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Optimal Auction Design and Irrelevance of Privacy of Information*

by Tymofiy Mylovanov† and Thomas Tröger‡

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Abstract

We consider the problem of mechanism design by a principal who has private information. We point out a simple condition under which the privacy of the principal's information is irrelevant in the sense that the mechanism implemented by the principal coincides with the mechanism that would be optimal if the principal's information were publicly known. This condition is then used to show that the privacy of the principal's information is irrelevant in many environments with private values and quasi-linear preferences, including the Myerson's classical auction environments in which the seller is privately informed about her cost of selling. Our approach unifies results by Maskin and Tirole, Tan, Yilankaya, Skreta, and Balestrieri. We also provide an example of a classical principal-agent environment with private values and quasi-linear preferences where a privately informed principal can do *better* than when her information is public.

Keywords: informed principal, strong solution, optimal auction, full-information optimum, quasi-linear payoff functions

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1 Introduction

The optimal design of mechanisms in the presence of privately informed market participants is central to economics. Under the assumption that all participants have quasi-linear preferences over market outcomes, a rich theory has emerged (see, e.g., the books by Krishna (2002) and Milgrom (2004)). A caveat in much of this theory is that the mechanism proposer (the principal) is assumed to have *no* private information although, in many applications, she is one of the market participants and, as such, should have private information. For example, often the designer of an auction is in fact the seller of the auctioned good and is privately informed about her opportunity cost of selling.[1]

Private information held by the principal changes her mechanism design problem into an "informed-principal problem." On the one hand, she may gain by withholding her private information at the mechanism proposal stage.[2] On the other hand, the privacy of the principal's information may harm her because she faces incentive constraints.[3] In this paper, we point out a simple condition (*), described below, which guarantees that the privacy of the principal's information is irrelevant, in the sense that a privately informed principal offers the same mechanism as when her information is public.

We show that condition (*) is satisfied in many quasi-linear environments. In particular, condition (*) is satisfied in Myerson's (1981) classical auction environments in which the seller is privately informed about her opportunity cost of selling, implying that the seller's optimal auction mechanism is the same as when her cost is publicly known;[4] this result holds even if Myerson's regularity condition (1981, p. 66) is not satisfied. Condition (*)

[1]Other examples abound. For instance, in settings of mechanism design with collusion (e.g., Laffont and Martimort (1997), p. 7, footnote 8, Che and Kim (2006), p. 1093, Quesada (2004), and Mookherjee and Tsumagari (2004), footnote 7, p. 1186) the proposer of a collusive side contract may act as an informed principal.

[2]Maskin and Tirole (1990) demonstrate this for a class of environments with *non-quasi-linear* preferences.

[3]To see this, consider Akerlof's (1970) lemons market with the seller being the principal. If the seller's quality type is public, then she can extract all rents, but this is not incentive compatible for low-quality types if the seller's type is her private information.

[4]This result can be used to justify *a posteriori* corresponding assumptions in a number of models of auctions with resale; see, e.g., Zheng (2002), p. 2201, Haile (2003), footnote 19, p. 13, Garratt et al. (2008), Hafalir and Krishna (2008).

is also satisfied in the classical principal-agent environments of Guesnerie and Laffont (1984) under a certain regularity assumption. Versions of condition (*) underly a number of earlier results involving informed principals (cf. Maskin and Tirole (1990, Proposition 11), Tan (1996), Yilankaya (1999), and Balestrieri (2008)).

To state condition (*), we need three standard concepts. A *full-information-optimal* allocation rule[5] (mechanism) is the collection of the allocation rules that are optimal for each (information) type of the principal when her type is publicly known. An allocation rule is *ex-ante optimal* if it maximizes the principal's ex-ante expected payoff (before she observes her own type). An environment has *private values* if the principal's type does not enter the payoff functions of the other players (the agents).[6]

The privacy of the principal's information is irrelevant under the following simple condition[7]

> (*): The environment has private values, and there exists a full-information-optimal allocation rule that is ex-ante optimal.

The crucial implication of condition (*) is that there exists a full-information optimal allocation rule that is a *strong solution* in the sense of Myerson (1983). If an allocation rule is a strong solution, then it should be considered a solution of the informed-principal problem; in particular, a strong solution is a perfect Bayesian equilibrium outcome of a non-cooperative mechanism-proposal game.[8] [9] Furthermore, if there are multiple strong solutions, they yield the same payoffs to the principal.

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[5]This term was coined by Maskin and Tirole (1990).

[6]Our definition of "private values" allows that the agents' types enter the principal's payoff function, and that the agents' payoff functions are interdependent.

[7]Condition (*) does not cover environments with non-private ("common") values. In common-value environments, full-information-optimal allocation rules are typically not incentive compatible for the principal. Maskin and Tirole (1992) compute perfect Bayesian equilibria of informed-principal games in various common-value environments.

[8]For environments where no strong solution exists, Myerson (1983) proposes a concept *neutral optimum* as a solution to the informed-principal problem. A neutral optimum exists in any environment with finite type spaces and a finite outcome space, and is a perfect Bayesian equilibrium outcome. (Note that, in contrast to an apparently widespread misunderstanding, strong solution and neutral optimum are not concepts of cooperative game theory. Rather, they are based on axioms that serve as a device for selecting among the non-cooperative equilibrium outcomes.)

[9]In the terminology employed by Maskin and Tirole (1992) for their treatment of common-value environments, a strong solution is an interim efficient Rothschild-Stiglitz-

The fact that condition (*) is satisfied in many environments with independent private values and quasi-linear preferences may lead to the conjecture that (*) is always satisfied in such environments. We provide a counterexample, using a principal-agent environment that belongs to the class of Guesnerie and Laffont (1984); in the example, "bunching" occurs in the full-information optimal allocation rule, but not in the ex-ante optimal allocation rule.[10] A failure of condition (*) can also occur if an agent is budget constrained: we provide an example where the principal extracts the entire surplus in the ex-ante optimal allocation rule, but not in the full-information optimal allocation rule.

Maskin and Tirole (1990, Proposition 11) were the first to point out that the privacy of the principal's private information is irrelevant in some quasi-linear environments with independent private values.[11] This result has also been obtained in a number of other quasi-linear environments: a procurement environment in which the buyer has private information about his marginal valuation (Tan 1996), the Myerson-Satterthwaite bargaining environment (Yilankaya 1999), an auction environment in which the auctioneer has private information about the bidders' valuations (Skreta 2008), and a procurement environment in which the buyer's private information is his compatibility with the suppliers' inputs (Balestrieri 2008). A version of condition (*) is satisfied in all of these environments and is explicit in the arguments in Tan (1996) and Yilankaya (1999).

The remainder of the paper is organized as follows. In Section 2 we present the model and condition (*). Applications of condition (*) are treated in Section 3. Section 4 contains (counter-)examples.

Wilson allocation rule. Lemma 6.3 in Tirole (2006) provides sufficient conditions for existence of a strong solution in such environments.

[10]The example also shows that Maskin and Tirole's (1990, Proposition 11) result does not generalize beyond the case of two agent types without further regularity assumptions.

[11]Maskin and Tirole allow for one agent and two types and impose conditions that restrict the set of relevant incentive and participation constraints. Their techniques are very different from ours. In particular, Maskin and Tirole refer to the shadow values (Lagrange multipliers) of the agent's incentive and participation constraints. Extending their approach to environments with continuous type spaces such as those commonly used in auction theory appears difficult.

2 Model

We consider the interaction of a principal (player 0) and n agents (players $i \in N = \{1, \ldots, n\}$). The players must collectively choose an outcome from a set

$$Z \;=\; A \times [-\hat{x}, \hat{x}]^n,$$

where $[-\hat{x}, \hat{x}]^n$ represents the set of feasible vectors of monetary transfers from the agents to the principal,[12] and the compact metric space A represents a set of verifiable collective actions.[13] For example, $A = \{0, 1, \ldots, n\}$ may represent an environment where the collective action is the allocation of a single unit of a private good among the principal and the agents.

Every player $i = 0, \ldots, n$ has a *type* t_i that belongs to a compact *type space* $T_i \subseteq \mathbb{R}$.[14] The product of agents' type spaces is denoted $\mathbf{T} = T_1 \times \cdots \times T_n$. Player i's payoff function is denoted

$$u_i :\; Z \times T_0 \times \mathbf{T} \to \mathbb{R},$$

We restrict attention to *quasi-linear* payoff functions: for all $i \in N$, $a \in A$, $\mathbf{x} \in [-\hat{x}, \hat{x}]^n$, $t_0 \in T_0$, and $\mathbf{t} \in \mathbf{T}$,

$$u_i(a, \mathbf{x}, t_0, \mathbf{t}) \;=\; v_i(a, t_0, \mathbf{t}) - x_i, \qquad (1)$$

$$u_0(a, \mathbf{x}, t_0, \mathbf{t}) \;=\; v_0(a, t_0, \mathbf{t}) + x_1 + \cdots + x_n, \qquad (2)$$

for some *values functions* $v_0, \ldots, v_n$. We assume that, for all $i = 0, \ldots, n$, the family of functions $(v_i(a, \cdot) : T_0 \times \mathbf{T} \to \mathbb{R})_{a \in A}$ is equi-continuous, that $v_i(\cdot, t_0, \mathbf{t}) : A \to \mathbb{R}$ is measurable for all $t_0 \in T_0$ and $\mathbf{t} \in \mathbf{T}$, and that v_i is a bounded function.

An environment has *private values* if the agents' payoff functions are independent of the principal's type, that is, if

$$\forall i \in N,\; a \in A,\; t_0, t_0' \in T_0,\; \mathbf{t} \in \mathbf{T} :\quad v_i(a, t_0, \mathbf{t}) = v_i(a, t_0', \mathbf{t}).$$

[12] The assumption that transfers are bounded by some (arbitrarily large) number $\hat{x}$ guarantees that stochastical expectations are finite throughout the analysis.

[13] For treatments of the informed-principal problem in settings with non-verifiable actions (that is, in moral-hazard settings), see Beaudry (1994), Bond and Gresik (1997), Chade and Silvers (2002), Jost (1996), and Mezzetti and Tsoulouhas (2000).

[14] One-dimensional type spaces are sufficient for all our applications. The results of Section 2 carry over to multi-dimensional type spaces.

According to this definition, in a private-value environment it is still possible that the agents' payoff functions are interdependent, and the principal's payoff function may depend on the agents' types.

We assume that the types $t_0, \ldots, t_n$ are realizations of stochastically independent[15] random variables with cumulative probability distribution functions $F_0, \ldots, F_n$, where the support of F_i equals T_i. We call F_i the *prior distribution* for player i's type. The joint distribution of agents' types (excluding the principal) is denoted $\mathbf{F}$. We will use the notation $\mathbf{t}_{-i}$ for the vector of types of the agents other than i (also excluding the principal), use $\mathbf{T}_{-i}$ for the respective product of type spaces, and use $\mathbf{F}_{-i}$ for the respective product of c.d.f.s.

The interaction leads to a probability distribution over outcomes. Any probability distribution over transfer vectors leads to a vector of expected transfers. Hence, if we identify any payoff-equivalent distributions, the set of probability distributions over outcomes is given by

$$\mathcal{Z} \;=\; \mathcal{A} \times [-\hat{x}, \hat{x}]^n,$$

where $\mathcal{A}$ denotes the set of probability measures on A; any element of $\mathcal{A}$ is also called a collective action.[16] We identify any $a \in A$ with the point distribution that puts probability 1 on the point a; hence, $A \subseteq \mathcal{A}$.[17] We

[15]If types are correlated, a rather different analysis is required: typically, a privately informed principal will be strictly better off than if her information is public; see Cella (forthcoming) and Severinov (2008).

[16]We endow $\mathcal{A}$ with the smallest σ-algebra such that, for every measurable set $B \subseteq A$, the mapping $m_B : \mathcal{A} \to [0,1]$, $\alpha \mapsto \alpha(B)$ is measurable. Given this σ-algebra, any uncertainty about outcomes in $\mathcal{A}$ can be equivalently described as uncertainty about outcomes in A. Formally, any probability measure P on $\mathcal{A}$ can be identified with a probability measure α_P on A, via the definition

$$\alpha_P(B) \;=\; \int_{\mathcal{A}} \alpha(B) P(\mathrm{d}\alpha) \quad \text{for every measurable } B \subseteq A.$$

[17]Observe that, if $\mathcal{M}$ is an arbitrary measurable space and if a mapping $f : \mathcal{M} \to Z$ is measurable with respect to the σ-algebra on A, then f is also measurable when viewed as a mapping into $\mathcal{A}$ (the reason is that the composite mapping $m_B f$ is measurable for every measurable $B \subseteq A$.

extend the definition of v_i via the statistical expectation: for all $\alpha \in \mathcal{A}$,[18]

$$\begin{aligned} v_i(\alpha, t_0, \mathbf{t}) &= \int v_i(a, t_0, \mathbf{t}) \; \alpha(\mathrm{d}a) \quad (i \in N), \\ v_0(\alpha, t_0, \mathbf{t}) &= \int v_0(a, t_0, \mathbf{t}) \; \alpha(\mathrm{d}a). \end{aligned}$$

Fixing some collective action $a_0 \in \mathcal{A}$, we normalize $v_i(a_0, t_0, \mathbf{t}) = 0$ for all $i \in I$, $t_0 \in T_0$, and $\mathbf{t} \in \mathbf{T}$. We call $z_0 = (a_0, 0, \ldots, 0)$ the *disagreement outcome*.

The interaction is described by the following *informed-principal game*. First, each player privately observes her type t_i. Second, the principal offers a mechanism M (chosen from some set of feasible game forms). Third, the agents decide simultaneously whether or not to accept M. If M is accepted unanimously, each player chooses a message in M, and the outcome specified by M is implemented. If at least one agent rejects M, the disagreement outcome z_0 is implemented.

An *allocation rule* is any measurable function

$$\rho : \quad T_0 \times \mathbf{T} \to \mathcal{Z}, \quad (t_0, \mathbf{t}) \mapsto \rho(t_0, \mathbf{t})$$

that assigns an outcome $\rho(t_0, \mathbf{t})$ to every type profile $(t_0, \mathbf{t})$. Thus, an allocation rule describes the outcome of the players' interaction as a function of the type profile. Alternatively, an allocation rule ρ can be interpreted as a *direct mechanism*, where the players $i = 0, \ldots, n$ simultaneously announce types $\hat{t}_i \in T_i$ and the outcome $\rho(\hat{t}_0, \ldots, \hat{t}_n)$ is implemented.

Strong solution

Myerson (1983) argues that a particular allocation rule, called *strong solution*, should be considered a solution of the informed-principal game whenever a strong solution exists. Myerson introduces the concept of a strong solution for environments with finite type spaces and finite outcome spaces, and shows that a strong solution always is a perfect Bayesian equilibrium outcome of an informed-principal game. We extend the concept of a strong solution to non-finite environments.

[18] Observe that the extended mapping $v_i : \mathcal{A} \times T_0 \times \mathbf{T} \to \mathbb{R}$ inherits the following properties: the family of functions $(v_i(\alpha, \cdot) : T_0 \times \mathbf{T} \to \mathbb{R})_{\alpha \in \mathcal{A}}$ is equi-continuous, the function $v_i(\cdot, t_0, \mathbf{t}) : \mathcal{A} \to \mathbb{R}$ is measurable for all $t_0 \in T_0$ and $\mathbf{t} \in \mathbf{T}$, and v_i is bounded.

A direct mechanism is called *safe* for the principal if no type of any player has an incentive to deviate from announcing her true type or can gain from refusing to participate, and if this would remain so even if all agents knew the principal's true type. To state this formally, define the agents' payoffs

$$U_i^\rho(\hat{t}_i, t_i, t_0) \;=\; \int_{\mathbf{T}_{-i}} u_i(\rho(t_0, \hat{t}_i, \mathbf{t}_{-i}), t_0, (t_i, \mathbf{t}_{-i}))\; \mathbf{F}_{-i}(\mathrm{d}\mathbf{t}_{-i})$$
$$(i \in N,\ \hat{t}_i, t_i \in T_i,\ t_0 \in T_0)$$

and the principal's payoff

$$U_0^\rho(\hat{t}_0, t_0) \;=\; \int_{\mathbf{T}} u_0(\rho(\hat{t}_0, \mathbf{t}), t_0, \mathbf{t})\; \mathbf{F}(\mathrm{d}\mathbf{t}) \qquad (\hat{t}_0, t_0 \in T_0).$$

A direct mechanism ρ is *safe* if

$$\forall i \in N,\ t_i, \hat{t}_i \in T_i: \quad U_i^\rho(t_i, t_i, t_0) \;\geq\; U_i^\rho(\hat{t}_i, t_i, t_0), \tag{3}$$
$$\forall i \in N,\ t_i \in T_i: \quad U_i^\rho(t_i, t_i, t_0) \;\geq\; 0, \tag{4}$$
$$\forall t_0, \hat{t}_0 \in T_0: \quad U_0^\rho(t_0, t_0) \;\geq\; U_0^\rho(\hat{t}_0, t_0). \tag{5}$$

A direct mechanism is called *incentive feasible* if no type of any player has an incentive to deviate from announcing her true type or can gain from refusing to participate, given the prior type distributions. To state this formally, define the agents' payoffs

$$U_i^\rho(\hat{t}_i, t_i) \;=\; \int_{T_0} U_i^\rho(\hat{t}_i, t_i, t_0)\; F_0(\mathrm{d}t_0) \quad (i \in N,\ \hat{t}_i, t_i \in T_i).$$

A direct mechanism ρ is called *incentive feasible* if it satisfies the condition (5) and the conditions

$$\forall i \in N,\ t_i, \hat{t}_i \in T_i: \quad U_i^\rho(t_i, t_i) \;\geq\; U_i^\rho(t_i', t_i), \tag{6}$$
$$\forall i \in N,\ t_i \in T_i: \quad U_i^\rho(t_i, t_i) \;\geq\; 0. \tag{7}$$

An incentive feasible direct mechanism ρ is called *dominated* if there exists an incentive feasible direct mechanism ρ' such that all types of the principal are at least as well off in ρ' as in ρ, and a positive mass of types of the principal is strictly better off in ρ'.[19] A safe direct mechanism that is not dominated is called a *strong solution*.

Strong solutions yield a unique payoff prediction for the principal: if there are multiple strong solutions, each type of the principal obtains the same payoff in any of these.[20]

Perfect Bayesian Equilibrium

Myerson (1983, Theorem 2) proves that in any environment with finite type spaces and a finite outcome space, a strong solution is a perfect Bayesian equilibrium outcome of an informed-principal game where any finite simultaneous-move game form is a feasible mechanism. As for extending the definition of the informed-principal game to environments with infinite type spaces, it is not obvious which game forms should be considered feasible mechanisms; note that a direct mechanism is not a finite game form.

Here we sketch a proof that, if all types of the principal offer a given strong solution as a direct mechanism, then, for any deviating finite (simultaneous-move or multi-stage) game form, we can construct off-path beliefs about the principal's type such that no type of the principal has an incentive to deviate by offering this game form as a mechanism. Formally, we compute perfect Bayesian equilibria under the assumption that the set of feasible mechanisms equals the set of finite game forms together with the set of direct mechanisms that are strong solutions.[21]

Let M denote any strong solution. The idea for constructing a perfect Bayesian equilibrium with outcome M is as follows. All types of the principal propose M as a direct mechanism; agents' retain their prior beliefs,

[19] The seemingly weaker alternative requirement that "a single type of the principal is strictly better off in ρ'" is in fact not weaker. Using the equi-continuity assumption and (6), it can be shown that the function $t_0 \mapsto U_0^{\rho}(t_0, t_0)$ is continuous. Hence, if some type t^* is strictly better off in ρ', then all types in some open neighborhood N of t^* are strictly better off in ρ'. Because the support of F_0 equals T_0, the F_0-probability of the set $T_0 \setminus N$ is less than 1.

[20] For any two strong solutions ρ_1 and ρ_2, one can construct a third strong solution ρ_3 by choosing for each type of the principal the better of the two allocation rules (because ρ_1 and ρ_2 are safe, ρ_3 is safe as well). If there was a type t^* that is better off in ρ_3 compared to ρ_1 or ρ_2, then ρ_3 would dominate ρ_1 or ρ_2, a contradiction.

[21] Allowing a larger set of feasible mechanisms may be desirable, but such an extension is beyond us: There are many general Bayesian Nash equilibrium existence results for non-finite incomplete-information games (see, e.g., Reny (2008)), but to the best of our knowledge virtually none about existence of perfect Bayesian equilibria.

accept M, and everybody reveals their true type. It remains to define the agents' beliefs about the principal's type, and everybody's actions, when the principal deviates by proposing any mechanism $M^d \neq M$.[22]

Consider an auxiliary game $G(M^d)$ where the principal chooses between either directly obtaining her strong-solution payoff and the game ends, or offering the mechanism M^d which may be accepted or rejected and is played if unanimously accepted. Because M^d is a finite game form, it can be shown that a perfect Bayesian equilibrium exists in the game $G(M^d)$.[23] We can construct actions and beliefs such that a deviation to M^d is not profitable. Simply define the beliefs and subsequent actions in the informed-principal game when M^d is proposed to be identical to the beliefs and subsequent actions when M^d is proposed in the equilibrium of $G(M^d)$.

To show that the described strategies and beliefs form an equilibrium of the informed-principal game, let $T(M^d)$ denote the set of types of the principal that by proposing M^d obtain a higher payoff than their strong-solution payoff. We have to show that $T(M^d) = \emptyset$.

Extend the perfect Bayesian equilibrium in $G(M^d)$ to a strategy profile in a *restricted* informed-principal game where the only feasible mechanisms are M and M^d, as follows. Every type of the principal proposes M in the restricted informed-principal game if and only if she chooses the strong-solution payoff in the equilibrium of the game $G(M^d)$, all types of all agents accept M if it is offered, and everybody reveals their true types in M. The so-constructed strategy profile is a perfect Bayesian equilibrium in the restricted informed-principal game because M is safe. The allocation rule implemented by this perfect Bayesian equilibrium would dominate M if $T(M^d)$ were non-empty. Because M is a strong solution, $T(M^d) = \emptyset$.

Next we introduce two definitions that are needed to state condition (*).

[22]In general, equilibrium requires that the agents switch away from prior beliefs when M^d is proposed. Yilankaya (1999) provides an insightful example involving the bilateral trade environment of Myerson and Satterthwaite (1983), with the seller being the principal. The strong solution is constructed from optimal take-it-or-leave-it offers by all types of the seller. If prior beliefs about the seller are retained, some seller types may have an incentive to deviate by proposing a double auction mechanism.

[23]In environments with finite type spaces, $G(M^d)$ is a finite game, so that equilibrium existence is well known. This fact is utilized in Myerson's (1983, Theorem 2) proof.

Full-information optimality

Consider the hypothetical environment where the principal's type is commonly known. If each type of the principal uses a payoff-maximizing mechanism, we obtain a *full-information optimal* allocation rule,[24] that is, an allocation rule that solves problem $P(t_0)$ for all $t_0 \in T_0$,

$$\begin{aligned} P(t_0): \quad & \max_{\rho} U_0^{\rho}(t_0, t_0) \\ \text{s.t.} \quad & (3), \quad (4). \end{aligned}$$

Ex-ante optimality

An allocation rule is ex-ante optimal if it maximizes the principal's expected payoff in the hypothetical environment where the players do not yet know her own type. Formally, ρ is called *ex-ante optimal* if it solves problem

$$\begin{aligned} E: \quad & \max_{\rho} \int_{T_0} U_0^{\rho}(t_0, t_0)\ F_0(\mathrm{d}t_0) \\ \text{s.t.} \quad & (5), \quad (6), \quad (7). \end{aligned}$$

We now state condition

> (*): The environment has private values, and there exists a full-information-optimal allocation rule that is ex-ante optimal.

To understand the significance of condition (*) for the informed-principal problem, observe that, firstly, an ex-ante optimal rule cannot be dominated, and, secondly, in private-value environments, any full-information optimal allocation rule is incentive feasible and, in fact, safe. Hence:[25]

Lemma 1. *If (*) is satisfied, then there exists a full-information-optimal allocation rule that is a strong solution.*

In Section 3 we provide applications of this lemma. In Section 4, we provide an example of a private-value environment where (*) is violated; in the example, the ex-ante optimal allocation rule is a perfect Bayesian equilibrium outcome that dominates any full-information-optimal allocation rule, and no strong solution exists.

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[24] The terminology follows Maskin and Tirole ((1990), Section 2.C)). Clippel and Minelli (2004) use the term "best safe."

[25] Lemma 1 extends straightforwardly to environments with arbitrary non-quasi-linear payoff functions, but in this paper we consider only quasi-linear applications.

3 Applications

In this section, we present two applications of Lemma 1. First, we consider an extension of Myerson's (1981) auction environments in which the auctioneer (seller) has private information. A single unit of a good is to be allocated among the players, $A = \{0, \ldots, n\}$. Initially, the good is owned by the principal, $a_0 = 0$. Hence, the principal is the "seller" and the agents are "buyers." Any distribution over A can be described by a vector listing the probability that each buyer gets the good; i.e.,

$$\mathcal{A} \;=\; \{(q_1, \ldots, q_n) \mid q_i \geq 0\ \forall i,\ \sum_{j \in N} q_j \leq 1\}.$$

Each buyer $i \in N$ has an interval type space $T_i = [\underline{t}_i, \overline{t}_i]$ (the seller's type space is arbitrary). The distributions F_i are continuously differentiable with strictly positive density f_i on T_i. Define $\mathbf{f}$ and $\mathbf{f}_{-i}$ analogously to $\mathbf{F}$ and $\mathbf{F}_{-i}$. Defining payoff functions as in (1) and (2), the value function of any player $i = 0, \ldots, n$ is given by

$$v_i(a, t_0, \mathbf{t}) \;=\; \begin{cases} t_i + \sum_{j \in N \setminus \{i\}} e_j(t_j) & \text{if } a = i, \\ 0 & \text{otherwise,} \end{cases}$$

where $e_1, \ldots, e_n$ are called "revision effect functions" (cf. 1981, p. 60).

Observe that this definition yields private-value environments, so that we can use the shorter notation $v_i(a, \mathbf{t})$ for all agents $i \in N$. Still, the buyers' valuations of the good can be interdependent, and the seller's valuation can depend on the buyers' types.

Myerson (1981, p. 68) defines functions $\overline{c}_i : T_i \to \mathbb{R}$ $(i \in N)$. Analogously to Myerson (1981, p. 68), but making the dependence on the principal's type explicit, we define a set

$$M(t_0, \mathbf{t}) \;=\; \{i \in N \mid t_0 \leq \overline{c}_i(t_i),\ i \in \arg\max_{j \in N} \overline{c}_j(t_j)\}.$$

From Myerson (1981, p. 69), a full-information optimal allocation rule $(\overline{p}, \overline{x}) = ((\overline{p}_1, \ldots, \overline{p}_n), (\overline{x}_1, \ldots, \overline{x}_n))$ is given by

$$\overline{p}_i(t_0, \mathbf{t}) \;=\; \begin{cases} 1/|M(t_0, \mathbf{t})| & \text{if } i \in M(t_0, \mathbf{t}), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\overline{x}_i(t_0, \mathbf{t}) \;=\; \overline{p}_i(t_0, \mathbf{t}) v_i(\mathbf{t}) - \int_{\underline{t}_i}^{t_i} \overline{p}_i(t_0, \mathbf{t}_{-i}, s_i) \mathrm{d}s_i. \tag{8}$$

Proposition 1. *In Myerson's auction environments, the full-information optimal allocation rule* $(\overline{p}, \overline{x})$ *is ex-ante optimal and is a strong solution.*

The proof relies on Lemma 2, which is an ex-ante version of Myerson's Lemma 3 (1981, p. 64). For any $i \in N$, $t_i \in T_i$, and $p = (p_1, \ldots, p_n)$, define

$$Q_i^p(t_i) \;=\; \int_{T_0}\int_{\mathbf{T}_{-i}} p_i(t_0, \mathbf{t})\mathbf{f}_{-i}(\mathbf{t}_{-i})\mathrm{d}\mathbf{t}_{-i}\;\mathrm{d}F_0(t_0).$$

Define functions $c_i : T_i \to \mathbb{R}$ $(i \in N)$ as in Myerson (1981, p. 66).

Lemma 2. *Suppose that* p *solves*

$$\max_{p:T_0\times\mathbf{T}\to\mathcal{A}} \int_{T_0}\int_{\mathbf{T}}\sum_{i\in N}(c_i(t_i) - t_0)p_i(t_0, \mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t}\,dF_0(t_0)$$

$$\textit{s.t.} \qquad t_i \mapsto Q_i^p(t_i) \qquad \textit{is weakly increasing on } T_i. \tag{9}$$

Suppose also that (p, x) *satisfies (8) with* $(\overline{p}, \overline{x})$ *replaced by* (p, x)*, and that* $\rho = (p, x)$ *satisfies (5). Then* (p, x) *is ex-ante optimal.*

Proof. Using an ex-ante version of Lemma 2 in Myerson (1981), one argues analogously to the proof of Myerson (1981, Lemma 3). The seller's objective in problem E can be rewritten as

$$\int_{T_0}\int_{\mathbf{T}}\sum_{i\in N}(c_i(t_i) - t_0)p_i(t_0, \mathbf{t})\mathbf{f}(\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}F_0(t_0)$$
$$+\int_{T_0}\int_{\mathbf{T}}\sum_{i\in N} v_0(t_0, \mathbf{t})\mathbf{f}(\mathbf{t})\mathrm{d}\mathbf{t}\mathrm{d}F_0(t_0) \;-\; \sum_{i\in N} U_i^{(p,x)}(\underline{t}_i, \underline{t}_i).$$

The constraints (6) and (7) can be rewritten as (9) and

$$\int_{T_0}\int_{\mathbf{T}}\left(v_i(\mathbf{t})p_i(t_0, \mathbf{t}) - \int_{\underline{t}_i}^{t_i} p_i(t_0, s_i, \mathbf{t}_{-i})\mathrm{d}s_i - x_i(t_0, \mathbf{t})\right)\mathbf{f}_{-i}(\mathbf{t}_{-i})\mathrm{d}\mathbf{t}\mathrm{d}F_0(t_0)$$
$$= U_i^{(p,x)}(\underline{t}_i, \underline{t}_i) \geq 0. \tag{10}$$

Given any p, if we choose x such that (p, x) satisfies (8) with $(\overline{p}, \overline{x})$ replaced by (p, x), then $U_i^{(p,x)}(\underline{t}_i, \underline{t}_i) = 0$, which is the best the principal can achieve, and (10) is also satisfied. Hence, choosing p as described in the statement of the lemma corresponds to a relaxed version of problem E, without constraint

(5). If the solution to the relaxed problem happens to satisfy (5), then (p,x) solves E. *QED*

Proof of Proposition 1. Because $(\overline{p},\overline{x})$ is full-information optimal and the environment has private values, constraint (5) is satisfied. It remains to show that $\overline{p}$ solves the program in Lemma 2.

Define functions H_i, G_i $(i \in N)$ as in Myerson (1981, p. 68). Analogous to an argument by Myerson (1981, (6.10)), the objective in Lemma 2 can be rewritten as

$$\int_{T_0}\int_{\mathbf{T}}\sum_{i\in N}(\overline{c}_i(t_i)-t_0)p_i(t_0,\mathbf{t})\mathbf{f}(\mathbf{t})\mathrm{d}t\mathrm{d}F_0(t_0) \\ -\sum_{i\in N}\underbrace{\int_{T_i}(H_i(F_i(t_i))-G_i(F_i(t_i)))\mathrm{d}Q_i^p(t_i)}_{=:\Delta^p(t_i)}. \quad (11)$$

Analogously to Myerson (1981, p. 69-70), one argues that the constraint (9) implies $\Delta^p(t_i)\geq 0$. Moreover, $\Delta^{\overline{p}}(t_i)=0$ and $p=\overline{p}$ is such that $\sum_{i\in N}(\overline{c}_i(t_i)-t_0)p_i(t_0,\mathbf{t})$ is maximal for each type profile $(t_0,\mathbf{t})$. Hence, (11) is maximized at $p=\overline{p}$, subject to the constraint (9). *QED*

Proposition 1 does not claim that the perfect Bayesian equilibrium outcome is unique. But the principal's payoff is uniquely determined if there is only one agent.

Remark 1. *Consider Myerson's auction environments with a single agent, $|N|=1$. Then in any equilibrium of an informed-principal game where any fixed-price offer is a feasible mechanism, each type of the principal obtains the same expected payoff as in $(\overline{p},\overline{x})$.*

Proof. As shown by Myerson (1981, p. 70), each type of the principal can obtain her full-information-optimum payoff by making an optimal fixed-price offer; hereby the agent's belief about the principal's type is irrelevant. Because the principal is free to deviate to any fixed-price offer, this yields a lower bound for her payoff in any equilibrium. There cannot be an equilibrium where some type of the principal obtains more, because the allocation rule induced by this equilibrium would dominate $(\overline{p},\overline{x})$, contradicting the fact that $(\overline{p},\overline{x})$ is a strong solution. *QED*

In the proof of Remark 1 we use the fact that the continuation game following the proposal of a fixed-price mechanism has a unique equilibrium independently of the agent's belief about the principal's type. With multiple agents, such uniqueness cannot be obtained, so that we cannot prove a result parallel to Remark 1.

As a second application, we consider Guesnerie and Laffont's (1984, case B) quasi-linear principal-agent environments (with the planner's shadow cost parameters being equal to 1).[26] We extend their model by allowing for a privately informed principal. In addition, we allow for multiple agents. For example, the principal may be a multi-product price-discriminating monopolist who is privately informed about the cost of production, while the agents are consumers who are privately informed about their preferences over the products.

We have a, possibly multi-dimensional, set of collective actions, $A \subseteq \mathbb{R}^L$ ($L \geq 1$) (for instance, a set of multi-product quantity vectors). We assume that A is a rectangle with non-empty interior. Agents' type spaces and beliefs are as in Myerson's (1981) model; in addition, we assume that F_i ($i \in N$) is twice differentiable and the hazard rate

$$\frac{f_i}{1 - F_i} \quad \text{is weakly increasing.} \tag{12}$$

We assume private values; accordingly, defining payoff functions as in (1) and (2), we drop the argument t_0 from the agents' value functions $v_1, \ldots, v_n$. We assume the players' value functions are once continuously differentiable in the action, twice continuously differentiable in the type vector, and *supermodular* (in the negatives of the actions): for all $t_0 \in T_0$, $\mathbf{t} \in \mathbf{T}$, $(a_1, \ldots, a_L) \in A$, $k = 1, \ldots, L$, $i \in N$, and $j \in N \cup \{0\}$,

$$\frac{\partial^2 v_j}{\partial a_k \partial t_i} \leq 0. \tag{13}$$

Our application of condition (*) in Proposition 2 relies on a *third-derivative condition* (cf. Fudenberg and Tirole, 1991, p. 263, l.h.s. in A8) that, in

[26] Our exposition is based on Fudenberg and Tirole (1991, Ch. 7) . In contrast to Fudenberg and Tirole, we apply monotone comparative statics (Milgrom and Shannon, 1994), which makes some of Fudenberg and Tirole's assumptions (1991, p. 263, A6, A9, r.h.s. in A8) obsolete.

particular, requires agents' marginal values to be concave in own types: for all $\mathbf{t} \in \mathbf{T}$, $(a_1, \ldots, a_L) \in A$, $k = 1, \ldots, L$, and $i, j \in N$,

$$\frac{\partial^3 v_i}{\partial a_k \partial t_i \partial t_j} \ \geq \ 0. \tag{14}$$

Without (14), condition (*) can fail (Proposition 3).

Finally, in order to be able to apply condition (*) in environments with multi-dimensional actions ($L > 1$), we need *action cross-derivative conditions*: for all $t_0 \in T_0$, $\mathbf{t} \in \mathbf{T}$, $(a_1, \ldots, a_L) \in A$, $k \neq l$, and $i \in N \cup \{0\}$,

$$\frac{\partial^2 v_i}{\partial a_k \partial a_l} \ \geq \ 0, \tag{15}$$

and, if $i \in N$,

$$\frac{\partial^3 v_i}{\partial a_k \partial a_l \partial t_i} \ \leq \ 0. \tag{16}$$

(Note that conditions (15) and (16) are empty if $L = 1$.)

To square the current model with our extension of Myerson, suppose for simplicity that there is only one agent ($n = 1$). Then the Myerson value functions can be equivalently written as $v_0(a, t_0, t_1) = a(t_0 + e_1(t_1))$ and $v_1(a, t_1) = (1 - a)t_1$, where the collective action $a \in A = [0, 1]$ is the probability that the seller keeps the good. Because (12) and (13) may be violated in our extension of Myerson, Proposition 1 is not a special case of Proposition 2 below.

Of importance for the analysis is the derivative of the value function of any agent $i \in N$ with respect to her own type,

$$Dv_i(a, \mathbf{t}) := \frac{\partial v_i}{\partial t_i}(a, \mathbf{t}) \quad (a \in A, \ \mathbf{t} \in \mathbf{T}).$$

It is useful to write any allocation rule ρ as a pair consisting of an *action allocation rule* $\mu : T_0 \times \mathbf{T} \to \mathcal{A}$ and a *transfer allocation rule* $\tau = (\tau_1, \ldots, \tau_n) : T_0 \times \mathbf{T} \to [-\hat{x}, \hat{x}]^n$; that is, $\rho = (\mu, \tau)$.

For all $a \in A$, $t_0 \in T_0$, and $\mathbf{t} \in \mathbf{T}$, define the *virtual surplus function*

$$V(a, t_0, \mathbf{t}) \ = \ v_0(a, t_0, \mathbf{t}) + \sum_{i=1}^{n} \left(v_i(a, \mathbf{t}) - \frac{1 - F_i(t_i)}{f_i(t_i)} Dv_i(a, \mathbf{t}) \right).$$

Define an action allocation rule μ^* via

$$\mu^*(t_0, \mathbf{t}) \in \arg\max_{a \in A} V(a, t_0, \mathbf{t}), \tag{17}$$

and a transfer allocation rule $\tau^* = (\tau_1^*, \ldots, \tau_n^*)$ via

$$\tau_i^*(t_0, \mathbf{t}) = v_i(\mu^*(t_0, \mathbf{t}), \mathbf{t}) - \int_{\underline{t}_i}^{t_i} Dv_i(\mu^*(t_0, s, \mathbf{t}_{-i}), (s, \mathbf{t}_{-i}))\mathrm{d}s \quad (i \in I). \tag{18}$$

We have the following result.

Proposition 2. *Suppose that the conditions (12)–(16) are satisfied. Then there exists* (μ^*, τ^*) *satisfying (17) and (18) that is full-information optimal and ex-ante optimal. Hence,* (μ^*, τ^*) *is a strong solution.*

For the proof, additional notation is needed. Let $\tilde{t}_0, \ldots, \tilde{t}_n$ denote stochastically independent random variables with c.d.f.s. $F_0, \ldots, F_n$. Let $\tilde{\mathbf{t}} = (\tilde{t}_i)_{i \in N}$ and $\tilde{\mathbf{t}}_{-i} = (\tilde{t}_j)_{j \in N \setminus \{i\}}$. For all $i \in N$, $t_i, t_i' \in T_i$, action allocation rules μ, and $t_0 \in T_0$, let

$$\overline{v}_i^\mu(t_i', t_i, t_0) = E[v_i(\mu(t_0, t_i', \tilde{\mathbf{t}}_{-i}), (t_i, \tilde{\mathbf{t}}_{-i}))]$$

and

$$D\overline{v}_i^\mu(t_i', t_i, t_0) = E[Dv_i(\mu(t_0, t_i', \tilde{\mathbf{t}}_{-i}), (t_i, \tilde{\mathbf{t}}_{-i}))].$$

Because Dv_i is bounded, Lebesgue's monotone convergence theorem implies

$$D\overline{v}_i^\mu(t_i', t_i, t_0) = \frac{\partial}{\partial t_i}\overline{v}_i^\mu(t_i', t_i, t_0). \tag{19}$$

Given any action allocation rule μ, we can ask whether μ satisfies the t_0-*monotonicity constraints*

$$\int_{t_i'}^{t_i} (D\overline{v}_i^\mu(s, s, t_0) - D\overline{v}_i^\mu(t_i', s, t_0))\,\mathrm{d}s \geq 0 \quad (t_i' \leq t_i), \tag{20}$$

$$\int_{t_i}^{t_i'} (D\overline{v}_i^\mu(s, s, t_0) - D\overline{v}_i^\mu(t_i', s, t_0))\,\mathrm{d}s \leq 0 \quad (t_i' \geq t_i). \tag{21}$$

Defining $D\overline{v}_i^\mu(t_i', s) = E[D\overline{v}_i^\mu(t_i', s, \tilde{t}_0)]$, we can also ask whether the following *average monotonicity constraints* are satisfied:

$$\int_{t_i'}^{t_i} \left(D\overline{v}_i^\mu(s,s) - D\overline{v}_i^\mu(t_i', s)\right) \mathrm{d}s \quad \geq \quad 0 \qquad (t_i' \leq t_i), \tag{22}$$

$$\int_{t_i}^{t_i'} \left(D\overline{v}_i^\mu(s,s) - D\overline{v}_i^\mu(t_i', s)\right) \mathrm{d}s \quad \leq \quad 0 \qquad (t_i' \geq t_i). \tag{23}$$

Lemma 3 below gives a sufficient condition for ex-ante optimality of an allocation rule. The condition requires that the action allocation rule maximizes the expected virtual surplus under the average monotonicity constraints. Choosing a transfer allocation rule such that the agents' incentive compatibility constraints are satisfied, we require that the principal's incentive constraints (5) are satisfied as well.

Lemma 3. *Suppose that μ solves*

$$\begin{array}{ll} U' & \max\limits_{\mu} \ E\left[V(\mu(\tilde{t}_0, \tilde{\mathbf{t}}), \tilde{t}_0, \tilde{\mathbf{t}})\right] \\ s.t. & (22),\ (23), \end{array}$$

formula (18) is satisfied with (μ^*, τ^*) replaced by (μ, τ), and $\rho = (\mu, \tau)$ satisfies (5).

Then (μ, τ) is ex-ante optimal. Moreover, the solution value of U' equals the solution value of problem E.

Proof. Step 1. Under the constraints of the ex-ante optimality problem E, its objective $U_0^\rho = E[U_0^\rho(\tilde{t}_0, \tilde{t}_0)]$ can be written as

$$U_0^\rho \quad = \quad E\left[V(\mu(\tilde{t}_0, \tilde{\mathbf{t}}), \tilde{t}_0, \tilde{\mathbf{t}})\right] - \sum_{i=1}^n U_i^\rho(\underline{t}_i, \underline{t}_i).$$

To see this, let $\rho = (\mu, \tau)$ and write

$$U_0^\rho \quad = \quad E\left[v_0(\mu(\tilde{t}_0, \tilde{\mathbf{t}}), \tilde{t}_0, \tilde{\mathbf{t}}) + \sum_{i=1}^n \left(v_i(\mu(\tilde{t}_0, \tilde{\mathbf{t}}), \tilde{t}) - U_i^\rho(\tilde{t}_i, \tilde{t}_i)\right)\right]. \tag{24}$$

Because of (6) and (19), the envelope theorem in integral form implies

$$U_i^\rho(t_i, t_i) \quad = \quad U_i^\rho(\underline{t}_i, \underline{t}_i) + \int_{\underline{t}_i}^{t_i} D\overline{v}_i^\mu(s,s)\mathrm{d}s. \ \ (t_i \in T_i) \tag{25}$$

Using integration by parts, (25) implies

$$\begin{aligned} E[U_i^\rho(\tilde{t}_i,\tilde{t}_i)] &= U_i^\rho(\underline{t}_i,\underline{t}_i) + E[\frac{1-F_i(\tilde{t}_i)}{f_i(\tilde{t}_i)} D\overline{v}_i^\mu(\tilde{t}_i,\tilde{t}_i)] \\ &= U_i^\rho(\underline{t}_i,\underline{t}_i) + E[\frac{1-F_i(\tilde{t}_i)}{f_i(\tilde{t}_i)} Dv_i(\mu(\tilde{t}_0,\tilde{\mathbf{t}}),\tilde{t})]. \end{aligned} \tag{26}$$

From (24) and (26),

$$\begin{aligned} U_0^\rho &= E\Bigg[v_0(\mu(\tilde{t}_0,\tilde{\mathbf{t}}),\tilde{t}_0,\tilde{\mathbf{t}}) \\ &\qquad + \sum_{i=1}^n \left(v_i(\mu(\tilde{t}_0,\tilde{\mathbf{t}}),\tilde{t}) - \frac{1-F_i(\tilde{t}_i)}{f_i(\tilde{t}_i)} Dv_i(\mu(\tilde{t}_0,\tilde{\mathbf{t}}),\tilde{t})\right)\Bigg] \\ &\quad - \sum_{i=1}^n U_i^\rho(\underline{t}_i,\underline{t}_i). \end{aligned}$$

Step 2. The constraint (6) implies (22) and (23).

By (6), for all $i \in N$ and $t_i, t_i' \in T_i$,

$$U_i^\rho(t_i,t_i) - U_i^\rho(t_i',t_i') \ + \ U_i^\rho(t_i',t_i') - U_i^\rho(t_i',t_i) \geq 0.$$

Hence, using (25), if $t_i' \leq t_i$,

$$\int_{t_i'}^{t_i} D\overline{v}_i^\mu(s,s)\mathrm{d}s \ + \ \overline{v}_i(\mu(\tilde{t}_0,t_i',\tilde{t}_{-i}),t_i') - \overline{v}_i(\mu(\tilde{t}_0,t_i',\tilde{t}_{-i}),t_i) \geq 0.$$

Hence, (22) is satisfied. The proof that the monotonicity constraint (23) is satisfied is analogous.

Step 3. If (18) is satisfied with (μ^*,τ^*) replaced by (μ,τ), then $U_i^\rho(\underline{t}_i,\underline{t}_i) = 0$ for all $i \in N$, and (6) and (7) are satisfied.

Verifying this is straightforward.

By *Step 1* and *Step 2*, if we choose (μ,τ) such that μ solves U' and such that $U_i^\rho(\underline{t}_i,\underline{t}_i) = 0$ for all $i \in N$, then we obtain an upper bound for the solution value of the ex-ante optimality problem. By *Step 3*, this upper bound is obtained. *QED*

Because Lemma 3 holds in particular if F_0 puts probability 1 on one point t_0, we have an analogous result concerning full-information optimal allocation rules.

Lemma 4. *Suppose that, for all $t_0 \in T_0$, μ solves*

$$\begin{array}{ll} P(t_0)' & \max_{\mu} \; E\left[V(\mu(t_0,\tilde{\mathbf{t}}),t_0,\tilde{\mathbf{t}})\right] \\ s.t. & (20),\ (21), \end{array}$$

and (18) is satisfied with (μ^*,τ^*) replaced by (μ,τ).

Then (μ,τ) is full-information optimal. Moreover, the solution value of problem $P(t_0)'$ equals the solution value of problem $P(t_0)$.

Proof of Proposition 2. We show that $(\mu,\tau)=(\mu^*,\tau^*)$ satisfies the conditions in Lemma 3 and in Lemma 4. Hence, condition (*) is satisfied and Lemma 1 applies.

By construction (17), $\mu=\mu^*$ maximizes the objective in U' and the objective in $P(t_0)'$ for all $t_0 \in T_0$. It remains to show that μ^* satisfies (20) and (21) for all $t_0 \in T_0$ (then constraints (22) and (23) are satisfied as well, and (5) is satisfied because (μ^*,τ^*) is full-information optimal).

A sufficient condition for (20) and (21) is that $D\overline{v}_i^{\mu^*}$ is weakly increasing in its first argument. Because, from (13), $Dv_i(a,\mathbf{t})$ is weakly decreasing in every component of a, it is sufficient to show that every component of $\mu^*(t_0,t_i,\mathbf{t}_{-i})$ is weakly decreasing in t_i. From Milgrom and Shannon (1994, Theorem 5, Theorem 6) , a sufficient condition for this is that, for all k and $l \neq k$,

$$\frac{\partial^2 V}{\partial t_i \partial a_k} \leq 0, \qquad \frac{\partial^2 V}{\partial a_k \partial a_l} \geq 0.$$

The left inequality follows from a straightforward computation using (12), (13), and (14). To see the right inequality, use (15) and (16). *QED*

4 Examples

In this section, we present two examples of quasi-linear environments in which condition (*) is violated. Consider the principal-agent environments of Guesnerie and Laffont (Guesnerie and Laffont 1984) as defined above. The third-derivative condition (14) of Guesnerie and Laffont, which is used for the main result, Proposition 2, appears strong. We show by example that it cannot be dropped. In the example, (14) is violated and no full-information optimal allocation rule is ex-ante optimal.[27]

[27] The example also satisfies the assumptions of Maskin and Tirole (1990, Proposition 11), except that there are more than two types of the agent. Hence, the example qualifies

Without condition (14), ex-ante optimal allocation rules and full-information optimal allocation rules can still be computed using Lemma 3 and Lemma 4. But the point-wise maximizer (17) may violate one of the monotonicity constraints (20)–(23), so that "bunching" becomes optimal.

In the example, there is a unique point-wise maximizer (17) of the virtual surplus function. This maximizer satisfies the average monotonicity constraints (22) and (23), but violates (20) for some type of the principal. Hence, the solution value of problem U' is strictly larger than the ex-ante expectation of the solution value of problem $P(t_0)'$. Moreover, it can be checked that (μ^*, τ^*) satisfies (5). Hence, by Lemma 3 and Lemma 4, in the ex-ante optimum some type of the principal is strictly better off than in the full-information optimum.

The example is as follows. Suppose that the support of F_0 is $T_0 = \{9, 49\}$. We denote the probability of the point 9 by $\pi = F_0(9)$. There is a single agent, $n = 1$, and F_1 is the uniform distribution on $T_1 = [0, 1]$. Observe that F_1 satisfies (12). The action space is $A = [0, 3]$, with disagreement action $a_0 = 0$. The example features private values:

$$\begin{aligned} v_0(a, t_0) &= -t_0 a^2 + 400a, \qquad (a \in A,\ t_0 \in T_0), \\ v_1(a, t_1) &= -350a - a^2 - a t_1 + \gamma(a) t_1^2, \qquad (a \in A,\ t_1 \in T_1), \end{aligned}$$

where we use the auxiliary function

$$\gamma(a) = \begin{cases} 0 & \text{if } a \in [0, 1], \\ 2a - a^2 - 1 & \text{if } a \in [1, 2], \\ 3 - 2a & \text{if } a \in [2, 3]. \end{cases}$$

Observe that γ is continuously differentiable, weakly decreasing, and weakly concave.

It is straightforward to check that (13) is satisfied. As additional regularity properties, the principal's value function is strictly increasing in the action, and the agent's value function is strictly decreasing in the action. Moreover, each player's value function is strictly concave in the action.

Proposition 3. *Consider the environment described above. Suppose that*

$$\frac{5}{2}\pi < \frac{1}{50}(1 - \pi). \tag{27}$$

a claim by Maskin and Tirole (1990, p. 384) that the "restriction of the agent's parameter to two values is not essential."

Then the allocation rule (μ^*, τ^*) defined in (17)–(18) is ex-ante optimal, and type $t_0 = 9$ obtains a higher payoff than in any full-information optimal allocation rule.

Proof. Observe that

$$Dv_1(a, t_1) = -a + \gamma(a)(2t_1) > 0. \tag{28}$$

The virtual surplus function is given by

$$\begin{aligned} V(a, t_0, t_1) &= -(t_0 + 1)a^2 + 50a - at_1 + \gamma(a)t_1^2 \\ &\quad -(1 - t_1)(-a + \gamma(a)(2t_1)). \end{aligned}$$

Using the formula

$$\frac{\partial V}{\partial a} = -(t_0 + 1)(2a) + 50 + 1 - 2t_1 + \gamma'(a) \cdot (3t_1^2 - 2t_1),$$

one can verify that $\partial V/\partial a$ is strictly decreasing in a. Hence, V is strictly concave in a. Using the first-order condition $\partial V/\partial a = 0$ to maximize V, we find

$$\mu^*(t_0, t_1) = \frac{50 + 1 - 2t_1 + \gamma'(\mu^*(t_0, t_1))(3t_1^2 - 2t_1)}{2(t_0 + 1)}. \tag{29}$$

Observe that this is an implicit equation because γ' is evaluated at the point $\mu^*(t_0, t_1)$. However, using the fact that $\gamma'(a) \in [0, -2]$ for all $a \in A$, it is straightforward to check that (29) implies

$$\mu^*(49, t_1) \in (0, 1), \tag{30}$$
$$\mu^*(9, t_1) \in (2, 3). \tag{31}$$

Hence,

$$\gamma'(\mu^*(49, t_1)) = 0, \tag{32}$$
$$\gamma'(\mu^*(9, t_1)) = -2. \tag{33}$$

Using (32) and (33) in (29), we find

$$\mu^*(49, t_1) = \frac{51 - 2t_1}{100}, \tag{34}$$
$$\mu^*(9, t_1) = \frac{51 + 2t_1 - 6t_1^2}{20}. \tag{35}$$

Using (28) and (31), for all $s, t_1' \in [0,1]$,

$$Dv_1(\mu^*(9, t_1'), s) \;=\; 24s - \mu^*(9, t_1')(1 + 4s). \tag{36}$$

Hence,

$$\begin{aligned} Dv_1(\mu^*(9,s),s) - Dv_1(\mu^*(9,t_1'),s) &= -(\mu^*(9,s) - \mu^*(9,t_1'))(1+4s) \\ &\overset{(35)}{=} -\frac{1+4s}{10}(s - t_1')(1 - 3(s + t_1')). \end{aligned} \tag{37}$$

For later use, observe that

$$|\; Dv_1(\mu^*(9,s),s) - Dv_1(\mu^*(9,t_1'),s) \;| \;\leq\; \frac{5}{2} \;|\; s - t_1' \;| \;. \tag{38}$$

If $1/6 \geq s > t_1' \geq 0$, then (37) implies

$$Dv_1(\mu^*(9,s),s) - Dv_1(\mu^*(9,t_1'),s) \;<\; 0,$$

implying that the t_0-monotonicity constraint (20) is violated at $t_0 = 9$, $t_1 = 1/6$, and $t_1' < 1/6$.

This shows that, at $t_0 = 9$, the solution value of problem $P(t_0)'$ must be strictly smaller than the value obtained from $\mu = \mu^*$. Hence, in the full-information optimal allocation rule, type $t_0 = 9$ is strictly worse off than with μ^*.

Analogously to (37), we find, for all $s, t_1' \in [0,1]$,

$$Dv_1(\mu^*(49,s),s) - Dv_1(\mu^*(49,t_1'),s) \;=\; \frac{s - t_1'}{50}. \tag{39}$$

Now we turn to the average monotonicity constraints (22) and (23). For all $s, t_1' \in [0,1]$, we have

$$D\overline{v}_1^{\mu^*}(t_1', s) = \pi Dv_1(\mu^*(9, t_1'), s) + (1-\pi) Dv_1(\mu^*(49, t_1'), s).$$

Therefore, if $s > t_1'$,

$$\begin{aligned} & D\overline{v}_1^{\mu^*}(s,s) - D\overline{v}_1^{\mu^*}(t_1', s) \\ = \;& \pi(Dv_1(\mu^*(9,s),s) - Dv_1(\mu^*(9,t_1'),s)) \\ & +(1-\pi)(Dv_1(\mu^*(49,s),s) - Dv_1(\mu^*(49,t_1'),s)) \\ \overset{(38),(39)}{\geq} \;& -\frac{5}{2}\pi(s - t_1') + \frac{1}{50}(1-\pi)(s - t_1'), \end{aligned}$$

which is greater than 0 because π satisfies (27). Hence, the average monotonicity condition (22) is satisfied. The proof that (23) is satisfied is analogous.

It follows that μ^* solves problem U'. Moreover, it can be verified that $\rho = (\mu^*, \tau^*)$ satisfies (5): $U_0^\rho(49, 49) = 7501/600 < -37523/200 = U_0^\rho(9, 49)$ and $U_0^\rho(9, 9) = 37523/600 > 22503/1000 = U_0^\rho(49, 9)$. Hence, (μ^*, τ^*) is ex-ante optimal by Lemma 3. *QED*

The allocation rule (μ^*, τ^*) is not a strong solution (because the agent's incentive constraints are violated if she believes to face type $t_0 = 9$ with a sufficiently high probability). Nevertheless, (μ^*, τ^*) is a perfect Bayesian equilibrium outcome of an informed principal game: extending Maskin and Tirole's (1990) concept of a Strong Unconstrained Pareto Optimum (SUPO) to the environment of the current example, it can be shown that any SUPO is a perfect Bayesian equilibrium outcome of an informed-principal game (with an appropriately restricted set of feasible mechanisms). By observing that (μ^*, τ^*) is an SUPO, we obtain the following result.

Remark 2. *Suppose that (27) is satisfied. Then* (μ^*, τ^*) *is a perfect Bayesian equilibrium outcome of an informed-principal game.*

Sketch of Proof. We want to show that (μ^*, τ^*) is an SUPO. Suppose not. Then there exists a belief F_0' about the principal's type, and an allocation rule ρ that satisfies the agent's constraints (6) and (7) with F_0 replaced by F_0', such that ρ leaves all types of the principal at least as well off as (μ^*, τ^*), and some types are strictly better off in ρ.

Hence, if ρ is used, then the principal's F_0'-ex-ante expected payoff is larger than if (μ^*, τ^*) is used. But μ^* is a point-wise maximizer of the virtual surplus function, and, by Lemma 3, yields an upper bound for the principal's F_0'-ex-ante expected payoff, a contradiction to the definition of SUPO. *QED*

The general principle at work in the example above is that the ex-ante optimal allocation rule satisfies certain constraints (here, the monotonicity constraints) on average over the principal's types, but not for each type separately. Hence, in the full-information-optimal allocation rule some type of the principal is necessarily worse off.

The same principle is sometimes at work when the agent is budget-constrained. What follows is an example where the principal extracts the entire surplus in the ex-ante optimal allocation rule, but not in the full-information-optimal allocation rule.[28] There is a seller (principal) and a buyer (agent), who may trade up to three units of some good. The marginal valuation is 1 for the buyer and 0 for the seller. The buyer has $\hat{x} = 2$ units of money. The seller has one of two equally likely types: he owns either $t_0 = 1$ units of the good or $t_0 = 3$ units of the good (formally, $A = \{1, 3\}$, and the seller's payoff is $v_0 = -\infty$ if she hands out more than what she has). In this example, the full-information-optimal allocation rule consists of allocating the entire available amount of the good, $a \in \{1, 3\}$, to the buyer, and the payment from the buyer to the seller is equal to $\min\{a, 2\}$. In the ex-ante optimal allocation rule, the buyer obtains the entire available amount of the good for a payment of 2 units of money.

From Maskin and Tirole's (1990) analysis it is clear that the techniques used in this paper do not work beyond the class of quasi-linear environments. Maskin and Tirole present a class of environments with private values where (*) is violated; for generic *non-quasi-linear* payoff functions the full-information optimal allocation rule is *not* a perfect Bayesian equilibrium outcome of a suitably defined informed-principal game.

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[28]Fleckinger (2007) provides another class of environments where the same conclusion holds. He considers quasi-linear environments where the agent's payoff has a fixed type-dependent term; his environments are as in Maskin and Tirole (1990), except for one assumption (see footnote 1 in Fleckinger (2007)).

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