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Simple Regression Based Tests for Spatial Dependence

by

Benjamin Born and Jörg Breitung

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Kaiserstrasse 1
D-53113 Bonn

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Simple Regression Based Tests for Spatial Dependence*

Benjamin Born
BGSE

Jörg Breitung
University of Bonn

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Abstract

We propose two simple diagnostic tests for spatial error autocorrelation and spatial lag dependence. The idea is to reformulate the testing problem such that the test statistics are asymptotically equivalent to the familiar LM test statistics. Specifically, our version of the test is based on a simple auxiliary regression and an ordinary regression t-statistic can be used to test for spatial autocorrelation and lag dependence. We also propose a variant of the test that is robust to heteroskedasticity. This approach gives practitioners an easy to implement and robust alternative to existing tests. Monte Carlo studies show that our variants of the spatial LM tests possess comparable size and power properties even in small samples.

1 Introduction

Recent years have seen an increasing availability of regional datasets leading to a growing awareness of spatial dependence (see Anselin 2007), an issue that can render ordinary least squares (OLS) estimation and inference inefficient or even biased and inconsistent (see Krämer and Donninger 1987, Anselin 1988b, Krämer 2003). Arguably the most commonly used test for spatial dependence is Moran's I (see Moran 1948, Cliff and Ord 1972, Cliff and Ord 1981), which is based on regression residuals and which has been shown to be best locally invariant by King (1981). In the maximum likelihood framework, Lagrange Multiplier test statistics were proposed by Burridge (1980) against a spatial error alternative and Anselin (1988a) against a spatial lag alternative.

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We show that after a minor reformulation of the model, we can test for spatial dependence by regressing the OLS residuals on their spatial lags and then testing the significance of the spatial coefficient by an asymptotic t-test. Our approach allows us to formulate the LM tests as NR^2 expressions based on auxiliary regressions, something that cannot readily be done with the standard formulation of the LM statistics (see Anselin 2001). This provides us with an easily implementable test that can be generalized straightforwardly to accommodate heteroskedastic and non-normal disturbances. In an alternative approach, Baltagi and Li (2001) use Davidson and MacKinnon's (1984, 1988) double length artificial regression approach to test for spatial error and spatial lag dependence but their approach is computationally more demanding and not robust to heteroskedasticity.

Monte Carlo simulations demonstrate that under standard assumptions our version of the test performs similarly to Moran's I and the LM test. However, if the errors are heteroskedastic the latter tests suffer from size distortions whereas the regression based test (using White's (1980) estimator of the standard errors) turns out to be robust against heteroskedastic errors processes. We can also confirm the results from other simulation experiments (e.g. Anselin and Florax 1995) showing that in small samples Moran's I is more powerful than the LM test. To improve the power of the regression test, we suggest a modification yielding a test that approaches the power of Moran's statistic.

The remainder of the paper is organized as follows. Section 2 discusses a t-test against a spatial error alternative and section 3 focuses on tests against a spatial lag alternative. Size and power of these tests are compared to Moran's I and LM tests via Monte Carlo simulations in section 4. Section 5 concludes.

2 Testing against spatial error alternatives

We consider the linear spatial first order autoregressive model with first order autoregressive disturbances (see e.g. Anselin 1988b), which is given by

$$\begin{aligned} y &= \phi W_1 y + X\beta + u \\ u &= \rho W_2 u + \varepsilon \ , \end{aligned} \tag{2.1}$$

where y is an $N \times 1$ vector of observations on a dependent variable, X is an $N \times k$ matrix of exogenous regressors, β is the associated $k \times 1$ parameter vector and ϕ and ρ are spatial autoregressive parameters with $|\rho| < 1$ and $|\phi| < 1$.

Following Kelejian and Prucha (1999, 2001), we make the following assumptions concerning model (2.1):

Assumption 1. *(i) The errors $\varepsilon_1, \ldots, \varepsilon_N$ are i.i.d. with zero mean, $E(\varepsilon_i^2) = \sigma^2$, and $E(|\varepsilon_i|^{2+\delta}) < \infty$ for some $\delta > 0$. (ii) The spatial weight matrices W_1 and W_2 are $N \times N$ matrices of known constants. The elements on the main diagonal of the matrices are zero and*

the matrices $(I - \rho W_2)$ and $(I - \phi W_1)$ are nonsingular for all $|\rho| < 1$ and $|\phi| < 1$. The row and column sums of the matrices W_1, W_2 are bounded uniformly in absolute value as $N \to \infty$.
(iii) The matrix X has full column rank and is independent of u.

The spatial error model is obtained by setting $\phi = 0$, yielding

$$\begin{aligned} y &= X\beta + u \\ u &= (I_N - \rho W)^{-1}\varepsilon \ , \end{aligned} \tag{2.2}$$

where W_2 is replaced by W to simplify the notation.

Moran's I-statistic is defined as

$$I = \frac{\widehat{u}' W \widehat{u}}{\widehat{u}'\widehat{u}} \ , \tag{2.3}$$

where $\widehat{u} = y - X\widehat{\beta}$ is the vector of OLS residuals. Under the null hypothesis, the standardized version $(I - \mu_I)/\sigma_I$ is standard normally distributed, where

$$\begin{aligned} \mu_I &= \mathrm{tr}(MW)/(N-k) \\ \sigma_I^2 &= [\mathrm{tr}(MWMW') + \mathrm{tr}(MW)^2 + (\mathrm{tr}(MW))^2]/d - \mu_I^2, \end{aligned}$$

$M = I - X(X'X)^{-1}X'$, and $d = (n-k)(n-k+2)$ (see Cliff and Ord (1972, 1981)). For $N \to \infty$, we have

$$\mu_I \to 0 \tag{2.4}$$

$$N^{-1}\sigma_I^2 \to \lim_{N\to\infty} N^{-1}\,\mathrm{tr}(W^2 + W'W) \ . \tag{2.5}$$

Burridge (1980) showed that the LM statistic results as

$$LM_\rho = \frac{(\widehat{u}' W \widehat{u})^2}{\widehat{\sigma}^4\,\mathrm{tr}(W^2 + W'W)} \ , \tag{2.6}$$

where $\widehat{\sigma}^2 = \widehat{u}'\widehat{u}/N$. Using (2.4) and (2.5), it is easy to see that the square of Moran's I and the LM statistic are asymptotically equivalent.

To motivate the regression version of the test, assume that β is known and consider the t-statistic for the null hypothesis $\rho = 0$ in the regression $u = \rho W u + \varepsilon$ which can be written as

$$t_\rho = \frac{u'Wu}{\widehat{\sigma}_\varepsilon \sqrt{u'W'Wu}} \ , \tag{2.7}$$

where $\widehat{\sigma}_\varepsilon^2$ is the usual variance estimator of ε. It is easy to see that under the null hypothesis, t_ρ is not asymptotically standard normally distributed. Defining $z = Wu$, for the numerator of t_ρ we obtain

$$
\begin{aligned}
u'z &= \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} 0 & w_{1,2} & \cdots & w_{1,n} \\ w_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & w_{n-1,n} \\ w_{n,1} & \cdots & w_{n,n-1} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \\
&= \sum_{i=1}^{N} \sum_{j \neq i} w_{ij} u_i u_j \\
&= \sum_{i=2}^{N} \sum_{j=1}^{i-1} (w_{ij} + w_{ji}) u_i u_j \\
&= \sum_{i=2}^{N} u_i \xi_i \ ,
\end{aligned}
$$

where

$$\xi_i = \sum_{j=1}^{i-1} (w_{ij} + w_{ji}) u_j \ .$$

Defining $\xi = (0, \xi_2, \ldots, \xi_N)'$, we can re-write the numerator as

$$u'z = u'(D_1 + D_2)u = u'\xi \ ,$$

where D_1 and D_2 are lower triangular matrices such that $W = D_1 + D_2'$ and $\xi = (D_1 + D_2)u$. Note that $u'(D_1 + D_2)u = u'(D_1 + D_2')u = u'z$. However, there is an important difference between the two formulations of the numerator. Whereas ξ_i is associated with an increasing σ-field generated by $\{u_1, \ldots, u_{i-1}\}$, this is not the case for $z_i = \sum_{j \neq i} w_{ij} u_j$, as this variable depends on $\{u_j | j \neq i\}$. This has important consequences for the variance. Specifically, under the null hypothesis we have

$$\mathrm{Var}(u'\xi) = \sigma^2 E(\xi'\xi) \qquad \text{but} \qquad \mathrm{Var}(u'z) \neq \sigma^2 E(z'z).$$

If W is symmetric, it is not difficult to show that $\mathrm{Var}(u'z) = 2\sigma^2 E(z'z)$. The factor 2 results from the fact that, due to the symmetric nature of the sum, the products $u_i u_j$ occur two times for each combination of i and j. We therefore suggest to use ξ instead of $z = Wu$ as the regressor in the test regression, where D_1 results from W by setting all elements above the main diagonal equal to zero. Analogously, D_2' is obtained from setting the elements below the main diagonal equal to zero.

If β is unknown, the errors u are replaced by $\widehat{u} = y - X\widehat{\beta}$ and the regression test for spatial error correlation is the t-statistic for the null hypothesis $\rho = 0$ in the regression

$$\widehat{u} = \rho\widehat{\xi} + \varepsilon \ . \tag{2.8}$$

That is,

$$\widetilde{t}_\rho = \frac{\widehat{u}'\widehat{\xi}}{\widehat{\sigma}\sqrt{\widehat{\xi}'\widehat{\xi}}} \ , \tag{2.9}$$

where $\widehat{\xi} = (D_1 + D_2)\widehat{u}$ and $\widehat{\sigma}^2$ is the usual estimator for the variance of the errors ε.

The following proposition considers the asymptotic properties of the test statistic.

Proposition 1. *(i) Under the null hypothesis $\rho = 0$ and $N \to \infty$, we have*

$$\widetilde{t}_\rho \xrightarrow{d} \mathcal{N}(0,1). \tag{2.10}$$

(ii) The regression test is asymptotically equivalent to Moran's I and the LM statistic in the sense that under the null hypothesis $\widetilde{t}_\rho - I \xrightarrow{p} 0$ and $\widetilde{t}_\rho^{\,2} - LM \xrightarrow{p} 0$.

A difference between the regression test and the LM test is that the latter estimates the variance of the numerator by imposing the null hypothesis. To simplify the discussion assume that β is known. Under the null hypothesis,

$$\text{Var}(u'\xi) = \sigma^2 E(\xi'\xi) = \sigma^4 \text{tr}[(D_1 + D_2)'(D_1 + D_2)] = \sigma^4 \text{tr}(W^2 + W'W) \ .$$

Under the alternative, we have

$$E(\xi'\xi) = \sigma^2 \text{tr}[(I_N - \rho W')^{-1}(D_1 + D_2)'(D_1 + D_2)(I_N - \rho W)^{-1}] \ .$$

Using the expansion $(I_N - \rho W')^{-1} = I_N + \rho W + (\rho W)^2 + (\rho W)^3 + \cdots$, it becomes clear that under the alternative $E(\xi'\xi)$ is larger than $\sigma^2\text{tr}(W^2 + W'W)$ which is used for the LM statistic. It follows that under the alternative the regression statistic $\widetilde{t}_\rho$ is usually smaller (in absolute value) than the LM statistic and, therefore, the power of the regression test tends to be smaller than the power of the LM statistic. This negative effect on the power of the test can be avoided by replacing $\widehat{u}$ in the denominator of $\widetilde{t}_\rho$ by the residual $\widehat{e} = \widehat{u} - \hat{\rho}\widehat{\xi}$, where $\hat{\rho}$ denotes the OLS estimator from a regression of $\widehat{u}$ on $\widehat{\xi}$. Thus, the modified regression statistic results as

$$\widetilde{t}_\rho^* = \frac{\widehat{u}'\widehat{\xi}}{\widehat{\sigma}\sqrt{\widehat{e}'(D_1 + D_2)'(D_1 + D_2)\widehat{e}}} \ . \tag{2.11}$$

Our Monte Carlos simulations presented in Section 4 suggest that this modification indeed yields a more powerful test statistic.

An important advantage of the regression test is that it can be made robust against heteroskedasticity by employing White's (1980) heteroskedasticity consistent covariance estimator. This yields the heteroskedasticity robust test statistic

$$\widetilde{t}_\rho = \frac{\widehat{u}'\widehat{\xi}}{\sqrt{\sum_{i=1}^{N} \widehat{u}_i^2 \widehat{\xi}_i^2}} \ . \tag{2.12}$$

Note that we have imposed the null hypothesis $\rho = 0$ in White's variance estimator. An alternative is to replace the residuals $\widehat{u}_i$ in the denominator by the OLS residuals $\widehat{e}$ of the auxiliary regression $\widehat{u} = \rho\widehat{\xi} + e$. However, Monte Carlo simulations suggest that the former estimators yields superior size properties of the test.

3 Testing against spatial lag alternatives

Setting $\rho = 0$, the linear spatial autoregressive model (2.1) with first order autoregressive disturbances becomes the spatial lag model

$$y = \phi W y + X\beta + \varepsilon \ , \tag{3.1}$$

where again we suppress the index for the weight matrix W_1. Anselin (1988a) derives the LM test statistic for the null hypothesis $\phi = 0$. The one-sided version of the test statistic results as

$$LM_\phi = \frac{y'MWy}{\sqrt{\widehat{\sigma}^4 \mathrm{tr}(W^2 + W'W) + \widehat{\sigma}^2 \widehat{\beta}' X'W'MWX\widehat{\beta}}} \ , \tag{3.2}$$

where $\widehat{\beta}$ is the OLS estimator from a regression of y on X and $\widehat{\sigma}^2$ is the usual variance estimator of the residuals.

The least squares estimator of ϕ from (3.1) is given by

$$\widehat{\phi} = \frac{y'MWy}{y'W'MWy} \ . \tag{3.3}$$

As in the case of the test for spatial autocorrelation, the numerator of this estimator is identical to the numerator of the LM statistic. This suggests that a regression test can be constructed that is asymptotically equivalent to the LM statistic (3.2).

To derive this estimator, we employ the same technique as in the previous section. First note that the numerator of the LM statistic can be re-written as

$$y'MWy = \widehat{u}'W\widehat{u} + \widehat{u}'W\widehat{y} \ ,$$

where $\widehat{y} = X\widehat{\beta}$. Using $W = D_1 + D_2'$ and $\widehat{u}'(D_1 + D_2')\widehat{u} = \widehat{u}'(D_1 + D_2)\widehat{u}$, we obtain

$$y'MWy = \widehat{u}'\widehat{\xi}^* \ ,$$

where

$$\widehat{\xi}^* = (D_1 + D_2)\widehat{u} + MW\widehat{y} \ .$$

Note that we do not need to decompose W in the last expression of this equation as X is assumed to be exogenous.[1] In the proof of Proposition 2, it is shown that the asymptotic properties are not affected by using $\widehat{y} = X\widehat{\beta}$ instead of $X\beta$.

The regression test for a spatial lag results as the ordinary t-statistic for the hypothesis $\phi = 0$ in the regression

$$\widehat{u} = \phi\widehat{\xi}^* + \eta \ ,$$

yielding the test statistic

$$\widetilde{t}_\phi = \frac{\widehat{u}'\widehat{\xi}^*}{\widehat{\sigma}\sqrt{\widehat{\xi}^{*\prime}\widehat{\xi}^*}} \ . \tag{3.4}$$

In the following proposition the limiting distribution of the test statistic is presented.

Proposition 2. *Assume that y can be represented as in* (3.1). *Under $H_0 : \phi = 0$ and Assumption 1 it follows that*

$$\widetilde{t}_\phi \ \stackrel{d}{\longrightarrow} \mathcal{N}(0,1)$$

and $\widetilde{t}_\phi - LM_\phi \stackrel{p}{\longrightarrow} 0$.

4 Monte Carlo Simulations

In our Monte Carlo study the data are generated according to the spatial error model (2.2) and the spatial lag specification (3.1). The regressor matrix, X, contains two regressors x_1 and x_2 with corresponding parameters β_1 and β_2. In both models, x_1 is a constant, the elements of x_2 are drawn independently from a standard normal distribution and arranged in ascending order, and $\beta = (1,1)'$. The disturbance term, ε, is generated as a vector of normally distributed random variables with $E(\varepsilon\varepsilon') = I$. We use a "3 ahead and 3 behind" spatial weight matrix in our simulations. In this design, the i-th row of the weight matrix, $3 < i < N-3$, has nonzero elements in positions $i-3, i-2, i-1, i+1, i+2$, and $i+3$, directly relating each element of the matrix to the three immediate neighbors ahead and behind. Adjusting the first and last three rows appropriately creates a circular world (see e.g. Kelejian and Prucha 1999, Kapoor, Kelejian and Prucha 2007). Following common practice in empirical applications, we row normalize the spatial weight matrix, yielding nonzero entries of 1/6 in this weight matrix design. In each experiment, we use 1000 replications.

Kelejian and Robinson (2004) show that Moran's I and the LM tests remain valid under heteroskedasticity as long the heteroskedasticity is not itself spatially correlated but, as they

[1] Note further that we have introduced the matrix M in the last term. Due to the idempotency of M this matrix does not affect the product $y'MWy$. However, the matrix M affects the denominator of the test statistic and is required to derive the results presented in Proposition 2.

	LM Test	Regr. Test	Mod. Regr. Test	Moran's I
$N = 50$	0.032	0.065	0.062	0.042
$N = 100$	0.033	0.059	0.046	0.043
$N = 150$	0.040	0.048	0.042	0.030
$N = 200$	0.043	0.051	0.041	0.032
$N = 300$	0.038	0.053	0.043	0.033
$N = 500$	0.041	0.041	0.045	0.033
$N = 1000$	0.044	0.045	0.045	0.032

Table 1: Errortest: Size under homoskedasticity

	LM Test	Regr. Test
$N = 50$	0.046	0.074
$N = 100$	0.044	0.056
$N = 150$	0.040	0.047
$N = 200$	0.038	0.046
$N = 300$	0.042	0.047
$N = 500$	0.038	0.038
$N = 1000$	0.047	0.042

Table 2: Lagtest: Size under homoskedasticity

argue, it is reasonable to assume that the heteroskedasticity possesses a spatial pattern. We therefore introduce a disturbance $\psi_i = \varepsilon_i x_{2i}$ with a "medium" extent of heteroskedasticity (see Kelejian and Robinson 1998), where the spatial correlation in the heteroskedasticity is induced by the sorted vector x_2.

Table 1 shows the simulation results for the spatial error specification with homoskedastic errors. All tests have approximately the correct size of 0.05 although the size of Moran's I is consistently below those of the other tests. The results of the spatial lag model with homoskedastic errors are similar. Both LM and regression test attain the correct size (see table 2).

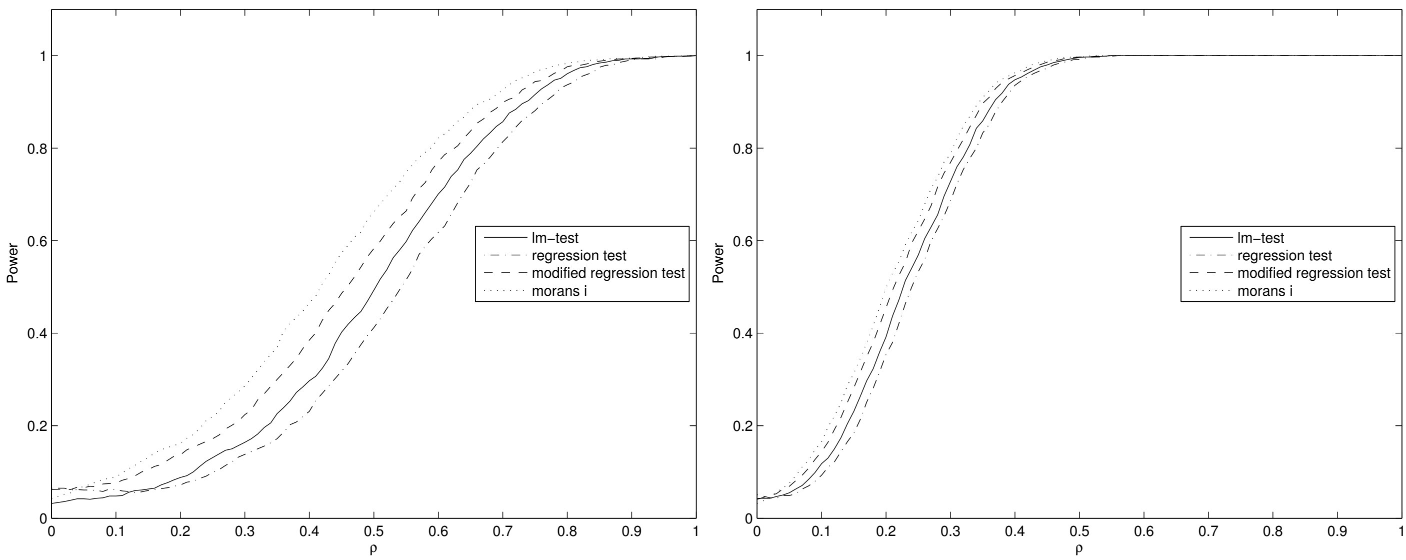
The results change considerably when we introduce heteroskedasticity. Moran's I and

	LM Test	Regr. Test	Moran's I
$N = 50$	0.088	0.027	0.094
$N = 100$	0.170	0.045	0.100
$N = 150$	0.172	0.038	0.111
$N = 200$	0.197	0.047	0.105
$N = 300$	0.220	0.046	0.129
$N = 500$	0.221	0.052	0.129
$N = 1000$	0.253	0.049	0.140

Table 3: Errortest: Size under heteroskedasticity

	LM Test	Regr. Test
$N = 50$	0.232	0.046
$N = 100$	0.313	0.032
$N = 150$	0.305	0.042
$N = 200$	0.280	0.046
$N = 300$	0.302	0.052
$N = 500$	0.331	0.050
$N = 1000$	0.327	0.048

Table 4: Lagtest: Size under heteroskedasticity



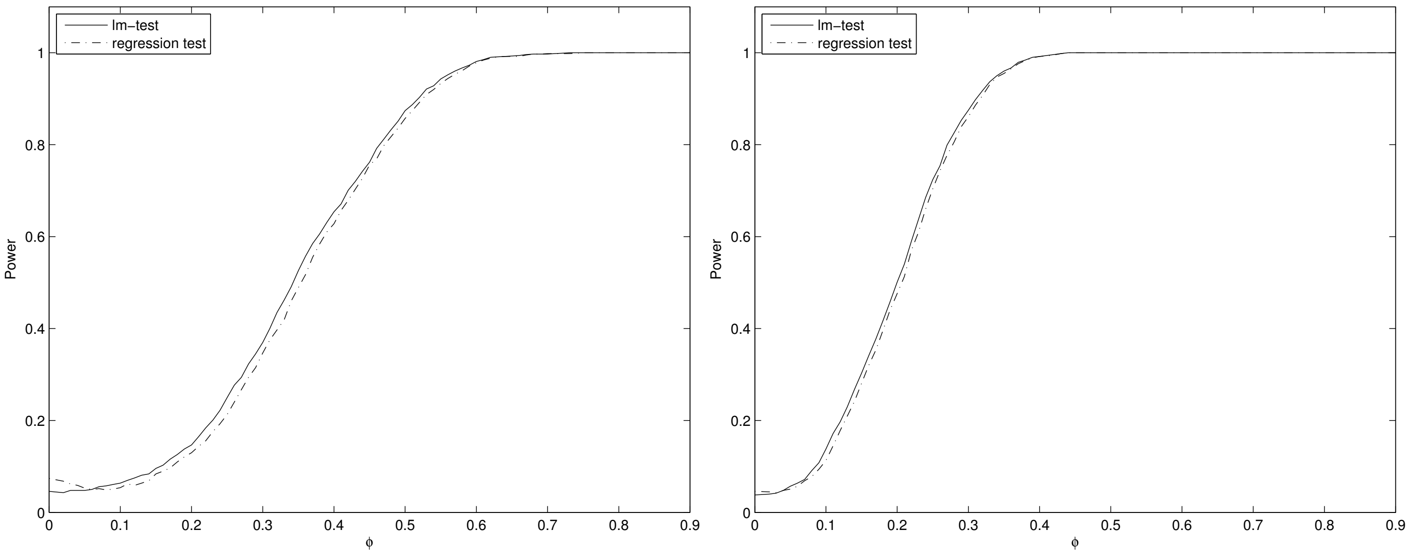
(a) $N = 50$ (b) $N = 200$

Figure 1: Errortest: power under homoskedasticity

the LM tests are now strongly oversized as can be seen in tables 3 and 4. Our regression test with the proposed White correction on the other hand has a correct size under spatial heteroskedasticity..

The two panels in figure 1 show that the regression test has slightly lower power than both the LM test and Moran's I, but with the modified statistic $\widetilde{t}^*_\rho$ suggested in section 2, the power approaches that of Moran's I. We also confirm the results from other simulation experiments (e.g. Anselin and Florax 1995) that Moran's I is more powerful than the LM test in small samples. With increasing sample size, the tests gain considerable power and their performances become very similar. Again, the results for the spatial lag case are comparable to those just reported. Here, the power of LM and regression test are very similar even in samples as small as $N = 50$.

In Figures 3 and 4, we plot the power of the regression test under spatial heteroskedasticity.



(a) $N = 50$ (b) $N = 200$

Figure 2: Lagtest: power under homoskedasticity

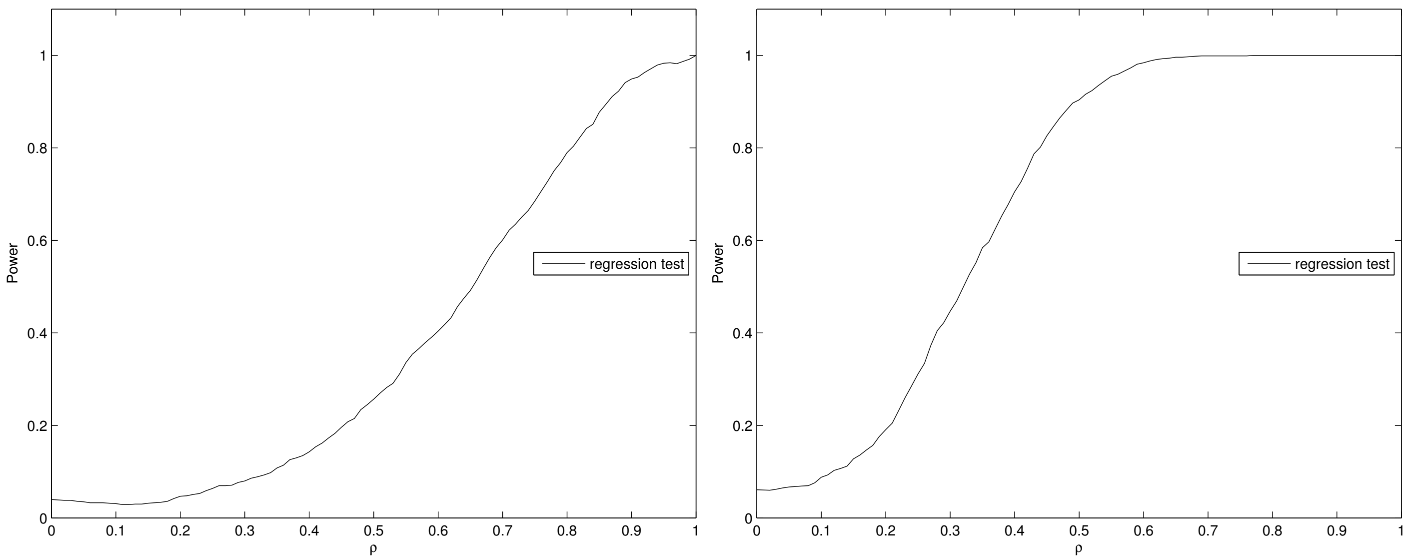
Due to the severe size distortions of the other tests ,we do not present their empirical power. The White correction leads to a loss of power but it is still reasonably powerful and, given that the other tests have large size distortions, it is clearly the best choice.

5 Conclusion

In this paper, we propose two new test procedures for spatial dependence. A reformulation of the model allows us to test against a spatial error or spatial lag specification by simply regressing the ordinary least squares residuals on their spatial lags and testing the significance of the spatial coefficient by an asymptotic t-test. We show that these tests are asymptotically equivalent to the existing Moran's I and LM tests, yet simpler to implement. Furthermore, using the approach of White (1980) it is straightforward to construct a test that is robust against heteroskedastic errors.

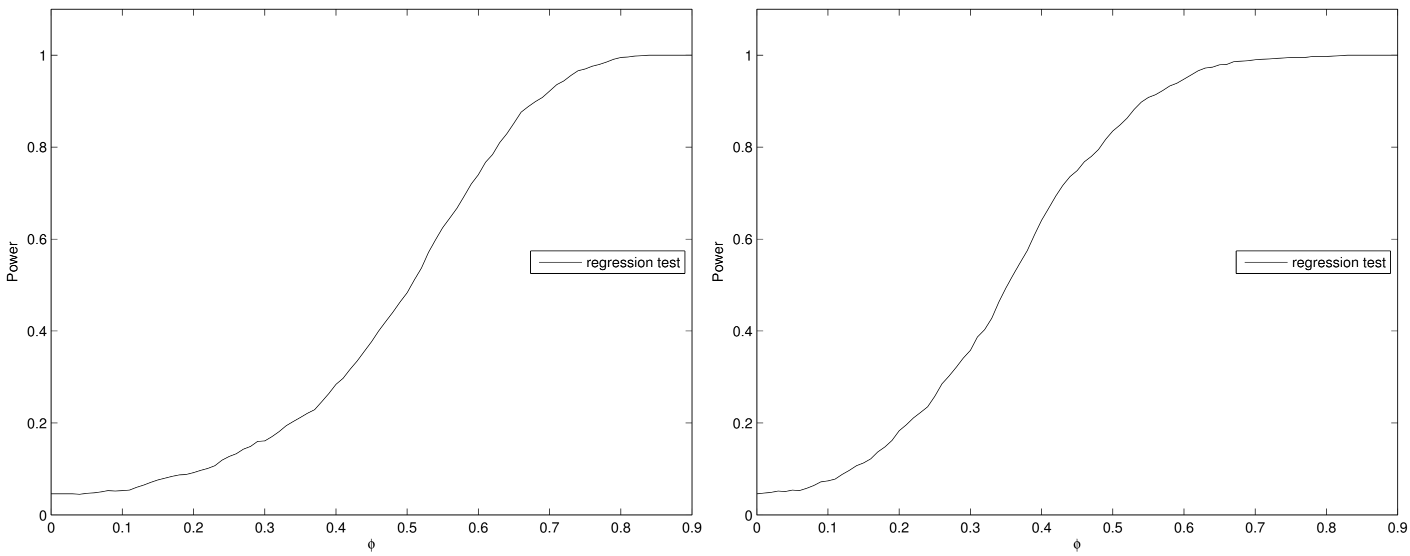
Monte Carlo simulations suggest that our new tests have good size properties, even under heteroskedasticity, where Moran's I and LM tests suffer from size distortions. A modification of the t-statistic is suggested that improves the size properties of the original test against the spatial error alternative. In medium and large samples, the performance of all tests becomes very similar.

Hence, we believe that the proposed tests will give researchers a robust and easily implementable tool for their applied work.



(a) $N = 50$ (b) $N = 200$

Figure 3: Errortest: power under heteroskedasticity



(a) $N = 50$ (b) $N = 200$

Figure 4: Lagtest: power under heteroskedasticity

A Proofs of Results

Proof of Proposition 1: (i) Under the null hypothesis, we have $\widehat{u} = \varepsilon - X(\widehat{\beta} - \beta)$. It follows that the numerator of $\widetilde{t}_\rho$ can be written as

$$\begin{aligned}\widehat{u}'\widehat{\xi} &= \widehat{u}'(D_1 + D_2)\widehat{u} \\ &= \varepsilon'(D_1 + D_2)\varepsilon - 2\varepsilon'(D_1 + D_2)'X(\widehat{\beta} - \beta) + (\widehat{\beta} - \beta)'X'(D_1 + D_2)X(\widehat{\beta} - \beta) \ . \qquad \text{(A.1)}\end{aligned}$$

Since ε_i is a martingale difference sequence with respect to ξ_i, for $i = 1, 2, \ldots$, where $E(\varepsilon_i|\xi_i) = 0$ and $\text{Var}(\xi) < \infty$ due to assumptions 1(i) and 1(ii), we obtain from the central limit theorem for martingale difference sequence (see White 2001, Corollary 5.26)

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{n} \varepsilon_i \xi_i \ \overset{d}{\to} \ \mathcal{N}(0, V_1) \ ,$$

where

$$\begin{aligned}V_1 &= \lim_{N\to\infty} E\left(N^{-1} \sum_{i=1}^{N} \varepsilon_i^2 \xi_i^2 \right) \\ &= \sigma^4 \lim_{N\to\infty} \frac{1}{N} \text{tr}[(D_1 + D_2)'(D_1 + D_2)] \ .\end{aligned}$$

Since D_1 and D_2 are lower triangular matrices with zeros on the leading diagonal, we have $D_s D_t = 0$ for $s, t \in \{1, 2\}$. Thus,

$$\begin{aligned}\text{tr}(W^2 + W'W) &= \text{tr}[(D_1 + D_2')(D_1 + D_2') + (D_1' + D_2)(D_1 + D_2')] \\ &= \text{tr}(D_1^2 + D_1 D_2' + D_2' D_1 + D_2' D_2' \\ &\quad + D_1' D_1 + D_2 D_1 + D_1' D_2' + D_2 D_2') \\ &= \text{tr}(D_1' D_1) + \text{tr}(D_2' D_2) + 2\text{tr}(D_2' D_1) \\ &= \text{tr}[(D_1 + D_2)'(D_1 + D_2)] \ . \qquad \text{(A.2)}\end{aligned}$$

From Assumption 1 (ii), it follows that $0 < N^{-1}\text{tr}(W'W) < \infty$ and $0 < N^{-1}\text{tr}(W^2) < \infty$ for all N and, therefore, $0 < V_1 < \infty$.

Since, under Assumption 1, $\widehat{\beta} - \beta$ is $O_p(N^{-1/2})$ and X is independent of u, we obtain for the other two terms in equation (A.1)

$$\begin{aligned}\varepsilon'(D_1 + D_2)'X(\widehat{\beta} - \beta) &= O_p(N^{1/2})O_p(N^{-1/2}) = O_p(1) \\ (\widehat{\beta} - \beta)'X'(D_1 + D_2)X(\widehat{\beta} - \beta) &= O_p(N^{-1/2})O_p(N)O_p(N^{-1/2}) = O_p(1)\end{aligned}$$

and it follows that

$$\frac{1}{\sqrt{N}} \widehat{u}'\widehat{\xi} \ \overset{d}{\longrightarrow} \mathcal{N}(0, V_1) \ .$$

To derive the asymptotic properties of the denominator, we first consider

$$\widehat{\xi}'\widehat{\xi} = \xi'\xi - 2\xi'(D_1 + D_2)X(\widehat{\beta} - \beta) + (\widehat{\beta} - \beta)'X'(D_1 + D_2)'(D_1 + D_2)X(\widehat{\beta} - \beta) \ .$$

Using the weak law of large numbers, we obtain

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N}\xi_i^2 \stackrel{p}{\longrightarrow} & \lim_{N\to\infty} E\left(\frac{1}{N}\sum_{i=1}^{N}\xi_i^2\right) \\ & = \sigma^2 \lim_{N\to\infty} N^{-1}\text{tr}[(D_1 + D_2)'(D_1 + D_2)] = \frac{1}{\sigma^2}V_1 \ . \end{aligned}$$

Furthermore,

$$\begin{aligned} 2\xi'(D_1 + D_2)X(\widehat{\beta} - \beta) &= O_p(1) \\ (\widehat{\beta} - \beta)'X'(D_1 + D_2)'(D_1 + D_2)X(\widehat{\beta} - \beta) &= O_p(1) \end{aligned}$$

and, hence,

$$\frac{1}{N}\widehat{\xi}'\widehat{\xi} \stackrel{p}{\longrightarrow} \frac{1}{\sigma^2}V_1 \ .$$

In a similar manner it can be shown that $\widehat{\sigma}^2 \stackrel{p}{\longrightarrow} \sigma^2$. From these results it follows that $\widetilde{t}_\rho$ has a standard normal limiting distribution.

(ii) Following Kelejian and Prucha (2001), Moran's I statistic can be written as

$$I = N^{-1/2}\widehat{u}'W\widehat{u}/\sqrt{\widehat{V}_1} \ ,$$

where $\widehat{V}_1$ is a consistent estimator of V_1. In particular,

$$\widehat{V}_1 = \frac{1}{N}\widehat{\sigma}^4 \,\text{tr}(W^2 + W'W) \ ,$$

in which case the LM statistic is the square of Moran's I. Note that the numerators of Moran's I and $\widetilde{t}_\rho$ are identical. The only difference is the denominator. However, since $\widehat{V}_1 \stackrel{p}{\to} V_1$ it follows that $\widetilde{t}_\rho - I \stackrel{p}{\longrightarrow} 0$ as $N \to \infty$. Similarly, $\widetilde{t}_\rho^{\,2} - LM \stackrel{p}{\longrightarrow} 0$. □

Proof of Proposition 2: As shown in the proof of Proposition 1, we have

$$\frac{1}{\sqrt{N}}\widehat{u}'W\widehat{u} = \frac{1}{\sqrt{N}}u'Wu + o_p(1) \stackrel{d}{\longrightarrow} \mathcal{N}(0, V_1) \ .$$

Furthermore, we obtain

$$\widehat{u}'MW\widehat{y} = \widehat{u}'MWX\beta + \widehat{u}'MWX(\widehat{\beta} - \beta) \ . \qquad \text{(A.3)}$$

Using

$$
\begin{aligned}
X'W'M\widehat{u} &= X'W'Mu \\
&= X'W'u - X'W'X(X'X)^{-1}X'u \\
&= O_p(N^{1/2}) - O_p(N)O_p(N^{-1})O_p(N^{1/2}) \\
&= O_p(N^{1/2})
\end{aligned}
$$

and $\widehat{\beta} - \beta = O_p(N^{-1/2})$, we obtain

$$
\frac{1}{\sqrt{N}}\,\widehat{u}'MW\widehat{y} = \frac{1}{\sqrt{N}}\,u'WX\beta + O_p(N^{-1/2}) \ .
$$

Since X is exogenous, it follows that

$$
E\left(\frac{1}{\sqrt{N}}\widehat{u}'\widehat{\xi}^*\right) \to 0 \tag{A.4}
$$

$$
\mathrm{Var}\left(\frac{1}{\sqrt{N}}\widehat{u}'\widehat{\xi}^*\right) \to \sigma^2 \lim_{N\to\infty} E\left(N^{-1}\xi^{*'}\xi^*\right), \tag{A.5}
$$

where $\xi^* = (D_1 + D_2)u + MWX\beta$. Since

$$
\begin{aligned}
E(\xi^{*'}\xi^*) &= \sigma^2\mathrm{tr}[(D_1 + D_2)'(D_1 + D_2)] + \beta'X'W'MWX\beta \\
&= \sigma^2\mathrm{tr}(W^2 + W'W) + \beta'X'W'MWX\beta
\end{aligned}
$$

and $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$, it follows that

$$
\widetilde{t}_\phi = \frac{\widehat{u}'W\widehat{u} + \widehat{u}'W\widehat{y}}{\sqrt{\sigma^4\mathrm{tr}(W^2 + W'W) + \sigma^2\beta'X'W'MWX\beta}} + o_p(1)
$$

and, therefore, $\widetilde{t}_\phi$ is asymptotically equivalent to the LM_ϕ statistic and possesses a standard normal limiting distribution. □

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