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Gains from second-order approximations

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Abstract

The benefit from using second-order approximations to stochastic dynamic rational expectations models is explained. By example of the neoclassical growth model, this note assesses the accuracy of the obtained approximation. The implications for optimal policy are discussed.

Keywords: Second-order approximation; accuracy; optimal policy

JEL classification: E0; C63

1 Introduction

Various authors have pointed to shortcomings in using first order approximations to dynamic general equilibrium models. Kim and Kim (2003) have shown that such a method can lead to spurious welfare reversals, i.e. full risk sharing appears inferior to autarky in a simple two country model. This note assesses the accuracy of the second-order approximation to the policy function developed by Schmitt-Grohé and Uribe (2004). Furthermore, it suggests a welfare measure usable for policy rankings that can be easily constructed and takes account of the effect of variances on means.

2 Model representation and form of the solution

To fix notation, consider the generic representation for rational expectations models introduced by Schmitt-Grohé and Uribe (2004)

$$E_t\, f(\boldsymbol{y}_{t+1}, \boldsymbol{y}_t, \boldsymbol{x}_{t+1}, \boldsymbol{x}_t) = 0. \tag{1}$$

f is a known function describing the equilibrium conditions of the model economy, $\boldsymbol{y}_t$ is a vector of co-state variables and $\boldsymbol{x}_t$ a vector of state variables partitioned as $\boldsymbol{x}_t = [\boldsymbol{x}_{1,t}; \boldsymbol{x}_{2,t}]$. $\boldsymbol{x}_{1,t}$ is a vector of endogenous state variables and $\boldsymbol{x}_{2,t}$ a vector of state variables following an exogenous stochastic process

$$\boldsymbol{x}_{2,t+1} = \boldsymbol{L}\boldsymbol{x}_{2,t} + \tilde{\boldsymbol{N}}\sigma\boldsymbol{\epsilon}_t. \tag{2}$$

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$\boldsymbol{L}$ and $\tilde{\boldsymbol{N}}$ are known coefficient matrices, $\boldsymbol{\epsilon}_t$ is a vector of innovations with bounded support, independently and identically distributed with mean zero and covariance matrix I. σ is a parameter scaling the standard deviation of the innovations. The solution to the model described by (1) is of the form

$$\boldsymbol{y}_t = g(\boldsymbol{x}_t, \sigma), \tag{3}$$

$$\boldsymbol{x}_{t+1} = h(\boldsymbol{x}_t, \sigma) + \sigma \boldsymbol{N} \boldsymbol{\epsilon}_{t+1}, \qquad \text{with: } \boldsymbol{N} = \begin{bmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{N}} \end{bmatrix}. \tag{4}$$

Schmitt-Grohé and Uribe (2004) derive the second-order Taylor approximation to the policy functions $g(\cdot)$ and $h(\cdot)$ and provide MATLAB codes for the numerical implementation. The approximate model dynamics obtained from their second-order approximation can be compactly expressed as

$$\boldsymbol{y}_t = \boldsymbol{G}\boldsymbol{x}_t + \frac{1}{2}\boldsymbol{G}^*(\boldsymbol{x}_t \otimes \boldsymbol{x}_t) + \frac{1}{2}\boldsymbol{g}\sigma^2, \tag{5}$$

$$\boldsymbol{x}_{t+1} = \boldsymbol{H}\boldsymbol{x}_t + \frac{1}{2}\boldsymbol{H}^*(\boldsymbol{x}_t \otimes \boldsymbol{x}_t) + \frac{1}{2}\boldsymbol{h}\sigma^2 + \sigma \boldsymbol{N}\boldsymbol{\epsilon}_{t+1}. \tag{6}$$

Here, the vectors $\boldsymbol{y}_t$ and $\boldsymbol{x}_t$ denote deviation or log-deviation from the steady state. $\boldsymbol{G}$ and $\boldsymbol{H}$ are coefficient matrices representing the linear part of the Taylor approximation. The matrices $\boldsymbol{G}^*$ and $\boldsymbol{H}^*$ form the second-order part jointly with the vectors $\boldsymbol{g}$ and $\boldsymbol{h}$.

3 Accuracy

The neoclassical growth model is employed to assess the accuracy of the second-order approximation . The model consists of the following equilibrium conditions

$$c_t^{-\gamma} = \beta E_t c_{t+1}^{-\gamma}[\alpha A_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)], \tag{7}$$

$$c_t + k_{t+1} = A_t k_t^{\alpha} + (1-\delta)k_t, \tag{8}$$

$$\ln A_{t+1} = \rho \ln A_t + \sigma \epsilon_{t+1}. \tag{9}$$

3.1 den Haan - Marcet χ^2 test

The den Haan and Marcet (1994) test exploits that for an exact solution the prediction error, $u_t = \beta c_{t+1}(\alpha k_t^{\alpha-1}\theta_{t+1} + 1 - \delta) - c_t^{-\gamma}$, must be orthogonal to any function $\phi(\boldsymbol{x}_t)$ of state variables included in period t information set. The test is carried out by constructing a simulated time series of length T for the model's variables and computing

$$\boldsymbol{B}_T \equiv \frac{\Sigma_{t=1}^{T}(u_t \phi(\boldsymbol{x}_t))}{T}. \tag{10}$$

The choice of $\phi(\boldsymbol{x}_t)$ in this analysis is the vector valued function of a constant, current period state variables and two lags of the states. den Haan and Marcet (1994) show how to construct from $\boldsymbol{B}_T$ a test statistic that has χ^2_{qm} distribution, where q is the number of instruments and m the number of Euler equations. Note that the growth model is solved in log-deviations using the choice of parameters in den Haan and Marcet (1994), i.e. $\gamma = 0.5$ and 3.0, $\alpha = 0.33, \beta = 0.99, \delta = 0.025, \rho = 0.95, \sigma = 0.03$. The following table shows the percentages of 500 repetitions of the test statistics for sample size $T = 3,000$ and $T = 10,000$ falling into the upper or lower 5% percentile of the χ^2 distribution. A test statistic belonging to this region is evidence against the accuracy of the solution. The χ^2-test delivers ample

		$\gamma = 0.5$		$\gamma = 3$	
	T	lower 5%	upper 5%	lower 5%	upper 5%
1^{st} order	$3,000$	0.0%	52%	0.3%	13.6%
2^{nd} order	$3,000$	5.6%	6.4%	5.6%	4.2%
1^{st} order	$10,000$	0.0%	98%	0.2%	33.6%
2^{nd} order	$10,000$	5%	5.2%	4%	4.2%

Table 1: χ^2 accuracy test

evidence for inaccuracy of the log-linear solution method, too much probability mass is in the upper tail. This is very clear for $T = 10,000$. As the sample size increases, the null will be rejected more often for any approximate solution method. However, even for this large sample size, there is little evidence for inaccuracy of the second-order approximation.

A more comprehensive way of illustrating the test results is the percentile plot in Figure 1. This is a plot of the analytical χ^2 - c.d.f . against the simulated distribution function of the test statistic. Up to sampling variability, if the test statistic follows the χ^2 distribution, the points should lie on the diagonal, which is plotted as a reference. For the second-order approximation, this is roughly the case. In contrast, the test statistic from the first order approximation is clearly not compatible with the χ^2 distribution.

However, the relationship between orthogonal prediction errors and deviation of the approximate policy function from the unknown exact policy function has not yet been sharply characterized. Therefore it is instructive to consider Euler equation residuals, which is done in the next subsection.

3.2 Euler equation residuals

Let $\boldsymbol{x}_{t+1} = h^s(\boldsymbol{x}_t)$ denote the transition function for the state variables obtained under solution method s. For notational simplicity, the dependence of this function on σ and ϵ_{t+1} is suppressed. The residual arising from the Euler equation is

$$R^s(\boldsymbol{x}_t) = 1 - \frac{\left\{\beta \mathrm{E}_t \left[c\left(h^s(\boldsymbol{x}_t)\right)^{-\gamma} \left\{\alpha A_{t+1} k(\boldsymbol{x}_t)^{\alpha-1} + 1 - \delta\right\}\right]\right\}^{-\frac{1}{\gamma}}}{c(\boldsymbol{x}_t)}. \tag{11}$$

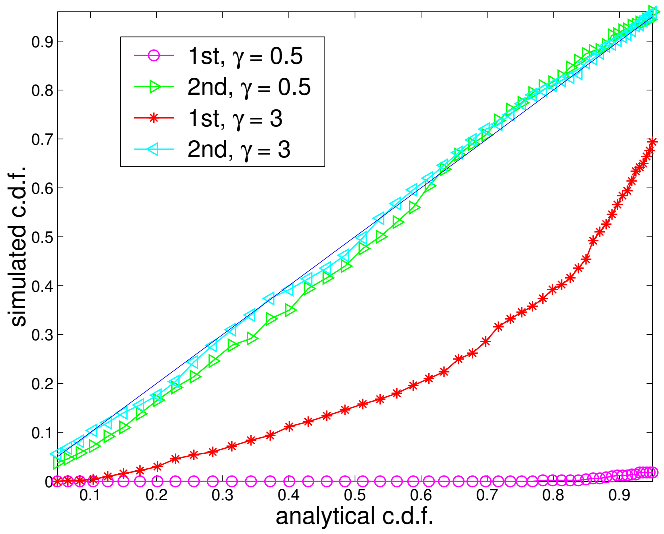

Figure 1: Percentile plot for T = 10,000

The Euler equation residual expresses the error from following the approximated policy rule as a fraction of current period consumption. Clearly, for an exact solution, the error is zero. Under certain conditions the approximation error of the policy function is of the same order of magnitude as the Euler equation residual as pointed out by Santos (2000).

Figures 2 and 3 plot the Euler equation residual as a function of the percentage deviation of capital and technology from the steady state for the calibration of the previous subsection with $\gamma = 0.5$. The Euler equation residuals for the linear solution method are of the order of magnitude of 10^{-3}, an error of $ 1 for every 1,000 dollars spent. However the error does not have mean zero, the residual function is positive everywhere on its domain.

The second-order approximation yields residuals which are of an order of magnitude smaller than those of the linear approximation. It also seems to center the residuals better around zero.

4 Welfare evaluation

Given the superior accuracy of the second-order approximation, it remains to be demonstrated in what way this matters for the economic analysis. This section shows that the effect of variances on means cannot be captured by a first order approximation and suggests a simple way to do undertake welfare ranking of alternative policies.

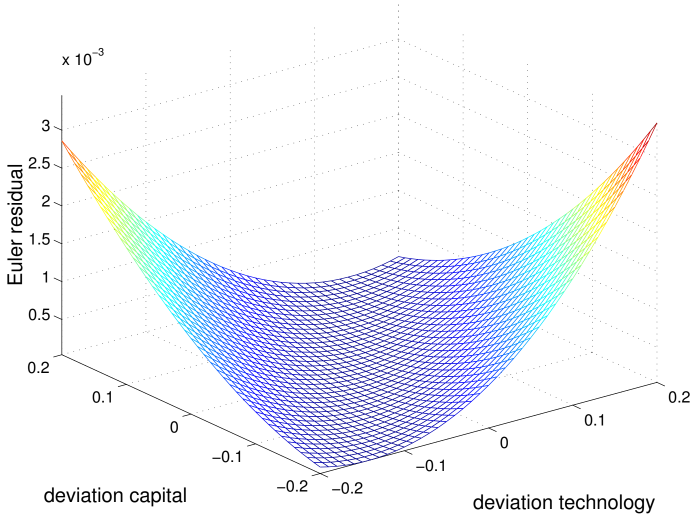

Figure 2: Euler equation error based on linear approximation

4.1 Unconditional welfare

A natural welfare measure for rankings of fiscal or monetary policy that can be easily constructed from the second-order approximation is the unconditional expectation of period utility.

The second-order approximation to an arbitrary utility function $u(\boldsymbol{y}_t)$ of co-states $\boldsymbol{y}_t$ is

$$u(\boldsymbol{y}_t) \approx u(\overline{\boldsymbol{y}}) + \nabla u(\overline{\boldsymbol{y}})\boldsymbol{y}_t + \frac{1}{2} vec\left(\nabla^2 u(\overline{\boldsymbol{y}})\right)' (\boldsymbol{y}_t \otimes \boldsymbol{y}_t), \tag{12}$$

such that upon taking expectations

$$E(u(\boldsymbol{y}_t)) \approx u(\overline{\boldsymbol{y}}) + \nabla u(\overline{\boldsymbol{y}})\boldsymbol{\mu_y} + \frac{1}{2} vec\left(\nabla^2 u(\overline{\boldsymbol{y}})\right)' vec(\boldsymbol{\Sigma_y} + \boldsymbol{\mu_y}\boldsymbol{\mu_y'}). \tag{13}$$

Here, $\boldsymbol{\mu_y}, \boldsymbol{\Sigma_y}$ denote unconditional mean and covariance matrix of $\boldsymbol{y}$, respectively. To construct first and second moments of the co-state variables assume covariance stationarity and take expectation of (5) and (6)

$$\boldsymbol{\mu_y} = \boldsymbol{G}\boldsymbol{\mu_x} + \frac{1}{2}\boldsymbol{G}^* vec(\boldsymbol{\Sigma_x} + \boldsymbol{\mu_x}\boldsymbol{\mu_x'}) + \frac{1}{2}\boldsymbol{g}\sigma^2, \tag{14}$$

$$\boldsymbol{\mu_x} = \boldsymbol{H}\boldsymbol{\mu_x} + \frac{1}{2}\boldsymbol{H}^* vec(\boldsymbol{\Sigma_x} + \boldsymbol{\mu_x}\boldsymbol{\mu_x'}) + \frac{1}{2}\boldsymbol{h}\sigma^2. \tag{15}$$

Note that while under the linear approximation unconditional means do not differ from the steady state values, the second-order approximation is able to capture

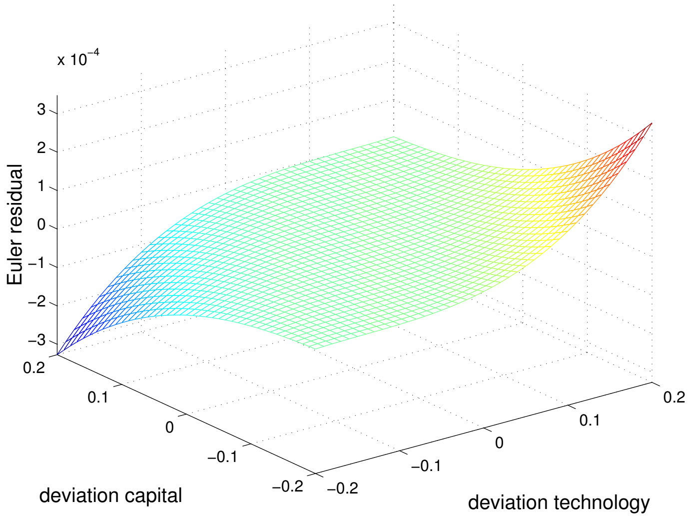

Figure 3: Euler equation error based on quadratic approximation

the effect of variances on means. Since variances can be computed accurately up to second-order from the linear part of the policy function, it is sufficient to approximate $vec(\boldsymbol{\Sigma_x} + \boldsymbol{\mu_x}\boldsymbol{\mu_x'}) \approx vec(\boldsymbol{\Sigma_x})$ and $vec(\boldsymbol{\Sigma_y} + \boldsymbol{\mu_y}\boldsymbol{\mu_y'}) \approx vec(\boldsymbol{\Sigma_y})$. It is then possible to construct these using the simple formulas

$$vec(\boldsymbol{\Sigma_y}) = (\boldsymbol{G} \otimes \boldsymbol{G}) vec(\boldsymbol{\Sigma_x}), \tag{16}$$

$$vec(\boldsymbol{\Sigma_x}) = \sigma^2 (\boldsymbol{I} - \boldsymbol{H} \otimes \boldsymbol{H})^{-1} (\boldsymbol{N} \otimes \boldsymbol{N}) vec(\boldsymbol{I}). \tag{17}$$

Given these approximations for the variances, the means can be computed from (14) and (15). The described welfare measure has the following compact representation, which can easily be verified by applying the rules of the partitioned inverse.

$$E(u(\boldsymbol{y}_t)) \approx u(\overline{\boldsymbol{y}}) + \left[\nabla u(\overline{\boldsymbol{y}}) \, , \, \frac{1}{2} vec\left(\nabla^2 u(\overline{\boldsymbol{y}})\right)' \right] \times \\ \left(\begin{bmatrix} \boldsymbol{G} & \frac{1}{2}\boldsymbol{G}^* \\ \boldsymbol{0} & \boldsymbol{G} \otimes \boldsymbol{G} \end{bmatrix} \left[\boldsymbol{I} - \begin{bmatrix} \boldsymbol{H} & \frac{1}{2}\boldsymbol{H}^* \\ \boldsymbol{0} & \boldsymbol{H} \otimes \boldsymbol{H} \end{bmatrix} \right]^{-1} \begin{bmatrix} \frac{1}{2}\boldsymbol{h} \\ \boldsymbol{N} \otimes \boldsymbol{N} vec(\boldsymbol{I}) \end{bmatrix} \sigma^2 + \begin{bmatrix} \frac{1}{2}\boldsymbol{g} \\ \boldsymbol{0} \end{bmatrix} \sigma^2 \right) \tag{18}$$

The task of computing optimal monetary or fiscal policy in dynamic general equilibrium models then amounts to numerically optimizing this welfare measure through choice of the coefficients in the policy rules.

If the second-order approximation to the welfare function can be re-written so as to involve quadratic terms only, then linear and quadratic approximations to the policy functions employed as suggested here will yield the same level of welfare.

This is the case for some models of optimal monetary policy, where welfare is equal to the weighted sum of the variances of inflation and the output gap. Up to second-order, there is no bias in welfare calculations based on linear policy rules in such a case.

4.2 Conditional welfare and transitional dynamics

The welfare measure employed so far is unconditional expectation of period utility. However this measure neglects the effects of transitional dynamics. The following paragraphs present a straightforward extension to the work of Kim et al. (2003) who develop a formula for conditional welfare. These authors assume that utility depends on state variables directly. Here, their approach is adapted to the case where utility depends on co-states and the second-order approximation is obtained using the code of Schmitt-Grohé and Uribe (2004).

In line with the approach used so far of evaluating second moments from the linear part of the policy function only, append to the system recursions for the variances of the states and co-states computed from the linear part of the policy function only

$$(\tilde{\boldsymbol{y}}_t \otimes \tilde{\boldsymbol{y}}_t) = (\boldsymbol{G} \otimes \boldsymbol{G})(\tilde{\boldsymbol{x}}_t \otimes \tilde{\boldsymbol{x}}_t), \tag{19}$$

$$(\tilde{\boldsymbol{x}}_t \otimes \tilde{\boldsymbol{x}}_t) = (\boldsymbol{H} \otimes \boldsymbol{H})(\tilde{\boldsymbol{x}}_{t-1} \otimes \tilde{\boldsymbol{x}}_{t-1}) + \sigma^2(\boldsymbol{N} \otimes \boldsymbol{N})(\boldsymbol{\epsilon}_t \otimes \boldsymbol{\epsilon}_t). \tag{20}$$

Rewrite the system in state space form using the above equations

$$\begin{bmatrix} \boldsymbol{y}_t \\ \tilde{\boldsymbol{y}}_t \otimes \tilde{\boldsymbol{y}}_t \end{bmatrix} = \boldsymbol{M_1} \begin{bmatrix} \boldsymbol{x}_t \\ \tilde{\boldsymbol{x}}_t \otimes \tilde{\boldsymbol{x}}_t \end{bmatrix} + \boldsymbol{K_1}, \tag{21}$$

$$\begin{bmatrix} \boldsymbol{x}_{t+1} \\ \tilde{\boldsymbol{x}}_{t+1} \otimes \tilde{\boldsymbol{x}}_{t+1} \end{bmatrix} = \boldsymbol{M_2} \begin{bmatrix} \boldsymbol{x}_t \\ \tilde{\boldsymbol{x}}_t \otimes \tilde{\boldsymbol{x}}_t \end{bmatrix} + \boldsymbol{K_2} + \boldsymbol{u}_{t+1}. \tag{22}$$

where

$$\boldsymbol{M_1} = \begin{bmatrix} \boldsymbol{G} & \frac{1}{2}\boldsymbol{G^*} \\ \boldsymbol{0} & \boldsymbol{G} \otimes \boldsymbol{G} \end{bmatrix}, \quad \boldsymbol{M_2} = \begin{bmatrix} \boldsymbol{H} & \frac{1}{2}\boldsymbol{H^*} \\ \boldsymbol{0} & \boldsymbol{H} \otimes \boldsymbol{H} \end{bmatrix},$$

$$\boldsymbol{K_1} = \begin{bmatrix} \frac{1}{2}\boldsymbol{g}\sigma^2 \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{K_2} = \begin{bmatrix} \frac{1}{2}\boldsymbol{h}\sigma^2 \\ \sigma^2(\boldsymbol{N} \otimes \boldsymbol{N})vec(\boldsymbol{I}) \end{bmatrix},$$

$$\boldsymbol{u}_t = \begin{bmatrix} \sigma \boldsymbol{N} \boldsymbol{\epsilon}_t \\ \sigma^2 \left(\boldsymbol{N} \otimes \boldsymbol{N} \left(\boldsymbol{\epsilon}_t \otimes \boldsymbol{\epsilon}_t - vec(\boldsymbol{I}) \right) \right) \end{bmatrix}.$$

Expected discounted lifetime utility conditional on an initial state vector with mean

$\boldsymbol{\mu_x}$ and covariance matrix $\boldsymbol{\Sigma_x}$ can then be expressed as

$$U(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \approx \frac{u\left(\overline{\boldsymbol{y}}\right)}{1-\beta} + E\left[\sum_{t=0}^{\infty} \beta^t \left\{ \nabla u\left(\overline{\boldsymbol{y}}\right) \boldsymbol{y}_t + \frac{1}{2} vec(\nabla^2 u\left(\overline{\boldsymbol{y}}\right))' \left(\tilde{\boldsymbol{y}}_t \otimes \tilde{\boldsymbol{y}}_t\right) \right\}\right]$$
$$= \frac{u\left(\overline{\boldsymbol{y}}\right)}{1-\beta} + \left[\nabla u(\overline{\boldsymbol{y}})\,,\, \frac{1}{2} vec\left(\nabla^2 u(\overline{\boldsymbol{y}})\right)'\right] \times$$
$$\left[\boldsymbol{M_1}\left[\boldsymbol{I} - \beta \boldsymbol{M_2}\right]^{-1} \left(\begin{bmatrix} \boldsymbol{\mu} \\ vec(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}') \end{bmatrix} + \frac{\beta}{1-\beta} \boldsymbol{K_2} \right) + \frac{\beta}{1-\beta} \boldsymbol{K_1} \right] \tag{23}$$

The difference to the case considered by Kim et al. (2003) manifests itself in the matrices $\boldsymbol{M_1}$ and $\boldsymbol{K_1}$.

5 Conclusion

This note has shown that the second-order approximation yields Euler equation residuals for the simple neoclassical growth model that are of an order of magnitude smaller than the residuals from the linear approximation. The den Haan and Marcet (1994) test confirms the superior accuracy of this solution method. For the purpose of second-order accurate welfare rankings of policy, a simple way to compute unconditional expectation of period utility is suggested. Such a measure is essential, whenever the welfare function cannot be re-written in terms of quadratic terms only.

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