(1|0) = \frac{1-r}{2}$, $g(0|0) = r$ and $f_2(0) = f_2(1) = f_2(2) = \frac{1}{3}$. Again, $g(2|0) < f_2(2)$ and bidder 1 takes the role of bidder 2 when we apply Proposition 16. The according probabilities in the asymmetric auction are thus given by $p^0 = p^1 = p^2 = \frac{1}{3}$ and $q^2 = q^1 = \frac{1-r}{2}$, $q^0 = r$. Hence, $q'' = q^0$ holds and the expected profit in this case amounts to

$$\begin{aligned}\pi_1(0,r) &= q^2\left(2-b^*\right)+q^1p''\\ &= \frac{1-r}{2}\left(r+\frac{1-r}{4}+\sqrt{\left(\frac{1-r}{4}\right)^2+\frac{2}{3}r}\right)+\frac{1-r}{2}\cdot\frac{2}{3}\cdot\frac{r}{\frac{1-r}{4}+\sqrt{\left(\frac{1-r}{4}\right)^2+\frac{2}{3}r}}\\ &= \frac{1-r}{4}\cdot\frac{3+32r-3r^2+(1+r)\sqrt{9+78r+9r^2}}{3-3r+\sqrt{9+78r+9r^2}}.\end{aligned}$$

Calculating

$$\pi_1^s(r) = \frac{1}{3}\left(\pi_1(0,r)+\pi_1(1,r)+\pi_1(2,r)\right)$$

and simplifying yields the result. □

Proof of Lemma 4

Suppose $\bar{s}$ wins with a probability less than 1 in equilibrium. Then, a set of types of the opponent with a positive mass must submit the same bid as $\bar{s}$ – their bid cannot be higher, as they had a profitable deviation to a lower bid in this continuous setting otherwise. As $\bar{s}$ makes positive profits (this e.g. follows from Lemma 3), he than would have a profitable deviation by slightly increasing his bid and win with probability 1. This deviation will increase his profit if the bid increase is chosen small enough, such that the gain in winning probability makes up for the loss coming from a higher bid. As this profitable deviation cannot exist in equilibrium, $\bar{s}$ must win with probability 1. □

Inverse bidding strategies according to Kaplan and Zamir (2007), Proposition 1. We assume that bidder i's valuation is uniformly distributed on $[\underline{v}_i, \bar{v}_i]$ with $\underline{v}_2 < \underline{v}_1$ and $\underline{v}_1 < 2\bar{v}_2 - \underline{v}_2$.[3] Without the latter regularity assumption, bidder 2 always loses in equilibrium and the analysis is trivial. Hence, in equilibrium, both bidders have a positive chance of winning on the same interval, $[\underline{b}, \bar{b}]$. These boundaries are given according to

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2} \qquad \text{and} \qquad \bar{b} = \frac{\bar{v}_1 \cdot \bar{v}_2 - \left(\frac{\underline{v}_1 + \underline{v}_2}{2}.\right)^2}{\bar{v}_1 - \underline{v}_1 + \bar{v}_2 - \underline{v}_2} \tag{20}$$

If bidder 2 has a value $v_2 < \underline{b}$ we assume that he bids truthfully. For all $b \in [\underline{b}, \bar{b}]$, the inverse bid functions $\beta_i^{-1}(b)$ are given by

$$\beta_1^{-1}(b) = \underline{v}_1 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_1 + \underline{v}_2 - 2b)c_1 e^{\frac{\underline{v}_2 - \underline{v}_1}{\underline{v}_1 + \underline{v}_2 - 2b}} + 4(\underline{v}_2 - b)} \tag{21}$$

$$\beta_2^{-1}(b) = \underline{v}_2 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_1 + \underline{v}_2 - 2b)c_2 e^{\frac{\underline{v}_1 - \underline{v}_2}{\underline{v}_1 + \underline{v}_2 - 2b}} + 4(\underline{v}_1 - b)} \tag{22}$$

with constants

$$c_1 = \frac{\frac{(\underline{v}_2 - \underline{v}_1)^2}{\bar{v}_1 - \underline{v}_1} + 4(\bar{b} - \underline{v}_2)}{-2(\bar{b} - \underline{b})} e^{\frac{\underline{v}_2 - \underline{v}_1}{2(\bar{b} - \underline{b})}} \qquad \text{and} \qquad c_2 = \frac{\frac{(\underline{v}_2 - \underline{v}_1)^2}{\bar{v}_2 - \underline{v}_2} + 4(\bar{b} - \underline{v}_1)}{-2(\bar{b} - \underline{b})} e^{\frac{\underline{v}_1 - \underline{v}_2}{2(\bar{b} - \underline{b})}}. \tag{23}$$

This solution does not cover the case $\underline{v}_1 = \underline{v}_2 = \underline{v}$, which was already solved by Griesmer et al. (1967) in the context of reverse auctions. A generalization is given by Plum (1992) for the class of power distributions. The inverse bid functions can be written as follows, for $b \in [\underline{b}, \bar{b}]$ as in (20):

$$\beta_1^{-1}(b) = \underline{v} + \frac{2(b - \underline{v})}{1 + b^2 c - 2bc\underline{v} + c\underline{v}^2} \tag{24}$$

$$\beta_2^{-1}(b) = \underline{v} + \frac{2(b - \underline{v})}{1 - b^2 c + 2bc\underline{v} - c\underline{v}^2}. \tag{25}$$

The constant c is defined by

$$c = \frac{1}{(\bar{v}_1 - \underline{v})^2} - \frac{1}{(\bar{v}_2 - \underline{v})^2}. \tag{26}$$

[3]Note that the roles of bidder 1 and 2 are exchanged compared to Kaplan and Zamir (2007) for consistency reasons with the rest of this paper.

Proof of Lemma 7

We first calculate $\underline{b}^+$ and $\bar{b}^+$ using (20):

$$\underline{b}^+ = \frac{\alpha\underline{v}_1 + k + \alpha\underline{v}_2 + k}{2} = \alpha\frac{\underline{v}_1 + \underline{v}_2}{2} + k = \alpha\underline{b} + k$$
$$\bar{b}^+ = \frac{(\alpha\bar{v}_1 + k)\cdot(\alpha\bar{v}_2 + k) - \left(\frac{\alpha\underline{v}_1 + k + \alpha\underline{v}_2 + k}{2}\right)}{\alpha\bar{v}_1 + k - \alpha\underline{v}_1 - k + \alpha\bar{v}_2 + k - \alpha\underline{v}_2 - k} = \alpha\frac{\bar{v}_1\cdot\bar{v}_2 - \left(\frac{\underline{v}_1+\underline{v}_2}{2}\right)}{\bar{v}_1 - \underline{v}_1 + \bar{v}_2 - \underline{v}_2} + k = \alpha\bar{b} + k.$$

Now consider the case $\underline{v}_1 < \underline{v}_2$. First note, using (23), that the constants c_1 and c_2 are invariant with respect to the transformation:

$$c_1^+ = \frac{\frac{(\alpha\underline{v}_2+k-\alpha\underline{v}_1-k)^2}{\alpha\bar{v}_1+k-\alpha\underline{v}_1-k} + 4(\alpha\bar{b} + k - \alpha\underline{v}_2 - k)}{-2(\alpha\bar{b} + k - \alpha\underline{b} - k)} e^{\frac{\alpha\underline{v}_2+k-\alpha\underline{v}_1-k}{2(\alpha\bar{b}+k-\alpha\underline{b}-k)}} = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\bar{v}_1-\underline{v}_1} + 4(\bar{b} - \underline{v}_2)}{-2(\bar{b} - \underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\bar{b}-\underline{b})}} = c_1.$$

A similar calculation is true for c_2. Hence, we can calculate the inverse bidding function for bidder 1 according to (21):

$$\begin{aligned}
&\left(\beta_1^+\right)^{-1}(b^+) \\
&= \alpha\underline{v}_1 + k + \frac{(\alpha\underline{v}_2 + k - \alpha\underline{v}_1 - k)^2}{(\alpha\underline{v}_1 + k + \alpha\underline{v}_2 + k - 2(\alpha b + k))c_1 e^{\frac{\alpha\underline{v}_2+k-\alpha\underline{v}_1-k}{\alpha\underline{v}_1+k+\alpha\underline{v}_2+k-2(\alpha b+k)}} + 4(\alpha\underline{v}_2 + k - \alpha b - k)} \\
&= \alpha\left(\underline{v}_1 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_1 + \underline{v}_2 - 2b)c_1 e^{\frac{\underline{v}_2-\underline{v}_1}{\underline{v}_1+\underline{v}_2-2b}} + 4(\underline{v}_2 - b)}\right) + k \\
&= \alpha\beta_1^{-1}(b) + k.
\end{aligned}$$

Again, the calculation for bidder 2, using (22), is similar.
Finally, consider the case $\underline{v}_1 = \underline{v}_2 = \underline{v}$. We first calculate the constant c^+ according to (26):

$$c^+ = \frac{1}{(\alpha\bar{v}_1 + k - \alpha\underline{v} - k)^2} - \frac{1}{(\alpha\bar{v}_2 + k - \alpha\underline{v} - k)^2} = \frac{1}{\alpha^2}\left(\frac{1}{(\bar{v}_1 - \underline{v})^2} - \frac{1}{(\bar{v}_2 - \underline{v})^2}\right) = \frac{c}{\alpha^2}.$$

Hence, with (24) the inverse bidding strategy for bidder 1 can be written as

$$\begin{aligned}
\left(\beta_1^+\right)^{-1}(b^+) &= \alpha\underline{v} + k + \frac{2(\alpha b + k - \alpha\underline{v} - k)}{1 + (\alpha b + k)^2\frac{c}{\alpha^2} - 2(\alpha b + k)\frac{c}{\alpha^2}(\alpha\underline{v} + k) + \frac{c}{\alpha^2}(\alpha\underline{v} + k)^2} \\
&= \alpha\left(\underline{v} + \frac{2(b - \underline{v})}{1 + b^2c - 2bc\underline{v} + c\underline{v}^2}\right) + k \\
&= \alpha\beta_1^{-1}(b) + k.
\end{aligned}$$

The inverse bidding strategy for bidder 2 can be derived in the same way using (25). □

Proof of Lemma 8

Using (11), the profit of bidder 1 for the transformed support can be written as

$$\pi_1(\underline{v}_1^+, \bar{v}_1^+, \underline{v}_2^+, \bar{v}_2^+) = \int_{\underline{b}^+}^{\bar{b}^+} \left(\left(\beta_1^+\right)^{-1}(b^+) - b^+ \right) \cdot \frac{\left(\beta_2^+\right)^{-1}(b^+) - \underline{v}_2^+}{\bar{v}_2^+ - \underline{v}_2^+} \cdot \frac{\left(\left(\beta_1^+\right)^{-1}\right)'(b^+)}{\bar{v}_1^+ - \underline{v}_1^+} \,\mathrm{d}b^+$$

$$= \int_{\underline{b}}^{\bar{b}} \left(\left(\beta_1^+\right)^{-1}(\alpha b + k) - \alpha b - k \right) \cdot \frac{\left(\beta_2^+\right)^{-1}(\alpha b + k) - \alpha \underline{v}_2 - k}{\alpha \bar{v}_2 + k - \alpha \underline{v}_2 - k} \cdot \frac{\left(\left(\beta_1^+\right)^{-1}\right)'(\alpha b + k)}{\alpha \bar{v}_1 + k - \alpha \underline{v}_1 - k} \cdot \alpha \,\mathrm{d}b \tag{27}$$

$$= \int_{\underline{b}}^{\bar{b}} (\alpha \beta_1^{-1}(b) - \alpha b) \cdot \frac{\alpha \beta_2^{-1}(b) - \alpha \underline{v}_2}{\alpha \bar{v}_2 - \alpha \underline{v}_2} \cdot \frac{\left(\beta_1^{-1}\right)' \left(\frac{(\alpha b + k) - k}{\alpha}\right)}{\alpha \bar{v}_1 - \alpha \underline{v}_1} \cdot \alpha \,\mathrm{d}b \tag{28}$$

$$= \alpha \int_{\underline{b}}^{\bar{b}} (\beta_1^{-1}(b) - b) \cdot \frac{\beta_2^{-1}(b) - \underline{v}_2}{\bar{v}_2 - \underline{v}_2} \cdot \frac{\left(\beta_1^{-1}\right)'(b)}{\bar{v}_1 - \underline{v}_1} \,\mathrm{d}b$$

$$= \alpha \pi_1(\underline{v}_1, \bar{v}_1, \underline{v}_2, \bar{v}_2).$$

(27) holds by using the substitution $b^+ = \alpha b + k$. (28) follows from Lemma 7 and

$$\left(\left(\beta_1^+\right)^{-1}\right)'(b^+) = \frac{\mathrm{d}\left(\alpha \beta_1^{-1}(b) + k\right)}{\mathrm{d}b^+} = \alpha \frac{\mathrm{d}\beta_1^{-1}(b)}{\mathrm{d}b} \frac{\mathrm{d}b}{\mathrm{d}b^+} = \alpha \cdot \left(\beta_1^{-1}\right)'(b) \cdot \frac{1}{\alpha} = \left(\beta_1^{-1}\right)' \left(\frac{b^+ - k}{\alpha}\right)$$

applied to $b^+ = \alpha b + k$. A similar calculation with changed indices gives the result for bidder 2. □

Proof of Lemma 9

Using (12) and (10), the expected profit after the transformation can be written as

$$\pi_1^s(\underline{v}^+, \bar{v}^+, d^+) = \int_{\underline{v}^+ - d^+}^{\bar{v}^+ + d^+} \frac{\bar{s}^+(s^+, d^+) - \underline{s}^+(s^+, d^+)}{(\bar{v}^+ - \underline{v}^+) 2 d^+} \pi_1(\underline{s}^+(s^+, d^+), \bar{s}^+(s^+, d^+), \underline{v}^+, \bar{v}^+) \,\mathrm{d}s^+$$

$$= \int_{\underline{v} - d}^{\bar{v} + d} \frac{\bar{s}^+(\alpha s + k, \alpha d) - \underline{s}^+(\alpha s + k, \alpha d)}{(\alpha \bar{v} + k - \alpha \underline{v} - k) 2 \alpha d} \pi_1(\underline{s}^+(\alpha s + k, \alpha d), \bar{s}^+(\alpha s + k, \alpha d), \alpha \underline{v} + k, \alpha \bar{v} + k) \cdot \alpha \,\mathrm{d}s \tag{29}$$

$$= \int_{\underline{v} - d}^{\bar{v} + d} \frac{\alpha \bar{s}(s, d) - \alpha \underline{s}(s, d)}{\alpha (\bar{v} - \underline{v}) 2 \alpha d} \pi_1(\alpha \underline{s}(s, d) + k, \alpha \bar{s}(s, d) + k, \alpha \underline{v} + k, \alpha \bar{v} + k) \cdot \alpha \,\mathrm{d}s$$

$$= \alpha \int_{\underline{v} - d}^{\bar{v} + d} \frac{\bar{s}(s, d) - \underline{s}(s, d)}{(\bar{v} - \underline{v}) 2 d} \pi_1(\underline{s}(s, d), \bar{s}(s, d), \underline{v}, \bar{v}) \,\mathrm{d}s \tag{30}$$

$$= \alpha \pi_1^s(\underline{v}, \bar{v}, d).$$

(29) follows from the substitution $s^+ = \alpha s + k$, (30) from Lemma 8. □

Proof of Proposition 11

The proof proceeds in several steps.

First step: the maximum bid increases, $\bar{b}^+ > \bar{b}$.

We use (20) to calculate the difference $\bar{b}^+ - \bar{b}$:

$$\bar{b}^+ - \bar{b} = \frac{\bar{v}_1^+ \cdot 1 - \left(\frac{\underline{v}_1^+ + 0}{2}\right)^2}{\bar{v}_1^+ - \underline{v}_1^+ + 1 - 0} - \frac{\bar{v}_1 \cdot 1 - \left(\frac{\underline{v}_1 + 0}{2}\right)^2}{\bar{v}_1 - \underline{v}_1 + 1 - 0}$$
$$= \frac{\left(\bar{v}_1 - \underline{v}_1 + 1\right)\left(\bar{v}_1^+ - \left(\frac{\underline{v}_1^+}{2}\right)^2\right) - \left(\bar{v}_1^+ - \underline{v}_1^+ + 1\right)\left(\bar{v}_1 - \left(\frac{\underline{v}_1}{2}\right)^2\right)}{\underbrace{\left(\bar{v}_1^+ - \underline{v}_1^+ + 1\right)}_{>0}\underbrace{\left(\bar{v}_1 - \underline{v}_1 + 1\right)}_{>0}}.$$

As the denominator is positive, we only need to calculate the sign of the numerator to see whether $\bar{b}^+ > \bar{b}$ or not:

$$\begin{aligned}
&\left(\bar{v}_1 - \underline{v}_1 + 1\right)\left(\bar{v}_1^+ - \left(\frac{\underline{v}_1^+}{2}\right)^2\right) - \left(\bar{v}_1^+ - \underline{v}_1^+ + 1\right)\left(\bar{v}_1 - \left(\frac{\underline{v}_1}{2}\right)^2\right)\\
&= \bar{v}_1^+\left(1 - \frac{1}{2}\underline{v}_1\right)^2 + \left(\underline{v}_1^+ - \underline{v}_1\right)\frac{\underline{v}_1^+\underline{v}_1}{4} - \frac{1}{4}\left((\underline{v}_1^+)^2 - \underline{v}_1^2\right) - \bar{v}_1\left(1 - \frac{1}{2}\underline{v}_1^+\right)^2\\
&\geq \bar{v}_1\left(1 - \frac{1}{2}\underline{v}_1\right)^2 + \left(\underline{v}_1^+ - \underline{v}_1\right)\frac{\underline{v}_1^+\underline{v}_1}{4} - \frac{1}{4}\left((\underline{v}_1^+)^2 - \underline{v}_1^2\right) - \bar{v}_1\left(1 - \frac{1}{2}\underline{v}_1^+\right)^2\\
&= \left(\underline{v}_1^+ - \underline{v}_1\right)\left(\bar{v}_1 + \frac{\underline{v}_1^+\underline{v}_1}{4} - \frac{1}{4}\left(\underline{v}_1^+ + \underline{v}_1\right)\left(1 + \bar{v}_1\right)\right)\\
&\geq \left(\underline{v}_1^+ - \underline{v}_1\right)\left(\bar{v}_1 - \frac{1}{2}\bar{v}_1\left(1 + \bar{v}_1\right)\right)\\
&= \left(\underline{v}_1^+ - \underline{v}_1\right)\left(\frac{1}{2}\bar{v}_1\left(1 - \bar{v}_1\right)\right)\\
&\geq 0.
\end{aligned}$$

The first inequality is strict if $\bar{v}_1^+ > \bar{v}_1$, the second inequality is strict if $\underline{v}_1^+ > \underline{v}_1$. As at least one of these two statements is true by assumption, we get $\bar{b}^+ - \bar{b} > 0$.
Second step: the bids of bidder 2 increase: for all $b \in [\underline{b}^+, \bar{b}]$ *it holds that* $\left(\beta_2^+\right)^{-1}(b) \leq \beta_2^{-1}(b)$.
First note that $\beta_2^{-1}(\bar{b}) = 1$ and $\left(\beta_2^+\right)^{-1}(b) < 1$ as $\bar{b} < \bar{b}^+$ by the first step. Hence, the assertion is true at the top. Now assume that the assertion fails for some lower b. Then, by continuity of the bid functions, there is a largest b^* in the interior of the interval where the two inverse bid functions cross,

$$b^* := \max_{b \in (\underline{b}^+, \bar{b})} \{b | \left(\beta_2^+\right)^{-1}(b) = \beta_2^{-1}(b)\}.$$

To come to a contradiction, we look at two different cases regarding the inverse bid function of bidder 1. The first case is $\left(\beta_1^+\right)^{-1}(b^*) < \beta_1^{-1}(b^*)$.
By the first-order conditions of the maximization problems of the two bidders, we get directly the following differential equations[4]:

$$\left(\beta_1^{-1}\right)'(b)\left(\beta_2^{-1}(b) - b\right) = \beta_1^{-1}(b) - \underline{v}_1$$
$$\left(\beta_2^{-1}\right)'(b)\left(\beta_1^{-1}(b) - b\right) = \beta_2^{-1}(b) - \underline{v}_2.$$

[4]see e.g. Kaplan and Zamir (2007), equation (2)

Applying this to our setting, as $\underline{v}_2 = 0$ the following equation holds at b^*:

$$\left(\left(\beta_2^+\right)^{-1}\right)'(b^*)\left(\left(\beta_1^+\right)^{-1}(b^*) - b^*\right) = \left(\beta_2^+\right)^{-1}(b^*) = \beta_2^{-1}(b^*) = \left(\beta_2^{-1}\right)'(b^*)\left(\beta_1^{-1}(b^*) - b^*\right). \tag{31}$$

By assumption, we have $\left(\beta_1^+\right)^{-1}(b^*) < \beta_1^{-1}(b^*)$. For (31) to hold, it is thus necessary that $\left(\left(\beta_2^+\right)^{-1}\right)'(b^*) > \left(\beta_2^{-1}\right)'(b^*)$. This leads to a contradiction: by construction of b^* we know that for all $\tilde{b} > b^*$ the inequality $\left(\beta_2^+\right)^{-1}\left(\tilde{b}\right) < \beta_2^{-1}\left(\tilde{b}\right)$ is true. Thus, at b^*, with $\left(\beta_2^+\right)^{-1}(b^*) = \beta_2^{-1}(b^*)$, we get that β_2^{-1} is at least as steep as $\left(\beta_2^+\right)^{-1}$. Consequently, $\left(\left(\beta_2^+\right)^{-1}\right)'(b^*) \leq \left(\beta_2^{-1}\right)'(b^*)$ holds, which contradicts the conclusion from above.

Thus, only the remaining case $\left(\beta_1^+\right)^{-1}(b^*) \geq \beta_1^{-1}(b^*)$ is possible. However, we will come to a contradiction in this case as well. We make use of an equilibrium condition derived by Kaplan and Zamir (2007) from the differential equations. This is equation (6) in their paper:

$$\beta_1^{-1}(b) = \frac{b\beta_2^{-1}(b) - (\underline{v}_1 + \underline{v}_2)b + \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{\beta_2^{-1}(b) - b}. \tag{32}$$

We apply this equation to our setting and conclude that at b^*

$$\frac{b^*\beta_2^{-1}(b^*) - \underline{v}_1 b^* + \frac{\underline{v}_1^2}{4}}{\beta_2^{-1}(b^*) - b^*} = \beta_1^{-1}(b^*) \leq \left(\beta_1^+\right)^{-1}(b^*) = \frac{b^*\left(\beta_2^+\right)^{-1}(b^*) - \underline{v}_1^+ b^* + \frac{\left(\underline{v}_1^+\right)^2}{4}}{\left(\beta_2^+\right)^{-1}(b^*) - b^*}.$$

As by assumption $\left(\beta_2^+\right)^{-1}(b^*) = \beta_2^{-1}(b^*)$, this reduces to

$$-\underline{v}_1 b^* + \frac{\underline{v}_1^2}{4} \leq -\underline{v}_1^+ b^* + \frac{\left(\underline{v}_1^+\right)^2}{4} \quad \Longleftrightarrow \quad b^*\left(\underline{v}_1^+ - \underline{v}_1\right) \leq \frac{1}{4}\left(\underline{v}_1^+ - \underline{v}_1\right)\left(\underline{v}_1^+ + \underline{v}_1\right).$$

In case the lower end of the interval strictly increases, $\underline{v}_1^+ > \underline{v}_1$, we conclude

$$b^* \leq \frac{1}{4}\left(\underline{v}_1^+ + \underline{v}_1\right) < \frac{\underline{v}_1^+}{2}.$$

This is a contradiction to the fact that $b^* > \underline{b}^+ = \frac{\underline{v}_1^+}{2}$. In case the lower end of the interval stays the same, $\underline{v}_1^+ = \underline{v}_1$, by (32) we can directly see that $\left(\beta_1^+\right)^{-1}(b^*) = \beta_1^{-1}(b^*)$ needs to hold. We look at the explicit solution of the equilibrium bid functions, (21) and (22) or, in case $\underline{v}_1^+ = \underline{v}_1 = 0$, (24) and (25). Using the fact that $b^* > \underline{b}^+$, it follows from $\left(\beta_1^+\right)^{-1}(b^*) = \beta_1^{-1}(b^*)$ that respectively $c_1 = c_1^+$ and $c_2 = c_2^+$ or $c = c^+$ need to hold. But this is not consistent with the true values of these constants – it would e.g. follow that the bid functions are the same for both intervals. We thus arrived at a contradiction and finished the proof of the second step.

Third step: the profit of bidder 1 with valuation v_1 is weakly decreasing.

Suppose to the contrary that the expected profit of bidder 1 with valuation v_1 is higher after the shift of the interval. Furthermore, assume b and b^+ are such that $\beta_1^{-1}(b) = v_1 =$

$\left(\beta_1^+\right)^{-1}(b^+)$. By the second step[5], we know that $\left(\beta_2^+\right)^{-1}(b^+) \le \beta_2^{-1}(b^+)$. Hence, as bidder 2's valuation is distributed uniformly on $[0,1]$, we conclude that

$$(v_1 - b)\beta_2^{-1}(b) < (v_1 - b^+)\left(\beta_2^+\right)^{-1}(b^+) \le (v_1 - b^+)\beta_2^{-1}(b^+).$$

This would be a profitable deviation for bidder 1 to b^+ in the case with the unshifted interval, a contradiction, as bidding b is equilibrium behavior by assumption. This concludes the proof. □

Proof of Result 13

For the second case, $d < 0.5$, we rewrite the expected profit with signaling as follows:

$$\begin{aligned}
\pi_1^s(0,1,d) &\overset{(12)}{=} \int_{-d}^{1+d} h_d(s)\pi_1(s,d)\,\mathrm{d}s \\
&\overset{(10)}{=} \int_{-d}^{1+d} \frac{\bar{s}(s,d) - \underline{s}(s,d)}{2d}\pi_1(\underline{s}(s,d), \bar{s}(s,d), 0, 1)\,\mathrm{d}s \\
&= \int_{-d}^{d} \frac{s+d}{2d}\pi_1(0, s+d, 0, 1)\,\mathrm{d}s + \int_{d}^{1-d} 1\cdot\pi_1(s-d, s+d, 0, 1)\,\mathrm{d}s \\
&\qquad + \int_{1-d}^{1+d} \frac{1-(s-d)}{2d}\pi_1(s-d, 1, 0, 1)\,\mathrm{d}s \\
&= \frac{1}{2d}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t + \int_0^{1-2d} \pi_1(t, t+2d, 0, 1)\,\mathrm{d}t + \frac{1}{2d}\int_{1-2d}^{1}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t.
\end{aligned}$$

We now check for all three summands whether they are increasing or decreasing in d by using Leibniz' rule and the assertion that an increase in the upper or lower end point of the support of bidder 1's uniform distribution also increases his expected profit. We start with the first one:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\frac{1}{2d}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t &= \frac{-1}{2d^2}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t + \frac{1}{2d}\cdot\frac{\mathrm{d}}{\mathrm{d}d}\int_0^{2d} t\pi_1(0,t,0,1)\,\mathrm{d}t \\
&\ge \frac{-1}{2d^2}\pi_1(0,2d,0,1)\int_0^{2d} t\,\mathrm{d}t + \frac{1}{2d}\cdot\left(\int_0^{2d}\frac{\mathrm{d}}{\mathrm{d}d}t\pi_1(0,t,0,1)\,\mathrm{d}t + 2d\pi_1(0,2d,0,1)\cdot 2\right) \\
&= \pi_1(0,2d,0,1) \\
&> 0.
\end{aligned}$$

The first summand is thus increasing. The second summand is decreasing:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\int_0^{1-2d}\pi_1(t,t+2d,0,1)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}d}\int_0^{1-2d}\pi_1(1-2d-t, 1-t, 0, 1)\,\mathrm{d}t \\
&= \int_0^{1-2d}\underbrace{\frac{\mathrm{d}}{\mathrm{d}d}\pi_1(1-2d-t,1-t,0,1)}_{<0}\,\mathrm{d}t + \pi_1(0,2d,0,1)\cdot(-2) \\
&< 0.
\end{aligned}$$

[5]Technically, we did not show $b^+ \le \bar{b}$, and in case $b^+ > \bar{b}$ the inverse $\beta_2^{-1}(b^+)$ is not well defined – no type of bidder 2 will bid so high. However, a bid of b^+ will win with probability 1, and it is thus sufficient to identify $\beta_2^{-1}(b^+)$ with the highest possible valuation of bidder 2, which is 1. The inequality is thus trivially fulfilled in this case.

The third summand is increasing:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d}\frac{1}{2d}\int_{1-2d}^{1}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t &= \frac{-1}{2d^2}\int_{1-2d}^{1}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t \\
&\quad + \frac{1}{2d}\left(\int_{1-2d}^{1}\frac{\mathrm{d}}{\mathrm{d}d}(1-t)\pi_1(t,1,0,1)\,\mathrm{d}t - 2d\pi_1(1-2d,1,0,1)\cdot(-2)\right) \\
&\geq \frac{-1}{2d^2}\pi_1(1,1,0,1)\int_{1-2d}^{1}(1-t)\,\mathrm{d}t + 2\pi_1(1-2d,1,0,1) \\
&\geq -\pi_1(1,1,0,1) + 2\pi_1(0,1,0,1) = -0.25 + \frac{1}{3} \\
&> 0.
\end{aligned}$$

To show that $\pi_1^s(0,1,d) < \pi_1(0,1,0,1) = \frac{1}{6}$, we calculate the summands for different d values and use the results from above for the values in between. The following table gives simple (rounded) upper bounds for the values of the summands.

d	summand 1	summand 2	summand 3
0.00	0	0.09	0
0.26	0.01	0.06	0.06
0.36	0.02	0.04	0.075
0.44	0.035	0.02	0.09
0.5	0.05	0	0.095

Given the fact that summands one and three are increasing, and summand 2 is decreasing, we can thus estimate:

- For $d \leq 0.26$: $\pi_1^s(0,1,d) < 0.01 + 0.09 + 0.06 < \frac{1}{6}$
- For $0.26 \leq d \leq 0.36$: $\pi_1^s(0,1,d) < 0.02 + 0.06 + 0.075 < \frac{1}{6}$
- For $0.36 \leq d \leq 0.44$: $\pi_1^s(0,1,d) < 0.035 + 0.04 + 0.09 < \frac{1}{6}$
- For $0.44 \leq d \leq 0.50$: $\pi_1^s(0,1,d) < 0.05 + 0.02 + 0.095 < \frac{1}{6}$. □