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Collusion via Resale*

by Rodney J. Garratt,† Thomas Tröger,‡ and Charles Z. Zheng§

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Abstract

The English auction is susceptible to tacit collusion when post-auction inter-bidder resale is allowed. We show this by constructing equilibria where, with positive probability, one bidder wins the auction without any competition and divides the spoils by optimally reselling the good to the other bidders. These equilibria interim Pareto dominate (among bidders) the standard value-bidding equilibrium, without requiring the bidders to make any commitment on bidding behavior or post-bidding spoil-division.

1 Introduction

In private-value English auctions that ban resale, it is a dominant strategy for each participant to bid up to her use value. With resale allowed, value-bidding remains an equilibrium outcome, but there is no dominant strategy. Resale opens the possibility that some bidders will optimally drop out at a price below their use values. They prefer to let a competitor win and buy from her in the resale market. The existence of non-value-bidding equilibria is important because the celebrated advantages of the English auction, in particular efficiency, are based on value-bidding, and because resale is possible in most applications.

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In this paper we construct a family of non-value-bidding equilibria for an English auction that allows inter-bidder resale. Such equilibria exist in any independent private value environment (symmetric or asymmetric) for any number of bidders (Proposition 1). Each equilibrium in this family is identified by the choice of a *designated bidder* and a *threshold* type, below which all bidders, except the designated bidder, bid zero. All bidders with types above the threshold bid up to their values. In cases where the designated bidder wins the initial auction and has a sufficiently low type, she will offer the item for resale instead of consuming it. Because the determination of the designated bidder does not depend upon her type and the resale market retains information asymmetry, the final outcome may be inefficient.

Since a designated bidder may win the initial auction at a low price, such equilibria provide an opportunity for a form of tacit collusion among the bidders. By using a publicly observed randomizing device (or sunspot) to choose the designated bidder, the surplus can be distributed in a way that makes every bidder of every type better-off than under the value-bidding equilibrium; i.e., the value-bidding equilibrium is interim (bidder-)Pareto dominated (Proposition 2).[1] The recommendation made by the sunspot device is not binding. Once the sunspot picks a designated bidder, it is in the interest of each bidder to bid accordingly in the initial auction based on the expectation that others will follow their assigned roles.

Previous models of collusion in second-price and English auctions (e.g., Graham and Marshall, 1987, Mailath and Zemsky, 1991, Marshall and Marx, 2007) rely on *pre-auction communication*, in which every colluding bidder reports her type to the bidding ring. By communicating, the colluding bidders determine side payments and designate a single bidder to participate in the auction and win at a low price. These papers specify mechanisms that achieve efficient collusion as an equilibrium. However, the proposed use of pre-auction communication is problematic because it is usually illegal and participating bidders risk being detected. Moreover, the proposed collusive schemes require the non-designated bidders in

[1] Readers who are familiar with U.S. litigation history might draw some parallels between our proposed use of a sunspots variable and the famous phases-of-the-moon bidding ring that was operated by electrical equipment suppliers in the 1950s. However, despite some reports, the phases-of-the-moon scheme earned its designation because it involved an explicit two-week rotation to determine the low bidder. While it perhaps could have been, bidding was not actually determined by the phase of the moon. See Smith (1961).

the bidding ring to bid below their values in the actual auction. Without resale, such bidding strategies are weakly dominated, and bidders may not be willing to play them. To make the expectation of a dominated strategies credible, the colluding bidders might require a commitment device.[2]

By introducing the possibility of resale after an English or second-price auction, our paper rationalizes collusion without pre-auction communication or dominated strategies. Instead of pre-auction communication of private information, a publicly observable sunspot selects a designated bidder in a manner commonly known to the colluding bidders, and the final owner of the good is decided through a resale mechanism. Before the auction, no one commits to what she will do in the auction or at resale. During the auction colluding bidders do not bid up to their values, however we prove that such strategies are not weakly dominated given the option for resale (Appendix B). The winner of the auction chooses a resale mechanism that is optimal for her given the posterior beliefs after the auction, and only after the initial auction has ended can she commit to the rules of her resale mechanism.

McAfee and McMillan (1992, p. 587) noted that in practice a bidding ring's own "knock-out auction" often happens after rather than before the legitimate auction. This practice is well represented in our equilibria.

Our Pareto-improving equilibria are, in contrast to the previously proposed collusive schemes, not ex-post efficient.[3] This is consistent with an impossibility result in Lopomo, Marshall, and Marx (2005), which shows that inefficiency is a quite general feature of Pareto-improving equilibria in English auctions without pre-auction communication. From the viewpoint of antitrust authorities, inefficient collusive equilibria are important precisely because of the distortion; efficient collusion involves merely a pure transfer from the seller to the bidders.

Blume and Heidhues (2004) completely characterize the Bayesian Nash equilibria for the second-price auction with three or more bidders with a common type space. These equilibria are valid for English auctions and have the same bidding structure as our equilibria.

[2]Sustaining collusion in first-price auctions is more difficult than in second-price auctions because non-designated bidders have a strict incentive to overbid the designated bidder whose bid in the main auction is below their value; see McAfee and McMillan (1992) and Marshall and Marx (2007).

[3]The payoff gains in our equilibria relative to value-bidding can still be substantial (Table 1, Section 5).

However, because there is no resale market, the Blume-Heidhues equilibria are in dominated strategies. Moreover, in some environments none of these equilibria Pareto dominates the value-bidding equilibrium, even if we give everyone a chance to be the designated bidder through sunspot coordination. This is because high-value bidders strictly prefer the value-bidding equilibrium; we show that for all symmetric environments with strictly concave value distributions (Proposition 3). But with resale, there always exists an equilibrium that makes every type of every bidder strictly better off than the value-bidding equilibrium.

In contrast to much of the earlier literature on auctions with resale, our model allows for any number of asymmetric bidders. With multiple bidders at the resale stage, the optimal resale mechanism typically is no longer a take-it-or-leave offer, as often assumed, but is an optimal auction as derived by Myerson (1981). A technical novelty of our paper is that we avoid complicated explicit computations of bidders' resale payoffs. We identify several structural properties of the resale continuation game that facilitate our perfect Bayesian equilibria for the entire auction-with-resale game. These structural properties do not appear to be specific to the Myerson optimal auction, suggesting that our qualitative results extend to other forms of the resale market.[4]

Our result that the value-bidding equilibrium is interim (bidder-)Pareto dominated is based on two regularity properties of resale that are satisfied in our equilibria. The properties ensure that, even when the threshold is arbitrarily close to zero, a non-vanishing fraction of the bidder-types below the threshold still engage in actual resale trade. The proof begins with the observation that our equilibria interim Pareto dominate the value-bidding equilibrium if the prior type distributions are uniform. Then we extend this dominance relation to arbitrary type distributions provided that the thresholds in our equilibria are sufficiently small. Sufficiently small thresholds allow us to approximate the expected payoffs by the ones in the uniform-distribution case. The aforementioned regularity properties imply that a bidder's gain from trade at resale outweighs the error of the approximation.

The threshold-bidding strategies in our equilibria are built upon Garratt and Tröger (2006)

[4]For instance, the period-2 seller may be restricted to use a second-price or English auction with an optimal reserve price, or the period-2 seller may use an English auction with the right to reject all bids (Haile, 2003), or, in two-bidder environments, a random draw may specify which bidder has the right to propose a resale price (Calzolari and Pavan, 2006).

for English and second-price auctions. However, there are nontrivial differences. In Garratt and Tröger, the bidder who will become the reseller has no private information, and the other bidders have identical prior value distributions. In contrast, in this paper every bidder has private information and bidders' value-distributions can differ. Our extension of the equilibrium construction to the case of asymmetric bidders is made possible by conditioning the designated bidder's bid on the identities of the bidders who stay in the auction. This information is not available in a sealed-bid format. Hence, in environments with three or more bidders our equilibrium construction applies to second-price auctions only under an additional symmetry assumption (Remark 3).

Recent works by Lebrun (2007) and Hafalir and Krishna (2007) compare revenue in first- and second-price auctions with resale in 2-bidder models. Hafalir and Krishna show that in a 2-bidder asymmetric model there exists a "general revenue ranking" in favor of first-price auctions, provided bidders play the value-bidding equilibrium in the second-price auction. Lebrun shows that, depending on the selected equilibria, this ranking does not necessarily hold when behavior (mixed) strategies are allowed. Our construction of equilibria for second-price auctions that interim Pareto-dominate value bidding does not challenge the revenue ranking established by Hafalir and Krishna. Because the value-bidding equilibrium is efficient, it yields a seller revenue that is necessarily greater than in any interim Pareto-dominating equilibrium.

Collusive equilibria have been constructed for multi-unit auctions by Milgrom (2000), Brusco and Lopomo (2002) and Engelbrecht-Wiggans and Kahn (2005), however resale does not play a role.[5] In these multi-unit environments bidders signal their preferences in early rounds and then optimally abstain from bidding on other bidders' preferred items. Interestingly, the open aspect of the ascending English auction is essential in their construction, as it is here in the case of ex-ante asymmetric bidders.

[5] Pagnozzi (2007) analyzes multi-unit auctions with resale in a complete information model.

2 Model

We consider environments with $n \geq 2$ risk-neutral bidders pursuing a single indivisible private good. Bidder $i \in N := \{1, \ldots, n\}$ has a privately known use value, or *type*, $t_i \in T_i = [0, \bar{t}_i]$ ($\bar{t}_i > 0$), for the good. The *type space* is denoted by $\mathbf{T} := T_1 \times \cdots \times T_n$.[6] From the viewpoint of the other bidders, t_i is independently distributed according to a probability distribution with cumulative distribution function F_i, called *prior belief*, with support T_i and Lipschitz continuous positive density f_i. We consider a 2-period interaction, which begins after each bidder i has privately observed her use value t_i.

In period 1, the good is offered via an English auction that is modelled as in Milgrom and Weber (1982). The auctioneer continuously raises the *current price* beginning at 0. Remark 1 extends the equilibrium construction to an English auction that starts at a positive price). Initially, all bidders are "in" or "active". As the current price rises, each bidder can irreversibly "drop out" at any point. When somebody drops out, other bidders may react by dropping out at the same price. Dropout decisions are publicly observed. The auction ends at the first current price where at most one bidder is active. The ending current price is called the *auction price*. If one bidder is active at the end, this is the auction winner; if all active bidders drop out at the auction price, there is a tie among the previously active bidders, and any tieing bidder becomes the winner with equal probability.

The auction winner either consumes the good in period 1, thereby ending the game, or becomes the *period-2 seller*. The period-2 seller proposes a sales mechanism to the losing bidders, called *period-2 buyers*. A sales mechanism is any game form to be played by the period-2 buyers. The mechanism is played if it is accepted by all period-2 buyers; otherwise the resale seller consumes the good in period 2.[7] Observe that the period-2 seller faces no restrictions: she can choose an arbitrary sales mechanism. No bidder can pre-commit in period 1 to use a particular mechanism in period 2.

Every bidder's discount factor is $\delta \in (0, 1]$. From the viewpoint of period 1, the payoff

[6] We shall use boldface letters to denote multidimensional quantities.

[7] We assume that there is no further resale after the re-seller's resale mechanism. However, for certain prior value distributions, this assumption can be weakened to allow the winner of the resale mechanism to offer further resale à la Zheng (2002).

of type t_i of bidder i is $t_i(q^1 + \delta q^2) - p^1 - \delta p^2$, where q^k $(k = 1, 2)$ denotes the probability of her consuming the good in period k, and p^k denotes her net expected monetary payment in period k. From the viewpoint of period 2, the bidders only care about their period-2 payments and period-2 allocation probabilities.

2.1 Histories, strategies, beliefs, and period-2 outcome

A *non-terminal history* lists the observed dropout decisions and the current price at any point before the period-1 auction ends. The set of non-terminal histories where bidder $i \in N$ is active is denoted $\mathcal{H}_i$. A *terminal history* lists the observed dropout decisions up to the end of the auction, the auction price, and the winner. The set of *terminal histories* is denoted $\mathcal{H}_{\text{term}}$. The set of all histories is denoted $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_n \cup \mathcal{H}_{\text{term}}$.

A *bidding strategy profile* $(\beta_i(\cdot \mid h))_{i \in N, h \in \mathcal{H}_i}$ determines, for any bidder $i \in N$, any history $h \in \mathcal{H}_i$ and any type $t_i \in T_i$, the current price $b = \beta_i(t_i \mid h)$ at which bidder i plans to drop out. We call b the *bid* of type t_i of bidder i at h. If b is smaller than or equal to the current price at h, then b is interpreted as dropping out immediately at h. Observe that, if at h a bidder $\neq i$ bids $b' < b$, then the plan to drop out at b becomes irrelevant once the current price b' is reached.

A *resale decision profile* is a vector $(\gamma_h)_{h \in \mathcal{H}_{\text{term}}}$, where $\gamma_h(t) = \text{resale}$ if type t of the winner offers the good for resale at the terminal history h, and $\gamma_h(t) = \text{consume}$ if the winner consumes the good in period 1. A *period-1 belief profile* $(\mathbf{A}_h)_{h \in \mathcal{H}}$ assigns to each history h a probability distribution $\mathbf{A}_h$ on $\mathbf{T}$ that represents the (common) belief at h about the bidders' types. A *period-2 belief profile* $(\mathbf{G}_h)_{h \in \mathcal{H}_{\text{term}}}$ assigns to each h a probability distribution $\mathbf{G}_h$ on $\mathbf{T}$ that represents the (common) belief about the bidders' types if the winner offers the good for resale, given the terminal history h.

Using the revelation principle, we can describe a *period-2 outcome* as a vector $(P_i(\mathbf{t}), Q_i(\mathbf{t}))_{i \in N,\ \mathbf{t} \in \mathbf{T}}$ such that, for any bidder i and type profile $\mathbf{t} \in \mathbf{T}$, the number $P_i(\mathbf{t})$ denotes the expected net period-2 monetary transfer of bidder i, and $Q_i(\mathbf{t})$ denotes the probability that bidder i consumes the good in period 2, where $\sum_{k \in N} Q_k(\mathbf{t}) = 1$ and $\sum_{k \in N} P_k(\mathbf{t}) = 0$.

2.2 The equilibrium concept

An *equilibrium* consists of a bidding profile $(\beta_i(\cdot \mid h))_{i\in N, h\in\mathcal{H}_i}$, a resale-decision profile $(\gamma_h)_{h\in\mathcal{H}_{\text{term}}}$, a period-1 belief profile $(\mathbf{A}_h)_{h\in\mathcal{H}}$, a period-2 belief profile $(\mathbf{G}_h)_{h\in\mathcal{H}_{\text{term}}}$, and a family of period-2 outcomes $(P_{i,h}(\mathbf{t}), Q_{i,h}(\mathbf{t}))_{i\in N,\ \mathbf{t}\in\mathbf{T},\ h\in\mathcal{H}_{\text{term}}}$ with the following properties:

a. for all $h \in \mathcal{H}_{\text{term}}$, the period-2 outcome $(P_{i,h}(\mathbf{t}), Q_{i,h}(\mathbf{t}))_{i\in N,\ \mathbf{t}\in\mathbf{T}}$ is induced by a perfect Bayesian equilibrium of the period-2 continuation game, given the period-2 seller from terminal history h and the (commonly known) period-2 belief $\mathbf{G}_h$;

b. the period-1 belief profile $(\mathbf{A}_h)_{h\in\mathcal{H}}$ obeys Bayes' rule with respect to the bidding profile $(\beta_i(\cdot \mid h))_{i\in N, h\in\mathcal{H}_i}$;

c. for all $h \in \mathcal{H}_{\text{term}}$, $j \in N$, and $t_j \in T_j$, if bidder j with type t_j is the period-1 winner at history h, then $\gamma_h(t_j) = \text{resale (resp., consume)}$ if t_j is strictly smaller (resp., greater) than j's discounted period-2 payoff, given the period-2 belief $\mathbf{G}_h$ and the expected period-2 outcome $(P_{i,h}(\mathbf{t}), Q_{i,h}(\mathbf{t}))_{i\in N,\ \mathbf{t}\in\mathbf{T}}$;

d. the period-2 belief profile $(\mathbf{G}_h)_{h\in\mathcal{H}}$ obeys Bayes' rule with respect to the period-1 belief profile $(\mathbf{A}_h)_{h\in\mathcal{H}}$ and the resale-decision profile $(\gamma_h)_{h\in\mathcal{H}_{\text{term}}}$;

e. for all $i \in N$, $t_i \in T_i$, and $h \in \mathcal{H}_i$, the bid $\beta_i(t_i \mid h)$ maximizes the expected payoff of type t_i of bidder i at the history h, given the belief $\mathbf{A}_h$, provided that everyone else abides to the bidding profile, bidder i abides to her bidding strategy after additional bidders drop out, and that the resale-decision profile and family of period-2 outcomes are implemented.

We construct equilibria where period-1 beliefs and period-2 beliefs are stochastically independent across bidders, and where any belief about bidder i's ($i \in N$) type is derived from the prior belief and the information that her type lies in a (possibly degenerate) interval $J_i \subseteq T_i$. Let $\mathcal{J}$ denote the set of interval products $\mathbf{J} = J_1 \times \cdots \times J_n \subseteq \mathbf{T}$. We identify any belief with an element of $\mathcal{J}$. That is, we will treat $\mathbf{J}$ variably as a product of intervals or as a cumulative distribution function on $\mathbf{T}$, and will treat J_i variably as an interval or a cumulative distribution function on T_i.

3 The period-2 continuation game

In view of equilibrium condition (a), we select a perfect Bayesian equilibrium of the period-2 continuation game for any period-2 seller $j \in N$ and any belief $\mathbf{J} = J_1 \times \cdots \times J_n \in \mathcal{J}$. We use a well-known selection: the mechanism proposed by the period-2 seller is Myerson's (1981) auction that is optimal for period-2 seller j based on period-2 belief profile $\mathbf{J}$, given the assumption that resale after period 2 is impossible.[8] Denote the period-2 outcome implemented by this auction by $(P_i^{j,\mathbf{J}}(\mathbf{t}), Q_i^{j,\mathbf{J}}(\mathbf{t}))_{i\in N,\ \mathbf{t}\in\mathbf{T}}$ for any $(j, \mathbf{J})$ specified above.[9]

Notation is needed to describe the properties of the Myerson optimal auction outcome that are used in the following analysis. For all $i \in N$, let $\mathbf{J}_{-i} = \prod_{k\neq i} J_k$ denote the marginal distribution induced by $\mathbf{J}$ on $\mathbf{T}_{-i} := \prod_{k\neq i} T_k$. The probability that type $t_i \in T_i$ of bidder i consumes the good in period 2 (by keeping it if $i = j$ and obtaining it if $i \neq j$) is denoted

$$q_{ij}(t_i, \mathbf{J}) = \int_{\mathbf{T}_{-i}} Q_i^{j,\mathbf{J}}(\mathbf{t})\mathrm{d}\mathbf{J}_{-i}(\mathbf{t}_{-i}),$$

and her period-2 expected payoff is denoted

$$l_{ij}(t_i, \mathbf{J}) = t_i q_{ij}(t_i, \mathbf{J}) - \int_{\mathbf{T}_{-i}} P_i^{j,\mathbf{J}}(\mathbf{t})\mathrm{d}\mathbf{J}_{-i}(\mathbf{t}_{-i}).$$

For the period-2 seller we will use the shortcuts $q_j(t_j, \mathbf{J}) = q_{jj}(t_j, \mathbf{J})$ and $w_j(t_j, \mathbf{J}) = l_{jj}(t_j, \mathbf{J})$.

A well-known implication of the incentive compatibility of the period-2 outcome, proved via the envelope theorem in integral form (Milgrom and Segal, 2002), are the following

[8] Our period-2 environment differs from Myerson's environment insofar as the period-2 seller may be privately informed about her type. This plays no role because by assumption the period-2 seller is not a player in her sales mechanism, so that the period-2 buyers' beliefs about the seller's type have no impact on their behavior. If we did allow mechanisms where the period-2 seller is a player, the Myerson optimal auction outcome would still be a "strong solution" defined by Myerson (1983); see Mylovanov and Tröger (2008).

[9] Observe that the period-2 outcome includes payments and allocation probabilities for types not in $\mathbf{J}$. The period-2 seller believes that types $t_i \notin J_i$ occur with probability 0. We assume that any type $t_i > \sup J_i$ obtains the good with the same probability as type $\sup J_i$, and any type $t_i < \inf J_i$ obtains the good with probability 0.

envelope formulas: for all $t_i, t_i' \in T_i$ $(t_i' < t_i)$ and $t_j, t_j' \in T_j$ $(t_j' < t_j)$,

$$l_{ij}(t_i, \mathbf{J}) - l_{ij}(t_i', \mathbf{J}) = \int_{t_i'}^{t_i} q_{ij}(x, \mathbf{J})\mathrm{d}x, \quad (1)$$

$$w_j(t_j, \mathbf{J}) - w_j(t_j', \mathbf{J}) = \int_{t_j'}^{t_j} q_j(x, \mathbf{J})\mathrm{d}x. \quad (2)$$

Period-2 payments can be defined such that the ex-post participation conditions are satisfied: for all $\mathbf{t} \in \mathbf{T}$, $j \in N$, and $i \neq j$,

$$t_j \leq t_j Q_j^{j,\mathbf{J}}(\mathbf{t}) - P_j^{j,\mathbf{J}}(\mathbf{t}), \quad (3)$$

$$0 \leq t_i Q_i^{j,\mathbf{J}}(\mathbf{t}) - P_i^{j,\mathbf{J}}(\mathbf{t}). \quad (4)$$

If there are expected gains from trade between the period-2 seller and all buyers, then by using an optimal auction the period-2 seller j captures a nonzero share of the gains: for all $t_j \in T_j$,

$$[\forall i \in N \setminus \{j\},\ \max J_i >> t_j] \implies w_j(t_j, \mathbf{J}) > t_j. \quad (5)$$

A further straightforward property of the Myerson optimal auction outcome is that the period-2 seller's period-2 payoff is continuous in the period-2 belief about her type (in fact, the payoff is independent of the belief). To state this property, let $[0, x] \times \mathbf{J}_{-j}$ denote the belief that bidder j's type is at most x and other bidders' types are in $\mathbf{J}_{-j}$. For all $j \in N$, $t_j \in T_j$, and $\mathbf{J} \in \mathcal{J}$, the map

$$x \mapsto w_j(t_j, [0, x] \times \mathbf{J}_{-j}) \text{ is continuous on } T_j. \quad (6)$$

The next property—that a bidder's period-2 payoff as a buyer is never larger than as a seller—follows from the fact that the bidder's type is a lower bound for her period-2 payoff as a seller and an upper bound for her period-2 payoff as a buyer. For all $i \in N$, $t_i \in T_i$, $j \in N \setminus \{i\}$, and $\mathbf{J} \in \mathcal{J}$,

$$w_i(t_i, \mathbf{J}) \geq l_{ij}(t_i, \mathbf{J}). \quad (7)$$

The next relevant property of the Myerson optimal auction outcome is proved in Appendix A: each bidder as a period-2 seller consumes the good with a weakly higher probability than obtaining it as a period-2 buyer, given the same period-2 beliefs.

Lemma 1 *Let $i \in N$, $t_i \in T_i$, $j \in N \setminus \{i\}$, and $\mathbf{J} \in \mathcal{J}$. Then*

$$q_i(t_i, \mathbf{J}) \quad \geq \quad q_{ij}(t_i, \mathbf{J}). \tag{8}$$

Our equilibrium construction in the next section applies to any resale market that satisfies properties (1)–(8). None of these properties appears specific to the Myerson optimal auction outcome. In particular, (5) simply reflects the facts that the period-2 seller has some bargaining power and there is some resale trade. Property (7) is implied by (3)–(4) if $n = 2$; otherwise (7) essentially provides an upper bound on an auction loser's ability to extract rents if two other bidders trade the good in period 2. Property (8) reflects the basic intuition that private information leads to less trade than efficiency requires.

4 Equilibria for English auctions with resale

In this section, we construct a family of equilibria for the English auction with resale. In each equilibrium, one of the bidders, say bidder 1, is commonly known to be the designated bidder of the period-1 auction. Bidding strategies depend upon a threshold t^*, which can take on any value in the interval $(0, \min_{i \in N} \bar{t}_i]$. Every bidder with type above t^* bids her own type. All designated-bidder types below t^* bid more than 0 and not more than t^*. Any other bidder with a type below t^* drops out at the beginning of the auction. If someone with type above t^* wins, resale does not occur. Otherwise, the designated bidder wins at price 0; if her type is sufficiently low, she offers the good for resale in period 2 according to a continuation equilibrium described in Section 3. Since the selection of the designated bidder does not depend upon her type and informational asymmetries remain at resale, these equilibria are inefficient, contrary to the value-bidding equilibrium of English auctions.[10]

Before we present the equilibria (Proposition 1), we state some results that are used to specify the period-1 strategy for bidder 1. First, we establish that bidder 1's resale decision is defined by a non-zero cutoff (Lemma 2) such that lower types of bidder 1 prefer offering resale to immediate consumption, and higher types have the reverse preference. Second,

[10] Haile (1999) proves that when resale after an English or second-price auction is allowed, the efficient value-bidding equilibrium remains valid.

for each bidder $i \neq 1$ we define a price at which type t^* is indifferent between winning the period-1 auction and waiting for resale, and we show that these prices are strictly positive (Lemma 3). These prices become the bids for all types below t^* of bidder 1 who offer resale. Fix a threshold $t^* > 0$. For any $x \in T_1$, let $\mathbf{J}^*_x = [0, x] \times [0, t^*]^{n-1}$ denote the belief resulting from the prior belief and the information that bidder 1's type is below x and the other bidders' types are below t^*.

If the period-2 belief is $\mathbf{J}^*_{t^*}$, then a period-2 buyer's highest possible use value is t^*, so that types just below t^* of bidder 1 prefer consuming the good in period 1 to offering resale in period 2 if period-2 payoffs are discounted ($\delta < 1$). To find a cutoff type between consumption and offering resale, let

$$\tau^* := \sup \{x \in T_1 \mid \delta w_1(x, \mathbf{J}^*_x) > x\} \quad \text{if } \delta < 1. \tag{9}$$

Observe that $\tau^* < t^*$, because by (3)–(5) the set in (9) contains a type below t^*. We define $\tau^* = \bar{t}_1$ if $\delta = 1$.

Lemma 2 *Let t^* be a threshold. Then $\tau^* > 0$. For all $t_1 \in T_1$,*

$$t_1 \geq \delta w_1(t_1, \mathbf{J}^*_{\tau^*}) \quad \textit{if} \quad t_1 > \tau^*, \tag{10}$$

$$t_1 \leq \delta w_1(t_1, \mathbf{J}^*_{\tau^*}) \quad \textit{if} \quad t_1 < \tau^*, \tag{11}$$

$$\tau^* \to t^* \quad \textit{as} \quad \delta \to 1, \ \delta \neq 1. \tag{12}$$

For all bidders $i \neq 1$, let b^*_i denote the price that makes type t^* of bidder i indifferent between (i) winning the auction at price b^*_i and consuming the good, and (ii) participating in a resale market where bidder 1 is the period-2 seller and the period-2 belief is $\mathbf{J}^*_{\tau^*}$:

$$b^*_i \quad := \quad t^* - \delta l_{i1}(t^*, \mathbf{J}^*_{\tau^*}). \tag{13}$$

The following lemma provides bounds for b^*_i.

Lemma 3 *Let t^* be a threshold. Then $0 < b^*_i < t^*$ for all bidders $i \neq 1$.*

The bidding strategy for designated-bidder types below t^* is as follows. If her type is below τ^*, then at any history $h \in \mathcal{H}_1$ she bids $\max_{i \in S_1(h)} b^*_i$, where $S_1(h)$ denotes the set of bidders other than 1 who are active at the history h; and she will offer resale if she wins. If her type is between τ^* and t^*, then she bids t^* and she does not offer resale if she wins.

Proposition 1 *For any threshold t^*, there exists an equilibrium with properties (i)-(iv).*
(i) For any history $h \in \mathcal{H}_1$, type $t_1 \in T_1$ of bidder 1 bids the maximum of the current price and

$$\beta_1(t_1 \mid h) \quad = \quad \begin{cases} \max_{i \in S_1(h)} b_i^* & \text{if } t_1 \leq \tau^*, \\ t^* & \text{if } \tau^* < t_1 \leq t^*, \\ t_1 & \text{if } t_1 > t^*. \end{cases} \tag{14}$$

(ii) For all bidders $i \neq 1$ and any history $h \in \mathcal{H}_i$, type $t_i \in T_i$ of bidder i bids the maximum of the current price and

$$\beta_i(t_i \mid h) \quad = \quad \begin{cases} 0 & \text{if } t_i \leq t^*, \\ t_i & \text{if } t_i > t^*. \end{cases} \tag{15}$$

(iii) Let $\hat{h} \in \mathcal{H}_{term}$ denote the history where all bidders other than bidder 1 have dropped out at the beginning of the auction. Bidder 1's resale decision at history $\hat{h}$ is

$$\gamma_{\hat{h}}(t_1) := \begin{cases} \text{resale} & \text{if } t_1 \leq \tau^*, \\ \text{consume} & \text{if } t_1 > \tau^*. \end{cases} \tag{16}$$

(iv) At history $\hat{h}$, the period-2 outcome is the Myerson optimal auction outcome given the period-2 seller 1 and the period-2 belief $[0, \tau^*] \times [0, t^*]^{n-1}$.

Any equilibrium that satisfies properties (i)-(iv) is called a t^**-equilibrium*.

Proposition 1 is proved in Appendix A. Here we explain heuristically why a bidder $i \neq 1$, who is not the designated bidder, would abide by the equilibrium strategy of dropping out of the auction when his type is below the threshold t^*. Bidder i can deviate in at least two ways. He can either try to outbid the designated bidder (bidder 1) and consume the good upon winning it. Or he can try to outbid bidder 1 and offer resale upon winning.

To explain why both kinds of deviation are unprofitable, let us suppose that every other bidder's type is below t^*. Otherwise, someone else is bidding above bidder i's use value and bidder i's deviation does not increase his payoff on the equilibrium path.

If bidder i manages to outbid bidder 1 and consumes the good upon winning, then bidder i's payoff is equal to $t_i - b_i^*$, as the low-type bidder 1 bids up to b_i^* against bidder i

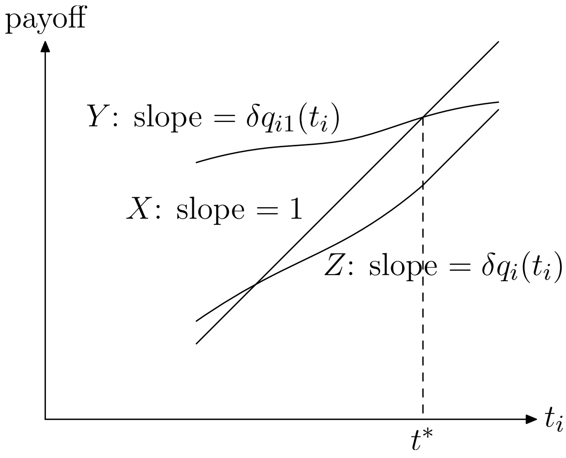

Figure 1: X: consume now; Y: buy at resale; Z: win now and offer resale

and everyone else, low-type and obedient, quits at zero price. This payoff is represented by the slope-1 straight line, labeled X, in Figure 1.

If bidder i plays the equilibrium strategy of dropping out of the auction and trying to buy the good at resale, then his expected payoff is equal to $\delta l_{i1}(t_i, \mathbf{J}^*_{\tau*})$. That is because, when everyone's type is below t^* and everyone abides by the equilibrium, bidder 1 wins the good at zero price and hence the post-auction belief is $\mathbf{J}^*_{\tau*}$. (Bidder i cannot profit from dropping out at a positive price given the off-path post-auction belief in our construction.) This expected payoff is represented by the curve labeled Y in Figure 1. Note that curve Y and line X intersect at the threshold t^*. That is because bidder 1's highest bid b^*_i against a deviant bidder i, defined by (13), makes bidder i of type t^* indifferent between X and Y.

An important point is that curve Y is less steep than line X, so that bidder i with types below t^* prefers Y (abiding by the equilibrium and waiting for resale) to X (outbidding bidder 1 and consuming the good). That follows from the envelope formula (1): When t_i changes from t^*, the payoff from X, $t_i - b_i(t^*)$, changes at the rate one while the payoff from Y, $\delta l_{i1}(t_i, \mathbf{J}^*_{\tau*})$, changes at the rate $\delta q_{i1}(t_i, \mathbf{J}^*_{\tau*}) < 1$.

Now consider the deviation where bidder i outbids the designated bidder and offers the good for resale upon winning. If bidder i manages to do that, his role in period 2 is switched from a bidder to a seller, and the post-auction beliefs $\mathbf{J}^*_{\tau*}$ are unchanged. His expected payoff at the start of period 2 will be equal to $w_i(t_i, \mathbf{J}^*_{\tau*})$, so his present expected payoff from such deviation is equal to $\delta w_i(t_i, \mathbf{J}^*_{\tau*}) - b_i(t^*)$.

This payoff is represented by the curve labeled as Z in Figure 1. Note that the position of the curve for $t_i \geq t^*$ is lower than that of line X. That is because a reseller whose type is above t^* cannot profit from resale, since everyone else's type is below t^*. Thus, this part of curve Z coincides with the corresponding part of X if there is no discounting, and it lies below X if there is discounting.

A crucial observation is that, for $t_i < t^*$, the curve Z is steeper than the curve Y. As Z is below Y at t^*, that means Z is always below Y for types below t^*, i.e., bidder i with types below t^* would rather be a period-2 buyer than a period-2 seller. To show this relationship between the slopes, recall the definition of $b_i(t^*)$ given in (13). We have

$$\begin{aligned}
Z^* &:= \delta w_i(t^*, \mathbf{J}^*_{\tau*}) - b_i(t^*) \\
&\leq w_i(t^*, \mathbf{J}^*_{\tau*}) - b_i(t^*) \\
&= t^* - b_i(t^*) \\
&= \delta l_{i1}(t^*, \mathbf{J}^*_{\tau*}) =: Y^*.
\end{aligned}$$

By the envelope formulae (1)–(2), $\frac{\partial w_i}{\partial t_i}(t_i, \mathbf{J}^*_{\tau*}) = q_i(t_i, \mathbf{J}^*_{\tau*})$ and $\frac{\partial l_{i1}}{\partial t_i}(t_i, \mathbf{J}^*_{\tau*}) = q_{i1}(t_i, \mathbf{J}^*_{\tau*})$. By the inequality (8), $q_i(t_i, \mathbf{J}^*_{\tau*}) \geq q_{i1}(t_i, \mathbf{J}^*_{\tau*})$. Thus, the expected payoff from achieving Z, $\delta w_i(t_i, \mathbf{J}^*_{\tau*}) - b_i(t^*)$, decreases from the level Z^* faster than the expected payoff from Y, $\delta l_{i1}(t_i, \mathbf{J}^*_{\tau*})$, decreases from the level Y^*. As $Z^* \leq Y^*$, the claim is established.

We provide some remarks on Proposition 1.

Remark 1. Proposition 1 can be extended to the case where the English auction in period 1 has a reserve price $r > 0$. Amend the English auction as follows. The auction starts with a current price lower than r (say zero price) that corresponds to "no sale." If someone drops out at no-sale, then the price clock pauses to give others a chance to drop out. Once no more bidders drop out at no-sale, the price clock jumps to the reserve price r.

An equilibrium can be constructed for any threshold $t^* > r$, provided the discount factor is sufficiently close to 1. Let $\hat{t} \in T_1$ be the type of bidder 1 such that her expected payoff (for the entire auction-resale game) is zero if she wins the good at price r and offers the good for resale, given the belief that the types in $[0, t^*]$ of other bidders participate in the resale market. Suppose δ is sufficiently close to 1, so that $\hat{t} < r$. According to the equilibrium, bidder 1 drops out at "no sale" if and only if her type is below $\hat{t}$. Once bidder 1 has dropped out at no-sale, other bidders play the value-bidding equilibrium; if bidder 1 does not drop out at no-sale, then the bidders' subsequent actions are analogous to the equilibria described in Proposition 1, where "dropping out at zero price" is replaced by "dropping out at no-sale." Resale occurs given the belief that (i) bidder 1's type is distributed on $[\hat{t}, \tau]$ for some $\tau \in (r, t^*)$, and (ii) the other bidders' types are distributed on $[0, t^*]$.

Remark 2. The t^*-equilibrium construction makes essential use of the transparent dynamic nature of an English auction, because the designated bidder's dropout price depends on the set of the other bidders who have not dropped out (the upper branch of (14)). This dependence is important in our construction because by (13), bidders drawn from different distributions need different prices b_i^* to be kept obedient to the threshold t^*. For exactly this reason, the t^*-equilibrium construction does not generally extend to second-price auctions. The construction does extend if $b_2^* = \cdots = b_n^*$, which holds if bidders 2 to n are ex ante symmetric.

Remark 3. The t^*-equilibria are not the only equilibria that differ from the value-bidding equilibrium. There also exist "extreme equilibria" where bidder 1's bid is so high that all types of all other bidders find it optimal to drop out at the beginning of the auction.[11] Extreme equilibria are conceptually simpler than t^*-equilibria, however there are practical reasons why extreme equilibria might not be played. First, in an extreme equilibrium, the good is always sold at zero price at the initial auction. That would make a regulator suspicious of collusion, which the bidders may want to avoid. Second, if low-type bidders have a budget constraint that prevents them from staying active up to very high prices, then a designated bidder's bidding strategy in an extreme equilibrium is not credible.[12] Third, extreme equilibria cannot generally be used to obtain the interim Pareto dominance property described below (see Section 5.1 for an example).

[11]Zheng (2000, Section 5.2) constructs an extreme equilibrium in a second-price-auction-type mechanism with reserve prices. See also Garratt and Tröger (2006, Section 4).

[12]Brusco and Lopomo (2006) made this point previously in a no-resale model.

Remark 4. One may drop the assumption that the period-2 seller can prevent further resale transactions. Suppose that, beginning with the period-2 seller, each current owner of the good designs a sales mechanism, given that the next owner will design her own sales mechanism, and so on. This amounts to using Zheng's (2002) repeated-resale game to describe period 2 of our model. For a certain class of period-2 beliefs,[13] Zheng's result shows that there exists a period-2 perfect Bayesian continuation equilibrium such that the final outcome is still the Myerson optimal auction outcome intended by the period-2 seller (though this outcome is achieved via intermediate sales mechanisms different from Myerson's). Accordingly, for certain prior beliefs our t^*-equilibrium construction extends to the repeated-resale market without any change; this is true in particular for symmetric environments (i.e., where prior beliefs are identical across bidders). One may conjecture that the properties (1)–(8) hold for the repeated-resale market for a larger class of prior beliefs, but proving this hinges on first solving Zheng's repeated-resale game for the corresponding beliefs.

5 The interim Pareto dominance of collusion

We assume that a sunspot (à la Shell, 1977 and Cass and Shell, 1983) with n equally likely states is commonly observed before the period-1 auction starts (and after the bidders have been privately informed). This extends the game so that actions can depend on the realization of the sunspot state. Given any strategy profile, the payoff of any type of a given bidder is defined via the expectation over the n sunspot states. We call an equilibrium in the extended game *Pareto improving* if it interim-Pareto-dominates the value-bidding equilibrium, that is, if every type of every bidder is strictly better-off than in the value-bidding equilibrium. We show that Pareto improving equilibria exist in any environment.

For any threshold t^*, we define a *t^*-collusive equilibrium*: if the realized state is $j = 1, \ldots, n$, then a t^*-equilibrium is played, with bidder j taking the role of the designated bidder. Clearly this constitutes an equilibrium.

[13] Mylovanov and Tröger (forthcoming) characterize the class of beliefs such that Zheng's construction applies.

Proposition 2 *The t^*-collusive equilibria are Pareto improving for all t^* sufficiently close to 0.*

Note the generality of this result. A Pareto improving equilibrium exists in any symmetric or asymmetric environment; in particular, strong bidders can gain from colluding with weak bidders. Moreover, the discount factor does not have to be close to 1; the result applies to any non-zero discount factor.

The proof of Proposition 2 utilizes two regularity properties of the period-2 outcome when the period-2 belief is concentrated on types close to 0. These properties, which are stated in Lemma 5 and Lemma 6, do not appear to be specific to the Myerson optimal auction outcome; the conclusion of Proposition 2 holds for any resale market outcome that satisfies these properties.

The first step towards the proof of Proposition 2 is to observe that it is sufficient to focus the payoff comparison on the types in the interval $[0, t^*]$.

Lemma 4 *If in a t^*-collusive equilibrium, type t^* of a given bidder is strictly better-off than in the value-bidding equilibrium, then all types above t^* of this bidder are strictly better-off.*

Proof. Consider any type $t_i \geq t^*$ of a bidder $i \in N$. Her payoff in the t^*-collusive equilibrium can be different from her value-bidding equilibrium payoff only in the event that the highest type among the other bidders $t_{-i}^{(1)} \leq t^*$. Then bidder i's payoff in the t^*-collusive equilibrium is $t_i - \frac{n-1}{n} b_i^*$, and her payoff in the value-bidding equilibrium is $t_i - t_{-i}^{(1)}$. The payoff difference is independent of t_i. ■

The idea behind using t^* close to 0 in Proposition 2 is to make the interval of relevant types $[0, t^*]$ small, so that approximations of payoff comparisons can be obtained using first-order Taylor expansions of the prior distributions,

$$F_i(x) \quad = \quad f_i(0)x + h_i(x), \qquad (x \geq 0, \ i = 1, \ldots, n), \tag{17}$$

where $h_i(x)/x \to 0$ as $x \to 0$.

The Taylor expansions (17) are exact ($h_i(x) = 0$) if the priors are (possibly asymmetric) uniform distributions. The uniform-priors example captures the rough idea why Proposition 2 is correct. In the value-bidding equilibrium with uniform priors, it is well-known that the payoff of type $t_i \leq t^*$ of bidder i is

$$U_i^{\text{val, uniform}}(t_i) = \int_0^{t_i} \prod_{k \neq i} F_k(x)\ \mathrm{d}x = \prod_{k \neq i} f_k(0)\ \frac{t_i^n}{n}. \tag{18}$$

For the t^*-collusive equilibrium payoff with uniform priors, denoted $U_i^{*,\text{uniform}}(t_i)$, we obtain a lower bound by not counting the gains from resale trade:

$$U_i^{*,\text{uniform}}(t_i) \geq \frac{1}{n} \prod_{k \neq i} f_k(0)\ (t^*)^{n-1}\ t_i, \tag{19}$$

because with probability $1/n$ bidder i is the designated bidder, in which case she gets the good for free if all others have valuations below t^*. Clearly, (18) and (19) imply

$$U_i^{*,\text{uniform}}(t_i) \quad > \quad U_i^{\text{val, uniform}}(t_i) \qquad \text{for all } t_i \in (0, t^*). \tag{20}$$

The strict inequality in (20) also holds at $t_i = 0$ and $t_i = t^*$, because in a t^*-collusive equilibrium type 0 makes a profit as a period-2 seller with positive probability, and type t^* gets an information rent as a period-2 buyer with positive probability. Thus, in the uniform-priors example, t^*-collusive equilibria are Pareto improving for all t^*. Using the Taylor expansion (17), we generalize this result to arbitrary prior distributions and *small* t^*, via several lemmas that are proved in Appendix A.

For all $i \in N$, let τ^{*i} be defined analogously to τ^*, with bidder i instead of bidder 1 taking the designated-bidder role. Lemma 5 states that, for small t^*, a non-vanishing fraction of the designated-bidder types below t^* offer the good for resale on the equilibrium path of a t^*-equilibrium.

Lemma 5 *There exists* $0 < \underline{\theta} < 1$ *such that, for all* t^* *sufficiently close to 0,*

$$\forall i \in N: \qquad \tau^{*i} > \underline{\theta} t^*. \tag{21}$$

The proof uses the fact that the period-2 seller, if she believes that each buyer's type belongs to $[0, t^*]$, is free to make a take-it-or-leave-it fixed-price offer at $t^*/2$ to any buyer. This lower bound on what the period-2 seller can achieve ensures that all types that are sufficiently small relative to t^* offer resale, thus bounding τ^{*i} from below.

For all $i \in N$, let $\mathbf{J}^{*i}$ denote the belief resulting from the prior belief and the information that bidder i's type is below τ^{*i} and the other bidders' types are below t^*. Lemma 6 states that, given the period-2 belief $\mathbf{J}^{*i}$ for any small t^*, a non-vanishing fraction of the period-2-buyer types below t^* buy the good with a non-vanishing probability; if $\delta = 1$, the probability is conditional on the seller's type being below t^*.

Lemma 6 *There exist* $0 < \bar{\xi} < 1$ *and* $\bar{\epsilon} > 0$ *such that, for all* t^* *sufficiently close to 0,*

$$\forall i \in N,\ j \in N \setminus \{i\},\ t_i \in [\bar{\xi} t^*, t^*]: \qquad \bar{\epsilon} \ < \ \begin{cases} q_{ij}(t_i, \mathbf{J}^{*j}), & \text{if } \delta < 1, \\ \frac{q_{ij}(t_i, \mathbf{J}^{*j})}{F_j(t^*)}, & \text{if } \delta = 1. \end{cases} \tag{22}$$

The intuition behind this result is that, because the conclusion (22) holds for any t^* in the uniform-priors example, one can use the Taylor expansion (17) to show the conclusion for arbitrary priors and small t^*. The proof uses the assumption that the prior densities are Lipschitz continuous. This implies that the virtual valuation functions (Myerson, 1981) for the period-2 beliefs about the period-2 buyers' types are strictly increasing if t^* is small. According to the Myerson optimal auction outcome, the good is then resold to the period-2 buyer with the highest virtual valuation, unless the period-2 seller's type is higher. This allocation rule yields explicit formulas for the buyer-allocation probabilities $q_{ij}(t_i, \mathbf{J}^{*j})$. The lower bound (22) is obtained via approximations for the virtual valuation functions that are obtained using (17).

The next lemma provides an approximation result for payoffs in the value-bidding equilibrium.[14]

Lemma 7 *The payoff of type* $t_i \leq t^*$ *of bidder* $i \in N$ *in the value-bidding equilibrium is*

$$U_i^{val}(t_i) \ = \ \prod_{k \neq i} f_k(0) \frac{1}{n} {t_i}^n + o((t^*)^n).$$

The next lemma states that, given the period-2 belief $\mathbf{J}^{*i}$ for any small t^*, a non-vanishing fraction of the period-2-seller types below t^* sell the good with a non-vanishing probability.

Lemma 8 *Let* $\underline{\theta}$ *be as in Lemma 5. There exist* $0 < \underline{\xi} < \underline{\theta}$ *and* $\underline{\epsilon} > 0$ *such that, for all* t^* *sufficiently close to 0,*

$$\forall j \in N,\ t_j \in [0, \underline{\xi} t^*]: \qquad q_j(t_j, \mathbf{J}^{*j}) < 1 - \underline{\epsilon}. \tag{23}$$

[14] For any $k \geq 0$, we will use $o((t^*)^k)$ to denote any function $h(x, t^*)$ (or $h(t, t^*)$ or $h(t_i, t^*)$) such that $\sup_{x \in [0,t^*]} |h(x, t^*)|/(t^*)^k \to 0$ as $t^* \to 0$.

To prove this, we observe that the upper bound (23) follows from the lower bound (22) because buyer- and seller-allocation probabilities add up to 1 in expectation over all types.

Let $U_{ij}^b(t_i)$ denote the payoff of type $t_i \leq t^*$ of bidder $i \in N$ in a t^*-equilibrium where $j \neq i$ is the designated bidder. Using the lower bound on the trading probability (22) and the envelope formula (1), we obtain a lower bound for her payoff.

Lemma 9 *Let $\overline{\xi}$ and $\overline{\epsilon}$ be as in Lemma 6. For all sufficiently small t^* and $t_i \in [0, t^*]$,*

$$U_{ij}^b(t_i) \quad \geq \quad \delta\overline{\epsilon}\max\{0, t_i - \overline{\xi}t^*\}F_j(\underline{\theta}t^*)\prod_{k\notin\{i,j\}} F_k(t^*).$$

Let $U_i^s(t_i)$ denote the payoff of type $t_i \leq t^*$ of bidder $i \in N$ in a t^*-equilibrium where i is the designated bidder. Using the upper bound on the no-trade probability (23) and the envelope formula (2), we obtain a lower bound for her payoff.

Lemma 10 *Let $\underline{\xi}$ and $\underline{\epsilon}$ be as in Lemma 8. For all sufficiently small t^* and $t_i \in [0, t^*]$,*

$$U_i^s(t_i) \quad \geq \quad \big(t_i + \delta\underline{\epsilon}\max\{0, \underline{\xi}t^* - t_i\}\big)\prod_{k\neq i} F_k(t^*).$$

Let $U_i^*(t_i)$ denote the payoff of type $t_i \leq t^*$ of bidder $i \in N$ in a t^*-collusive equilibrium. Combining Lemma 9 and Lemma 10, we get, for all sufficiently small t^* and $t_i \in [0, t^*]$,

$$\begin{aligned}
U_i^*(t_i) \quad &\geq \quad \frac{1}{n}\big(t_i + \delta\underline{\epsilon}\max\{0, \underline{\xi}t^* - t_i\}\big)\prod_{k\neq i} F_k(t^*) \qquad (24)\\
&\qquad + \frac{n-1}{n}\delta\overline{\epsilon}\max\{0, t_i - \overline{\xi}t^*\}F_j(\underline{\theta}t^*)\prod_{k\notin\{i,j\}} F_k(t^*)\\
&\overset{(17)}{=} \quad \left(\frac{1}{n}t_i + \frac{1}{n}\delta\underline{\epsilon}\max\{0, \underline{\xi}t^* - t_i\} + \frac{n-1}{n}\delta\overline{\epsilon}\max\{0, t_i - \overline{\xi}t^*\}\underline{\theta}\right) \qquad (25)\\
&\qquad \cdot\left(\prod_{k\neq i} f_k(0)(t^*)^{n-1} + o((t^*)^{n-1})\right)\\
&= \quad \prod_{k\neq i} f_k(0)g(\frac{t_i}{t^*})(t^*)^n + o((t^*)^n), \qquad (26)
\end{aligned}$$

where

$$g(x) \quad = \quad \frac{1}{n}x + \frac{1}{n}\delta\underline{\epsilon}\max\{0, \underline{\xi} - x\} + \frac{n-1}{n}\delta\overline{\epsilon}\max\{0, x - \overline{\xi}\}\underline{\theta} \qquad (x \in [0,1]).$$

Because $g(0) > 0$, $g(1) > 1/n$, and $x > x^n$ if $0 < x < 1$,

$$\forall x \in [0,1]: \quad g(x) \quad > \quad \frac{x^n}{n}.$$

Hence, combining (26) with Lemma 7,

$$\frac{U_i^*(t_i) - U^{\text{val}}(t_i)}{(t^*)^n} \geq \prod_{k \neq i} f_k(0) \underbrace{\min_{x \in [0,1]} (g(x) - \frac{x^n}{n})}_{>0} + o(1).$$

Hence, for sufficiently small t^*,

$$\min_{t_i \in [0,t^*]} (U_i^*(t_i) - U^{\text{val}}(t_i)) \quad > \quad 0.$$

This completes the proof of Proposition 2.

We provide some remarks on Proposition 2:

Remark 5. The uniform-priors example shows that the gains to playing a Pareto improving equilibrium can be quite large. Table 1 shows the gains to a bidder with type $t^* = .9$ in an environment with F_i $(i \in N)$ uniform on $[0,1]$, for various numbers of bidders n. The gains to type t^* are the minimum gains over all types in this example.

n	$U_i^{\text{val}}(.9)$	$U_i^*(.9)$	% increase
2	0.405	0.50625	25
5	0.1181	0.19043	61.24
10	0.03487	0.06277	80.01

Table 1: $F_i(t) = t$, $t^* = .9$

Remark 6. Proposition 2 extends to an English auction with a small reserve price. This follows from Remark 1, by continuity. For larger reserve prices r, the question is whether bidders can collude so that the payoff of any bidder-type above r is larger than in the value-bidding equilibrium with reserve price r (where bidders with types below r abstain). We have three results for environments where the prior beliefs F_i $(i \in N)$ are uniform on $[0,1]$.

First, if $n = 2$, then a Pareto improving equilibrium exists for any reserve price below 1. Second, if $n \geq 4$, then a Pareto improving equilibrium exists if the optimal reserve price under value-bidding, 1/2, is used. Third, a Pareto improving equilibrium exists for any reserve price arbitrarily close to 1 if n is sufficiently large.

Proposition 2 establishes that the bidders can always achieve some, possibly small, Pareto improvement over value-bidding. This raises two questions. First, is the restriction to small t^* needed for this result? Second, is resale trade needed?

5.1 Not all t^*-collusive equilibria are Pareto improving

In this section we provide an example showing that the Pareto improvement can break down if t^* is not sufficiently small. We consider a symmetric 2-bidder environment without discounting ($\delta = 1$). Let $t^* = 1$.[15] We will construct a prior belief $F := F_1 = F_2$ with support $[0, 1]$ such that the payoff of type $t^* = 1$ of any bidder in a 1-collusive equilibrium is smaller than in the value-bidding equilibrium.[16] Hence, by continuity, a positive mass of types prefers value bidding.

Let $\tilde{t}$ denote a random variable with cumulative distribution function F. In the value-bidding equilibrium, type 1 of any bidder obtains the payoff $1 - \mathbb{E}[\tilde{t}]$, where $\mathbb{E}[\cdot]$ denotes the expected-value operator. In a 1-collusive equilibrium, her payoff is

$$\frac{1}{2} + \frac{1}{2}\left(1 - \mathbb{E}[p^*(\tilde{t})]\right),$$

where $p^*(t)$ ($t \in [0, 1]$) denotes an optimal resale price of type t of any bidder given the period-2 belief $[0, 1]$ about the other bidder. Thus, type 1 strictly prefers the value-bidding

[15] The example can be easily generalized to show that for any $t^* > 0$ there exists an F such that a t^*-collusive equilibrium is not Pareto improving.

[16] Obtaining a Pareto improvement remains impossible if arbitrary probabilities are allowed for the sunspot states. Let u_1 and u_2 denote the two bidders' type-t^* payoffs in a t^*-equilibrium. Let u^{val} denote the type-t^* payoff in the value-bidding equilibrium. Let σ denote the sunspot probability that bidder 1 is the designated bidder. If $(u_1 + u_2)/2 < u^{\text{val}}$, then we cannot have that both bidder 1 is better off ($\sigma u_1 + (1 - \sigma)u_2 \geq u^{\text{val}}$) and bidder 2 is better off ($(1 - \sigma)u_1 + \sigma u_2 \geq u^{\text{val}}$). To see this, add the inequalities.

equilibrium if

$$\mathbb{E}[p^*(\tilde{t})] > 2\mathbb{E}[\tilde{t}]. \tag{27}$$

We construct a distribution F such that (27) holds. F is piecewise linear with a single kink at some point $\alpha \in (0, 1/3)$. We compute an explicit solution for the resale price function p^*. This allows us to verify (27). For all $t \in [0, 1]$, let[17]

$$F(t) := \begin{cases} \frac{1-\alpha}{\alpha} t & \text{if } 0 \le t \le \alpha, \\ 1 - \alpha + \frac{\alpha}{1-\alpha}(t - \alpha) & \text{if } \alpha \le t \le 1. \end{cases}$$

Straightforward calculations show that $\mathbb{E}[\tilde{t}] = \alpha$.

Let $t \in (0, 1)$. To find the optimal resale price $p^*(t)$ for a type-t seller, observe that $p = p^*(t)$ maximizes the period-2 payoff $\pi(p, t) := (p - t)(1 - F(p))$ (written net of own type t) among all $p \in [0, 1]$.

Suppose that $p^*(t) < \alpha$. Because $p^*(t) > t > 0$, the first-order condition

$$0 = \frac{\partial}{\partial p} \pi(p, t) \bigg|_{p = p^*(t)} = 1 - \frac{1-\alpha}{\alpha}(2p^*(t) - t)$$

holds. This implies $p^*(t) = \frac{\alpha}{2(1-\alpha)} + \frac{t}{2}$. Using $p^*(t) < \alpha$ we find

$$t \;<\; \frac{\alpha}{1-\alpha}. \tag{28}$$

Comparing the payoff obtained from $p^*(t)$ with the payoff obtained from the price $p = (1 + t)/2$ contradicts the optimality of $p^*(t)$:

$$\pi(p^*(t), t) = \frac{\alpha}{2(1-\alpha)} \left(1 - \frac{1-\alpha}{\alpha} t\right) 2 \overset{(28)}{<} \frac{\alpha}{2(1-\alpha)} (1 - t)\, 2 = \pi(p, t).$$

Hence, $p^*(t) \in [\alpha, 1]$. For all $p \in (\alpha, 1)$,

$$\frac{\partial}{\partial p} \pi(p, t) \;=\; \frac{\alpha}{1-\alpha} (1 - 2p + t)\,.$$

[17] To simplify the exposition, the example uses a distribution with a discontinuous density. There exists an approximating distribution with a Lipschitz continuous density such that the conclusion of the example still holds. In fact, the conclusion holds for any distribution function on $[0, 1]$ that is sufficiently close, in the L_1-topology on densities, to the distribution in the example. This follows from Berge's Theorem of Maximum because in the example all non-zero types have a unique optimal resale price.

Because $\pi(\cdot, t)$ is strictly concave on $[\alpha, 1]$, the first-order condition $0 = (\partial/\partial p)\pi(p, t)$ implies

$$p^*(t) = \frac{1+t}{2}.$$

Now (27) follows because

$$\mathbb{E}[p^*(\tilde{t})] \;=\; \int_0^\alpha \frac{1+t}{2}\frac{1-\alpha}{\alpha}dt + \int_\alpha^1 \frac{1+t}{2}\frac{\alpha}{1-\alpha}dt \;=\; \frac{1+\alpha}{2} \;\overset{\alpha<1/3}{>}\; 2\alpha \;=\; 2\mathbb{E}[\tilde{t}].$$

In this example, even though the 1-collusive equilibrium does not Pareto dominate value-bidding, a Pareto dominating t^*-collusive equilibrium still exists by Proposition 1.

5.2 There may be no Pareto improvement without resale

As pointed out by Blume and Heidhues (2004), the second-price auction without resale has equilibria (in dominated strategies) with bidding profiles similar to our t^*-equilibria. Given any $t^* \geq 0$, there exists a *no-resale t^*-equilibrium* in which bidders whose use values are above t^* bid their use values, all bidders except a designated bidder bid 0 if their use values are below t^*, and the designated bidder bids t^* if her use value is below t^*. These equilibria remain valid if the second-price auction is replaced by the English auction. In fact, bidding in any of our t^*-equilibria converges to the no-resale t^*-equilibrium with designated bidder 1 in the limit $\delta \to 0$ because $\tau^* \to 0$.

Is it possible to construct a Pareto improving equilibrium without resale based on the no-resale t^*-equilibria, using a sunspot as in the model with resale? In many environments it is not possible because high-type bidders are better off with value-bidding. To state this result, label the no-resale t^*-equilibrium with designated bidder i by $(i, t^*) \in N \times \mathbb{R}_+$. For any probability distribution D on $N \times \mathbb{R}_+$, call the equilibrium obtained by playing a no-resale equilibrium according to the distribution D a *no-resale collusive equilibrium.*

Proposition 3 *Suppose that the prior belief is strictly concave and identical for all bidders. Then in any no-resale collusive equilibrium there exists a bidder, the highest possible type of which, is strictly worse off than in the value-bidding equilibrium.*

Proof. Let $F = F_i$ $(i \in N)$ denote the prior belief. Let $\bar{t}$ denote the highest possible type. Let u^{val} denote the payoff of type $\bar{t}$ in the value-bidding equilibrium. For any no-resale equilibrium $(i, t^*) \in N \times \mathbb{R}_+$ and any $k \in N$, let u_k^{i,t^*} denote the payoff of type $\bar{t}$ of bidder k.

We first show that

$$u^{\text{val}} \quad > \quad \frac{1}{n}\sum_{k=1}^{n} u_k^{i,t^*}. \tag{29}$$

Because the designated bidder gets the good for free if all others have types below t^*,

$$\frac{1}{n}\sum_{k=1}^{n} u_k^{i,t^*} \quad = \quad F(t^*)^{n-1}\frac{1}{n}t^* + \int_{t^*}^{\bar{t}} F(t)^{n-1}\mathrm{d}t.$$

It is well-known that

$$u^{\text{val}} \quad = \quad \int_0^{\bar{t}} F(t)^{n-1}\mathrm{d}t.$$

Strict concavity of F implies

$$\forall 0 < t < t^* : \frac{F(t)}{t} > \frac{F(t^*)}{t^*}.$$

Therefore,

$$u^{\text{val}} - \int_{t^*}^{\bar{t}} F(t)^{n-1}\mathrm{d}t > \frac{F(t^*)^{n-1}}{(t^*)^{n-1}}\int_0^{t^*} t^{n-1}\mathrm{d}t = \frac{F(t^*)^{n-1}}{(t^*)^{n-1}}\frac{1}{n}(t^*)^n = \frac{1}{n}\sum_{k=1}^{n} u_k^{i,t^*} - \int_{t^*}^{\bar{t}} F(t)^{n-1}\mathrm{d}t,$$

implying (29).

In a no-resale collusive equilibrium based on a probability distribution D, the payoff of type $\bar{t}$ of bidder $k \in N$ is

$$u_k^D \quad := \quad \int u_k^{i,t^*}\mathrm{d}D(i,t^*).$$

Suppose that the highest possible type of each bidder is at least as well off as in the value-bidding equilibrium: $u_k^D \geq u^{\text{val}}$ for all $k \in N$. Then

$$u^{\text{val}} \leq \frac{1}{n}\sum_{k=1}^{n} u_k^D = \int \frac{1}{n}\sum_{k=1}^{n} u_k^{i,t^*}\mathrm{d}D(i,t^*) \overset{(29)}{<} u^{\text{val}},$$

a contradiction. ■

The proof works by showing that, in any no-resale collusive equilibrium, the payoff of the highest type averaged over all bidders, is smaller than her value-bidding payoff. Hence, somebody must be worse off in any no-resale collusive equilibrium. Of course, by continuity, the result can be extended to types close to the highest possible type.

Blume and Heidhues (2004) show that in environments where the priors $(F_i)_{i\in N}$ have a common support and there are at least three bidders, the no-resale t^*-equilibria, with t^* ranging in $[0, \max_{i\in N} \bar{t}_i]$, are the only equilibria of the second-price auction without resale.[18] Thus, the set of no-resale collusive equilibria span all of the possible equilibrium utility profiles in this environment. Hence we have the following additional result.

Corollary 1 *Suppose that the prior belief is strictly concave and identical for all of the $n \geq 3$ bidders. Then, assuming no resale, the value-bidding equilibrium of the second-price auction is not Pareto dominated by any equilibrium of the second-price auction with a public randomization device.*

6 Appendix A

Proof of Lemma 1. From Myerson (1981, p. 68–69), there exist weakly increasing functions $\bar{c}_i : T_i \to \mathbb{R}$ $(i \in N)$ such that, given any type profile $\mathbf{t} \in \mathbf{T}$, the period-2 seller $j \in N$ optimally assigns the good with equal probability to any one of the buyers in the set

$$\{i \in N \setminus \{j\} \mid t_j < \bar{c}_i(t_i) = \max_{k\in N\setminus\{j\}} \bar{c}_k(t_k)\},$$ [19] [20]

and consumes the good if the set is empty. Hence, for all $i \in N$, $t_i \in T_i$, and $j \in N \setminus \{i\}$,

$$q_{ij}(t_i, \mathbf{J}) \quad \leq \quad \Pr[\bar{c}_i(t_i) > \tilde{t}_j] \cdot \Pr[\bar{c}_i(t_i) \geq \max_{k\in N\setminus\{i,j\}} \bar{c}_k(\tilde{t}_k)], \tag{30}$$

where $(\tilde{t}_1, \ldots, \tilde{t}_n)$ denotes a random vector with distribution $\mathbf{J}$. Similarly,

$$\begin{aligned} q_i(t_i, \mathbf{J}) &= \Pr[t_i \geq \max_{k\in N\setminus\{i\}} \bar{c}_k(\tilde{t}_k)] \\ &= \Pr[t_i \geq \bar{c}_j(\tilde{t}_j)] \cdot \Pr[t_i \geq \max_{k\in N\setminus\{i,j\}} \bar{c}_k(\tilde{t}_k)]. \end{aligned} \tag{31}$$

[18] They also show that with any positive reserve price only the value-bidding equilibrium ($t^* = 0$) remains. This is in contrast to the "with resale" case where t^*-equilibria with $t^* > 0$ are robust to reserve prices (Remark 1).

[19] Writing "$t_j < \ldots$" instead of "$t_j \leq \ldots$", we deviate from Myerson's original definition while retaining optimality for the seller.

[20] For all $i \in N$, we extend $\bar{c}_i$ to T_i via $\bar{c}_i(t_i) = -\infty$ if $t_i < \inf J_i$ and $\bar{c}_i(t_i) = \bar{c}_i(\sup J_i)$ if $t_i > \sup J_i$. Cf. footnote 9.

From Myerson' construction, $\bar{c}_i(t_i) \le t_i$ and $\bar{c}_i(\tilde{t}_j) \le \tilde{t}_j$. Hence, (30) and (31) yield (8). ■

Proof of Lemma 2. For any x, the function $\phi(x,t) := \delta w_1(t, \mathbf{J}^*_x) - t$ is Lipschitz continuous in t, where the Lipschitz constant is independent of x by (2). By (6), for any t the function $\phi(x,t)$ is continuous in x. Hence, ϕ is continuous. By (5), $\phi(0,0) > 0$. Moreover, $\phi(t,t) \le 0$ for all $t \ge t^*$. Hence, $\phi(\tau^*, \tau^*) = 0$ and $\tau^* > 0$.

To prove (10) and (11) if $\delta < 1$, observe that (2) together with $q_1(s, \mathbf{J}^*_{\tau^*}) \le 1$ $(s \in T_1)$ implies that $\phi(\tau^*, t)$ is weakly decreasing in t.

To prove the limit result (12), consider the correspondence

$$\psi : \quad \delta \quad \mapsto \{t \in [0, t^*] \mid \phi(t,t) \le 0\}.$$

Because ϕ is continuous, ψ is upper-hemicontinuous. From (5), $\psi(1) = \{t^*\}$. Hence, $\tau^* = \min \psi(\delta) \to \min \psi(1) = t^*$ as $\delta \to 1$, $\delta \ne 1$. ■

Proof of Lemma 3. By Lemma 2, $\tau^* > 0$. Hence, (3)–(4) implies $l_{i1}(t^*, \mathbf{J}^*_{\tau^*}) < t^*$, so (13) implies $0 < b^*_i$. By (13), the remaining claim $b^*_i < t^*$ is implied by the claim $l_{i1}(t^*, \mathbf{J}^*_{\tau^*}) > 0$, which we establish now. Suppose that $l_{i1}(t^*, \mathbf{J}^*_{\tau^*}) \le 0$. Then $l_{i1}(t_i, \mathbf{J}^*_{\tau^*}) \le 0$ for all $t_i < t^*$ by (1). However, $l_{i1}(t_i, \mathbf{J}^*_{\tau^*}) \ge 0$ by (4). Hence, $l_{i1}(t_i, \mathbf{J}^*_{\tau^*}) = 0$. Thus, $q_{i1}(t_i, \mathbf{J}^*_{\tau^*}) = 0$ by (1). Hence, $Q_i^{1,\mathbf{J}^*_{\tau^*}}(\mathbf{t}) = 0$ for almost all $\mathbf{t} \in \mathbf{J}^*_{\tau^*}$. Thus, $P_i^{1,\mathbf{J}^*_{\tau^*}}(\mathbf{t}) = 0$ by (4). Because probabilities sum up to 1 and payments sum up to 0, $Q_1^{1,\mathbf{J}^*_{\tau^*}}(\mathbf{t}) = 1$ and $P_1^{1,\mathbf{J}^*_{\tau^*}}(\mathbf{t}) = 0$ for almost all $\mathbf{t} \in \mathbf{J}^*_{\tau^*}$. Hence, $w_1(t_1, \mathbf{J}^*_{\tau^*}) = t_1$ for almost all $t_1 \le \tau^*$, contradicting (5). ■

Proof of Proposition 1. We begin with a complete description of period-1 beliefs and period-2 beliefs. For any $i \in N$ and $p \ge 0$, let $\mathcal{H}^a_{i,p}$ denote the set of histories with current price p (= auction price if the history is terminal) where bidder i is active; let $\mathcal{H}^d_{i,p}$ denote the set of histories where bidder i has dropped out at price p (while the current price is $\ge p$). Observe that, for any $i \in N$, these sets cover the set of all histories: $\mathcal{H} = \cup_{p \ge 0}(\mathcal{H}^a_{i,p} \cup \mathcal{H}^d_{i,p})$. Hence, to describe the equilibrium period-1 belief profile $(\mathbf{A}_h)_{h \in \mathcal{H}}$, it is sufficient to specify, for all $i \in N$, $p \ge 0$, and $h \in \mathcal{H}^a_{i,p} \cup \mathcal{H}^d_{i,p}$, the period-1 belief $\mathbf{A}_{h,i}$ about bidder i at history h. In the following Tables 2–4, we shall identify the posterior distributions $\mathbf{A}_{h,i}$ and $\mathbf{G}_{h,i}$ with their posterior supports.

Let $i \in N$. At the initial history h_0, all bidders are active and $\mathbf{A}_{h_0,i} = F_i$. Table 2 provides information on period-1 beliefs about bidder i at any history where bidder i has dropped out at price p.

i	i's dropout price p	$\mathbf{A}_{h,i}$ at $h \in \mathcal{H}^d_{i,p}$
≥ 2	$\leq t^*$	$[0, t^*]$
≥ 1	$\in (t^*, \bar{t}_i]$	$\{p\}$
≥ 1	$> \bar{t}_i$	$\{\bar{t}_i\}$
$= 1$	$< t^*$	$[0, \tau^*]$
$= 1$	$= t^*$	$[\tau^*, t^*]$

Table 2: Period-1 beliefs about bidder i at any history where bidder i has dropped out.

Table 3 provides information on period-1 beliefs about bidder i at any non-initial history where bidder i is active. Let $\hat{\mathcal{H}}$ denote the set of histories such that bidder 1 is active while, according to β_1, she would not be active if her type were $\leq \tau^*$.

i	current price p	$\mathbf{A}_{h,i}$ at $h \in \mathcal{H}^a_{i,p}$
≥ 2	$\leq t^*$	$[t^*, \bar{t}_i]$ if $h \neq h_0$
≥ 1	$\in [t^*, \bar{t}_i]$	$[p, \bar{t}_i]$
≥ 1	$> \bar{t}_i$	$\{\bar{t}_i\}$
$= 1$	$< t^*$	$[\tau^*, \bar{t}_1]$ if $h \in \hat{\mathcal{H}}$ T_1 if $h \notin \hat{\mathcal{H}}$

Table 3: Period-1 beliefs about bidder i at any non-initial history where bidder i is active.

Equilibrium condition (b) can be verified in a straightforward manner using (14), (15), Table 2 and Table 3. For example, if bidder 2 drops out at the initial history h_0, then (15) implies the period-1 belief $[0, t^*]$. The event that she drops out at a price in $(0, t^*]$ has probability 0; hence, Bayes rule allows an arbitrary period-1 belief such as $[0, t^*]$.

Next we specify period-2 beliefs for any terminal history $h \in \mathcal{H}_{\text{term}}$. Let $\omega(h) \in N$ denote the winner at the terminal history h. Period-2 beliefs about losing bidders are identical to the beliefs at the end of the auction, because the decision whether to offer the good for resale is not made by the losing bidders.

$$\mathbf{G}_{h,i} \quad = \quad \mathbf{A}_{h,i} \quad \text{for all } h \in \mathcal{H}_{\text{term}} \text{ and } i \neq \omega(h).$$

In the no-discounting case $\delta = 1$, the same equation $\mathbf{G}_{h,i} = \mathbf{A}_{h,i}$ is assumed for the winner $i = \omega(h)$, and $\gamma_h(t_i) = \text{resale}$, so that equilibrium conditions (c) and (d) are clearly satisfied.

Suppose that $\delta < 1$. Consider a terminal history h where any bidder $i \geq 2$ wins at a price $p \leq t^*$, that is, $h \in (\mathcal{H}^a_{i,p} \cup \mathcal{H}^d_{i,p}) \cap \mathcal{H}_{\text{term}}$ and $i = \omega(h)$. As in the proof of Lemma 2, there exists $\tau_h \in (0, t^*)$ such that

$$\delta w_i(\tau_h, \mathbf{G}_h) = \tau_h \quad \text{ if } \ \mathbf{G}_{h,i} = [0, \tau_h].$$

Table 4 specifies, for any $\delta < 1$, information on the period-2 beliefs about the period-1 winner (non-zero probability types) and the resale decision profile, where h' is any history in $(\mathcal{H}^a_{i,p} \cup \mathcal{H}^d_{i,p}) \cap \mathcal{H}_{\text{term}}$.

i	auction price p	$\mathbf{G}_{h,i}$ at $h \in (\mathcal{H}^a_{i,p} \cup \mathcal{H}^d_{i,p}) \cap \mathcal{H}_{\text{term}},\ i = \omega(h)$	$\gamma_h(t_i) = \begin{cases} \text{resale} & \text{if } t_i \leq x, \\ \text{consume} & \text{if } t_i > x. \end{cases}$
≥ 2	$\leq t^*$	$[0, \tau_{h'}]$ if $h \in \mathcal{H}^a_{i,p}$	$x = \tau_{h'}$
		$[0, \tau_h]$ if $h \in \mathcal{H}^d_{i,p}$	$x = \tau_h$
≥ 1	$\in (t^*, \bar{t}_i]$	any belief	any optimal $x < p$
≥ 1	$> \bar{t}_i$	$\{\bar{t}_i\}$	any optimal x
$= 1$	$\leq t^*$	$[0, \tau^*]$	$x = \tau^*$

Table 4: Period-2 beliefs about the period-1 winner and the resale decision profile.

Equilibrium condition (d) can be verified using Table 4. To understand the row with h', observe that at history $h \in \mathcal{H}^a_{i,p}$ the belief about bidder i is $[t^*, \bar{t}_i]$, by Table 3. According to γ_h, no type in $[t^*, \bar{t}_i]$ chooses "resale" at h. Hence, Bayes rule allows an arbitrary period-2 belief such as $[0, \tau_{h'}]$. Verifying equilibrium condition (d) for the next row is analogous to Lemma 2.

To verify equilibrium condition (d) for the row with "any belief", observe that by the second row of Table 2 losing bidders are believed to have types $\leq p$, and by the second row of Table 3 the winner is believed to have a type $\geq p$. Because none of the winning types chooses "resale", Bayes rule allows for any belief.

To verify equilibrium condition (d) for the next row, observe that either $\gamma_h(\bar{t}_i) = \text{resale}$ or $\gamma_h(\bar{t}_i) = \text{consume}$. In the "resale" case, $G_{h,i} = \{\bar{t}_i\}$ follows from $A_{h,i} = \{\bar{t}_i\}$ (the third row of Table 3) by Bayes rule. In the "consume" case, the event that bidder i offers resale has probability 0, so Bayes rule allows for any period-2 belief.

To verify equilibrium condition (d) for the last row of Table 4, we distinguish two cases. If $A_{h,1} \in \{T_1, [0, \tau^*]\}$ then, according to γ_h, Bayes rule implies $G_{h,1} = A_{h,1} \cap [0, \tau^*] = [0, \tau^*]$; if $A_{h,1} \in \{[\tau^*, \bar{t}_1], [\tau^*, t^*]\}$, then the event that bidder 1 chooses "resale" has probability 0, so Bayes rule allows for any belief, such as $G_{h,1} = [0, \tau^*]$.

The proof of equilibrium condition (c) for the first two rows of Table 4 is analogous to the corresponding argument in the proof of Lemma 2. To verify equilibrium condition (c) for the row with "any belief", observe that type p of the winner strictly prefers "consume" because $\delta < 1$ and the highest type among the losing bidders is $\leq p$ by Table 2. The exact value of the optimal cutoff type x in this row and the next one plays no role. Equilibrium condition (c) for the last row of Table 4 follows from Lemma 2.

Equilibrium condition (a) follows by construction of the Myerson optimal auction outcome because all beliefs in Table 4 belong to $\mathcal{J}$.

To verify equilibrium condition (e), write $b' \succeq_{h,i,t_i} b$ if at history h the expected payoff of type t_i of bidder i when she bids b' is not smaller than when she bids b, given that other bidders stick to their candidate equilibrium strategies from h onwards, and bidder i sticks to her candidate equilibrium strategy after additional bidders drop out. We write $b' \simeq_{h,i,t_i} b$ if $b' \succeq_{h,i,t_i} b$ and $b \succeq_{h,i,t_i} b'$. We will sometimes omit the lower indices.

First we consider bidder-1 types above the threshold t^*,

$$\forall h \in \mathcal{H}_1,\ t_1 \geq t^*,\ b \geq 0: \quad t_1 \succeq_{h,1,t_1} b. \tag{32}$$

We prove (32) for the initial history $h = h_0$; other histories are treated similarly. Fix $b \geq 0$. One of the events I-IV occurs; in each event, the payoff from bid t_1 is not lower than from bid b.

Event I: "some bidder $\neq 1$ bids $< \min\{b, t_1\}$." Then the bid b leads to the same ending history (thus, same payoff) as the bid t_1 because bidder 1's bid at the initial history h_0 becomes irrelevant.

Event II: "all bidders $\neq 1$ bid $> b$." Then the highest type among the bidders $\neq 1$ is some $t' > \max\{b, t^*\}$. With discounting ($\delta < 1$), the bid b yields the payoff 0 because by Table 4 the winner consumes the good. Without discounting ($\delta = 1$), the good is offered for resale, but the bidders' period-2 payoffs add up to at most $\max\{t_1, t'\}$ (= the highest type among all bidders), and the winner obtains a period-2 payoff of $\geq t'$ by (3), implying that bidder 1 obtains from bid b a payoff of $\leq \max\{t_1 - t', 0\}$ by (4). But the bid t_1 yields a payoff equal to $\max\{t_1 - t', 0\}$ because bidder 1 pays t' if she wins. Hence, t_1 is weakly better than b.

Event III: "all bidders $\neq 1$ bid $> \min\{b, t_1\}$, and the first dropping out of a bidder $\neq 1$ occurs at a price $t' \leq b$." Then $t_1 < b$. Hence, $t_1 < t' \leq b$ and the highest type among the bidders $\neq 1$ is $t' > t^*$. The bid b of bidder 1 becomes irrelevant once the current price t' is reached because a bidder has dropped out. Either bidder 1 wins at price t' or, given her candidate equilibrium strategy (14), she drops out at price t'. In both cases, payoffs are bounded due to (3) and (4). If she wins, then the bidders' period-2 payoffs add up to at most t', hence bidder 1's payoff is ≤ 0 due to the auction price t'. If she loses at price t', then her payoff is 0. In any case, the bid b yields a payoff ≤ 0 so that the bid t_1 is weakly better.

Event IV: "at least one bidder $\neq 1$ bids $\min\{b, t_1\}$, and all others bid more." The probability of IV is positive (and hence IV is payoff-relevant) only if $b = 0$. If all bidders $\neq 1$ bid 0, then bidder 1's type $t_1 \geq t^*$ is the highest among all bidders, implying that bidder 1's payoff from the bid $b = 0$ is $\leq t_1$ (whether or not she wins the tie at 0), so that the bid t_1, which yields payoff t_1, is weakly better. If some bidders $\neq 1$ bid more than 0, then the highest type among bidders $\neq 1$ is some $t' > t^*$, implying that bidder 1's payoff from bid 0 is $\leq \max\{t_1 - t', 0\}$ by (3) and (4), while her payoff from bid t_1 equals $\max\{t_1 - t', 0\}$. This completes the proof of (32).

For bidder-1 types below the threshold any bid between 0 and the threshold is optimal,

$$\forall h \in \mathcal{H}_1,\ t_1 \leq t^*,\ b \in (0, t^*], b' \geq 0 : \quad b \succeq_{h,1,t_1} b'. \tag{33}$$

The proof of (33) uses similar arguments as the proof of (32). The only essentially new aspects are that the proof of $b \succeq 0$ uses the property (7), and that bidder 1 is indifferent in the range $(0, t^*]$ where no other bidder is expected to drop out. We omit the details.

Turning to the non-designated bidder-types above the threshold, observe that value-bidding is at least as good as any bid above the designated bidder's competing bid,

$$\forall i \neq 1,\ h \in \mathcal{H}_i,\ t_i \geq t^*,\ b > b_i^* : \quad t_i \succeq_{h,i,t_i} b. \tag{34}$$

The proof of (34) uses similar arguments as the proof of (32). One defines events I'–IV' analogous to I–IV, with bidder 1 replaced by bidder i, and t_1 replaced by t_i. One of the events I'–IV' occurs; in each event, the payoff from bid t_i is not lower than from bid b. The only essentially new aspect is that Event IV' has positive probability (and, hence, is payoff-relevant) only if $b = t^*$ and $\delta < 1$, in which case bidder 1 bids t^*. Consider Event IV'. Suppose that $n = 2$. Then the bid $b = t^*$ ends the auction with a tie between bidder 1 and bidder $i = 2$. If bidder 1 wins the tie, then by Table 2 and Table 4 bidder 1 consumes the good so that bidder i obtains the payoff 0; if bidder i wins the tie, then she obtains $t_i - t^*$. In any case, the bid t_i is weakly better than b. Suppose that $n > 2$. Then bidder i loses with bid b and obtains the payoff 0, so that the bid t_i is weakly better. This completes the proof of (34).

For non-designated-bidder types below the threshold, any bid between the designated bidder's competing bid and t^* is at least as good as any bid above t^*,

$$\forall i \neq 1,\ h \in \mathcal{H}_i,\ t_i \leq t^*,\ b \in (b_i^*, t^*],\ b' > t^* : \quad b \succeq_{h,i,t_i} b'. \tag{35}$$

To prove (35), first consider the initial history $h = h_0$. If at least one bidder $\neq i$ has a type above t^*, bidder i obtains payoff 0 with any b. Otherwise all bidders $\neq i$ have types below t^*. If $n > 2$, then some bidder drops out at price 0, so that bidder i's bid at the initial history becomes irrelevant and any $b > 0$ leads to the same ending history. Suppose that $n = 2$. If $t_1 \leq \tau^*$, then bidder 1 drops out at price b_i^*, which ends the auction, so that all b lead to the same ending history. If $t_1 > \tau^*$, then bidder 1 drops out at price t^* and bidder i obtains payoff 0 with any b. Arguments are similar for $h \neq h_0$. This completes the proof of (35).

For non-designated-bidder types above the threshold, value-bidding is at least as good as any bid below the designated bidder's competing bid,

$$\forall i \neq 1,\ h \in \mathcal{H}_i,\ t_i \geq t^*,\ b \leq b_i^* : \quad t_i \succeq_{h,i,t_i} b. \tag{36}$$

We prove (36) at $h = h_0$ (other histories are treated similarly). Also, we do not consider the bid $b = b_i^*$; its treatment combines the arguments used to prove (34) with the arguments below, depending on whether or not bidder i wins the tie against bidder 1.

Suppose first that $n = 2$. One of the events V–VII occurs; in each event, the payoff from bid t_i is not lower than from bid b.

Event V: "bidder 1 bids b_2^*." Then with bid t_i bidder i wins at price b_2^*, the period-2 belief is $\mathbf{J}_{\tau^*}^*$, and bidder i's payoff is

$$\max\{\delta w_i(t_i, \mathbf{J}_{\tau^*}^*), t_i\} - b_i^* \quad = \quad t_i - b_i^*.$$

With any bid $b < b_i^*$, bidder i loses, the period-2 belief is $\mathbf{J}_{\tau^*}^*$, and her payoff is

$$\begin{aligned} \delta l_{i1}(t_i, \mathbf{J}_{\tau^*}^*) &\overset{(1)}{\leq} \delta(t_i - t^*) + \delta l_{i1}(t^*, \mathbf{J}_{\tau^*}^*) \\ &\overset{(13)}{=} \delta(t_i - t^*) + \delta(t^* - b_i^*) \\ &= \delta(t_i - b_i^*) \leq t_i - b_i^*. \end{aligned}$$

Hence, the bid t_i is weakly better.

Event VI: "bidder 1 bids t^*." This event has positive probability only if $\delta < 1$, in which case bidder 1 consumes the good if she wins, so that any bid $b < b_i^*$ yields the bidder-i payoff 0. But the bid t_i yields the payoff $t_i - t^* \geq 0$.

Event VII: "bidder 1 bids $> t^*$." Then the proof that t_i is weakly better than any bid $b < b_i^*$ uses similar arguments as the proof of (32).

Now suppose that $n \geq 3$. Consider first the bid $b = 0$. If all bidders $\notin \{1, i\}$ bid 0, then the proof that bid t_i is weakly better is analogous to the treatment of Event V. If some bidder $\notin \{1, i\}$ does not bid 0, then this bidder bids more than t^* and this case is analogous to the treatment of Event VII.

Now consider bids $b > 0$. If some bidder $\notin \{1, i\}$ bids 0, then all bids $b > 0$ (in particular, the bid t_i) yield the same ending history and hence same payoff (because the initial-history bid becomes irrelevant). Otherwise all bidders $\notin \{1, i\}$ bid $> t^*$, so that any bid $b < b_i^*$ yields the payoff 0 and the bid t_i is weakly better (similar arguments as in the proof of (32)). This completes the proof of (36).

For non-designated-bidder types below the threshold, the bid 0 is at least as good as any bid below the designated bidder's competing bid,

$$\forall i \neq 1,\ h \in \mathcal{H}_i,\ t_i \leq t^*,\ b < b_i^* : \quad 0 \succeq_{h,i,t_i} b. \tag{37}$$

To prove (37), observe that the bid 0 yields the same period-2 belief as any bid $b < b_i^*$ in any event, and bidder i always loses the auction. Hence, payoffs are the same.

Finally, for non-designated-bidder types below the threshold, the bid 0 is at least as good as any bid between the designated bidder's competing bid and t^*,

$$\forall i \neq 1,\ h \in \mathcal{H}_i,\ t_i \leq t^*,\ b \in [b_i^*, t^*] : \quad 0 \succeq_{h,i,t_i} b. \tag{38}$$

We prove (38) at $h = h_0$ (other histories are treated similarly). Also, we do not consider the bid $b = b_i^*$; its treatment combines the arguments used to prove (37) with the arguments below, depending on whether or not bidder i wins the tie against bidder 1.

Suppose first that $n = 2$. One of the events V–VII defined above occurs; in each event, the payoff from bid t_i is not lower than from bid b.

Suppose Event V occurs. Then with bid b bidder i wins at price b_2^*, the period-2 belief is $\mathbf{J}^*_{\tau^*}$, and bidder i's payoff is

$$\hat{U}_i(t_i) \quad = \quad \max\{\delta w_i(t_i, \mathbf{J}^*_{\tau^*}), t_i\} - b_i^*$$

With bid 0, bidder i loses, the period-2 belief is $\mathbf{J}^*_{\tau^*}$, and her payoff is

$$U_i^*(t_i) \quad = \quad \delta l_{i1}(t_i, \mathbf{J}^*_{\tau^*}).$$

Observe that

$$\hat{U}_i(t^*) = t^* - b_i^* \overset{(13)}{=} U_i^*(t^*).$$

From (1), U_i^* is Lipschitz continuous and hence is differentiable almost everywhere, and the derivative is

$$U_i^{*\prime}(t_i) \quad = \quad \delta q_{i1}(t_i, \mathbf{J}^*_{\tau^*}).$$

Similarly, the derivative of $\hat{U}_i$ is

$$\hat{U}_i{}'(t_i) \quad \geq \quad \delta q_i(t_i, \mathbf{J}^*_{\tau^*}).$$

Hence,

$$\begin{aligned} U_i^*(t_i) &= U_i^*(t^*) - \int_{t_i}^{t^*} U_i^{*\prime}(s)\mathrm{d}s \\ &= \hat{U}_i(t^*) - \delta \int_{t_i}^{t^*} q_{i1}(s, \mathbf{J}_{\tau^*}^*)\mathrm{d}s \\ &\overset{(8)}{\geq} \hat{U}_i(t^*) - \delta \int_{t_i}^{t^*} q_i(s, \mathbf{J}_{\tau^*}^*)\mathrm{d}s \\ &\geq \hat{U}_i(t^*) - \int_{t_i}^{t^*} \hat{U}_i{}'(s)\mathrm{d}s \\ &= \hat{U}_i(t_i). \end{aligned}$$

Hence, the bid 0 is weakly better.

If Event VI or Event VII occurs, bidder i's payoff is 0 with either bid 0 or any $b \in (b_i^*, t^*]$.

Now suppose that $n \geq 3$. If all bidders $\notin \{1, i\}$ bid 0, then the proof that bid 0 is weakly better than $b \in (b_i^*, t^*]$ is analogous to the treatment of Event V. If some bidder $\notin \{1, i\}$ does not bid 0, then bidder i's payoff is 0 anyway. ■

Proof of Lemma 5. If $\delta = 1$, the result is obvious because $\tau^{*i} = \bar{t}_i$. Let $\delta < 1$ and let

$$\underline{\theta} \;<\; \min\{\frac{1}{4}, \frac{\delta}{1-\delta}\frac{1}{12}\}. \tag{39}$$

Let $i = 1$ (the proof is analogous for other bidders). From (17), $F_2(t^*/2)/F_2(t^*) \to 1/2$ as $t^* \to 0$. Hence, for all t^* sufficiently close to 0,

$$1 - \frac{F_2(t^*/2)}{F_2(t^*)} \;>\; \frac{1}{3}. \tag{40}$$

A lower bound for the period-2 payoff of any type $x \leq \underline{\theta} t^*$ of bidder 1 is the payoff from a take-it-or-leave-it fixed-price offer at $t^*/2$ to bidder 2,

$$w_1(x, \mathbf{J}_x^*) \;\geq\; x + (1 - \frac{F_2(t^*/2)}{F_2(t^*)})(\frac{t^*}{2} - x) \overset{(39),\,(40)}{>} x + \frac{1}{3}\frac{t^*}{4}.$$

Therefore, for all $x \leq \underline{\theta} t^*$,

$$\delta w_1(x, \mathbf{J}_x^*) - x \;>\; \frac{\delta}{12} t^* - (1-\delta)x \overset{(39)}{\geq} 0,$$

implying $\tau^* \geq \underline{\theta} t^*$ by (9). ■

Proof of Lemma 6. For any threshold t^*, $i \in N$, and $t_i \in [0, t^*]$, let[21]

$$V_i(t_i) \quad := \quad t_i - \frac{F_i(t^*) - F_i(t_i)}{f_i(t_i)}. \tag{41}$$

Step 1. If t^* is sufficiently close to 0, then V_i is strictly increasing.

To show this, consider any t, t' such that $0 \leq t < t' \leq t^*$. Then

$$\begin{aligned}
V_i(t') - V_i(t) &= t' - t - \frac{F_i(t^*) - F_i(t')}{f_i(t')} + \frac{F_i(t^*) - F_i(t)}{f_i(t)} \\
&= t' - t - \frac{F_i(t^*) - F_i(t')}{f_i(t')} + \frac{F_i(t^*) - F_i(t)}{f_i(t')} \\
&\quad - \frac{F_i(t^*) - F_i(t)}{f_i(t')} + \frac{F_i(t^*) - F_i(t)}{f_i(t)} \\
&= t' - t + \frac{F_i(t') - F_i(t)}{f_i(t')} + (F_i(t^*) - F_i(t)) \frac{f_i(t') - f_i(t)}{f_i(t) f_i(t')} \\
&\geq t' - t + (F_i(t^*) - F_i(t)) \frac{f_i(t') - f_i(t)}{f_i(t) f_i(t')} \\
&\geq (t' - t) \left(1 - F_i(t^*) \frac{L}{f_i(t) f_i(t')} \right),
\end{aligned} \tag{42}$$

where L is a Lipschitz constant for f_i.

Let t^* be so close to 0 that $f_i(0)/2 < f_i(s) < 2 f_i(0)$ for all $s \in [0, t^*]$. Then $F_i(t^*) = \int_0^{t^*} f_i(s) \mathrm{d}s < 2 f_i(0) t^*$ and (42) implies

$$V_i(t') - V_i(t) \quad \geq \quad (t' - t) \left(1 - 2 f_i(0) t^* \frac{4L}{f_i(0)^2} \right).$$

The right-hand side is strictly positive if $t^* < f_i(0)/8L$, showing that V_i is strictly increasing.

The next step is to provide lower bounds for the virtual valuation functions V_i ($i \in N$) if t^* is small. To this end, define

$$\kappa_i(t^*) := \frac{f_i(0)}{\min_{x \in [0, t^*]} f_i(x)}. \tag{43}$$

Observe that $\kappa_i(t^*) \to 1$ as $t^* \to 0$. Hence, if t^* is small, the lower bound established in *Step 2* approximates the virtual valuation function for a uniform distribution on $[0, t^*]$.

[21] This is the virtual valuation function (cf. Myerson, 1981) for the belief $[0, t^*]$ about bidder i.

Step 2. For all $i \in N$, $t \in [0, t^*]$, and any threshold t^*,

$$V_i(t) \quad \geq \quad t - \kappa_i(t^*)(t^* - t) + o(t^*).$$

Using (17) and (41),

$$\begin{aligned} V_i(t) &= t - \frac{f_i(0)t^* + h_i(t^*) - f_i(0)t - h_i(t)}{f(t)} \\ &= t - \frac{f_i(0)}{f_i(t)}(t^* - t) + h_i^1(t, t^*) \\ &\geq t - \kappa_i(t^*)(t^* - t) + h_i^1(t, t^*), \end{aligned} \tag{44}$$

where

$$h_i^1(t, t^*) \quad := \quad \frac{h_i(t^*) - h_i(t)}{f_i(t)}. \tag{45}$$

Observe that (17) implies

$$\frac{\sup_{x\in[0,t^*]} \mid h_i(x) \mid}{t^*} \leq \sup_{x\in[0,t^*]} \frac{\mid h_i(x) \mid}{x} \to 0 \quad \text{as} \quad t^* \to 0.$$

Hence, defining $\underline{f}_i(t^*) := \min_{x\in[0,t^*]} f_i(x)$,

$$\frac{\sup_{x\in[0,t^*]} |h_i^1(x, t^*)|}{t^*} \quad \leq \quad \frac{1}{\underline{f}_i(t^*)} \left(\frac{|h_i(t^*)|}{t^*} + \frac{\sup_{x\in[0,t^*]} |h_i(x)|}{t^*} \right) \quad \to 0 \quad \text{as} \quad t^* \to 0.$$

The next step is to provide formulas for period-2-buyer allocation probabilities. From now on, suppose that t^* is sufficiently close to 0 so that the conclusion of *Step 1* holds.

Step 3. For any $j \in N$, $i \in N \setminus \{j\}$, and $t_i \in [0, t^*]$,

$$q_{ij}(t_i, \mathbf{J}^{*j}) \quad = \quad \frac{F_j(V_i(t_i))}{F_j(\tau^{j*})} \prod_{k\in N\setminus\{i,j\}} \frac{F_k(V_k^{-1}(V_i(t_i)))}{F_k(t^*)}. \tag{46}$$

According to the Myerson optimal auction outcome given the period-2 seller $j \in N$ and the belief $\mathbf{J}^{*j}$, the good is assigned to the buyer with the highest virtual valuation, unless j's use value is higher (cf. Myerson, 1981). From this the allocation probabilities (46) are straightforward.

Step 4. Proof of (22).

Define $\overline{\xi} = 4/5$. Using *Step 2*, for any $i \in N$,

$$\frac{V_i(\overline{\xi}t^*)}{t^*} = \frac{4}{5} - \kappa_i(t^*)\frac{1}{5} + o(1).$$

Hence, $V_i(\overline{\xi}t^*)/t^* \to 3/5$ as $t^* \to 0$. Thus, using *Step 1*, if t^* is sufficiently close to 0,

$$\forall t_i \in [\overline{\xi}t^*, t^*]:\ V_i(t_i) \geq \frac{t^*}{2}. \tag{47}$$

Using (46), for any $j \neq i$, if $\delta < 1$,

$$\begin{aligned}
q_{ij}(t_i, \mathbf{J}^{*j}) \quad &= \quad \frac{F_j(V_i(t_i))}{F_j(\tau^{*j})} \prod_{k \in N\setminus\{i,j\}} \frac{F_k(V_k^{-1}(V_i(t_i)))}{F_k(t^*)} \\
&\overset{(47),\ \tau^{*j}<t^*}{\geq} \quad \frac{1}{\prod_{k\neq i} F_k(t^*)}\ F_j(\frac{t^*}{2}) \prod_{k\in N\setminus\{i,j\}} F_k(\underbrace{V_k^{-1}(\frac{t^*}{2})}_{\geq t^*/2}) \\
&\geq \quad \prod_{k\neq i} \frac{F_k(\frac{t^*}{2})}{F_k(t^*)} \\
&= \quad \prod_{k\neq i} \frac{f_k(0)\frac{t^*}{2} + o(t^*)}{f_k(0)t^* + o(t^*)} \\
&= \quad \prod_{k\neq i} \frac{f_k(0)\frac{1}{2} + o(1)}{f_k(0) + o(1)} \quad \to \frac{1}{2^{n-1}} > 0 \quad \text{as } t^* \to 0.
\end{aligned}$$

If $\delta = 1$, similar arguments show that $q_{ij}(t_i, \mathbf{J}^{*j})/F_j(t^*) \to 1/2^{n-1}$ because $\tau^{j*} = \overline{t}_j$. Hence, we can choose any $\overline{\epsilon} < 1/2^{n-1}$. ■

Proof of Lemma 7. Using the envelope theorem,

$$\begin{aligned}
U_i^{\text{val}}(t_i) \quad &= \quad \int_0^{t_i} \prod_{k\neq i} F_k(x)\ \mathrm{d}x \\
&\overset{(17)}{=} \quad \int_0^{t_i} \prod_{k\neq i} (f_k(0)x + h_k(x))\ \mathrm{d}x \\
&= \quad \int_0^{t_i} (\prod_{k\neq i} f_k(0)x^{n-1} + h^1(x))\ \mathrm{d}x \quad \text{where } \frac{h^1(x)}{x^{n-1}} \to 0 \ \text{ as } x \to 0. \\
&= \quad \prod_{k\neq i} f_k(0)\frac{1}{n}{t_i}^n + \int_0^{t_i} h^1(x)\ \mathrm{d}x.
\end{aligned}$$

Let $\epsilon > 0$. If t^* is sufficiently small, then $|h^1(x)| \leq \epsilon\ x^{n-1}$ for all $x \leq t^*$. Therefore

$$\left| \int_0^{t_i} h^1(x)\mathrm{d}x \right| \leq \int_0^{t_i} |h^1(x)|\ \mathrm{d}x \leq \epsilon \int_0^{t_i} x^{n-1}\mathrm{d}x \leq \epsilon\ (t^*)^n,$$

which completes the proof. ■

Proof of Lemma 8. First let $\delta < 1$. Let $\underline{\xi} < \underline{\theta}$ and $\underline{\epsilon} > 0$ be so close to 0 that

$$1 - (n-1)\overline{\epsilon}(1-\overline{\xi}) \quad < \quad (1 - \underline{\xi}/\underline{\theta})(1-\underline{\epsilon}). \tag{48}$$

Because probabilities add up to 1,

$$\int_0^{\tau^{*j}} q_j(t_j, \mathbf{J}^{*j}) \frac{\mathrm{d}F_j(t_j)}{F_j(t^*)} + \sum_{i \neq j} \int_0^{t^*} q_{ij}(t_i, \mathbf{J}^{*j}) \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \quad = \quad 1. \tag{49}$$

Using Lemma 5,

$$\liminf_{t^* \to 0} \frac{F_j(\tau^{*j}) - F_j(\underline{\xi} t^*)}{F_j(\tau^{*j})} \;\geq\; \liminf_{t^* \to 0} \frac{F_j(\underline{\theta} t^*) - F_j(\underline{\xi} t^*)}{F_j(\underline{\theta} t^*)} \;\overset{(17)}{=}\; 1 - \underline{\xi}/\underline{\theta}. \tag{50}$$

It is well-known that the incentive compatibility constraints for period 2 imply that (*) the functions $q_j(\cdot, \mathbf{J}^{*j})$ are weakly increasing. We find

$$\begin{aligned}
(1 - \underline{\xi}/\underline{\theta}) \limsup_{t^* \to 0} q_j(\underline{\xi} t^*, \mathbf{J}^{*j}) &\overset{(50)}{\leq} \liminf_{t^* \to 0} \frac{F_j(\tau^{*j}) - F_j(\underline{\xi} t^*)}{F_j(\tau^{*j})} \; \limsup_{t^* \to 0} q_j(\underline{\xi} t^*, \mathbf{J}^{*j}) \\
&\leq \limsup_{t^* \to 0} \frac{F_j(\tau^{*j}) - F_j(\underline{\xi} t^*)}{F_j(\tau^{*j})} q_j(\underline{\xi} t^*, \mathbf{J}^{*j}) \\
&\overset{(*)}{\leq} \limsup_{t^* \to 0} \int_0^{\tau^{*j}} q_j(t_j, \mathbf{J}^{*j}) \frac{\mathrm{d}F_j(t_j)}{F_j(\tau^{*j})} \\
&\overset{(49)}{=} 1 - \liminf_{t^* \to 0} \sum_{i \neq j} \int_0^{t^*} q_{ij}(t_i, \mathbf{J}^{*j}) \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \\
&\leq 1 - \liminf_{t^* \to 0} \sum_{i \neq j} \int_{\overline{\xi} t^*}^{t^*} q_{ij}(t_i, \mathbf{J}^{*j}) \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \\
&\overset{(22)}{\leq} 1 - \overline{\epsilon} \; \liminf_{t^* \to 0} \sum_{i \neq j} \int_{\overline{\xi} t^*}^{t^*} \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \\
&= 1 - \overline{\epsilon} \; \liminf_{t^* \to 0} \sum_{i \neq j} \frac{F_i(t^*) - F_i(\overline{\xi} t^*)}{F_i(t^*)} \\
&\overset{(17)}{=} 1 - (n-1)\overline{\epsilon}(1 - \overline{\xi}) \\
&\overset{(48)}{<} (1 - \underline{\xi}/\underline{\theta})(1 - \underline{\epsilon}).
\end{aligned}$$

Dividing both sides by $1 - \underline{\xi}/\underline{\theta}$ yields (23).

Now let $\delta = 1$. Define $\underline{\xi} < 1$ and $\underline{\epsilon} > 0$ as in (48) with $\underline{\theta} = 1$. Using (3) and (4), $q_j(t_j, \mathbf{J}^{*j}) = 1$ for all $t_j > t^*$. Hence, because $\tau^{*j} = \bar{t}_j$ and probabilities sum up to 1,

$$\int_0^{t^*} q_j(t_j, \mathbf{J}^{*j}) \mathrm{d}F_j(t_j) + (1 - F_j(t^*)) + \sum_{i \neq j} \int_0^{t^*} q_{ij}(t_i, \mathbf{J}^{*j}) \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \;=\; 1.$$

Rearranging yields

$$\int_0^{t^*} q_j(t_j, \mathbf{J}^{*j}) \frac{\mathrm{d}F_j(t_j)}{F_j(t^*)} + \sum_{i \neq j} \int_0^{t^*} \frac{q_{ij}(t_i, \mathbf{J}^{*j})}{F_j(t^*)} \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \;=\; 1. \tag{51}$$

Similar to the cases $\delta < 1$, we obtain (23) because

$$\begin{aligned}
(1 - \underline{\xi}) \limsup_{t^* \to 0} q_j(\underline{\xi} t^*, \mathbf{J}^{*j}) &\overset{(17)}{=} \limsup_{t^* \to 0} \frac{F_j(t^*) - F_j(\underline{\xi} t^*)}{F_j(t^*)} q_j(\underline{\xi} t^*, \mathbf{J}^{*j}) \\
&\overset{(*)}{\leq} \limsup_{t^* \to 0} \int_0^{t^*} q_j(t_j, \mathbf{J}^{*j}) \frac{\mathrm{d}F_j(t_j)}{F_j(t^*)} \\
&\overset{(51)}{=} 1 - \liminf_{t^* \to 0} \sum_{i \neq j} \int_0^{t^*} \frac{q_{ij}(t_i, \mathbf{J}^{*j})}{F_j(t^*)} \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)} \\
&\overset{(22)}{\leq} 1 - \bar{\epsilon}\, \liminf_{t^* \to 0} \sum_{i \neq j} \int_{\bar{\xi} t^*}^{t^*} \frac{\mathrm{d}F_i(t_i)}{F_i(t^*)}.
\end{aligned}$$

■

Proof of Lemma 9. If $t_i < \bar{\xi} t^*$, the claim $U_{ij}^b(t_i) \geq 0$ is immediate from (4). Let $t_i > \bar{\xi} t^*$. Using (1) and $l_{ij}(0, \mathbf{J}^{*j}) \geq 0$ (from (4)), if $\delta < 1$,

$$\frac{1}{F_j(\tau^{*j})} \frac{1}{\prod_{k \notin \{i,j\}} F_k(t^*)} U_{ij}^b(t_i) \;\geq\; \delta \int_0^{t_i} q_{ij}(x, \mathbf{J}^{*j}) \; \mathrm{d}x \;\overset{(22)}{\geq}\; \delta \int_{\bar{\xi} t^*}^{t_i} \bar{\epsilon} \; \mathrm{d}x \;=\; \delta \bar{\epsilon} (t_i - \bar{\xi} t^*),$$

which together with Lemma 5 implies the claim. If $\delta = 1$, similar arguments imply

$$\frac{1}{\prod_{k \notin \{i,j\}} F_k(t^*)} U_{ij}^b(t_i) \;\geq\; \bar{\epsilon} F_j(t^*)(t_i - \bar{\xi} t^*).$$

■

Proof of Lemma 10. If $t_i > \underline{\xi} t^*$, the claim $U_i^s(t_i) \geq t_i$ is immediate from equilibrium condition (c). Let $t_i \leq \underline{\xi} t^*$. First let $\delta < 1$. Then $t_i < \underline{\theta} t^* \leq \tau^{*i}$ by Lemma 5. Using (2) and the fact that $\delta w_i(\tau^{*i}, \mathbf{J}^{*i}) = \tau^{*i}$ (from the definition of τ^{*i}),

$$\begin{aligned}
\frac{1}{\prod_{k\neq i} F_k(t^*)} U_i^s(t_i) &= \tau^{*i} - \delta \int_{t_i}^{\tau^{*i}} q_i(x, \mathbf{J}^{*i})\ \mathrm{d}x \\
&= \tau^{*i} - \delta(\tau^{*i} - t_i) + \delta \int_{t_i}^{\tau^{*i}} (1 - q_i(x, \mathbf{J}^{*i}))\ \mathrm{d}x \\
&\overset{(23)}{\geq} t_i + \delta\underline{\epsilon}(\underline{\xi} t^* - t_i),
\end{aligned}$$

as was to be shown. If $\delta = 1$, then the claim follows from analogous arguments with τ^{*i} replaced by t^*. ■

7 Appendix B

We wish to show that the strategies used in t^*-equilibria are undominated. To do so we utilize the "type-player interpretation" of the auction-with-resale game, where different types of a bidder represent different players (see, e.g., Osborne and Rubinstein, 1994, p. 26).[22] A pure strategy s for type $t_i \in T_i$ of bidder i is *undominated* if, for any strategy $s' \neq s$, either s' yields the same (expected) payoff as s no matter what other bidders do, or

> (*) there exists a profile of strategies s_{-i} for the bidders other than i such that s yields a strictly higher payoff than s' against s_{-i}.

Assume that there are $n = 2$ bidders, so that the period-2 seller makes a fixed-price take-it-or-leave-it offer according to the Myerson optimal auction outcome; at the end of this section we comment on the extension of the arguments to environments with $n \geq 3$ bidders. A pure strategy s for type t_i of bidder i is described by a bid $\beta_i(t_i)$ and a resale price function $r_i(t_i, \cdot)$, where $r_i(t_i, p) \in [0, \infty) \cup \{\text{consume}\}$ denotes her resale price or consumption decision when she wins the auction at price p. The domain of the function r_i can be restricted to the set $\{(t_i, p) \mid t_i \in [0, \bar{t}_i],\ p \in [0, \beta_i(t_i)]\}$.[23] Any resale price is accepted by all losing-bidder types greater than or equal to this price.

[22] This corresponds to the natural viewpoint that the bidders select their strategies after they have learned their private information.

[23] We do not have to specify a bidder's resale price if she wins at a price higher than her bid, because different prices yield equivalent strategies.

The t^*-equilibrium strategies can be defined such that, for all p,[24]

$$r_i(t_i, p) \in \begin{cases} (t_i/\delta, \bar{t}_{-i}) \cup \{\text{consume}\} & \text{if } \delta < 1, \\ (t_i, \bar{t}_{-i}) & \text{if } \delta = 1 \text{ and } t_i < \bar{t}_{-i}, \\ t_i & \text{if } \delta = 1 \text{ and } t_i \geq \bar{t}_{-i}. \end{cases} \tag{52}$$

Proposition 4 shows that pure strategies satisfying (52) are undominated, except possibly for 0-type bidders.[25] The proof proceeds in two steps. First, we show that a strategy of any type of bidder i that involves a bid b cannot be weakly dominated by an alternative strategy where the bidder submits a bid $\neq b$. This is true because assuming the other bidder bids so aggressively that bidder i must wait for a resale offer, it may be the case that a favorable resale offer is made only if bidder $-i$ wins at price b, making the bid b uniquely optimal. In the second step we show that a strategy of any type of bidder i that involves a bid b and a certain resale price $x \neq$ consume if the bidder wins at a given price $p \leq b$ cannot be weakly dominated by a strategy where bidder i sticks to the bid b, but changes her resale behavior upon winning at p. Here we use the assumption that bidder $-i$'s valuation density is positive at x. Indeed, if, for some small $\epsilon > 0$, types $[x, x + \epsilon]$ of bidder $-i$ bid p and all other types bid more than b, then it is uniquely optimal to offer resale at price x upon winning at p (and the resale decisions when winning at a price $\neq p$ are irrelevant). The argument is slightly different if $x =$ consume.

Proposition 4 *Let $n = 2$. For all types $t_i \neq 0$ of all bidders i, any pure strategy satisfying (52) is undominated.*

Proof. Consider any pure strategy s for type t_i of bidder i consisting of a bid b and a resale price function $r_i(t_i, \cdot)$. Let $V(b)$ denote the set of all (pure or mixed) strategies of type t_i of bidder i where she bids b with certainty.

[24] If $\delta = 1$ and $t_i \geq \bar{t}_{-i}$, all resale prices $\geq t_i$ yield the same payoff as "consume", so we can as well fix the price at t_i. Otherwise, any strategy not satisfying (52) is dominated. If $\delta < 1$, any price $\leq t_i/\delta$ is no better, and sometimes worse, than consuming the good. If $\delta = 1$, any price $\leq t_i$ is no better, and sometimes worse, than any price in $(t_i, \bar{t}_{-i})$. Any price $\geq \bar{t}_{-i}$ will be accepted with probability 0 from an ex-ante viewpoint.

[25] The 0-types' t^*-equilibrium strategies are in fact dominated. Each 0-type bidder may switch to any undominated strategy at the cost of complicating the definition of t^*-equilibria and leaving the rest of the analysis unchanged because the 0 types occur with probability 0.

Step 1. (*) holds if $s' \notin V(b)$.

Define strategies $s_{-i} = (\beta_{-i}(t_{-i}), r_{-i}(t_{-i}, \cdot))$ for the various types $t_{-i} \in T_{-i}$ of bidder $-i$ as follows. Bidder $-i$ of type t_{-i} bids $\beta_{-i}(t_{-i}) > \max\{b, t_{-i}\}$. If she wins at any price $p \neq b$, then she chooses a resale price $r_{-i}(t_{-i}, p)$ optimally given the belief that bidder i's type is in $[t_i - \epsilon, \bar{t}_i]$, where $\epsilon > 0$ is defined below.[26] She chooses $r_{-i}(t_{-i}, b)$ optimally given the belief that bidder i's type is in $[0, t_i]$.

Now consider bidder i of type t_i who is bidding against s_{-i}. Any bid $\neq b$ yields a payoff $\leq \epsilon$, while the bid b yields a positive expected payoff $\hat{u} > 0$ because with positive probability bidder i gets a resale offer priced below t_i. Choosing $\epsilon < \hat{u}$, the strategy s is strictly better than any strategy not in $V(b)$. This completes *Step 1.*

If $\delta = 1$ and $t_i \geq \bar{t}_{-i}$, we are done because $V(b) = \{s\}$. From now on, assume that $\delta < 1$ or $t_i < \bar{t}_{-i}$.

Given any $p \leq b$, let $W(p, b)$ denote the set of all (pure or mixed) strategies where type t_i of bidder i chooses the resale price $r_i(t_i, p)$ with certainty when she wins at price p.

Step 2. (*) holds if $s' \in V(b) \setminus W(p, b)$.

First assume that $r_i(t_i, p) =: x \neq$ consume. Observe that $t_i < \bar{t}_{-i}$ by (52). Define strategies $s_{-i} = (\beta_{-i}(t_{-i}), r_{-i}(t_{-i}, \cdot))$ such that $\beta_{-i}(t_{-i}) = p$ if $t_{-i} \in [x, x+\epsilon]$, and $\beta_{-i}(t_{-i}) > b$ otherwise, where $\epsilon > 0$ is chosen so small that

$$y - \frac{F_{-i}(x+\epsilon) - F_{-i}(y)}{f_{-i}(y)} \quad > \quad t_i \qquad \forall y \in [x, x+\epsilon]. \tag{53}$$

Resale prices $r_{-i}(t_{-i}, \cdot)$ are arbitrary. Against s_{-i}, if type t_i of bidder i wins at price p, then the resale price x is uniquely optimal for her (her period-2 payoff from resale price y is strictly decreasing for $y \in [x, x+\epsilon]$, which can be seen by computing the derivative of the payoff and using (53)). Because the price $x > t_i/\delta$ is accepted with certainty, it is also better than "consume". Hence, s is strictly better than any s'.

[26] The argument works with $\epsilon = 0$ if $t_i < \bar{t}_i$. But the belief $\{\bar{t}_i\}$ implies $r_{-i}(t_{-i}, p) = \bar{t}_i$ if t_{-i} is small, which violates (52).

Now assume that $r_i(t_i, p)$ = consume. Define strategies $s_{-i} = (\beta_{-i}(t_{-i}), r_{-i}(t_{-i}, \cdot))$ such that $\beta_{-i}(t_{-i}) = p$ if $t_{-i} \in [0, \epsilon]$, and $\beta_{-i}(t_{-i}) > b$ otherwise, where $\epsilon < t_i$. Resale prices $r_{-i}(t_{-i}, \cdot)$ are arbitrary. If type t_i of bidder i wins at price p against s_{-i}, then "consume" is better than offering resale. Hence, s is strictly better than any s'. This completes *Step 2.*

Step 1 and *Step 2* show that s is undominated because[27]

$$\{s\} \quad = \quad \cap_{p \leq b} W(p, b) \ \cap \ V(b).$$

■

Now consider environments with $n \geq 3$ bidders. The crucial complication in extending Proposition 4 occurs in *Step 1* of the proof. Below we outline a proof that strategies that involve arbitrary bid functions are undominated, so long as the resale behavior is appropriately restricted in a manner similar to (52).

Any type of a given bidder must choose a planned dropout price (bid) at each history during the English auction. First one shows that a strategy that involves some bid b *at the initial history* cannot be weakly dominated by a strategy that involves a bid $\neq b$ at the initial history. The resale mechanisms involved in this argument are second-price auctions with reserve prices similar to the resale prices used in the 2-bidder proof. Next, a strategy that involves a bid b at the initial history and a bid b' at the history h' reached when a certain bidder drops out at a certain price cannot be dominated by a strategy where she submits the bid b at the initial history and a bid $\neq b'$ at h'; to show this, one constructs strategies of the other bidders such that history h' is reached and then, as before, the remaining active rivals bid so aggressively that the bidder must wait for a resale offer. Continuing inductively to the end of the auction, one sees that a strategy cannot be dominated by any strategy that involves a bidding structure different from the one used in the original strategy.

[27]Observe that, for the verification of (*) in steps 1 and 2, the strategy profile s_{-i} can itself be taken to be a profile of strategies satisfying (52). In this sense, any pure strategy satisfying (52) survives any iterated elimination of dominated strategies.

8 References

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