Bonn Econon Discussion Papers

Discussion Paper 19/2009

FAILURE OF SADDLE-POINT METHOD IN THE PRESENCE OF DOUBLE DEFAULTS

by

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August 2009

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Financial support by the
Deutsche Forschungsgemeinschaft (DFG)
through the
Bonn Graduate School of Economics (BGSE)
is gratefully acknowledged.

Deutsche Post World Net is a sponsor of the BGSE.

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Abstract. We show that the saddle-point approximation method to quantify the impact of undiversified idiosyncratic risk in a credit portfolio is inappropriate in the presence of double default effects. Specifically, we prove that there does not exist an equivalent formula to the granularity adjustment, that accounts for guarantees, in case of the extended single-factor CreditRisk$^+$ model. Moreover, in case of the model underlying the double default treatment within the internal ratings based (IRB) approach of Basel II, the saddle-point equivalent to the GA is too complex and involved to be competitive to a standard Monte Carlo approach.

Key words: analytical approximation, Basel II, double default, granularity adjustment, IRB approach, saddle-point approximation

JEL Codes: G31, G28

1. Introduction

When applying a single-factor credit portfolio model such as the one-factor Vasicek model, the portfolio loss variable can be written as a function of the risk factor representing the systematic risk and a residual characterizing the idiosyncratic risk. To determine explicitly the contribution of the idiosyncratic component to credit Value-at-Risk (VaR), a standard method is to apply Monte Carlo simulation. This can be computationally quite time consuming, especially because we are interested in events far in the tail of the loss distribution. Thus, one often seeks for analytical alternatives. The impact of undiversified idiosyncratic risk on portfolio VaR can be approximated analytically for example by means of a granularity adjustment (GA). This idea has first been introduced in Gordy [2003] for application in Basel II. It was then substantially refined and put on a more rigorous foundation by Martin and Wilde [2003].[1] Recently, Gordy and Lütkebohmert [2007] introduced a GA that is suitable for application under Pillar 2 of Basel II as it is parameterized w.r.t. the inputs of the internal ratings based (IRB) risk weight functions.

An alternative to the GA approach is the saddle-point approximation method. Martin et al. [2001] first proved that this method allows to derive a portfolio loss distribution from a model of loss events without simulation. The approach is based on the moment generating function (MGF) of the portfolio loss distribution. The stationary points of the MGF (which can be regarded as saddle-points) contain lots of information about the shape of the loss distribution. The saddle-points of the MGF can be computed via a simple formula, and can be used to derive the shape of the tail of the loss distribution without using time-consuming Monte Carlo simulation. Based on this work, Martin and Wilde [2003] apply the saddle-point approximation method to assess the amount by which percentiles change when a new risk is added to a portfolio. Results in that paper suggest that it would be quite similar to the GA in performance and pose a similar tradeoff between fidelity to the IRB model and analytical tractability. The main advantage of the saddle-point approximation method compared to the GA approach is that it provides an approximate loss distribution over the entire loss interval.

[1]See Lütkebohmert [2009] for more details on the GA and a comparison with other methods for the quantification of undiversified idiosyncratic risk in a credit portfolio.

Recently, Ebert and Lütkebohmert [2009] extended the GA methodology of Gordy and Lütkebohmert [2007] to account also for double default effects induced by guarantees in a credit portfolio. They derive a simple analytic formula that includes additional terms compared to the GA in Gordy and Lütkebohmert [2007] which quantify the capital reducing effect of guarantees within a credit portfolio. Accounting for double default effects is a relevant issue since it is not at all rare that credit exposures are hedged in some way. Double default effects, however, are neglected in the existing saddle-point methods for the quantification of name concentration risks in loan portfolios.

It is the aim of this article to show that the saddle-point approximation technique to quantify the impact of undiversified idiosyncratic risk on portfolio VaR leads to numerically highly involved problems or even fails when some exposures in the portfolio are hedged by certain guarantors. In case of the single-factor CreditRisk$^+$ model, we explicitly show that the equation for the saddle-point equivalent to the GA can no longer be solved neither analytically nor numerically in the presents of guarantees in a credit portfolio. This is, in particular, interesting as it has been proved in Martin and Wilde [2003] that both methods agree in the single-factor CreditRisk$^+$ setting when neglecting double default effects. We also show that in other model settings, as e.g. in the extended Merton model underlying the double default treatment of Basel II, the saddle-point approximation technique is unsatisfactory as its numerical evaluation is too expensive and involved to be a reasonable alternative to Monte Carlo simulation.

2. CreditRisk$^+$ Framework

The credit portfolio model, on which our analysis is based, is an extended version of the single-factor CreditRisk$^+$ model that accounts for guarantees. The unidimensional systematic risk factor X in our model is $\Gamma(\alpha, \beta)$-distributed, i.e. the mean of X equals $\alpha\beta$ and its variance is $\alpha\beta^2$. Denote the probability density function of X by $h(X)$. We consider a portfolio consisting of N obligors indexed by $n = 1, 2, \ldots, N$. Suppose that exposures to each obligor have been aggregated so that there is a unique position for each obligor in the portfolio.[2] Denote by E_n the exposure at default of obligor n and let $s_n = \mathrm{E}_n / \sum_{i=1}^N \mathrm{E}_i$ be its share on total exposure. Applying an actuarial definition of loss as in the CreditRisk$^+$ model, we define the loss rate of obligor n as $U_n = \mathrm{LGD}_n \cdot \mathrm{D}_n$, where D_n is a default indicator equal to 1 if obligor n defaults and 0 otherwise. Here $\mathrm{LGD}_n \in [0, 1]$ denotes the loss given default rate of obligor n which for simplicity is assumed to be non random.[3] The systematic risk factor X generates correlation across obligor defaults by shifting the default probabilities. Conditional on $X = x$ the default probability of obligor n is

$$\mathrm{PD}_n(x) = \mathrm{PD}_n \cdot x \tag{2.1}$$

where PD_n is the unconditional default probability.[4] Hence default probabilities in CreditRisk$^+$ are proportional to the systematic risk factor X.

When there are no guarantees in the portfolio we have conditional independence between obligors. Thus we can express the portfolio loss as

$$L = \sum_{n=1}^{N} s_n U_n. \tag{2.2}$$

[2] This assumption is motivated in Gordy and Lütkebohmert [2007].
[3] LGD can in principal also be modeled as a random variable. For details we refer to Ebert and Lütkebohmert [2009].
[4] This corresponds to the setting in Ebert and Lütkebohmert [2009] when assuming a factor loading of 1 and non random LGDs.

When the exposure to obligor 1 is hedged by guarantor g_1, the loss rate of obligor 1 changes to $\hat{U}_1 = U_1 \cdot U_{g_1}$ (where U_{g_1} denotes the loss rate of the guarantor) as the exposure to obligor 1 is only lost in the case when the guarantor also defaults. In general we denote the guarantor of obligor n by g_n. Assume for simplicity that guarantors are not obligors in the portfolio themselves, that different obligors are hedged by different guarantors and that we only have full guarantees.[5] In that case we still have conditional independence between the loss rates of the individual positions in the portfolio. Using this setting the conditional mean $\mu(X)$ and the conditional variance $\sigma^2(X)$ in CreditRisk$^+$ can be expressed in terms of the scaled variable $X/\alpha\beta$ as

$$\begin{aligned} \mu(X) &= \sum_{n=1}^{N} s_n \cdot \mu_n(X)\mu_{g_n}(X) \\ \sigma^2(X) &= \sum_{n=1}^{N} s_n^2 \cdot \sigma_n(X)\sigma_{g_n}(X) \end{aligned} \tag{2.3}$$

respectively with

$$\begin{aligned} \mu_n(X) &:= \mathrm{LGD}_n \cdot \mathrm{PD}_n \cdot X \\ \text{and} \quad \sigma_n(X) &:= \mathrm{LGD}_n^2 \cdot \mathrm{PD}_n \cdot \frac{X}{\alpha\beta}. \end{aligned}$$

Here quantities with a subindex g_n are one in case the exposure to obligor n is unhedged.

3. Failure of Saddlepoint Approximation Technique in CreditRisk$^+$ Setting

In this section we want to show that there does not exist a saddle-point equivalent to the GA in case of the extended single-factor CreditRisk$^+$ model introduced above where double default effects are respected. Therefore, let us briefly review how this method is used to measure name concentration risk within a one-factor framework. In analogy to Martin and Wilde [2003] we use the following procedure to construct a series of portfolios which become more and more fine grained and such that the expected loss in the portfolios remains constant. We start with a portfolio of M risky loans indexed by $m = 1, \ldots, M$.[6] Denote by $s_m = \mathrm{E}_m / \sum_{i=1}^{M} \mathrm{E}_i$ the exposure share of obligor m. Starting with this portfolio of M risky loans, we construct a sequence of new portfolios by dividing the exposures in the former portfolio by the iteration step number, i.e. in step N there exist NM loans in the portfolio and each N of them are identical in size, namely s_m/N for $m = 1, \ldots, M$. Let L_N denote the loss distribution of the N^{th} portfolio constructed in this way. Each of these portfolios satisfies the assumptions underlying the Asymptotic Single Risk Factor (ASRF) model that underpins the IRB approach of Basel II. Thus, it can be shown that the portfolio loss variable L_N tends to the systematic loss distribution almost surely for N large, i.e.[7]

$$L_N - \mathbb{E}[L_N|X] \longrightarrow 0 \quad \text{almost surely as } N \to \infty.$$

Here we used the simplifying assumption that guarantors are external, distinct and hedges are full guarantees in order to ensure conditional independence. Note, however, that it is in principle possible to generalize this setting. If guarantors are obligors in the portfolio themselves, one guarantor hedges exposures to several obligors or exposures are only partially hedged, we assume that the total exposure share that is hedged is sufficiently small such that the assumptions underlying the ASRF approach are satisfied. The expression for the conditional variance in equation (2.3) is then more complicated but can still be computed analytically as in Ebert and Lütkebohmert [2009]. For the message of this paper, however, it suffices to consider the simplified situation.

[5] A generalization of this assumption is discussed in Section 3.

[6] The variable N which described the number of loans in our former portfolio will be used later on in the construction of new portfolios which become more and more fine-grained as N increases.

[7] Compare Gordy [2003], Proposition 1.

The difference in terms of percentiles between L_N and the conditional expected loss $\mathbb{E}[L_N|X]$ is the granularity adjustment. In the generalized CreditRisk$^+$ framework of Section 2 we can rewrite the expectation $\mu(x)$ and the variance $\sigma^2(x)$ of portfolio loss variable L_N of the N^{th} portfolio conditional on the systematic factor X as

$$\begin{aligned}\mathbb{E}[L_N|X] &= \sum_{m=1}^{M} s_m \mu_m(X)\mu_{g_m}(X) = \mu(X),\\ \mathbb{V}[L_N|X] &= N\sum_{m=1}^{M}\left(\frac{s_m}{N}\right)^2 \sigma_m^2(X)\sigma_{g_m}^2(X) = \frac{\sigma^2(X)}{N},\end{aligned}$$

for $N \geq 1$. Note that the conditional mean stays constant for all portfolios while the conditional variance gets divided by the number N of the iteration step. Setting $u = 1/N$ we can express the loss distribution of the N^{th} portfolio as

$$L_u = \mu(X) + \sqrt{u}\sigma(X)\cdot Y, \tag{3.1}$$

where Y is a standard normal random variable.

We now apply the saddle-point approximation to this setting in order to derive a granularity adjustment that takes into account double default effects. Notice that the derivation of the saddle-point approximation is exactly the same as in Martin and Wilde [2003] due to equation (3.1) where only the expectation $\mu(X)$ and the variance $\sigma^2(X)$ conditional on X are different in the double default setting. Hence the result in Martin and Wilde [2003] can be applied such that the saddle-point equivalent to the granularity adjustment is

$$\left.\frac{\partial\alpha_q(L_u)}{\partial u}\right|_{u=0} = \frac{\hat{t}}{2M_{\mu(X)}(\hat{t})}\int_0^{+\infty}\sigma^2(x)e^{\hat{t}\mu(x)}h(x)dx \tag{3.2}$$

where the saddle-point $\hat{t}$ is the solution to

$$\left.\frac{\partial M_{\mu(X)}(s)}{\partial s}\right|_{s=\hat{t}} = \alpha_q(\mu(X))\cdot M_{\mu(X)}(\hat{t}), \tag{3.3}$$

and $M_{\mu(X)}(s)$ is the moment generating function of the random variable $\mu(X)$ evaluated at s. The expressions, which have to be inserted for $\mu(X)$ and $\sigma^2(X)$, are given in equation (2.3) and respect double default effects in our setting.

The MGF of $\mu(X)$ in our extended CreditRisk$^+$ model is given by

$$M_{\mu(X)}(s) = \int_0^{+\infty} e^{s(A\cdot x^2+B\cdot x)}\frac{1}{\beta^\alpha\Gamma(\alpha)}e^{-x/\beta}x^{\alpha-1}dx \tag{3.4}$$

where $A := \sum_{n=1}^{K} s_n\cdot \mathrm{LGD}_n\,\mathrm{LGD}_{g_n}\cdot \mathrm{PD}_n\cdot\mathrm{PD}_{g_n}$ and $B = \sum_{n=K+1}^{N} s_n\cdot\mathrm{LGD}_n\cdot\mathrm{PD}_n$. Here K is the number of hedged positions in the portfolio.[8] Note that the function $M_{\mu(X)}(s)$ and also its derivative with respect to s is positive for all s. Moreover, for $K \geq 1$ this function does not converge for positive s. From equation (3.2) we can deduce that the saddle-point $\hat{t}$ (if it exists) will be positive as $M_{\mu(X)}(\hat{t})$ is positive and also the term

$$\int_0^{+\infty}\sigma^2(x)e^{\hat{t}\mu(x)}h(x)dx > 0$$

since $\sigma^2(x)$ and the density $h(x)$ of the Gamma distribution are always positive. Thus $\hat{t}$ has to be positive since the GA and its saddle-point equivalent only make sense if they take positive values. For positive s, however, the MGF $M_{\mu(X)}(s)$ and its derivative do not converge. Hence the equation (3.3) for the saddle-point $\hat{t}$ cannot be solved in case of the single-factor CreditRisk$^+$ model when double default effects are respected. Thus, there does not exist a saddle-point equivalent to the GA in the presence of double defaults in this model framework.

[8] Without loss of generality we assume these to be the first K positions.

4. Saddle-Point Method Within the Double Default Treatment of Basel II

In this section we show that the saddle-point approximation method for the quantification of undiversified idiosyncratic risk in credit portfolios also leads to unsatisfactory results when based on the model underlying the double default treatment in Basel II (see Basel Committee on Bank Supervision [2006], paragraph 284). The latter credit portfolio model to incorporate double default effects is based on a conditional independence framework. Specifically, conditional on a (possibly multidimensional) systematic risk factor X with probability density function $h(x)$, default events are assumed to be independent. Consider a portfolio with N obligors indexed by $n = 1, \ldots, N$. Denote the exposure to counterparty n by E_n and its conditional default probability (PD) given the systematic risk factor X by $p_n(X)$. The unconditional PD of counterparty n is the average over X and thus equals

$$\mathrm{PD}_n = \mathbb{E}[p_n(X)] = \int_{-\infty}^{+\infty} p_n(x)h(x)dx.$$

Denote by $\mathrm{LGD}_n \in [0,1]$ the loss given default (LGD) of counterparty n. In the classical default-mode Merton model the asset return of counterparty n is given by

$$r_n = \sqrt{\rho_n}X + \sqrt{1-\rho_n}\epsilon_n$$

where X and ϵ_n are independent, standard normally distributed random variables with $h(\cdot) = \varphi(\cdot)$, the standard normal density function. ρ_n denotes the asset correlation of counterparty n. Then the default threshold t_n can be calculated from counterparty n's PD as

$$t_n = \Phi^{-1}(\mathrm{PD}_n)$$

and the conditional PD given X for counterparty n is

$$p_n(X) = \mathbb{P}(r_n < t_n | X) = \Phi\left(\frac{t_n - \sqrt{\rho_n}X}{\sqrt{1-\rho_n}}\right)$$

where Φ is the cumulative standard normal distribution function and Φ^{-1} its inverse.

Now suppose the exposure to obligor n is hedged by a guarantor g_n who is not an obligor of the portfolio himself. The double default treatment under the IRB approach of Pillar 1 in Basel II is based on the model framework of Heitfield and Barger [2003].[9] They introduce additional risk factors Z_{n,g_n} which only affect the obligor n and its guarantor g_n. Then asset returns can be modeled as

$$r_n = \sqrt{\rho_n}X + \sqrt{1-\rho_n}\left(\sqrt{\psi_{n,g_n}}Z_{n,g_n} + \sqrt{1-\psi_{n,g_n}}\epsilon_n\right)$$

where all random variables are standard normally distributed and stochastically independent. Here ψ_{n,g_n} specifies the sensitivity of obligor n to the factor Z_{n,g_n}. Then the correlation between two obligors can be computed as[10]

$$\mathrm{Corr}(r_n, r_m) = \sqrt{\rho_n \rho_m},$$

while the correlation between two guarantors is

$$\mathrm{Corr}(r_{g_n}, r_{g_m}) = \sqrt{\rho_{g_n}\rho_{g_m}}.$$

Finally, the correlation between an obligor and its guarantor is given by

$$\rho_{n,g_n} = \mathrm{Corr}(r_n, r_{g_n}) = \sqrt{\rho_n \rho_{g_n}} + \sqrt{(1-\rho_n)(1-\rho_{g_n})}\,\psi_{n,g_n}$$

[9] For a detailed description of the theoretical model framework we also refer to Grundke [2008].

[10] Here it is also assumed that different obligors have different external guarantors.

and the joint default probability of an obligor and its guarantor is

$$\mathbb{P}\left(\{r_n < t_n\} \cap \{r_{g_n} < t_{g_n}\}\right) = \Phi_2\left(\Phi^{-1}(\mathrm{PD}_n), \Phi^{-1}(\mathrm{PD}_{g_n}), \rho_{n,g_n}\right).$$

Thus, the conditional expected loss function $\mu_n(X)$ for an unhedged exposure equals

$$\mu_n(X) = \mathbb{E}\left[\mathbf{1}_{\{r_n<t_n\}}\,\mathrm{LGD}_n\,|X\right] = \Phi\left(\frac{t_n - \sqrt{\rho_n}X}{\sqrt{1-\rho_n}}\right)\cdot \mathrm{LGD}_n$$

and for a hedged exposure

$$\begin{aligned}\mu_n(X) &= \mathbb{E}\left[\mathbf{1}_{\{r_n<t_n\}}\mathbf{1}_{\{r_{g_n}<t_{g_n}\}}\,\mathrm{LGD}_n\,\mathrm{LGD}_{g_n}\,|X\right]\\ &= \mathrm{LGD}_n\,\mathrm{LGD}_{g_n}\cdot\Phi_2\left(\frac{\Phi^{-1}(\mathrm{PD}_n)-\sqrt{\rho_n}X}{\sqrt{1-\rho_n}}, \frac{\Phi^{-1}(\mathrm{PD}_{g_n})-\sqrt{\rho_{g_n}}X}{\sqrt{1-\rho_{g_n}}}, \frac{\rho_{n,g_n}-\sqrt{\rho_n\rho_{g_n}}}{\sqrt{(1-\rho_n)(1-\rho_{g_n})}}\right)\end{aligned}$$

where $\Phi_2(\cdot,\cdot,\rho)$ denotes the (cumulative) bivariate normal distribution function with correlation parameter ρ. Similarly, we can compute the conditional variance $\sigma_n^2(X)$ for an unhedged and for a hedged exposure. The resulting expressions are quite complex. Thus, we omit them at this point and only mention that the computation is similar to the corresponding calculations in Ebert and Lütkebohmert [2009]. Note that in this framework we still have conditional independence between obligors as every hedged obligor has a different guarantor and guarantors are not part of the portfolio themselves.

To compute the saddle-point equivalent to the GA we have to plug these expressions into equations (3.2) and (3.3). The latter then first has to be solved for $\hat{t}$. The resulting value then has to be inserted into equation (3.2) which finally would provide the GA. This method, however, is neither analytically solvable nor can it be solved with numerical standard software. For its solution (which might not exist or be unique) an involved numerical algorithm would have to be designed. In the most outer loop of such an algorithm, a Newton-Raphson method or another numerical equation solver has to be applied. In each iteration step an infinite dimensional non standard integral has to be computed using an appropriate numerical integration scheme.[11] Finally, for each point evaluation required by the integration scheme a cumulative bivariate normal distribution function has to be evaluated which by itself is already a nontrivial task. While it might be in principle possible to compute the saddle-point GA in this way, however, it should be clear that this approach is not competitive anymore to standard Monte Carlo simulation. Let us also note that there do exist other analytical approaches, like the GA in Ebert and Lütkebohmert [2009], which are far less complicated and which can be computed also in the setting used for the double default treatment of Basel II.

5. Conclusion

We showed that the saddle-point approximation technique is inappropriate to quantify the impact of idiosyncratic risk on portfolio VaR when some exposures in the portfolio are hedged by certain guarantors (even if the latter are not part of the portfolio themselves). We explicitly proved that the method fails in case of the extended single-factor CreditRisk$^+$ model as the equation for the saddle-point is no longer solvable since it contains a non-converging integral. Moreover, we showed that the saddle-point method leads to a highly complicated numerical problem in case of the extended default-mode Merton model underlying the double default treatment in Basel II. Let us finally note that there are other analytical approaches such as the GA method of

[11] An efficient approach would be to use a locally adaptive Gauss-Legendre method combined with a Moro scheme to treat the Gauss kernel accordingly.

Ebert and Lütkebohmert [2009] which account for double default effects and thus outperform the saddle-point approximation in this respect.

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